# New Approaches to Visual Reasoning in Mathematics and Kantian Characterization of Mathematics

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Özge Ekin

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Erstgutachterin: Prof. Dr. Sybille Krämer

Zweitgutachter: Prof. Dr. Konrad Polthier

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**CHAPTER**

# Acknowledgment

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# Abstract

My aim in this thesis is to reveal the true nature of mathematical reasoning and to show the necessity of pictorial representations in mathematics. In order to support my thesis I argue that using only the formal language, without fully integrating visual reasoning in mathematical communication, mathematics operates insufficiently and it can never reach its full potential.

First of all, I provide a short historical approach to philosophy of mathematics as well as a summary of current state of visualization in mathematics. Then, I bring forth a complete picture of philosophy of mathematics and mathematical objects, which consistently allows visual reasoning, intuitions and constructions in mathematical communication. This interpretation makes it easy to see the pictorial representations as a natural part of mathematical reasoning. Moreover, I lay out the medium for visual reasoning in mathematics: Space. After providing several definitions of space I conclude that Kant's characterization is the finest one as a base for mathematical reasoning, in particular for visual reasoning. Furthermore, I bring out the method of visual reasoning: Synthetic *a priori*. In order to show the link between visualization and synthetic a priori I provide hitherto definitions of visualization and argue that these lead up to Kantian characterization of constructions and synthetic *a priori*. Then, I provide several interpretations to show that Kant's characterization

of mathematics has elements valuable for modern mathematics that modern logic cannot capture.

Finally, I offer three main arguments leading to the main thesis "pictorial representations/ diagrams are necessary parts of mathematical reasoning": (1) There are formal systems with diagrams. Hence, not all uses of diagrams are synthetic. (2) There are necessary non-formal uses of diagrams in mathematical proofs. (3) There is a strong link between visualization and Kantian philosophy of mathematics. As a result, I claim that the real capacity of mathematics is not in turning it into a machine language but integrating and appreciating the aesthetic part of it as well. I also indicate the necessity of having a theory of diagrammatics, which should be set up as a branch of mathematics.

# Zusammenfassung

Das Ziel dieser Dissertation ist es, die wahre Natur mathematischen Denkens offenzulegen und die Notwendigkeit bildlicher Repräsentationen in der Mathematik aufzuzeigen. Zur Begründung meiner These werde ich wie folgt argumentieren: Sofern sie allein formale Sprachen verwendet und es unterlässt, visuelles Denken vollständig in die mathematische Kommunikation zu integrieren, operiert Mathematik ungenügend und kann ihr volles Potential nicht entfalten.

Ich beginne mit einen kurzen Abriss zur Geschichte der Mathematikphilosophie, sowie einem Überblick zum gegenwärtigen Status von Visualisierungen in der Mathematik. Dann zeichne ich ein Gesamtbild der Philosophie der Mathematik und der mathematischen Objekte, die durchweg visuelles Denken, Anschauungen und Konstruktionen als zur mathematischen Kommunikation gehörend ansieht. Diese Interpretation macht auf einfache Weise einsichtig, inwiefern bildliche Repräsentationen einen natürlichen Teil mathematischen Denkens darstellen. Des Weiteren lege ich das Medium visuellen Denkens in der Mathematik dar: den Raum. Nach der Darstellung verschiedener Definitionen des Raums folgere ich, dass die Kantische Auffassung sich am besten als eine Grundlage mathematischen Denkens eignet, insbesondere visuellen Denkens. Im Weiteren erörtere ich die Methode visuellen Denkens: das synthetische Apriori. Um die Verbindung zwischen Visualisierung und synthetischem Apriori aufzuzeigen, gebe ich Definitionen von Visualisierung und argumentiere dafür, dass

diese zu Kants Charakteristik von Konstruktion und synthetischem Apriori hinführen. Dann gebe ich mehrere Interpretationen, die zeigen, dass Kants Mathematikauffassung Elemente enthält, die sich für die moderne Mathematik als wertvoll erweisen und zugleich von moderner Logik nicht erfasst werden können.

Schließlich führe ich drei zentrale Argumente an, die die Ausgangsthese dieser Dissertation stützen, die lautet: „bildliche Repräsentationen bzw. Diagramme bilden einen notwendigen Teil mathematischen Denkens“. (1) Es gibt formale Systeme auf diagrammatischer Basis. Daher sind nicht alle Verwendungen von Diagrammen synthetisch. (2) Es gibt notwendige nicht-formale Verwendungen von Diagrammen in mathematischen Beweisen. (3) Es gibt einen starken Zusammenhang zwischen Visualisierung und der kantischen Mathematikphilosophie. Als Resultat behaupte ich, dass die wirkliche Kapazität der Mathematik nicht ausgeschöpft ist, wenn sie in eine Maschinensprache überführt wird, sondern nur dann, wenn ihr ästhetischer Anteil ebenso einbezogen und gewürdigt wird. Ich weise ebenfalls auf die Notwendigkeit einer Theorie der Diagrammatik hin, die als ein Zweig der Mathematik etabliert werden sollte.

# CHAPTER I

# Introduction

"We are not very pleased when we are forced
to accept a mathematical truth by virtue
of a complicated chain of formal conclusions
and computation, which we traverse blindly,
link by link, feeling our way by touch.
We want first an overview of the aim
and of the road; we want to understand
the *idea* of the proof, the deeper context" [1]
-Hermann Weyl

## 1.1 Intuition, Diagram, Visual Reasoning in Mathematics

Since 19th century the general approach to mathematics has been to formalize and axiomatize arguments in mathematics in order to have strong foundations and utmost

[1]([142], p.453) Original version: "Wir geben uns nicht gerne damit zufrieden, einer mathematischen Wahrheit überführt zu werden durch eine komplizierte Verkettung formeller Schlüsse und Rechnungen, an der wir uns sozusagen blind von Glied zu Glied entlang tasten müssen.Wir möchten vorher Ziel und Weg überblicken können, wir möchten den inneren Grund der Gedankenführung, die Idee des Beweises, den tieferen Zusammenhang verstehen." ([141], p.348)

rigor in it. For that reason, intuition and, with it, the visual reasoning were excluded from mathematics. My main aim in this thesis is to introduce visual reasoning with an approach of 21st century practice and discuss the advantages of visual reasoning combined with formal sentential reasoning. This combined reasoning is also called heterogeneous reasoning and recently getting a lot of attention from mathematicians and philosophers of mathematics due to emerging technologies in representing visual reasoning in more accurate and less time consuming ways.

What I mean by visual reasoning in this thesis is not limited only to physically drawn or computer generated figures but includes also the so-called mental images - those representations which are seen as psychological tools and deemed not belonging to mathematics. I group these as internal diagrams and demonstrate the objectivity of both internal and external diagrams by using Kant's characterization of mathematics. Hence, I argue that Kantian characterization of mathematics is useful for formulating and understanding heterogeneous reasoning. That is to say, firstly, that methods and representations Kant uses are helpful for understanding visual or diagrammatical reasoning. Secondly, I believe that the way Kant describes and uses synthetic method can explain the validity of visual or heterogeneous reasoning. There are many different interpretations of Kant's characterization of mathematical reasoning[2]. In this thesis, I also outline these approaches and offer a fresh reading of Kantian characterization of "synthetic and *a priori*", which provides a consistent explanation to intuitions of mathematical objects in Kantian philosophy of mathematics.

Moreover, based on the introduced approach and recent practices in mathematics, I explain the use of visual and diagrammatical reasoning in mathematics in detail. Indeed, mathematicians in practice use diagrams either as helpful tools in proofs, or

[2]Some of the important works which will be also discussed in this work, contains but not limited to: "Kant on the Mathematical Method" [53], *Logic, Language-Games and Information* [52], *Introduction to Mathematical Philosophy* [113], *Studies in the Philosophy of Kant* [6], "Mathematical Intuition" [94]

as preferable means to formal sentential tools, or use them essentially as in Euclid's proofs. Thus, I argue that by integrating visual reasoning or intuitions in mathematical practice "legitimately", the nature of mathematical reasoning can be explained more efficiently. It is only in prescription that this visual or heterogeneous reasoning is not allowed in mathematics. This was affected by both counter intuitive findings in mathematics and also because of the vast developments in modern logic since 19th century.

I am not suggesting that one should use only visual reasoning in mathematics. This claim would be meaningless, since without formal sentential reasoning one cannot have the explanatory power and rigor in mathematics. My approach to prescriptive mathematical reasoning, by no means, underestimates the value of modern logic or formal sentential reasoning. However, insisting on the usage of *only* formal sentential reasoning in mathematics is meaningless as well, since communicating in mathematics or learning higher mathematics becomes much more difficult as Poincaré also points out:

> "Without intuition young minds could not make a beginning in the understanding of mathematics; they could not learn to love it and would see in it only a vain logomachy; above all, without intuition they would never become capable of applying mathematics."([104], p. 21)

With all above considerations, this work will demonstrate how pictorial representations play an essential role in mathematical communication and proofs. It will be revealed that, although Hilbert's methods in *Grundlagen der Geometrie* [50] and similar formalization methods that are developed extensively later on, can eliminate the physical drawings and constructions used in proofs, they do not dispose of the visualization part in mathematical proofs. Adopting only Hilbert's methods for geometrical proofs causes to report the mathematical process incompletely using only the symbolic, sentential and formal part. In this thesis, with the analysis of different types of proofs from geometry and mathematics, we will see that how this pictorial process intervenes and proceeds in mathematical methods. In conclusion, the importance of aesthetic/intuitive/ visual part of mathematics will be demonstrated for both context of discovery and, most significantly, in context of justification. With the development of new technologies, the visual communication came to a turning point. It is fast, its evolution is fast and it's getting more and more accurate. As a result, researchers who understand the value of visual communication are looking for every possible way to integrate pictorial representations in every branch of sciences, as mathematics was looking for new ways to integrate formal logic to its fullest in the 19th century.

However, many mathematicians and philosophers of mathematics, still, cannot use these representations in published mediums and there are still many who are reluctant to accept that dehumanization process of mathematics is outdated and should come to an end with this visual turn. This realization is only possible with the acceptance of the aesthetic part of mathematics in contrast to insisting on dehumanizing attempts to have a "stronger" system. If mathematics were built on finite structures and based only on machinery, iterative processes, how, then, the proofs of many theorems were not self evident since they were first proposed? Why do we not have only one science that is called logic but we have mathematics also?

Mathematics is considered to be the highest form of the sciences/practices that captures the structure of valid reasoning in order to apply formal logical structures by mathematical models to the empirical world. Until the end of 19th century, in addition to this view, it was also widely accepted that mathematics was the way to capture the truth about the universe, to penetrate the God's design of the universe. However, because of the foundational crisis in mathematics with the emergence of Gödel's Incompleteness Theorem (cf. [42]), this view lost many defenders due to the fact that one cannot capture the truth about a sufficiently complex formal system by using the rules (inference rules) and evident truths about the system (axioms), which, also indicates that **one cannot reduce the mind's process to finitely many steps with machinery process**. This last result has two more corollaries: Rationality will not capture all the truths of the universe and mind's process does not only consist of rational, iterative thinking even in sciences and mathematics.

In other words, according to this formal view, if universe is a mathematically designed system, then its rules has to be formally structured; but then one can never have all the truths about the system (completeness) if this system is a consistent mathematical structure. Hence, Gödel's Incompleteness Theorem affected not only the logical and axiomatic foundations of mathematics but also the belief that all the truths of the universe were available to the human kind if the highest form of rational reasoning is practiced to its end.

However, despite Gödel's discoveries, there were only few mathematicians and philosophers of mathematics ready to admit the importance of having other methods than logic for the practice and communication of mathematics. One of the deserters of the crowd was the most talented mathematicians of 20th century: Hermann Weyl (cf. [140]). He argued that mathematics can be free creative art, like music and language, and its originality and indispensable value comes exactly from its being an art, from its carrying an aesthetic value.

Now, between the two opposing views of mathematics, one defending that mathematical expressions and objects are ever non-changing entities that reveal us ever-non-changing structures in the world and universe, and the other defending that mathematics can be an art that reveals the world and universe in front of our eyes with a different point of view than other arts, which one explains adequately the structure of mathematics, mathematical objects and reasoning? Before giving a detailed answer to this question let us first briefly look at the historical reasons for formalization and axiomatization process in mathematics and discuss, shortly, whether the supporters for the mere logical foundations of mathematics had their correct reasons.

## 1.2 The History and the Critique of Logicism

When foundations of mathematics are considered, first tool comes to mind is logic. Logic has been used for formulating and improving the foundations of different areas of mathematics. Exclusion of geometrical intuition and turning to logical analysis of mathematical foundations has been at stake since late 19th century, especially with Frege's *Begriffsschrifft*[35] (1878) and counter intuitive findings in mathematics such as nowhere differentiable everywhere continuous functions (known as Weierstrass function) [139] (1872).

Logicist period in philosophy of mathematics was most powerful when the works of Frege [34] in addition to Russell and Whitehead [143] appeared. Although logicism was weakened soon after the publication of *Principia Mathematica* [143] [3], logic as

[3]In 1930, Gödel announced to the whole mathematical community that there were formally undecidable propositions in *Principia Mathematica* [143]; meaning that there were sentences in arithmetic and in sufficiently strong formal systems, which their truth cannot be derived from the axioms and inference rules of the system. Moreover he also showed as a corollary that the consistency of the whole system cannot be proven within the system. That was a ground breaking result for both meta-mathematics and philosophy. Philosophically these results entail that there might not be any absolute truth. What is more is that the method Gödel used for proof and the motivation he started with were supposed to be for just the opposite purpose: the completeness of axiomatic, formal mathematical systems. When Hilbert formulated meta-mathematics to analyze mathematics with its own methods he wanted to show that mathematics can be based on finite number of axioms and these axioms are consistent. Although for many people Gödel's results indicated incompleteness of mathematical systems, for Gödel it only meant that logic was not enough to deal with all truths of mathematics and one needed intuition and perception to complete and free mathematics from the inconsistencies.

a tool to analyze mathematical methods did only get stronger. The most important contributions to analysis of mathematics by its own analytic methods (metamathematics) were made by the works of Gödel [42] and Tarski [130] [131]. Moreover, by these developments, the disciplines such as model theory, proof theory and set theory were formed, which in turn assisted the formal philosophical reasoning as foundations of mathematics at great length.

The crisis of intuition was declared by mathematicians and philosophers of mathematics as a result of raising power of logic and increasing number of counter intuitive findings in mathematics. The mathematician Hans Hahn[4], who had an interest in logical positivism, declared that:

> "Because intuition turned out to be deceptive in so many instances, and because propositions that had been accounted true by intuition were repeatedly proved false by logic, mathematicians became more and more skeptical of the validity of intuition. They learned that it is unsafe to accept any mathematical proposition, much less to base any mathematical discipline on intuitive convictions. Thus a demand arose for the expulsion of intuition from mathematical reasoning, and for the complete formalization of mathematics." ([47], p.93)

One of the instances that intuition turned out to be deceptive, as Hahn points out, is continuous functions. The common belief among most mathematicians before 1872 was that everywhere continuous functions cannot be nowhere differentiable[5]. Geometrical intuition tells us that, if there are no jumps or breaks in a function, the

[4]He was also Gödel's Ph.D adviser.

[5]$f$ is differentiable at $a$ ($f$ has a derivative at $a$) if $\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ exists.

curve of the function would have tangents in most parts of the curve. Hence, a graph and the intuitive definition of a continuous function would look like the following [6]:

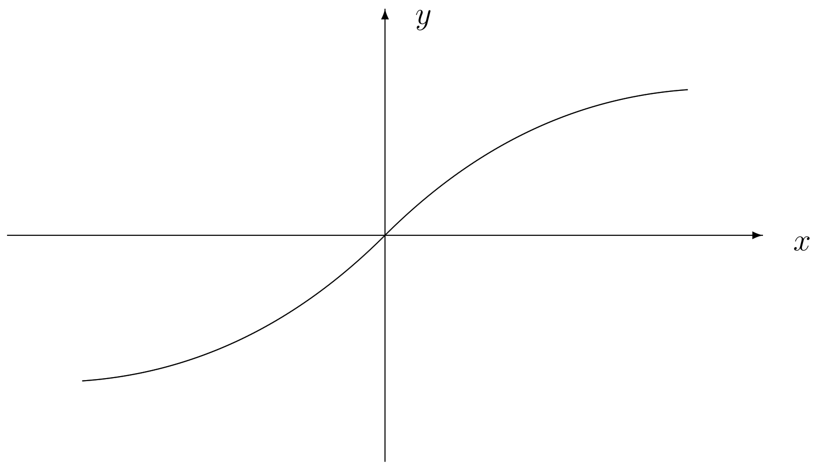


Figure 1.1: A visual representation of a continuous function

*Intuitive Definition:* If we choose a small enough neighborhood in the domain set of the function and if the image of the function stays in a small neighborhood for each such interval; which means the function has no breaks and jumps, then this function is continuous in that range.

Accordingly, if one only uses the graph of the function or the intuitive definition, it is possible to conclude that everywhere continuous functions cannot be nowhere differentiable, which means that the curve of the function has tangent lines at most points of the curve. For example, a continuous function that is not differentiable in some parts was not counter intuitive, however a function that is continuous everywhere but differentiable *nowhere* came as a surprise and gave the logicists more reasons to

[6] The formally accepted definition of a continuous function is as follows: Let $f$ be a function, $f : A \to B$ where $A, B \subset \mathbb{R}$ and let $c \in A$. Then $f$ is continuous at $c$ if, $\forall \epsilon > 0$, $\exists \delta$ such that $\forall x \in A$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

exclude intuition in mathematical reasoning. These counter intuitive examples are called *monsters*. Feferman points out that:

> "The appearance of monsters was a direct result of the nineteenth century program for the rigorous foundation of analysis and its arithmetization, i.e. for the triumph of number over geometry, at the hands most notably of Bolzano, Cauchy, Weierstrass, Dedekind and Cantor. That program grew in response to the increasing uncertainty as to what it was legitimate to do and say in mathematics, and especially in analysis." ([32], p. 320)

An example of nowhere differentiable everywhere, continuous function is called Weierstrass function[7] . A representative of a function Weierstrass constructed can be formulated as:

$$f_a(x) = \sum_{k=1}^{\infty} \frac{sin\pi k^a x}{\pi k^a}$$

where $a$ is an integer.

A graph of a Weierstrass function looks like the following in the interval [-2, 2]. (cf. [144]):

[7]Riemann also constructed a continuous function that is not differentiable on a dense set in real numbers 17 years before Weierstrass did (1861). However there is not enough information whether he has given the proof of the result. (cf. [136])

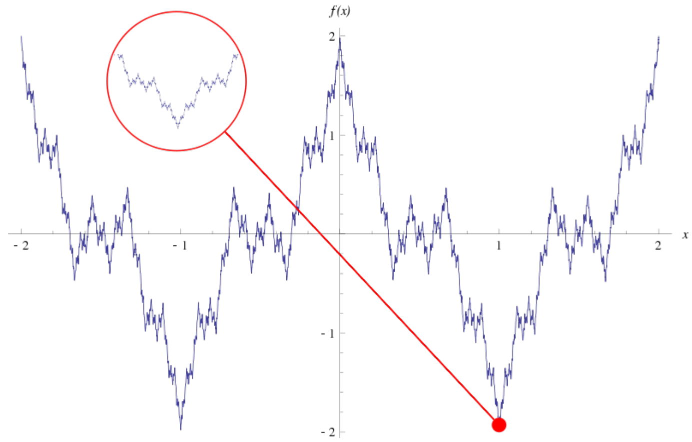


Figure 1.2: Graph of a Weierstrass function in the interval [-2, 2] (cf. [144])

Since there are no breaks in the function, it is clear that this function is continuous from the graph. However, because of the fractal nature of the function, i.e. that the jumps repeat themselves infinitely many times- the pattern is never-ending-, one cannot take the derivative of this function and one cannot see this from the graph of a continuous function. As in the above graph, one should represent the fractal nature of the function with additional information and drawings. However, since the visual representation and the intuition related to it that are used until this one caused false conclusions, they were classified as untrustworthy tools of mathematics.

On the other hand, Feferman argues that:

> "Without in the least bit denying the necessity of developing mathematics in particular analysis and topology in a rigorous manner, evidently I disagree with those who, like Hahn and others, believe that intuition has no value and that it must be expelled from mathematics. What, then, is one to say about the geometrical and topological monsters that are supposed to demonstrate the unreliability of intuition? The answer is simply

> that these serve as counterexamples to intuitively expected results when certain notions are used as explications which serve various purposes well enough but which do not have all expected properties." ([32], p. 322)

I also disagree that counter intuitive examples should be an indicative that intuition should lose its usage in mathematics. However, the prevalent view is the opposite so far. For example, Tennant representing the general approach since 19th century, argues that intuition is "dispensable as a proof theoretic device; indeed,..it has no proper place in a proof as such". ([132], p.304) Hence, the diagrams or intuitions, and in general visual reasoning were excluded from mathematics as much as possible. I believe this should not be so. Mainly because, deciding what is legitimate to use in mathematics affects how mathematics is taught and how published media appears. The manner introductory higher mathematics is taught nowadays is far from being able to establish objective, intuitive foundations because of the insistence on using only formal sentential tools. Eliminating the visual reasoning and starting, for example, teaching analysis on purely formal grounds is not the best way to start higher mathematics. Moreover, mathematicians in practice use diagrams frequently, but these diagrams disappear in the published media. When these papers are studied by other mathematicians than their authors, the diagrams are drawn again, to understand, or to read many facts at once from the diagrams. As a result communication for mathematical results and findings becomes much harder and more time consuming than it should have been.

## 1.3 Why is the Change in Perception of How One Practices Mathematics Necessary?

In order to answer this question we first need to understand what happened when the constructions were banned from legitimate mathematical communication. Once we reveal what is lost with the exclusion of construction from the context of justification, which means communication of proofs, we begin to comprehend which approach explains the true nature of mathematics.

**Why are constructions indispensable?** Mathematics, by then Geometry, first became a science in the practice of the elites of Ionian Greeks. Since elites of Ionian Greeks despised the trade but praised the beauty and structure of numbers and shapes, which were coming first from Egyptians and Babylonians as *numbers of* something and, *shapes of* something, they found a way to deal with this promising beautiful collection of properties, without using them in practical matters and trade. The tool they found to deal with this collection was "abstraction". For example, a border of an olive farm, the edge of a room, a string in a violin have all become the mathematical straight line, simply known as "line" in this newly arising structure. Then, they thought about the self-evident propositions about this abstraction that no rational person can doubt, such as: the shortest distance between two objects is a line, and then they only allowed the deductive reasoning for new constructions since deductive reasoning would only allow true conclusions from true statements. Greek mathematicians have to test these results, either by applying them to experience or by construction. It seems, even applied and pure mathematics arised immediately with the first steps of construing this new science: when they could apply their results to experience they were practicing applied mathematics and when they were using construction they were practicing pure mathematics. To this day, although mathematics evolved immensely, and currently there are many branches of mathematics; above described process always played a role in all. Namely, every branch of mathematics has its own abstract objects, axioms and its deductive method. Of course, we now have more abstract objects of mathematics than the Ancient Greeks discovered or created. Moreover, the deductive method is so much more powerful than the one used by Ancient Greeks. We have now very strong logics, including quantifier logic that proved to be extremely powerful for the developments of mathematics. In current practice of mathematics only one thing is expelled from mathematical practice: construction. As I mentioned earlier, when the deductive method flourished and it was possible not to apply to construction and empirical evidence in order to justify the results, and after finding couple of examples that showed themselves to be susceptible to error because of the construction activity, the intuitive construction process was expelled from mathematical communication. Hence, the dehumanization process of mathematical method started

However, one should first note that without the construction none of the mathematical process could be made possible since it was these constructions that allowed pure mathematics to flourish starting from the practice in the Ancient Greece. (cf. also [125]) Without construction there would be no number pi, no infinity, and no ideal circles. Moreover, construction is the tool that binds humans with mathematics in pure mathematics and goes hand in hand with intuition which is intrinsic to human beings.

As it is apparent from above summary of construing of mathematical objects and statements, mathematics is not independent from humans, although this connection seemed lost with the enthusiasm of the logical era. Mathematics is a human made science, which idealizes, abstracts, approximates some structures in the world. However, despite this fact there has been only few mathematicians and philosophers of mathematics defending the importance of aesthetic side of mathematics, and only recently, with the realization of the importance of visualization and with the visual turn in philosophy, we see more philosophers of mathematics emphasizing the role of human beings and visual constructions again, which can be much more than only a straight edge and compass constructions.

**Is there intuition in the purely mechanical part of mathematics, "Context of justification"?** Most philosophers of mathematics acknowledge that intuition and non-deductive structures play role in context of discovery, however, none would admit that context of justification, or repetition of a proof, in contrast to discovery of the proof, requires intuition. But this is not the case. Every time there is "..." in a proof, or a symbol that needs to be visualized we apply to non-deductive process. However, due to in-practice-recording-process, these steps are not recorded in a medium. As we will see in detail in Chapter V, mathematicians use expressions such as "as this picture explains", although the whole system is explained formally and sententially. In other words, we see only part of the process on a published media, and without realizing we fill in between the lines with many intuitive processes.

I argue that if proving a theorem were only a machinery process, everyone who is capable of thinking rationally and able to perform mechanically would be able to practice it without much effort. If it were only a data processing and following some basic rules based on deductive structures, there would be nothing interesting about it and no difficulty in seeing solutions of the problems.

Mathematics is not a collection of deductive formal structures, although, all proofs seem to be so in current mathematical communication. Moreover, when interpreting the structure and property of these proofs, one point is always being missed. Yes, the mathematical models consist of deductive structures that P then Q. However, if mathematics is a deductive formal structure that was only involved deducing Q from the premises P when they were the case, why is it so hard to prove some statements? Why was everything not self-evident from the beginning? It is also not the new knowledge or new methods that helps to come up with a proof of a true statement, but one person seeing one side of P, an aspect of P that is P* and finally the way to the Q is open. Hence, although P then Q seems to be a deductive reasoning, it needs some special ability to see the P* in a newly solved proof and that is what makes mathematics special. That is why one needs the special talent, the mathematical intuition, which is intrinsic in mathematicians, which goes hand in hand with constructions. Moreover, that is why, even though it is not recorded on a medium in context of justification, there is still more than deductive formal process in context of justification.

In this thesis, we will see, with the examples of the theorems and proofs from Euclidean Geometry, from applied mathematics and also from pure mathematics how this intuitive process not only occupies the discovery of a proof but also appears the steps in context of justification.

**Mathematical objects and mathematical practice.** With all above considerations, I defend that mathematical objects are both man made artful abstractions and self-existing entities at the same time. Moreover, in some cases mathematicians invented or discovered abstract objects to make some structures more harmonious, complete and useful (i.e. complex numbers). In this sense, mathematics, as Weyl argues in *Das Kontinuum* ([142], [140]), is much like an art. When its structure is carefully designed according to some rules it becomes a beautiful harmonious structure like one of Bach's preludes, and if one piece were missing it would not be the correct structure. If one piece were to be played wrong all the musicians would realize it as do mathematicians in a theory if one piece were false. Although notes and sounds have physical structure in some sense and mathematical objects do not, I will explain why mathematical objects can be as real as notes and sounds for musicians and why in a sense mathematical objects are discovered rather than being created.

I claim that mathematical intuition, with which mathematicians create/ discover the mathematical objects or a solution of a problem before moving on to the formal solution, works as a sense for the mathematician. It is similar to hearing, for example, C-sharp for a musician to intuit the $\sqrt{2}$ for a mathematician, or hearing "fugue in G minor" to a pianist and intuiting a Fourier Series to a mathematician working on mathematical analysis. They are beautiful in themselves and in their applications. They can even be surprising as Juliet Floyd suggests in "Das Überraschende" [33]. Whereas the application of the former can be playing it in piano in Berliner Philharmonie, the second can be used for interpreting the wavelengths in MR machines in Charite. So, if we add the mathematical intuition as a sense for the mathematical practice -which for mathematicians is really like a physical sense anyway-, then we can see that in which sense the mathematical objects are real, as if they are spatio-temporal objects, only, one may call them intuitive-objects. They are almost visible to the mind's eye, which has been argued for, for hundreds of years. This mind's eye, is the intuition; another sense that makes seeing the aspects, meta levels, missing entities possible in a mathematical theory.

Mathematicians intuit the complex numbers, as composers can hear very complex melodies without playing them and can find what is missing in a piece without even completing this piece. Mathematicians intuit that there is a need for complex numbers even before discovery or creation of them. A mathematician intuits these whereas a composer hears the missing part of a piece to be completely beautiful and harmonious in a master piece. In this sense mathematicians discover mathematical objects. Although when looked from outside, if one does not have this intuition, it looks like they are being created, and as if mathematics is a fiction. However, when one has the intuition and can use it as a sense, similar to hearing, seeing, mathematics become real, its objects become real. Moreover, mathematical objects and methods evolve just as human beings evolve. Evolving of mathematical methods can even be regarded as the mirrors of the intellectual evolution of human beings. So far, all the accounts of mathematics, to name a few, empirical, platonistic, logical, formal, structural, fictional might have explained one side of mathematics. Mathematics traveled through all these paths, and has all of it in itself. Some objects of it are abstracted from experience, some were discovered or created with the intuition to complete a structure or to have the limit of a construction, some were only place holders in a formal structure, in a model.

There have been many mathematical objects since from Ancient Greeks that could not be abstracted from, approximated to, such as number pi, the continuum, the infinity. Seeing a unit diameter circle and seeing the periphery as pi cannot ever happen without the Euclidean constructions, since one can never measure or approach to the number pi, it exists only conceptually by way of constructions.

Moreover, temporal or spatial continuity cannot give us continuum, since first, these temporal and spatial points are not individual physical points, and such things cannot be independent entities, as Weyl argues in *Das Kontinuum*, and, second, even if one takes them as really existing entities as Hilbert does, one has to take infinitely many limits to arrive at the concept of continuum from physical, temporal or spatial points, which makes the definition attempt already circular since taking infinitely many limits presupposes that concept of infinity is already defined.

If mathematical reasoning stays in the empirical world or only in the formal structure there is no way to know about the infinity. One acquires knowledge always in parts, thinks always in parts, since as human beings, we are finite beings with finitely many thoughts. In order to have an idea of infinity or continuum or the number pi, one has to go one step higher from the given structure and look at the system from above, as a whole.

This ability of us alone makes us able to think seemingly non-possible things in one level higher structures. If mathematics is a practice that allows access to the knowledge of concepts such as infinity, it cannot be only one level structure that consists of only deduction with formal logical rules and methods. In order to reach to the knowledge of the number pi, one must be able to go above experience and start the construction.

In order to reach the knowledge of the continuum, one must see that there is more than natural numbers, then, there is more than rational numbers, then, when the real numbers are reached, one must be able to go above and see the structure, the denseness, the possibility of infinity, continuity. This is not only logic. Logic provides every answer *in* the system; however, for the information about the system it falls short. For that we need intuition, another sense, that elevates us to see the meta level in any given problem, or making us able to see the aspect in a problem that finally, the proof that is searched for comes out.

If we look at the bigger picture in the practice and history of mathematics and see the meaning of foundational crisis, we can realize that mathematics is teaching us a very important structural lesson. If we investigate the meaning of Russell Paradox and Gödel's Incompleteness theorem, they teach us that if we stay on one level in examining structures we will not have complete systems and the method of the science will never provide the meaning of the science, however strong and however detailed the method is.

If we take logic to be the ultimate method of mathematics and continue with the dehumanizing process of mathematics by expelling constructions and intuitions from mathematics I fear that the real power and authenticity of mathematics, and especially of the pure mathematics, will be lost.

By accepting the human factor of mathematics one can capture the full power of the mathematical system. This acceptance also opens the way for the motivation to develop visual tools of mathematics that serves for the improvement of intuition.

## 1.4 Contemporary State of Visualization in Mathematics

There have been vast developments in technologies that allow visual representations in mathematics. Especially in mathematical and differential geometry, **computer generalized** visual representations are significant tools and allow mathematicians to work simultaneously on proofs from different locations. (cf. [105]) Receptions of these visual representations can be grouped under three views: (1) There is a loss of rigor with the usage of these visual representations, hence mathematicians should stick to the formal sentential tools. (2) They are part of the mathematical practice, however, the properties and results, which can be easily read off from the graphs or graphic animations, should be formally proven. (cf. [54]) (3) Emphasis on extreme rigor in this case is overrated: Modern practice of mathematics and insistence on formal practice are comparable to a society "in which music can be only written and read but never be 'listened to or performed' ". ([84], p. 20, [87], vii) The debate on this matter is inconclusive, however, the benefits mathematics is getting from these visualizations are immense, especially in the areas of chaos theory and differential geometry. (cf. [31], [92])

Another fruitful approach to visualization in mathematics is providing an **epistemological** background to the mathematical practice, mostly based on Kantian characterization of mathematics and partially inspired by recent cognitive findings as Giaquinto offers (cf. [39], [40]). As I mention in Chapter V in more detail, the results of such research suggest that mathematical thinking, at least for discoveries, is non-evidential and diagrammatical, hence *a priori* and synthetic ([40], pp.67-68).

I find the **cognitive** approaches combined with philosophical considerations such as Kosslyn's, Deheane's and Izard et.al.'s. (cf. [74], [19], [59]) very promising, as we will see in more detail in Chapter V. These are genuinely beneficial researches since, once our natural relationship with visual representations are revealed, it is easier to show that they are in the *actual* mathematical practice, especially, given that these results point at the direction that Kant was right about intuitions being *a priori* and objective in mathematical practice. A new interpretation of Kant's approaches in this light provides an objective ground for visual representations as intuitions both as inner (eg. mental images) and outer representations (eg. computer generated or drawn on paper images).

There is also significant work on visual representation as **formal** tools as I show in the Chapter V. (cf. also [4], [5], [120], [119], [123]) This practice is beneficial for demonstrating that we *can* have rigorous visual formal systems in mathematics. However, as Mancousu points out "there are several philosophers of mathematics who are opposed to this traditional approach and are interested in visual reasoning as an essential factor in providing a more 'realistic' philosophy of mathematics, sensitive to its practice and its cognitive roots. I believe that many of them would find the work by Barwise and Etchemendy on diagrammatic reasoning insufficient at best and misguided at worst. The problem, for many people in this tradition, is that exclusive attention to the goal of justification is unacceptable". ([84], pp.25-26) On the other hand, I should point out that although I believe "exclusive attention to the goal of justification is unacceptable", I still find the work of Barwise and Etchemendy pedagogically and conceptually very beneficial. For example, pedagogically, as teaching logic to freshmen as Barwise and Etchemendy did with their Hyperproof ([4]), and conceptually the idea that: *we can communicate information formally and rigorously with tools other than linguistic ones.* However, of course, I believe that goldmine in visualization in mathematics does not lie in the formalization of it but in the ways of representing the actual human mathematical activity.

Mathematical activity is a human activity, but this affects little the insistence on absolute rigor in mathematics. When emphasis put on to the rigor only, disregarding the actual practice of mathematics- the part that provides the beauty and harmony-, mathematics becomes something unnatural and unaccessible to most, and this should not be the case. One should also keep in mind that how mathematics *should* be practiced and mathematical notions are revisable since “behind such seemingly unshakable mathematical works are visions of mathematicians and that these visions can change in the course of time and that there is discussion about these changes”. ([11], p. 152) Hence, while insisting on rigor, it is wise to keep these considerations and history of mathematics in mind. Similarly, Brown argues that:

> “Our concepts of electron, gene and subconscious evolved considerably over the years. Mathematical concepts are similarly revisable, and therein has a major source of fallibility in mathematics- perhaps the most interesting” ([12], p. 23)

and he adds:

> “The intuitive truths of mathematics need not be certainties any more than ordinary empirical observations must be incorrigible to be confidently used by scientists. The parallel postulate [8] need not be embraced in spite of its intuitive character. And Russell’s paradox [9] shows us that some things which seem highly evident (i.e. that sets exactly correspond to properties) are, in fact, downright false. Still, we can use these intuitions, just as we can use our ordinary eyesight when doing physics, even though we sometimes suffer massive illusions” ([12], p. 33)

[8]Parallel axiom (also known as Euclid’s fifth postulate): ”If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles.” A detailed discussion is provided in Chapter 2 for the discovery of non-Euclidean geometries.

[9]Russell paradox is a famous paradox in set theory. Defined as follows: Let $R = \{x \mid x \notin x\}$, then $R \in R \iff R \notin R$ Namely, if we let R be the set of all sets that is not a member of itself we end up with the paradox: If R is not a member of itself, it must be a member of itself. If it contains itself, then it contradicts its definition.

For example, as I mentioned in the last section, visual representation of continuous functions caused an illusion that Weierstrass function revealed: There can be functions that can be nowhere differentiable and everywhere continuous. However, this was because there was no concept of the representation neither visually nor sententially. When Weierstrass discovered a continuous function that was nowhere differentiable and everywhere continuous he did not have a visual representation and Weierstrass school did not even care about it. For a while no one had an idea what the function would look like until von Koch (cf.[138]) discovered the pattern of snowflakes: The fractal nature. Main point for this fallacy was that the concept was not accessible to anyone at the time, neither sententially nor visually. If, maybe, the fractals were discovered earlier, then the fallacy could be fixed visually as well and the visual representation of continuous functions would not be one of the reasons that intuition and pictorial representations are categorized as mischievous tools of mathematics

Moreover, as Mancosu also draws attention, another aspect to consider is the following:

> "... the reasons for why such [visual] tools are problematic is not necessarily an account of some intrinsic feature of the visual medium. It is rather that one must always check that the visual medium does not introduce constraints of its own on the representation of the target area. And I doubt this is an issue that can be settled a priori rather than by a detailed case by case analysis of such uses." (Mancosu, p.26)

I am a strong defender of visual tools and I believe, there is a great benefit of providing epistemological roots of visualization in actual mathematical reasoning by integrating cognitive findings. I find it especially fruitful to show how visualization actually appears in justification steps without having it made into a formal tool.

Hence, my approach to visualization in mathematics aims at demonstrating visualization steps both in discovery and justification contexts, as non-formal but necessary and valid tools. Namely, I emphasize the importance of revealing the actual process of mathematical thinking by cognitive and epistemological approaches in both discovery and justification steps. Therefore, I argue throughout this thesis that visualization is a necessary part of proofs by offering a new definition for *necessity* outside of formal framework[10] in the sense that "primarily and naturally using visual representations before needing to apply to any other representation", since one cannot use formal definition in a heterogeneous system as Barwise and Etchemendy also point out with their *validity* definition. [11]

Formal logic is empty symbolism but helps us to write algorithms that machines can process. Once we start to apply these symbols to the models, such as mathematical models, then human intervention and interpretation begins and the aesthetics starts to play role. This is not something one should avoid or get rid of. This is where the beauty and essence of mathematics lies. It is too much of a cost to insist on making mathematics a dehumanized practice to make it free from fallacies. There will be fallacies and they will be addressed when the right tools and methods for them are discovered. This is the natural course of any science.

The formal tools are good for checking out the errors, advancing step by step securely, however, the mathematical process is more than that. It includes discoveries, decisions of which problems to focus on, visualization and aesthetics. As Poincareé points out: "It is by logic that we prove, but by intuition that we discover. To know

[10]Necessity in logic implies that in the assertion "Q is necessary for P", "P cannot be true unless Q is true".

[11]Barwise and Etchemendy's *validity* definition: "extraction or making explicit of information that is only implicit in information already obtained". The formal definition of *validity*: An argument is *valid* if it is impossible for the premises to be true and the conclusion to be false. (cf. ([5], p.4 and more on this in Chapter V)

how to criticize is good, to know how to create is better. ([102], p.129) [12]

Due to above considerations, in this work I show how pictorial representations play an essential role in mathematical communication and proofs. I also reveal that, although Hilbert's methods in *Grundlagen der Geometrie* can get rid of the physical drawings and constructions used in proofs, it does not get rid of the visualization part. I point out that adopting only Hilbert's methods for geometrical proofs and formalization methods developed extensively later on, causes to report the mathematical process incompletely. The new tools that are developing as a result of visual turn of this era can help this missing part to be represented iteratively in mathematical communication if we start to appreciate the importance of this process. Because, only with this appreciation the visual tools and methods specific to mathematics can be developed continuously and in unison.

Hence, with my analysis of different type of proofs from geometry, pure and applied mathematics I show how the pictorial methods are necessarily used in mathematics by giving a strong philosophical foundation based on Kantian characterization of mathematics . As a result, I draw attention to the aesthetic part of mathematics and suggest the importance of recording this pictorial process in mathematical practice and communication. By adopting this approach, I defend throughout this work that heterogeneous reasoning which uses both visual and formal sentential reasoning has many advantages over using only formal reasoning. Finally, I conclude that the heterogeneous reasoning is the natural representation of mathematical reasoning and validity of this heterogeneous reasoning can be established by analyzing Kantian characterization of mathematics.

[12] Original version: "C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente. Savoir critiquer est bon, savoir créer est mieux. " ([103], Part II. Ch. II)

# CHAPTER II

# The Medium for Visual Reasoning:

> The mere fact that there can be alternative geometries was in itself a shock. But the greater shock was that one could no longer be sure which geometry was true or whether any one of them was true. It became clear that mathematicians had adopted axioms for geometry that seemed correct on the basis of limited experience and had been deluded into thinking that these were self-evident truths.[1]
>
> -Morris Kline

The notion of space can differ depending on the discourse it is defined. It might suffice to describe it as "boundless three-dimensional extent in which objects and events have relative position and direction" [147] or one might need to analyze deeply to come up with a clear declaration until some useful definition is formed for the subject purpose. In our case we are faced with the latter. Especially in philosophy,

[1]([71], p. 88)

the notion of space is a topic that is discussed with polar opposite ideas. One can define it as a physical reality. On the other hand, one can also defend with equal sound arguments that it is a void or can only be defined as relation of objects to each other.

In this chapter, we will discover many notions of space and its classification in philosophy, mathematics and physics. From Aristotle to this day, there have been fascinating discussions on what space is and how best to use this notion for sciences.

For our purposes, defining space has utmost importance, since this is the first step any theory or practice needs: Clearly setting up its medium and foundations. Once we have the properties and rules of the space we are using for this purpose, we can define how visualization or visual reasoning can function in a rigorous and valid way. In order to reveal these properties and rules, we need to form the right questions and give our detailed and accurate answers to these questions.

In this chapter, I will discuss following questions and provide historical answers along with my interpretations of them. As a result, we will have a unique and operational definition of space for the purpose of having objective and adequate foundation of visualization in mathematics:

Question 1: What is space?

Question 2: How does Kant's characterization of space can be related to visual reasoning in mathematics?

Question 3: Why did I chose to explain visual reasoning in relation to Kant's characterization of space?

## 2.1 What is space?

We have mathematical space, physical space, cognitive space and all these notions have different properties and definitions. We can have curved or non-curved geometry of space. How space is defined, depends much on the area we are discussing it. One of the visual representation that comes from formulating space as three dimensional, boundless void between planets and objects is the following[2]:

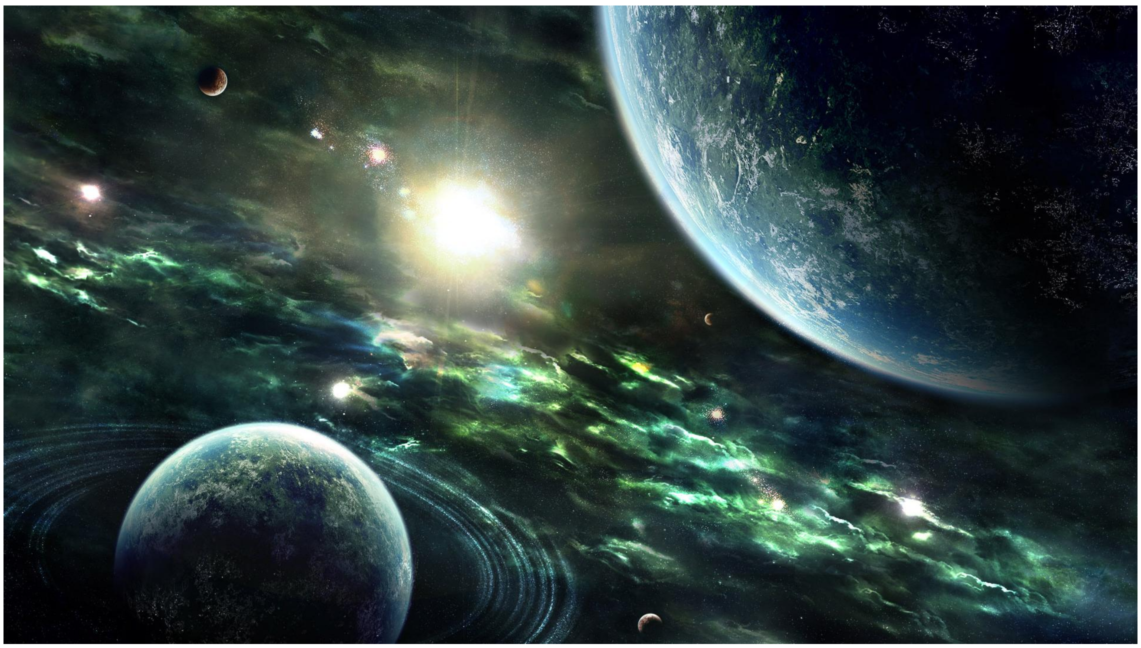

Figure 2.1: An image of space.

However, in philosophy, space has much more abstract definitions and visual representations. I will first reveal two rival characterization of space which affected much of the philosophical discussions for thousands of years till 18th century: Plato and Aristotle.

[2]Image "space" as first comes to mind when it is mentioned independent from any theory. Image from [27]

**Aristotle's and Plato's Characterization of Space.** In Timaeus 48e-53c [101] **Plato** characterizes space as receptacle, where everything comes into being. It is neither a form nor a matter, since *form* is "from which it is becoming" and *matter* is "from which it is constituted". Space is neither "from which it is becoming" nor it constitutes another entity as a matter. Therefore, it is receptacle for entities to come into being. **Aristotle** defines space as "a limit of the containing body at which it is in contact with the contained" ([1], 4.212a, Book IV Delta ). Space or "place" is different from form, matter or an empty interval between the matter. Plato's and Aristotle's notions of space set the course of the discussion over "space"; dividing the theories in "absolute" (can be attributed to Plato) and "relational" (can be attributed to Aristotle) until Kant offered a completely new approach to the notion of space.

**Other Notions of Space till 18th century: Newton, Leibniz and Kant.** Based on Aristotle's and Plato's notions of space two kind of theories dominated the discussions over space: relational and absolute. These polar formations came into their final form in Newton's and Leibniz's characterizations. While Newton construed an absolute notion of space in *Mathematical Principles of Natural Philosophy* [89], Leibniz defended a relational view of space in his exchanges of papers with Samuel Clarke [77].

**Leibniz** defines space as something merely relative (as time is). He asserts that space is not a substance and gives a proof of it with the need for sufficient reason whereby defending the applicability of mathematical reasoning to physical and metaphysical subjects. He concludes that (siding with Aristotle) "Space is nothing but an order or set of relations among bodies, so that in absence of bodies space is nothing at all except the possibility of placing them" ([77], pp.9-10)

**Newton,** on the other hand, defends that the space is a physical reality and absolute. He argues that motion is the only medium which space can be explored. His argument is as follows: There is a real force causing real motion, which is the rotational absolute motion. If there is an absolute motion this requires the existence of absolute space. ([89], pp.11,12)

Till 18th century these notions were discussed by Plato, Aristotle, Proclus, Descartes, Newton, Leibniz, Clark and many others. **Relational view (Leibnizian-Wolfian)** defended that space was not an individual thing and there can be only spatial properties of material things. Space referred only to spatial relationships of objects. This view denied the concept of void space and classified forces such as gravity as imaginary things. **Absolute view (Newton)** defended that space is an individual entity and it is something like a container, existing independently from material objects, hence making void space necessary for motion, where motion and forces are also defined as real and absolute entities.

**Kant** was a close follower of the lively discussions between Newton and Leibniz. He analysed the opponent thought systems of the two throughout his whole philosophy career and gave an exceptional synthesis of these two systems. (cf. [69], [37], [61]) In the end of 17th century and through first half of the 18th century the changes and developments in mathematical thinking, mechanics and metaphysics were just evolving in a very fluctuating manner that provided Kant to work on a great synthesis and a philosophical system.

Kant's thesis is different from previous theories about space. According to him, space and time are conditions for one's perceiving. They are the forms of sensibility. Kant also characterizes space and time as pure intuitions, meaning that they are independent from experience.[3]

I believe, Kant's notion of space lays out the necessary foundation for visual reasoning in mathematics, therefore, deserves a detailed analysis. In the next section I will reveal his characterization and its relation to the visual reasoning in mathematics. However, before going into his theory in more detail, I would like to close the historical summary of space by briefly mentioning the modern theories of space that are developed during and after 18th century.

**Discovery of Non-Euclidean Spaces: Space becoming Spaces.** Most differentiated theories of space surfaced in 18th century with the discussion on Euclid's fifth postulate. Moreover, even before mathematical discussions, Thomas Reid (1764), a Scottish philosopher, offered a spherical characterization of space using common sense. As of now -2015-, we have not only Euclidean or spherical definition of space but many spaces with all kind of curvatures that can be defined by setting the angle in the Euclid's first postulate (eg. Riemannian Space) in addition to the spacetime, a mathematical model combining space and time into a single continuum (see below for some representations).

[3]For our evaluation of space in this section it suffices to note that *intuition* according to Kant is a singular representation and *pure* means acquired or generated independent from experience. I will give a detailed analysis of the notions in following sections.

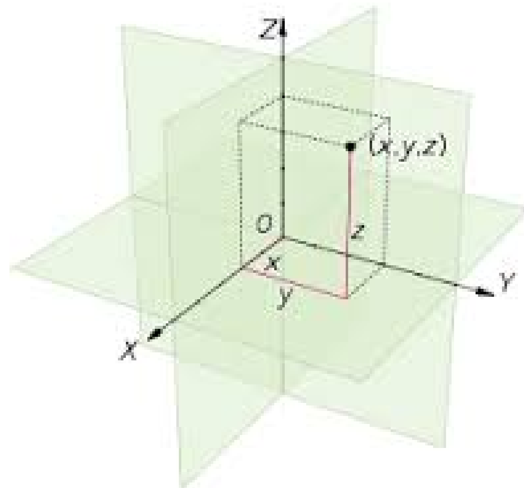


Figure 2.2: A representation for Euclidean Space [145]

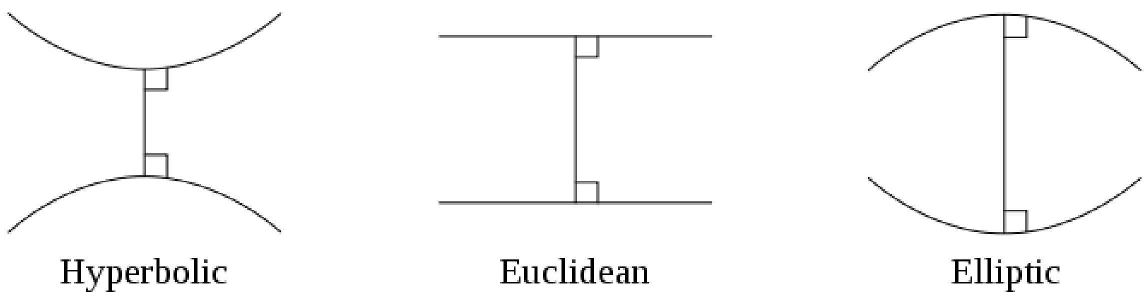


Figure 2.3: Euclidean vs. Non-Euclidean Geometry Representations [146]

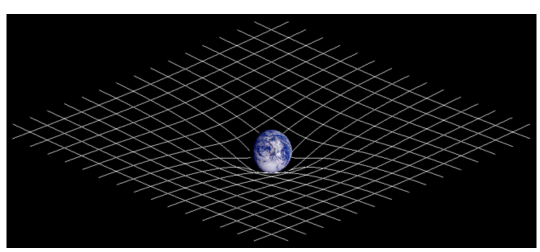

Figure 2.4: A representation of Spacetime [148]

In the history of mathematics the discovery of non-Euclidean geometries is related to the parallel axiom.[4] The parallel axiom seemed too complicated to be an axiom. For that reason many mathematicians either tried to reformulate it in a simpler way or tried to prove it from other axioms of the Euclidean geometry. As an attempt to establish the Euclidean characterization of the real space Leibniz tried to prove the parallel axiom in *In Euclidis Prota* [78] (beginning of 18th century). He was aware of the possibilities of spaces other than the Euclidean one. However, he did not doubt that the real space has the form of the Euclidean geometry. Then, Giovanni Girolamo Saccheri in *Euclides Vindicatus* [114]) tried to prove the parallel axiom. The procedure common to both was to adopt a contradictory assumption to the parallel postulate and derive a contradictory result whereby showing that the parallel axiom follows from the other axioms. Saccheri took the results as being contradictory since the results were really strange, as not having all angles as right angles in a quadrilateral. Hence, he concluded that that he found a contradiction and the parallel axiom must be true. However, another mathematician Johann Heinrich Lambert realized that one can have a consistent system with a contradictory assumption to the Euclid's parallel axiom and he acknowledged the possibility of new spaces, namely that one can use a spherical conceptualization of space (non-Euclidean) consistently instead of using a flat space (Euclidean). Lambert was the first one who realized that a new geometrical system different from the plane geometry, namely, the spherical geometry was possible.

Around the same decade another independent formation of a non-Euclidean geometry, which does not concern the parallel axiom, had been developed by the Scottish philosopher Thomas Reid in *Inquiry into the Human Mind on the Principles of Common Sense* [111]. Although his motivation was different from Lambert, his results

[4]Parallel axiom (also known as Euclid's fifth postulate): "If a line segment intersects two straight lines forming two interior angles on the same side that sum to less than two right angles, then the two lines, if extended indefinitely, meet on that side on which the angles sum to less than two right angles."

were similar, on the construction of his "geometry of visibles", claiming that perceptual shapes have spherical characters. According to Reid, by sight one can only have a tangible figure, and a real figure is possible by touch. Hence, there could be different conceptions of the same space.

**Euclidean and Non-Euclidean spaces combined for visual reasoning:** In late 18th century Immanuel Kant was articulating the approach that shapes have to conform to the axioms and rules of Euclidean space. According to Kant, space is a pure intuition, meaning that it is a result of the formation of the human mind and is acquired independently from experience. Moreover, this space is the form of sensibility and it is Euclidean. For this reason all figures that are exhibited in this space should conform to the Euclidean rules.

The visual reasoning in mathematics and the requirement of diagrams in mathematical discovery go hand in hand with the notion of space Kant proposed. In the following sections and chapters I establish a connection between visualization in mathematics and Kantian interpretation of intuitions which also consists of the pure intuition of space. It is also interesting to find out that Kant's thesis is consistent with the current cognitive findings as described in "Flexible Intuitions of Euclidean Geometry in an Amazonian Indigene Group" [59]. They argue that our conceptualization of space is Euclidean whereas our perception conforms to the rules of spherical geometry. Their results provide an approach that combines Kant's and Reid's views on the conceptualization and perception, respectively.

## 2.2 Kant's Notion of Space

Geometry plays an essential role in Kant's construction of transcendental philosophy in the *Critiqe* and his natural philosophy. The idea of space in Kant's theory is based on the method and content of geometry. The objectivity and validity of synthetic *a priori* judgments as well rest on the objectivity and validity of geometrical judgments and geometrical representations. For that reason, explaining geometrical foundations in connection with space and with his transcendental philosophy by emphasizing the strength of synthetic method is quite central to Kant's thesis.

Kant lived in an era where the synthetic method and geometry were the unshakable grounds of scientific development. The method of geometry in 17th and 18th century was synthetic and the representations used for the proofs were visual representations (intuitions). The tools were not developed enough to represent the basic notions used in proofs conceptually and the method of analysis were not developed enough to arrive at the results by only using conceptual analysis. Therefore, the only valid method available to Kant was exhibited intuitions as singular representations and synthetic method that makes this exhibition and construction possible. Most of the critiques against the validity of Kantian approach to mathematics rest on the idea that polyadic logic was not available to Kant and Hilbert's axiomatization as formal representation of Euclidean geometry was not foreseen by the era Kant was developing his thesis.

One of the important notions that will be dealt in this thesis is to evaluate whether the carrying out proofs in a synthetic way and using singular intuitive representations can be classified as outdated and invalid just because formal sentential method, which proved itself to be rigorous and free from fallacies is favored in the current practice of mathematics.

It is important to remember that every method which was used in mathematics, whether it was construction with compass and ruler, or using algebraic methods and curves, or using polyadic logic and formal concepts; these methods were treated by the mathematicians and philosophers of mathematics of the era as fixed and unchangeable methods. As Henk Bos points out: "behind such seemingly unshakable mathematical works are visions of mathematics and that these visions can change in the course of time and that there is discussion about these changes" ([11], p. 152). I will return to discuss these issues in detail in Chapter V. I mention this point to remind us that, Kant's great synthesis of his time's natural philosophy, based on the theories of distinguished physicists, mathematicians and philosophers, could be more than a study of history of philosophy. His approach could lead the way of re-analyzing the thought system in mathematics, by re-evaluating the advantages of synthetic method and intuitions in mathematics.

### 2.2.1 Kant's Historical Background

As I mentioned earlier, Kant provided a great synthesis of the notion of space based on Newton's and Leibniz's conceptualizations.

He characterized two opponent thought systems of his era as follows:[5]

| (1) *Leibnizian-Wolffian System* | (2) *Newtonian System* |
|---|---|
| Denies that space is infinitely divisible | Asserts that space is infinitely divisible |
| Denies the void space | Void space is necessary for free motion |
| Attraction or universal gravitation | Attraction or universal gravitation |
| [or geometry or mechanics] is | is explained by the forces, which are in the bodies |
| imaginary things. | that are active at a distance and at rest |

[5]Referred by ([37], p. 4): Leibniz and later Wolff (Elementa Matheseos [149]), as a follower and allocator of Leibniz's thought system as opposed to Newton's, described a system as Kant characterizes it in (AKK. 475.22-476.2 [60] ). Also note that the interpretation on Leibniz denying infinite divisibility of space that Kant reveals in *Physical Monadology (1756)* is not a commonly accepted view. Most of the Leibnizian philosophers in the 18th century considered space to be infinitely divisible.

Kant accepts the Newtonian characterization of gravity and attraction between masses, however, thinks that Newton's philosophy of nature did not reach to its full potential. (cf.[37], p.1) Kant's aim was to give a better metaphysical foundation to Newton's natural philosophy by improving it with a Leibnizian thought system. These thought systems handled geometry and metaphysics separately, whereas Kant, from the beginning of his philosophical career set himself the task to unite the geometry and transcendental philosophy. According to him, space, along with time is the condition for one's perceiving. It is a pure intuition and a representation as I will explain in more detail in the next section.

### 2.2.2 Space as Pure Intuition: The Metaphysical Exposition

Kant gives the argumentation of space being the representation, as pure intuition, in four steps (five steps in the first edition) in Metaphysical Exposition in the *Critique*. In Metaphysical Exposition, Kant describes the *extension*[6] of the concept space by analytic method, (1) as a representation, (2) as a necessary *a priori* representation, (3) as a single representation which is a pure intuition, and finally (4) as a representation given in infinite magnitude. In Transcendental Exposition, Kant describes the intension of the concept, space, by synthetic method, by constructing the definition of it as the form of sensibility. Moreover, in this way, the description of space can explain the possibility of synthetic *a priori* judgments in geometry. Furthermore, since

[6]In modern formulation extension of a concept denotes what falls under a concept and intension of a concept determines its description. Kant's characterization is slightly different from this modern formulation. He argues that "Every concept, *as a partial concept*, is contained *in* the presentation of things; as a *ground of cognition*, i.e. *a characteristic*, it has these things contained *under it*. In the former regard, every concept has an *intension*; in the latter, it has an *extension*". ([63], p.101 §7) (Ein jeder Begriff, als Teilbegriff, ist in der Vorstellung der Dinge enthalten; als Erkenntnisgrund, d. i. als Merkmal sind diese Dinge unter ihm enthalten. -In der erstem Rücksicht hat jeder Begriff einen Inhalt; in der andern einen Umfang.) ([63], Allgemeine Elementarlehre, §7) Hence, descriptions or definitions that are used for presenting the properties of the concept can be thought as the intension of a concept. In particular, intension of a concept is the partial concepts that are used to describe a concept whereas extension is the sphere of the concepts with respect to what falls under a concept.

this approach makes the explanation of geometrical judgments possible and since geometry is the solid ground of all knowledge, given that geometrical judgments are synthetic *a priori*, it must be a true explanation according to Kant, .

Let us now look closely how he constructs his argument in the *Critique* and let us analyze critically how sound his argumentation is.

Kant denies the previous theories of space to build his own characterization of it. He argues that:

> "Space represents no property at all of any things in themselves nor any relation of them to each other, i.e., no determination of them that attaches to objects themselves and that would remain even if one were to abstract from all subjective conditions of intuition. For neither absolute nor relative determinations can be intuited prior to the existence of the things to which they pertain, thus be intuited *a priori*." ([65], A26/ B42)

Kant is denying both Newton's and Leibniz' approach to space in this paragraph and he concludes that:

> "Space is nothing other than merely the form of all appearances of outer sense, i.e., the subjective condition of sensibility, under which alone outer intuition is possible for us. Now since the receptivity of the subject to be affected by objects necessarily precedes all intuitions of these objects, it can be understood how the form of all appearances can be given in the mind prior to all actual perceptions, thus *a priori*, and how as a pure intuition, in which all objects must be determined, it can contain principles of their relations prior to all experience. " ([65], A26/ B42)

Accordingly, all the empirical intuitions are possible because the space is the form of sensibility and space is the underlying pure intuition where all the intuitions are exhibited. It plays the role of "basis" for the intuitions, as the Euclidean space plays the role of "basis" for the figures to be exhibited in geometry. Moreover, this "prior to all appearances space" contains necessarily the geometrical principles and their relations. Space is the subjective condition for one's experience and perception. However, by exposing it as prior to all experience and intuitions, it becomes an objective, universal rule carrying ground. Space (and time), is the natural consequence of the formation of the human mind, and space carries the principles of geometry which grants its objective validity. Since geometry is an established solid ground in 18th century and Euclidean space is the only accepted form of real space, by asserting the Euclidean space, as the space which already constituted by the rules of Euclidean geometry, Kant objectively reveals space as the pure basis of his transcendental philosophy.

An exposition, according to Kant can be analytic and synthetic. (cf. [63], p.142, §102, p.143 §105) If it is analytic, it analyzes given concepts (cf. [63], p.143, §105); if it is synthetic, it is the synthesis of appearances. [7]

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[7] "§102 **Synthetische Definitionen durch Exposition oder Konstruktion** (cf. [63], p.143 §102)

Die Synthesis der gemachten Begriffe, aus welcher die synthetischen Definitionen entspringen, ist entweder die der Exposition (der Erscheinungen) oder die der Konstruktion. - Die letztere ist die Synthesis willkürlich gemachter, die erstere die Synthesis empirisch - d.h. aus gegebenen Erscheinungen, als der Materie derselben, gemachter Begriffe (conceptus factitii vel a priori vel per synthesin empiricam). - Willkürlich gemachte Begriffe sind die mathematischen.

Anmerk. Alle Definitionen der mathematischen und - wofern anders bei empirischen Begriffen überall Definitionen statt finden könnten - auch der Erfahrungsbegriffe müssen also synthetisch gemacht werden. Denn auch bei den Begriffen der letztern Art, z. B. den empirischen Begriffen Wasser, Feuer, Luft u. dgl. soll ich nicht zergliedern, was in ihnen liegt, sondern durch Erfahrung kennen lernen, was zu ihnen gehört. - Alle empirische Begriffe müssen also als gemachte Begriffe angesehen werden, deren Synthesis aber nicht willkürlich, sondern empirisch ist."

§105 "**Erörterungen und Beschreibungen**

Nicht alle Begriffe können also, sie dürfen aber auch nicht alle definiert werden. Es gibt Annäherungen zur Definition gewisser Begriffe; dieses sind teils Erörterungen (expositiones), teils Beschreibungen (descriptiones). Das Exponieren eines Begriffs besteht in der an einander hangenden (sukzessiven) Vorstellung seiner Merkmale, so weit dieselben durch Analyse gefunden sind. Die Beschreibung ist die Exposition eines Begriffs, so fern sie nicht präzis ist. Anmerk. 1. Wir können entweder einen Begriff oder die Erfahrung exponieren. Das erste geschieht durch Analysis, das zweite durch Synthesis. 2. Die Exposition findet also nur bei gegebenen Begriffen statt, die dadurch deutlich

In the exposition of space, Kant is exercising the analytic exposition, in order to connect "(successive) presentation of its [given concept's] characteristics so far as these have been found by analysis".(cf. [63], §105 for the German translation cf. fn. 7) Space is not a made or a constructed concept as in the case of geometrical concepts. Hence, exposition of the concept of space is an analytic exposition with the given concept of space. Moreover, note that, there are two types of pure entities in Kant's thought system. These are pure concepts of understanding and pure intuitions. The former is categories, the latter is space and time. Whereas the original representation of categories are concepts, the original representation of space and time are intuitions as Kant concludes at the end of Metaphysical Exposition.

### 2.2.3 Space as Form of Sensibility: The Transcendental Exposition

Kant's synthesis on space being the form of sensibility is as follows:

> "Geometry is a science that determines the properties of space synthetically and yet *a priori*. What then must the representation of space be for such a cognition of it to be possible? It must originally be intuition; for from a mere concept no propositions can be drawn that go beyond the concept, which however, happens in geometry. But this intuition must be encountered in us *a priori*, i.e., prior to all perception of an object, thus it must be pure, not empirical intuition. For geometrical propositions are all apodictic, i.e., combined with consciousness of their necessity, e.g., space

gemacht werden; sie unterscheidet sich dadurch von der Deklaration, die eine deutliche Vorstellung gemachter Begriffe ist. - Da es nicht immer möglich ist, die Analysis vollständig zu machen; und da überhaupt eine Zergliederung, ehe sie vollständig wird, erst unvollständig sein muß: so ist auch eine unvollständige Exposition, als Teil einer Definition, eine wahre und brauchbare Darstellung eines Begriffs. Die Definition bleibt hier immer nur die Idee einer logischen Vollkommenheit, die wir zu erlangen suchen müssen. 3. Die Beschreibung kann nur bei empirisch gegebenen Begriffen statt finden. Sie hat keine bestimmten Regeln und enthält nur die Materialien zur Definition." ([70], II. Allgemeine Methodenlehre, §102, §105)

> has only three dimensions, but such propositions cannot be empirical or judgments of experience, nor inferred from them.
>
> Now how can an outer intuition inhabit the mind that precedes the objects themselves, and in which the concept of the latter can be determined *a priori*? Obviously not otherwise than insofar as it has its seat merely in the subject, as its formal constitution for being affected by objects and thereby acquiring immediate representation, i.e., intuition, of them, thus only as the form of outer sense in general.
>
> Thus our explanation alone makes the possibility of geometry as a synthetic *a priori* cognition comprehensible. Any kind of explanation that does not accomplish this, even if it appears to have some similarity with it, can most surely be distinguished from it by means of this characteristic." ([65],B40/41)

It is clear from above quotations for Kant two things are assumed here: that he had already given a clear proof in the Metaphysical Exposition that space is pure intuition, and that propositions of Euclidean geometry are synthetic *a priori*.

**Is space form of sensibility?** Kant takes Euclidean geometry as the model for his transcendental philosophy. Note that, there was no theories of "spaces" while Kant's thought system was maturing. The other non-Euclidean spaces were considered at most as logically consistent mathematical theories. There were almost no questioning about the nature of real space being Euclidean. For that reason, Euclidean space and Euclidean geometry were considered as the most trustable and unshakable ground for basing theories on, just as modern logic has been considered to be so since 1920s.

Moreover, the most respected method till the beginning of 20th century was considered to be synthetic method, although, mathematicians and philosophers like Leibniz and Descartes started integrating analytic method to mathematics as early as 17th century and claiming themselves to be rediscovering the analytic method that the ancients hid from them. ([11], p. 26) Moreover, the method of geometry was the representation with compass and ruler, or giving a description with a pointwise drawing and providing the solution with a geometrical construction. Analytical, meaning that, algebraical solutions solely were not accepted in proofs. So, synthetic nature of Euclidean geometry and requirement of drawn figures were not questionable in 17th and 18th century..

The model of geometry, therefore, had the basis as Euclidean space which allows synthetic judgments to be derived, according to Kant. Moreover, this space has to be prior to the shapes that will be used in order for them to be exhibited.

With these considerations in mind let us now analyze other building blocks of the *Critique*, namely, representations and schema that will help us to connect visual reasoning in mathematics and Kant's notion of space.

### 2.2.4 Representations (*Vorstellungen*)

In Kantian thought system a representation is either a concept or an intuition. When the representation is general and mediate, meaning that it is inferred from other concepts or intuitions, this is called *concept*. When the representation is singular and immediate, meaning that it is not inferred from other concepts or intuitions, this is called *intuition*. According to Kant, one can have only the intuitions of objects that are out in the world. One cannot know what *is* the object in the world, however, she can have an immediate or mediate representation of it. This characterization seems to lead that the knowledge of a person must be subjective, since everyone has her own representations. However, Kant's theory suggests that one can have objective knowledge despite this subjective construction. He provides this objectivity with the notion of *a prioriness*.

According to Kant, our forms of sensibility do not borrow anything from experience. These forms of sensibility, pure intuitions, space and time are natural consequences of the formation of human mind, and these are the same for everyone. Although we might not have an objective experience for colors for example, our knowledge of geometrical objects, as *a priori* exhibited intuitions are objective. Kant's construction for this objective knowledge is based on, first, that one has pure intuitions, space and time as the ground for this objective exhibition. Second, one can exhibit intuitions of mathematical objects *a priori*. If the ground is independent from experience and if the exhibition, which progresses through construction, is independent from experience, then this knowledge is necessary according to Kant, and this gives the objective validity of this knowledge.

Moreover, formal conditions of sensibility are contained in pure concepts, categories[8] and these concepts are acquired independent from experience. Without the *formal conditions of sensibility* and without the *pure concepts*, experience is not possible. On the other hand, without experience these concepts cannot get activated. The link between experience and categories are provided by *schema*. According to Kant, without a rule to decide the intension of a concept one would not be able to have a general representation, the concept. Since some knowledge is objective, then the origins of this knowledge from the mind must come from an objective ground again with an objective, certain rule making mechanism.

With respect to acquiring the senses and dealing with singular representations, space and time provide the objective ground. This ground is also where the mathematical notions and proofs are carried out. Moreover, the rules of the space conform

[8]Quantity, Quality, Relation, Modality ([65], A80/B106)

to the Euclidean geometry, where the synthesis of objective intuitions are made into concepts, which is the construction of mathematical concepts. For the general representation and thinking, the objective ground is categories, and the rules are given by schema. This duality is between synthesis and analysis, Euclidean rules and schema, sensing and thinking, constructing a concept and analyzing a concept, deciding the intension of a concept and finding the extensions of a concept. The last step is the end of duality, since deciding the intension is done by schema, and although up to the work of schema, starting from the role of categories, analytic method is at issue, now the synthesis for the first time starts from within an analytical structure, and it is provided by schema.

### 2.2.5 Method for Representing: Schema

Pure intuitions (space and time), pure concepts of understanding (categories), and pure sensible concepts (shapes) are acquired or generated independently from experience, according to Kant. Pure intuitions and pure sensible concepts can be applied to experience since they are the forms of sensibility. Pure concepts require a rule making procedure to be applied to experience, which is called *schema*.

Kant talks about schema as a process, something that acts on empirical intuitions to connect them with the categories. (cf. [66], B179) It is actually contained in the categories, it is possessed by the *a priori* concepts and used as "adapters". (cf. [73], p. 71) Hartman and Schwarz interpret this as the end of analysis and beginning of synthesis (cf. [63], p. lxi). They also point out that Kant's notion of schema is similar to Descartes' "simples" (cf. [63], p. lxiii).

The reason Hartman and Schwarz find a similarity between the simples and schemata are that the method that is used to reveal them is similar. They both mark the beginning of the synthesis. In Descartes: “The simples are ... the axioms of the system. The system is applicable to reality because the simples were analyzed out of it” ([63], p. lxiii). In other words, the abstraction from the given reality provides the simples and then the symbolization starts with the synthesis. This is similar to Kant’s system in the sense that, schema is the result of analysis but also, as an intension determination rule, it marks the start of synthesis.

These *a priori* formal conditions “contain the general condition that has to be satisfied if the category is to be applied to any object.” ( [68], B179) These formal conditions are products of imagination [*Einbildung*] but they are not images. Imagination in Kantian doctrine is a technical term and corresponds to a faculty in human mind. Moreover, this faculty of imagination gives the synthesis of representations. Its main act is to synthesize the synopsis that comes from senses.(cf. [65], A95/B127) Its *a priori* work is to produce schema and its empirical task is to reproduce images, which in turn provides recognition.

**Categories and schema.** Categories are not directly applied to things in themselves. They are applied to appearances. Appearances are given in sensibility, that is why the application of the categories to experience must conform to the formal and pure condition of sensibility (cf. [65], A138/B179) and in this case this is *time*. One needs to make a category temporal (cf. [65], A139/B178). Only by this way the category has something common with the appearance, the temporality. Kant calls this transcendental time-determination, *transcendental schema*:

> “Now it is clear that there must be a third thing [apart from pure concepts and the appearances they are applied to], which must stand in homogeneity with the category on the one hand and the appearance on the other,

> and makes possible the application of the former to the latter. This mediating representation must be pure (without anything empirical) and yet **intellectual** on the one hand and **sensible** on the other. Such a representation is the **transcendental schema**." ([65], A138/B177)

Kant's theory of schema is easier to understand in generation of *a priori* concepts, such as shapes than application of pure concepts to experience. His argument for the latter is restricted by the following paragraph:

> "The concept of a dog signifies a rule in accordance with which my imagination can specify the shape of a four-footed animal in general, without being restricted to any single particular shape..." ([65], A141/B180).

Although Kant does not articulate the application of schema to the empirical appearances and only says that nature will not easily reveal how this connection is set up (cf. [65], A141/B180), it is important to understand the essential parts of this theory of schema, since only by way of schema the objectivity of mathematical knowledge can be explained. To clarify let us call "schema", $S$, that helps for the production of pure sensible concepts and "transcendental schema", $T$, that helps for the application of the categories to experience . This will make it easier to categorize when the schema gives the output for pure sensible concepts as shapes (S) and when it gives the application of the categories to the appearances (T). We can think of them as mathematical functions having a domain and range. Now first consider $S$ : $S : D \to R$ **where** $D = \{$**Space**$\}$ **and** $R = \{$**Pure Sensible Concepts**$\}$.

An application of this would be as follows:

**S(Euclidean Rules for Triangle) = Triangle $\in$ (Pure Sensible Concepts).**

These above formulations are representations of Kant's following description:

> "The pure image of all magnitudes [*Größen*] (quantorum) is space and pure sensible concepts are shapes." (cf. [66], A142/B182).

Here, we can say that, the domain for the production of pure sensible concepts is "Space" and **schema** $S$ **works as propositions of Euclidean geometry** and produces the *a priori* concepts of shapes, in other words, it provides the *intension* to construct these concepts. (cf. also [38] and [66] A234/B287)

Now, consider $T$, the transcendental schema:

$T : D \to R$ **where** $D = \{$**Categories**$\}$ **and**

$R = \{$**time series, content of time, order of time, sum total of time**$\}$**.**

Kant explains this as:

> "...the schema of each category contains and makes representable: in the case of magnitude, the generation (synthesis) of time itself, in the successive apprehension of an object; in the case of the schema quality, the synthesis of sensation (perception) with the representation of time, or the filling of time; in the case of the schema relation, the relation of the perceptions among themselves to all time (i.e., in accordance with a rule of time-determination); finally, in the schema of modality and its categories,

time itself, as the correlate of the determination of whether and how an object belongs to time. The schemata are therefore nothing but *a priori time-determinations* in accordance with rules, and these concern, according to the order of the categories, the **time-series**, the **content of time**, the **order of time** and finally the **sum total of time** in regard to all possible objects" (cf. [65], A145/B185)

Hence, the domain for application of transcendental schema is the pure concepts of understanding: "Categories" (magnitude/ quantity, quality, relation, modality). By application of the transcendental schema to the category quantity the rule for generating the concept "number" is constructed. And, similarly, this condition is applied to other categories:

$\mathbf{T(Quantity) = Number \in (Time\text{-}Series).}$

$\mathbf{T(Quality) = Intensity \in (Content\ of\ Time).}$

$\mathbf{T(Relation) = Difference \in (Order\ of\ Time).}$

$\mathbf{T(Modality) = Existence \in (Sum\ Total\ Of\ Time).}$[9]

By the application of transcendental schema, one can decide, for example, **schema of Reality, a subcategory of Quality** in the sense of **persistence of the real in time**. None of the schema of categories can explain the schema of the concept of an empirical object, for example *the concept dog.* However, it can explain the persistence of dog through time, etc. One cannot go further from the schema of categories more than saying that it is time determination and by this way it shares something common

[9] Intensity, Difference, Existence are my terms summarizing Kant's longer definitions. They stand for: Intensity:=Synthesis of Sensation, Difference:= Perception Among Themselves, Existence:= How an Object Belongs to Time.

with appearances and it somehow determines the temporal conditions and connection between the category and appearances.

For our purposes in this work, it suffices to conclude from Kant's complicated theory of schema that it is important to realize image is different than schema, and schema determines the intension of the concept which can be used to construct the image. By this way, a concept can be applicable to the extensions of the concepts. Schema of sensible concepts can be brought to an image, but the schema of pure concepts of understanding never can be brought to an image. ([65], A142/B182)

For example, Number as a schema of Quantity (not a particular number but the concept of number in general) is what Kant describes as a *transcendental schema* as I have represented as a function above. Hence, only temporality will be applied to the category of Quantity in this case and this will be a transcendental synthesis, a work of pure imagination in production. In a closer look as in the form of above function this will look like:

**T(Quantity) = Number ∈ (Time-Series).** Which is a "representation that summarizes the successive addition of one (homogeneous) unit to another" ( [65], A143/B182)) The reason Kant calls number as time series is that "I" generate the time in each apprehension of the intuition (cf. [65], A143/B182), hence it's a rule of succession, a rule for counting the homogeneous magnitudes.

On the other hand, when Kant gives the example of number 5, he explains that when he places five dots next to each other that is an image, which means an empirical intuition. (cf. [66], B179) However when one just thinks about the number 5 in a general way, only by the help of categories, by the pure possibility of it, it is a representation of variety of things the term 'number 5' applies. It is a synthetic construction similar to mathematical construction. The difference is that, synthetic construction that is done by schemata is a general representation, and mathematical construction adds singular representations in the construction of the concepts with respect to shapes. Now let us investigate the application of schema/synthesis on the construction of pure sensible concepts.

**Pure sensible concepts and schema.** According to Kant the pure sensible concepts, shapes, are not abstracted from experience. If they were abstracted from experience, there would be no way of explaining their objectivity.

Friedman also comments on Kant's concept, arguing that:

> "Schemata of geometrical concepts ... not only serve to contribute towards the objective reality of such concepts, but are also essential to our rigorous representation of the concepts themselves." ([37], p.123)

In a recent paper of his, "Kant on Geometry and Spatial Intuition" [38], he develops this idea and reveals that:

> "... these concepts [all geometrical concepts (of triangle, circle, and so on)] are "generated" by particular geometrical constructions in accordance with their schemata." ([38] and cf. also A234/B287)

Kant indicates this also in the Axioms of Intuition, while arguing for all intuitions being extensive magnitudes:

> "On this successive synthesis of the productive imagination, in the generation of shapes, is grounded the mathematics of extension (geometry) with its axioms, which express the conditions of sensible intuition *a priori*, under which alone the schema of a pure concept of outer appearance can come about; e.g. between two points only one straight line is possible; two straight lines do not enclose a space, etc. These are the axioms that properly concern only magnitudes (*quanta*)[10] as such." ([65], A 163/ B204)

Friedman suggests that the reason Kant used the Euclidean construction for schemata of pure sensible concepts, is because Kant needed something that does the work of quantifier logic.(cf. [65], B28; [37], p.124) For example, for a construction of a circle as pure sensible concept, providing the rule with the given line and point for constructing a circle should be as follows: construct a closed plane curve, every point of which is equidistant from a given fixed point, the center M(h,k). Equation: $(x-h)^2+(y-k)^2=r^2$ where r is the radius and M( h, k) are the coordinates of the centre .

Next, let us analyze the following questions to understand clearly the *a priori* construction of mathematical objects:

What is the difference between Number and number 3?
What is the difference between shape and triangle?

**A case study: Transcendental Schema of Quantity, Schema of Number 3 and Concept of Number 3.** Whereas Transcendental schema of quantity is

[10]According to Kant these kind of magnitudes, *quanta*, belongs to geometry wherea *quantitas* refers to the magnitude of arithmetic and algebra. (cf. [65], A204-205 and cf. [129], p.548)

number, schema of number 3 is the rule for the determination of the intension of the number 3. This corresponds to end of analysis and beginning of synthesis. The concept of number 3, which is determined by the schema can be used for analysis and synthesis. If it is handled by the analysis, one finds infinitely many extensions, infinitely many group of 3 units, that can be applied to every three appearances by this schema; if on the other hand, it is handled by the synthesis, it is used by mathematics and it is 3 homogeneous magnitude.

By the transcendental schema of Quantity, Number, we are able to count, namely by uniting homogeneous representations as long as we generate the time.

Thinking of a particular number initiates the rule schema. In the case of number 3, the representation of a method for representing a the multitude 3 (cf. [65], A140/B179) is in question. This is the schema for number 3. This could be approximately= (generating time during the unity of holding together "homogeneous magnitude", "homogeneous magnitude", "homogeneous magnitude").

How mathematical concepts can be applied to experience by way of judgments, schema and categories can be explained as follows: When one has the empirical intuition of 7 houses, the judgment is "There are seven houses". The Quantity of the Judgment is Particular, the Quality is Affirmative, Relation is Categorical and Modality is Assertoric. In the categories these correspond to: Plurality, Reality, Subsistence and Existence, respectively. In order for one to hold this unity of Judgment of Particular Quantity, one needs the unity of synthetic representations through time, which is made possible by transcendental schema and apperception.

The judgment for "There are 7 houses" is formed by two different processes. First, there must be an empirical intuition of 7 houses. This is the empirical part which comes through sensation. Then, one must have the category of quantity *a priori*, which needs to be there in order for one to schematize the plurality of entities in relation to time determination. One must have the time determination, transcendental schema, since successor relation can be formulated only by the apperception and generation of time. (cf. [65], A95/B127) Thus, time, with its unity given by apperception, as a successor relation, makes grounds possible for cognizing plurality, where plurality as a sub category provides the *a priori* ground for generating the corresponding *a priori* concept of number 7.

**A case study: Shape, Schema of Triangle, Concept of Triangle.** Shapes reside in pure intuition Space and pure image of all magnitudes (quantorum) for outer sense is also Space. (cf. [66], A142/B182) Moreover, "Shape" is not a transcendental schema of any category as Number. The temporality, the work of Schema, plays role in generating the concepts such as divisibility, however for generating pure sensible concepts, a different kind of synthesis, a different kind of schematism applies.

Friedman argues that while the infinite divisibility can be represented in polyadic logic statically by existential instantiation and universal generalization, this is not possible for monadic logic. (cf. [37], pp. 70-71) Hence, for Kant, infinite divisibility must be constructed as a dynamic intuitive procedure. In detail Friedman describes this as:

> "...whereas we can represent infinite divisibility by "$\forall x \exists y$ (y is a proper part of x)", Kant would formulate this proposition by "$f_B(x)$ is a proper part of x", where $f_B(x)$ is the operation of bisection, say. And, in this representation, the idea of infinity is conveyed not by logical features of the relational concept "y is a proper part of x" but by the well definedness and iterability of the function $f_B(x)$: our ability, for any given line segment

x, to construct (distinct) $f_B(x)$ $f_B(f_B(x))$, ad infinitum. This, which all parts or spaces must be "cut out" by intuitive construction ("limitation"). Only the unbounded iterability of such constructive procedures makes the idea of infinity, and therefore all "general concepts of space," possible. And, of course, it is this very same constructive iterability that underlies the proof-procedure of Euclid's geometry." ([37], p. 71)

For the construction of general shapes, however, as Friedman points out "we can take the Euclidean construction corresponding to the fundamental concepts (line, circle, triangle, and so on) as what Kant means by schemata of these concepts" [38]. This plays role in the construction of concepts in the proofs. However, note that, the schema of Quality or schema of Relation makes application of concepts to appearances possible, such as the application of the pure concept of circle to a round plate.

In this sense,"we can understand the schema of the concept of triangle as a function or constructive operation which takes three arbitrary lines (such that two together are greater than the third) as input and yields the triangle constructed out of these three lines as output (in accordance with Proposition I.22..."([38]) and "the empirical concept of a **plate**, has homogeneity with the pure geometrical concept of a **circle**, for the roundness that is thought in the former can be intuited in the latter." ([65], A137/B176)

We can, therefore conclude that, in mathematics, what constructs the pure sensible concepts is the synthesis of pure imagination, schema with respect to Euclidean principles as their schemata. From these concepts alone the *a priori* intuition of a triangle can be exhibited *a priori* in pure intuition space, for its mathematical usage. One should also note that schemata are the reason one can imagine the possibilities of the application of a word 'triangle' in all triangular appearances. Moreover, schemata are the reason that one can exhibit intuition objectively and *a priori* corresponding to the concept triangle, without borrowing anything from experience, in geometrical proofs.

### 2.2.6 The General Representation: Concept

According to Kant there are pure concepts of understanding, categories; there are pure sensible concepts, shapes; and there are empirical concepts. Pure concepts are acquired or generated *a priori*, independent from experience, independent from sensations. Categories reside in human mind, as being the the consequence of the formation of human mind. They carry the formal conditions of sensibility, schemata in them. By way of schemata, other *a priori* concepts, such as shapes, can be generated and categories can be applied to experience.

A concept, in general, is defined as a general representation and relates to the object through thinking, through inference or uniting different intuitions. This relating process, when empirical, starts with an object affecting the mind in a certain way. (cf. [65], B33) Kant further argues that "[w]hen an object affects us, its effect on our capacity for representation is sensation". ([68], B34) Empirical concepts always have *a priori* parts in them, since they will include shapes for one reason and for another, they are only known to us through these *a priori* concepts of the understanding since they are the conditions of any knowledge.

According to Kant when you consider a cone, for example, the shape of this cone resides in the *a priori* part, whereas the color of it resides always in the empirical part. He argues that:

> "The shape of a cone can be made intuitive without any empirical assistance, merely in accordance with the concept, but the color of this cone must first be given in one experience or another." ([65], A715/B743)

For that reason, the pure intuition can come from pure sensible concept, namely the shape, in other words, the general representation of an object (concept) can be used to derive the particular representation of an object (intuition). This is the construction in the mathematical proofs, exhibiting *a priori* the intuition corresponding to the concept. However, for a color this is not possible. For that reason, when I consider a green cone, I cannot construct the concept of it without having an experience of it. There has to be effect of an object to my senses. This is similar to getting the sensation of a green cone hat. The judgment could be similar to this: "There is a green cone hat on this man's head". In this case, the green cone is an empirical concept as a general representation, that leads to the empirical intuition, after the concept makes the experience possible. However, this concept also contains the *a priori* intuition, namely the shape of the cone.

### 2.2.7 The Singular Representation: Intuition

According to Kant, an intuition is a singular and immediate representation of an object. (cf. [68], B33, B376-377) *A priori* intuitions are the mere forms of sensibility, they are the correspondents of pure sensible concepts. Whereas these concepts are the general representations and allow for analysis, intuitions corresponding to them are used for constructions and allow for synthesis.

Kant explains construction with intuitions by analyzing how Thales (or someone thought to be Thales) as a geometer, works with the properties of the isosceles triangle (cf. ([68], Bxi- xii). This geometer does not analyze a particular empirical concept as drawn to somewhere, or does not analyze merely the concept *qua* isosceles triangle. But he let's "his *a priori* concept of the isosceles triangle guide him in constructing such a triangle in his mind, and then to attribute to isosceles triangles only such properties as followed necessarily from what he had put into his construct." ([68], Bxii) This *a priori* intuition is a particular representation of a general concept, that only necessary properties from the concept are brought together for the purposes of mathematician, hence, it is not an empirical intuition, neither a concept that has each and all the properties of the triangle that reason can generate.

Kant also talks about these kind of intuitions in connection with their concept in more detail in the Doctrine of Method. He argues in the following paragraph that:

> "There is, to be sure, a transcendental synthesis from concepts alone, with which in turn only the philosopher can succeed, but which never concerns more than a thing in general, with regard to the conditions under which its perception could belong to possible experience. But in mathematical problems the question is not about this nor about existence as such at all, but about the properties of the objects in themselves, solely insofar as these are combined with the concept of them." ([65], A719/ B747)

Kant informs us that these intuitions "in question can themselves be given *a priori*, which makes it hard to distinguish them from mere pure concepts" ([68],B9). These intuitions are only used in construction in mathematics. (cf. [68], B99; [65], A715/B743) They are not the ones that make experience possible, they can be used to make proofs and propositions in geometry or mathematics objective and make the concept possible for visualization for the geometer. So that geometer without borrowing anything from experience can carry out the proof, **by seeing the universal in particular**. (cf. [65], A714/B742) Moreover, as the above quotation indicates, mathematician does not concern the existence of these pure concepts and pure intuitions. If the mathematician prefers, she can get help from experience as drawing a shape, but that's only a choice and not a necessary process for a construction or a proof. Thus, in the mathematical process it is possible to work with the intuitions purely *a* *priori*, without borrowing anything from experience. However, Kant also points out that these *a priori* representations would mean nothing if one cannot present their meaning in empirical objects. (cf. [68], B299) Accordingly, "we are required to take the bare concept and make it sensible, i.e., present a corresponding object in intuition." ([68],B299) Kant further argues that although mathematicians produce these figures *a priori*, once they are produced they are appearances present to senses.The mathematical construction process is reversed compare to the cognition of the object itself. (cf.[68], B 376) Namely, for cognition, all intuitions must be empirical and must come through sensations. However, for construction, one needs the *a priori* forms of sensibility, and one does not need any sensory data, or empirical intuition. These representations become empirical intuitions only when they are applied in experience. Hence, we both have a particular triangle as *a priori* intuition for a certain construction or proof, having the necessary properties of the concept of the triangle, and also have an empirical triangle as found in the objects having the properties of this individual triangle.

Kant at some point believed that numbers were *a priori* intuitions too. (cf. Kant, Inaugural Dissertation, §23, referred by [106], p. 42) Later, in the *Critique* and in his letter to Schultz he talks about numbers as only general concepts that can be applied to empirical world. Kant also says that "[s]pace and time are only forms of our sensible intuition, so that the only things that exist in space and time are *things as appearances*"([68], Bxxvi). Extensions and shapes are also derived from the pure intuition, space, and thus also are *a priori* intuitions. One should also note that time is not related to numbers as space is related to the shapes. Kant talks about it in his letter to Schultz (25 November 1788; [60] AKK. 10: 554-58) and argues that:

> "Time, as you quite rightly remark, has no influence on the properties of numbers (as pure determinations of magnitude), as it does on the property

> of any alteration (as a quantum), which itself is possible only relative to a specific condition of inner sense and its form (time); and the science of number, in spite of succession, which every construction of magnitude requires, is a pure intellectual synthesis which we represent to ourselves in our thoughts." (reffered by Parsons in [94], p. 63)

Hence, Kant does not think that numbers are *a priori* intuitions. This is because he believes that numerical equations are immediate and indemonstrable. (cf. [65],A 164/ B204) They are immediate in the sense geometrical shapes are immediate, so immediacy is not the issue for numbers not being intuitions. But they are indemonstrable according to Kant, so, one cannot construct the concept by exhibiting the intuition *a priori* corresponding to that concept as in the mathematical proofs. Geometrical concepts are generated according to Euclidean principles as their schemata. Moreover, by way of their a priori generated concepts, the a priori intuitions as singular representations can be exhibited and these singular representations are objective for that reason. For arithmetic, there were no axioms or principles, at the time Kant was constructing his thesis. Hence, there are no rule making mechanisms for the particular numbers according to Kant. There is only a schema of Quantity as the concept of general number and there is a way to construct particular numbers as generation of time, enabling one to count homogeneous magnitudes.

Note that, Kant uses 'intuition' as a technical term and although the term in philosophy of mathematics was introduced by Kant ([65], [64], [67]) it did not become popular until after the importance of Gödel's incompleteness proof was realized. Moreover, requirement of intuitions were admitted by some very powerful logicians as empirical representations (cf. [36]) or as a propositional attitude ([43]).

Emily Carson and Renate Huber also draw attention to the usage of intuition in different meanings among mathematicians, physicians and philosophers of mathematics. Accordingly, although they use the term *intuition* in different context, they were influenced by Kantian notion of intuition according to Carson and Renate:

> "Following developments in modern geometry, logic and physics, many scientists and philosophers in the modern era considered Kant's theory of intuition to be obsolete... But this only represents one side of the story concerning Kant, intuition and twentieth century science. Several prominent mathematicians and physicists were convinced that the formal tools of modern logic, set theory and the axiomatic method are not sufficient for providing mathematics and physics with satisfactory foundations. All of Hilbert, Gödel, Poincaré, Weyl and Bohr thought that intuition was an indispensable element in describing the foundations of science. They had very different reasons for thinking this, and they had very different accounts of what they called intuition. But they had in common that their views of mathematics and physics were significantly influenced by their readings of Kant." ([15], pp. vii-viii)

Most common usage of intuition in the literature is as a propositional attitude, something similar to perception, which is a rare usage in Kantian thought system. Kant uses intuition as a singular representation. Parsons suggest the usage of the terms 'intuition *that*' to refer to the propositional attitude and 'intuition *of*' to refer to the usage in relation to objects (cf. [95], p.139), and Kantian intuitions for mathematical objects are intuitions *of* mathematical objects.

Parsons, moreover supports the claim that the main use of intuition according to Kant is not as a propositional attitude but as a representation. He argues that:

> “In Kant, intuition as a propositional attitude plays no explicit role. By definition, an intuition is a singular representation that is a representation of a single object. When Kant in the *Critique of Pure Reason* says that it is through intuition that knowledge has “immediate relation” to objects (A19/ B33), this immediacy seems to be a direct presence of the object to the mind, as in perception. At all events, intuition gives “immediate evidence” to propositions of, for example, geometry. Thus intuition that seems to be present in Kant, although his official use of “intuition” is only for intuition of.” ([93],p.147)

In the paragraph Parsons refers, Kant is employing the first usage of “intuition”in the sense of a cognitive tool similar to perception or similar to the capacity of sensibility. However, his usage of intuition apart from this paragraph is almost always as a representation of an object. Hence, from here on, unless stated otherwise, when we use “intuition” it will refer to the singular representation of an object.

Now that we have clarified the notion for space and representations according to Kant, we are one step closer to showing how Kant’s characterization of mathematics will help to provide the foundations for visual reasoning in mathematics.

## 2.3 Kant’s Characterization of Space and Visual Reasoning in Mathematics

Kant bases his notion of space on the premise that Euclidean propositions are synthetic *a priori* and the objections to Kant’s theory of space mainly attack to this premise. The defenders of this view argue that the premise is not valid anymore since we can now formally represent the Euclidean geometry by using Hilbert's formulations. (cf. [133], pp. 65-66; [45], p. 367 and for Hilbert's method cf. [51]) As I have indicated in Introduction and argue in detail in the next chapter, having a formal finitary system for Euclidean geometry does not have to make the non-axiomatic and visual one invalid and does not make Euclidean propositions analytic or not-synthetic-*a priori*.

I argued in the Introduction that mathematics has an aesthetic value and essential part of mathematics depends on the ability of mathematician. Therefore, it is only natural that mathematics evolves as the human beings evolve. Hence, before accepting the exile of visual representations from mathematics, I suggest to consider Bos' observations I mentioned in the beginning of this chapter, namely his significant evaluation that "behind such seemingly unshakable mathematical works are visions of mathematicians and that these visions can change in the course of time and that there is discussion about these changes" ([11], p. 152). Moreover, we also need to keep in mind the amazing invention and developments in visualizations in the last decade as I will explain in more detail in Chapter V. It would only hinder the practice and teaching of mathematics to ignore these considerations and developments. Trying to eliminate the usage of visual tools in proofs is damaging to mathematics' progress and practice. In formal mathematical communication the process is slow and the visual representations, in most cases, are communicated with formal sentences. This has two detrimental effects for the practice in mathematics: First, it is time consuming for the author to formalize every visual representation and for the reader to reconstruct them again and second, the development of visual tools in mathematics is far from being a continuous natural process due to prejudice against them. The latter has a huge negative impact for the usage of visual tools too. Apart from some coding tools like LaTeX, javascript and some locally used programs, there are not many tools that make visual communication easy and fast in mathematics in an era where visual tools are used in every second in our lives, and this is not natural at all. Therefore, I suggest that we take a close look how Kant sets up the **objectivity for constructions**. These constructions and synthetic *a priori* method are building blocks of what we call visual reasoning nowadays.

Firstly, one of the crucial reasons that intuitive representations such as geometrical shapes in Euclidean geometry were attempted to be eliminated from mathematics is that two persons' intuitive representations may differ. Moreover, mathematics is a science that is trusted for its rigor and objectivity. If two persons' intuitive representations may differ and since mathematics should hold the objective ground it follows that intuitive representations should not be used in mathematics.

I believe that two persons' intuitive representations may differ because we did not excel in the external representations of the visualization and we do not have a complete knowledge of our visual and conceptual spaces.[11] Kant provides us the theory for the objective ground of conceptual/ cognitive space by formulating space as a pure intuition and necessary objective foundation for constructions to be carried out. As I will explain in more detail in the next chapter, although the external representations when taken as instances on a medium like a paper may exhibit accidental features that might cause invalid inferences, the internal representations do not have to have these drawbacks. Mathematical intuitions, which are exhibited *a priori*, as described by Kant, are the building blocks of visual reasoning in mathematics as we will see in the following chapters.

As I mentioned earlier in the previous section, with the development in cognitive sciences as well, now we are in a position to compare the visual space that we perceive in with the conceptual space that we visualize. Moreover, according to Izard et. all

[11] I call conceptual space where we represent our pictorial mental representations or visualize the geometrical figures, whereas visual space is where we perceive objects.

([59]), our conceptualization of geometrical objects is Euclidean whereas our perception conforms to the rules of spherical geometry. Their results provide an approach that validates Kant's characterization of space and mathematical figures as *a priori* entities. [12]

Understanding and revealing the nature of our conceptual and visual space clearly sheds light on the real representation of mathematical practice. Moreover, it can help immensely in the development of pedagogical tools. Last but not the least, this research, offers news ways of communicating in philosophy of mathematics and can take it further than its present position.

In mathematical practices there have been an illusion that we do not use visual representations or visualization in the proofs but this is not true. If we delete the external diagram we do not eliminate the visualization we just record the procedure incompletely. Kant's characterization of space and constructions help us to see how this visualizations are taking place objectively and necessarily in mathematical reasoning as we have seen in the previous sections and continue to analyze in more detail in the following chapters.

To recapitulate: Kant's theory of our cognition of empirical objects in relation to space we posses cognitively, is based on the relation between Euclidean figures and Euclidean space. This pure intuition, space, is equivalent to Euclidean space in Euclidean geometry. Kant's aim in the Aesthetic is to show this equivalence. In this way, he proves our **structure for cognition as objective, since the form of sensible intuitions must conform to the *a priori* synthetic Euclidean propositions**. Moreover, since Euclidean propositions cannot be deduced by mere analysis of space, similarly, our cognition requires the internal *a priori* space for carrying out something more than analysis, namely to carry out the *a priori* constructions and to inhibit forms of the empirical objects.

[12] More an more research show similar results cf. [19]

We have now seen how Kantian characterization of space and representations can provide an objective ground for visual reasoning in mathematics, namely, by offering a universal conceptual space formed by universal geometrical rules and a universal way of generating the visual representations. Moreover, the conceptual space where we visualize has the same properties for everyone since this space is the result of the formation of human mind, according to Kant. Hence, this space conforms to the universal rules when constructing mathematical visual representations. We will now discuss how constructions are exhibited objectively with synthetic method in this space as the *method of visualization* in more detail in the next Chapter.

# CHAPTER III

# Methods for Visual Reasoning: Synthetic *a priori*, Construction and Abstraction

Man is the religious animal.
He's the only one who's
got the true religion-
several of them.[1]
-Mark Twain

In this Chapter, I will clarify the meaning and method of visual reasoning, which will help us to identify the specific methods of visualization in mathematics. For many years mathematicians and philosophers of mathematics argued for the elimination of visual tools, constructions and intuitions. But this was only eliminating what we "see". It did not eliminate the visualization from the process. One cannot carry out a proof without a visualization and we will see why in this chapter.

[1]([135], p. 32)

## 3.1 The Definitions of Visualization and Methods Associated with these Definitions

There is not a unified definition of visualization so far in the history. Phillips, Norris, and Macnab in their book *Visualization in Mathematics, Reading and Science Education* compiled several definitions of visualization, that have been used since 1974. ([98], p. 23) Below are the prominent ones that are helpful for classifying visual reasoning in mathematics:

- 1982 Hortin: "Visual literacy is the ability to understand and use images and to think and learn in terms of images, i.e., to think visually" ([55], p. 262)
- 1983 Nelson: "Visualization is an effective technique for determining just what a problem is asking you to find. If you can picture in your mind's eye what facts are present and which are missing, it is easier to decide what steps to take to find the missing facts" ([88], p. 54)
- 1985 Sharma: "Visualization (mental imagery) serves as a kind of 'mental blackboard' on which ideas can be developed" ([118], p. 1)
- 1989 Ben-Chaim, Lappan, and Houang: "Visualization is a central component of many processes for making transitions from the concrete to the abstract modes of thinking. It is a tool to represent mathematical ideas and information, and it is used extensively in the middle grades" ([7], p. 50)
- 1989 DeFanti, Brown, and McCormick: "Visualization is a form of communication that transcends application and technological boundaries" ([18], p. 12)
- 1995 Rieber: "Visualization is defined as representations of information consisting of spatial, nonarbitrary (i.e. 'picture-like' qualities resembling actual objects

or events), and continuous (i.e. an 'all-in-oneness' quality) characteristics. Visualization includes both internal (for example, mental imagery) and external representations (for example, real objects, printed pictures and graphs, video, film, animation)" ([112], p. 45)

- 1999 Liu, Salvendy and Kuczek: "Visualization is the graphical representation of underlying data. It is also the process of transforming information into a perceptual form so that the resulting display makes visible the underlying relation in the data."([79], pp. 289-290)

- 2001 Presmeg and Balderas-Canas: "The use of visual imagery with or without drawing diagrams is called visualization" ([107], p. 2)

- 2001 Strong and Smith: "... spatial visualization is the ability to manipulate an object in an imaginary 3-D space and create a representation of the object from a new viewpoint" ([127], p. 2)

- 2008 Gilbert, Reiner, and Nakhleh: "Visualization is concerned with External Representation, the systematic and focused public display of information in the form of pictures, diagrams, tables, and the like ([134], 1983). It is also concerned with Internal Representation, the mental production, storage and use of an image that often (but not always...) is the result of external representation" ([134], p. 4). "A visualization can be thought of as the mental outcome of a visual display that depicts an object or event. ([41], p. 30)

- 2009 Deliyianni, Monoyiou, Elia, Georgiou, Zannettou: "Particularly, in the context of mathematical problem solving, visualization refers to the understanding of the problem with the construction and/or the use of a diagram or a picture to help obtain a solution" ( [9], 1989) ([20], p. 97)

As it is obvious from above compilation there is not one common definition of visualization. This lack of common ground, partially, is due to visualization being re-introduced quite recently. Especially after the abandonment of the visual representations in mathematics. The trend up to last 20 years was to eliminate visual representation from mathematical practice and hence from scientific thinking since mathematical method decides the rigor and method in sciences as well. However, recently, there have been many attempts to re-introduce the visual representations as I classify them in detail in Chapter V and also as it comes out in recent studies (cf. [49], [4], [5]). In these proofs that I introduce in Chapter V, although visual representations are referred as only "helpful tools" to obtain a solution in most cases, I point out that this approach only considers the external visual representations. We will see that this account does not represent true nature of mathematical reasoning, and we will also understand *why* this account is not a true representation of complete mathematical reasoning. Now, to understand fully what visual reasoning is let us combine the most prominent characteristics of the above definitions.

The patterns that appear repetitively with the visual reasoning in the above definitions are: "mind's eye", "thinking visually", "mental blackboard", "concrete to abstract", "form of communication", "internal and external representations", "resulting display", "making visible the underlying relation in the data", "use of visual imaging with or without drawing diagrams", "display of information", "mental production", "construction of a diagram", "use of a picture". These can be now grouped under: construction (synthetic -a priori- method) vs. abstraction and both of these can be inner and outer representations. Now, let us look closer to these methods and what falls under each group.

**Construction** I will use Kant's definition of construction since it gives a good foundation for visual reasoning and encompasses many other definitions of the term.

According to Kant, *construction or intuition* is a particular representation where mathematician chooses the necessary properties for the proof from the general *a priori* concept. This is construed partly from the rules and space as a consequence of the formation of the human mind and partly from the concepts generated according to the rules. This does not correspond to an empirical particular representation since it will only have some properties belonging to the concept, and this intuition or construction resides *a priori* in human mind. The process here is from *a priori* to *a priori* and respectively from thinking to construction. (cf.[65], B299) This process does not need any external representation and it exists in all the proofs whether it contains a geometrical representation or not.

The second kind of construction or intuition for a mathematical object is generated when mathematician decides to use an empirical representation such as a drawing of a triangle or fingers in terms of counting, which becomes then an empirical intuition. This is adding one more step to the above construction steps and exhibiting it to a physical medium. However, note that, this process is different than seeing a drawing and making an abstraction. In the construction we are embodying the rules in a visual representation, which is a similar process described by Kant as schema.

Our main focus on this chapter will be on constructions since they are the valid tools that explain the visual reasoning in mathematics. Before going deeper into constructions, intuition and synthetic method, I would like to clear out one more classification for the method of visual reasoning that caused all the nuisance in philosophy of mathematics.

**Abstraction** Abstraction differs from construction. It is a singular representation that comes through senses such as seeing 7 houses, or a square table. This intuition is empirical and can be generalized to a concept with the right methods, such as omitting accidental properties of a diagram and using the general ones. ( cf. [119], [85]) When one makes an abstraction in a geometrical proof from a drawn triangle, if the rules of abstraction are not determined or misused then the reasoning based on the diagram can be false. So, the reason mathematicians and philosophers of mathematics wanted to eliminate intuitions or visual representation from mathematics are"abstractions" and not "constructions". For example, let us check where it all begin, with Weierstrass function:

One of the instances that intuition or abstraction turned out to be deceptive, is continuous functions as I mentioned in the Introduction.

The belief wast that, every continuous function was differentiable except on a set of isolated points. In other words, if we choose a small enough neighbourhood in the domain set of a continuous function, the image of the function stays in a small neighborhood too; which means the function has no breaks or jumps. However, Weirstrass found a function that was everywhere continuous but nowhere differentiable. Below is a graph of the function as it was also shown in the Introduction.

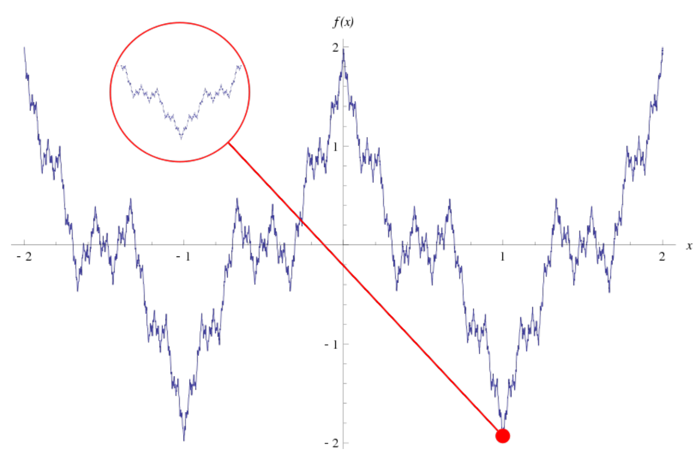



Figure 3.1: Graph of a Weierstrass function in the interval [-2, 2]

Recall that, since there are no breaks in the function, it is clear that this function is continuous from the graph. However, because of the fractal nature of the function, i.e.

that the jumps repeat themselves as if the line has a depth, one cannot take derivative of this function and one cannot see this from a graph of a continuous function. As in the above graph, one should represent the fractal nature of the function with additional information and drawings. Since the graph and/or intuition caused false conclusions, they were classified as untrustworthy tools in mathematics.

It is easy to see that the unreliable information comes from abstraction and not from the intuition or construction itself. If the concept and tools were available one could, from the beginning, create the second figure that shows the Weierstrass function, which then would not lead to the false conclusion. The visual tools were not sufficient to construct this concept at that time, but the fault is not on the intuition or construction. The problem with this is not seeing the difference between the two different processes: construction and abstraction. For these kinds of representations and constructions one needs interactive tools and even visual inference or guidance steps.

It is obvious that abstractions are inevitable in mathematical communication since writer's construction has to become reader's abstraction. Hence, one has to create rules for abstraction such as Manders' guidelines[2]. Moreover, visual constructions not only should adhere certain rules but also be in the form of inferential steps as much as possible, as if they are embodying the rules to allow the abstraction process to be flawless. Then operability of these representations will provide an objective communication similar to the symbols in algebra and analysis. Hence, to see which kinds of constructions lead to flawless abstractions, let us analyze the methods that admit valid visual representations in mathematics.

[2]Ken Manders is the first philosopher naming the distinction, exact and co-exact properties of diagrams. He characterizes *co-exact* properties of diagrams as "part-whole relations of regions, segments bounding regions,and lower-dimensional counterparts". ([85], p.69) This characterization is similar to Kantian claim "seeing the general in particular". In contrast, the *exact* properties of diagrams consist of their side lengths, the measurement of angles; namely, the properties that belong to a particular diagram.

## 3.2 "Construction" and "Synthetic *a priori*" method

I believe, Kant's approach to synthetic *a priori* method is underestimated in current philosophical practice for the explanation of mathematical process and also misunderstood in some parts. For example, it is often overlooked that in the *Critique* he disclosed the method in a way that explained even symbols such as $x$, $y$ as constructions as much as a geometrical figure of a triangle is a construction.

Kant took synthetic method to be central in mathematical reasoning. This was not only because the method was central to mathematics in 17th and 18th century and not only that practices in mathematics were carried out with constructions but also that there was this element: **the ability of a mathematician to carry out the right construction**. Kant disclosed this as an indispensable part of mathematical reasoning. And, this is the reason Kant says that "the sum of 7+5" is not contained in "12"; or, similarly, in Euclid's Proposition I. 32, geometrician knows which line to extend, which intuition to exhibit. According to Kant there is more than a mechanic procedure and there is more than what logic is capable of. Brilliant mathematicians and philosophers of mathematics, among few are Poincare, Weyl, Gödel, Feferman, Stekeler-Weithofer also point out the importance of intuition and construction in mathematics in this sense. Combination of construction and intuition was the starting point of the method that Kant disclosed as the synthetic *a priori* method.

Kantian characterization of mathematics provides us a perspective in current philosophy of mathematics that shows the possibility of another kind of valid reasoning other than formal sentential one. Formal sentential reasoning, which uses polyadic logic, proved itself to be an excellent method for providing the rigor, practicality and efficiency needed in mathematics and mathematical proofs. However, this does not mean that there should be only one way of reasoning in mathematics. Synthetic reasoning, or reasoning with visual tools, constructions, diagrams can prove itself to be beneficial rather than being superfluous and even harmful in mathematical proofs and communication. It is important to understand the difference between formal/ analytic reasoning and constructive/ synthetic reasoning to see the advantages and disadvantages of both. In order to comprehend fully what falls into these two categories, I will first give a brief overview of *analytic method* in mathematics, then, move onto the usage of analytic and synthetic method in mathematics which will shed light into the role and validity of visual reasoning in recent practices of mathematics.

The prominent and most respected method until the beginning of 20th century was synthetic method. Descartes introduced algebra as an analytic method in geometry and, in general, used the analytic method to assume the result and go backwards to find foundations in a science. (cf. [21], [23], [22]) The method of geometry was the representation with compass and ruler, or giving a description with a pointwise drawing and providing the solution with a geometrical construction. Analytical, in other words, algebraical solutions, when used without construction were not accepted in proofs. So, synthetic nature of Euclidean geometry and requirement of drawn shapes or thinking in diagrams were not questionable in 17th and 18th century, even though analytic methods were introduced by Descartes, Leibniz, Newton and others.

The common procedure was to take up a geometrical representation for a problem, construct the equation that is assumed to solve the problem algebraically, then construct the new geometrical representation by using higher degree curves to give the solution of the algebraic equation. This construction belongs to synthetic method- there are no constructions in the analytic method, only assumptions to be analyzed. Leibniz also used both methods, but favored the synthetic method. Analytic method was only to make the foundation stronger. It was more important for him to construct a science which would represent the reality.

Synthetic method in mathematics is based on geometric constructions as representations and as solutions. Henk Bos, in his article "The Concept of Construction and the Representation of Curves in Seventeenth-Century Mathematics" [11], points out to the tension, when Leibniz first introduced exponential equation to represent a curve. (cf. [11], p.23) Until then, the only equations to represent curves were the algebraic ones with a constant power such as $x^3 - 3x + q = 0$. However the one Leibniz proposed had a variable in the exponential, which was not an algebraic equation but a transcendental one. He gave as an example "the curve representing the relation between the time $t$ and the velocity $v$ of a body falling in a medium with resistance proportional to $v^2$ " and explained that the curve was represented by the equation $b^t = \frac{1+v}{1-v}$ . ([11], p. 28) His correspondent, Huygens' reaction was to classify this as empty symbolism.(cf. ([57] , Referred by [11], p.28)) Leibniz tried to convince Huygens by giving a pointwise construction and description of the curve. However, still, Huygens did not see the point of introduction of the symbolism for such a transcendental curve since **the curve was already "known" to him**.[3]

At these times, there were two important issues regarding geometrical construction with respect to the method of practice: (cf. [11], p. 26)

1. What means of construction should be used if problems cannot be constructed by ruler and compass?

2. What kinds of analytic methods can be integrated for geometrical construction?

For example, according to Descartes the construction applied was "the intersection of the higher curves" and the analytical method was algebra. This type of geometrical construction, constructing circles and straight lines to solve the degree 1 and 2

[3]It is worth to note that the description of a curve counted as the representation of the curve, and by this way it was "known", such as a pointwise construction as the method introduced by Descartes and verbal description such as saying "sequence of equidistant points". Nowadays a curve considered to be "known" mathematically if the equation of it is given. (cf. [90]. vol. 5 pp. 425-427 and [21], pp. 297-413; Eng. [23])

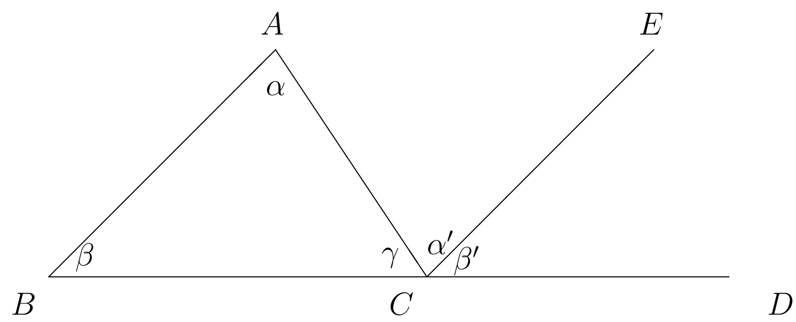
equations, were at issue almost for hundred years. The more progress was made in calculus and in analysis, the harder it was to keep up with the construction of equations geometrically. Then, the effort was not found useful and subject's importance faded off around 1750s. (cf. [11], p.30) Henk Bos explains the reason in more detail as:

> "The construction of equations originated as a sensible procedure within geometry. Purely algebraically, however, it does not make much sense. If a problem consists of a polynomial in one unknown, why should two polynomials in two unknowns constitute a solution? As the theory progressed, the techniques to find constructing curves became more and more algebraic. But the geometrical meaning of the subject -exact construction- and the geometrical criteria of adequacy -simplicity of the curves- refused translation into algebra in a natural way. The subject had a tendency to become algebraic, but its aims, criteria, and meaning proved untranslatable into algebra it succumbed to this internal contradiction." ([11], pp. 30-31)

Although geometric construction for solving equations faded, the ideal representation, which was different from constructions by ruler and compass became central in the mathematical practice, which is also present in Kant's approach. One does not have to give exact constructions by ruler and compass but an exhibited intuition, a single representative of a concept would suffice to represent a figure in a proof. Friedman, referring to A 715-717/ B743-745 [4] also reveals that:

[4] "Give a philosopher the concept of a triangle, and let him try to find out in his way how the sum of its angles might be related to a right angle. He has nothing but the concept of a figure enclosed by the three straight lines, and in it the concept of equally many angles. Now he may reflect on this concept as long as he wants, yet he will never produce anything new. He can analyze and make distinct the concept of a straight line, or of angle, or of the number three, but he will not come

"Kant is here outlining the standard proof of the proposition that the sum of the angles of a triangle = 180° = two right angles ([28], Book I, Prop. 32). Given a triangle $ABC$, one prolongs the side $BC$ to $D$ and then draws $CE$ parallel to $AB$ (see below figure). One then notes that $\alpha = \alpha'$ and $\beta = \beta'$, so $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 180°$




In contending that construction in pure intuition is essential to this proof, Kant is making two claims that strike us as quite outlandish today. First, he is claiming that (an idealized version of) the figure we have drawn is necessary to the proof. The line $AB$, $BD$, $CE$, and so on are indispensable constituents; without them the proof simply could not proceed." ([37], p.57)

Although Friedman categorizes these notions as outlandish, two points are crucial for us to note here: the usage of "idealized version of the figure" in mathematics and noticing that constructions were abandoned from mathematics before it came to the full fruition. All the fallacies that I mentioned, which were used for blaming visual representations were due to not having the right way of constructing or abstracting.

upon any other properties that do not already lie in these concepts. But now let the geometer take up this question. He begins at once to construct a triangle. Since he knows that two right angles together are exactly equal to all of the adjacent angles that can be drawn at one point on a straight line, he extends one side of his triangle, and obtains two adjacent angle which is equal to an internal one, etc. In such a way, through a chain of inferences that is always guided by intuition, he arrives at a fully illuminating and at the same time general solution of the question." ([65], A 715-717/ B743-745)

Another point to keep in mind is that analytic method or analysis is only one method which does not have to exclude another method in mathematics. We will see the importance of these emphases in the next section.

### 3.2.1 The Method of Visual Reasoning in Mathematics

In Euclidean proofs there is a motivation to use the particular instances of general concepts and that is in the setting out part. (cf. [53], p.28) The geometer almost creates his own figure, by choosing necessary properties from the general concept. When the geometer says "Let ABC be a triangle" she does not have to draw a figure in experience, but she needs to exhibit the figure *a priori*, so that she can have a singular representation that carries general properties and so that she can extend one edge of the triangle etc. by using construction. One cannot add lines only by logical rules and axioms, that is why Kant thought that in construction for Euclidean proofs geometer needs the *a priori* intuition. Moreover, these *a priori* intuitions are instantiated from one of only two forms of *a priori* intuitions, namely space -a Euclidean space- and rules of Euclidean geometry. This is the visual reasoning and constructions we want to start with. As I mentioned in the previous chapter, cognitive findings even sugguest that we conceptualize in this manner and construct *a priori* some of the properties of Euclidean geometry (cf. [59])

The importance of *a priori* exhibited intuitions are that although they are particular representations that still admit the general properties that is required in the mathematical reasoning, they are **idealized singular representations**. The visualization is independent from experience in the sense that one does not have to apply to experience to visualize a triangle, it is not an abstraction, **it is displaying the information and relations/ rules in the mind's eye**.

Moreover, Kantian intuition of mathematical objects are representations both in sentential and visual reasoning. Namely, linguistic symbols and diagrams are particular representations of general concepts which still carry the general properties by only this particular representation, intuition.

For example, in an informal proof of "There are infinitely many prime numbers." as I will show in the below, the symbols function as intuitions, embodying the necessary parts of a general concepts in idealized singular representations.

*Proof.* Proof by Contradiction:

Assume that there are finitely many primes. Let $p_1, \ldots, p_n$ be the list of all primes in an increasing order and $p_n$ be the greatest prime number. Now set

$$p = p_1 \cdots p_n + 1.$$

Then $p$ is a prime since none of the previous prime numbers $p_1, \ldots, p_n$ divides $p$ .

$\therefore$ There is no greatest prime number.

$\therefore$ There are infinitely many prime numbers.

□

Above is a construction with symbols that conceptually can be written as:

"Suppose there are only finitely many primes, and we have an ordered list of all of them. Now consider the numbers that is the product of all these primes plus one. Either this new number is prime or it is not. If it is prime then we have a prime number that is larger than all those originally listed; and if it is not prime then, because none of the numbers on our list divide this number without remainder, this new number must have a prime divisor larger than any of the primes on our list. Either way there is a prime number larger than any with which we began. Q.E.D." (cf. [83])

In this proof $p_1, \ldots, p_n$ represent prime numbers in a singular way. The embodiment and iterative process make this reasoning and representation objective and valid.

Note that, the embodiment of the rules in a representation, be it a symbol or a geometrical figure, can occur both internally and externally.

In the ***external visualization*** one draws the construction on a paper or a medium, this construction can be a representation of a sitting system for combinatorial analysis as below:

| | | | |
|---|---|---|---|
| ABCD | BACD | CABD | DABC |
| ABDC | BADC | CADB | DACB |
| ACBD | BCAD | CBAD | DBAC |
| ACDB | BCDA | CBDA | DBCA |
| ADBC | BDAC | CDAB | DCAB |
| ADCB | BDCA | CDBA | DCBA |

It can also be a Venn diagram, a graph, a triangle, sequence of symbols... All these represent a construction of a rule according to a problem and the ability of the constructor/ mathematician.

In the ***internal visualization*** one uses the so called mind's eye, constructs the visual representation in the imagination. This is partially what Kant calls the work of schema and construction, and partially the representation of exhibiting the intuition. Once this figure is constructed it can stay in this level as an internal representation or can be made an external representation by drawing it on a medium. Mathematicians use their hands to represent the internal visualizations in space as well, by drawing circles in the air or representing a Brownian motion by using their fingers and an imaginary space on a surface to communicate the visual representations. So, internal visualizations' medium can be decided by the individual, but in any case this is an imaginary space used by the minds eye. One can imagine a screen in the mind, in front of the eyes around 20 cm away, or between two communicators, no matter where this space is constructed there is no usage of external 3D figures and no usage of external drawn visual representations.

Now that we reviewed the synthetic method and representations, I will provide a detailed analysis of synthetic and analytic method and how Kant arrived at these notions to strengthen the groundwork of setting up synthetic method as a legitimate method in mathematics.

### 3.2.2 Analytic and Synthetic

Analytic as a method goes as far back to Plato [13], and introduced with a rigorous approach by Descartes[22], Leibniz [77] and Kant. Hartman and Schwarz point out that:

> "All three of these were philosophers [Descartes, Leibniz, Kant] as well as scientific creators. Their philosophy of science was part of their creating a science - analytic geometry, calculus, theory of Heavens, and scientific metaphysics, respectively..." ([63], p. lviii)

As I mentioned in the previous sections, the philosophers who used analytic method with synthetic method were aiming to establish or develop a science that will both do the analysis for the building block of this science and also to create new concepts, establish its boundaries, apply the synthesis for creating this science's own concepts. Descartes introduced algebra as an analytic method in geometry and, in general, used the analytic method to assume the result and go backwards for discovering the foundations in a science. The procedure was to take up a geometrical representation for a problem, construct the equation that is assumed to solve the problem algebraically, then construct the new geometrical representation by using higher degree curves to give the solution of the algebraic equation. The construction belongs to synthetic method here, there are no constructions in the analytic method, there are only assumptions to be analyzed. Leibniz also used both methods, but favored the synthetic method. Analytic method was only to make the foundation stronger. It was more important for him to construct a science which would represent the reality. Vincenzo de Risi points out the importance of synthetic method for Leibniz in his book *Geometry and Monadology* as follows:

> "Leibniz recognized the shortfalls of algebraic calculus in geometry at least as knowingly as Newton did – he never was fully satisfied with it. On the contrary, he insisted that the synthetic method of the elders was a great deal more elegant than the analysis of the moderns. Also, he stressed again and again that many results were obtained through algebra only at the price of extravagantly demonstrational rigmaroles, most of them arbitrary and misleading if compared to the simplicity of the problem. Here as elsewhere, Leibniz wanted to be both an innovator in method and a champion of the old masters' excellence. On this point, apparently, he completely agreed with Newton." ([17], p. 11)

Where Descartes applied the method to Geometry and Leibniz to his natural philosophy, Kant, by following their step, applied this method to his philosophy and has established a new kind of metaphysics, which he called transcendental philosophy.

Kant first found out the least simple notions of pure understanding, by analytic method, by abstraction. Then, he used synthetic method to unite these simple notions to construct new synthetic concepts. In other words, by the method of analysis, he established the pure concepts of understanding, categories. Then he used these formal conditions of sensibility, categories, to be the root of the sensible construction. The link was *schema*, "the end of analysis and beginning of synthesis" ([63], p. lxi).

### 3.2.3 Analytic and Synthetic in the Kantian Transcendental Philosophy

The *Critique* is a work *in* philosophy and *about* philosophy where the two methods, analysis and synthesis are used together interchangeably. Kant uses analysis to clarify the meanings of the elements of pure reasons, to approach to a distinct and clear definition of them. This is the analytic method, which belongs to the philosophy and is used within philosophy. Then he uses synthetic method to construct a new science, the metaphysics, the philosophy of philosophy. He arrives at new definitions. It is similar to the science of mathematics according to Kant: The concepts arise at the same time with the definition, objectively and definitely.

The definitions are certain and ready to establish a new science. This is the synthetic method, namely arriving at new definitions, from pieces of information. Analysis and synthesis can go hand in hand together. One can always analyze the concepts constructed by synthetic method, to understand them better. On the other hand, when a new science arises, one can synthesize concepts that are defined by the analytic method, to arrive at new concepts. Hartman and Schwarz also point out that:

> "Philosophy, according to Kant, makes concepts given in everyday discourse analytically distinct and articulates the notes or predicates of conceptual intensions or contents. Science, on the other hand, interrelates synthetically concepts not given but made, *the definitions of which are at the same time the constructions of the corresponding objects*." ([63], p.xix)

**Definitions vs. Expositions.** Kant explains the possibility of synthetic *a priori* judgments with the judgments and concepts used in mathematics. These concepts are equivalent to their definitions. Since the method of mathematic is synthesis, and since the constructions are carried out on *a priori* means, at the end of the construction, a new objective concept arises and the definition also arises with the constructed concept. Kant calls these constructions also *declarations* "since in them one declares one's thoughts or renders account of what one understands by a word". ([63], §103) [5]

A definition according Kant "is a sufficiently distinct and delimited [precise] concept". ([63], §99) [6] Kant believes that only mathematics and natural sciences have definitions. (cf. [65], A730/B758) Everyday discourse and philosophy has exposition, explication and description. (cf. [65], A730/B758 and [63],§105 for [63])

In philosophy one can start with a complex concept and divide it into parts to get a better understanding of the concept (cf. [65], A717/B745). According to Kant,"[s]ynthetic empirical definitions are impossible and analytic definitions are uncertain. The only kind of definitions that are logically both definite and certain are

[5] "... und vor alle dem, was vermittelst eines willkürlichen Begriffs gesagt wird, vorangehen müssen, könnte man auch Deklarationen nennen, so fern man dadurch seine Gedanken deklariert oder Rechenschaft von dem gibt, was man unter einem Worte versteht." ([70], Allgemeine Methodenlehre, §103 )

[6] Eine Definition ist ein zureichend deutlicher und abgemessener Begriff (conceptus rei adaequatus in minimis terminis; complete determinatus). ([70], Allgemeine Methodenlehre, §99 )

constructive, synthetic definitions. Analytic definitions can only be approximate, and these approximations to definitions Kant calls *expositions* and *descriptions*." ([63], p.xxvii) Moreover, "[d]efinition, then, for given concepts, is a goal which we asymptotically approach."([63], p.xxvii)

In other words, the synthetic method starts with simple concepts and construct new concepts. Analytic method starts with given simple or complex concepts and is concerned with the clarification of these concepts. There is qualitatively improved knowledge in the analytic method. On the other hand, there is a quantitative new knowledge in synthetic method unlike analytic method. (cf. [63],p.xxxix) The judgments created through these processes have the same properties, namely, synthetic judgments have new information about the subject and analytic judgments helps only to clarify the subject.

Analytic method helps one to understand the concept at stake better. With the analytic judgments, one *approaches* to the complete definition of the concept. However, one can never have the complete definition. The definition can be almost complete, but the concept and the definition can never be equivalent. Analysis is about finding the extension of a concept which is infinitely many. Kant reveals this as:

> "Distinctness of cognitions and joining them into a systematic whole depends on distinctness of concepts both in respect of what is contained *in* them and in regard to what is contained *under* them. The distinct consciousness of the *intension* of concepts is furthered by their *exposition* and *definition*; the distinct consciousness, however, of their *extension*, by their *logical division*." ([63], §98) [7]

[7]Die Deutlichkeit der Erkenntnisse und ihre Verbindung zu einem systematischen Ganzen hängt

For example, if one analyzes the *a priori* concept triangle, she will arrive at the properties (Teilbegriffe): lines, angles, number 3, 3 lines, 3 angles and so on and the extensions are: isosceles triangle, acute triangle, etc. However, if one constructs the *a priori* concept triangle with the synthesis, she will start with lines, angles, number 3 and arrive at the concept triangle, this is **declaring of the concept**.

Synthetic method constructs new concepts which are called synthetic concepts. In the case of mathematics these concepts are always possible and they exist due to the rules they posses in their generation as I explained in detail in the "Schema" section and due to their exhibition in pure intuition. Synthetic concepts, other than these mathematical concepts, do not have to "be possible". One can construct a unicorn by synthesizing a horn and a horse, however this is a concept with no possibility of existence.

**To make a distinct concept vs. to make a concept distinct.** Kant characterizes synthetic- analytic difference as the distinction between *to make a distinct concept* and *to make a concept distinct*:

> "For when I make a distinct concept, I begin with the parts and proceed from these to the whole. There are not characteristics present here; I obtain them first by synthesis. From this synthetic procedure then results synthetic distinctness, which actually expands my concept as to content by what is added as a characteristic *over* and *above* the concept in intuition (pure or empirical). This synthetic procedure in making distinct concepts

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ab von der Deutlichkeit der Begriffe sowohl in Ansehung dessen, was in ihnen, als in Rücksicht auf das, was unter ihnen enthalten ist. Das deutliche Bewußtsein des Inhalts der Begriffe wird befördert durch Exposition und Definition derselben; - das deutliche Bewußtsein ihres Umf anges dagegen durch die logische Einteilung derselben.- Zuerst also hier von den Mitteln zu Beförderung der Deutlichkeit der Begriffe in Ansehung ihres Inhalts. ([70], Allgemeine Methodenlehre, §98 ) is employed by the mathematician and also by the philosopher of the nature. For all distinctness of mathematical as well as of experiential cognition rests on expansion through synthesis of characteristics.

But when I make a concept distinct, then my cognition does not in the least increase in its content by this mere analysis. The content remains the same; only the form is changed, in that I learn to distinguish better or with greater clarity of consciousness what already was lying in the given concept. Just as by the mere illumination of a map nothing is added to it, so by the mere elucidation of a given concept by means of analysis of its characteristics no augmentation is made to this concept itself in the least." ([63], p.70) [8]

Accordingly, *to make a distinct concept* is to construct a concept which is exhibiting intuition *a priori* corresponding to the concept. The end result is synthetic *a priori* concept. *To make a concept distinct* is to analyze a concept, there is no new product, there is an abstraction and finding the building blocks of the concept. It is for understanding the given concept better, to clarify it.

Kant, moreover claims that:

[8] Denn wenn ich einen deutlichen Begriff mache: so fange ich von den Teilen an und gehe von diesen zum Ganzen fort. Es sind hier noch keine Merkmale vorhanden; ich erhalte dieselben erst durch die Synthesis. Aus diesem synthetischen Verfahren geht also die synthetische Deutlichkeit hervor, welche meinen Begriff durch das, was über denselben in der (reinen oder empirischen) Anschauung als Merkmal hinzukommt, dem Inhalte nach wirklich erweitert. - Dieses synthetischen Verfahrens in Deutlichmachung der Begriffe bedient sich der Mathematiker und auch der Natur-Philosoph. Denn alle Deutlichkeit des eigentlich mathematischen so wie alles Erfahrungserkenntnisses beruht auf einer solchen Erweiterung desselben durch Synthesis der Merkmale.

Wenn ich aber einen Begriff deutlich mache: so wächst durch diese bloße Zergliederung mein Erkenntnis ganz und gar nicht dem Inhalte nach. Dieser bleibt derselbe; nur die Form wird verändert, indem ich das, was in dem gegebenen Begriffe schon lag, nur besser unterscheiden oder mit klarerem Bewußtsein erkennen lerne. So wie durch die bloße Illumination einer Karte zu ihr selbst nichts weiter hinzukommt: so wird auch durch die bloße Aufhellung eines gegebenen Begriffs, vermittelst der Analysis seiner Merkmale, dieser Begriff selbst nicht im mindesten vermehrt.([70], VIII. C) Logische Vollkommenheit des Erkenntnisses der Qualität nach - Klarheit, p. 70 )

> "All *given* concepts, be they given *a priori* or a posteriori, can only be defined through *analysis*. For given concepts can only be made distinct by making their characteristics successively clear. If *all* characteristics of a given concept are made clear, the concept becomes *completely* distinct; and if it does not contain too many characteristics, it is at the same time precise, and from this springs a definition of the concept. **Note.** Since one cannot become certain by any proof whether all characteristics of a given concept have been exhausted by complete analysis, all analytic definitions must be held uncertain." ([63], §104) [9]

For that reason, the definition of *a priori* concepts when approached by analysis can only be an approximation. This is because, the method requires finding all the parts, and one can never be sure to find "all the parts" in a given concept. With a new knowledge one can possibly find more information about the concept. That is why it is an approximation. However, when one constructs an *a priori* concept it is a made concept, the constructor decides when the concept is complete, hence, gives a new definition together with creating the concept. On the other hand, this process obeys the rules of synthetic *a priori* rules of Euclidean geometry. With the empirical synthetic concepts, the case is different.

For example, number 4, gray, head, feet, can make up the concept dog, the description of a dog can be given this way, but it will never have the certainty of the definition of a synthetic *a priori* concept as, say, triangle. One can only "learn

[9] Alle gegebene Begriffe, sie mögen a priori oder a posteriori gegeben sein, können nur durch Analysis definiert werden. Denn gegebene Begriffe kann man nur deutlich machen, so fern man die Merkmale derselben sukzessiv klar macht. - Werden alle Merkmale eines gegebenen Begriffs klar gemacht: so wird der Begriff vollständig deutlich; enthält er auch nicht zu viel Merkmale, so ist er zugleich präzis und es entspringt hieraus eine Definition des Begriffs. Anmerk. Da man durch keine Probe gewiß werden kann, ob man alle Merkmale eines gegebenen Begriffs durch vollständige Analyse erschöpft habe: so sind alle analytische Definitionen für unsicher zu halten. ([70], Allgemeine Methodenlehre, §104)

through experience what belongs *to*" the empirical concepts and "not analyze what lies *in* them". ([63], §102) [10]

By understanding the 18th century characterization of analytic and synthetic methods, and by revealing clearly Kantian characterization of these methods and the concepts arose from them, we now have clear notions to analyze Kantian characterization of mathematics. In this section, I examined the analytic and synthetic method with a brief historical approach in connection with its influence on Kant's thought system and how Kant uses them in his critical philosophy.

This analysis also shows us that the synthetic method Kant took central in his philosophy, is not about synthetic *judgments* only, which is a popular approach in recent "interpretations of Kantian synthetic *a priori*". I believe, **the analysis given to Kant's analytic and synthetic notions, starting by only the analytic and synthetic *judgments* and meaning analysis cannot give adequate analysis of Kant's thought system**. With this line of reasoning, let us look at the analytic and synthetic *judgments* in Kant's transcendental philosophy.

### 3.2.4 Analytic and Synthetic as *Judgments*

According to Kant in analytic judgments the concept of predicate is contained in the subject. These kind of judgments are affirmative. They clarify the meaning of a subject. In synthetic judgments the predicate has quantitatively more information than subject. (cf. [65], A7/B11) This is the most common definition that is used to evaluate Kantian analytic and synthetic judgments. The examples Kant use to clarify this description are:

[10] Denn auch bei den Begriffen der letztern Art, z. B. den empirischen Begriffen Wasser, Feuer, Luft u. dgl. soll ich nicht zergliedern, was in ihnen liegt, sondern durch Erfahrung kennen lernen, was zu ihnen gehört. - Alle empirische Begriffe müssen also als gemachte Begriffe angesehen werden, deren Synthesis aber nicht willkürlich, sondern empirisch ist. ([70], Allgemeine Methodenlehre, §102)

"All bodies are extended" (analytic)

"All bodies are heavy" (synthetic)

In the first case all one needs to do is to analyze the concept "body" to reach to the concept "extended". Hence the concept "extended" is contained in the concept "body" but this is not the case in the second example.

Historically, there are at least two main approaches to analytic and synthetic judgments which critically take into consideration Kant's above description: The most popular two are Frege' and Quine's approaches. According to Frege when a proof of a true proposition is carried on by purely logical means and when the premises and definitions of the terms can be given logically, then this proposition is analytic. If one has to use, in the argumentations of a true proposition, rules belonging to some special science apart from logic and/or terms that are not logically described then this proposition is synthetic. (cf. [34], §3) Frege took the analytic method to be logic and gave an extensively developed logical system. Although the descriptions of analytic and synthetic seem quite different from Kant's and 18th century philosophy, the method used is very similar: **Find an analytic method, [11] then apply this method to newly created science to make it rigorous**.

Quine talks about the distinction between the analytic and synthetic propositions in "Two Dogmas of Empiricism" [110]. He defines analytic propositions as based on the meanings and independent from the facts, and the synthetic propositions as the propositions that are grounded on the facts. After arguing whether there can be any distinction between analytic and synthetic propositions he concludes that "For all *a priori* reasonableness, a boundary between analytic and synthetic statements simply has not been drawn. That there is such a distinction to be drawn at all is

[11]Example of analytic methods: Algebra (Descartes and Leibniz), finding extensions by analysis (Kant) and quantificational logic (Frege).

an unempirical dogma of empiricists, a metaphysical argument of faith" ([110], p. 37). Moreover, Quine finds Kant's definition of analytic, as predicate contained in the subject, as metaphorical, therefore, too vague.

If one starts to evaluate analytic judgments only by this description "predicate contained in the subject" and does not take into consideration the importance of the method and structure that was prevalent since 17th century, the description of course looks vague or metaphorical. However, modern analysis of Kantian synthetic and analytic judgments have been focusing only on this aspect.

I believe, it is easier and more secure to look at the method used in order to distinguish synthetic *a priori* judgments from analytic ones as I have explained throughout previous section. If one is working with the analytic method and with the purpose of clarifying concepts with the judgments, then these judgments are analytic judgments. They are used for breaking down the concepts. If one is using the synthetic method, then the judgments are about constructing a new concept. Then these judgments are synthetic judgments.

**An analytic judgment, when it is used in mathematics with the synthetic method becomes a synthetic judgment.**[12] For that reason it does not matter whether a pure logical statement is used in the argumentation steps. When this general analytic proposition is used in mathematics it becomes synthetic according to Kant. When general concepts are used, they are used with the corresponding intuitions. This interpretation makes the Kantian characterization of mathematics immune to the developments from syllogistic logic to quantification logic and axiomatization in geometry and mathematics. This is an overlooked issue in many interpretations of Kant.

[12] However, note that, a synthetic judgment when analyzed with analytic method still remains as a synthetic judgment.

To summarize, synthetic *a priori* reasoning is objective according to Kant since *a priori* intuitions are instantiated from general concepts, which are synthesized by the rule schema in a way independent from experience. These intuitions are also exhibited *a priori* in pure intuition, space. Both the process and the basis borrow nothing from experience in this case. Then, intuitions exhibited in this *a priori* ground are objective. Hence, synthetic *a priori* judgments are necessarily true. Moreover, since intuitions exhibited independent from experience on the objective grounds and by carrying the general properties of the concepts by synthetic *a priori* propositions, synthetic a priori reasoning is valid. Furthermore, the reasoning style of visual reasoning is essentially synthetic *a priori*, and one can see the validity and objectivity of the method by reflecting on Kant's arguments on the synthetic *a priori* method.

Kant argues that, mathematical reasoning can be practiced *only* in synthetic *a priori* way and he constructs from within his theory how synthetic *a priori* judgments can be formulated as necessary and objective judgments, and how this reasoning can be taken as a valid and legitimate one, as I have disclosed in this chapter. My results in this work, although based on Kant's characterization of mathematics, does not, of course, defend that synthetic *a priori* reasoning or visual reasoning in mathematics should replace formal sentential reasoning in mathematics or the only way to reason in mathematics should be synthetic *a priori.* The purpose of this work is to generate sufficient motivation in philosophy of mathematics for the refinement of visual reasoning in mathematics. Formal sentential method has many advantages, including practicality in proofs in mathematical analysis, and rigor it provides. In Chapter V, I will discuss in more detail how a legitimate account for visual reasoning can be given and how this method can be used together with formal sentential reasoning or in some areas as an alternative to the formal one.

The analysis given in this chapter help us to comprehend in which ways visual reasoning in mathematics can be constructed as a legitimate theory. In the next chapter, I will give interpretations of synthetic *a priori* characterization of mathematics according to Kant, comparing it to my account. I believe, most of the prejudice towards visual reasoning and/ or synthetic reasoning can be seen in these interpretations and, in a way, these interpretations are also the ground for generating more prejudice against visual reasoning in mathematics. Moreover, next chapter will provide us tools to evaluate Kant's philosophy of mathematics in connection to current philosophy of mathematics.

## CHAPTER IV

# Interpretations on Kant's notion of Space and Synthetic *a priori* Method

As far as the laws of mathematics
refer to reality, they are not certain;
and as far as they are certain,
they do not refer to reality. [1]

-Albert Einstein

The previous chapter examined Kantian characterization of mathematics pointing out the importance of synthetic method and *a priori* characterization of intuitions and concepts with the aim of bringing clarification to the visual reasoning in mathematics. This chapter is concerned with analyzing several illuminating interpretations on Kantian mathematical intuitions, constructions, judgments and also his notion of space.

One of the questions in interpretations of Kant's philosophy of mathematics and epistemology is, "Why does Kant construct space and time as pure intuitions?" (cf.

[1]([26]) Original version: "Insofern sich die Sätze der Mathematik auf die Wirklichkeit beziehen, sind sie nicht sicher, und insofern sie sicher sind, beziehen sie sich nicht auf die Wirklichkeit." ([25], p.3)

[133], pp. 65-66, [116]) Some argue that construction of space as intuition, as revealed by Kant, is not a sound theory. (cf. [133], p.57, pp. 65-66) Moreover, some even claim that construction of space is actually a concept according to Kant. (cf. [116]) My aim in this section is to discuss why categories or concepts were not enough for Kant's theory, or why space was not one of the concepts or categories, by examining different interpretations given to Aesthetic and thereby strengthening my argument in the previous chapters.

In this section, by revealing different interpretations, I aim to present, in Kantian thought system, space as an *a priori* concept could not lead to a complete transcendental philosophy and epistemic theory. Kant needed some ground to exhibit *a priori* intuitions, so that his theory could allow for objective synthetic constructions by the use of intuitions. Kant's theory of our cognition of empirical objects in relation to space we posses cognitively, is based on the relation between Euclidean figures and Euclidean space. This pure intuition, space, is equivalent to Euclidean space in Euclidean geometry. His aim in the Aesthetic is to show this equivalence. In this way, he can prove our structure for cognition as objective, since the form of the sensible intuitions must conform to the *a priori* synthetic Euclidean propositions. Moreover, since Euclidean propositions cannot be deduced by mere analysis of space, similarly, our cognition requires the internal *a priori* space for carrying out something more than analysis, namely to carry out the *a priori* constructions and to inhibit forms of the empirical objects.

The objections to Kant's theory of space mainly attacks to his basing the notion of space on Euclidean propositions and these propositions being synthetic *a priori*. They argue that this approach is not valid anymore since we can formally represent the Euclidean geometry by using Hilbert's formulations. (cf. [133], pp. 65-66; [45], p. 367 and for Hilbert's method cf. [51]) As I have indicated in the previous chapter and argue in detail in the next chapter, to have a formal finitary system for Euclidean geometry does not have to make the non-axiomatic and visual one invalid. Keeping these details in mind, in this section, I first examine an unconventional approach to Kant's exposition of space by Shabel as it is revealed in "Reflections on Kant's concept (and intuition) of space"[116], "Kant's 'Argument from Geometry"'[117] and a response to it by Friedman as it is revealed in "Kant on Geometry and Spatial Intuition" [38] to connect it to my own argument in the previous chapter. In the following section I aim to show that space is not constructed as a concept in the Aesthetic.

Next, I analyze Sutherland's interpretation on the homogeneity of space and the requirement of intuitions in arithmetic in Kantian characterization of mathematics as they are revealed in "Kant's Philosophy of Mathematics and the Greek Mathematical Tradition" [128] and "Kant on Arithmetic, Algebra, and the Theory of Proportions" [129]. Sutherland argues that intuition is required for geometry, arithmetic and algebra and for that reason mathematics require space as pure intuition. He discusses the strict homogeneity of space versus homogeneity of counting concept in arithmetic and connects these notions by Eudoxian theory of proportions, arguing that arithmetic also needs strict homogeneity and, therefore a spatial differentiation, by using pure intuitions and exhibiting quantitative differences. This section aims to show that Kant's argument of requirement of intuitions in mathematics and his construction of space as pure intuition are sound arguments. I also indicate the validity of internal representations as *a priori* intuitions in current philosophy of mathematics. I believe this particular Kantian approach, space as pure intuition, can help us to see internal diagrams[2], as *a priori* intuitions.

[2] I am not arguing that all internal diagrams should be *a priori* intuitions, however all *a priori* intuitions, in this case, should be internal diagrams.

## 4.1 Spaces's Original Representation is not a Concept

In Transcendental Aesthetic, Kant provides two expositions for space being a pure intuition and form of sensibility as I have discussed in the previous chapter. Contrary to my interpretation Lisa Shabel characterizes Metaphysical Exposition as a step for Transcendental Exposition. (cf.[116], p.47) I have argued that Metaphysical Exposition's task was to find the properties of the concept space by analytic method. Moreover, by finding the characteristics that do not have accustomed characterizations of a general concept, Kant concludes that original representation of space is not a concept. The reason he says "concept of space" is a matter of naming and giving a meta analysis of the notion of space. So, in the Metaphysical Exposition, only for mentioning purposes he chooses to call the representation of space as the "concept of space". The difficulty of Aesthetic lies in it being a meta-analysis: by using the methods of itself, it aims to lay out the foundations of its science.

Shabel sees the difficulty of Aesthetic in elsewhere, and gives a reversed interpretation starting from constructions of pure spaces to construe SPACE (concept of space).[3] She also categorizes pure intuitions as empirical intuitions, which differ only in their usage from empirical intuitions. She argues that "the pure intuition is distinguished on the basis of *how* we construct and attend to the individual drawn figure"([115], p. 94). However, in Kant's doctrine the pure intuition is the basis and the source of the generation of these drawn figures. I believe, in this case Friedman rightly summarizes the Kantian argument for pure intuitions and suggests that:

[3]She uses the capital lettered SPACE to designate the concept of space. She argues that: "The Kantian way to put this point would be to say that our concept of SPACE unifies the pure sensible manifold in a formal intuition, thus representing space as an object. See the infamous footnote at B160, as well as one less infamous at B136. In what follows I will emphasize Kant's distinction between the concept of SPACE and the construction of individual spaces represented as intuitions. In order to track this distinction, I will use 'SPACE' when I wish to designate the concept of space and 'space' or 'spaces' to designate intuitions of finite spatial regions. In what follows, I hope to articulate the sense in which SPACE is a 'principle' that governs the construction of spaces."([116], p.46)

> “Kant begins with general concepts as conceived within the Leibnizean (logical) tradition and then shows how to “schematize” them sensibly by means of an intellectual act or function of the pure productive imagination. Both the general concepts in question and their corresponding general schemata are pure rather than empirical representations; and a particular concrete figure occurs, as it were, only incidentally for Kant, at the end of a process of intellectual determination of pure (rather than empirical) sensibility. The more general point underlying these considerations is that pure intuition, for Kant, is the form of (empirical) intuition: it lies in wait prior to the reception of all sensations- the corresponding matter of (empirical) intuition- as an *a priori* condition of the possibility of all sensory perceptions and their objects.” [38]

On the other hand, with her unconventional interpretation Shabel has to allege that the arguments in the Metaphysical Exposition are partial and does not explain the *a priority* of space. She states that “[t]he exposition is partial because these claims are not sufficient on their own to describe the *a priority* of space, but only its intuitivity.”([116], p. 51, n.17)

As Friedman, I also find these emphases misplaced[4] (cf. [38]) and I will give my reasons in the following.

Shabel interprets Kantian space to be an object of our cognition and the concept of space as the sum total of partial representations of space which she identifies as intuitions. (cf. [116], p. 46) She does not consider the space as a basis for intuitive representations as I have argued in detail in the previous chapter. Therefore,

[4]Friedman only talks about her approach to pure intuition as being another form of empirical intuition. But with this line of reasoning, the criticism naturally applies to her interpretation on the concept of space and her reassessment of the Metaphysical and Transcendental Exposition of space.

her interpretations result estimating Kant's arguments as partial in the Metaphysical Exposition for establishing the *a priority* of space. Kant merely uses the term 'concept' in his analysis of space for talking about the the representation of space, for meta- analysis purposes. That is why this section is confusing. Once he reveals space as pure intuition, by arguing that it does not carry the properties of a concept, it is clear that space is a singular representation independent from experience in its generation and exhibition, allowing to represent magnitudes intuitively for the synthetic method and synthetic constructions.

Moreover,she argues that:

> "A concept of SPACE provides us with a 'principle' that governs our pure intuitions of individual spatial regions, which it is the business of (pure Euclidean) geometry to construct and investigate; pure geometry in turn provides us with the means for structuring our (spatial) experience of empirical objects." ([116], p. 46)

Shabel differentiates Kantian cognitive space and the Euclidean space, and takes cognitive space prior to Euclidean space. (cf. [117]) However, Kant aims to show the equivalence of the cognitive space to the Euclidean space. He injects Euclidean space that was argued as external into the subject by stating that it is "nothing other than merely the form of all appearances of outer sense, i.e., the subjective condition of sensibility" ([65], A26/B42).

The difficulty with her interpretation arises in many places in her texts. She argues that:

> "Given that the science of sensibility nevertheless begins with examination of a concept, it is worth asking what role there is for a concept to play

> in explaining the uniquely intuitive contribution that sensibility makes to our cognition of objects." ([116], p. 49)

She interprets space as a "concept" and for that reason, keeps encountering conflicting interpretations difficult to resolve.

Another difficulty arises when she argues:

> "Presumably, then, where Kant identifies SPACE as a concept, he means to identify it as a pure aesthetic concept...Now, one might be tempted to suppose that Kant was merely careless to describe SPACE as an aesthetic concept, since he clearly argued that our representations of space are intuitive." ([116], p. 49)

She alleges that the reason Kant presents the argument in this way "is to isolate a 'principle' of sensibility". ([116], p. 50) Moreover, she supports this claim by arguing that "the aesthetic concept of SPACE conditions or governs our capacity to form intuitions of any and all particular spatial regions". ([116], p. 50) Kant never argues for a requirement of concept of space to intuit space. This emphasis is truly misleading and makes difficult to have a complete theory of Kant's characterization of mathematics. Moreover, she says that we have a "capacity to represent space intuitively, and which serve thereby as a partial exposition of our concept of SPACE"([116], p. 51) according to Kant. Space, for Kant, is never an object of intuition. Space is a singular representation, but it precedes all the outer representations/intuitions as it is made clear in the Aesthetic. Kant states that "[s]pace is necessary representation, *a priori*, which is the ground of all outer intuitions. " ([65], A24/ B38-39)

Shabel, lastly, bears on that:

> "On first reading, these remarks seem to imply that our representation of SPACE is a representation of a single (and unique) whole, which is infinite, or unbounded in extent. And on first reading, it is difficult to know what to make of this. It is impossible that such features be represented in a concept"([116], p. 52)

Her supportive claim for this argument is that "it seems equally impossible to intuit a single infinitely large object". ([116], p. 52)

The difficulty with this interpretation is that she takes a reverse approach of what Kant originally argues for. Therefore, she has to conclude that space is a concept and it is impossible to intuit a single infinite object. She takes space to be the object of intuition, and characterizes empirical intuitions as the source of pure intuition. However, Kant's main argument develops just the opposite way: Space is the source of *a priori* intuitions as a basis and space is the natural consequence of the formation of human mind just as the categories. This is the reason Kant says that these notions can be confused with one another. (cf. [65], A51-2/B75-6 and A719/B747) Moreover, it is indeed the Euclidean space which contains the rules of Euclidean principles. Hence, the cognitive space, which is equivalent to the Euclidean space, contains the Euclidean rules, as the forms of sensibility.

Moreover, although one should not disregard the function of the schema and the role of categories in generation of pure sensible concepts, there is one more issue in the generation of these concepts. As Emily Carson points out in "Kant on the Method of Mathematics" [14] and as I argued in the previous chapter, synthetic definition arises with the *a priori* synthetic concept and it is complete definition of the concept since it is constructed in a synthetic *a priori* way.

Carson discloses that:

> "... the concept of a square is the arbitrary combination of the concepts four-sided, equilateral, and rectangle. This is not the result of an analysis of some concept given in another way it is not, for example, abstracted from our experience of squares in nature; the concept is, as Kant says, first given by the definition itself." ([14], p.633)

It is a very distinguished fact, contrary to what Shabel argues, that pure sensible concepts, or pure intuitions are generated independently from experience, and there is no abstraction process in their construction as Carson and Friedman rightly point out. By taking up the interpretation I suggest, which sides more with Friedman and Carson on these issues, Kant's theory in Metaphysical Exposition is freed from being partial, and by this way explains both syntheticity and *a priority* of space.

It is a natural consequence of Shabel's interpretation, to find the the argument in A25/B40 puzzling, as she mentions below, since she takes the emphasis on the empirical intuitions generating pure intuitions, rather than pure intuitions preceding them, and being the form of them. Shabel argues that:

> "He [Kant] proceeds to argue, famously, that 'the original representation of space is an a priori intuition, not a concept' (A25/B40). This is puzzling given Kant's own view that the elements of our cognition are divided exhaustively among concepts and intuitions: it is strange at best, and contradictory at worst, to suppose that we can understand the content of our concept of SPACE only by seeing that we in fact use intuitions, and not concepts, to represent that content. It is likewise puzzling that 'a science of all principles of *a priori* sensibility' (A21/B35) should begin with an exposition of a concept at all: the representations received by the faculty of sensibility are all and only intuitions, while those thought by the faculty of understanding are all and only concepts." ([116], p. 49)

We now see, from Shabel's point of view, why so many puzzling, contradictory and partial accounts result from this interpretation of Kant's Aesthetic in the *Critique*. It is because space is not a concept and empirical intuitions are not the source of pure intuitions as she proposes. If space were to be a concept, then Euclidean propositions could not be the rule giving principles of the construction in space, since then these propositions had to be analytic resulting from mere analysis of the concept space. However, this is not the case. The Euclidean propositions are synthetic rule givers, that allow more synthetic *a priori* constructions in space, based on space being *a priori* and a singular representation. Shabel's claims do not square with Kant's general characterization of mathematics, space and pure intuitions since Kant's aim does not appear in a natural way if the space's original representation is taken to be a concept.

I will now proceed securely to the interpretation of space as pure intuition and explain why it is important to have space as a singular *a priori* representation for the mathematical characterization Kant aims at.

## 4.2 Space's Original Representation is Pure Intuition

Sutherland argues that numbers are intuitions because they exhibit quantitative differences and if they were only concepts they could not do that. He proposes a structure that unites Kantian theory of geometry and arithmetic through homogeneous magnitudes.

Sutherland argues for the requirement of intuitions in arithmetic, algebra and geometry in Kantian characterization of mathematics. He points out a particularly important issue in Kant's mathematical arguments and proposes that Kant's characterization of mathematics was mostly based on the ancients' view of mathematics rather than his modern contemporaries. (cf. [129], p.552)

Sutherland argues that Kant analyzes primarily geometry in his questioning of synthetic *a priori* judgments since Kant is influenced by the Greek conception of number. So, Kant gives priority to geometry over arithmetic as ancient Greeks do. (cf. [129], p. 537) Moreover, Kant claims that, although geometry is not an empirical science, foundations of it cannot be reduced to logic. For this reason, the judgments of geometry as well as its objects are synthetic *a priori*.

Kant suggests that objects of arithmetic, namely the numbers, are constructed by successively adding and placing homogeneous units immediately one after another as I have argued in detail in the previous chapter. These units differ only quantitatively and not qualitatively from each other. So, the homogeneity of space is used in differentiating numbers. (cf. [68], B320) Kant argues that:

> "...the pure schema of magnitude, as a concept of the understanding, is number, which is a representation that comprises the successive addition of similar units to one another. So number is simply the unity of the synthesis of the manifold of a homogeneous intuition as such." ([68],B182)

Kant's philosophy of mathematics, notoriously, depends on Euclidean theory of magnitudes and part of Euclidean theory of magnitudes and proportions rest on Eudoxian theory of proportions as Sutherland argues. (cf. [129], p . 539) Moreover, Sutherland points out that: "Kant states that in a lecture in metaphysics that the homogeneous manifolds of space and time allow us to grasp the part-whole relations of magnitude. (29: 979, 1794-95)" ([129], p. 552) In addition, what Eudoxus manages by proportions and part whole relations Kant manages it with category of quantity (unity, plurality, totality) (cf. [65], A67/ B92), getting his roots from the Eudoxian theory of proportions, according to Sutherland.[5]

The connection between arithmetic and theory of proportions have to be linked by algebra, Sutherland argues. (cf. [129], p.551) Both arithmetic and algebra use the same kind of magnitude (*quantitas*) as opposed to the magnitude of geometry (*quanta*). (cf. [65], B204-05, A717/B745) Moreover, Kant claims that algebra is a general form of arithmetic and arithmetic is based on theory of magnitudes. Further-

[5]In Greek tradition from Plato onwards numbers were defined as pure units, either having their own independent existence (Plato) or being abstracted from empirical countable collection of objects (Aristotle). Until and including Pythagoreans, numbers were considered as indivisible pure units and ratios were commensurable. However, they encountered a problem with the incommensurable magnitudes such as $\sqrt{2}$. Eudoxos offered a solution to the problem of incommensurability between numbers and suggested that the "sameness of two ratios" should be taken into account and "two pairs of magnitudes that have the same ratio are said to be proportional" ([124], p. 338) Euclid also takes up the Eudoxian theory of proportions in his Book V (cf. [29], p. 17). Stein claims that there is a relation between Eudoxian notion of ratio in the definition 4 of Book V and Dedekind's construction of real numbers. Dedekind used Dedekind Cut in order to partition two subsets of rational numbers. The cut does not include any of the rational numbers and creates an irrational number. The irrational number $\sqrt{2}$ is represented by the cut of rational numbers divided in two subsets of rational numbers as follows:

$$A = \{a \in \mathbb{Q} : a^2 < 2 \vee a \leq 0\}$$

$$B = \{b \in \mathbb{Q} : b^2 \geq 2 \wedge b > 0\}$$

In the following I will iterate Stein's argument of the connection between the Dedekind Cut and Eudoxian notion of ratio.(cf. [124], p.342):

If $a$ and $b$ are two quantities and have a ratio and if $m$ and $n$, $m$' and $n$' are positive whole numbers; then if $mb \leq na$ and $m/n = m'/n'$, then $m'b \leq n'a$. Then there is a well defined partition of rational numbers in two subsets, say A and B where

$$m/n \epsilon A, if mb \leq na$$
$$m/n \epsilon B, if mb > na$$

By definition 4 in Book V, if $a$ and $b$ have a ratio then $A \neq \emptyset$ and $B \neq \emptyset$. And one can observe that $\forall r, s \epsilon \mathbb{R}$ if $r \epsilon A$ and if $s \epsilon B$ then $r < s$. This creates a well defined partition as in Dedekind's cut of the set off all positive rational numbers into two subsets by allowing a unified theory of geometry and arithmetic due to "homogenous magnitude" being defined with relation $=, >, <$ (cf. [129], p. 537)

This examination shows that Eudoxian notion of ratio is allowing irrational numbers, using homogeneous magnitude. Moreover, as Sutherland argues, using homogeneous magnitude this way allows a unified theory of geometry and arithmetic since part whole relation with magnitudes can be established with being smaller portion of some magnitude and, hence, *smaller* in this sense could be used both for discrete and continuous magnitudes.

more, since Kant's theory of magnitudes is based on Eudoxian theory of proportions, as Sutherland argues, Kant's characterization of algebra and arithmetic is also based on the Eudoxian theory of proportions.

Sutherland concludes that:

> "Kant does not simply assimilate arithmetic into the theory of magnitudes by thinking of numbers as represented by lengths and thinking of arithmetic as a kind of measuring. He retains the traditional conception of numbers as requiring a unit. Kant's account draws on the development of arithmetic into algebra and an interpretation of algebra as expressing the Eudoxian theory of proportions." ([129],p.557)

Basing his theory of geometry and algebra, and hence arithmetic on Eudoxian theory, Sutherland is able to give a unified account of geometry and arithmetic in Kantian framework, namely a unified account of continuous and discrete magnitudes.

Moreover, another important supporting claim of Sutherland for this argument comes from his showing the similarity between homogeneity of counting concept and the strict homogeneity. Sutherland alleges that geometrical figures require strict homogeneity according to Kant since Kant follows ancients' view of geometry, and ancients had a notion of "mathematical homogeneity" meaning that "a composition in which parts of the same kind can be composed to make more of exactly the same kind". ([129], p. 539) Moreover, for Kant, this mathematical homogeneity goes one step further and becomes strict homogeneity, a property that presents "numerical difference without qualitative difference". ([129], p. 539) Furthermore, according to Sutherland, ancients' conception of counting also required some sort of homogeneity, which is "homogeneity of merely falling under a counting concept". ([129], p.

545) Sutherland's ingenious attempt to show homogeneity of falling under a concept and strict homogeneity can be linked by strict homogeneity of space, allowing "to differentiate these discrete and disconnected, qualitatively identical objects". ([129], p. 556) Eudoxian theory of proportions allows one to unite magnitudes falling under the counting concept and the continuous magnitudes, namely, *quanta* (magnitude of geometry) and *quantitas* (magnitude of arithmetic and algebra). As Sutherland points out, in Kant's theory of magnitudes, the homogeneous magnitudes are numerically different without qualitative differences. Sutherland, by drawing attention to the need for intuition in arithmetic, points out that:

> "On Kant's view, concepts on their own can represent only qualitative differences, and hence on their own they cannot represent a homogeneous manifold. By contrast, intuition can represent numerical difference without qualitative difference, and as a consequence, intuition allows us to represent magnitudes -a previously neglected role for the intuitions of space and time in Kant's philosophy of mathematics." ([129], p. 539)

Significantly, Sutherland concludes that: "The fact that space consists of pure numerical difference without qualitative difference explains the special composition of which spatial magnitudes are capable." ([129], p. 540)

As I have argued for the geometrical case in the previous chapter, and as Sutherland argues, by uniting geometry, arithmetic and algebra, space is required in Kantian characterization of mathematics **as an *a priori* singular representation, ready to receive all other intuitions and exhibit quantitative differences, which concepts are *not* capable of.** We now see, from Kant's point of view, why space should be constructed as pure intuition and not a concept. Moreover, we also realize that this requirement of space as pure intuition is not only limited to geometry but also the case for arithmetic and algebra.

There arises also another conclusion from Kant's characterization of space as pure intuition and its role as containing synthetic *a priori* Euclidean principles and also *a priori* intuitions corresponding to the pure sensible concepts, shapes. This particular Kantian characterization, space as pure intuition, and form of sensibility, combined with Kant's mathematical characterization can bring **new approaches for examining internal diagrams, as *a priori* intuitions or as visualizations**. In this way, **internal diagrams do not have to characterized as psychological tools** and can be admitted and expressed in philosophy of mathematics. I am not suggesting one should integrate Kantian characterization of space and mathematics to current philosophy of mathematics completely, but I argue that internal diagrams exist in mathematical practice as I discuss in the next chapter, and Kant once gave an objective and rigorous argumentation of them as we have seen in detail. (cf. also [40]) Moreover, only *a priori* intuitions can be objective according to Kant, and empirical intuitions are only accidental imprints of them in mathematics. Current approach to internal diagrams is just the opposite way, characterizing them as subjective cognitive tools, despite the developments in computer technologies which allow these visualizations, internal diagrams to be exhibited in iterative steps where objectivity can be preserved. (cf. [92]) To give a complete account of Kant's characterization of mathematics and its possible relation to current philosophy of mathematics, let us now turn our attention to the interpretations on synthetic *a priori* character of mathematics according to Kant.

## 4.3 Where does Synthetic and *a priori* Character of Mathematics Appear in Kant's Doctrine? Different Interpretations.

There are many attempts to explain the *a priori* and/or synthetic character of Kantian characterization of mathematics.[6] The synthetic character of Euclidean proofs that Kant uses for his account is evaluated differently by different interpreters.

The common difficulty in interpretations of Kantian intuitions for mathematical intuitions comes from the different readings of Transcendental Aesthetics since this division emphasizes the sensibility and immediacy parts of the intuitions of mathematical objects. For example, Hintikka argues that Kant's views about intuitions when considered for mathematical objects in Transcendental Aesthetic should be disregarded since Transcendental Aesthetic was written before, namely was logically prior to Kant's methodology of mathematics. (cf. [53], p.33) However, as we will see in the following, the reason Hintikka had to disregard the Transcendental Aesthetic was because he took the dependence of intuitions to sensibility literally. Kant uses sensibility as a faculty; as the structure we perceive in. Sensible in this sense does not always mean coming from senses and this is a crucial point for understanding *a priori* intuitions.

In the case of Hintikka, the reason he prefers to deal only with Transcendental Deduction is, in Transcendental Aesthetic, Kant argues that all the intuitions used in mathematics were sensible and Hintikka interpreted "sensible" as coming from senses. (cf.[53], p. 33) However, as Potter points out for Kantian intuitions: "Our sensibility ... supplies us not only with empirical but with pure intuitions, i.e. intuitions deriving

[6] The most known interpretations being Hintikka's (cf. [53], [52], Russel's [113], Beck's [6], and Parsons' [94]).

not from experience itself but from its structure." ([106], p. 41) What Potter means by the structure is the mere forms of sensibility, which are space and time, but also extensions and shapes, since they are generated *a priori* in the pure intuition space. So, when Kant claims that all the intuitions used in mathematics were sensible, this does not mean that they come through senses, rather they need to conform to the mere forms of sensibility and they cannot be represented without the faculty of sensibility.

Recall that according to Kant intuitions can be categorized in three groups, excluding the pure intuitions- space and time-:

1. The *a priori* intuition mathematician uses (cf. [65], B299): This is a particular representation which is immediate and singular, and the mathematician chooses the necessary properties for the construction/ proof from the general *a priori* concept. This does not correspond to an empirical particular representation since it will only have some properties belonging to the concept, and this intuition resides *a priori* in human mind. The process here is from *a priori* to *a priori* and respectively from thinking to construction.

2. The second kind of intuition for mathematical object is when mathematician decides to use an empirical representation such as a drawing of a triangle or fingers in terms of counting, which becomes then an empirical intuition. So, the process in this kind of intuition is from *a priori* to empirical and respectively from construction to sensibility.

3. The third one is when this intuition comes through senses such as seeing 7 houses, or a square table. In this kind of intuition the process is from empirical to empirical and respectively from sensibility to sensibility. This intuition is empirical and can be used to make judgments.

It is important to notice that there is a necessary usage for *a priori* intuitions of geometrical figures in Kant's characterization of geometry. On the other hand, it is true that many significant commentators of Kant argued that these geometrical figures in Euclid's proofs were presented only to help the geometer in the proof. But, that was not Kant's argument. Yes, they were there *also* to help the geometer when they were drawn empirically in a piece of paper etc., however, this is not their *only* task.

In Euclidean proofs there is a motivation to use the particular instances of general concepts and that is in the setting out part. (cf. [53], p.28) The geometer almost creates his own figure, by choosing necessary properties from the general concept. When the geometer says "Let ABC be a triangle" she does not have to draw a figure in experience, but she needs to exhibit the figure *a priori*, so that she can have a singular representation that carries general properties to extend one edge of the triangle etc. by using construction. One cannot add lines only by logical rules and axioms, that is why Kant thought that in construction for Euclidean proofs geometer needs the *a priori* intuition. Moreover, these *a priori* intuitions are instantiated from one of only two forms of *a priori* intuitions, namely space.

The reason intuitions are used for constructing concepts is that, only in this way one is able to admit quantitative differences. (cf. [129], p. 539 and [65], A10/B14) Concepts cannot be used to admit quantitative differences, they can only be used to differentiate qualitative properties. That is why Kant constructed space and time as pure intuitions. Exhibition of intuitions in pure intuitions allows one to differentiate quantitative properties.

When an intuition is exhibited, this is done in an *a priori* way in geometry for the construction of concepts. One can recall the intuition in pure intuitions, space and time, and exhibit it to the corresponding concept and with this process constructs a new concept. The importance of *a priori* exhibited intuitions are that although they are particular representations they still admit the general properties that is required in the mathematical reasoning, namely, they are idealized singular representations. This is similar to visualization for shapes rather than drawing the shape on a paper. The visualization is independent from experience in the sense that one does not have to apply to experience to visualize a triangle. Moreover, one needs the intuition to admit the quantitative differences. Otherwise, a triangle would not be a shape that some constructions can be carried out but only a concept that can be analyzed according to its qualitative differences with respect to its sides, vertexes and angles. The synthetic character of mathematics comes from the synthesis done with the intuition and concept for constructing the new concept. The *a priori* part comes from the synthesis of the geometrical concepts by schema, the a priori concepts and exhibition of intuitions *a priori* corresponding to these concepts. For that reason, mathematical propositions are synthetic and *a priori* according to Kant.

Moreover, Kantian intuition of mathematical objects are representations both in sentential and visual reasoning. Namely, linguistic symbols and diagrams are particular representations of general concepts which still carry the general properties by only this particular representation, intuition. As I will discuss in more detail in the next sections, I believe there are use of intuitions in axioms, in application of general propositions and in argumentation steps according to Kant. Kant argues that general analytic propositions when used in mathematics become synthetic a priori since one has to exhibit the corresponding intuition to the concept a priori. For example, the rule of identity is an analytic general proposition. However, when it is used in algebra as $a = a$, the letters are attributed for the equation. The symbols become particular intuitions that also represent general properties. (cf. [65], B17)

This characterization will help us to see why in mathematics we actually cannot and should not exclude visual representations in proofs and practice. In Chapter

V, we will analyse several mathematical proofs and see how this characterization comes out in every step in the proofs and how actually mathematicians are using visual reasoning even in the context of justification. Before starting this analysis I aim to reply several objections to Kant's thesis and interpretations in order to erase any remaining doubts that we can use Kant's approach to synthetic method as a legitimate tool in mathematics.

There is a prolific debate started in the mid 1960s between Parsons and Hintikka and more work followed by other scholars based on this debate. Hintikka, in his paper "Kant on the Mathematical Method" [53] refers to Kantian intuitions as particulars and claims that there is nothing intuitive about these intuitions (cf. [53],p.23). He uses Kant's definition in *Logic* and also refers to other sources which carry similar meanings.(cf. ([62], Sec.2 §10), ([65], A320/ B376-7) and ([67], §8))

He claims that if we read Kant's Transcendental Aesthetic in the Critique, the intuitions are referred as "mental images or an image before our minds eye" ([53], p.26), but then he argues, in connection with Kant's mathematical views, the definition of intuition, which should be taken into account, is the one at the end of the Critique, in the Transcendental Doctrine of Method. Then, he suggests that, we will have no problems with Kant basing arithmetic and algebra on intuition, if we take his -Hintikka's- account into consideration. Otherwise, if the intuitions are taken as mental images, there is no way to interpret arithmetic and algebra being based on intuitions. (cf. [53], p.26) However, as we have seen above, it is possible to give a sound interpretation of intuitions that are bases of geometry, arithmetic and algebra by using the Eudoxian Theory of Proportions without disregarding the Aesthetic and without having contradictions with Transcendental Doctrine of method. But, according to Hintikka, singularity of intuitions are essential and immediacy is a byproduct of singularity.

It follows that intuitions can be given logical formulations with instantiations and existential quantifier eliminations according to Hintikka. Parsons disagrees and argues that immediacy of intuitions are essential and in interpreting Kantian intuitions for his philosophy of mathematics, he claims that one needs more than logic, a method similar to perception. Towards the end of this debate, Parsons reveals that: "I do not think that either of us [Hintikka and Parsons] has undertaken the task of constructing a truly Kantian explanation of the *a priori* character of mathematics." ([94], p. 75)

As I have pointed out earlier, there have been numerous attempts to explain synthetic *a priori* character of mathematics according to Kant and as Parsons quote indicates clearly, most of them failed to explain the *a priori* character of it but instead focused on the synthetic character of Kant's mathematical account.

Hintikka, in his reconstruction of Kantian characterization of mathematics, argues that the synthetic character of the Euclidean proofs Kant uses in his argumentations lie in their setting out part and in argumentation steps, where new individuals are introduced. Russell suggests that the synthetic propositions are in the inference steps. (cf.[113]) Beck and Parsons have more phenomenological approaches and consider that these propositions are in the axioms. (cf.[6], [94]) I defend that they are used in setting-out, in axioms and in inference steps.

Moreover, Friedman, analyzing Kantian mathematics from a Russellian point of view[7] argues that "construction in pure intuition is primarily intended to explain mathematical proof of reasoning, a type of reasoning which is therefore distinct from logical or analytic reasoning". ([37], p.80) I talked about Friedman's argumentation in the previous chapter where he points out the requirement of quantifier logic in explaining infinite divisibility without applying to any intuition, whereas, with the

[7]This view indicates that the reason Kant preferred synthetic method in mathematics is because polyadic logic was not available to him.

monadic logic available to Kant, it can only be represented as a dynamic procedure, with the generation of time and with the exhibition of intuitions. (cf. [37], pp. 70-71)

Friedman's view, based on the Russell's argument, can be summarized in his following words:

> "Kant, having observed that the geometers of his day could not prove their theorems by unaided argument, but required an appeal to the figure, invented a theory of mathematical reasoning according to which the inference is never strictly logical, but always requires the support of what is called "intuition". " ([113], p.145)

Contrary to this idea, Lewis White Beck, by pointing out the §3 in *Prologemena*, which is similar to B14 in the *Critique*[8], argues that: "The theorems, therefore, can be called synthetic even though they are strictly (analytically, in modern usage) demonstrable". Moreover, he also suggests that "intuitive construction enters into the theorem itself and its proof." ([6], p.89)

With this line of reasoning, in this chapter, I aim to show, formalist attacks to Kant's theory, asserting non-availability of modern logic and axiomatic method to Kant, are not justified and do not affect the current validity of Kant's thesis. I also defend that synthetic method that is used by Euclidean geometry, by way of exhibiting geometrical constructions and diagrams, does not have to be classified as an invalid method in the face of modern axiomatic and logical method. Now, for the purposes of current work, to complete our analysis on the different perspectives of the Kantian

[8]§3 "... as it was found that the conclusions of mathematicians all proceed according to the law of contradiction... men persuaded themselves that the fundamental principles were known from the same law. This was a great mistake, for a synthetical proposition can indeed be established [*eingesehen*] by the law of contradiction, but only by presupposing another synthetical proposition from which it follows, never by that law alone." (Quoted as referred by [6], p. 88)

account of syntheticity in mathematics and also see why Kant's characterization of mathematics cannot be categorized as outdated despite the developments in modern logic, let us examine Hintikka's reconstruction of Kant's account of mathematics in detail.

### 4.3.1 Hintikka's Reconstruction of Kant's Mathematical Intuition

Hintikka has an interesting and different interpretation of Kantian intuitions from most commentators. He discusses singularity and intuitions with a new approach from quantifier logic (cf. [52] and [8]) and offers that "searching" and "looking for" from a domain that is defined for a predicate is similar to exhibiting intuition in Kantian sense. (cf. [53], p. 40; [52], pp.109-111; [65], B741). He, moreover, argues that "the notion of *ecthesis* [setting out] offers a very good reconstruction of Kant's notion of construction, i.e., of the notion of the exhibition of a general concept by means of particular representatives" ([53], p. 34) He suggests that by using modern logic the notion of *ecthesis* can be written explicitly; in particular, by using existential instantiation for introducing new individuals. (cf. [53], p. 35; [52])

Hintikka develops his thesis for formalizing intuition in first order logic in "Quantifiers, Language-Games, and Transcendental Arguments" ([52], pp. 98-122). He argues that with a game theoretical interpretation of quantifiers $E$ and $U$ , with predicates $F$, individual variables, individual constants, connectives '&', '$\vee$', '$\neg$' and players '*myself*' and '*Nature*' a winning strategy can be developed for *myself* since I will be able to *look in/search for* in the domain $D$ of individuals on which all the relations and properties used in $F$ are defined". ([52], p. 100)

In terms of this game, one can define the truth of $F$ in $D$. For that reason, this game theoretical interpretation of quantifiers yields a complete semantical theory of first order logic. In other words $F$ is true, if $myself$ has a winning strategy. (cf.[52],p. 101)

This is an interpreted language and the rules can be described as follows: First of all, one should note that $F$ is defined on $D$ of individuals and all $F$'s relations and properties are defined on this domain. Then, Hintikka defines rules, letting substitutions instances $G$ of a subformula of $F$ and allows the game to begin with $F$. In the existential instantiation rule, "I" choose a member of $D$ and in the universal generalization rule "Nature" chooses a member of $D$. The cases are defined for exclusive, inclusive and negation options. The key idea is that a winning strategy can be developed in this game. One has to find a suitable (true) substitution instance from the domain. (cf.[52], p.101, p.109) These are the game rules, $(G)$. He, moreover, defines modal set construction rules, $(A)$. (cf.[52],p. 108) In $G$, while "I am trying to win in the game of exploring the world", in $(A)$ "I am asking whether the world could be such that I can win in a game of this sort" ([52], p. 111). According to Hintikka, in $(A)$, method of expositions, namely setting out in Kantian sense, is defined with existential instantiation and with seeking and finding in place of construction. (cf. [52], p. 110) Hintikka introduces intuitions as free singular terms in this process. Moreover, he later concludes that:

> "My attempted partial reconstruction of the main point of Kant's philosophy of mathematics as applied to modern symbolic logic instead of mathematics thus gives rise to an interesting suggestion for our present-day philosophy of logic. This suggestion is to consider the logic of quantification as being essentially logic of the notions of searching for and finding (suitably generalized)." ([53], p.40)

Hence, for Hintikka, this partial reconstruction, which I have given the summary above, is a method for searching for the existence of the intuitions and finding them as in the case of construction where Kant argues for the exhibition of intuitions corresponding to the concept. Moreover, in his analysis of Kantian characterization of mathematics, Hintikka reveals that there are 5 steps in an Euclidean proof and these are (cf. [53], pp.28-29):

1. declaration of a general proposition
2. *ecthesis*/setting out
3. *auxiliary construction*/ preparation/ machinery/ drawing additional lines etc.
4. *apodeixis*/ proof proper/ using inferences, axioms, earlier propositions
5. general declaration again: e.g. "Therefore, in any triangle ..."

In this structure 3. and 4. are attributed to the mathematician by Kant (cf. [65], A716-7, B744-5, referred by [53], p. 30). Furthermore, by revealing this structure in connection to Kantian characterization of mathematics Hintikka draws the conclusion that "[w]e can see here that according to Kant the peculiarity of mathematics does not lie in the axioms and postulates of the different branches of mathematics, but in the mathematical mode of argumentation and demonstration." ([53], fn.11, p. 41) According to Hintikka's general theory (cf. [53], [52]), whenever a new individual is introduced with incrementing the degree of existential instantiation "$\exists$" this corresponds to the exhibiting intuition in the argumentation steps. Moreover in the setting out part when, for example, the proposition "Let $x_0$ be a triangle" is used, $x_0$ also corresponds to the intuition.

According to Hintikka when Kant talks about the use of constructions, that entails the use of *a priori* intuitions in mathematical sense. That is because, constructing a concept is exhibiting the intuition *a priori* corresponding to that concept. ([68], B741) Moreover this construction, which is the part (3) of an Euclidean proof makes geometry and arithmetic synthetic. ([53], p. 31) He argues that, the reason Kant says $7 + 5 = 12$ immediate and indemonstrable is (cf. [65], A 164/ B204) because it does not use the part (4) in Euclid's proof structure. Kant claims that in mathematics "we are required to take the bare concept and make it sensible, i.e. present a corresponding object in intuition. The mathematician meets this demand by constructing the figure corresponding to the concept; it is produced *a priori*, but all the same it is an appearance present to senses." ([68], B 299) Hintikka emphasizes this definition in terms of construction and then suggests that Euclid's procedure "encourages the idea that mathematics is based on the use of particular instances of general concepts". ([53], p. 28)

Hintikka claims that in Euclid's proofs, principle of contradiction or inferences are carried out in the *apodeixis* part ([53],pp 32-33), where only analytic propositions are dealt with . *Ecthesis* and *auxiliary construction* are where the synthetic propositions are dealt with, and $7 + 5 = 12$ falls only in this part. For that reason it is immediate and indemonstrable - not because it is a mental image. (cf. [53], p. 32)

I agree that these propositions are not mental images and Kant does not say that. However, Hintikka's argument for arithmetical propositions does not represent Kant's views on the arithmetical propositions. Kant alleges that since numbers are either general concepts or empirical intuitions, one cannot give a mathematical proof of them. One can only get help from empirical intuition and make them particular instances, and this is the only way to exhibit them. The *a priori* part of these arithmetical propositions comes from using *a priori* concepts, the immediacy comes from being related to experience immediately as intuitions. They are synthetic because the calculation cannot be an analysis since it does not use concepts.

The calculation process is a synthetic process, that is why the resulted propositions are synthetic. It requires an ability which is not contained in analytic process. In particular, Kant suggests that if one looks closely at $7 + 5 = 12$, the concept *sum of 7 and* 5 only contains the number in which 7 and 5 are united and not number 12. (cf. [65], B16) Hence, one cannot find number 12 by analyzing the concept *sum of* 7 *and* 5, which according to Kant's definition not analytic, therefore, synthetic. Kant claims that "[t]o arrive at 12 we have to go beyond these concepts, we have to get help from an intuition that corresponds to one of the concepts (an intuition of five fingers, for instance) and add the units of the intuited five, one by one, to the concept of 7". (cf. [68], B16)

Moreover, in Euclid's proofs, postulates are the principles of the constructions, Hintikka argues, and the principles of proof proper are axioms (common notions). ([53], p. 36) The examples Kant talks about in B17 for analytic principles used in geometry are among the common notions. (cf.[53], p. 36) In B17 Kant argues that:

> "To be sure, a few principles that the geometers presuppose are actually analytic and rest on the principle of contradiction; but they also only serve, as identical propositions, for the chain of method and not as principles, e.g. $a = a$, ... And yet even these, although they are valid in accordance with mere concepts, are admitted in mathematics only because they can be exhibited in intuitions."([65], B17)

Therefore, Hintikka concludes that:

> "Hence, the distinction between intuitive and logical ways of reasoning was for Kant, within geometry at least, equivalent with the distinction between the use of postulates, i.e., principles of construction, and the use of axioms, i.e., principles of proof." ([53], p. 36)

Moreover, the distinction lies, Hintikka argues, in the history of philosophy and mathematics, where postulates were known as assumptions of existence. Hence, Hintikka claims that "Kant's problem of justification of constructions, therefore, amounts to the problem of justifying the use of existential assumptions in mathematics." ([53], p.36)

However, note that, Kant indicates more in B17. Namely, that the same proposition can be synthetic and analytic according to its representation method. When the judgment only rests on concepts and analysis, this judgment is analytic. But the same judgment when used in mathematical proofs, it becomes synthetic since intuition corresponding to the concept must be exhibited. In this sense even $a = a$ is synthetic when used in mathematical proofs. The method makes the difference not the content or justification of the judgment.

I believe Hintikka reveals some unique facts about Kant's theory by differentiating the language games (G) and model set constructions (A), and by claiming that method of expositions in the latter corresponds to the setting out part in an Euclidean proof according to Kant. This part captures Kantian notion of exhibiting intuitions distinctly but partially. Revealing that expositions in model construction correspond to setting out and that searching for and finding individuals correspond to exhibiting intuitions in the mathematical constructions is an ingenious way of explaining the exhibiting individuals and the mathematical construction in the *ecthesis* and *auxiliary* part respectively, however, it does not explain the Kantian counterparts of these notions completely.

First, as Parsons indicates, it does not explain the *a priori* part of the construction, and, second, it fails to explain what is implied by the synthetic method, namely going beyond the concept. Hence, introduction of new individuals and searching and finding them in a domain do not explain the Kantian synthetic character of mathematics In this section, I have shown the departure points of Hintikka's reconstruction from Kant's account of mathematics. In the next section, I will reveal why this departure is quite essential and emphasize the importance of synthetic method for grounding a different kind of reasoning in mathematics.

## 4.4 Kant's Characterization of Mathematics and Modern Logic

In Kant's characterization of mathematics syntheticity of mathematical propositions does not come from their justification method or from their content, but merely from the method itself. Moreover, this synthetic method contains implicitly the ability of the mathematician's carrying out the correct constructions[9] in geometrical proofs and in finding "the sum of 5+7" being "12". Carson, pointing out to A711/B739[10] also argues that "[o]nly this 'fundamental transcendental doctrine' allowed Kant to substantiate his distinction between the methods of mathematics and philosophy and to explain both the objective content and the nature of our knowledge of mathematics... there is a substantial philosophical role for Kant's doctrine of intuition

[9]A similar critique is raised by Shin contrasting Peircian Theorematic Reasoning and Hintikka's reconstruction. She argues that: "What makes theorematic reasoning non-trivial is not only that new individuals should be introduced by EI [Existential Instantiation] (in the case of modern logic) but also that it is not obvious which new individuals need to be introduced. Hintikka, recognizing the former, correctly points out that EI, which introduces a new object, increases the number of individuals in reasoning. Therefore, whether EI is used in an argument becomes a criterion for whether the argument is synthetic. But this definition takes into account only one condition for Peirce's theorematic reasoning, that is, that theorematic reasoning requires some experiment, i.e., construction. However, Hintikka's definition does not consider the other, more important, aspect of Peirce's theorematic reasoning, that is, the fact that the choice of the correct new individual requires ingenuity. ([121], p. 28)

[10]"No critique of reason in empirical use was needed, since its principles were subjected to a continuous examination on the touchstone of experience; it was likewise unnecessary in mathematics whose concepts must immediately be exhibited *in concreto* in pure intuition, though which anything unfounded and arbitrary instantly becomes obvious. But where neither empirical nor pure intuition keeps reason in a visible track, namely in its transcendental use in accordance with mere concepts, there it so badly needs a discipline to constrain its propensity to expansion beyond the narrow boundaries of possible experience and to preserve it from straying and error that the entire philosophy of pure reson is concerned merely with this negative use." ([65], A 711/B739)

over and above the logical role which various commentators have emphasized." ([14], pp.651-652)

This line of reasoning, siding with Carson, shows that Hintikka's approach, although a distinguished partial reconstruction of Kantian mathematical characterization, does not capture the essential argument in the Kantian account. In particular, there is more than what quantifier or polyadic logic can capture in the Kantian account of mathematics and even if modern logic were available to Kant, that would not have changed his mind about the need for intuitions in mathematics and the requirement of the ability of a mathematician that is implicitly contained in the synthetic method.

In this chapter, I examined the requirement of space as pure intuition in Kantian characterization of mathematics in connection to my interpretation of the notion of space in the Aesthetic. In order to give an account on this requirement I first pointed out the discrepancies in the case of admitting space as a concept in Kantian doctrine. Moreover, I analyzed Sutherland's argumentation on the importance of space being a pure intuition, which gives a unified theory of geometry, arithmetic and algebra. By concluding that Kant's argumentation in the *Critique* on space is a sound one in the previous chapter and by pointing out the importance of constructing space as pure intuition in Kantian doctrine in this chapter, I indicated the necessity[11] of *a priori* intuitions in mathematical practice, which in this thesis, I also characterize as internal diagrams. Moreover, I pointed out that the prevalent criticism against Kant's argumentation in the Aesthetic comes from Kant's grounding his argumentation on the premise that Euclidean propositions are synthetic *a priori*. This view suggests that, this premise is false since we can now formalize Euclidean propositions in pure

[11] Necessity in the sense that "primarily and naturally using visual representations before applying to any other representation".

sentential forms, therefore, Kant's account of mathematics is outdated and his construction, space as pure intuition is fallacious. I argued that having one valid formal representation system does not have to make other representation systems invalid, hence, pointed out the invalidity of this criticism.

I also revealed the approaches to Kant's syntheticity in mathematics and analyzed Hintikka's reconstruction of Kant's characterization of mathematics as applied to quantifier logic. I argued that although this interpretation ingeniously reconstructs Kant's account of mathematics and intuitions *partially*, it does not capture two important facts in the Kantian doctrine. (1) It does not explain the *a priori* character of mathematics. (2) It does not take into consideration the synthetic method providing the correct constructions in mathematical proofs. Therefore, the most important conclusion from this analysis is that there is more to Kant's characterization of mathematics which modern logic can capture, and there is an added value to mathematics, mathematical proofs and communication if pictorial representations are used.

Even though this mathematical reasoning becomes synthetic along the way, by way of departing from formal analysis, I do not believe that that would reduce the validity of the proof if the correct representational schema kept in mind, as seeing the general in particular. At the very least, this argumentation, combined with Kantian synthetic method in mathematics can help us to appreciate a different kind of reasoning in mathematics, which can prove itself quite beneficial in mathematics and also in philosophy of mathematics.

I believe, I have so far provided the groundwork for the next chapter which examines the reasoning with pictorial representations in mathematics.

# CHAPTER V

# Visual Reasoning

A scientific truth does not triumph
by convincing its opponents and making
them see the light, but rather
because its opponents eventually die
and a new generation grows up
that is familiar with it. [1]
-Max Planck

The key aim of this chapter is to explore the valid use of diagrams in mathematical reasoning. Pictorial representations or diagrams can be used as formal tools, similar to sentential symbols. They can even be employed for constructing formal representational systems as we will see in the Section 1 of this chapter. On the other hand diagrams can be used as heuristic tools as well, namely, as intuitive tools for their visual properties rather than their formal representative attributes. In prevalent mathematical practice almost all usage of the diagrams in mathematics has been in

[1]([100], pp. 33-34) Original version: "Eine neue wissenschaftliche Wahrheit pflegt sich nicht in der Weise durchzusetzen, daß ihre Gegner überzeugt werden und sich als belehrt erklären, sondern vielmehr dadurch, daß ihre Gegner allmählich aussterben und daß die heranwachsende Generation von vornherein mit der Wahrheit vertraut gemacht ist." ([99], p. 22)

heuristic ways and all diagrams in mathematics can now be sententially formalized. Currently, the formalized mathematical reasoning is the only accepted legitimate reasoning in mathematics. Moreover, diagrams have no role in constructions and proofs in this reasoning system. However, as I will argue in this chapter, formalized reasoning does not adequately represent the actual mathematical reasoning.

This chapter will also focus on the current research on diagrams as a medium for valid reasoning. One cannot use the definition of validity and valid reasoning that are constructed for the formal sentential systems.[2] For that reason, I will adopt Barwise and Etchemendy's definition for valid reasoning, which is "extraction or making explicit of information that is only implicit in information already obtained", when evaluating the validity of visual reasoning in mathematics. ([5], p.4)

The first section of this chapter will focus on diagrams as formal tools. Sun-Joo Shin [120] shows that reasoning with Venn diagrams can be a sound and complete formal representational system.[3] I will reveal her construction and summarize her proofs of completeness and soundness.

The second section will be concerned with the diagrams as intuitive tools. First, I will discuss their usage in some proofs and draw attention to their roles and validity in these proofs. After discussing the proofs we will see that although diagrams are

[2] *Sentential valid reasoning* can be defined as carrying out valid and sound arguments. An argument is *valid* if it is impossible for the premises to be true and the conclusion to be false, and an argument is *sound* if the premises are true and the argument is valid. These definitions work for sentential representational systems such as propositional logic or formal sentential proofs in mathematics. There were not many significant attempts to define valid reasoning beyond sentential reasoning until Barwise and Etchemendy revealed their "heretical approach" to mathematics and logic.

[3] Soundness of a system shows the reliability of it. Namely, if there is a sentence that can be deduced from the system then this sentence is true if the system is sound. A system is semantically complete if a sentence in the system is true then it has a proof in the system. A system is syntactically complete if there is a proof of the sentence or its negation in the system. Although an incomplete system is not desirable in general, unsound system is a more serious issue. A system that is unsound can produce false conclusions from true assumptions, which cannot be considered as a valid reasoning tool.

not sufficient for the proofs that involve diagrammatical reasoning, they are valid essential parts of the reasoning.

In the third section, I will analyze Giaquinto's account [39] of visual reasoning in mathematics. He reveals that mathematical knowledge is synthetic *a priori* as Kant once claimed. This line of reasoning is related to one of my claims in this work, namely that there is a close connection between Kantian synthetic *a priori* reasoning and the validity of using diagrams in mathematics. I will discuss Giaquinto's approach in connection with this argument and formulations I disclosed in previous chapters.

## 5.1 Diagrams as Formal Representation Systems

Until recently, predominant view among most logicians has been that diagrams were only heuristic tools that were used as helpful mediums in proofs, discovery and learning. Although one can say this is *still* the predominant view, recent studies started to surface claiming the role of diagrams as valid reasoning tools.

In this section, I will present diagrammatical representation systems that are also valid formal representation systems. Barwise and Etchemendy are the pioneers of this development and created a computer program called Hyperproof [4] that helps students to carry out proofs with diagrams. Although this program is not purely formal, it was the inspiration for developing formal visual tools in logic. They claim that "visual forms of representation can be important, not as heuristic or pedagogic tools, but as legitimate elements of mathematical proofs." ([5], p.3) They continue by saying that "[a]s logicians, we recognize that this a heretical claim, running counter to centuries of logical and mathematical tradition" ([5], p.3).

In this section, I will first reveal Barwise and Etchemendy's pioneering ideas [5] on diagrams as valid reasoning tools. The diagrams as logical tools were introduced by Euler [30], Venn [137] and Peirce ([97], [122]). As an example of a diagrammatic system that has the equivalent rigor and validity to a propositional logic system I will present a case study by Shin [120] that manipulates Venn diagrams with Peircian existential import x and shading from Venn. Shin's system ([120], [119]) keeps the visual clarity and explanatory power for complex logical statements such as existential statements and disjunctive information that were not present in the systems postulated by Euler, Venn and Peirce [4]. This case study shows that diagrams can be sound and complete representation systems in logic.

### 5.1.1 Valid Reasoning with Diagrams

It is a speculation that if the same attention were given to diagrams for excelling their use in valid reasoning, they would be as strong a system as the formal linguistic systems. However, I find this speculation a very valuable one to consider. Isabel Luengo, a defender of a similar view also points out the misunderstanding of pictorial representations by emphasizing a curicial point: "the problem is not with diagrams, but having bad semantics and syntax". ([80], p.150) Logicians and philosopher of mathematics, who aim to keep up with the visualization needs in mathematics, like in every science, have done valuable work to show sound use of pictorial representations in logic. (cf. [120], [86], [80])

Etchemendy and Barwise started a serious inquiry on diagrams as formal representation tools. They pointed out that the reasoning with Venn diagrams can be valid reasoning in set theoretical proofs, since they "provide us with formalism that consists of a standardized system of representations, together with rules manipulating them." ([5], p. 9)

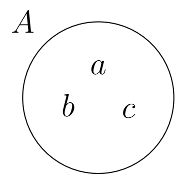
[4] Peirce's system fulfills the explanatory part but with sacrificing visual clarity. (cf.[97],4.365)

Note that with the operations of Venn diagrams there is a homomorphic (structure preserving) relation between the drawn circles covering some certain regions and sets that are representative of elements together with their corresponding operations. This structure preserving map allows the proofs, which are carried out with Venn diagrams, to be valid ones. In other words, the homomorphic relation between regions and sets preserves the relation between "operations of additions and intersection of regions" and, "operations of union and intersection of sets". ([5], pp. 9-10) If homomorphic relation is to be disregarded then the invalid reasoning appears.

For example, Barwise and Etchemendy point out that "if we were to take the nonemptiness of a region as indicating nonemptiness of the represented set, we would be led to conclude that the intersection of A, B and C is nonempty, which it might not be. Likewise, the size of a region carries no representational significance whatsoever about the represented set, any more than the size of the letter, "A" indicates that the set denoted is larger than the set denoted by "a". The use of diagrams, like the use of linguistic symbols, requires us to be sensitive to the representational scheme at work."([5], p. 10)

Surely, one should first understand or decide the representational schema of the diagrams that are in question. For instance, a shading can be used to represent the emptiness of a region. As in the linguistic systems, first, the objects' meanings should be made clear.

Venn diagrams represent sets by circular shapes and letter next to each circular shape. In formal representation the practice is to use the linguistic symbols only.




Venn Diagram with 3 members {a,b,c}

Formal representation of the above figure is the following:

$$A = \{\text{a,b,c}\}$$

Moreover, Venn diagrams can be primitive visual analogs of this linguistic formal system. In general, manipulation rules of Venn diagrams are as follows :

- Circular regions on a page represent sets.
- Addition and intersection of regions correspond to union and intersection of sets.
- Shading a region serves to focus attention on that region.
- There is a homomorphic relationship between regions and sets. (cf. [5], pp. 9-10)
  (From operations of addition and intersection of regions to operations of union and intersection of sets).

If one abides the above manipulating rules, generality and homomorphic relation diagrams are valid reasoning tools. However, if one attempts to use the area of a Venn diagram, or the measurement of the circumference, then obviously the reasoning will be invalid. Note that these kind of reasoning failures can be done with formal sentential reasoning. For example, as in the quotation above from Barwise and Etchemendy;

if one thinks that the capital letter “A” represents a bigger set than “a” while carrying out the formal reasoning, same invalid reasoning will appear.

Barwise and Etchemendy point out that visual reasoning can be part of valid reasoning in following three ways:

1. Visual information is part of the given information from which we reason: extraction from visual scene and represent it linguistically.

2. Visual information can be integral to the reasoning itself in an actual diagram.

3. Visual representations can play a role in the conclusion of a piece of reasoning. (cf. [5], p. 13)

Similar to what I have been arguing throughout this thesis, Barwise and Etchemendy suggest that (2) plays more role in mathematical proofs than it is realized. I will come back to this argument in detail in the next section. In addition to the use of diagrams as an integral part of the reasoning, I will reveal some proofs where (1) is also used and also where the diagrammatical reasoning used as a preferable tool to formal sentential reasoning. For this section's purposes though, to have a complete view of the formal usage of diagrams, I will discuss Shin's approach to diagrams next. Then we will go back to the general use of diagrams in proofs as in (1) and (2).

### 5.1.2 Venn Diagrams as an Information Manipulating Formal Representation System

Etchemendy and Barwise point out that distributive law can be proven by the use of Venn diagrams as follows ([5], pp. 9-10):

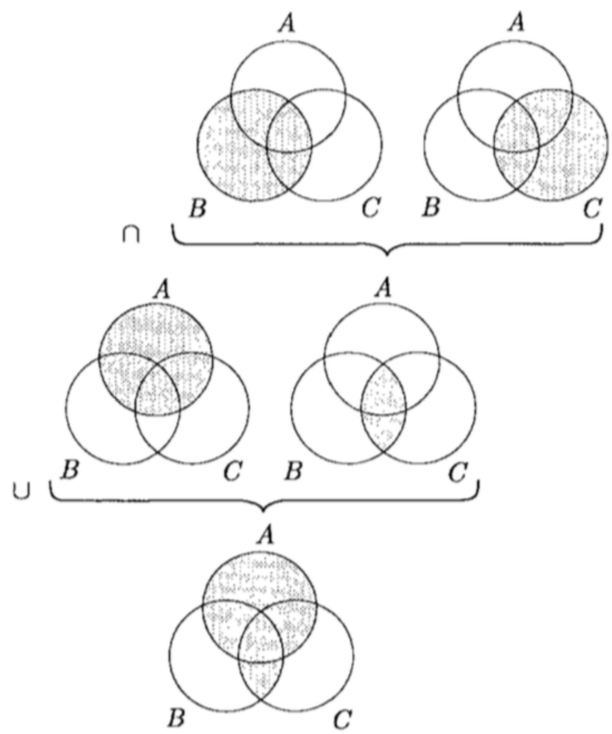



Figure 5.1: Derivation of $A \cup (B \cap C)$

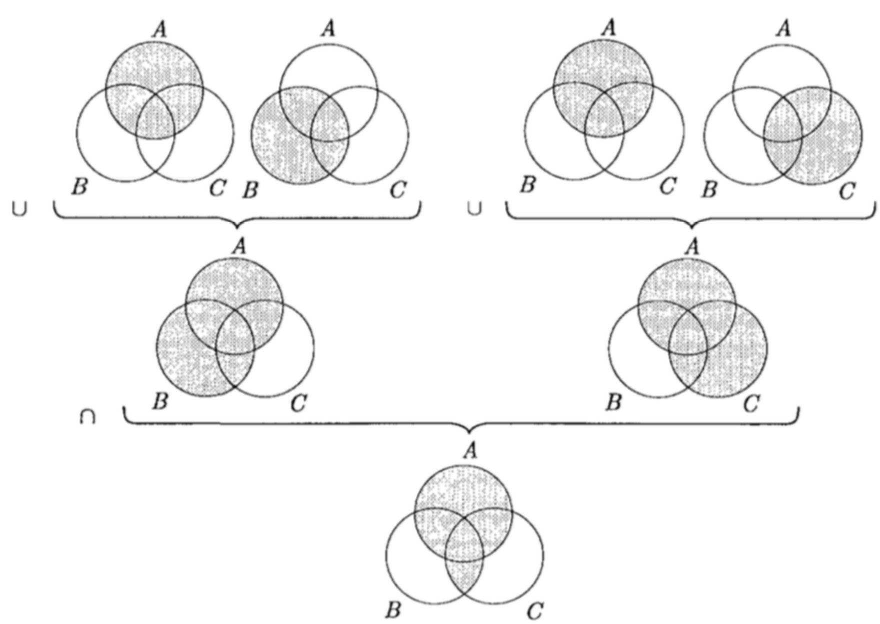



Figure 5.2: Derivation of $(A \cup B) \cap (A \cup C)$

Shin [120] demonstrates that these kind of proofs are valid with an improved system of Venn diagrams. She gives a "Situation- Theoretic Account of Valid reasoning with Venn diagrams" [120] by using Venn diagrams and Venn's shading property combined with Peircian existential import x's. Her aim is to provide the visual clarity of the Venn diagrams -missing part of Peirce's own theory- but still keep the strong explanatory power of diagrams.

In using sets as diagrams the aim has been to represent sentences as sets with a visual clarity. Shin's aim is to go one step further, namely that, to establish a formal representative logical system with Venn diagrams.

To set up foundations she first defines the primitive objects of her system. These are a closed curve, a rectangle, a shading and X's. Then she sets up a rule similar to quotation marks tool in logic, for "use/ mention" distinction in this diagrammatical representational system.[5] Then, she uses capital letters for closed curves and rectangles. Moreover, since these shadings and X's will be represented by the minimal region of a set, she chooses to mention them with the minimal region name. For example, if the shading is in A intersection B then she mentions this shading as the "shading in $A \cap B$". Shading, in this system, means that the region has no members, and X-sequence represents the areas with members. In comparison to Peircian diagrams where $\circ$ represents "region has not members" Shin's diagrams look as follows:

[5] There is a convention in logical syntax to use single quotation marks when an object is mentioned and double quotation marks when it is used, such as: The word 'dog' refers to "dog".

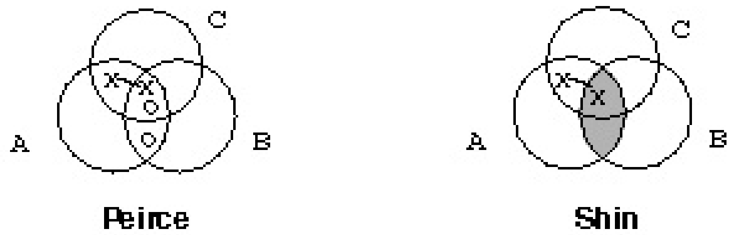



Figure 5.3: Comparison of Peirce's and Shin's Venn Diagrams.

Moreover, she defines the equivalence relation as the counterpart relation *cp*. This relation can be a relation on the set of basic regions of $D_1, ...., D_n$ satisfying the following: (cf. [120], p. 89)

1. If $\langle A, B \rangle \in cp$ then both $A$ and $B$ are either closed curves or rectangles.
2. If $\langle A, B \rangle \in cp$ then <u>either</u> $A$ is identical to $B$ <u>or</u> $A$ and $B$ are in different diagrams.

So, either one has two diagrams represented with rectangles and closed circles and have the same set in two diagrams, or these two diagrammatical objects should overlap.

Next, she defines rules for well-formed diagrams (*wfd*), an anologue of well-formed formulas (*wff*) in propositional logic. Well formed formulas in traditional sense are the finite sequences of the alphabet of the system that obey the formation rules. For example $(P\neg P)$ is not a *wff* in propositional logic. In the case of diagrams, the set of *wfd*, $\mathcal{D}$, is the smallest set satisfying the following rules (cf. [120], p. 90):

1. Any rectangle drawn in the plane is in set $\mathcal{D}$.

2. If $D$ is in the set $\mathcal{D}$, then if $D'$ results by adding a closed curve interior to the rectangle of $D$ by the partial-overlapping rule (described below), then $D'$ is in set $\mathcal{D}$.

   Partial-overlapping rule: A new closed curve should overlap *every* existent minimal region, but *only* part of each minimal region.

3. If $D$ is in the set $\mathcal{D}$, and if $D'$ results by shading some entire region of $D$, then $D'$ is in set $\mathcal{D}$.

4. If $D$ is in the set $\mathcal{D}$, and if $D'$ results by adding an X to a minimal region of $D$, then $D'$ is in set $\mathcal{D}$.

5. If $D$ is in the set $\mathcal{D}$, and if $D'$ results by connecting existing X's by lines (where each X is in different regions), then $D'$ is in set $\mathcal{D}$.

According to the above definitions below are some examples of well formed and ill formed diagrams: (cf. [120], pp. 90-93)

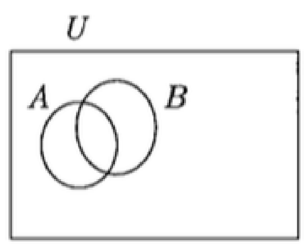



Diagram $c$

Figure 5.4: Well formed Diagrams according to the Definition 1.

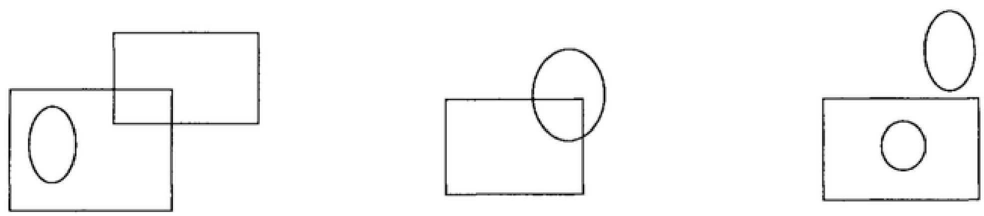


Figure 5.5: Ill formed Diagrams according to the Definition 1.

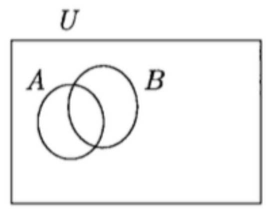



Figure 5.6: Well formed diagram according to Definition 2.

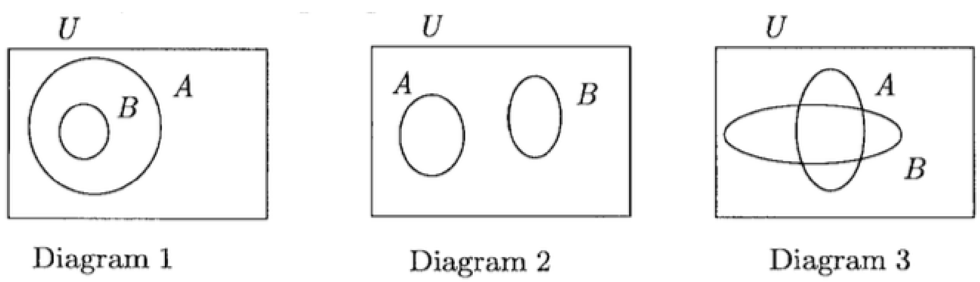



Figure 5.7: Ill formed Diagrams according to the above recursive Definition 2.

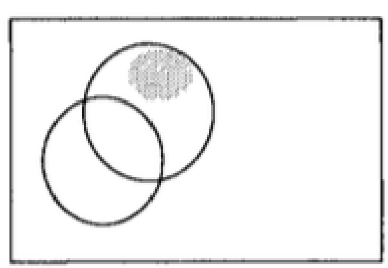


Figure 5.8: Ill formed diagram according to Definition 3.

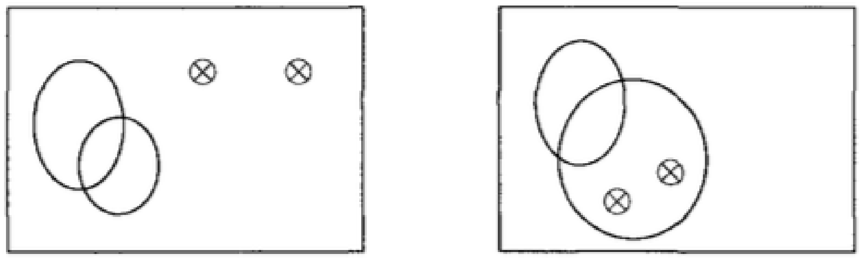


Figure 5.9: Well formed Diagrams according to Definitions 4 and 5.

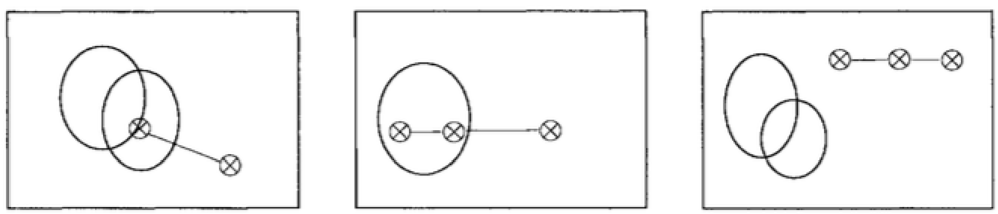


Figure 5.10: Ill formed diagrams according to Definitions 4 and 5.

Then, her next step is to define semantics from regions to sets, which have the homomorphic relation. The content of a diagram can represent certain facts of a situation which one aims to reason. Recall the requirement of the homomorphy revealed above: "Shading a region serves to focus attention on that region [and] there is a homomorphic relationship between regions (with the operations of addition and intersection of regions) and sets (with the operations of union and intersection of sets)". ([5], pp. 9-10). Accordingly, drawn regions represent sets; shading and X represent facts about the sets when they are placed in these regions specified by rectangles and circles. To prove the soundness and completeness of the system she needs to show that facts about a diagram follows from facts about a set of diagrams, which means that a certain diagram follows from other diagrams if and only if the following rules of transformations entails the diagram, which means the diagrams can be obtained from other diagrams. Formally this is represented as: $\mathcal{D} \vdash D$ if and only if $\mathcal{D} \models D$. Notice that, she replaces the terms from formal logic "provable" by "obtainable" and "logically valid" by "facts following from facts".

In this system the rules of transformations are as follows:

1. The rule of erasure of a diagrammatic object
2. The rule of erasure of the part of an X-sequence
3. The rule of spreading X's
4. The rule of conflicting information
5. The rule of unification of diagrams

Examples and explanation for the rules:
(1) If for example there is one set filled with shading this region can be deleted.(cf. [120], p. 98)

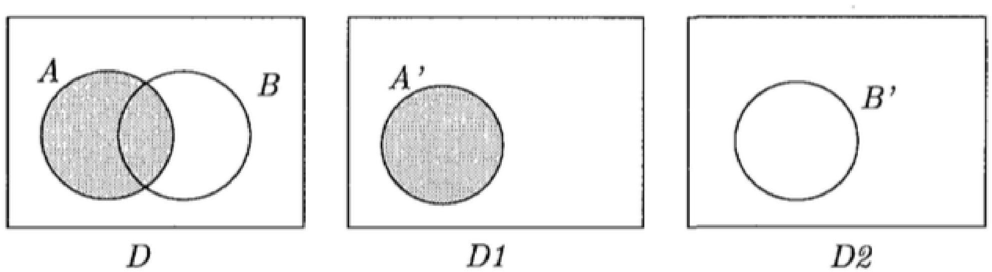



Figure 5.11: An example of this rule.

(2) If X is in the shaded region it can be deleted.(cf. [120], p. 100)

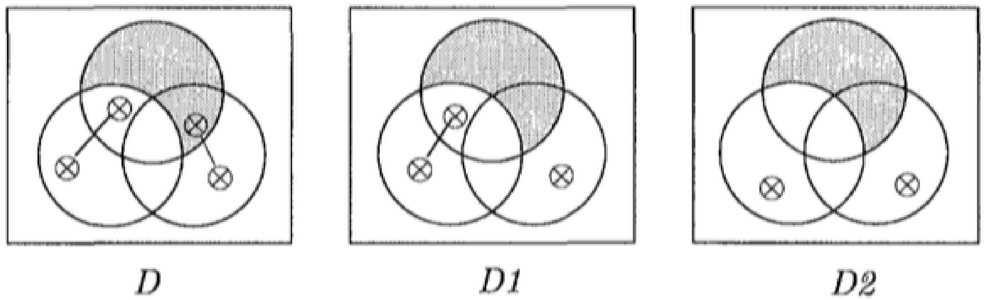



Figure 5.12: An example of this rule.

(3) If *wfd* $D$ has an X-sequence, then the copy $D$ with $\otimes$ can be drawn in some other region and connected to to the existing X-sequence.(cf. [120], p. 100)

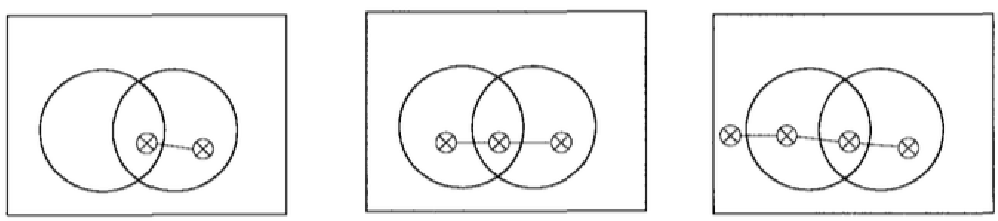


Figure 5.13: An example of this rule.

(4) If a diagram has a region with both a shading and an X-sequence then this diagram can be transformed to any diagram. (cf. [120], p. 101)

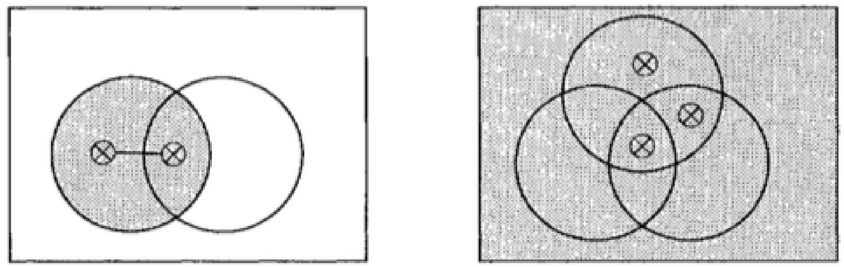


Figure 5.14: An example of this rule: We can transform the diagram on the left to the one on the right.

(5) It is similar to unification of Venn diagrams.(cf. [120], p. 103)

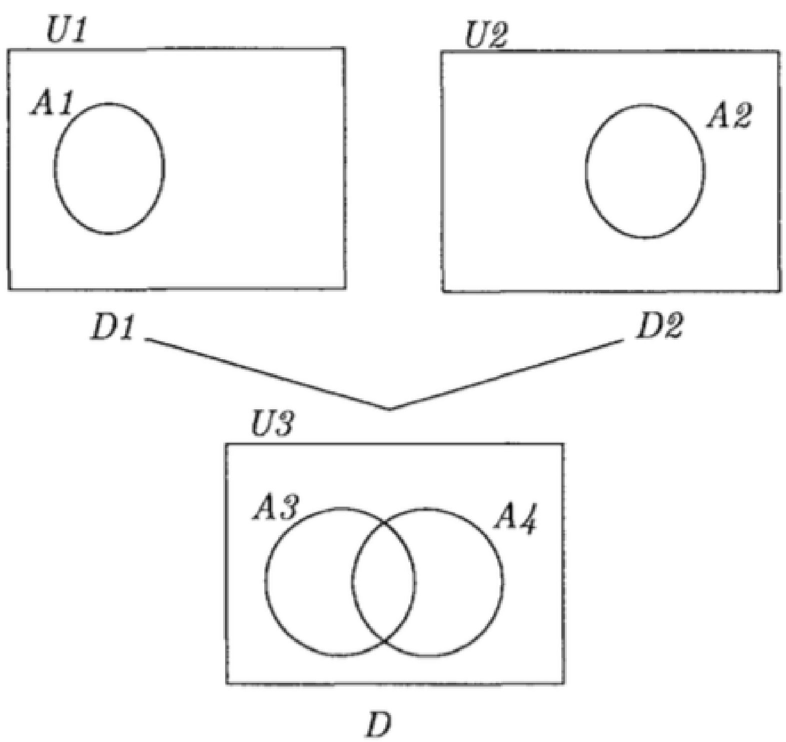



Figure 5.15: An example of this rule.

With these transformation rules and with the semantics of the diagrams she shows that a diagram is obtainable from other diagrams if and only if a diagram follows from other diagrams. Hence, the representation system she proposed is complete and sound.

We have seen that a formal representational system with diagrams can be constructed as a result of Shin's characterization of Venn diagrams with shading and Peircian existential import x. Her system is built in the way a formal valid linguistic system is constructed.

The use/ mention rules were first created. Formation rules for well formed diagrams were designated. Then after setting up the syntax, semantics of the system were built. Finally transformation rules were exhibited and the completeness and soundness of the system were proved. That makes this visual system a formal and valid visual representation system.

This system is a counter example for the claim that all diagrammatical thinking is synthetic or non-logical. It is evident that the explanatory power of Shin's system is not even comparable with a formal sentential system. However, I believe that this is a good example for showing that formal systems can be constructed using diagrams. This and similar works (cf.[86], [80] ) are only the initial attempts for such systems that can be representative of much stronger and efficient formal systems with diagrams.

Although these formal systems with diagrams show the legitimacy of using diagrams as formal tools, one question arises naturally: "Is constructing formal systems with diagrams or pictorial representations a prerequisite to be able to use them as valid tools in mathematical reasoning?". My answer is no. Although it is an advantage to have such a system and would make skeptical formalists more open to idea of using diagrams in proofs, I do not believe that it is a prerequisite to use diagrams as valid tools in proofs. After introducing other uses of diagrams in the next section, I will return to this claim and show the necessity of diagrams in mathematics and also show that why mathematical communication without pictorial representations does not represent the actual mathematical reasoning.

## 5.2 Diagrams as Valid Reasoning Tools

I believe, it is important to notice that although there could be invalid use of diagrams in proofs, the conclusion that "all diagrammatical reasoning is either heuristic, or invalid or superfluous" does not necessarily follow. We have seen that, diagrams could be formally valid, sound representation systems in the last section. Now, let us analyze in which other usages they can be valid tools in mathematics.

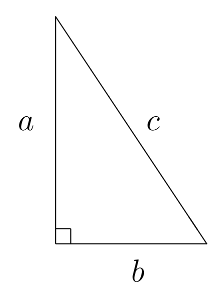
### 5.2.1 Diagrams in Proofs

In certain mathematical proofs diagrams maybe superfluous, in some they can be replaceable with sentences, sometimes at a cost: the clarity and practicality. However in some of the proofs diagrams are "neither superfluous nor replaceable". ([39], p. 86) In this section, I will introduce certain studies that point out the need for diagrams in some mathematical proofs. First, I will reveal a proof of Pythagorean Theorem that uses the geometrical manipulation with algebraic manipulation. Second, I will represent a classical Euclidean argument as it is analyzed by Jesse Norman, in his book *After Euclid* [91]. Then, I will discuss the permutational reasoning with diagrams. In the first two cases, the proofs do not even make sense without the diagram and in the third case the proof that use the diagram is a preferable alternative to the formal one. As a last example I will analyze diagrams that are used as helpful tools for visualizing in proofs. Although the proof makes sense without the use of diagrams, and not having the diagrams does not reduce the explanatory power of the proof, diagrams seem to be naturally used, helpful tools for the reasoning in this proof.

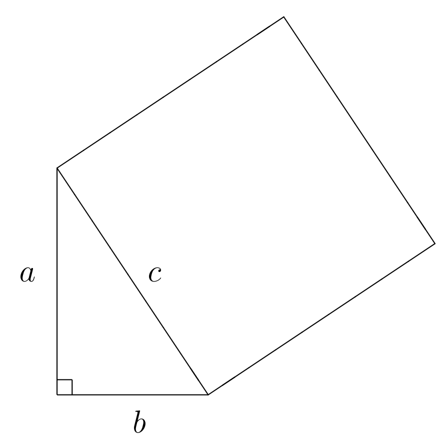
**Pythagorean Theorem.** Barwise and Etchemendy analyze the proof of Pythagorean Theorem since "it is an interesting combination of both geometric manipulation of a diagram and algebraic manipulation of nondiagrammatic symbols." ([5], p. 8) which is called the heterogeneous reasoning.

The theorem states that:

Given any right angle triangle as $b \leq a \leq c$, that are the side-lengths of a right triangle, then $a^2 + b^2 = c^2$.

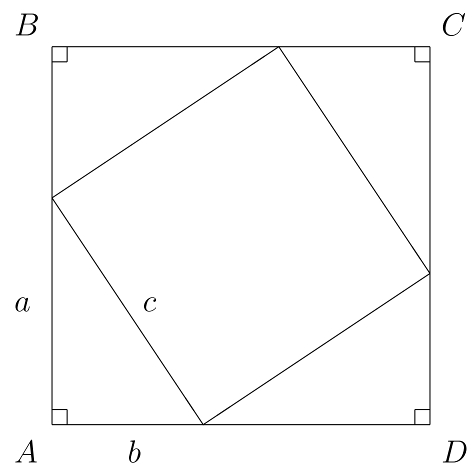



One of the proofs[6] of the theorem constructs a square on the hypotenuse as follows:




[6] There are 93 known ways of proving the Pythagorean theorem: cf. [10]

Then the original triangle is replicated three times as follows:




$ABCD$ is a square since the sum of angles of a triangle is a straight line (Prop. I. 32 below). The area of the square is equal to both of the following:

$$AREA(ABCD) = (a+b)^2$$

$$AREA(ABCD) = 4(ab/2) + c^2$$

Hence:

$$(a+b)^2 = 4(ab/2) + c^2$$

$$a^2 + b^2 + 2ab = 2ab + c^2$$

Therefore:

$$a^2 + b^2 = c^2$$

This proof makes no sense without the diagram. Trying to get rid of the pictorial part is absurd since we are talking about the pictorial representation. We can remove the actual drawing and use letters to represent the triangle, call the the first triangle $ABC$, and then move on to describe the whole proof by letters and name the straight lines $AB = a$ and then continue in this fashion to define all the triangles, but what good does it provide for the proof or mathematical reasoning apart from showing that these kind of proofs can be written down formally? We still construct the image in our mind. We still do not reason by only using these symbols. **These symbols refer to visual entities**. Using only symbols without the figures makes the process slower and records the process insufficiently.

Etchemendy and Barwise also argue that "no one would discover or remember this proof without the use of diagrams" ([5], p. 8). Moreover, as in all geometrical proofs, once one finds the diagrams to reason, the rest of the proof is easy to carry on. In this proof, by manipulating shapes and using algebraic properties such as distribution law, one can arrive at the conclusion. Although one cannot conclude only from the diagram that $a^2 + b^2 = c^2$, since algebraic manipulation is necessary too; without the diagram $a^2 + b^2 = c^2$ does not make sense either. This is an example where diagram is not a sufficient but essential part of the proof.

In the following, I will introduce another Euclidean proposition and its proofs as in the Euclid's *Elements* and Hilbert's *Grundlagen der Geometrie.* But, before continuing to the proofs, to understand Hilbert's system better, let me summarize his formalization steps briefly:

Hilbert, sets up his axiomatic and formalized system by defining points (A, B,C), straight lines (a, b, c) and planes ($\alpha$, $\beta$, $\gamma$) and axioms connecting these. He formalizes Euclidean geometry and provides a logical and rigorous foundation. He introduces his aim in the introduction of *Grundlagen der Geometrie* as follows:

> "Geometry, like arithmetic, requires for its logical development only a small number of simple, fundamental principles. These fundamental prin-

> ciples are called the axioms of geometry. The choice of the axioms and the investigation of their relations to one another is a problem which, since the time of Euclid, has been discussed in numerous excellent memoirs to be found in the mathematical literature. This problem is tantamount to the logical analysis of our intuition of space." ([51], p. 1) [7]

With this motivation, he introduces "simple and complete set of independent axioms" and derives all propositions of Euclidean geometry only with axioms, logical inference rules and definitions. In Hilbert's *Grundlagen der Geometrie* Pythagorean Theorem is just a "simple consequence of the theorems". ([50], p. 42)

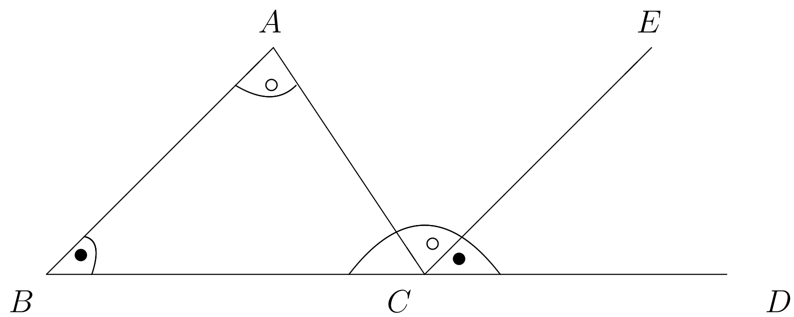
In the following, I will compare proofs of Proposition 32 as they are in Euclid's *Elements* and Hilbert's *Grundlagen der Geometrie*. We will clearly see the difference between formal and synthetic proof of an Euclidean proposition and also will be able to conclude that although pictorial representations seem to be excluded from Hilbert's proof, one still has to construct the diagrams internally, if not externally, in order to understand or create the proof.

**Euclidean Proposition 32.** The proposition states that: "the sum of interior angles of any triangle is equal to two right angles".[8]

$$\left.\begin{array}{l}\text{For any triangle } ABC \text{ on line } BCD\\ (1)\ \angle ACD = \angle ABC + \angle BAC\\ (2) \angle ABC + \angle BAC + \angle ACB = \text{two right angles}\end{array}\right\} \textbf{Claims}$$

[7] "Die Geometrie bedarf- ebenso wie die Arithmetik - zu ihrem folgerichtigen Aufbau nur weniger und einfacher Grundsätze. Diese Grundsätze heißen *Axiome* der Geometrie. Die Aufstellung der Axiome der Geometrie und die Erfroschung ihres Zusammenhanges ist eine Aufgabe die seit *Euclid* in zahlreichen vortrefflichen Abhhandlungen der mathematischen Literatur sich erörtert findet. Die bezeichnete Aufgabe läuft auf die logische Analyze unserer räumlchen Anschauung hinaus."([50], p. 1)

[8] Reiterated from Jesse Norman's book *After Euclid* [91].




Let ABC be a triangle [by Definition 19] } **Setting-Out**

Let BC be produced to D [by Postulate 2]

Let CE be drawn through C parallel to AB

E to lie on the same side of BCD as A [by Prop. I. 31]

} **Construction**

**Demonstration:**

1. $\angle BAC$ and $\angle ACE$ are alternate [from the diagram]

2. Alternate angles are equal [by Prop. I. 29]

3. $\angle BAC = \angle ACE$ [1, 2: by substitution]

4. $\angle ECD$ is exterior and opposite to interior $\angle ABC$ [from the diagram]

5. Exterior and interior opposite angles are equal [by Prop. I.29]

6. $\angle ABC = \angle ECD$ [4, 5: by substitution]

7. $\angle ACD = \angle ECD + \angle ACE$ [from the diagram]

8. $\angle ACD = \angle ECD + \angle BAC$ [3, 7: by substitution]

9. $\angle ACD = \angle ABC + \angle BAC$ [6, 8: by substitution]

   Line 9 proves claim 1.

10. $\angle ACD + \angle ACB = \angle ABC + \angle BAC + \angle ACB$ [9: by Common Notion 2 [9], adding $\angle ACB$ to both sides]

11. $\angle ACD + \angle ACB$ is the sum of all the angles on BCD [from the diagram]

12. BCD is a straight line [by the first Construction step]

13. $\angle ACD + \angle ACB$ is the sum of all the angles on a straight line [11, 12: by substitution]

14. The sum of all angles on a straight line = two right angles [by Prop. I. 13]

15. $\angle ACD + \angle ACB =$ two right angles [13, 14: by substitution]

16. $\angle ABC + \angle BAC + \angle ACB =$ two right angles [10, 15: by substitution]

    Line 16 proves claim 2.

Norman points out that the lines 1, 4, 7 and 11 are justified by the diagram since it has representational content. This proof does not only use, common notions, previous propositions and inference rules, it also requires at least the identification of a previous propositions on the diagram. For example, how can one use the alternate angle proposition, without specifying where to use it. It is clear that even though one does not draw the diagram empirically on a paper or a board, one has to visualize it. **Which brings us back to the Kantian argument. The geometer has to exhibit the intuition corresponding to the concept *a priori*. It can be exhibited a posteriori as a drawing on a paper but this is voluntary**

[9]"if equals be added to equals, the wholes are equal

**and hence does not make the reasoning empirical according to Kant.** Note that, if one carries out the proof by visualizing, with pictorial representations, in the Kantian terms this corresponds to exhibiting the intuition *a priori* corresponding to the concept, which produces internal diagrams. In current philosophy of mathematics and logic, only external diagrams are used. The internal diagrams are classified as psychological tools and completely excluded from official mathematical practice. Only in Kantian characterization of mathematics we see internal diagrams categorized as valid and objective tools in mathematics and also this claim is asserted earlier by Proclus in *A Commentary on the First Book of Euclid's Elements* [108]. He argues that geometers "do not make use of the particular qualities of the subjects [diagrams] but draw the angle or the straight line in order to place what is given before our eyes, they consider that what they infer about the given angle or straight line can be identically asserted for every similar case. They pass therefore to the universal conclusion in order that we may not suppose that the result is confined to the particular instance."[10] ([108], p. 207)

Now, let me refer to the **the formal proof of the proposition** corresponding to Theorem 20 in Hilbert's system to see the differences between these two methods. The whole written part of the theorem and proof are as follows in *Grundlagen der Geometrie*:

> "Theorem 20. The sum of the angles of a triangle is two right angles.
>
> **Definitions:** If M is an arbitrary point in the plane $\alpha$, the totality of all points A, for which the segments MA are congruent to one another, is called a *circle*. M is called the *centre of the circle*. Then the proof follows

[10] Proclus argues that Platonic shapes are in the imagination, however a "true geometer should ... make it his goal to arouse himself to move from imagination to pure and unalloyed understanding", thereby placing the universals of mathematical figures prior to their instances .(cf. [108], p. 51, p. 56)

> from the axioms: From this definition can be easily deduced, with the help of the axioms of groups III and IV, the known properties of the circle; in particular, the possibility of constructing a circle through any three points not lying in a straight line, as also the congruence of all angles inscribed in the same segment of a circle, and the theorem relating to the angles of an inscribed quadrilateral." ([51], p. 15)

That is the whole proof. One does not need to draw shapes in Hilbert's system, since points, lines and planes are represented by sentential symbols as I have pointed out earlier. The Euclidean geometry, hence, was formalized in this way. However, can one claim that we do not need pictorial representation in this proof?

The symbols Hilbert uses refer to pictorial representations, and not having these representations in the written medium does not mean that we do not need them. Moreover, there is not one single reproduction of this proof with Hilbert's methods in history of mathematics after he has given the formulation. Hence, Hilbert's method shows that Euclidean proofs can be axiomatized and formalized but this method has no practical use since it is against the nature of this kind of mathematical reasoning.

The general conclusion formalists derive from Hilbert's formalization of Euclidean proofs is that that there is no need for pictorial representations in geometrical proofs since we can formalize them as Hilbert shows in *Grundlagen der Geometrie*. This is one extreme example of formalism going as far as defending the formalization of visual representations when all proof is about a visual representation.

This example shows us the motivation of the formalists of the era and the prejudice against pictorial representations. Even Hilbert himself did not claim one should get rid of the pictorial representations. His aim was to show that Euclidean system can be adequately axiomatized and formalized with the new formal tools. He also wanted to give "logical analysis of our intuition of space" ([51], p. 1), He did not have the purpose of completely eliminating pictorial representations from mathematical reasoning.

As I have repeated several times in this thesis, not drawing the diagram next to the proof physically does not get rid of the pictorial representation that is constructed during the mathematical reasoning in the proof. We still have to construct the diagram at least in our mind's eye, since the symbols refer to visual representations. Of course formalist influence who even defended the exclusion of diagrams from Euclidean geometry was even stronger in other areas of mathematics. So, the belief that "every visual representation we can formalize we should get rid of" became the prevalent practice in mathematics.

Is this the right way of approaching the matter? Sentential formalization provides the rigor, but with today's technology in visualization can we not provide a similar rigor with visual representations? Are we not missing out much by not using visualization in mathematical communication? I will get back to these questions in the next sections. To see some more examples of visual proofs let me now go on to introduce the ones where the visual proof has more clarity over the formal one and still keeping the rigor and validity as well.

**Permutation Proofs** Barwise and Etchemendy discuss a problem solving situation in permutational seating examples. ([5], pp. 11-12) The set up is that there are four people $A$, $B$, $C$, and $D$ that must be seated in a row of five chairs. $A$ and $C$ must wing the empty chair. $C$ must be closer to the center than $D$, who is to sit next to $B$. From this information we want to show:

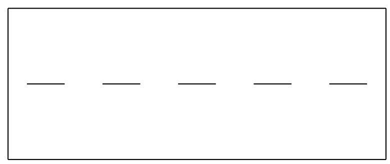
1. The empty chair is not in the middle or on either end.

2. It is possible to tell who must be seated in the center.

3. Who the specific people are to be seated on the two ends.

The diagrammatical proof is as follows:

Let the following diagram represent five chairs:



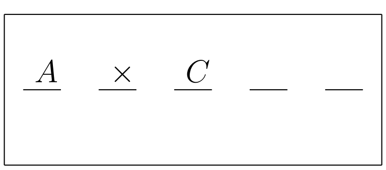
We have:

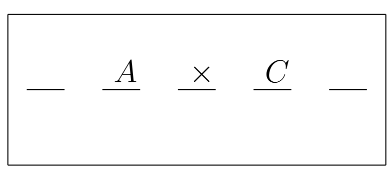
1. $A$ and $C$ must wing the empty chair.

2. $C$ must be closer to the center than $D$.

3. $D$ sits next to $B$.

From 1. one can split the situation in three cases since there is no difference between left and right since they mirror each other.



$\underline{\quad}\ \ \underline{\quad}\ \ \underline{A}\ \ \underline{\times}\ \ \underline{C}$

From 2. one can eliminate the case where $C$ is at the end. Which leaves the cases:

$\underline{A}\ \ \underline{\times}\ \ \underline{C}\ \ \underline{B}\ \ \underline{D}$

$\underline{A}\ \ \underline{\times}\ \ \underline{C}\ \ \underline{D}\ \ \underline{B}$

One of the set ups for the formal proof can be as follows:

Let $T : \{A, B, C, D, X\} \to \{-2, -1, 0, 1, 2\}$ be 1-1 and onto map such that:

i) $T(A) + 1 = T(X) = T(C) - 1$ or $T(A) - 1 = T(X) = T(C) + 1$

ii) $|T(C)| < |T(D)|$

iii) $|T(B) - T(D)| = 1$

Then show that $T(X) \neq 0$.

*Proof.* Assume $T(X) = 0$ then by $i)$ either $T(A) = 1$ and $T(C) = -1$ or $T(A) = -1$ and $T(C) = 1$. Assume $T(A) = 1$ and $T(C) = -1$ then $|T(C)| = 1$ and by $ii)$ this implies $|T(D)| = 2$. Hence, either $T(D) = 2$ or $T(D) = -2$. If $T(D) = 2$ by $iii)$

$T(B) = 1$ but we have $T(A) = 1$ which contradicts with $T$ being 1-1. If $T(D) = -2$ by $iii)$ $T(B) = -1$ but we have $T(C) = -1$ which again contradicts $T$ being 1-1. Now assume $T(A) = -1$ and $T(C) = 1$ then $|T(C)| = 1$ and $ii)$ implies that $|T(D)| = 2$. If $T(D) = 2$ by $iii)$ $T(B) = 1$ but we have $T(C) = 1$ which contradicts with $T$ being 1-1. If $T(D) = -2$ by $iii)$ $T(B) = -1$ but we have $T(A) = -1$ which again contradicts $T$ being 1-1. Therefore $T(X) \neq 0$. Hence the middle chair cannot be empty. □

That was only the proof of the first of the following:

1. The empty chair is not in the middle or on either end.
2. It is possible to tell who must be seated in the center.
3. Who the specific people are to be seated on the two ends.

One can prove 2. and 3. in a similar fashion by exhausting the cases with contradiction. Moreover, Barwise and Etchemendy draw attention to even more important part that "Some parts of this proof are missed in traditional logic" ([5], p. 12).

There are three tasks that are asked to solve the problem (cf. [5], p.12) :

1. Prove that specific facts follow from the given information.
2. Answer whether certain sort of information is implicit in the given proof.
3. Show that something does not follow from the given, namely, who is seated on the end opposite A

Traditional formal account cannot give the proof of all three cases simultaneously as the diagrammatical proof provides in this problem since there is both a model building and deduction in this problem. One should notice that model building is different than deduction and both of these exist in this proof but not essentially separately. Deduction is showing from some given premises, according to inference rules in the system, a conclusion follows from these assumptions and inference rules. Model building is showing that something does not follow from the assumptions and the inference rules. In this case we are asked to show that the "*nonconseqence* result by coming up with models of the given, one of which has B on the end, while the other has D" ([5], p.12) The advantage of reasoning with the diagram is that we can consider all the given cases at once and show that some things followed and somethings did not follow from the reasoning, however, giving the proof sententially will lack this power and, moreover, it will lack the practical clarity.

In this case the proof with diagram is preferable to the linguistic proof, despite the fact that diagram can be formulated completely with formal sentences and linguistic notation. However, diagrammatic proof has more clarity and practicality in a valid sense than the linguistic proof.

**Combinatorial Expressions.** There is also a fourth case where the diagrams are not needed strongly for the proof but makes the reasoning easier in the case of, for example, "analysis using combinatorics, where diagrams were used to obtain certain results" ([16], p. 2). In these proofs diagrams are used to "break down proofs into manageable parts and thus to focus on certain details of a proof, by removing irrelevant information." ([16], p. 3).

Jessica Carter discusses a case study in the free probability theory. (cf. [16], pp. 4-8 and for the original proof cf. [46]). Briefly, the aim of the theory is "to formulate a noncommutative analogue to classical probability theory hoping that this would lead to new results in analysis" ([16], p. 4). She discusses a part of a paper published by Haagerup and Thorbjørnsen [46]. In this proof the notion of "non-crossing and crossing diagrams" are firstly explained by diagrams, but it is not in the published paper. Carter notes that "the definitions of a pair of neighbours, a non-crossing and a crossing permutation, as well as cancellation of pairs, are inspired by the diagrams." ([16], p. 9)

She also argues that from the beginning of this proof the terms such as "successive cancellation" and similar visualization demanding concepts are used. (cf. [16], p. 10)

> "First, certain diagrams are used to represent permutations (indicating which matrices are identical) and similar diagrams represent equivalence classes under the discussed equivalence class. Experimenting with these diagrams points to the fact that the number of equivalence classes depends on whether the permutation has crossings or not. The definition of a crossing is then given an algebraic formulation. Likewise the concepts of a neighbouring pair and of removing pairs (from the diagram) are translated to an algebraic setting." ([16], p. 11)

After these steps, the full algebraic part of the proof is given. The following diagram is used for reducing the permutation until there are no more pairs to arrive at the result that "all equivalence classes, except possibly the class containing 1, contains at least 2 members". In general, the number of equivalence classes must be $p+1$ where $p$ is the index of the permutation.[11]

[11] The figure is taken from Carter's paper: ([16], p. 9)

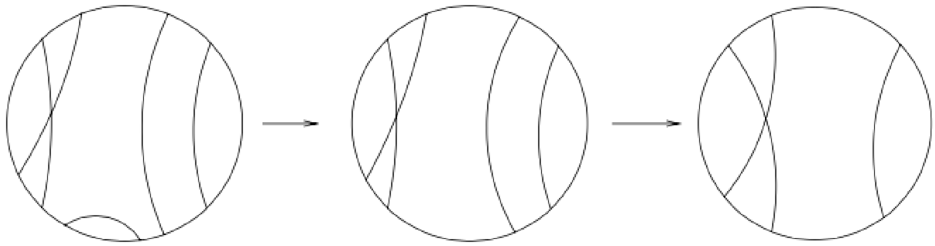


Figure 5.16: Successive cancellations of pairs of a crossing permutation. It is possible to remove one more pair in the last diagram.

The figure is to support the general claim that "the relations used in the proof based on the diagram represent relations that also hold in the algebraic setting"([16], p. 11) Since this diagram has the reliable representational properties "it is possible to translate the diagrammatic proof to an algebraic proof." ([16], p. 11) Therefore, in a legitimate way the diagram corresponds to the algebraic formulations.

Carter also draws attention to a notion introduced by Jody Azzouni in his paper "Is There Still a Sense in Which Mathematics Can Have Foundations?" [3]. Carter reveals that:

> "Azzouni introduces the notion of inference packages that encapsulate a body of assumptions used in a particular inference. As examples, he mentions inferences drawn about properties of triangles as they are moved on the plane, a sphere, and an ellipsoid. Inference packages need not involve the use of diagrams, but Azzouni explains that a diagram may 'trigger the employment of one or another inference package that we have already (largely) got available, and that enables us to package a number of assumptions together that are separately operating in the proof, and quickly see what they imply'." ([3], p. 25)(referred in: [16], p. 12)

In this case, the usage of diagrams is similar to the one Azzouni talks about: it is possible to do the reasoning without diagrams, however, in the discovery and preparation of the proof, diagrams enable one to see quickly what algebraic formulations imply. This case is different from the previous three cases we have seen.

In the first two cases (Pythagorean Theorem and Proposition 32) when diagrams are replaced by algebraic formulations the proofs are not used in mathematical practice, which indicated the preferability of the diagrammatic proof. The proofs are about the diagrams, hence, apart from showing that they are translatable to formal sentential representations, these proofs do not have practical usage in mathematics at all.

In the third case (Permutation Proofs), the diagrammatical proof is preferable to the algebraic one. In the fourth case (Combinatorics) the diagrams make the reasoning easier and used as a corresponding counterpart for the algebraic formulation. In this proof, the pictorial representations are used purely in operable and iterative way. That makes the proof with the diagram a valid one as well. Indeed almost all diagrams/ pictorial representations are used in operable way in mathematical proofs. Measuring with compass and ruler are long gone ancient methods of geometry. One does not use the "exact" properties of diagrams in mathematical reasoning. As I have mentioned earlier, Ken Manders is the first philosopher naming these distinction as exact and co-exact properties of diagrams. [85]

Manders characterizes *co-exact* properties of diagrams as "part-whole relations of regions, segments bounding regions, and lower-dimensional counterparts". ([85], p .69) This characterization is similar to Kantian claim "seeing the general in particular". In contrast, the *exact* properties of diagrams consist of their side lengths, the measure of angles; namely, the properties that belong to a particular diagram.

Using the co-exact properties in operable way makes the use of diagrams in proofs a valid usage and even a necessary one in some cases. I believe, no one uses the sentential formal proof of the third case (Permutation Proofs), no one even bothers to write it down since the pictorial proof is practical, clear and unquestionably free from fallacies whereas the sentential formal proof is long, unpractical and hard to understand for the reader.

When I argue for the necessity of diagrams, some objections arise as I do not use necessity in the logical sense. Necessity in logic implies that in the assertion "Q is necessary for P", "P cannot be true unless Q is true". Of course, in this sense, from within logic one cannot prove the necessity of diagrams in mathematics. The statement "diagrams are necessary for mathematics" will not be true in current situation of mathematics where only available and valid method is formal sentential reasoning. Because, in this state mathematics *can* operate without diagrams and diagrams are not necessary. Every diagram can be turned into sentential formal operators, and that is the argument formalists have been using to show that diagrams are not "necessary".

When we use the terminology of the prevalent formal method, which does not already allow any pictorial representations, we cannot show the "necessity" of diagrams in mathematics. We have to use another terminology as Barwise and Etchemendy also points out by giving a different definition for validity as "extraction or making explicit of information that is only implicit in information already obtained", rather than the formal counterpart "an argument is *valid* if it is impossible for the premises to be true and the conclusion to be false. I suggested the definition of necessity in heterogeneous reasoning as "primarily and naturally using visual representations before needing to apply to any other representation".

It is not possible to argue the formal system needs pictorial representations from within the system and that is why it is so hard to change the current mathematical practices. But by using the sentential formal method merely, which instead should be one of the methods in mathematics, mathematics loses a lot of value and does not fulfill its potential. By not allowing diagrams fully, mathematics does not represent actual mathematical reasoning. It limits itself as much as a machine can operate. However, the real capacity of mathematics is not in turning it into a machine language but seeing and appreciating the aesthetic part of it as well. Moreover, in current analysis of mathematical reasoning emphasis is mostly placed on the justification part of it as we have seen in Barwise's, Etchemendy's and Shin's works. Context of justification is only a part of mathematics and I find the extreme emphasis placed on it a little bit misguided. Therefore, in the next section, I will analyze another approach to diagrams, valuing the epistemic discovery part by Marcus Giaquinto [39]. His investigation of visual reasoning in mathematics also points out a connection between Kantian characterization of mathematics and diagrams.

## 5.3 Visual Reasoning with Diagrams

In this section, I will discuss Giaquinto's arguments on diagrams as they are revealed in his book *Visual Thinking in Mathematics* [39]. Giaquinto has an epistemic approach to diagrams and Kantian characterization of mathematics. Therefore, his first key aim is to find an answer to the following question: "Can a visual way of acquiring a mathematical belief justify our believing it?" ([39], p. 1) He applies standard *justified true belief* approach, combining it with reliability and rationality of his knowledge thesis. Accordingly, a mathematical belief is knowledge, if it is true, if it does not admit "a violation of epistemic rationality in the way it is acquired and maintained", and if this belief is justified. ([39], p. 40) In addition to these, beliefs that are obtained through reasoning with diagrams, namely through visual thinking, are also knowledge, according to Giaquinto.

Giaquinto, then, aims at analyzing purely visual knowledge and exhibits the following question: "Can a visually acquired mathematical belief be knowledge in the absence of independent non-visual grounds?" ([39], p. 1) In the process of answering this question, he also establishes that mathematical knowledge is diagrammatical and non-evidential, in other words, synthetic *a priori*. He borrows the latter terms from Kant, and argues that: " Given that "non-analytic and non-empirical" translates as "synthetic *a priori*", we have arrived at a view that is at least close to Kant's often dismissed view that there can be synthetic *a priori* knowledge."([39], p. 47)

His general thesis for this approach is that mathematical thinking, at least for discoveries, is non-evidential and diagrammatical, hence *a priori* and synthetic. (cf. [39], pp. 67-68)

Giaquinto's arguments in depth are fascinating in showing the different parts of mathematical knowledge. Understanding his arguments in detail will clear any doubts against the existence and usage of synthetic *a priori* reasoning in mathematics. Therefore, I would like to give a detailed analysis of his arguments to connect what we have been discussing throughout the thesis: Namely, that synthetic *a priori* or visual reasoning exist in actual mathematical thinking and it is very similar to how Kant characterized it in the *Critique*.

I will first discuss his approach to seeing and visual imagining of diagrams in two dimensional space. Then, I will reveal a case study on an arbitrary square to understand perceptual and geometrical concepts for a square. Moreover, I will summarize how he establishes mathematical knowledge as synthetic *a priori*, based on these concepts, and how he connects his thesis with the Kantian characterization of mathematics. Finally, I will compare and discuss my interpretation of diagrams and Kantian philosophy of mathematics in connection to Giaquinto's thesis.

### 5.3.1 Perception and Recognition

According to Giaquinto visual imagining and seeing triggers "belief-forming dispositions" and that is how the geometrical knowledge is gained - given that these "dispositions are reliable". ([39], p. 12). He is mainly concerned with two dimensional basic geometrical concepts which depend on our perception.

**Visual Perception.** Giaquinto first observes that "[v]isually perceiving edges or borders of surfaces ... does not necessarily involve seeing lines that mark those borders" ([39], pp. 13-14). In order to reveal his approach let us examine one of his examples, Kanizsa figure as triangle:

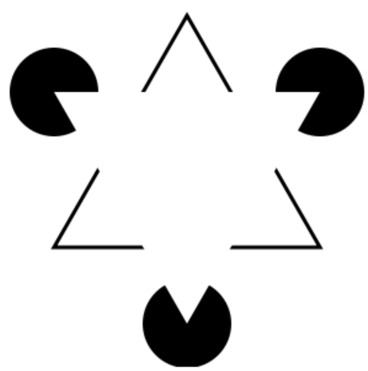


Figure 5.17: Kanizsa figure

Giaquinto argues that, although there are no edges drawn in Kanizsa figure as a triangle, one's response to it is to detect straight edges. He also reveals that: "..in seeing the Kanizsa figure as a triangle,the visual system responds as though detecting straight edges from vertex to vertex when no such edges occur in the retinal image". ([39], p. 13)

The reason he brings out this example is to challenge the conclusion of some cognitive findings. ([44], [56], [24]) Namely, "that the processing in visual perception produces representations only of features of the retinal image"([39], p. 14). By analyzing Kanizsa figure, and explaining how one can have the experience of seeing a line without actually seeing the boarders in the figure, **he claims that this can be the skeleton of geometrical knowledge being synthetic *a priori*.**

Another similar idea to Giaquinto is revealed by Izard, et al. [59] that even if human perception of space "violate Euclidean principles in several ways" (cf. [48], [72]) (for example, by imposing a curvature to the space (cf. [58]) or failing to unify different scales (cf. [81])), the way that we conceive space is not necessarily constrained by our perception. They argue, following Kant's proposal, that "the axioms of Euclidean geometry may constitute the most intuitive conceptualization of space not only in adults educated in the tradition of Euclidean geometry (cf. [2]) but also in cultures where this tradition is absent." ([59], p. 1)

Experiment of Izard, et al. focuses on Indigenous people in Amazon, called Munducuru, with no access to education in geometry. It examines whether the Munducuru people "possess an intuition that the sum of three angles in a triangle is constant". ([59], p. 1) The results do not change between Munducuru group, children who did not get education in geometry and the adults who got the education. ( [59], p. 3) From their experiment Izard, et al. derive the conclusion that: "Euclidean geometry, inasmuch as it concerns basic objects such as points and lines on the plane, is a crosscultural universal that results from inherent properties of the human mind as it develops in its natural environment". ([59], p. 4)

I will get back to Giaquinto's and Izard's results after explaining in detail how a person's conceptualization of geometrical shapes is acquired and exhibited according to Giaquinto.

**Reference System and Description Sets.** Giaquinto defines *reference system* as the "orthogonal axes, one which has an assigned "up" direction"([39], p.15). He argues that the orientation of a figure effects the perception of an object and therefore the reference system plays an important role in perception of the object. He also defines a *description set* as "set of associated feature representations" One thing to note is that "perceivers do not have conscious access to description sets of their perceptual representations" ([39], p. 17).

Giaquinto also notes that: " A description set is a set of associated feature representations, and it has much the same role as a category pattern in the category pattern activation subsystem postulated by Kosslyn."[12] ([39], p. 17) The link between perception and orientation, accordingly, is decided by description sets and the reference system. For example, according to Giaquinto, one's perceiving a square or a diamond shape is dependent on the description set. For example, in the case of a diamond shape, the description set is the following: "symmetrical about the vertical axis with one vertex at its top, and another at its bottom, and one vertex out to each side". In the case of a square these specifications are; "symmetrical about the vertical axis with one horizontal edge at its top, another at its bottom, and vertical edge out to each side". ([39], p. 16) This is how one distinguishes between a diamond figure and a square.

According to Giaquinto, perceivers have both **stored descriptions sets** and **generated descriptions sets**. (cf. [39], p. 17) This is what makes the difference

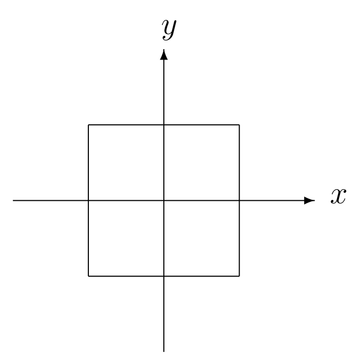
[12]Kosslyn's theory is concerned with neural patterns as category pattern activation subsystems and exemplar pattern activation subsystems which classifies and registers the instances of visual categories respectively. The full discussion can be found in ([74], p. 182ff, p. 231ff. p. 402ff).

between **perceptual recognition** and **perception** respectively. In the case of **perceiving**, only **generating a description set** is at stake, which does not have to be conscious. In the case of **perceptual recognition**, perceiver *finds* a "best match between the generated description set and a stored description set for the conventional appearance of the figure"([39], p. 17), which is a conscious procedure.

So far we have seen Giaquinto's general theory about shapes in connection with reference system, description sets and the difference between perception and perceptual recognition. I will now analyze his case study on squares. This will help us to understand how these basic terms are applied in detail in the case of acquiring perceptual and geometrical concepts of squares, which will eventually lead to the explanation of *a priori* and synthetic character of geometrical knowledge.

**Category Specification of a Square.** If $x$ and $y$ are the orthogonal axes, $x$ being the horizontal and $y$ being the vertical axis, the category specification for a square will consist of the following features according to Giaquinto:

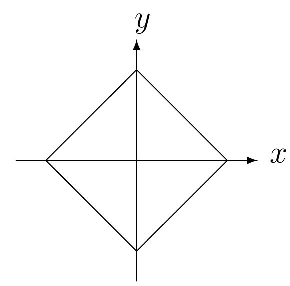
Plane surface region enclosed by straight edges:
edges parallel to $x$, one above and one below;
edges parallel to $y$, one each side;
symmetrical about $x$;
symmetrical about $y$;
symmetrical about each axis bisecting angles of $x$ and $y$.
([39], p. 23)




Giaquinto argues that although these category specifications may differ from person to person, the difference is not substantial.

This category specification can be applied to other shapes with changes to descriptions sets and reference system. One will get a better understanding how this category specification is used in contrast to the specification of a diamond shape:

Plane surface region enclosed by straight edges:
with vertices on $x$, one to each side of $y$,
symmetrical about $y$;
with vertices on $y$, one to each side of $x$,
symmetrical about $x$;
symmetrical about each axis bisecting angles of $x$ and $y$.
([39], p. 32, fn. 30)

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**Perceptual Concept for Squares.** A concept according to Giaquinto is a "constituent of thought", and he defines thought as "the content of a possible mental state which may be correct or incorrect and which has inferential relations with other such contents." ([39], p. 24) Giaquinto uses perceptual concept of square to explain how experiences with non-conceptual content is possible in mental states. (cf. [39], p. 26) Moreover, passing from one state to another is the key for "the individuation of a concept" ([39], p. 26).

Perceiving has less requirements than perceptual recognition. In order to be able to perceive something as a square it will be enough "if visual processing generates the descriptions of the category specification for squares; in fact no antecedently stored representation of any kind (for squares) needs to be accessed." ([39], pp. 26-27)

Giaquinto uses Peacocke's nonconceptual content explanation (cf. [96] and ([39], pp. 25-26)) and explains that what seeming to be "substantial innate stock of concepts" could be explained by the "non-conceptual content" that enter into the specification of the conditions for possessing a concept. ([39], pp. 25-26) He gives the example of the concept *uncle* and he argues that in order to acquire the concept of *uncle* one should already have the concepts *brother* and *parent*.

**Non-conceptual Content in Experience.** Giaquinto further argues that visual processing follows below actions *recognizing* a figure as a square ([39], p. 26) :

- generating the descriptions in the category specification for squares;
- making a best match between the set of generated descriptions and the previously stored category specification.

Visual processing follows below action when *perceiving* a figure as a square :

- generating the descriptions of the category specification for squares.([39], p. 26)

Then Giaquinto defines the perceptual concept of a square [13] by using the category specification of a square. This specification in the perceptual case does not have to be perfect, but at least should be congruent to the perfect case. The reason he allows this is that, one can perceive some figure as a square even the lines are not completely straight, or angles are slightly distorted for example. (cf. [39], p.28)

**A perceptual concept for a square** is "the concept C that one possesses if and only if both of the following hold" ([39], p. 23):

[13] Even though the full account of perceptual concept for squares can change from person to person, one can still give an approximate account for perceptual concept for a square according to Giaquinto. (cf. [39], p. 28)

1. When an item $x$ is represented in one's perceptual experience as a

    - (nearly)Plane surface region (nearly) enclosed by (nearly)straight edges:
        - edges (nearly) parallel to x, one above and one below;
        - edges (nearly)parallel to y, one each side;
    - (nearly) symmetrical about x;
    - (nearly) symmetrical about y;
    - (nearly) symmetrical about each axis bisecting angles of x and y.

    (cf. [39], p. 27)

    In addition, if one trusts the experience, then one believes that $x$ has C.

    Conversely, one trusts the experience and one believes that $x$ has C then an item $x$ is represented in one's perceptual experience as a:

    - (nearly)Plane surface region (nearly) enclosed by (nearly)straight edges:
        - edges (nearly) parallel to x, one above and one below;
        - edges (nearly)parallel to y, one each side;
    - (nearly) symmetrical about x;
    - (nearly) symmetrical about y;
    - (nearly) symmetrical about each axis bisecting angles of x and y.

    (cf. [39], p. 27)

2. The set of all shapes S, that can have the figure in (1.) can be found in the inferences that one makes even this item $x$ is not perceived. These inferences are:

    - $x$ has S. Therefore $x$ has C.

- $x$ has C. Therefore $x$ has S.

Accordingly, one has a perceptual concept C for a square if (1.) and (2.) hold. This is the explanation of perceptual concept of a square which is different than geometrical concept of a square. Once this notion of perceptual concept understood it is easy to comprehend what a geometrical concept is.

**Geometrical Concept for Squares.** The difference between geometrical concept and perceptual concept lies in being using "perfectly" instead of "nearly" in their category specification properties. So, from the category specification of the perceptual concept of a square the definition of the geometrical concept of square will consist of "perfectly" instead of "nearly". Then the conditions are the same as in the perceptual concept for squares.

Now, let us see how Giaquinto uses perceptual and geometrical concepts in his knowledge thesis.

### 5.3.2 Diagrams and Mathematical Knowledge

Giaquinto argues that a belief becomes knowledge if it is reliable, rational and justifiable.

He defines **reliability** as follows:

> "If the output belief would be true for any input that satisfied the given condition, the disposition is reliable. If in a significant number of possible cases the output belief would be false, the disposition is unreliable." ([39], p. 40)

He correlates this thesis for the mathematical arguments being sound. (cf. [39], p. 41)

His **rationality** argument is even more flexible. He allows reliable beliefs count as knowledge without their rationality is accounted. In other words his rationality condition depends on the reliability condition. Having inconsistent beliefs does not imply irrationality according to Giaquinto. These beliefs, if they are reliable, are still epistemically valuable and can constitute knowledge. Namely, "one may come to believe the target geometrical truth in a reliable way and keep that belief without irrationality." ([39], p. 42)

For the **justification** criteria he proposes implicit justification and argues that this condition is enough for account of knowledge. For example, if one can provide a justifying argument for the belief "the parts of any perfect square is either side of a diagonal are congruent" ([39], p. 43), then, that will lead to the safe knowledge of the belief. His example for the implicit justification of a belief is as follows:

1. x is a perfect square. [Assumption]
2. For any y perceived as perfectly square, x is in shape as y appears.
3. *Figure a* is perceived as perfectly square. [Perceptual state]
4. Anything perceived as perfectly square appears symmetric about its diagonals. [Category specification for squares]
5. *Figure a* appears symmetric about its diagonals. [3, 4]
6. x is symmetrical about its diagonals. [2, 3, 5]
7. The parts of x either side of a diagonal are congruent. [6, by concepts for symmetry and congruence]

8. If x is a perfect square, the parts of x either side of a diagonal are congruent. [7, discharging the assumption]
9. The parts of any perfect square either side of a diagonal are congruent.[8, universal generalization] ([39], p.43)

Moreover, he concludes that: "the kind of implicit justification that is available, on top of the satisfaction of the conditions of truth, reliability, and rationality, is enough to make the attribution of knowledge safe". ([39], p.43) Hence, above iterated **implicit justification** leads for a **safe knowledge of the belief that "the parts of any perfect square is either side of a diagonal are congruent"**.

So far, we have seen the epistemic account of Giaquinto on geometrical knowledge. In the following, I will show the connection between his account and Kantian characterization of mathematics according to Giaquinto and then evaluate the adequacy of this connection.

#### 5.3.2.1 Basic Geometrical Knowledge

Giaquinto forms famous Kantian question "How is synthetic *a priori* knowledge possible?" as "How is it possible to have basic geometrical knowledge?" since Kant claims geometrical propositions to be synthetic *a priori*. ([39], p. 40) Giaquinto believes that one acquires "general geometrical belief without inference or external written or spoken testimony". ([39], p. 39) He defines basic knowledge as the knowledge that is acquired not by inference from previous knowledge or not through an authority. This, in Kantian terminology, corresponds to immediate and direct way of acquiring knowledge through intuitions.

Giaquinto also points out another Kantian remark that:

> "[I]f the mind is equipped with the appropriate concepts, a visual experience of a particular figure can give rise to a general geometric belief. In short, having appropriate concepts enables one to "see the general in particular"." ([39], pp. 38-39)

Kant attributes "seeing the universal in particular" to mathematical cognition, (cf. [65], A714/B742) whereas philosophical cognition is concerned by particular in universal. This is important, as I have also discussed in earlier chapters, since, what makes Kantian intuitions objective and valid, for using them in proofs in geometry, is the "seeing the universal in particular". Moreover, Kant gives another argument in *Logic*, which makes this notion more clear:

> "...when a savage sees a house in distance, the use of which he does not know, he has the same object before him as another who knows it as a dwelling furnished for men. But as to form, this cognition of one and the same object is different in both. In the one it is *mere intuition*, in the other *intuition* and *concept*." ([63], p. 37-38) [14]

What Kant calls form becomes stored description sets in Giaquinto's theory. Moreover, Kant distinguishes, having the mere intuition vs. having both the intuition and the concept. According to Giaquinto, these can be characterized as the difference between perceiving and perceptual recognition. Recall that, while perceiving needs only generating a description set, perceptual recognition requires finding the best match between a generated description set and stored description set. Both Kant's and

[14] Sieht z. B. ein Wilder ein Haus aus der Ferne, dessen Gebrauch er nicht kennt: so hat er zwar eben dasselbe Objekt, wie ein anderer, der es bestimmt als eine für Menschen eingerichtete Wohnung kennt, in der Vorstellung vor sich. Aber der Form nach ist dieses Erkenntnis eines und desselben Objekts in beiden verschieden. Bei dem einen ist es bloße Anschauung, bei dem andern Anschauung und Begriff zugleich. ([70], V. Erkenntnis überhaupt)

Giaquinto's explanations are very good common sensical accounts for the difference between perception and recognition. As we will see below, because of these similarities, Giaquinto is able to establish his characterization of geometrical knowledge as synthetic *a priori* .

Giaquinto argues that the basic geometrical knowledge is *a priori*, since, although the perceptual concept is triggered by visual experience, the evidence is not provided by the experience. (cf. [39], p. 44) In order to defend his thesis, he replies two famous objections.

The first objection is by Quine [109] that "all beliefs are vulnerable to empirical overthrow". ([39], p. 48) Giaquinto points out that since no evidence is used in "acquiring the belief", "the belief about squares" relates *a priori* "to its genesis".([39], p. 44) So, in this case, the argument is immune to the Quinean objection. Therefore, the belief acquired in this way could be knowledge according to Giaquinto. He also answers the second objection that there is no conceptual knowledge. The objection is based on the claim that each constituent of a true proposition must have a referent or a meaning. Giaquinto refutes this by claiming that, if one knows the meaning of the sentence "S and T" one has already the concept of "and" without thinking about its meaning or reference. (cf. [39], pp. 45-46) He finds the objection too strong for any knowledge thesis. Moreover for his case he defends that "the concept perfect square arises naturally from perceptual experience." ([39], p. 46) and therefore is not used for postulating something, and hence not all constituents of it must have a reference.

Moreover, in reply to the first objection, Giaquinto also points out to an important issue for the claim "no belief whatever is immune from empirical overthrow". ([39], p. 44) He finds this argument inconclusive. He brings out the seemingly overthrown non-empirical belief that "the shortest distance between two physical points is a straight line" and he argues further that "it is not clear that the Parallels Postulate has been or could be overthrown empirically. What can be, and perhaps has been, overthrown is the claim that the Parallels Postulate is true of physical space" ([39], pp. 44-45)

Pirmin Stekeler- Weithofer also points out a similar issue in his book manuscript *Hegel's Analytic Pragmatism*. He argues that:

> "No axiomatic, algebraic or analytic geometry, be it Euclidean, non-Euclidean, Riemannian or Einsteinian (Minkowskian), does have a direct model in 'real space'. Or rather, it is fairly unclear what people refer to when they talk about 'real space'. Applications of geometrical statements presuppose different levels of projective de-idealization by judgments about the real properties of the method of measurement used ... the Euclidean concepts of spatial directions (straight lines and angles) still play an important role. The norms tell us how to judge ... *how plain* a surface of a body, *how straight* a line on such a surface is or *how well formed* a cube or a rectangular solid or rectangular wedge is. In other words, even after the developments of non- Euclidean mathematical models of geometry, we still work with synthetic *a priori* 'truths' that govern our interpretations of certain classes of measurement and judgments about the quality of a corresponding apparatus." ([126], p. 92)

As I have pointed out earlier, with the experiments of Izard, et al. [59], the cognitive experiments also supports that human conceptualization is based on the Euclidean truths on an Euclidean plane. I believe, to claim that the Euclidean concepts do not play role in current theories of mathematics or mathematical reasoning due to the discovery of non-Euclidean systems, is not taking into account that our reasoning can be based on many models depending on the situation. It is also possible that in some cases human cognition use conceptualizations from non-Euclidean systems. The claims that human cognition is based on Euclidean conceptualization is not a contradictory statement to that in some situations non-Euclidean reasoning works better. Moreover, statements from one model (non-Euclidean) cannot be used to refute the claims about another model (Euclidean). For that reason, as Giaquinto points out too, the claim that "all beliefs are vulnerable to empirical overthrow" is not a criteria in the case of geometrical reasoning where one deals with different models in different cases.

**Visualization as *a priori* and Synthetic.** Having replied the possible objections to his thesis, Giaquinto argues that, visualization is a "non-superfluous part of the process" in acquiring and discovering the synthetic *a priori* knowledge for the basic geometrical concepts. (cf. [39], p.47 and p. 67) For, firstly, this knowledge is non-empirical since **experience is not the evidence but only a trigger**. Second, it is non-analytic, since **it does not involve: "unpacking definitions, conceptual analysis, or logical deductions"** and **"some visual experience is essential for activating the relevant belief-forming dispositions"**([39], p. 47). Therefore, he concludes that "we have arrived at a view that is at least close to Kant's often dismissed view that there can be synthetic *a priori* knowledge. "([39], p. 47)

To summarize, we have seen that by using the detection of symmetries that are "built into a stored category specification for squares" Giaquinto explained the perceiving squareness. ([39], p. 46) Category specification for squares is used for the perceptual concept, and geometrical concept is finally built on perceptual square with restrictions. These perceptual and geometrical concepts produce some "certain general belief forming dispositions". These dispositions are "triggered by activating the stored category for squares either in seeing something as square or by visualizing a square". ([39], pp. 46-47) However, note that these visual experiences are only triggers and not evidences for the belief. The belief formed is still "reliable", there is no "violation of epistemic rationality" and "the believer has an implicit justification for the belief"([39], p. 46) Hence, Giaquinto establishes the following thesis on strong grounds: **basic geometrical knowledge is synthetic *a priori* as Kant claims**.

Moreover, for perceptual recognition and perceiving, Giaquinto distinguishes the two in the sense of finding a best match from the generated description set and from a stored description set for the conventional appearance of the figure. (cf. [39], p. 17) In Kantian terms, this is similar to one having only intuition (perception) and one having the intuition and the concept together (perceptual recognition) as I quoted earlier from *Logic*. (cf. [63], p. 37)

Another important claim of Giaquinto is that beliefs based on diagrams can constitute knowledge. In the proof of the Euclidean Proposition I. 32, as I mentioned previously, Norman argues in support of Giaquinto's view that diagrammatical thinking is used for the justification of the beliefs. Recall that Norman points out that the lines 1, 4, 7 and 11 of the proof are justified by the diagram since the diagram has a representational content. This does not mean that the diagram or any diagram has the propositional content. However, he argues that "in certain contexts a proposition can be inferred or known by observing a diagram: that someone can reliably infer the truth of a given claim about a situation represented by a diagram from observing such a diagram" ([91], p. 36) Hence, **if the diagram represents the situation correctly, which means if the diagram is reliable, then it leads to the justification of the proposition.**

### 5.3.3 Diagrams and Kantian Mathematical Characterization

Although the connection Giaquinto builds with the Kantian thesis and exhibiting his own account of visual reasoning are of great value for philosophy of mathematics, one of the drawbacks of his project is not to give an extensive analysis of Kantian characterization of mathematics. Another drawback is that he disregards the formal diagrammatic systems similar to the one we have seen in the first section. Recall that his argument for synthetic knowledge is the following: It does not involve "unpacking definitions, conceptual analysis, or logical deductions" since "some visual experience is essential for activating the relevant belief-forming dispositions" ([39], p. 47). I do not believe that one can conclude the synthetic character of a system just because the visual experience is used in the reasoning. Firstly, as Kant mentions, a proposition like $a$=$a$ can be synthetic *a priori* when used in mathematics. Secondly, as we have seen, the reasoning with Venn diagrams can consist of "unpacking definitions, conceptual analysis, or logical deductions" *with* the visual experience, but this does not make the system non-analytic in the sense Giaquinto uses the term.

In this thesis, I provided both an extensive analysis of Kant's characterization of mathematics and space, and also established the notion of synthetic *a priori* character in detail. Giaquinto's approach, however, demonstrates one additional step in addition to my research. The epistemic background that synthetic *a priori* knowledge *exists* in mathematical reasoning. Furthermore, whereas I find the connection in relation to usage of intuitions and diagrams in proofs and mathematical reasoning, he reveals the link between visual reasoning and Kantian philosophy of mathematics in terms of getting to know mathematical objects and mathematical propositions. I find his work very valuable since it proposes a new approach to the epistemic understanding of basic mathematical objects[15] in connection with cognitive findings and with Kantian characterization.

[15]In this work only his approach to basic geometrical objects were analyzed. He also gives a similar account for objects of arithmetic and analysis. (cf. [39], pp. 90-213)

## 5.4 Valid Use of Diagrams in Mathematics

In this chapter I showed several proofs that use diagrams. I first discussed about the Pythagorean theorem. In this proof there are geometrical manipulations in addition to algebraic manipulations. It, for example, requires the knowledge of distributive law for $(a+b).(a+b)$ in addition to placing the triangles appropriately so that one can have a square in the middle with a side length $c$. Second, I discussed the Euclidean proof of Proposition I. 32. The usage of diagrams in its proof is to apply the propositions to visually suitable conditions. When one is carrying out the proof of Euclidean Proposition I. 32, one first manipulates the diagram, then finds specific propositions on the diagram in order to apply them. The third example I discussed is an example from permutation. I have claimed, along with Barwise and Etchemendy, that the proof using the diagram is a valid one. The advantageous part of the reasoning with diagrams in this case is that admitting to the fallacy has a small chance because of the homomorphism as in the case of Venn diagrams. If one takes homomorphic relation into consideration in both of these cases the chances of committing a fallacy are greatly reduced. Fourth example uses diagrams as corresponding counterparts of the algebraic formulations to make the reasoning easier. The diagrams are not in the published papers, although some argue [16] that they should be in order to make the proof complete, more understandable.

When reliable diagrams as representations are used in the proofs they are perfectly legitimate tools to carry on mathematical reasoning. There is a danger in reasoning with diagrams, when accidental features are used instead of the general property of the diagram. Barwise and Etchemendy also point out that one of the reasons diagrams can be misleading "stems from the possibility of appealing to accidental features of the specific diagram in the proof" ([5], p. 8) For example, in the proof of the Pytagorean Theorem if one is not concerned with the general properties that belong to the representational schema for this proof and somehow deduces also that one perpendicular side of the triangle is greater than the other one, then one would of course commit a fallacy. As Kant argues, mathematical cognition sees general in the particular. ([66], A714/B742). For these reasons, when one must use a diagram for the proof, the properties that one should consider are only the general ones belonging to the representational schema: How this triangle can be completed to a square, or the sum of each triangle's interior angles is equal to two right angles.

In order to know the differences between the general and accidental properties of the diagrams, one should already know about the basic features of the diagrams and the homomorphy between the region the diagram uses and the diagram itself.

Moreover, one must note that a formal proof, which uses linguistic tools can also have accidental features. There can be an ambiguous sentence among the constituents and there could me many errors just because of this ambiguous sentence. (cf. [5], p.8) Barwise and Etchemendy draw attention to that: "The potential error in diagrammatic reasoning is real. But ... it is no more serious than the sorts of fallacies that can occur in purely linguistic forms of reasoning. The tradition has been to address these latter fallacies by delving into the source of the problem, developing a sophisticated understanding of linguistic proofs. It is not obvious that an analogous study of diagrammatic reasoning could not lead to an analogous understanding of legitimate and illegitimate uses of these techniques." ([5], p.8).

One should also keep the advantages of the diagrams in mind. First, "the reader can simply read off facts from the diagram as needed. This situation is in stark contrast to sentential inference, where even the most trivial consequence needs to inferred explicitly." ([5], pp. 23-24). Second, "a simple diagram can support countless facts, facts that can be read off from the diagram" ([5], p. 24).

If we go back to our analysis of proofs, the reason these examples are discussed is not to assert that diagrams should be in mathematical reasoning. They already are. Rather, it is to show how these examples can be more practical and can be shown as a part of valid reasoning; namely, how it is possible to establish a system that can use diagrams, where this reasoning is valid. In other words, I believe, if enough attention were given to reasoning with diagrams as much as to reasoning with sentences, there could be valid representational systems that use diagrams in much more areas in mathematics and logic. As a result, the learning, discovery and proofs could be in some cases much more practical if this line of practice is to be followed. Moreover, as Barwise and Etchemendy argues:

> "... we can hope to do something analogous to what Frege and his followers did for reasoning based on linguistic information. Frege made great strides in studying linguistically based inference by carving out a simple, formal language and investigating the deductive relationships among its sentences. Our hope is that the tools we have begun to develop will allow something similar to be done with information presented in more than one mode." ([5], p. 23)

There have been great developments with visualization tools, however, only few methods were developed in mathematics: Java applets, online videos, interactive thesis started to surface in the last 15 years. (cf. [105]) Due to the prevalent formal method and prejudice against visual tools, mathematics fell behind in catching up with visual developments of 21st century.

Polthier, one of the leading figures in visualization of mathematics, argues that "mathematical visualization has proven to be an efficient tool for analyzing complex mathematical phenomena, and it has given decisive hints leading to rigorous mathematical proofs of long-standing problems. Visualization is not only a tool to visualize complex objects but in combination with modern numerical methods allows to perform mathematical experiments and simulations in an artificially clean environment." (cf. [105], p. 3)

For example, the following is a visual representation of a "Discrete Minimal Catenoid" and "[t]he explicit representation allows to generate exact discrete minimal surfaces without numerical errors which are especially useful for index computations." ([105]) [Following is an image from this paper]:

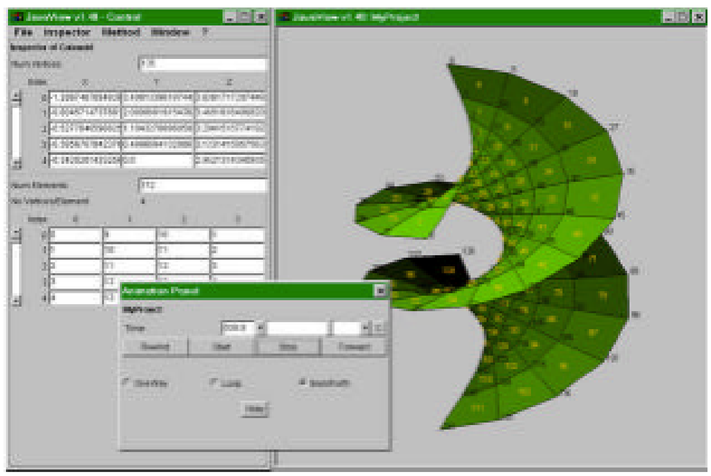
**Theorem 1.** *There exists a four-parameter family of embedded and complete discrete minimal catenoids $C = C(\theta, \delta, r, z_0)$ with dihedral rotational symmetry and planar meridians. If we assume that the dihedral symmetry axis is the $z$-axis and that a meridian lies in the $xz$-plane, then, up to vertical translation, the catenoid is completely described by the following properties:*

1. *The dihedral angle is $\theta = \frac{2\pi}{k}$, $k \in \mathbb{N}$, $k \geq 3$.*
2. *The vertices of the meridian in the $xz$-plane interpolate the smooth* cosh *curve*
$$x(z) = r \cosh\left(\frac{1}{r} a z\right),$$
*with*
$$a = \frac{r}{\delta} \operatorname{arccosh}\left(1 + \frac{1}{r^2} \frac{\delta^2}{1 + \cos\theta}\right),$$
*where the parameter $r > 0$ is the waist radius of the interpolated* cosh *curve, and $\delta > 0$ is the constant vertical distance between adjacent vertices of the meridian.*
3. *For any given arbitrary initial value $z_0 \in \mathbb{R}$, the profile curve has vertices of the form $(x_j, 0, z_j)$ with*
$$z_j = z_0 + j\delta$$
$$x_j = x(z_j)$$

*where $x(z)$ is the meridian in item 2 above.*

4. *The planar trapezoids of the catenoid may be triangulated independently of each other.*

The applet below allows to modify all parameters of the catenoid and study their geometric meaning. Such a web-based applet allows to simulateneously work on this example while one authors is in Germany and the other in Japan.



Although there are some great tools such as web applets, the development of visualization tools in mathematics are not universal and not constant with discontinuation and changes in the software applets. The hitherto attempts in the mathematical reasoning have been in the direction of elimination of visual tools to keep the fallacies out of mathematics. The approach that is in the opposite direction has not been fully explored yet and this is causing harm to mathematics in terms of not catching up with the visual developments of 21st century.

If formalists look at the issue from a broader perspective, they can see that there is not so much to object to the visual forms of thinking, and formalization and visualization are not actually so different from each other. Krämer outlines this as "...

visualization in the medium of schematizable line configurations and computability in the medium of symbolic calculations are related to one another. Computability is based on the two-dimensional embodiment of abstract relations, which are both represented and managed in graphic symbolism. Formalism, therefore, is not the opponent of perceptivity, but rather it presupposes and generates a radical form of perceptivity by referring to the surface of simultaneous visibility." ([76], p. 354)

Other sciences apart from mathematics are looking for ways to integrate images as legitimized objects. "They are considered not merely a means to illustrate and popularize knowledge but rather a *genuine* component of the discovery analysis and justification of scientific knowledge." ([75], cf also. [82])

In conclusion, defending the claim that intuitions, in the sense of diagrams or pictorial representations, should play no role in proofs or mathematical reasoning have a great prejudice against the idea that diagrams can be valid tools in mathematical reasoning. As we have seen, they can be perfectly valid tools as representational systems and they can be necessary part of the proofs. If there is a homomorphic relation between the shapes and representations of them; diagrams are no different than linguistic symbols, in terms of being valid. Hence, the danger of committing fallacies in diagrammatical reasoning is no greater than the one in linguistic reasoning. In the case of using diagrams as what they are, without any homomorphic relation, committing a fallacy by using accidental features is caused just because the knowledge about the reasoning is not complete. If one concludes from a diagram, which is used in Pythagorean Theorem, that one perpendicular side is greater than the other, this has nothing to with the general property in the particular diagram that is used. If a triangle is used as a diagram, the representational properties should be made clear. As Kant argues, one should see the general in particular. Hence, I believe, with enough research and study on reasoning with diagrams; their use in mathematical reasoning can be accounted as legitimate. Moreover, admitting them officially in the mathematical reasoning and in proofs can only make current official systems that are used in publishing and teaching mathematics much stronger with the combination of linguistic systems.

# CHAPTER VI

# Conclusion

It is by logic that we prove, but
by intuition that we discover.
To know how to criticize is good,
to know how to create is better. [1]
-Henry Poincaré

In this work, I revealed the advantages of visual reasoning in connection to Kantian characterization of mathematics. I argued that Kantian intuition of mathematical objects are representations both in sentential and visual reasoning. Namely, that linguistic symbols and diagrams are particular representations of general concepts which still carry the general properties by only this particular representation, intuition. In this sense, it's obvious that if the use of sentential representations are valid in mathematics then the use of visual representations should be valid as well.

Kantian characterization of mathematics, especially geometry, gives us a good basis for the valid application of diagrams in proofs. The synthetic *a priori* nature

[1]([102], p. 129) Original version: "C'est par la logique qu'on démontre, c'est par l'intuition qu'on invente. Savoir critiquer est bon, savoir créer est mieux. " ([103], Part II. Ch. II)

of geometrical propositions comes from nature of diagrams being exhibited *a priori*. Kantian intuition of mathematical objects correspond to diagrams and these intuitions are objective and valid in the constructed characterization.

I explained that synthetic *a priori* reasoning is objective according to Kant since *a priori* intuitions are instantiated from general concepts, which are synthesized by the rule schema in a way independent from experience. These intuitions are also exhibited *a priori* in pure intuition, **space**. Both the process and the basis borrow nothing from experience in this case. Then, intuitions exhibited in this *a priori* ground are objective. Hence, synthetic *a priori* judgments are necessarily true in Kantian sense.

Kant argues that, mathematical reasoning can be practiced *only* in synthetic *a priori* way and he constructs from within his theory how synthetic *a priori* judgments can be formulated as necessary and objective judgments, and how this reasoning can be taken as a valid and legitimate one. My results in this work, although based on Kant's characterization of mathematics, do not suggest that synthetic *a priori* reasoning or visual reasoning in mathematics should replace formal sentential reasoning in mathematics or the only way to reason in mathematics should be synthetic *a priori*. I pointed out that formal sentential method has many advantages, including practicality in proofs in mathematical analysis, and rigor it provides Moreover, I outlined the history of expulsion of diagrams from mathematics. I introduced and answered the following key question:"Why is the change in perception of how one practices mathematics necessary"? My answer indicated my objection to de-humanizing mathematics that has been in practice and popular approach to mathematics since 1920s. Therefore, I explained"the aesthetic part of mathematics" and showed that one should not expect mathematics to be only consisting of formal sentential structures. I explained how constructions (visual representations that are constructed based on the rules or concepts) played an essential role in the development of mathematics and concept building. I brought forth a complete picture of philosophy of mathematics and mathematical objects, which consistently allowed visual reasoning, intuitions and constructions in mathematical communication.

This interpretation makes it is easy to see the pictorial representations as a natural part of mathematical reasoning and also a natural part of proofs. I answered common objections to such pictorial methods and offered that a Kantian philosophical groundwork can help us clear these doubts and set a solid philosophical foundation for the visual methods in mathematics. Finally, I pointed out that with the analysis of different kinds of proofs afforded in this thesis from geometry and mathematics I showed how the pictorial methods are necessarily used in mathematics by giving a strong philosophical foundation based on Kantian characterization of mathematics. As a result, I drew attention to the aesthetic part of mathematics and suggested the importance of recording this pictorial process in mathematical practice and communication.

Moreover, I analyzed Kant's approach to space with a brief historical background. After giving several definitions of space I concluded that Kant's characterization is the finest one as a base for mathematical reasoning and, in particular, for visual reasoning with pictorial representations. His notion of space allows us to build the theoretical foundations of visual reasoning because he suggests the difference between external and internal representations, and also distinguishes constructions from abstractions, which is very crucial for the validity of pictorial representations.

I analyzed Kant's technical explanations and clarified them for our purposes. I also explained the Schema, Concept, Intuition according to Kant and revealed the link between Kantian characterization of space and visual reasoning in Mathematics. I also explained how the objectivity is provided by his characterization. In this way

I provided an answer to one of the most common attacks to diagrams being not-objective tools.

Furthermore, I examined Kant's approach to analytic and synthetic method, and explained how these notions of analytic and synthetic function in establishing his characterization of mathematics. I examined how he revealed the objectivity argument for the synthetic *a priori* judgments and discussed the *a priori* exhibited intuitions and pure intuitions in relation to synthetic *a priori* reasoning. I provided the arguments for, why his characterization of synthetic method is the best explanation for the method of visual reasoning in mathematics.

Then, I examined the requirement of space as pure intuition in Kantian characterization of mathematics in connection with my interpretation of the notion of space in the Aesthetic. In order to give an account on this requirement I first pointed out the discrepancies in the case of admitting the original representation of space as a concept in Kantian doctrine. Moreover, I analyzed Sutherland's argumentation on the importance of space being a pure intuition, which provided a unified theory of geometry, arithmetic and algebra for the Kantian characterization of Mathematics. By concluding that Kant's argumentation in the *Critique* on space was a sound one and by pointing out the importance of constructing space as pure intuition in Kantian doctrine, I indicated the objective usage of *a priori* intuitions in mathematical practice.

Moreover, I pointed out that the prevalent criticism against Kant's argumentation in the Aesthetic comes from Kant's grounding his argumentation on the premise that Euclidean propositions are synthetic *a priori*. The critique suggests that, this premise is false, since we can now formalize Euclidean propositions in pure sentential forms and, so, Kant's account of mathematics is outdated and his construction, space as pure intuition is fallacious. I argued that having one valid formal representation system does not have to make other representation systems invalid, hence, pointed out the questionability of this criticism. I also revealed the approaches to Kant's syntheticity in mathematics and analyzed Hintikka's reconstruction of Kant's characterization of mathematics as applied to quantifier logic. I argued that although this interpretation ingeniously reconstructed partially Kant's account of mathematics and intuitions, it did not capture two important facts in the Kantian doctrine. (1) It did not explain the *a priori* character of mathematics. (2) It did not take into consideration the synthetic method providing correct constructions in mathematical proofs.

Finally, I offered three main arguments leading to the main thesis "pictorial representations/ diagrams are necessary part of mathematical reasoning":

1. There are formal systems with diagrams. Hence, not all uses of diagrams are synthetic: To show that this claim is true I analyzed Shin's Venn Diagrams System that she proved to be sound and complete. This is a counter example to the claim that "All diagrammatical thinking is synthetic or non-logical." On the other hand, I also asked the question: "Is constructing formal systems with diagrams or pictorial representations a prerequisite to be able to use them as valid tools in mathematical reasoning?" My answer was no. I pointed out that, although it is an advantage to have such a system and would make skeptical formalists more open to idea of using diagrams in proofs, I do not believe that it is a prerequisite to use diagrams as valid tools in proofs.

2. I showed different non-formal uses of diagrams in mathematical proofs and argued for the "necessity" of pictorial representations, by defining necessity as "primarily and naturally using visual representations before needing to apply to any other representation". I argued that the statement "diagrams are necessary for mathematics" would not be true in current practice of mathematics where

only available and valid method is formal sentential reasoning. When we use the terminology of the prevalent formal method in mathematics, which does not already allow any pictorial representations, we cannot show the "necessity" of diagrams in mathematics. We have to use another terminology, which is built for the actual mathematical reasoning -including diagrams- and not only for its formal part. Moreover, I argued that by using the sentential formal method merely, which instead should be one of the methods in mathematics, mathematics loses a lot of value and does not fulfill its potential. By not allowing diagrams fully, mathematics does not represent actual mathematical reasoning. It just limits itself as much as a machine can operate. However, the real capacity of mathematics is not in turning it into a machine language but integrating and appreciating the aesthetic part of it as well.

3. One of my aims was to show the strong link between visualization and Kantian philosophy of mathematics. Therefore, I analyzed another approach to this link by Giaquinto and revealed the similarities and differences in our accounts. His main result is: Visualization is "non-superfluous part of the process in acquiring and discovering the synthetic a priori knowledge for the basic geometrical objects" which backs up my claim throughout this thesis. Moreover, in this section, to strengthen the claim of visual reasoning in mathematics, I offered solutions to fill the gaps in his theory with the notions I introduced in previous chapters.

I also explored the valid usage of diagrams in mathematics and argued that by understanding Kantian characterization of mathematics it was possible to see the objectivity of diagrams and validity of heterogeneous reasoning.

This argumentation, combined with Kantian characterization of synthetic method helps us to appreciate a different kind of reasoning in mathematics, which can prove itself quite beneficial.

The diagrams or intuitions, and in general visual reasoning were excluded from mathematics as much as it was possible, after the developments in modern logic and with discovery of counter intuitive fallacies. I discussed that counter intuitive examples should not be an indicative that intuition should loose all its usage in mathematics. However, I explained that, the prevalent view was the opposite so far. I suggested that the exercise should not be so. Mainly, because, deciding what is legitimate to use in mathematics affects how mathematics is taught and how published media appears.

Moreover, I argued that mathematicians in practice used diagrams frequently, but these diagrams disappeared in the published media. I pointed out that excluding visual reasoning and representations from formal communications only causes to record the process of mathematical reasoning in an incomplete way. Hence, I concluded that communication for mathematical results and findings became much harder and more time consuming than it should have been. By taking this approach, I defended throughout this work that integrating visual reasoning in official mathematical practice has many advantages than using only formal reasoning in mathematical practice.

As a closing remark I would like to add, as Krämer also points out in her work '"Epistemology of the Line" Reflections on the Diagrammatical Mind' ([75]), that there is a lack of *theory of diagrammatics*. I suggest that it would be a great starting point to integrate visual reasoning in mathematics to set up this theory as a branch of mathematics. In this way, a great help to all sciences would be provided by establishing the ground rules of valid visual reasoning with the foundations of mathematics.

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