The linear instability of uniform Black Strings

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Zusammenfassung

Black strings sind schwarze Löcher und Lösungen der einsteinschen Feldgleichungen in mehr als vier Dimensionen. In Raumzeitdimension $n+1$ hat ihr Ereignishorizont die Topologie $S^{n-1} \times S^1$. Durch numerische Simulationen wurde in den 1990ern entdeckt, dass *black strings* linear instabil sind. Ausgehend von diesem Resultat lag der Schwerpunkt der Forschung darauf, die physikalischen Aspekte dieser Instabilität zu verstehen. In dieser Arbeit werden wir zu dem Problem der linearen Instabilität zurückkehren und die Existenz einer speziellen Familie von Lösungen beweisen. Dies ist ein wichtiger Schritt für den Beweis der linearen Instabilität.

Abstract

Black strings are black hole solutions of Einstein's equations in more than four dimensions. In spacetime dimensions $n + 1$ they have horizon topology $S^{n-1} \times S^1$. Using numerical simulations, it was discovered in the 1990s that black strings are linearly unstable. Since then, most research surrounding this result was focused on understanding physical aspects of the instability. In this thesis we will return to the original problem of linear stability and prove an existence result for a special class of mode solutions. This constitutes an important step towards proving linear instability.

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Contents

1 Introduction

The time-independent vacuum black hole solutions in classical four dimensional general relativity are believed to be stable. They are also unique. Specifically, the Maxwell-Einstein equations describing electromagnetism in general relativity have only one asymptotically flat black hole solution for given charge, angular momentum and mass. This so called *no-hair conjecture* has been proven under regularity restrictions by Stephen Hawking, Brandon Carter, and David C. Robinson. Stability and uniqueness of solutions are closely related. If the exact black hole solutions were not stable and would - after being perturbed - evolve into a different stable state, the known exact solutions could not be unique. It is therefore generally believed that the known exact solutions represent the equilibria which generically other black hole solutions settle to. Stability properties of solutions are therefore essential for understanding the global dynamics of the Einstein equations.

This situation changes when considering general relativity in more than four dimensions. The interest in higher dimensional gravitational theories began already in the 1920s. Back then Kaluza and Klein developed a way to reinterpret solutions to the five dimensional vacuum Einstein equations as a four dimensional spacetime filled with electromagnetic waves and a scalar field. Klein combined this correspondence with quantum theory in order to conclude that electric charges in the four dimensional theory are quantized. String theory contains the higher dimensional Einstein equations as a formal limit. It is therefore important to understand their dynamics in more than four dimensions.

In higher dimensions the questions of uniqueness and stability of black hole solutions becomes much more complex. First of all, uniqueness does not hold. There exist many different known families of black vacuum solutions with vastly different event horizon topologies. Famous examples feature disconnected horizons, like the *black saturn* solution. Furthermore, the stability properties of these different solutions are usually more complicated and often unknown. For instance, a simple higher dimensional black hole is the Schwarzschild-Tangherlini solution, which - as its name suggests - is a higher dimensional analog of the Schwarzschild black hole. It is a static vacuum solution with rotational symmetry and event horizon topology S^{d-2} and it is believed to be stable [26]. On the other hand, the uniform black string, which is another simple higher dimensional black hole, is linearly unstable. The stability of uniform black strings was originally studied by Ruth Gregory and Raymond Laflamme who showed in their seminal work [18] using numerical techniques that black strings are linearly unstable. The linear instability signals the existence of families of solutions to the nonlinear equations and opens up the question of the dynamical endpoint of the instability. The linearized equations also seem to have time independent solutions, which suggests the existence of an unknown family of static black string solutions. The instability of uniform black strings is therefore an important result for understanding the global dynamics of the Einstein equations in higher dimensions.

The discovery of the *Gregory-Laflamme instability* has led to extensive research into the evolution of the unstable solution. Most of the results have been based on numerical simulations and focus on understanding the global picture of the instability. Among them are investigations into the existence of static solutions *close* to the uniform black string and also of the full dynamics of the Einstein equations starting from the instability. They have uncovered many interesting phenomena, for instance the possible violation of strong cosmic censorship. Some more details and references to the relevant literature can be found in section 2.9.

The purpose of this thesis is to go back to the initial problem and show rigorously that the perturbation equations considered by Gragory and Laflamme [18] in five dimensions admit modes which grow in time, thereby turning their numerical results into mathematical proofs. The main result of this thesis is Theorem 3.17, which proves the existence of growing modes. The structure of this thesis is as follows. Chapter 2 contains the necessary mathematical tools needed and reviews properties of the Einstein equations, which allow understanding how the results presented here fit into the bigger picture. It also tries to elucidate how the linear instability of solutions is related to the dynamical properties of solutions to the Einstein equations. Chapter 3 is mainly concerned with proving Theorem 3.17 while also providing a self contained description of the problem of linear stability for black string solutions. To that end, it repeats calculations and statements, in particular about the asymptotic behavior of solutions, which have already been presented in [18] and elsewhere. In some cases this has been done for the sake of completeness, and in other cases, because important steps and intermediate statements have not been published before. It should be noted here, that Gregory and Laflamme originally studied this problem for any dimension greater than four and later extended their numerical analysis to also include charged black strings. The author believes that the results and techniques presented here could be generalized.

2 Generalities

In this chapter mathematical concepts will be introduced, which form the basis for understanding the results presented in chapter 3. The purpose is not to give a complete introduction to mathematical relativity, but rather to introduce the necessary notation and to fix conventions.

We will start by explaining what a spacetime is and how some of its features can be understood physically. When studying the stability of solutions of the Einstein equations it is necessary to understand what stability actually means in this context. Physically, the notion seems to be clear; a solution is stable if, after being perturbed, it settles back to its original state. However, the mathematical formulation of this intuitive idea requires defining an evolution problem in order to make sense of what an *original state* is and how it changes under the evolution.

The Einstein equations do not trivially fall into a standard category of partial differential equations. Therefore, the formulation of the initial value problem for the Einstein equations is not obvious and will be given a short introduction. This will allow us to understand what initial data (i.e. the *original state*) is and its role in formulating stability problems. Next, we will introduce the standard mathematical framework for studying linear perturbations of solutions, which will allow us to define *linear instability*. Following that, we will briefly discuss the connection between *linear* and *nonlinear* stability in the context of the Einstein equations.

After all these general ideas and concepts have been introduced, we will focus on two special solutions and what is known about their stability. The first solution which we will look at is the well-known Schwarzschild solution. The reason for this choice is the fact that the uniform black string solution can be understood as a higher dimensional generalization to the Schwarzschild black hole. While the stability properties of the Schwarzschild solution are very different from those of uniform black strings, many mathematical concepts are similar and can be understood as historical parallels. Reconstructing how our understanding of the stability of the Schwarzschild solution has developed can therefore help to put results on the instability of the uniform black string into their historical context.

The last part of this chapter will then focus on the uniform black string solution. It will start by explaining how the uniform black string metric in arbitrary dimensions can be derived from the generalized Schwarzschild black hole. Finally, some important results on the stability of the uniform black string solution will be reviewed, in order to explain how the proof of linear instability in chapter 3 fits into recent research.

2.1 Spacetime and the Einstein Equations

The fundamental object of study in general relativity is a spacetime, which is a $(n+1)$-dimensional manifold M together with a Lorentzian metric g.

Let $\gamma : I \to M$ be a smooth curve on the real interval I with tangent vector v with components $v^\alpha = \frac{\mathrm{d}\, x^\alpha}{\mathrm{d}\, \lambda}$. A curve is called timelike, spacelike or null, if its tangent vector is everywhere timelike, spacelike or null, respectively.

The arc length of a spacelike curve is defined by the integral of $\sqrt{v^\alpha v_\alpha}$ over I and correspondingly that of a timelike curve by the integral of $\sqrt{-v^\alpha v_\alpha}$.

The movement of freely falling particles is described by timelike geodesics, which are a special class of curves in M. A curve γ is a geodesic if there exists a smooth function f such that $v^\alpha \nabla_\alpha v^\beta = f v^\beta$ for all $\lambda \in I$. If f is zero, then γ is said to be affinely parametrized. Null geodesics describe light rays. The arc length of a timelike curve is called *proper time* and can be physically understood as the time measured by a clock traveling on that curve.

Geodesics are called complete, if they are defined globally in an affine parameter. For a timelike geodesic this means that it exists for an infinite amount of proper time. A spacetime is called geodesically complete if all geodesics are complete. If a spacetime has incomplete geodesics, it is said to be singular. It is often important to differentiate different types of incompleteness because they can have different physical interpretations. For instance all incomplete geodesics may be of the same causal character, say timelike. Then a spacetime is called timelike geodesically incomplete. Also, all incomplete geodesics may exist for infinite affine parameter in *one* direction, e.g. a spacetime may be future geodesically complete. For instance, we hope that our universe is future geodesically complete, and we believe it to be past geodesically incomplete and that all causal geodesics start in the big bang.

There is a class of hypersurfaces which are essential when studying the evolution problem of the Einstein equations. A smooth hypersurface S is called a Cauchy hypersurface, if every causal curve intersects S exactly once. Spacetimes which contain a Cauchy hypersurface are called *globally hyperbolic*. Globally hyperbolic spacetimes are those which can be uniquely determined by initial data, as will be discussed in the next chapter.

The mathematical description of General Relativity consists of a spacetime (M, g) which satisfies the Einstein equations

$$\mathrm{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta}\, \mathrm{R} = \kappa T_{\alpha\beta}. \tag{1}$$

Here $\mathrm{R}_{\alpha\beta}$ denotes the Ricci curvature and R its scalar curvature of g, $T_{\alpha\beta}$ is the energy-momentum tensor, which describes the matter content of the universe. In four dimensions the constant κ is $8\pi G c^{-4}$, where G is the gravitational constant and c the speed of light. In mathematical treatments of general relativity both G and c are often chosen to be one to simplify calculations. However, in higher dimensions it is not generally clear what the value of κ is. Some of the issues of fixing the values of these constants in higher dimensions will be discussed in chapter 2.8, but for now we will just assume κ to be some constant.

The Einstein equations are often written in the above form, which highlights the fact that matter is the source of gravity. However, in contrast to the Newtonian theory of gravity, specifying the matter source does not uniquely determine the geometry. In particular, there are large classes of nontrivial vacuum solutions, for example those describing gravitational waves. When considering vacuum solutions with $T_{\alpha\beta} = 0$, the Einstein equations simplify to

$$\mathrm{R}_{\alpha\beta} = 0. \tag{2}$$

The solutions we are about to study, namely the uniform black strings, are vacuum solutions. We will therefore restrict the discussion to the vacuum case and only consider equation (2) henceforward.

The Einstein equations are invariant under diffeomorphisms of M. This is a consequence of the fact that equation (1) is a tensor equation. Two solutions which are related by a diffeomorphism describe the same physical configuration, which merely appear to be different in two different coordinate systems. This also means that nature does not prefer any particular choice of coordinates, which gives us the freedom to pick one which simplifies mathematical arguments.

2.2 The Cauchy–Problem of General Relativity

This chapter will explain very briefly how the Einstein equations can be formulated as an evolution problem. This brief outline will therefore only attempt to give a basic overview of the problem and sketch some results. For complete and mathematically rigorous treatments of the subject the reader is referred to [16].

The task of establishing and studying the correspondence between data on hypersurfaces and the corresponding solutions is called the Cauchy problem.

Assume that (Σ_0, g_0) is a spacelike hypersurface, i.e. a Riemannian manifold which is embedded in M, such that g_0 is the restriction of g to Σ_0. Denote the embedding by $i : \Sigma_0 \to M$ and its second fundamental form by k_0. Using the Gauß and Codazzi equations it can be shown that if (M, g) is a solution to the vacuum Einstein equations, then g_0 and k_0 satisfy

$$R_0 - |k_0|^2 + (\operatorname{tr} k_0)^2 = 0 \tag{3}$$

$$\nabla^j (k_0)_{ij} - \nabla_i \operatorname{tr} k_0 = 0 \tag{4}$$

on Σ_0. Here all quantities are to be understood with respect to the induced metric g_0 on Σ_0, i.e. R_0 is its scalar curvature and ∇ its covariant derivative. These equations are called the *constraint* equations; equation (3) is called the *Hamiltonian* constraint and (4) the *vector* constraint equation.

The above explains how a solution to the Einstein equations can be reduced to a solution on a spatial hypersurface. A solution to the Cauchy problem describes the reverse procedure: given a data set $(\Sigma_0,\, g_0,\, k_0)$, where Σ_0 is a n-dimensional Riemannian manifold with metric g_0 and a symmetric 2-tensor k_0, constructing a solution $(M,\, g)$ to the Einstein equations such that there exists an embedding $i : \Sigma_0 \to M$ such that g_0 coincides with the restriction of g to Σ_0 and k_0 is the second fundamental form of the embedding. We cannot expect to find solutions $(M,\, g)$ for any choice of $(\Sigma_0,\, g_0,\, k_0)$; the constraint equations have to be satisfied. We will therefore in what follows denote data $(\Sigma_0,\, g_0,\, k_0)$ satisfying the constraint equations as *initial data*. However, it is not obvious if that is sufficient, i.e. if given initial data satisfying the constraint equations, a solution to the Einstein equations can be constructed. The solution to this problem is due to Y. Choquet-Bruhat [6], who proved a local existence and uniqueness result. It makes use of the fact that in special coordinates the Einstein equations in vacuum can be written as a nonlinear symmetric hyperbolic system. These coordinates are called *wave* or *harmonic* coordinates. They do not necessarily cover the whole manifold, but can always be constructed locally. By using a local existence theorem for symmetric hyperbolic systems one can prove existence for solutions to the Einstein equations locally in space and time. Using gluing techniques the different local solutions in space can be combined to *one* solution which is global in space. The function spaces in which hyperbolic equations are naturally solved are the L^2 Sobolev spaces H^s. The original proof requires that $s > \frac{n}{2} + 2$, however there have been more recent results with weaker conditions. For simplicity one can assume that if s is big enough, then initial data $(g_0, k_0) \in (H^s, H^{s-1})$ gives rise to a solution g in H^s. If the initial data is smooth, then so is the solution. Furthermore, the solution depends continuously on the initial data.

It was later shown by Yvonne Choquet-Bruhat and Robert Geroch [10] that there exists a unique *maximal* development of any given initial data. A solution $(M,\, g)$ with initial data $(\Sigma_0,\, g_0,\, k_0)$ and embedding i is called maximal if for any other development $(\tilde{M},\, \tilde{g})$ of the same initial data with embedding $\tilde{i}$ there exists a diffeomorphism $\varphi : M \to \tilde{M}$ such that $\varphi^* \tilde{g} = g$ and $\tilde{i} = \varphi \circ i$. The existence of the maximal Cauchy development therefore answers the question of uniqueness of solutions of the Einstein equations.

The fact that the evolution problem in general only gives rise to solutions which exist locally in time is not surprising. Solutions to the Einstein equations can develop singularities, for instance in situations of collapse when black holes are formed.

The absence of general global existence results makes studying the stability of individual solutions important. However, the problem of nonlinear stability seems to be quite difficult and only very few solutions have been shown to be stable. Some examples are the proof of stability of Minkowski space [11] and the stability of de Sitter space [15].

2.3 Linearization and Perturbation

There are different reasons for studying the linearized Einstein equations. One is that it allows studying the linear stability of solutions, when the question of nonlinear stability seems intractable. The other is that relatively few physically relevant exact solutions to the Einstein equations are known, most of which have a great degree of symmetry. We know that many physical situations are not perfectly symmetrical. It is therefore of great relevance to study solutions to the Einstein equations which are in some sense *close* to known exact solutions in order to understand physical phenomena. It seems hopeless to construct these *almost* symmetrical solutions explicitly and it therefore makes sense to study perturbations of exact solutions.

We will in this chapter go along with the derivation and notation in [47]. We start off with Einsteins equations (2) for vacuum solutions which we will denote by

$$\mathcal{E}\left[g\right] = 0, \tag{5}$$

where g denotes the spacetime metric. Assume g_0 is a known exact solution and that there exists a family of solutions $g\left(\lambda\right)$ which depends smoothly on the parameter λ and satisfies $g\left(0\right) = g_0$. The family of solutions $g\left(\lambda\right)$ is not known but merely assumed to exist. It satisfies

$$\mathcal{E}\left[g\left(\lambda\right)\right] = 0 \tag{6}$$

and by differentiating by the parameter λ we arrive at

$$\left.\frac{\mathrm{d}}{\mathrm{d}\,\lambda}\mathcal{E}\left[g\left(\lambda\right)\right]\right|_{\lambda=0} = 0 \tag{7}$$

This is a linear equation in

$$h = \left.\frac{\mathrm{d}\, g}{\mathrm{d}\, \lambda}\right|_{\lambda=0}. \tag{8}$$

which we will abbreviate as $\mathcal{L}[h] = 0$. It is called the *linearization* of equation (5) and the solution h is called a *perturbation*. The metric g_0 is usually called the *background metric*. Since $g(\lambda)$ is differentiable in λ, we can approximate this exact family near g_0 by $g_0 + \lambda h$. Linear equations are easier to solve than their nonlinear counterparts and it is therefore often more feasible to gain information about the solutions $g(\lambda)$ by solving the linearized equation (7).

As we have seen, the existence of a differentiable one parameter family of solutions $g(\lambda)$ to the Einstein equations give rise to a solution to the linearized equations, which is tangent to $g(\lambda)$. However, in practice it is not necessarily known how the space of solutions looks like near a given solution g_0. More specifically, solutions to the Einstein equations need not be a manifold near g_0 and therefore a smooth one parameter family $g(\lambda)$ needs not exist. Furthermore, solutions to the linearized equations might exist which are not tangent to a family of solutions to the nonlinear equations, which makes them physically irrelevant. The problem of if and how solutions to the linearization are related to solutions of the nonlinear equations is called the problem of *linearization stability* and will be discussed in the next chapter.

In the following we will give explicit expressions for the linearization of the Einstein equations about a vacuum solution g_0. The derivation will be omitted here, details can be found in [47].

For a family of solutions to the Einstein vacuum equations $R_{ab} = 0$ the linearized equation (7) takes the form

$$-\frac{1}{2}\nabla_a \nabla_b h - \frac{1}{2}\nabla^c \nabla_c h_{ab} + \nabla^c \nabla_{(a} h_{b)c} = 0, \tag{9}$$

where the covariant derivative ∇ and contraction are with respect to the background metric g_0. This equation can be simplified further by choosing *transverse traceless* gauge, i.e.

$$\nabla^a h_{ab} = 0 \quad \text{and} \tag{10}$$

$$h^{cd}\ (g_0)_{cd} = h = 0. \tag{11}$$

Note that if the background is not a vacuum solution, the perturbation cannot be assumed to be traceless.

Using the properties of the Riemann tensor of the background g_0, equation (9) then becomes

$$\nabla^c \nabla_c h_{ab} - 2R^c{}_{ab}{}^d h_{cd} = 0. \tag{12}$$

The corresponding linear operator Δ_L is commonly called the Lichnerowicz operator. The background solution g is said to be linearly stable if all solutions of equations (12) remain bounded in forward time direction. If on the other hand a solution h exists, which grows without bounds, g_0 is *linearly unstable*. Of course, when speaking about bounded solutions it is important to also specify the norms. It is not generally clear which norms should be considered, however one natural choice are those norms which are compatible with the initial value problem.

2.4 Linearization stability

In the previous chapter we have explained how continuously differentiable families of exact solutions give rise to solutions to the linearized equations. It has been long known that the reverse is not true in general and in particular it does not hold for the Einstein equations. There exist solutions to the linearized equations which cannot possibly be tangent to a curve of exact solutions. The concept of linearization stability for the Einstein equations was introduced by Choquet-Bruhat and Deser [9]. Around the same time it was observed by Dieter Brill and Stanley Deser [5] that the flat 3-torus is not linearization stable. Since that beginning, the question of linearization stability has been studied for many different background metrics. In the following we will introduce the problem briefly and review some of the known results. Finally, we will discuss what can be currently said about the linearization stability of the Einstein equations at the uniform black string solution.

Linearization stability for mappings between Banach manifolds is defined as follows.

Definition 2.1. *Let X,Y Banach manifolds and $F : X \to Y$ a differentiable mapping. F is* linearization stable *at $x_0 \in X$, if and only if for all $h \in T_{x_0}X$ such that $\mathrm{D}\, F|_{x_0} \cdot h = 0$ exists a C^1 curve $x(t) \in X$ and some $\epsilon > 0$ such that $x(0) = x_0$, $F(x(t)) = F(x_0)$ for all $t \in (-\epsilon, \epsilon)$ and $\frac{\mathrm{d}}{\mathrm{d}\,t} x(t)\big|_{t=0} = h$.*

The two main results on the linearization stability of the Einstein equations are concerned with two distinct classes of solutions. The first are theorems by Moncrief [35][36] which treat solutions with initial data on a compact hypersurface Σ_0. One main observation is that linearization stability holds if the solution has no symmetries.

Theorem 2.2 (Moncrief)**.** *The constraint equations are linearization stable at* (Σ_0, g_0, k_0) *with* $(g_0, k_0) \in M^p_{s+2} \times W^p_{s+1}$ *and* $p > \frac{n}{2}$ *if its evolution admits no Killing vector fields.*

As usual W^p_s denotes the L^p-Sobolev space of functions with weak derivatives up to order s.[1] Similarly, M^p_s is the space of Riemannian metrics in W^p_s.

In the presence of symmetries the situation becomes more complicated, however Moncrief gave the exact conditions under which linearization stability still holds.

If Σ_0 is not compact, the situation is very different. For instance, in case of asymptotically Euclidean initial data, the following theorem holds. It was proven by Yvonne Choquet-Bruhat for traceless second fundamental form k_0 in [7] (see also [8]) and in the general case recently by Corvino and Schoen [12].

Theorem 2.3 (Choquet-Bruhat, Corvino and Schoen)**.** *The constraint equations are linearization stable at* (Σ_0, g_0, k_0) *with* $(g_0, k_0) \in M^p_{s+2,\delta} \times W^p_{s+1,\delta+1}$ *and* $p > \frac{n}{2}$ *if* $\delta > -\frac{n}{p}$.

Similarly to the above definitions $M^p_{s,\delta}$ denotes the space of Riemannian metrics g, such that $g - e$ is in the weighted Sobolev space $W^p_{s,\delta}$, where e is a smooth Riemannian metric which is euclidean at infinity. The weighted Sobolev spaces $W^p_{s,\delta}$ contain those s–times weakly differentiable functions u such that the norm

$$\|u\|^p_{s,\delta} = \left(\sum_{|\alpha|<s} \int \left| 1 + |x|^{\delta+|\alpha|} \right| |D^\alpha u|^p \, dx \right)^{\frac{1}{p}} \tag{13}$$

is bounded. The effect of these weights are to enforce a certain fall-off at infinity. The use of weighted Sobolev spaces is essential here in order to guarantee that the constraint equations are Fredholm.[2]

It might seem peculiar that the existence of Killing vector fields is irrelevant in this case. However, the solutions which correspond to Killing vector fields do not decay at infinity and are therefore not in the weighted Sobolev spaces for $\delta > -\frac{n}{p}$. The above result applies for instance to Minkowski space.

However, the uniform black string solution does not fall into either one of the above two categories. It neither has compact hypersurfaces, nor is it

[1].e.g. Adams [1] uses the different notation $W^{s,p}$

[2]See [34] for an introduction of weighted Sobolev spaces and their application in studying the properties of elliptic operators.

asymptotically flat and therefore none of the two results discussed here apply. Establishing the linearization stability of so called *asymptotically cylindrical* solutions remains an open problem. The asymptotically flat case suggests that it would be necessary to require similar asymptotic decay conditions for the solutions in the non-compact directions. The solutions studied here decay exponentially (see chapter 3.3) and it is therefore very likely that they decay fast enough. However, at the moment it is not possible to draw any conclusions about solutions of the nonlinear equations.

2.5 The Schwarzschild Solution

The Schwarzschild solution describes the exterior of a spherical source of mass M in an otherwise empty vacuum. The line element of the Schwarzschild solution can be written as

$$\mathrm{d}\,s^2 = -\left(1 - \frac{r_0}{r}\right)\mathrm{d}\,t^2 + \left(1 - \frac{r_0}{r}\right)^{-1}\mathrm{d}\,r^2 + r^2\left(\mathrm{d}\,\theta^2 + \sin^2\theta\,\mathrm{d}\,\varphi^2\right), \qquad (14)$$

where $t \in (-\infty, +\infty)$ is a time coordinate which can be interpreted physically as the time measured by a clock at infinity, $r \in (r_0, \infty)$ is a radial coordinate, $\varphi \in (0, 2\pi)$ the longitude and $\theta \in (0, \pi]$ the co-latitude. The quantity r_0 is called the Schwarzschild radius and is related to the mass of the source by $2M = r_0$. The Schwarzschild metric is a solution to the vacuum Einstein equations and as such only describes the outside of the massive source in its center.

Objects which are dense enough such that they are contained within a sphere which is smaller than the radius r_0 are called black holes. The boundary at $r = r_0$ is a null hypersurface and is called the event horizon. It prevents any light to emanate from the interior of the black hole. Here, we will only be interested in black holes, not in extended spherical objects which are bigger than their Schwarzschild radius.

The Schwarzschild metric has two singularities, one at the event horizon and one at $r = 0$. Only the singularity at $r = 0$ is a singularity of the spacetime. There exist coordinate systems which are regular at the event horizon and the singularity at r_0 is only due to the choice of coordinates. The most commonly used one are Kruskal–Szekeres coordinates, which we will introduce now.

Let us start off by defining the Tortoise–coordinate [3] r^* as

$$r^* = \frac{r}{r_0} - 1 + \log\left|\frac{r}{r_0} - 1\right|. \tag{15}$$

Then, using the quantities

$$P_+ = \exp\left(\frac{t + r^*}{4}\right) \tag{16}$$

$$P_- = -\exp\left(-\frac{t - r^*}{4}\right) \tag{17}$$

we can define the Kruskal–Szekeres coordinates T and R as

$$R = P_+ - P_-, \tag{18}$$

$$T = P_+ + P_-. \tag{19}$$

Using these new coordinates the line element of the Schwarzschild solution can be written as

$$\mathrm{d}\, s^2 = 4{r_0}^2 \frac{r_0}{r}\left(-\mathrm{d}\, T^2 + \mathrm{d}\, R^2\right) + r^2\left(\mathrm{d}\,\theta^2 + \sin^2\theta\, \mathrm{d}\,\varphi^2\right). \tag{20}$$

As discussed at the beginning of this chapter, the coordinate singularity at the event horizon is absent from the metric here.

The Schwarzschild time and radial coordinate can be expressed using Kruskal–Szekeres coordinates as

$$t = \log\left(\frac{R + T}{R - T}\right) \text{ and} \tag{21}$$

$$e^{\frac{r}{r_0} - 1}\left(\frac{r}{r_0} - 1\right) = \frac{1}{4}\left(R - T\right)\left(R + T\right). \tag{22}$$

The mapping $(r, t) \mapsto (R, T)$ between Schwarzschild and Kruskal–Szekeres coordinates is defined on the whole Schwarzschild chart, which corresponds to the exterior of the black hole. It maps

$$\{(r, t)\,|r \in (r_0, \infty) \wedge t \in \mathbb{R}\} \mapsto \{(T, R)\,|R > 0 \wedge T \in (-R, R)\} \tag{23}$$

and is bijective. Furthermore, this mapping is analytic and so is its inverse. The Schwarzschild coordinates as analytic functions of R and T can be extended up to the singularities at $T^2 - R^2 = 4e$. In this extended region the

[3] The name refers to Zeno's paradox. The event horizon at $r_0 = 2M$ corresponds to $r^* = -\infty$ and therefore the tortoise will never reach it. Note that the Schwarzschild time t is a Tortoise–coordinate, as well, when approaching the horizon. More specifically, a geodesic will reach the event horizon in finite proper time while t diverges to ∞.

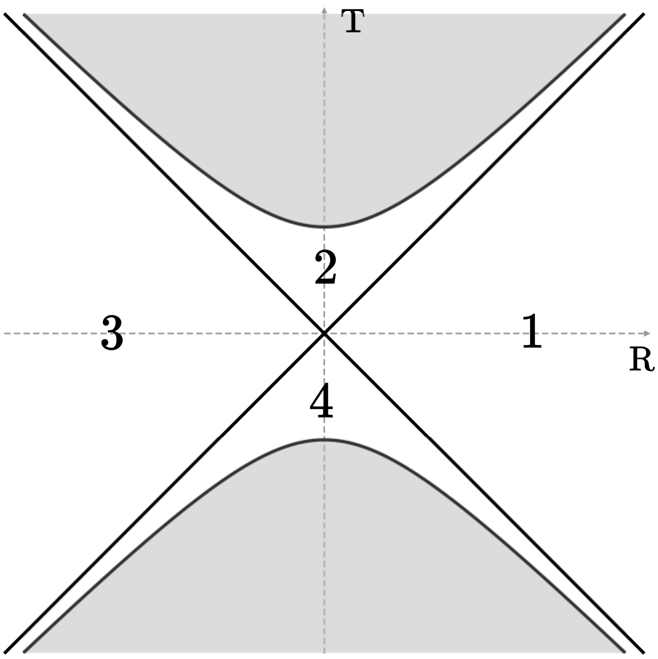

Figure 1: A diagram of the Schwarzschild spacetime in Kruskal–Szekeres coordinates depicting the four different regions.

metric in Kruskal–Szekeres coordinates (20) is regular and a solution to the vacuum Einstein equations. This extension is called the Kruskal–Szekeres extension. It is maximal in the sense that all geodesics are either complete or end in the spacetime singularity at $T^2 - R^2 = 4e$. The extended spacetime consists of four different regions, which can be individually covered by Schwarzschild coordinates. These regions are disconnected and separated by event horizons. The first region is the usual black hole exterior with the transformations as mentioned before in equations (21) and (22). The second region is the interior of the black hole, which corresponds to

$$\left\{(T, R)\,|0 < T < \sqrt{R^2 + 4e} \wedge -T < R < T\right\} \tag{24}$$

and

$$\{(r, t) \in (0, r_0) \times (-\infty, \infty)\} \tag{25}$$

in Schwarzschild coordinates. In this region the Schwarzschild time coordinate is given by

$$t = \log\left(-\frac{R + T}{R - T}\right) \tag{26}$$

while the radial coordinate r is defined as in the exterior region. The third and fourth region are mirror images of the first and the second. The third region is equivalent to the first when replacing R by $-R$. The forth is equivalent to the second when replacing T by $-T$. These two regions can be understood as the exterior and the interior of a *white hole*. The white hole behaves causally as a black hole under time reversal. All timelike geodesics starting from its interior will cross the event horizon into its exterior in finite proper time. Physically this means that particles can leave the interior region of the white hole but they can never return. The point $R = T = 0$ which connects the white hole and black hole interior is called the Schwarzschild *wormhole*.

2.6 The Stability of the Schwarzschild black hole

Uniform black strings can be constructed by extending Schwarzschild black holes to higher dimensions. It is therefore not surprising that the research on the stability properties of black strings bears many similarities with that on the Schwarzschild black hole. Similarities can be seen both in broad physical ideas and mathematical details. Despite the many similarities, the Schwarzschild solution is stable, while the uniform black string is unstable.

In this chapter we will highlight some result that have contributed towards proving the linear stability of the Schwarzschild solution. The purpose of reviewing them is to put the *instability* of black strings into a historical context and to explain where certain ideas originate from.

There are physical reasons to believe that the Schwarzschild solution is stable, because it describes very accurately many objects which can be observed in our universe. This is true both for stars and black holes, and more generally for all almost spherical, slowly rotating and almost uncharged objects. If the Schwarzschild solution was unstable it would be impossible to observe them, as they would have settled to different stable configurations long ago.

The first major step towards mathematically understanding the stability problem of the Schwarzschild solution are results published by T. Regge and J. A. Wheeler in 1957 [39]. In their paper they presented a novel approach towards understanding the behavior of solutions to the perturbation equations. Regge and Wheeler introduced simplifying gauge conditions and studied perturbations on the initial hypersurface $t = 0$. They decomposed possible perturbations into spherical harmonic modes and were able to derive their asymptotic behavior at the event horizon and spatial infinity. Regge and Wheeler argued that physically acceptable perturbations of the background metric must decay at infinity and must have a smooth continuation across the event horizon. This restriction of the perturbations makes them compatible with the Cauchy problem. By analyzing the asymptotic behavior of the modes, they were able to conclude that the perturbation equations do not admit physically acceptable solutions. Therefore, they concluded, the Schwarzschild solution must be linearly stable, because it does not admit solutions which grow in time.

In 1970 C. V. Vishveshwara [45] reinvestigated the mode solutions previously proposed by Regge and Wheeler. He transformed the solutions into Kruskal–Szekeres coordinates, which had become known around 1959. This allowed him to study the behavior of the perturbations in a coordinate system which is regular at the event horizon. It turned out that the mode solutions which Regge and Wheeler had found to be bounded at the horizon in Schwarzschild coordinates, were divergent in Kruskal–Szekeres coordinates. Since the metric itself remains bounded, these modes are not physically acceptable perturbations. However, Vishveshwara found that by superposing oscillating solutions, it is possible to construct *wave packets* which are sufficiently regular at the event horizon. Vishveshwara also pointed out that the only permitted time–independent perturbations correspond to rotations of the black hole. Since the black hole itself has rotational symmetry, these perturbations are not physically relevant. Therefore, Vishveshwara concluded that the Schwarzschild solution is stable. The statement that all mode solutions remain bounded is usually referred to as *mode stability*.

Both the result of Regge and Wheeler and that of Vishveshwara are only concerned with the behavior of modes. Additionally, the modes were not proven to be a complete set of eigenfunctions, as Vishveshwara points out in [45]. As such their results do not allow to draw any conclusions about arbitrary perturbations and therefore do not constitute stability theorems. It is generally accepted that the treatment of modes is not sufficient for proving linear stability. This is exemplified by the instability of the Einstein-Yang-Mills black hole, which is unstable even though there are no modes which grow in time [4][48]. On the other hand, the existence of modes which grow without bound in time is sufficient to prove linear instability.

In 1979 R. Wald approached the problem from a different direction and proved that solutions to the scalar wave equation on the Schwarzschild background must remain uniformly bounded for all time [46]. The scalar wave equation is considered to be a toy model for the linearized Einstein equations. This statement holds both outside and on the event horizon. However, this result uses energy integrals and assumes that the perturbation has compact support in the Tortoise–coordinate r^* away from the event horizon on the initial hypersurface. Later, in 1987 R. Wald and B. S. Kay improved upon this result and showed that it is possible to extend it to initial data which does not vanish on the event horizon [27]. Thereby, Kay and Wald proved the *mode stability* of the Schwarzschild solution, i.e. all modes of the linearized equations decay in forward time direction. Extending this result to arbitrary solutions of the linearized Einstein equations in order to show the linear stability remains an open problem. It is believed to be a crucial first step towards understanding the problem of non-linear stability.

2.7 The Black String solution

In four dimensional general relativity the possible solutions with event horizon are very limited. The only possible horizon topology is a sphere, with the Schwarzschild solution being the prototypical example. In higher dimensions the situation is very different and the Einstein equations admit a rich variety of solutions with many different horizon topologies. Among those are solutions with complicated horizon structures, for instance the *black saturn* solution [13] and many others. See for instance [14] for a recent survey of black holes in higher dimensions. One simple black hole in $d+1$ dimensions is the Schwarzschild–Tangherlini solution, which is the higher dimensional equivalent of the Schwarzschild solution. In d spacetime dimensions the line element of the Schwarzschild–Tangherlini metric is given by

$$\mathrm{d}\,s^2 = -\left(1-\left(\frac{r_0}{r}\right)^{d-3}\right)\mathrm{d}\,t^2 + \left(1-\left(\frac{r_0}{r}\right)^{d-3}\right)^{-1}\mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\Omega^2_{d-2}. \quad (27)$$

The term $\mathrm{d}\,\Omega^2_{d-2}$ denotes the standard metric on S^{d-2}. It can be defined in the following way. Let $\varphi \in (0, 2\pi]$, $\theta_n \in (0, \pi]$ coordinates on S^{d-2}. Then the standard metric can be defined recursively as

$$\begin{aligned} \mathrm{d}\,\Omega^2_1 &= \mathrm{d}\,\varphi^2 \text{ and} \\ \mathrm{d}\,\Omega^2_{n+1} &= \mathrm{d}\,\theta_n{}^2 + \sin(\theta_n)^2\,\mathrm{d}\,\Omega^2_n \text{ for } n > 1. \end{aligned} \quad (28)$$

The quantity r_0 is the Schwarzschild radius and denotes the location of the event horizon. As in four dimensions it is proportional to the mass of the black hole.

The uniform black string is another simple *black* solution in more than four dimensions, which can be constructed by *adding* one flat direction to the Schwarzschild–Tangherlini solution. The line element of the uniform black string metric in $d+1$ dimensions thus becomes

$$\mathrm{d}\,s^2 = -\left(1-\left(\frac{r_0}{r}\right)^{d-3}\right)\mathrm{d}\,t^2 + \left(1-\left(\frac{r_0}{r}\right)^{d-3}\right)^{-1}\mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\Omega^2_{d-2} + \mathrm{d}\,z^2. \quad (29)$$

Mathematically the extra dimensions could be compact or not, either way this solution is Ricci-flat and therefore a solution to the Einstein equation in

$d+1$ dimensions. For physical applications, in particular in string theory, it is usually considered to be compact.

The investigation of the stability properties of the uniform black string solution was initiated by R. Gregory and R. Laflamme in 1988 [17]. The authors extended the methods used by V. Vishveshwara [45] for the Schwarzschild solution to the 5–dimensional uniform black string. They studied the perturbation equations and concluded that they do not admit modes which could generate an instability. The arguments are in many ways similar to those in the case of the Schwarzschild solution, albeit with more complicated equations. For instance, they use the fact that the individual modes need to satisfy boundary conditions in order to be eligible for the perturbation problem. Using the asymptotic behavior of solutions they concluded that the perturbations equations do not admit solutions which are sufficiently regular at the event horizon and at infinity. However, the calculations contained errors which led them to overlook the possibility of modes with sufficiently regular asymptotic behavior. They noticed this shortcoming which led them to reinvestigate the problem and publish their new results in 1993 [18]. There they studied the behavior of the tensorial modes numerically and were able to establish that the perturbations equations do indeed admit solutions which satisfy the boundary conditions. This result was later extended by the same authors to uniform black strings in dimensions greater than six and solutions with an electromagnetic field [19].

2.8 A thermodynamical argument for instability

The 1960s and early 1970s saw the introduction of a series of thermodynamical thought experiments about systems containing black holes, which seemed to contradict the laws of thermodynamics. For instance, the Geroch–process describes a way to extract *work* from a black hole with unit efficiency by dropping radiation into it, thereby creating a *perpetuum mobile*, which contradicts the second fundamental law of thermodynamics. A much simpler example is that of dropping an object into a black hole, by which the entropy of that object is *lost*. These and other thought experiments made clear that in order to make sense of black holes physically required assigning them thermodynamical properties. This led to the development of a theory of black hole thermodynamics, which postulates a series of fundamental laws which are akin to those of classical thermodynamics. It was able to solve the paradoxes which had been brought up. Equipped with this theory, it is then possible to argue about the dynamics of systems containing black holes much in the same way as it is in classical physics. One such argument about the dynamical instability of black strings is common throughout the literature, especially in publications taking a physical point of view (c.f. [24, chapter 2.3], [30]). In order to explain this argument, some aspects of black hole thermodynamics need to be introduced. It should be noted that black hole thermodynamics does extend far into quantum gravity, in fact, many inconsistencies and flaws cannot be resolved without the incorporation of quantum theory. However, for the presentation in this thesis, the quantum aspects will not be relevant and therefore not be discussed. There are results on how to turn the thermodynamical argument into a mathematically rigorous statement about the instability of black strings. They will be discussed briefly at the end of this chapter.

As we have already mentioned, black holes must have an entropy, otherwise they could violate the second law of thermodynamics by lowering the entropy of the universe. This is also intuitively clear, just by reconsidering that entropy can be understood as the *unpredictability* of the internal state of a system. It follows from black hole uniqueness that the outside of a black hole is uniquely identified by its mass, charge and angular momentum. Furthermore, no information can be gained from the interior of a black hole by an external observer, due to the nature of its event horizon. In this sense, the interior of a black hole is perfectly unpredictable and therefore should have a certain entropy. Furthermore, the amount of entropy can only depend on the parameters observable from the outside.

One result which initiated the theory of black hole thermodynamics was the *Area Theorem* proven by Stephen W. Hawking [21], [22, Proposition 9.2.7, p. 318]. It states that the area of black hole event horizons is nondecreasing in time under very general conditions on the matter content and assuming that the spacetime does not contain any naked singularities. Hawking furthermore considered the situation of two black holes merging in a collision during which energy is emitted as gravitational waves. He showed that the amount of energy which can be radiated away to infinity is bounded and that due to this bound the total area of the resulting black hole cannot be smaller than the sum of the areas of the two original black holes. The behavior of the event horizon area of a black hole therefore is strikingly similar to the behavior of the entropy of a isolated systems in classical thermodynamics. This analogy was soon realized and Jacob D. Bekenstein [3] proposed that the entropy of a black hole is proportional to its horizon area. While this relationship started out as a mere analogy, Bekenstein gave information theoretical arguments for why this is actually plausible and constructed an argument to fix the proportionality factor. The exact relation was later fixed by Hawking, who determined the correct [4] factor. This relation is usually referred to as the

[4] Bekenstein's and also Hawking's results use thermodynamical arguments to fix the

Bekenstein–Hawking–formula and states that the entropy of a black hole with horizon area A is given by

$$S_{BH} = \frac{kA}{4l_p^2} \tag{30}$$

where k is the Boltzmann–constant and l_p the Planck–length. The Area Theorem of Hawking thereby becomes the *Second Law* of black hole thermodynamics.

Black holes can also be assigned a temperature. However, it is not to be understood in the classical sense, as black holes cannot emit black body radiation. Instead, the temperature of a black hole leads to *Hawking*–radiation, which has the same energy as the radiation of an ideal black body with the *same* classical temperature. Furthermore, using this temperature, it is possible to formulate a *Zeroth*, *First* and *Third* law of black hole thermodynamics (e.g. [2]). In the context of the black string instability and for the argument that follows, mainly the Second Law is required and we will therefore not provide further details here. While all the results quoted here have been concerned with black holes in 4 spacetime dimensions, they are in general also applicable to higher dimensional settings (see e.g. [37] for a discussion).

We are now ready to explain the argument, which - on thermodynamical grounds - suggests that black strings are unstable. More specifically, the argument assumes that given a fixed mass, the solution with higher entropy is the preferred one. In the case at hand the entropy of the uniform black string is compared to that of a *caged black hole* of equal mass. In both cases we consider solutions in five dimensions and that one dimension is compact and of length L. Caged black holes are spherical solutions which have a horizon radius, which is smaller than the length L. This solution is not known exactly. If their horizon radius is much smaller than the length L they can be well approximated by the standard Schwarzschild–Tangherlini solution. However, if the horizon radius is comparable to L, this approximation breaks down due to the self-interaction in the periodic dimension. It is conjectured that the instability of uniform black strings leads to a transition to caged black holes. This aspect will be explained in more detail in the next chapter. More information about these solutions can be found in [31], [42] and [30].

For fixed mass $M = \frac{r_0}{2}$ the entropy of the 5–dimensional uniform black string solution is given by

$$S_{BS} = 4\pi M^2. \tag{31}$$

---

proportionality factor. In analogy to statistical physics the *right* way to derive the entropy of a system would be by associating it with the micro states of the system. However, this is not possible without a theory of quantum gravity. In String Theory, for instance, the entropy of many black hole solutions can be calculated and all results agree with the Bekenstein–Hawking–formula (see e.g. [43]).

The entropy of a caged black hole can be approximated by that of the 5–dimensional Schwarzschild–Tangherlini black hole. If we assume that the horizon radius of the black hole is denoted by r_5, its entropy is given by

$$S_{BH} = \frac{\pi^2 r_5^3}{2L}. \tag{32}$$

The appearance of L in this formula is surprising at first. It is due to the fact that in higher dimensionsal models with compact dimensions the gravitational constant contains factors of the length L (see e.g. [49][section 3.9]). Using the fact that the black hole mass is related to its radius by

$$M = \frac{3\pi r_5^2}{8L} \tag{33}$$

we arrive at

$$S_{BH} = 4\pi M^2 \sqrt{\frac{8L}{27\pi M}}. \tag{34}$$

It follows that if L is large compared to the mass M, the entropy of the caged black hole is bigger than that of the black string. This simple comparison itself however does not provide any further information about the instability, it simply suggests that following a collapse, if L is sufficiently big, black holes should be thermodynamically preferred over black strings.

This simple comparison of entropies might seem superficial. However, there seems to be a deep connection between the thermodynamics of black hole solutions in higher dimensions and their stability. This connection was formalized into an exact statement by S. S. Gubser and I. Mitra [20] and became known as the Mitra–Gubser–conjecture. It states that for a black brane[5] solution to be dynamically stable, thermodynamical stability is both necessary and sufficient. Gubser and Mitra define a solution to be *thermodynamically stable* if its entropy as a function of the parameters (mass, charge, angular momenta) is concave. If it is not, i.e. if its Hessian matrix has a negative eigenvalue, it is said to be thermodynamically unstable. Very recently, Stefan Hollands and Robert M. Wald presented a proof to the Mitra–Gubser–conjecture [23]. It therefore offers a very general criterion to determine the stability of black brane solutions. However, according to [23][p. 7] the proof given there is not complete, rather the "[...] argument for instability has a status closer to that of a plausibility argument than a complete proof. [...]" It does therefore currently not replace the proof presented in this thesis. Furthermore, the *direct* method of proving linear instability can give more information about the mode solutions which lead to the instability.

[5] Brane is a term from string theory, which describes an object which has additional compact dimensions. Black brane refers to a solution which is the product of a black hole with an n–dimensional torus. The black string is the simplest example of this.

2.9 The endpoint of the instability

If a solution turns out to be unstable, the natural next question to ask is where the solution, after being disturbed, will evolve to. For the instability of black strings this question has been an active topic of research since the discovery of the instability in 1993. However, the question of what the endpoint of the evolution actually is has not been conclusively answered. In fact, since the beginning different possibilities have been proposed, some of which were contradicting previous ideas. Despite the uncertainty, we will try to highlight some results here, in order to attempt to draw a global picture of the dynamics.

The first proposal was given in [18] and [19] where Gregory and Laflamme suggested that the uniform black strings will, after becoming increasingly *thin* at some places on the horizon, fragment into several spherical black holes. It was also mentioned that this change of horizon topology would lead to a violation of cosmic censorship, since the horizon topology cannot change without the formation of naked singularities [22]. The proposal of the topology change later seemed to be proven wrong when in 2001 G. T. Horowitz and K. Maeda [25] showed that during the evolution the horizon of a black string solution cannot *pinch-off* in finite affine time. It was therefore conjectured that the *actual* end-state must be a stable nonuniform black string solution. However, despite the numerical discovery of stationary nonuniform solutions, at first none of them had enough entropy to be a suitable candidate.

Later, there have been many numerical studies trying to find proposed nonuniform black string solutions and to study their properties (see [41] and references therein). It was also found that the nature of the instability could depend on the dimension of the spacetime, since some nonuniform black string solutions appear to have smaller entropy than the initial uniform string which they evolve from. Other possibilities which have been considered were that the topology change does happen in infinite affine time, thereby being compatible with the result of Horowitz and Maeda.

The most recent result which drew a considerable amount of attention was a numerical study by L. Lehner and F. Pretorius [33]. They simulate the evolution of perturbed black string solutions in 5 dimensions using the full Einstein equations. What they observe is the formation of a series of spherical black holes connected by thin string segments. These segments are locally very similar to the original black string. This leads to the process repeating recursively, which forms a self-similar structure. Based on this observation, they conclude that the process, if continued, would lead to the development of string segments which become infinitely thin in finite Schwarzschild time. Using this conclusion they conjecture that the evolution would lead to a space-time which contains a naked singularity. If it turns out to be true, it would violate the cosmic censorship hypothesis, which states that generically singularities are hidden behind an event horizon.

3 Linear instability

In this chapter we will study a system of equations satisfied by a special family of perturbations of the black string metric in $4+1$ dimensions. We will show that this system admits regular solutions and thereby proof a linear instability result. The linearized Einstein equations in transverse trace-less gauge (12) as described in chapter 2.3 will be considered for solutions of a special form. This ansatz is adapted to the symmetry of the background metric similar to the usual Fourier decomposition is to a flat background. It assumes that perturbations are spherically symmetric in the three non-compact dimensions, behave sinusoidally in the compact dimension and either grow or decay exponentially in time. Solutions of this form are similar to s-waves of the Hydrogen atom and hence in the literature sometimes referred to as s-wave perturbations. Equation (12) is then reduced to a linear system of ordinary differential equations in the radial coordinate with two parameters, a frequency and a growth rate. Linear instability is equivalent to the existence of modes that grow in time and are sufficiently regular. In this case this boils down to decay properties that need to be satisfied both at the horizon and infinity. The structure of this chapter is as follows. First, the equations governing the perturbation modes will be derived following the original literature [18]. Then, the exact boundary conditions necessary for a perturbation to be *regular* will be discussed. By studying the asymptotic behavior of solutions at the horizon and infinity, the existence of regular perturbations will then be reduced to a shooting problem in the parameters. Using estimates for those solutions which are regular at the horizon, it can be shown that there are values of the parameters that admit solutions which decay fast enough both at the horizon and infinity. The main result of this chapter, which is the existence of regular unstable modes, can be found in Theorem 3.17.

3.1 The perturbation equations

In this chapter we derive the equations that govern the linear perturbations of the uniform black string metric. The general idea here is to consider equation (12) for solutions of a certain form. By doing so, the equations simplify to a system of ordinary differential equations in one variable. All necessary details and explicit expressions which are necessary for this derivation, such as the components of the Riemann and Ricci tensor of the background metric, can be found in [18]. Furthermore, due to the similarity of the black string and the Schwarzschild metrics, many of the quantities involved are identical for the Schwarzschild metric and can be found in standard text books. We will therefore not repeat individual steps here and restrict ourselves to giving an outline of how the perturbation equations can be derived. Furthermore, unlike in [18], we will not consider the problem in arbitrary dimensions, but instead only in the simplest case in five dimensions. All equations presented here have higher dimensional analogues, which can also be found in the original literature.

The uniform black string solution has been introduced in section 2.7. In five dimensions the line element (27) of the uniform black string metric $\bar{g}$ is given by

$$\mathrm{d}\,s^2 = -f\,(r)\,\mathrm{d}\,t^2 + f\,(r)^{-1}\,\mathrm{d}\,r^2 + r^2\,\mathrm{d}\,\Omega^2 + \mathrm{d}\,z^2, \tag{35}$$

where

$$f\,(r) = 1 - \frac{r_0}{r} \tag{36}$$

and r_0 is the Schwarzschild radius and $\mathrm{d}\,\Omega^2$ denotes the standard metric on S^2. As before, $r = r_0$ is the location of the event horizon.

If we require the family $g\,(\lambda)$ to exhibit the same spherical symmetry as the background, the linear equation for h_{ab} simplify further. When denoting the coordinates on S^2 as θ and φ, this manifests in the metric components as

$$\begin{aligned} h^{a\theta} &= 0 \quad \text{and} \\ h^{a\varphi} &= 0 \end{aligned} \tag{37}$$

for all off-diagonal a. We further assume the linear perturbation takes the form

$$h^{ab}\,(t,r,z) = \Re\left[e^{i\mu z} e^{\Omega t} H^{ab}\,(r)\right]. \tag{38}$$

Solutions with $\Omega > 0$ grow exponentially in time and correspond to *unstable* perturbations. For $\mu = 0$ and $\Omega = 0$ solutions are perturbations *within* the family of uniform black string solutions. By assuming spherical symmetry it also follows that the components $h^{\theta\theta}$ and $h^{\varphi\varphi}$ satisfy

$$h^{\varphi\varphi}\ \sin^2(\theta) = h^{\theta\theta}. \tag{39}$$

Therefore, the components of H^{ab} take the form

$$H^{ab} = \begin{pmatrix} H^{tt} & H^{tr} & 0 & 0 & H^{tz} \\ H^{tr} & H^{rr} & 0 & 0 & H^{rz} \\ 0 & 0 & K & 0 & 0 \\ 0 & 0 & 0 & K\ /\sin(\theta)^2 & 0 \\ H^{tz} & H^{rz} & 0 & 0 & H^{zz} \end{pmatrix} \tag{40}$$

The perturbation equations can now be derived by evaluating equation (12) for a metric of the above form. The nonzero components of H^{ab} can be decomposed into three groups, whose equations decouple from each other. These are the scalar–type variable H^{zz}, the vector–type variables H^{tz} and H^{rz} and the tensor–type variables H^{rr}, H^{tr} and H^{tt}. This so called tensorial decomposition of the metric perturbation can be applied in more general settings and results in an analogous decoupling of the corresponding perturbation equations. The main ingredient for this decomposition to be applicable is the symmetries of the background spacetime; it needs to be static and locally the product of a complete Einstein space $\mathcal{K}$ with a 2–dimensional space $\mathcal{N}$ with metric $-fdt^2 + f^{-1}dr^2$. This decomposition has been used in studying the stability properties of higher dimensional black holes (e.g. [26], [29]).

Due to the separation of variables ansatz, the derivatives with respect to t and z can be evaluated directly. The resulting equations are a coupled second order system of ordinary differential equations in r for each of the three groups of variables. They contain the growth rate Ω and the frequency μ as parameters. The full list of equations can be found in [18]. We assume that the perturbation h^{ab} has vanishing trace (c.f. equations (10)) and therefore the variables K can be written as a function of the other metric components

$$K = \frac{1}{2r^2}\left(f\left(r\right)h^{tt} - \frac{1}{f\left(r\right)}h^{rr}\right). \tag{41}$$

The scalar– and vector–type variables vanish for the instability we are about to study here. In fact, the perturbation equations do not admit non-vanishing solutions which satisfy the boundary conditions. This has been studied before and the corresponding arguments can be found among other places in [19]. We will therefore in the following assume that H^{zz}, H^{zt} and H^{zr} are zero. One is left with the tensor–type components H^{tt}, H^{tr} and H^{rr}. Using different combinations of the remaining equations and the gauge conditions the equations can be further simplified. There is a certain freedom as to which variables to eliminate and which equations to subsequently work with. Two main choices have been employed in the original literature, which will be presented here. Re-scaled versions of both of these two possible systems of equations will be used when describing the behavior of the perturbation in the following sections. The two different systems of differential equations are equivalent and one can be transformed into the other, as will be shown later in this section. We are therefore free to work with either of the two choices, as solutions of one correspond to solutions to the other. It should also be noted that in more recent publications on black strings several authors have proposed other choices.

The first possibility of simplifying the equations is to use the two gauge conditions (10) in order to eliminate all but one variable H^{tr}. By doing so, one is left with one second order equations which reads

$$
\begin{aligned}
&\left(\frac{1}{4r^2}\left(\frac{r_0}{r}\right)^2-\Omega^2-\mu^2 f\left(r\right)\right)\frac{\mathrm{d}^2 H^{tr}}{\mathrm{d}\, r^2}\\
-&\left(\frac{2\mu^2}{r}+\frac{\Omega^2}{rf\left(r\right)}\left(2+\frac{r_0}{r}\right)+\frac{3}{4r^3 f\left(r\right)}\left(\frac{r_0}{r}\right)^2\left(2-\frac{r_0}{r}\right)\right)\frac{\mathrm{d}\, H^{tr}}{\mathrm{d}\, r}\\
+&\left(\left(\mu^2+\frac{\Omega^2}{f\left(r\right)}\right)+\Omega^2\frac{8-16\frac{r_0}{r}+3\left(\frac{r_0}{r}\right)^2}{4r^2 f\left(r\right)^2}\right.\\
+&\left.\mu^2\frac{8-20\frac{r_0}{r}+13\left(\frac{r_0}{r}\right)^2}{4r^2 f\left(r\right)}+\left(\frac{r_0}{r}\right)^2\frac{6-6\frac{r_0}{r}+\left(\frac{r_0}{r}\right)^2}{4r^2 f\left(r\right)^2}\right)H^{tr}=0.
\end{aligned}
\tag{42}
$$

This equation can be found in [18] for any spacetime dimension.

Alternatively, the gauge conditions can be used to derive a first order system for two variables (c.f. [18] or [24, p. 36]). This is done by differentiating the gauge condition $\nabla^a h_{ab}=0$ with respect to r, which results in two equations containing second derivatives of the unknowns. Then, using the second order equations for h^{tr}, h^{tt} and h^{rr}, these equations become a first order system. Together with the gauge conditions $\nabla^a h_{ab}=0$ themselves we are left with four first order equations for three unknowns. It is possible to reduce them to a second order system for two unknowns and one constraint equation. The details of this derivation are omitted here and we rely on the resulting equations found in the literature (e.g. [18]).

The first order system can be expressed using the quantities

$$
H=-H^{tr} \tag{43}
$$

and

$$
H_\pm=f\left(r\right)H^{tt}\pm\frac{1}{f\left(r\right)}H^{rr}. \tag{44}
$$

The resulting equations are a system of first order equations for H and H_- given by

$$
\begin{aligned}
\frac{\mathrm{d}\, H}{\mathrm{d}\, r}&=\frac{\Omega}{2f\left(r\right)}\left(H_+ + H_-\right)-\frac{1+f\left(r\right)}{rf\left(r\right)}H,\\
\frac{\mathrm{d}\, H_-}{\mathrm{d}\, r}&=\frac{\mu^2}{\Omega}H+\frac{H_+}{r}+\frac{1-5f\left(r\right)}{2rf\left(r\right)}H_-.
\end{aligned}
\tag{45}
$$

Given a solution to the above equations, H_+ can then be expressed as a function of H and H_- as

$$\begin{aligned} H_+ =& \frac{H_-}{f(r)} \frac{2r^2\Omega^2 + r^2\mu^2 f(r) - (1 - f(r)^2)/2}{r^2\mu^2 + 1 - f(r)} \\ & - \frac{rH}{\Omega} \frac{4\Omega^2 + \mu^2 (1 - 3f(r))}{r^2\mu^2 + 1 - f(r)}. \end{aligned} \tag{46}$$

It is important to understand if solutions to this reduced system correspond to solutions of the linearized Einstein equations which satisfy the constraints. It is relatively straighforward to see that the condition that $h = 0$ can indeed be used to eliminate K from the equations. This can be done by checking manually that a solution of the form (41) will satisfy the second order equation for K. However, for the other constraint equations and the resulting reduced system of equations this is not as easy to decide. It should therefore be noted here that the existence proofs for equation (45) presented in the following chapters do not by themselves imply the existence of a solution to the linearized Einstein equations.

It is possible to eliminate the Schwarzschild radius r_0 from the equations by rescaling r, Ω and μ. To equations (42) and (45) this has the net effect of setting $r_0 = 1$. The rescaled quantities are

$$\tilde{r} = \frac{r}{r_0}, \quad \tilde{\Omega} = \frac{\Omega}{r_0} \quad \text{and} \quad \tilde{\mu} = \frac{\mu}{r_0}. \tag{47}$$

We will in the following employ this rescaling albeit dropping the ˜ in order to simplify the notation. After applying this rescaling equation (42) takes the form

$$\begin{aligned} & \left(\frac{1}{4r^4} - \Omega^2 - \mu^2 f(r)\right) \cdot \frac{\mathrm{d}^2 H^{tr}}{\mathrm{d}\, r^2} \\ - & \left(\frac{2\mu^2}{r} + \Omega^2 \frac{2r+1}{r^2 f(r)} - \frac{3(1+f(r))}{4r^5 f(r)}\right) \cdot \frac{\mathrm{d}\, H^{tr}}{\mathrm{d}\, r} \\ + & \left(\left(\mu^2 + \frac{\Omega^2}{f(r)}\right)^2 + \Omega^2 \frac{8r^2 - 16r + 3}{4r^4 f(r)^2} \right. \\ + & \left. \mu^2 \frac{8r^2 - 20r + 13}{4r^4 f(r)} + \frac{6r^2 - 6r + 1}{4r^8 f(r)^2}\right) H^{tr} = 0. \end{aligned} \tag{48}$$

The first order system (45) contains r_0 only in the form of $f(r)$, which simplifies to $f(r) = 1 - \frac{1}{r}$. Using these conventions the horizon of the black string is located at $r = 1$ and hence r is in $(1, \infty)$.

We will now rewrite equations (45) into the standard form

$$u' = A \cdot u \tag{49}$$

for the vector unknown $u = (H, H_-)$. This will make reasoning about solutions easier, especially when studying their behavior in regions in which the elements of A do not change sign. The matrix A has entries

$$\begin{aligned}
A_{11} &= -\frac{4r - 2 + \mu^2 r^3 + 4r^4\Omega^2 + 2\mu^2 r^4}{2r\,(r-1)\,(1+\mu^2 r^3)}, \\
A_{12} &= \frac{\Omega r\,(-1 + 4r^3\mu^2\,(r-1) + 4\Omega^2 r^4)}{4\,(r-1)^2\,(1+\mu^2 r^3)}, \\
A_{21} &= \frac{2\mu^2\,(r-1) + \mu^4 r^3 - 4\Omega^2 r}{\Omega\,(1+\mu^2 r^3)} \quad \text{and} \\
A_{22} &= \frac{-6\,(r-1) + 3\mu^2 r^3 + 4r^4\Omega^2 - 2\mu^2 r^4}{2r\,(r-1)\,(1+\mu^2 r^3)}.
\end{aligned} \tag{50}$$

The equivalence between the second order equation (48) and the first order system can be shown as follows. First, by differentiating the first order equation for H with respect to r, we arrive at a second order equation for H. This equation still contains H_- and its derivative, which in turn can be eliminated by using the two first order equations. The resulting equation is

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\,r^2} + p\frac{\mathrm{d}\,H}{\mathrm{d}\,r} + qH = 0 \tag{51}$$

where the coefficient functions p and q are given by

$$\begin{aligned}
p &= -\left(A_{22} + A_{11} + A_{12}^{-1}\partial_r A_{12}\right), \\
q &= A_{11}A_{22} - A_{12}A_{21} - \partial_r A_{11} + A_{11}A_{12}^{-1}\partial_r A_{12}.
\end{aligned} \tag{52}$$

It is possible to verify that (51) is identical to (48) up to a rescaling factor. Both p and q become singular where A_{12} has a zero.

It is convenient to introduce the rescaled quantities $\tilde{\mathrm{H}}_- = H_-\Omega$ and $\tilde{\mathrm{H}} = (r-1)\,H$, which (unlike H_- and H) have the same leading order behavior at the horizon. Then, for $\tilde{u} = \left(\tilde{\mathrm{H}}, \tilde{\mathrm{H}}_-\right)$ equation (49) leads to

$$\tilde{u}' = \tilde{A}\cdot\tilde{u} \tag{53}$$

where

$$\begin{aligned}
\tilde{A}_{11} &= -\frac{2\,(r-1) + \mu^2 r^3 + 4r^4\Omega^2}{2r\,(r-1)\,(1+\mu^2 r^3)}, \\
\tilde{A}_{12} &= \frac{r\,(-1 + 4r^3\mu^2\,(r-1) + 4\Omega^2 r^4)}{4\,(r-1)\,(1+\mu^2 r^3)}, \\
\tilde{A}_{21} &= \frac{2\mu^2\,(r-1) + \mu^4 r^3 - 4\Omega^2 r}{(r-1)\,(1+\mu^2 r^3)} \quad \text{and} \\
\tilde{A}_{22} &= \frac{-6\,(r-1) + 3\mu^2 r^3 + 4r^4\Omega^2 - 2\mu^2 r^4}{2r\,(r-1)\,(1+\mu^2 r^3)}.
\end{aligned} \tag{54}$$

In the proof of linear instability we will mainly work with equation (53) and its solutions.

$\tilde{A}_{12}$ has a zero r_{12}. Several arguments will depend on the location of r_{12} and we therefore state the following proposition here. Since $A_{12} = \Omega \tilde{A}_{12}$, the same statements holds for A_{12}.

Proposition 3.1. *Let $4\Omega^2 < 1$. Then $\tilde{A}_{12}$ has exactly one zero r_{12}. $\tilde{A}_{12}$ is negative for $1 < r < r_{12}$ and positive for $r > r_{12}$. Furthermore the zero r_{12} satisfies*

$$\frac{1}{\sqrt{2}} \left(\mu^2 + \Omega^2\right)^{-\frac{1}{4}} \leq r_{12} \leq 1 + \frac{1}{\sqrt{2}} \left(\mu^2 + \Omega^2\right)^{-\frac{1}{4}} . \tag{55}$$

Proof. Since the denominator of $\tilde{A}_{12}$ is positive for $r > 1$, $\tilde{A}_{12} = 0$ is equivalent to $\lambda(r) := -1 + 4r^3\mu^2 (r-1) + 4\Omega^2 r^4 = 0$. Since $\lambda(r)$ is negative close to the horizon for $4\Omega^2 < 1$ and positive for large r, a zero r_{12} exists. Since λ is strictly monotonic in r, r_{12} is unique. Furthermore it follows that $1 \leq 4r_{12}^4 (\mu^2 + \Omega^2)$ which leads to the first inequality. The second inequality follows similarly by replacing r_{12} by $(r_{12} - 1)$. □

3.2 Asymptotic behavior at the horizon

Equation (53) has a first order pole at $r = 1$. For fixed parameters μ and Ω, the matrix $(r-1)\tilde{A}(r)$ is analytic in r on some open neighborhood of $r = 1$. Denote by $\tilde{A}_n$ the coefficients of the power series of $\tilde{A}$ at $r = 1$ and $\tilde{A}_0$ its residue. Since the order of the singular term and the derivative are of the same order, equation (53) has a *regular* singularity. Solving equations with regular singularities follows a standard procedure. First, one generates formal series solutions using the method of Frobenius. These formal solutions can be expressed as series in powers of $(r-1)$ and $\log(r-1)$. Whether or not logarithmic terms are present in the series depends on the eigenvalues of $\tilde{A}_0$. When the formal solutions have been constructed, their convergence can then be shown using the theorem of Fuchs. This can be summarized by the following theorem.

Theorem 3.2 (c.f. [44][Theorem 4.11 and 4.13])**.** *If $A(z)$ has a simple pole at $z_0 = 0$ with residue A_0, then every solution of*

$$w' = A(z)\, w \tag{56}$$

is of the form

$$\begin{aligned} w(z) &= z^{\alpha} \sum_{k=0}^{l} u_{l-k}(z) \frac{\log(z)^k}{k!}, \\ u_l(z) &= \sum_{j=m_l} u_{l,j} z^j, u_{l,m_j} \neq 0, \end{aligned} \tag{57}$$

where $-m_l \in \mathbb{N}_0$ and $m_l \leq m_{l-1} \leq \ldots \leq m_0 = 0$. The vectors u_{l,m_l} are eigenvectors, $(A_0 - \alpha + m_l)\, u_{l,m_l} = 0$, if $m_l = m_{l-1}$ (set $m_{-1} = 0$) or generalized eigenvectors, $(A_0 - \alpha + m_l)\, u_{l,m_l} = u_{l,m_{l-1}}$, if $m_l < m_{l-1}$. Furthermore, the u_{l,m_l} have the same radius of convergence as the power series for $zA(z)$ around z_0.

However, for the purpose of our problem, constructing solutions for fixed values of the parameters is not sufficient. Specifically, it will be essential for the arguments in chapter 3.6 to understand the asymptotic behavior of solutions for a range of parameters and to know that they depend continuously on the parameters. Using Frobenius' method and the Theorem of Fuchs this is not obvious. This is highlighted by the fact that depending on the parameters the solution contains logarithms of $(r-1)$ or not. Furthermore the radius of convergence of the formal solutions depends on the parameters. In order to construct solutions which are analytic both in r and in the parameters we will use the following approach. First, we will explicitly construct series solutions of the form (57) for our equation. For fixed values of the parameters the convergence on a certain neighborhood will then follow from theorem 3.2. Using the solutions we can then transform equation (53) into an equation of *Fuchsian* form. Then, the following result for Fuchsian partial differential equations applies, which implies that the formal solution depends analytically on its parameters.

Theorem 3.3 (c.f. [28][Theorem 4.3])**.** *Consider the initial-value problem for a vector unknown* $u : \mathbb{C}^m \times \mathbb{R}_{>0} \to \mathbb{C}^n$ *of the form*

$$\left(t\partial_t + A\left(x\right)\right) u\left(x,t\right) = tf\left(t, x, u, \partial_x u\right), \tag{58}$$

where $A\left(x\right)$ *is a* $n \times n$*-matrix that depends analytically on the complex variables* x *on some neighborhood of the origin and that* f *depends analytically on all its arguments including at* $t = 0$*. Assume that all eigenvalues of* $A\left(x\right)$ *have nonnegative real part. Then, equation (58) has exactly one solution near the origin* $x = 0$*, which is analytic in* x*, continuous in* t *and tends to zero as* $t \to 0$*.*

Using theorem 3.3 we will now establish the following result for solutions of equation (53) and their behavior near the horizon. It should be noted here that parts of the following theorem are not needed for the shooting argument in chapter 3.6 and are therefore not necessary to prove linear instability. More specifically, the behavior of the unstable branch u_0^- is not used in the shooting argument and its treatment could therefore have been omitted. However, for the sake of completeness we have decided not to do so, even though the irregular branch u_0^- is responsible for most of the complexity in the following proof.

Corollary 3.4. *All solutions of equation (53) can be written as a linear combination of* $u_0^{\pm}\left(r, \mu, \Omega\right)$ *and*

$$u_0^{\pm}\left(r, \mu, \Omega\right) \cdot \left(r - 1\right)^{\mp\Omega} \tag{59}$$

are analytic functions in all parameters for $r^3 < \mu^2$*. Furthermore the solution* u_0^+ *behaves as*

$$u_0^+\left(r, \mu, \Omega\right) \sim \left(r - 1\right)^{\Omega} \sum_{n=0}^{\infty} a_n^+\left(\mu, \Omega\right)\left(r - 1\right)^n. \tag{60}$$

Proof. To begin with, note that theorem 3.2 and theorem 3.3 are stated in the complex setting. We will therefore assume for the moment that r, μ and Ω are complex. More specifically we assume that $\mu \in U_\mu$ and $\Omega \in U_\Omega$ where

U_μ and U_Ω are small neighborhoods of the real axis. The matrix $(r-1)\,\tilde{A}$ is an analytic function of all arguments for $|r|^3 < |\mu|^2$ and it can be written as a power series as

$$(r-1)\,\tilde{A}\,(r) = \sum_{i=0}^{\infty} \tilde{A}_i \cdot (r-1)^i\,. \tag{61}$$

To start things off, we will review the method of Frobenius to understand how formal solutions to equation (53) can be constructed near $r = 1$. Assume an ansatz of the form

$$u\,(r) = \sum_{i=0}^{\infty} a_i\,(r-1)^{\lambda+i}\,, \tag{62}$$

where a_i are coefficient vectors, which only depend on the parameters. Using term-wise differentiation this ansatz can be substituted into equation (53) which leads to

$$\sum_{i=0}^{\infty} (\lambda+i)\,a_i\,(r-1)^{\lambda+i} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \tilde{A}_j a_i\,(r-1)^{\lambda+i+j}\,. \tag{63}$$

By equating coefficients for identical powers of $(r-1)$ on both sides of the equation we arrive at the recurrence relation

$$\left(\lambda + i - \tilde{A}_0\right) a_i = \sum_{j=0}^{i-1} \tilde{A}_{i-j} a_j \tag{64}$$

for $i > 0$. The coefficients of the smallest power of $(r-1)$ in equation (63) satisfy

$$\left(\lambda - \tilde{A}_0\right) a_0 = 0. \tag{65}$$

and therefore λ needs to be chosen an eigenvalue of $\tilde{A}_0$ and a_0 a corresponding eigenvector. If a choice has been made, equations (63) can be solved iteratively in order to determine all other coefficients a_i. However, this is not always possible, since $\left(\lambda + i - \tilde{A}_0\right)$ need not be invertible. This is exactly the case when two eigenvalues of $\tilde{A}_0$ differ by an integer. This is when logarithms of $(r-1)$ need to be included in the ansatz (62) in order to construct two linearly independent formal solutions.

In the problem at hand, the matrix $\tilde{A}$ has residue

$$\tilde{A}_0 = \frac{1}{1+\mu^2}\begin{pmatrix} -\frac{1}{2}\left(4\Omega^2+\mu^2\right) & \frac{1}{4}\left(4\Omega^2-1\right) \\ -\left(4\Omega^2-\mu^4\right) & \frac{1}{2}\left(4\Omega^2+\mu^2\right) \end{pmatrix}, \tag{66}$$

which has eigenvalues $\lambda_\pm = \pm\Omega$. We can choose two eigenvectors as holomorphic functions of the parameters as

$$a_0^\pm = \begin{pmatrix} \pm\Omega - \frac{1}{2} \\ \mu^2 \pm 2\Omega \end{pmatrix}. \tag{67}$$

For the eigenvalue λ_+ the recurrence relation (64) can be solved for all $i \in \mathbb{N}$ and hence the corresponding formal solution u_0^+ can be constructed. The matrix $\left(\lambda_+ + n - \tilde{A}_0\right)$ is invertible for all n and its inverse depends analytically on the parameters. It thus follows from equation (64) that all coefficients of u^+ are analytic functions of the parameters.

The eigenvalues $\lambda_\pm$ differ by an integer n if $\Omega = \frac{n}{2}$. In order to understand how the method of Frobenius applies in this case, we will first construct a solution which is valid on some neighborhood of $\Omega_0 = \frac{n}{2}$ for fixed integer n_0. This can be generalized later to a solution which is valid for all values of the parameter. On some pointed neighborhood of Ω_0 we can construct a formal solution with $\lambda = \lambda_-$ as before by choosing $a_0 = v_-$. Let

$$B_n = \left(\lambda_- + n - \tilde{A}_0\right) \tag{68}$$

and

$$q_n = \sum_{j=0}^{i-1} \tilde{A}_{i-j} a_j, \tag{69}$$

then the recurrence relation can be written as

$$B_n a_n = q_n. \tag{70}$$

We can express q_{n_0} as a linear combination of the eigenvectors $v_\pm$ as $q_{n_0} = q_{n_0}^+ v_+ + q_{n_0}^- v_-$. If $q_n^+(\Omega_0)$ vanishes, equation (70) has the solution $a_n = q_{n_0}^- {n_0}^{-1} v_-$, which would yield a formal solution for $\Omega = \Omega_0$ which is linearly independent of u^+. We will therefore exclude those values of n_0 and assume that $q_{n_0}^+(\Omega_0)$ does not vanish.

Even though equation (70) cannot be solved for a_{n_0}, we can construct a second formal solution by setting the lowest order coefficient to

$$a_0^- = \det(B_n) v_-. \tag{71}$$

The recurrence relation for $\tilde{a}_n$ then becomes

$$a_n^- = B_n^{-1} \det(B_n) q_n. \tag{72}$$

Since the matrix A is two dimensional, this reduces to

$$a_n^- = \text{adj}\,(B_n) q_n. \tag{73}$$

The adjugate matrix has the same eigenvalues as B_n, albeit with eigenvectors interchanged. Hence, in the limit $\Omega \to \Omega_0$ the coefficient a_n^- satisfies

$$\lim_{\Omega \to \Omega_0} a_n^- = n q_n^+ v_+ \tag{74}$$

and therefore the corresponding formal solution ϕ_n satisfies

$$\lim_{\Omega \to \Omega_0} \phi_n = n q_n^+ u^+. \tag{75}$$

A second solution u_n^- which is valid across $\Omega = \Omega_0$ can then be constructed as

$$u_n^- = \frac{\phi_n - n q_n^+ u^+}{\det\,(B_n)}. \tag{76}$$

This is a formal solution for $\Omega \neq \Omega_0$ and it remains to show that the limit $\lim_{\Omega \to \Omega_0} u_n^-$ exists and solves the equation. This is indeed the case and will be shown later.

Constructing one formal solution u^- which is valid for all values of the parameters can be done using interpolation. To simplify notation we denote the set of values such that the eigenvalues $\lambda_\pm$ differ by an integer by $\Pi = \{\Omega \in U_\Omega | 2\Omega \in \mathbb{N}\}$. We furthermore exclude those points, for which the recurrence relation can be solved nonetheless, as discussed in the previous paragraph and denote the resulting subset by $\tilde{\Pi}$. The subset Π is discrete and so is $\tilde{\Pi}$.

The key ingredient in this interpolation argument is the Weierstrass factorization theorem (e.g. [40][Theorem 15.10]). It states that given a discrete subset $\tilde{\Pi}$ of $\mathbb{C}$ and a mapping $N : \tilde{\Pi} \to \mathbb{N}$, there exists a holomorphic function χ all of whose zeroes are in $\tilde{\Pi}$ and such that χ has a zero of order $N(\Omega)$ at each $\Omega \in \tilde{\Pi}$. This function χ replaces the determinant of B_n in the above construction of u_n^-. We choose the order of all zeroes of χ to be one. Using this function χ we can construct a formal solution ϕ similar to ϕ_n above, by setting the lowest order coefficient a_0^- to $\chi \cdot v_-$. As before this implies that for a given $\Omega_0 \in \tilde{\Pi}$

$$\lim_{\Omega \to \Omega_0} \phi = \lambda_{\Omega_0} u^+ \tag{77}$$

for some $\lambda_{\Omega_0} \neq 0$.

A combination of the Weierstrass factorization theorem and the Mittag-Leffler theorem (e.g. [40][Theorem 15.13]) show that there exists a holomorphic function φ which satisfies $\varphi(\Omega) = \lambda_\Omega$ for all $\Omega \in \tilde{\Pi}$. A proof for this interpolation problem can be found in [40][Theorem 15.15]. The formal solution u^- can then be defined as

$$u^- = \frac{\phi - \varphi u^+}{\chi}. \tag{78}$$

As mentioned before, it now remains to show that $u^-(\Omega_0)$ exists for all $\Omega_0 \in \tilde{\Pi}$ and that it solves the equation. This can be done by first reexpressing the limit $\lim_{\Omega\to\Omega_0} u^-$ using l'Hopital's rule as

$$\lim_{\Omega\to\Omega_0} u^- = \frac{\mathrm{d}}{\mathrm{d}\,\Omega}\left(\phi - \varphi u^+\right)\bigg|_{\Omega=\Omega_0} \left(\frac{\mathrm{d}\,\chi}{\mathrm{d}\,\Omega}\right)^{-1}_{\Omega=\Omega_0}. \tag{79}$$

By construction χ has a zero of first order in $\Omega = \Omega_0$ and therefore the right hand side of this equation exists as a formal power series. Furthermore the rule of l'Hopital applies, since all coefficients are analytic functions of Ω. In order to see that this limit solves the equation, we differentiate equation (53) with respect to Ω and arrive at

$$\frac{\mathrm{d}}{\mathrm{d}\,r}\frac{\mathrm{d}\,u}{\mathrm{d}\,\Omega} = \frac{\mathrm{d}\,\tilde{A}}{\mathrm{d}\,\Omega}u + \tilde{A}\frac{\mathrm{d}\,u}{\mathrm{d}\,\Omega}. \tag{80}$$

Setting $u = (\phi - \varphi u^+)$ and taking the limit $\Omega \to \Omega_0$ this reduces to

$$\frac{\mathrm{d}}{\mathrm{d}\,r}\frac{\mathrm{d}\,u}{\mathrm{d}\,\Omega} = \tilde{A}\frac{\mathrm{d}\,u}{\mathrm{d}\,\Omega} \tag{81}$$

and consequently $\lim_{\Omega\to\Omega_0} u^-$ is a solution to equation (53) for $\Omega = \Omega_0$. Equation (79) also explains the occurrence of logarithms. Evaluating the derivative on ϕ using equation (62) leads to

$$\frac{\mathrm{d}\,\phi}{\mathrm{d}\,\Omega} = \sum_{i=0}^{\infty} (r-1)^{\lambda_- + i}\left(a_i^- \log(r-1)\frac{\mathrm{d}\,\lambda_-}{\mathrm{d}\,\Omega} + \frac{\mathrm{d}\,a_i^-}{\mathrm{d}\,\Omega}\right). \tag{82}$$

For all values of the parameters the convergence of the formal solutions $u^\pm$ follows from [44][Theorem 4.13] and the limit is a product of $(r-1)^{\pm\Omega}$ and a holomorphic function of r. However, this convergence is only pointwise in the parameters, and therefore not enough to conclude that the limit is holomorphic everywhere. In order to show that the limit is holomorphic, it would be sufficient to show that the series converges locally uniformly[6] This might be a more direct approach, however theorem 3.3 offers a convenient way to obtain the analytic dependence on the parameters.

[6]This is a simple corollary of Morera's theorem, see e.g. [40][Theorem 10.27].

We now want to show that the solutions $u^{\pm}$ can be used to transform equation (53) into an equation for which Theorem 3.3 holds. Using the notation of Theorem 3.3 let $U \subset U_\mu \times U_\Omega$ be bounded and let $x = (\mu, \Omega) \in U$. Then there exists some integer n_0 such that $2\,|\Omega| < n_0$ in U. Denote by $U_n^{\pm}$ the partial sums given by terminating the solutions $u^{\pm}$ after finitely many terms as

$$U_n^{\pm}(r,x) = \sum_{i=0}^{n} a_i^{\pm}(x)\,(r-1)^{\lambda_{\pm}+i}\,. \tag{83}$$

We denote the remainders as

$$(r-1)^{\lambda_{\pm}+n}\,v_n^{\pm} = u^{\pm} - U_n^{\pm}, \tag{84}$$

where $v_n^{\pm}$ is analytic in r. Since the coefficients satisfy the recurrence relation (64), $U_n^{\pm}$ satisfies

$$\begin{aligned} &(r-1)\left(\partial_r - \tilde{A}(r,x)\right) U_n^{\pm}(r,x) \\ &= \sum_{i=0}^{n} \sum_{j=n-i+1}^{\infty} \tilde{A}_j(x) a_i^{\pm}(x)\,(r-1)^{\lambda_{\pm}+j+i+1} \end{aligned} \tag{85}$$

$(r-1)\,\tilde{A}$ depends analytically on the parameters and r and therefore the right hand side of (85) is the product of an analytic function and $(r-1)^{\lambda_{\pm}+n+1}$. Using this fact, it follows that (84) leads to an equation for $v_n^{\pm}$ of the form

$$\left((r-1)\,\partial_r + \left(\lambda_{\pm} + n - \tilde{A}_0\right)\right) v_n^{\pm}(r,x) = (r-1)\,f\,(r,x,v) \tag{86}$$

where f is analytic in all its arguments. If n has been chosen big enough, for instance such that $n \geq n_0$, then all eigenvalues of

$$\left(\lambda_{\pm} + n - \tilde{A}_0\right) \tag{87}$$

have nonnegative real part and therefore Theorem 3.3 applies. Because it is unique, it has to be such that $u^{\pm} = U_n^{\pm} + v_n^{\pm}$ and therefore we have shown that the remainders are analytic in the parameters. Since the coefficients are analytic functions of the parameters, so are the finite sums $U_n^{\pm}$, which implies that the solutions $u^{\pm}$ are analytic in the parameters *and* in r. □

Using the recurrence relation (64) we can determine the coefficients of the expansions of $u_0^{\pm}$. In the following we will mainly be interested in the behavior of u_0^{+}, which is the solution generating the instability. In order to solve equation (64), it is necessary to understand the asymptotic behavior of $\tilde{A}$ at the horizon. The first orders are

$$\tilde{A}_0 = \frac{1}{\mu^2+1} \begin{pmatrix} -\frac{1}{2}\left(4\Omega^2+\mu^2\right) & \frac{1}{4}\left(4\Omega^2-1\right) \\ \mu^4-4\Omega^2 & \frac{1}{2}\left(\mu^2+4\Omega^2\right) \end{pmatrix} \tag{88}$$

and

$$\tilde{A}_1 = \begin{pmatrix} -\frac{4\mu^2-\mu^4+12\Omega^2+2}{2\mu^4+4\mu^2+2} & \frac{4\mu^4+8\mu^2\Omega^2+6\mu^2+20\Omega^2-1}{4\mu^4+8\mu^2+4} \\ \frac{4\mu^4+8\mu^2\Omega^2+2\mu^2-4\Omega^2}{\mu^4+2\mu^2+1} & -\frac{3\mu^4+6\mu^2-12\Omega^2+6}{2\mu^4+4\mu^2+2} \end{pmatrix} \tag{89}$$

As has been pointed out before, the eigenvalues of $\tilde{A}_0$ are $\pm\Omega$ and two corresponding eigenvectors

$$a_0^{\pm} = \begin{pmatrix} \pm\Omega-\frac{1}{2} \\ \mu^2\pm 2\Omega \end{pmatrix}. \tag{90}$$

For the solution u_0^+ we choose the eigenvalue Ω and proceed to calculate a_1^+ by solving the equation

$$\left(1+\Omega-\tilde{A}_0\right) a_1^+ = \tilde{A}_1 a_0^+. \tag{91}$$

The matrix $\left(1+\Omega-\tilde{A}_0\right)$ is invertible and

$$\left(1+\Omega-\tilde{A}_0\right)^{-1} = \frac{1}{\mu^2+1} \begin{pmatrix} \frac{\mu^2}{2}-\Omega+1 & \frac{2\Omega-1}{4} \\ \frac{\mu^4-4\Omega^2}{2\Omega+1} & \frac{2\mu^2\Omega+3\mu^2+4\Omega^2+2\Omega+2}{4\Omega+2} \end{pmatrix}. \tag{92}$$

Therefore a unique a_1^+ exists which is

$$a_1^+ = \frac{1}{2\Omega+1} \begin{pmatrix} \left(\Omega+\frac{1}{2}\right)\left(\mu^2+2\Omega^2-\Omega+2\right) \\ \mu^4+2\mu^2\Omega^2+\mu^2\Omega-4\mu^2+4\Omega^3-6\Omega^2-6\Omega \end{pmatrix}. \tag{93}$$

This implies that the quantities H and H_- for the solution u_0^+ behave as

$$\begin{aligned} H =& \left(\Omega-\frac{1}{2}\right)(r-1)^{\Omega-1} \\ &+\frac{1}{2}\left(\mu^2+2\Omega^2-\Omega+2\right)(r-1)^{\Omega}+\mathcal{O}\left((r-1)^{\Omega+1}\right), \\ H_- =& \left(\frac{\mu^2}{\Omega}+2\right)(r-1)^{\Omega} \\ &+\frac{\mu^4+2\mu^2\Omega^2+\mu^2\Omega-4\mu^2+4\Omega^3-6\Omega^2-6\Omega}{2\Omega^2+\Omega}(r-1)^{\Omega+1} \\ &+\mathcal{O}\left((r-1)^{\Omega+2}\right). \end{aligned} \tag{94}$$

Using the algebraic equation for H_+ we can derive

$$H_+ = (2\Omega - 1)\,(r-1)^{\Omega-1} + \left(\mu^2 + 2\Omega^2 - \Omega + 1\right)^{\Omega} + \mathcal{O}\left((r-1)^{\Omega+1}\right). \quad (95)$$

We can now use these to recover the asymptotic behavior of the metric components at the horizon. The traceless gauge condition is equivalent to $K = -\frac{1}{2r^2} H_-$, which implies

$$\begin{aligned} K = &-\frac{1}{2}\left(\frac{\mu^2}{\Omega} + 2\right)(r-1)^{\Omega} \\ &-\frac{\mu^4 + 2\mu^2\Omega^2 + \mu^2\Omega - 4\mu^2 + 4\Omega^3 - 6\Omega^2 - 6\Omega}{4\Omega^2 + 2\Omega}\,(r-1)^{\Omega+1}\,. \\ &+\mathcal{O}\left((r-1)^{\Omega+2}\right). \end{aligned} \quad (96)$$

The remaining quantities H^{tt} and H^{rr} can then be found to behave as

$$\begin{aligned} H^{tt} &= \left(\Omega - \frac{1}{2}\right)(r-1)^{\Omega-2} + \frac{1}{2}\left(\mu^2 + 2\Omega^2 - \Omega + \frac{\mu^2}{\Omega} + 3\right)(r-1)^{\Omega-1} \\ &\quad+\mathcal{O}\left((r-1)^{\Omega}\right), \\ H^{rr} &= \left(\Omega - \frac{1}{2}\right)(r-1)^{\Omega} + \frac{1}{2}\left(\mu^2 + 2\Omega^2 - \Omega - \frac{\mu^2}{\Omega} - 1\right)(r-1)^{\Omega+1} \\ &\quad+\mathcal{O}\left((r-1)^{\Omega+2}\right). \end{aligned} \quad (97)$$

The vectors $a_0^{\pm}$ are linearly independent and therefore any solution u can be written as a linear combination of $u_0^{\pm}$ near the horizon. We define a mapping $\Phi_1 : \mathbb{R}^2 \times \mathbb{R}^2_{>0} \to \mathbb{R}^2$ as

$$(a, b, \mu, \Omega) \mapsto a\; u_0^{+} + b\; u_0^{-}. \quad (98)$$

Since the solutions $u_0^{\pm}$ depend continuously on the parameters, the mapping Φ_1 is continuous.

3.3 Asymptotic behavior at infinity

In order to study the asymptotic behavior of the solutions as $r \to \infty$ we consider the linear system of differential equations for the unrescaled quantities $u = (H, H_-)$

$$u' = Au \tag{99}$$

given in equations (49). We can study its behavior at ∞ by transforming it into equations of a new variable $s = \frac{1}{r}$. This leads to

$$-s^2 \frac{\mathrm{d}\, u}{\mathrm{d}\, s} = Au. \tag{100}$$

The components of A behave as $A_{11} = \mathcal{O}(s)$, $A_{22} = \mathcal{O}(s)$, $A_{12} = \mathcal{O}(1)$ and $A_{21} = \mathcal{O}(1)$ for $s \to 0$. Unlike the singularity at the horizon, the singularity at infinity is irregular. This means that the singular term is of higher order than the derivative. Therefore, the approach of the previous chapter does not apply here. However, there are results for second order equations with certain irregular singularities, which can be applied to this type of equation.

The following theorem can be found in [38]. In [38] it is stated using cross–references to previous theorems in the same chapter and other abbreviations.

Theorem 3.5 (c.f. [38][chapter 7, Theorem 3.1]). *Let $f(z,u)$ and $g(z,u)$ be analytic functions of the complex variable z for any given $u \in U \subset \mathbb{C}$ having a convergent series expansion*

$$f(z,u) = \sum_i f_i(u) z^{-i}, \quad g(z,u) = \sum_i g_i(u) z^{-i} \tag{101}$$

in a given annulus $A = \{z : |z| > a\}$. Assume that $\lambda^2 + f_0\lambda + g_0 = 0$ has two distinct solutions $\lambda_\pm$. Then for fixed u the equation

$$\frac{d^2 w}{dz^2} + f(z,u)\frac{dw}{dz} + g(z,u) = 0 \tag{102}$$

has unique solutions $w_\pm$, such that $w_\pm$ are holomorphic in the respective intersections of A with the sectors

$$S_\pm = \{z \in \mathbb{C} : |\mathrm{ph}\,(\pm(\lambda_- - \lambda_+)\, z)| \le \pi\}^{7} \tag{103}$$

and

$$w_\pm(z) \sim e^{\lambda_\pm z} z^{\gamma_\pm} \sum_i a_i^\pm z^{-i}, \tag{104}$$

where $\gamma_\pm$ and $a_i^\pm$ can be found by substituting into the equation. If in addition the following conditions are satisfied

[7] These sectors are not maximal, as noted in [38][chapter 7, Theorem 2.2].

1. *f_0 and g_0 are independent of u and all higher order coefficients are holomorphic functions of u.*

2. *If u restricted to any compact domain $U_c \subset U$, then $|f_i(u)| \leq F_i^c$ and $|g_i(u)| \leq G_i^c$, where F_i^c and G_i^c are independent of u and the series $\sum_i F_i^c z^{-i}$ and $\sum_i G_i^c z^{-i}$ converge absolutely in A.*

3. *$a_0^{\pm}$ are chosen holomorphic functions of u.*

then at each point z in A each branch of $w_{\pm}$ and their first two partial z derivatives are holomorphic functions of u.

We now want to apply this result to equation (51). However, there are several simplifying assumptions in the above theorem which are not satisfied for our problem. If those assumptions are lifted, for instance by *not* requiring that the lowest order coefficients of f and g are independent of the parameter, the region of validity of the solutions will depend on the parameters. More specifically, we will show the following corollary.

Corollary 3.6. *Let μ_-, μ_+, Ω_- and Ω_+ positive constants. Then there exists a positive constant a such that for $\mu_- < \mu < \mu_+$, $\Omega_- < \Omega < \Omega_+$ and any $a_{\pm}^0 \in \mathbb{R}$ equation (51) has two unique solutions $H_{\infty}^{\pm}(r)$ which are analytic for $r > a$. Furthermore the solutions and their first two derivatives depend analytically on μ and Ω and*

$$H_{\infty}^{\pm}(r) = e^{\pm r\sqrt{\Omega^2+\mu^2}} r^{\gamma_{\pm}} \sum_i a_i^{\pm} r^{-i} \tag{105}$$

where $\gamma_{\pm}$ and $a_i^{\pm}$ can be found by substituting into the equation.

Before presenting the proof of corollary 3.6 we prove an inequality for rational functions. It will be used in order to show that – under certain assumptions – condition 2 of theorem 3.5 is satisfied for our equation.

Proposition 3.7. *Let U, V open subsets of $\mathbb{C}$ and $Q : U \times V \to \mathbb{C}$, $P : U \times V \to \mathbb{C}$ polynomials and $f = Q \cdot P^{-1}$ a rational function. If P has no zeroes in $U \times V$, then for $v_0 \in V$ all coefficients f_n of*

$$f(u, v) = \sum_n f_n(u)(v - v_0)^n \tag{106}$$

are holomorphic functions of u. Assume further there exist positive constants C_1 and C_2 such that

$$\inf_{U \times V} |P| \geq C_1 \tag{107}$$

and

$$\sup_{u\in U}|Q_n| \le C_2 \text{ and } \sup_{u\in U}|P_n| \le C_2 \tag{108}$$

is satisfied for all coefficients Q_n and P_n. Then, there exists a constant F such that

$$\sup_{u\in U}|f_n(u)| \le F^n. \tag{109}$$

Proof. The coefficients f_n are given by Taylor's formula as

$$f_n(u) = \frac{1}{n!}\frac{\mathrm{d}^n f}{\mathrm{d}\, v^n}(u, v_0)\,. \tag{110}$$

By the product rule the derivatives of f are of the form

$$\frac{\mathrm{d}^n f}{\mathrm{d}\, v^n} = Q_n \cdot P^{-(n+1)} \tag{111}$$

where $Q_0 = Q$ and

$$Q_{n+1} = \frac{\mathrm{d}\, Q_n}{\mathrm{d}\, v}P - (n+1)\, Q_n \frac{\mathrm{d}\, P}{\mathrm{d}\, v}. \tag{112}$$

Both Q and P are polynomials and so are all Q_n. Since P has no zeroes in $U \times V$, the f_n are holomorphic in U.

Let $P_i(u)$ and $Q_{ni}(u)$ the coefficients of P and Q_n as polynomials in $(v - v_0)$. The coefficients of P and Q_0 are bounded by assumption. For the sake of brevity we use the notation

$$A_{Q_n} = \sup_i \sup_{u\in U} |Q_{ni}(u)|\,. \tag{113}$$

Denote by $\deg(P)$ and $\deg(Q_n)$ the degree of P and Q_n as functions of v, respectively. Then we can show by induction that the coefficients of Q_n are bounded and that

$$A_{Q_n} \le (n-1)! \cdot [\deg(Q_0) + 3\deg(P) - 1]^n\, A_p^n A_{Q_0} \tag{114}$$

holds. For convenience of notation we set $n! = 1$ for $n \le 0$. Using equation (112) we see that the degree of Q_{n+1} satisfies

$$\deg(Q_{n+1}) \le \deg(Q_n) + \deg(P) - 1 \tag{115}$$

and therefore

$$\deg(Q_n) \le \deg(Q_0) + n\,(\deg(P) - 1)\,. \tag{116}$$

Obviously, for $n = 0$ equation (114) is satisfied. For Q_{n+1} equation (112) leads to

$$\begin{aligned} A_{Q_{n+1}} \leq\ & \sup_i \sup_{u\in U} \left| \frac{\mathrm{d}\, Q_n}{\mathrm{d}\, v} \right| A_P + (n+1) \sup_i \sup_{u\in U} \left| \frac{\mathrm{d}\, P}{\mathrm{d}\, v} \right| A_{Q_n} \\ \leq\ & (\deg(Q_n) + (n+1)\deg(P))\, A_{Q_n} A_P \\ \leq\ & (\deg(Q_0) + n(\deg(P) - 1) + (n+1)\deg(P))\, A_{Q_n} A_P \\ \leq\ & n(\deg(Q_0) + 3\deg(P) - 1)\, A_P A_{Q_n}. \end{aligned} \tag{117}$$

By using equation (114) this leads to

$$A_{Q_{n+1}} \leq\ n!\,(\deg(Q_0) + 3\deg(P) - 1)^{n+1} A_P^{n+1} A_{Q_0}. \tag{118}$$

By induction this implies the desired inequality for all n. This implies that

$$\sup_{u\in U} \left| \frac{\mathrm{d}^n f}{\mathrm{d}\, v^n} \right| \leq (n-1)! \cdot [\deg(Q_0) + 3\deg(P) - 1]^n\, C_2^{n+1} C_1^{-(n+1)} \tag{119}$$

which implies

$$\sup_{u\in U} |f_n(u)| \leq F^n \tag{120}$$

where

$$F = [\deg(Q_0) + 3\deg(P) - 1] \left(\frac{C_2}{C_1} \right)^2 \tag{121}$$

□

Proof of corollary 3.6. Even though we are interested in solutions on the real axis depending on positive and real parameters, it is convenient here to work in the complex setting. After having proven the result, we can restrict it to the real axis.

Assume that μ and Ω are in small open neighborhoods of the intervals (μ_-, μ_+) and (Ω_-, Ω_+). We denote these neighborhoods by U_μ and U_Ω, respectively. The functions p and q can be written as quotients

$$p = \frac{F_p}{G_p} \text{ and } q = \frac{F_q}{G_q} \tag{122}$$

where the F_p, F_q, G_p and G_q are polynomials in r, μ and Ω. The full expressions are

$$\begin{aligned}
F_p &= 8\left(\Omega^2+\mu^2\right)r^5+\left(4\Omega^2-\mu^2\right)r^4-6r+3,\\
G_p &= 4\left(\Omega^2+\mu^2\right)r^6+\left(-4\Omega^2-8\mu^2\right)r^5+4\mu^2r^4-r^2+r,\\
F_q &= 4\left(\Omega^2+\mu^2\right)^2r^8+\left(-8\mu^2\Omega^2-8\mu^4\right)r^7\\
&\quad+\left(8\Omega^2+4\mu^4+8\mu^2\right)r^6+\left(-16\Omega^2-28\mu^2\right)r^5\\
&\quad+\left(3\Omega^2+33\mu^2\right)r^4-13\mu^2r^3+6r^2-6r+1 \text{ and}\\
G_q &= 4\left(\Omega^2+\mu^2\right)r^8+\left(-8\Omega^2-12\mu^2\right)r^7+\left(4\Omega^2+12\mu^2\right)r^6\\
&\quad-4\mu^2r^5-r^4+2r^3-r^2.
\end{aligned} \tag{123}$$

If G_p and G_q do not vanish, the functions p and q are holomorphic. It is evident from equations (123) that there exists a constant C_1 depending on μ_+ and Ω_+ which bounds all coefficients of G_p and G_q in magnitude from above on $U_\mu \times U_\Omega$. Furthermore, U_μ and U_Ω can be chosen small enough such that the highest order coefficients are bounded below in magnitude by $\left(\Omega_-^2+\mu_-^2\right)$. Then the roots of G_p and G_q are bounded in magnitude by

$$a = 1+\frac{C_1\left(\mu_+,\Omega_+\right)}{\mu_-^2+\Omega_-^2}. \tag{124}$$

This is a simple and well-known corollary of Rouché's theorem. Therefore the functions p and q are holomorphic in r, μ and Ω for $\mu \in U_\mu$, $\Omega \in U_\Omega$ and $|r| > a$. Furthermore, p and q remain bounded for $r \to \infty$ and

$$p = 4\frac{\Omega^2}{\mu^2}r^{-1}+O(r^{-2}) \text{ and} \tag{125}$$

$$q = -\left(\Omega^2+\mu^2\right)+\left(2\Omega^2+\mu^2\right)r^{-1}+O(r^{-2}). \tag{126}$$

Equation $\lambda^2+p_0\lambda+q_0=0$ has distinct solutions

$$\lambda_\pm = \pm\sqrt{\Omega^2+\mu^2}. \tag{127}$$

Then, for fixed $\mu \in U_\mu$ and $\Omega \in U_\Omega$ and all nonzero $a_0^\pm$ equation (51) has two linearly independent solutions $H_\infty^\pm$ which are analytic in r and behave like

$$H_\infty^\pm(r) = e^{\pm r\sqrt{\Omega^2+\mu^2}}r^{\gamma_\pm}\sum_i a_i^\pm r^{-i} \tag{128}$$

on the intersection of the annulus A with the sectors $S_\pm$.

In order to prove that the solutions depend analytically on μ and Ω, we need to show that conditions 1 through 3 are satisfied. The first difficulty is that in the above theorem the equation only depends on one complex parameter u. However, this is not a restriction. If the conditions are satisfied for both parameters μ and Ω separately, we can then conclude by Hartog's theorem [32][Theorem 1.2.5] that the solutions are holomorphic in both μ and Ω.

Assumption 3 is trivially satisfied here, since we chose $a_0^{\pm}$ to be constant and independent of the parameters.

Assumption 1 requires the coefficients p_0 and q_0 to be independent of the parameter. By rescaling r we can transform equation (51) into a new equation which does satisfy this condition. Let $\rho = \sqrt{\Omega^2 + \mu^2} r$. Then we arrive at

$$\frac{\mathrm{d}^2 H}{\mathrm{d}\rho^2} + \tilde{p}\frac{\mathrm{d}H}{\mathrm{d}\rho} + \tilde{q}H = 0 \tag{129}$$

where $\tilde{p} = p\sqrt{\mu^2 + \Omega^2}^{-1}$ and $\tilde{q} = q\left(\mu^2 + \Omega^2\right)^{-1}$. Then $\tilde{p}_0 = 0$ and $\tilde{q}_0 = -1$. Due to the above assumptions on U_μ and U_Ω, the factor $\sqrt{\mu^2 + \Omega^2}$ does not vanish. Therefore, both $\tilde{p}$ and $\tilde{q}$ are holomorphic functions in r, μ and Ω.

By increasing a, we can assume that G_p and G_q are both bounded below in magnitude by a positive constant which only depends on $\mu_\pm$ and $\Omega_\pm$. Then we can apply proposition 3.7 to p and q. Since we are interested in bounds on the coefficients of the Taylor series at infinity, we rewrite p and q as quotients of polynomials in r^{-1}. The previously mentioned bounds on the coefficients of these polynomials and their coefficients remain unchanged. In particular, the denominators are still bounded from below on A. Assume that $\Omega \in U_\Omega$ is fixed and denote by p_n and q_n the coefficients of the Taylor series at infinity. Then, using inequality (109) we can conclude that there exist constants F_p and F_q which depend only on $\mu_\pm$ and $\Omega_\pm$ such that

$$\sup_{\mu \in U_\mu} \left|p_n\left(\mu, \Omega\right)\right| \leq F_p^n \tag{130}$$

and

$$\sup_{\mu \in U_\mu} \left|q_n\left(\mu, \Omega\right)\right| \leq F_q^n. \tag{131}$$

The power series $\sum_i F_p r^{-i}$ and $\sum_i F_q r^{-i}$ converge absolutely on

$$\tilde{A} = \left\{r : |r| > \max\left(a, F_p, F_q\right)\right\}. \tag{132}$$

The same is true for $\tilde{p}$ and $\tilde{q}$, since the factor $\sqrt{\mu^2 + \Omega^2}$ is bounded from below in magnitude by $\frac{1}{2}\sqrt{\mu_-^2 + \Omega_-^2}$. This implies that condition 2 is satisfied for $\tilde{p}$ and $\tilde{q}$ and U_μ. Therefore, theorem 3.5 applies and we can conclude that $H_\infty^\pm\left(r\right)$ and their first two partial r derivatives are holomorphic functions of

μ on the intersection of A with the sectors $S_\pm$. The same argument can be repeated for fixed $\mu \in U_\mu$, which shows that $H_\pm$ are holomorphic in both parameters simultaneously. We can then restrict the solution to real μ and Ω. Then both sectors $S_\pm$ contain the real axis and this implies the result for all real $r > a$. $\square$

We now want to use this result to construct two linearly independent solutions $u_\infty^\pm$ to the first order system (53). We start by choosing the coefficients $a_0^\pm = 1$. The coefficients a_i and $\gamma_\pm$ can be determined by plugging the ansatz into the equation, balancing powers of r and solving the resulting algebraic equations. We can then recover the corresponding solutions $H_{-,\infty}^\pm$ using the first order equation for H

$$H_- = \tilde{A}_{12}^{-1}\left(H' - \tilde{A}_{11}H\right). \tag{133}$$

For r, μ and Ω as in corollary 3.6 both A_{12}^{-1} and A_{11} are analytic and their Taylor series at infinity behave as

$$\begin{aligned} A_{12}^{-1} &= \frac{1}{\Omega}\frac{\mu^2}{\Omega^2+\mu^2} - \frac{1}{\Omega}\frac{2\mu^2\Omega^2+\mu^4}{\left(\Omega^2+\mu^2\right)^2}r^{-1} + \mathcal{O}\left(r^{-2}\right) \text{ and} \\ A_{11} &= -\frac{2\Omega^2+\mu^2}{\mu^2}r^{-1} + \mathcal{O}\left(r^{-2}\right). \end{aligned} \tag{134}$$

The first derivative of $H_\infty^\pm$ is

$$\partial_r H_\infty^\pm = \pm\sqrt{\Omega^2+\mu^2}H_\infty^\pm + e^{\pm r\sqrt{\Omega^2+\mu^2}}r^{\gamma_\pm}\sum_i a_i^\pm\left(\gamma_\pm - i\right)r^{-i-1}. \tag{135}$$

Therefore it is easy to see that the solutions $H_{-,\infty}^\pm$ are of the form

$$H_{-,\infty}^\pm(r) = e^{\pm r\sqrt{\Omega^2+\mu^2}}r^{\gamma_\pm}\sum_i b_i^\pm r^{-i} \tag{136}$$

and that the lowest order coefficient b_0 is

$$b_0 = \pm\frac{\mu^2}{\sqrt{\mu^2+\Omega^2}}\frac{1}{\Omega} \tag{137}$$

This leads to two linearly independent solutions for the first order system with asymptotic behavior at infinity given by

$$u_\infty^\pm(z) = e^{\pm r\sqrt{\Omega^2+\mu^2}}r^{\gamma_\pm}\sum_i w_i^\pm r^{-i}, \tag{138}$$

where

$$w_0^{\pm} = \begin{pmatrix} 1 \\ b_0^{\pm} \end{pmatrix}. \tag{139}$$

For any given $\mu_{\pm}$ and $\Omega_{\pm}$ we can find an r such that the mapping $\Phi_3 : \mathbb{R}^2 \times (\mu_-, \mu_+) \times (\Omega_-, \Omega_+) \to \mathbb{R}^2$ given by

$$(a, b, \mu, \Omega) \mapsto a\ u_{\infty}^{+}(r) + b\ u_{\infty}^{-}(r). \tag{140}$$

is analytic in all arguments. Furthermore, $w_{\pm}^{0}$ are linearly independent for all μ and Ω and so are $u_{\infty}^{\pm}(r)$ for sufficiently big r. We can then apply the implicit function theorem and conclude that there exists a map $\overline{\Phi_3} : \mathbb{R}^2 \times (\mu_-, \mu_+) \times (\Omega_-, \Omega_+) \to \mathbb{R}^2$ given by

$$(u, \mu, \Omega) \mapsto (a, b). \tag{141}$$

Furthermore, $\overline{\Phi_3}$ is analytic, which means that the asymptotic data (a, b) depends analytically on solutions and the parameters.

3.4 Boundary conditions

Studying a perturbation problem requires choosing an initial data surface, on which the perturbation equations are considered. The natural choice compatible with the Cauchy-problem is a spacelike hypersurface. When considering solutions in the exterior of the black string, all spacelike hypersurfaces will necessarily touch the event horizon and extend towards spacelike or null infinity. In principle one would like to require perturbations to remain bounded at the horizon and decay at infinity. This would imply that the solution can be extended smoothly beyond the event horizon, which would provide data for a well-posed initial value problem for the Einstein equations. The boundary condition at infinity, namely that regular perturbations are those which decay for $r \to \infty$, is satisfied by the decaying branch u_∞^- of (138) and clearly violated by the branch u_∞^+. The boundary condition at the horizon is a bit harder to check. This is because Schwarzschild coordinates are singular at the horizon, and it is not obvious what the correct boundary conditions at the horizon are for solutions expressed in Schwarzschild coordinates. By transforming solutions to a regular coordinate system, this question can be more easily dealt with. This procedure was used in [18] to derive asymptotic boundary conditions for this problem on a particular spacelike hypersurface. It can then be shown that the solution u_0^+ of (60) satisfies these boundary conditions at the horizon. This is what was studied by R. Gregory and R. Laflamme in [18]. Unfortunately, in [18] most of the calculations and asymptotic terms, which are necessary to show that u_0^+ remains bounded when approaching the horizon, have been omitted. In particular, to the authors knowledge, the details of the following argument have not been published before. We will therefore repeat the argument here and include all necessary steps and intermediary results.

The regular coordinate system used is a higher dimensional equivalent of Kruskal–Szekeres coordinates. They are completely analogous to the usual Kruskal–Szekeres coordinates for the Schwarzschild spacetime as introduced in chapter 2.6.

The initial hypersurface which was chosen for this problem by Gregory and Laflamme is any surface of constant and positive T and therefore does not touch the wormhole at $T = R = 0$. There are several motivations for avoiding the Schwarzschild wormhole, which is located at $R = T = 0$. Typically black holes are created by the collapse of a star, in which case no wormhole is present. If this assumption remains valid for the creation of black strings, it provides a physical reason for choosing T to be positive. In the rest of this chapter, we will assume this choice of the initial hypersurface, i.e. T is constant and positive and $R \in (T, \infty)$. The event horizon is located

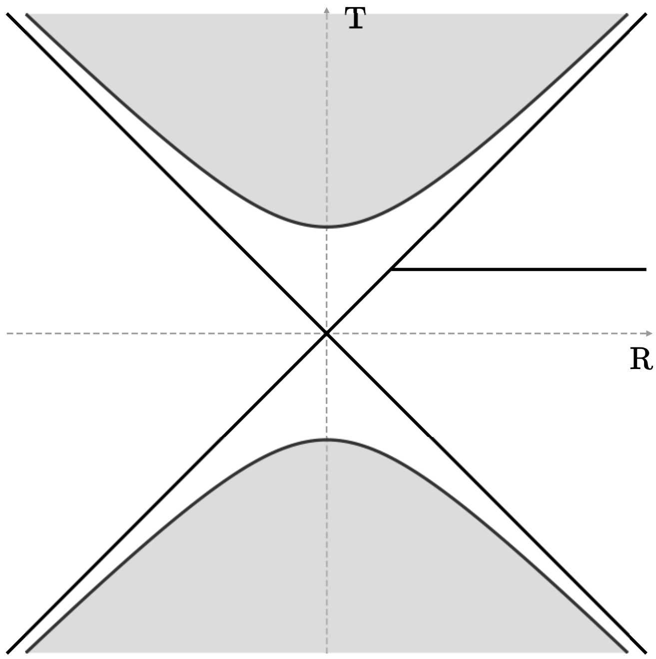

Figure 2: A diagram of the uniform black string spacetime in Kruskal–Szekeres coordinates. The horizontal line depicts an initial hypersurfaces of constant T.

at $R = T$.

In order to study the asymptotic behavior, we have to express t and r as functions of R and T. Then we can study the behavior of (60) as $R \to T$ from above.

$$t = \log\left(\frac{R+T}{R-T}\right) \tag{142}$$

$$e^{r-1}\left(r-1\right) = \frac{1}{4}\left(R-T\right)\left(R+T\right) \tag{143}$$

The event horizon consists of two null hypersurfaces, the *future* and *past* event horizon represented by $R = T$ and $R = -T$, respectively. In Schwarzschild coordinates these two surfaces are located at $t = \pm\infty$ and $r = 1$. Points on the event horizon for finite Schwarzschild time lie in the intersection of the future and past horizon at $T = R = 0$. The spacetime singularity at $r = 0$ corresponds to the hyperbola $R^2 - T^2 = 4e$ in Kruskal–Szekeres coordinates.

The perturbation h_{ab} is a symmetric 2–tensor. Therefore, the components of h in the coordinates T and R are given by

$$\begin{aligned} h^{TT} =& \left(\frac{\partial T}{\partial t}\right)^2 h^{tt} + \left(\frac{\partial T}{\partial r}\right)^2 h^{rr} + 2\frac{\partial T}{\partial r}\frac{\partial T}{\partial t}h^{tr} \\ h^{TR} =& \frac{\partial T}{\partial t}\frac{\partial R}{\partial t}h^{tt} + \frac{\partial T}{\partial r}\frac{\partial R}{\partial r}h^{rr} + \left(\frac{\partial T}{\partial t}\frac{\partial R}{\partial r} + \frac{\partial T}{\partial r}\frac{\partial R}{\partial t}\right) h^{tr} \\ h^{RR} =& \left(\frac{\partial R}{\partial t}\right)^2 h^{tt} + \left(\frac{\partial R}{\partial r}\right)^2 h^{rr} + 2\frac{\partial R}{\partial r}\frac{\partial R}{\partial t}h^{tr} \end{aligned} \tag{144}$$

The values of the partial derivatives with respect to r and t are

$$\frac{\partial R}{\partial r} = \frac{1}{2}Rf^{-1} \qquad \frac{\partial R}{\partial t} = \frac{1}{2}T \tag{145}$$

$$\frac{\partial T}{\partial r} = \frac{1}{2}Tf^{-1} \qquad \frac{\partial T}{\partial t} = \frac{1}{2}R \tag{146}$$

and therefore equations (144) can be written as

$$\begin{aligned} h^{TT} =& \frac{1}{4}\left(R^2 h^{tt} + T^2 f^{-2} h^{rr} + 2RTf^{-1}h^{tr}\right) \\ h^{TR} =& \frac{1}{4}\left(RTh^{tt} + RTf^{-2}h^{rr} + \left(R^2 + T^2\right) f^{-1}h^{tr}\right) \\ h^{RR} =& \frac{1}{4}\left(T^2 h^{tt} + R^2 f^{-2} h^{rr} + 2RTf^{-1}h^{tr}\right). \end{aligned} \tag{147}$$

For a solution to be regular it would be sufficient to show that its components in the coordinate system (R, T) remain bounded when approaching the event horizon on the initial hypersurface. By reorganizing equations (147) we arrive at

$$\begin{aligned} 4h^{TT} &= R^2 f^{-1}\left(fh^{tt} + f^{-1}h^{rr} + 2h^{tr}\right) \\ &+ \left(T^2 - R^2\right) f^{-2}h^{rr} + 2R\left(T - R\right) f^{-1}h^{tr}, \\ 4h^{TR} &= RTf^{-1}\left(h^{tt}f + f^{-1}h^{rr} + 2h^{tr}\right) \\ &- \left(R - T\right)^2 f^{-1}h^{tr} \text{ and} \\ 4h^{RR} &= T^2 f^{-1}\left(h^{tt}f + f^{-1}h^{rr} + 2h^{tr}\right) \\ &+ \left(R^2 - T^2\right) f^{-2}h^{rr} + 2T\left(R - T\right) f^{-1}h^{tr}. \end{aligned} \tag{148}$$

The regularity of h^{TT}, h^{TR} and h^{RR} at the horizon can now be studied using the asymptotic behavior of h^{ab} in Schwarzschild coordinates.

Lemma 3.8. *Consider a spatial hypersurface of constant, positive T. Then the quantities h^{TT}, h^{TR} and h^{RR} corresponding to the solution u_0^+ of (53) remain bounded on any subset $R < R_0$ up to and including the horizon $R = T$.*

Proof. It is sufficient to show that the three terms

$$
\begin{aligned}
I_1 &:= f^{-1}\left(fh^{tt} + f^{-1}h^{rr} + 2h^{tr}\right), \\
I_2 &:= \left(R^2 - T^2\right) f^{-2}h^{rr} + 2T\left(R - T\right) f^{-1}h^{tr} \text{ and} \\
I_3 &:= \left(R - T\right)^2 f^{-1}h^{tr}
\end{aligned}
\tag{149}
$$

remain bounded on the horizon. This is because h^{TT}, h^{TR} and h^{RR} can be rewritten as

$$
\begin{aligned}
4h^{TT} &= R^2 I_1 - I_2, \\
4h^{TR} &= RTI_1 - I_3 \text{ and} \\
4h^{RR} &= T^2 I_1 + I_2.
\end{aligned}
\tag{150}
$$

We start by listing some useful identities. By definition of R and T

$$
h^{ab}\left(R, T, z\right) = e^{i\mu z}\left(\frac{R+T}{R-T}\right)^{\Omega} H^{ab}\left(R, T\right). \tag{151}
$$

Furthermore equation (143) implies that

$$
(r - 1) = \mathcal{O}\left(\left(R+T\right)\left(R-T\right)\right) \tag{152}
$$

as $R \to T$. f satisfies

$$
f = \frac{1}{4}\frac{e^{1-r}}{r}\left(R+T\right)\left(R-T\right) \tag{153}
$$

which also implies

$$
f = \mathcal{O}\left(\left(R+T\right)\left(R-T\right)\right) \tag{154}
$$

as $R \to T$.

Regularity of I_1:

$$
fh^{tt} + f^{-1}h^{rr} + 2h^{tr} = e^{i\mu z}\left(\frac{R+T}{R-T}\right)^{\Omega}\left(fH^{tt} + f^{-1}H^{rr} + 2H^{tr}\right) \tag{155}
$$

$$
= e^{i\mu z}\left(\frac{R+T}{R-T}\right)^{\Omega}\left(H_+ - 2H\right) \tag{156}
$$

Equations (94) and (95) show that $H_+ - 2H = \mathcal{O}\left(\left(r-1\right)^{1+\Omega}\right)$ which implies that

$$
f^{-1}\left(fh^{tt} + f^{-1}h^{rr} + 2h^{tr}\right) = \mathcal{O}\left(1\right) \tag{157}
$$

for $R \to T$.

Regularity of I_2:

$$\begin{aligned}
& \left(R^2-T^2\right) f^{-2} h^{rr} + 2T\left(R-T\right) f^{-1} h^{tr} && (158)\\
= \;& \left(R-T\right) f^{-1}\left(\left(R+T\right) f^{-1} h^{rr} + 2T h^{tr}\right) && (159)\\
= \;& \left(R-T\right) f^{-1}\left(2T\left(f^{-1} h^{rr} + h^{tr}\right) + \left(R-T\right) f^{-1} h^{rr}\right) && (160)
\end{aligned}$$

Using equations (94) and (97) we see that $H^{rr} f^{-1} + H^{tr} = \mathcal{O}\left(\left(r-1\right)^{\Omega}\right)$ and consequently

$$\left(R-T\right) f^{-1}\left(2T\left(f^{-1} h^{rr} + h^{tr}\right)\right) = \mathcal{O}\left(1\right) \tag{161}$$

as $R \to T$. The second term $\left(R-T\right)^2 f^{-2} h^{rr}$ is regular because $h^{rr} = \mathcal{O}\left(1\right)$ at the horizon.

Regularity of I_3:

I_3 can be written terms of H as

$$\left(R-T\right)^2 f^{-1} h^{tr} = -e^{i\mu z}\left(R-T\right)^{2-\Omega}\left(R+T\right)^{\Omega} H. \tag{162}$$

Using the asymptotic behavior (94) of H and equation (152) we can conclude that $I_3 = \mathcal{O}\left(1\right)$ as $R \to T$. □

3.5 Gauge considerations

In general relativity physically identical systems can be described by different mathematical configurations. These different configurations can be transformed into one another by gauge transformations. Therefore it is important to make sure that the solutions to be studied here cannot be transformed *away* by a gauge transformation. If they were pure gauge, they would not be physically relevant and hence make the instability meaningless. This question was studied already by Gregory and Laflamme in [18], who concluded that the solutions considered here cannot be pure gauge. Their argument considers general gauge transformations and their effect on the components of h_{ab}. By doing so they are able to derive quantities, which are invariant under gauge transformations. Using the asymptotic behavior of solutions they can conclude that these invariant quantities do not vanish. They thereby show that the instability is not an artifact of a poor choice of gauge and therefore physical.

3.6 Shooting argument

In this chapter we will present an argument which proves the existence of solutions to (53) which satisfy the boundary conditions at the horizon and infinity for positive values of Ω. Therefore, the linearized Einstein equations at the uniform black string admit solutions which grow in time unboundedly and hence the uniform black string is linearly unstable.

In chapters 3.2 and 3.3 we have studied the asymptotic behavior of the general solution at the horizon and infinity. We know that, for any solution u with fixed parameters (μ, Ω) there exist coefficients (a_0, b_0) and (a_∞, b_∞) such that

$$u = a_0 u_0^- (r) + b_0 u_0^+ (r) \tag{163}$$

at the horizon and

$$u = a_\infty u_\infty^+ (r) + b_\infty u_\infty^- (r) \tag{164}$$

at infinity. In chapter 3.4 we have shown that the branches $u_0^+ (r)$ and $u_\infty^- (r)$ satisfy the boundary conditions at the horizon and infinity, respectively. We are therefore looking for values of the parameters such that there exist solutions for which both a_0 and a_∞ are zero. This is an *asymptotic* boundary value problem.

This problem was studied numerically by R. Gregory and R. Laflamme in [18], who concluded that solutions to the boundary value problem can be found. In this chapter, the existence of solutions will be shown rigorously. Furthermore, we will also present and discuss a reimplementation of the original numerical investigation. This code will be used to illustrate how solutions behave and in order to discuss which aspects of the numerical results can be proven rigorously.

The standard technique for solving boundary value problems numerically is the shooting method. The basic idea of this technique is to start with a desired initial value at the first boundary and then integrate numerically up to the second. Using the boundary condition one can determine if a suitable solution has been found. By varying the parameters or initial value and repeatedly performing the integration one might hope to find the parameters one was looking for. Solving an *asymptotic* boundary value problem using the shooting method is more complicated, but the basic concept remains the same.

Formally, the shooting method is usually stated as follows. For a given boundary value problem, a mapping Φ of the parameters is defined, which takes a particular value Φ_0 if the boundary conditions are satisfied. For instance, for a standard boundary value problem this could be the difference of the solution at the second boundary and the desired boundary value. Then, the boundary conditions are satisfied if Φ vanishes and therefore solutions to the boundary value problem are equivalent to zeroes of Φ.

For the asymptotic boundary value problem studied here, Φ can be defined as

$$\begin{aligned} \Phi : \mathbb{R}^2 \times \mathbb{R}^2_{\geq 0} &\to \mathbb{R}^2, \\ (a_0, b_0, \mu, \Omega) &\mapsto (a_\infty, b_\infty) . \end{aligned} \tag{165}$$

As stated above, we are looking for nonzero solutions for which a_0 and a_∞ vanish. Both b_0 and b_∞ are nonvanishing but arbitrary. Because equation (53) is linear, we can set one of them to 1 without loss of generality. Therefore, we can solve the asymptotic boundary value problem by finding values of μ and Ω such that Φ satisfies

$$\Phi\left(0, 1, \mu, \Omega\right) = \left(0, b_\infty\right) \tag{166}$$

for any nonzero b_∞. In other words, we are looking for values of the parameters such that the solution $u = u_0^+(r)$ is proportional to $u_\infty^-(r)$ at infinity.

We will start off by showing in lemma 3.9 that the mapping Φ as defined above is continuous. This is important when searching for zeroes numerically, as it allows for using efficient algorithms for finding them. However, more importantly here, it can be used to show the existence of solutions using the intermediate value theorem. More specifically, by proving that there exist values of μ and Ω such that a_∞ is negative and values such that a_∞ is positive, the existence of solutions follows by the continuity of Φ. This is the basic idea of the proof presented in this chapter. Showing that a_∞ can be both positive and negative constitutes the majority of the work.

Before proceeding to present the proofs, we will review how the shooting method can be implemented numerically and discuss some of its results. We will also use the numerical results to illustrate which qualitative features beyond the existence of solutions will be proven rigorously.

The mapping Φ can be implemented numerically as follows. First, one uses the leading order behavior of u_0^+ to approximate the value of the solution close to the horizon at $r_0 = 1$. Here the starting radius $r = 1.00001$ has been used. Then, $u_0 \approx u_0^+(r)$ is used as the initial value to solve the first order system (53) and calculate the value of the solution at some large $r = 200$. The asymptotic behavior at infinity is then matched against the value of the numerical solution in order to determine the constants a_∞ and b_∞. The numerical techniques used here are essentially a reimplementation of the ones described in [18]. They have been written in C and use a classical 4th order Runge-Kutta integration algorithm from the GNU Scientific library[8].

[8] `http://www.gnu.org/software/gsl/`

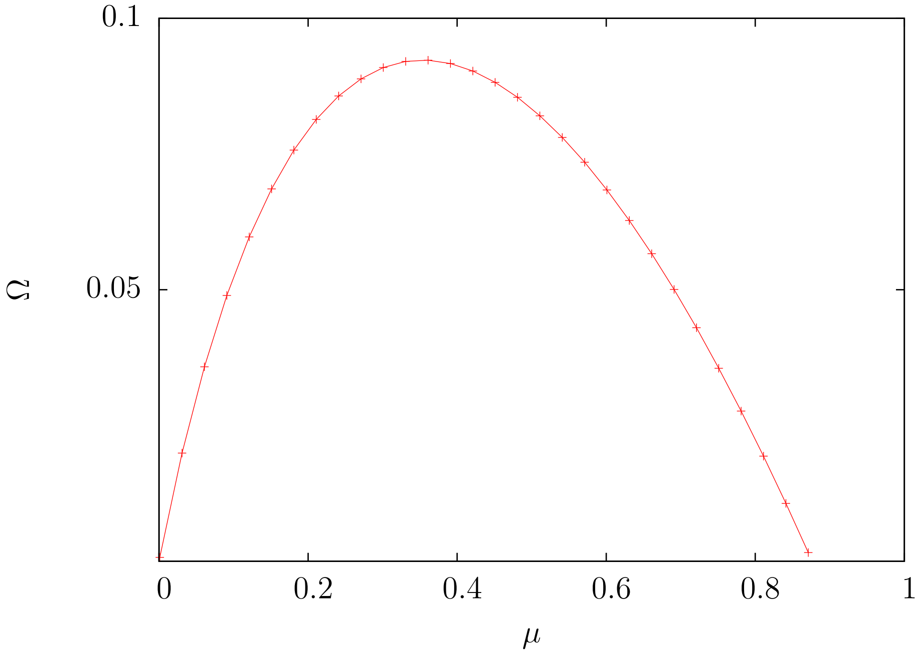

Figure 3: Values of Ω and μ for which u_0^+ satisfies $a_\infty = 0$.

Figure 3 shows those values of μ and Ω for which a_∞ is zero. For all values below the curve, a_∞ is negative and it is positive for all values above. The zeroes have been located by choosing a fixed value of μ and using bisection to determine that value of Ω for which a_∞ becomes zero. This is the same graph that has already been presented in [18]. It has several important features. First of all, it shows numerically that there exist values of (μ, Ω) such that equation (53) admits solutions which are regular both at the horizon and infinity. It therefore suggests that black strings are unstable. Secondly, the instability does not seem to be present for arbitrarily large values of μ. This has important implications when assuming that the extra dimension with coordinate z is compact. This is easy to realize when remembering that μ is the frequency of oscillation of the perturbation in z. If the z-dimension is compact, perturbations are required to satisfy an additional periodic boundary condition. If the extra dimension is too small, this periodic boundary condition would exclude solutions with small frequencies μ. Even though the absence of perturbations of this special form does not imply linear stability, it might still be a hint that long uniform black strings are stable. The existence of a maximal μ_C for which this instability is present is proven in lemma 3.19. The third important feature of figure 3 is that the zero set seems to include a point on the axis $\Omega = 0$ for positive μ. This corresponds to a solution which does not depend on time. It therefore suggests the existence of a family of *static* nonuniform solutions.

The first step of the proof is to show that Φ is continuous. Then, it will be shown that there exist nonempty subsets of the parameter space in which the solution u_0^+ has a positive and negative coefficient a_∞, respectively. The first case, the existence of a subset with positive a_∞ is shown in Lemma 3.10. The second case is considerably longer and is proven in several steps in Lemmata 3.13 through 3.16. Splitting the proof of the second case into smaller steps was not done arbitrarily. It highlights the logical structure of the proof. Lemmata 3.13 through 3.16 cover disjoint regions, in which the differential equations have different qualitative behavior. Together, these steps trace the behavior of the regular branch u_0^+ from the horizon at $r = 1$ to infinity. In each of these statements inequalities for u_0^+ are proven, which are used as input for the next lemma. Conceptually this is very similar to the integration in the numerical shooting method. Indeed, some of the steps in this proof have been inspired by looking at the behavior of u_0^+ numerically and tested using the numerical results. Therefore, understanding the problem numerically was helpful in guiding the construction of the arguments.

Lemma 3.9. *Φ is continuous in μ and Ω for fixed values of b_0 and a_0.*

Proof. Φ can be written as the composition $\Phi = \overline{\Phi_3} \circ \Phi_2 \circ \Phi_1$, where Φ_1 is defined as in (98), Φ_2 is defined by integrating the system of ODEs from $r = 1 + \epsilon$ to $r = \epsilon^{-1}$ and $\overline{\Phi_3}$ defined as in (141). It is shown in chapters 3.2 and 3.3 that Φ_1 and $\overline{\Phi_3}$ are continuous. The continuity of Φ_2 follows from standard theory of ordinary differential equations, since the right hand side of (53) is smooth for $r \in [1 + \epsilon, \epsilon^{-1}]$. Therefore, Φ is continuous as the composition of continuous functions. □

Lemma 3.10. *If $\Omega > \frac{1}{2}$ and $\mu^2 > 2\Omega$ holds, the solution u_0^+ satisfies $a_\infty > 0$.*

Proof. From the asymptotic behavior of solutions we know that close to the horizon u_0^+ behaves like

$$u_0^+ = a_0^+ (r-1)^\Omega + \mathcal{O}\left((r-1)^{\Omega+1}\right). \tag{167}$$

Using equation (90) it then follows that both $\tilde{H}$ and $\tilde{H}_-$ are positive there. Furthermore, from the assumptions follows that $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are everywhere positive. This implies, that $\tilde{H}$ and $\tilde{H}_-$ will be positive everywhere, as can be seen by the following argument. Assume this was not the case. Then there exists one smallest r_0 where either $\tilde{H}$, $\tilde{H}_-$ or *both* are zero. The last case is excluded by the uniqueness of solutions, since then u_+ would have to be the trivial solution $\tilde{H} = \tilde{H}_- = 0$. Assume that $\tilde{H}(r_0)$ vanishes and $\tilde{H}_-(r_0)$ is positive. Using the differential equation this leads to

$$\tilde{H}'(r_0) = \tilde{A}_{12}\tilde{H}_-(r_0) > 0. \tag{168}$$

By continuity, this also holds for all $r \in (r_0 - \epsilon, r_0]$ for some positive ϵ. This contradicts $\tilde{H}(r_0) = 0$ and therefore there is no such r_0. An analogous argument shows that $\tilde{H}_-$ cannot vanish and therefore both $\tilde{H}$ and $\tilde{H}_-$ remain positive for all r.

In order to see that this implies that the coefficient a_∞ is positive, we use the asymptotic behavior of solutions near infinity. It follows from equations (138) and (139) that for any solution with positive a_∞ the quantities $\tilde{H}$ and $\tilde{H}_-$ are both negative for sufficiently large r. Similarly, for all solutions with a_∞ vanishing, $\tilde{H}$ and $\tilde{H}_-$ have opposite sign for sufficiently large r. Consequently, for the solutions under consideration a_∞ is positive. □

In the following four lemmas we will show that the solution u_0^+ has a negative coefficient a_∞ for $2\Omega = \mu$ and sufficiently small μ. As can be seen from figure 3, there are other regions in the space of parameters, in which the same statement holds. However, the assumptions on the parameters made here simplify the equations considerably.

Figure 4 displays the behavior of the solution u_0^+ for two different values in the (μ, Ω)-plane. Both points lie on the line $2\Omega = \mu$, one below and the other above the curve depicted in figure 3. It shows how the qualitative behavior differs between the two cases and that a_∞ changes sign.

For certain ranges of the parameters the functions $\tilde{A}_{ij}$ have zeroes. The following lemmas will describe the behavior of u_0^+ in–between these zeroes. They therefore depend on the sign of the functions $\tilde{A}_{ij}(r)$ and how their zeroes depend on the parameters. The following two statements and proposition 3.1 will provide the necessary details.

Proposition 3.11. *Let $2\Omega = \mu$. If μ is sufficiently small, then $\tilde{A}_{21}$ has one zero r_{21}. $\tilde{A}_{21}$ is negative for $1 < r < r_{21}$ and positive for $r > r_{21}$. Furthermore, there exists a positive constant C such that*

$$2 - C\mu^2 \leq r_{21} \leq 2. \tag{169}$$

Proof. Under the above assumptions $\tilde{A}_{21} = 0$ is equivalent to $\lambda(r) := r - 2 + \mu^2 r^3 = 0$ for $r > 1$. Since $\mu^2 < 1$ holds, λ is negative for r sufficiently close to 1, strictly monotonic and positive for $r = 2$. Consequently, a unique zero r_{21} exists and $r_{21} < 2$, which in turn implies that $0 = \lambda(r_{21}) \leq r_{21} - 2 + C\mu^2$. □

Proposition 3.12. *Let $2\Omega = \mu$. If μ is sufficiently small, then $\tilde{A}_{22}$ has one zero r_{22}. $\tilde{A}_{22}$ is positive for $1 < r < r_{22}$ and negative for $r > r_{22}$. Furthermore, there exists a positive constant C such that*

$$r_{22} \leq 1 + C\mu^2. \tag{170}$$

Proof. If $2\Omega = \mu$ and $r > 1$, $\tilde{A}_{22} = 0$ is equivalent to $\lambda(r) := -6\,(r-1) + \mu^2 r^3\,(3-r) = 0$. For r sufficiently close to 1, $\lambda(r)$ is positive. Furthermore $\lambda(3) < 0$ and consequently a zero exists. By straightforward calculation one finds that $\lambda'(r)$ has a maximum at $r = \frac{3}{2}$ which is $\lambda'(\frac{3}{2}) = -6 + \frac{27}{4}\mu^2$. This implies that for sufficiently small μ, λ is monotonically decreasing and r_{22} is unique. Using the fact that $r > 1$, we arrive at $\lambda(r) \leq 2r^3\mu^2 - 6\,(r-1)$. Combining this with $r_{22} \leq 2$ leads to $0 \leq C\mu^2 - (r_{22} - 1)$ which implies the desired inequality. □

Remark. *Under the assumption that $2\Omega = \mu$ and that μ is sufficiently small, the three zeroes r_{12}, r_{21} and r_{22} satisfy*

$$1 < r_{22} < r_{21} < r_{12}. \tag{171}$$

Lemma 3.13. *Let $2\Omega = \mu$, r_{21} the zero of $\tilde{A}_{12}$ and $u = u_0^+$. If μ is sufficiently small, then $\tilde{H}_-\,(r_{21})$ is nonnegative, $\tilde{H}\,(r_{21})$ is negative and*

$$\left(\tilde{H}_- \tilde{H}^{-1}\right)(r_{21}) \geq -C\mu. \tag{172}$$

for some positive constant C.

Proof. The assumption that $\Omega < \frac{1}{2}$ and the asymptotic behavior of u_0^+ implies that $\tilde{H}_-$ is positive and $\tilde{H}$ is negative for r close to 1. Neither $\tilde{H}_-$ nor $\tilde{H}$ can change sign in the interval (r, r_{21}). This can be shown by contradiction similarly to an argument used in lemma 3.10 as follows. Assuming that this was not true, there exists some smallest $r_0 < r_{21}$ such that either $\tilde{H}$, $\tilde{H}_-$ or both are zero. Since u is not the trivial solution, they cannot be both zero. If μ is sufficiently small, propositions 3.1 and 3.11 state that both $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are negative for $r < r_{21}$. Assume that $\tilde{H}_-\,(r_0)$ vanishes. Then by using the differential equation it follows that

$$\tilde{H}'_-\,(r_0) = \tilde{A}_{21}\tilde{H}\,(r_0) \tag{173}$$

is positive. By continuity of $\tilde{H}'_-$ this holds on an open interval. Using integration this leads to a contradiction. An identical argument applies to $\tilde{H}$. Therefore, $\tilde{H}$ is negative and $\tilde{H}_-$ is positive for all $r < r_{21}$. Note that this argument for $\tilde{H}_-$ does not hold at r_{21} because $\tilde{A}_{21}$ vanishes there, while that for $\tilde{H}$ does. Therefore, we can conclude that $\tilde{H}(r_{21})$ is negative and that $\tilde{H}_-(r_{21})$ is nonnegative.

Let

$$F = -\mu(\mu+1)\tilde{H} - \frac{1}{2}(1-\mu)\tilde{H}_-. \tag{174}$$

Note that in case of the regular solution both $\tilde{H}$ and $\tilde{H}_-$ are continuous even at $r = 1$ and so is F. Furthermore, F satisfies $\lim_{r\to 1} F = 0$, which follows from the asymptotic behavior of u_+ near the horizon. Using equation (53) we arrive at

$$F' = \left[\frac{\mu}{2}\tilde{H}G + \frac{1}{4}\tilde{H}_-G_-\right]\left[r(r-1)\left(1+\mu^2r^3\right)\right]^{-1}, \tag{175}$$

where

$$\begin{aligned} G &= 2(r-1) + \mu^3r^3 + \mu^4r^4 + \mu\left(4r - r^2 - 2\right) \\ &\quad + \mu^2 r\left(r + r^2 + r^3 - 2\right) \text{ and} \\ G_- &= 6(r-1) - (r-1)\left[4\mu^4r^5 + 6\mu + 3\mu^2r^2 + 4\mu^3r^5 + \mu^3r^3\right] \\ &\quad + \mu\left[r^2 + \mu r^4 + 2\mu^2r^3 - 2\mu r^2 - \mu^2r^6 - \mu^3r^6\right]. \end{aligned} \tag{176}$$

Note that G is positive for $r \in (1,2)$ and hence also for all $r \le r_{21}$. We will now show that F is nonnegative for all $r \le r_{21}$. Assume that F is negative. By the definition of F this implies that G and G_- satisfy

$$\frac{\mu}{2}\tilde{H}G > -\frac{1}{4}\frac{1-\mu}{1+\mu}\tilde{H}_-G_- \tag{177}$$

which in turn implies

$$F' > \frac{\tilde{H}_-}{4(1+\mu)}\left[G_- - G + \mu(G_- + G)\right]\left[r(r-1)\left(1+\mu^2r^3\right)\right]^{-1}. \tag{178}$$

Furthermore, since r_{21} is smaller than 2, it follows from equations (176) that

$$G_- - G = 4(r-1) - \mu^2r^2 + \mathcal{O}(\mu(r-1)) + \mathcal{O}\left(\mu^3\right) \tag{179}$$

and

$$\mu(G_- + G) = \mu^2\left(r^2 + r\right) + \mathcal{O}\left(\mu^3\right) + \mathcal{O}(\mu(r-1)). \tag{180}$$

Combining these two equations we arrive at

$$G_- - G + \mu(G_- + G) = 4(r-1) + \mu^2 r + \mathcal{O}\left(\mu^3\right) + \mathcal{O}(\mu(r-1)), \tag{181}$$

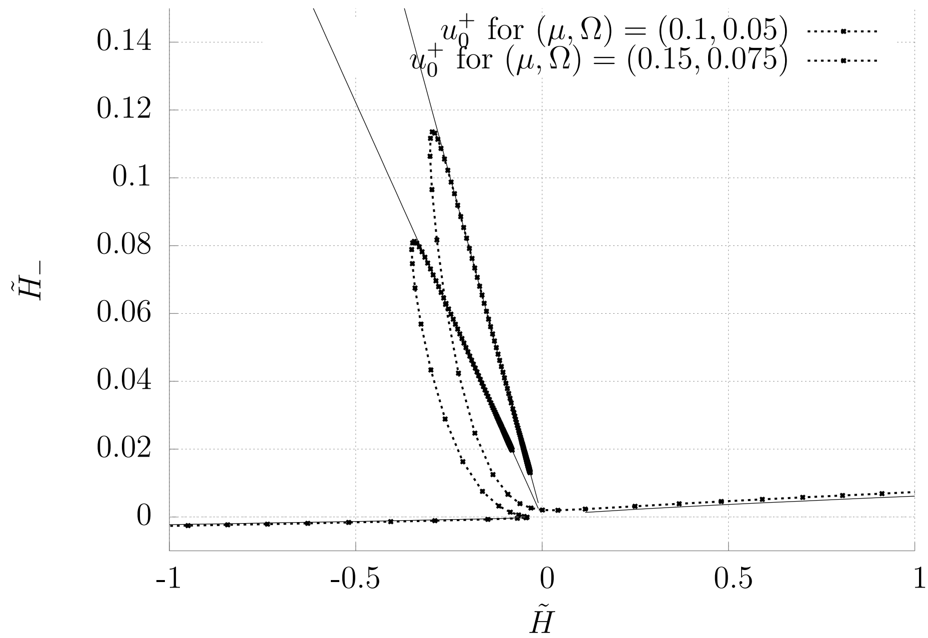

Figure 4: Phase plot of the regular branch u_0^+ for two different values of μ and Ω. Solid lines depict the asymptotic behavior at the horizon and infinity.

which is positive for sufficiently small μ and all $r < r_{21}$. Consequently, $F' > 0$ for $F < 0$, which together with $\lim_{r\to 1} F = 0$ implies that F is nonnegative for all $r < r_{21}$ by the following argument.

Assume there exists some $\tilde{r}$ such that $F(\tilde{r}) < 0$. Then there also exists some $r_0 \in [1, \tilde{r})$ such that $F(r_0) = 0$ and $F(r) < 0$ for all $r \in (r_0, \tilde{r}]$. By integrating the above inequality for F' it follows that

$$F(\tilde{r}) = \int_{r_0}^{\tilde{r}} F' dr > 0, \tag{182}$$

which is a contradiction.

Hence, $F(r) \geq 0$ for all $r < r_{21}$ which by continuity of F implies that $F(r_{21}) \geq 0$. By the definition of F this is equivalent to

$$\left(\tilde{H}_- \tilde{H}^{-1}\right)(r_{21}) \geq -2\mu \frac{1+\mu}{1-\mu}. \tag{183}$$

□

Lemma 3.14. *For $2\Omega = \mu$ and μ sufficiently small, the solution $u = u_0^+$ satisfies $\tilde{H}_-(r_0) = 0$ for some $r_0 \in [r_{21}, r_{12}]$. Furthermore $\tilde{H}(r_0)$ is negative.*

Proof. If $\tilde{H}_-(r_{21})$ vanishes, we are done. If not, we know from the previous lemma that $\tilde{H}(r_{21})$ is negative and $\tilde{H}_-(r_{21})$ is positive. Suppose that $\tilde{H}_-$ has *no* zero in $(r_{21}, r_{12}]$. Then $\tilde{H}_-$ is positive for all $r \in (r_{21}, r_{12}]$. Since $\tilde{A}_{12}$ is nonpositive,

$$\tilde{H}' = \tilde{A}_{11}\tilde{H} + \tilde{A}_{12}\tilde{H}_- \leq \tilde{A}_{11}\tilde{H} \tag{184}$$

holds. Consequently, since $\tilde{H}$ is negative close to r_{21}, it remains negative for all $r \in (r_{21}, r_{12}]$. Furthermore, by integration this leads to

$$\tilde{H}(r) \leq \tilde{H}(r_{21}) \exp\left(\int_{r_{21}}^{r} \tilde{A}_{11} dr\right) \tag{185}$$

for $r \in (r_{21}, r_{12}]$. Since $\tilde{A}_{21}$ is positive, combining this inequality with the differential equation for $\tilde{H}_-$ implies that

$$\tilde{H}_-' \leq \tilde{A}_{21}\tilde{H}(r_{21}) \exp\left(\int_{r_{21}}^{r} \tilde{A}_{11} dr\right) + \tilde{A}_{22}\tilde{H}_-. \tag{186}$$

Recall that the linear first order differential equation $y'(r) = f(r)y(r) + g(r)$ for arbitrary f and g can be solved exactly. Its solution for given initial value $y(r_0)$ is

$$y(r) = e^{\alpha(r)}\left(\int_{r_0}^{r} g\, e^{-\alpha} dr + y(r_0)\right), \tag{187}$$

where

$$\alpha(r) = \int_{r_0}^{r} f dr. \tag{188}$$

Applying this to equation (186) with $f = \tilde{A}_{22}$ and

$$g = \tilde{A}_{21}\tilde{H}(r_{21}) \exp\left(\int_{r_{21}}^{r} \tilde{A}_{11} dr\right) \tag{189}$$

leads to

$$\tilde{H}_-(r)\, e^{-\alpha(r)} \leq \tilde{H}(r_{21}) \int_{r_{21}}^{r} \tilde{A}_{21} \exp\left(\int_{r_{21}}^{r} \tilde{A}_{11} - \tilde{A}_{22} dr\right) dr + \tilde{H}_-(r_{21}), \tag{190}$$

where

$$\alpha(r) = \int_{r_{21}}^{r} \tilde{A}_{22} dr. \tag{191}$$

Since $2\Omega = \mu$, from proposition 3.1 follows that

$$r_{12} \leq 1 + \frac{1}{\sqrt{2}}\left(\frac{5}{4} \cdot \mu^2\right)^{-\frac{1}{4}} = 1 + 5^{-\frac{1}{4}}\sqrt{\mu}^{-1} \leq \sqrt{\mu}^{-1} \tag{192}$$

for sufficiently small μ. Similarly we can also conclude that

$$r_{12} \geq C\sqrt{\mu}^{-1}. \tag{193}$$

For $r \leq r_{12}$ this implies that $\mu^2 r^3 \leq \sqrt{\mu}$. Assume that μ is small enough such that $r_{21} \geq \frac{3}{2}$ holds. Then $\tilde{A}_{11} - \tilde{A}_{22}$ can be bounded from below as

$$\begin{aligned}\tilde{A}_{11} - \tilde{A}_{22} &= 2\frac{(r-1) - \mu^2 r^3}{r(r-1)(1+\mu^2 r^3)} \\ &= \frac{2}{r}\left(\frac{1}{1+\mu^2 r^3} - \frac{\mu^2 r^3}{(r-1)(1+\mu^2 r^3)}\right) \\ &\geq \frac{2}{r}\left(\frac{1}{1+\sqrt{\mu}} - 2\sqrt{\mu}\right).\end{aligned} \tag{194}$$

This leads to

$$\exp\left(\int_{r_{21}}^{r} \tilde{A}_{11} - \tilde{A}_{22} dr\right) \geq \left(\frac{r}{r_{21}}\right)^{\lambda} \tag{195}$$

where $\lambda = 2\left(\frac{1}{1+\sqrt{\mu}} - 2\sqrt{\mu}\right)$. Then

$$\begin{aligned}\tilde{A}_{21} &= \mu^2 \frac{r - 2 + \mu^2 r^3}{(r-1)(1+\mu^2 r^3)} \\ &\geq \frac{\mu^2}{r}\frac{1}{1+\sqrt{\mu}}\left(r - 2 + \mu^2 r^3\right) \\ &\geq C\mu^2 \frac{r - r_{21}}{r},\end{aligned} \tag{196}$$

where the last step uses the fact that $(r - 2 + \mu^2 r^3)$ is zero at r_{21} and its derivative is always greater than one. Combining these estimates with equation (190) for $r \in (r_{21}, r_{12}]$ leads to

$$\tilde{H}_-(r)\, e^{-\alpha(r)} \leq C\mu^2 \tilde{H}(r_{21}) \int_{r_{21}}^{r} \left(\frac{r}{r_{21}}\right)^{\lambda} - \left(\frac{r}{r_{21}}\right)^{\lambda-1} dr + \tilde{H}_-(r_{21}). \tag{197}$$

Evaluating the integral in this expression at $r = r_{12}$ leads to

$$\begin{aligned}&\int_{r_{21}}^{r_{12}} \left(\frac{r}{r_{21}}\right)^{\lambda} - \left(\frac{r}{r_{21}}\right)^{\lambda-1} dr \\ &= r_{21}\left[\frac{1}{\lambda+1}\left(\frac{r}{r_{21}}\right)^{\lambda+1} - \frac{1}{\lambda}\left(\frac{r}{r_{21}}\right)^{\lambda}\right]_{r_{21}}^{r_{12}} \\ &= r_{21}\left(\frac{r_{12}}{r_{21}}\right)^{\lambda+1}\frac{1}{\lambda+1}\left[1 - \frac{\lambda+1}{\lambda}\left(\frac{r_{21}}{r_{12}}\right)\right] + \frac{r_{21}}{\lambda(\lambda+1)}.\end{aligned} \tag{198}$$

As stated above, we know that for μ sufficiently small

$$C\sqrt{\mu}^{-1} \leq r_{12} \leq \sqrt{\mu}^{-1} \tag{199}$$

holds. Furthermore we can assume that $1 < \lambda < 2$ and $\frac{3}{2} \leq r_{21} \leq 2$ and consequently

$$\begin{aligned} &\int_{r_{21}}^{r_{12}} \left(\frac{r}{r_{21}}\right)^{\lambda} - \left(\frac{r}{r_{21}}\right)^{\lambda-1} dr \\ \geq\ & C\left(\frac{r_{12}}{r_{21}}\right)^{\lambda+1} [1 - C\sqrt{\mu}] \\ \geq\ & C\mu^{-\frac{1}{2}(\lambda+1)} \end{aligned} \tag{200}$$

We can now combine this with inequality (197) and arrive at

$$\tilde{H}_-(r_{12})\, e^{-\alpha} \leq C\mu^{2-\frac{1}{2}(\lambda+1)}\tilde{H}(r_{21}) + \tilde{H}_-(r_{21})\,. \tag{201}$$

Since $\tilde{H}(r_{21})$ is negative, lemma 3.13 implies that $\tilde{H}_-(r_{21}) \leq -C\mu\tilde{H}(r_{21})$ and consequently

$$\tilde{H}_-(r_{12})\, e^{-\alpha} \leq C\tilde{H}(r_{21})\mu\left(\mu^{1-\frac{1}{2}(\lambda+1)} - C\right). \tag{202}$$

Using $1 - \frac{1}{2}(\lambda+1) < 0$ and assuming that μ is sufficiently small this implies

$$\tilde{H}_-(r_{12})\, e^{-\alpha} < 0, \tag{203}$$

which is a contradiction. Consequently, $\tilde{H}_-$ has a zero in $(r_{21}, r_{12}]$. □

Lemma 3.15. *For $2\Omega = \mu$ and μ sufficiently small, for the solution $u = u_0^+$ both $\tilde{H}(r_{12})$ and $\tilde{H}_-(r_{12})$ are negative.*

Proof. Using lemma 3.14 we can conclude that there exists some smallest $r_0 \in (r_{21}, r_{12}]$ such that $\tilde{H}_-(r_0) = 0$. Since the equation is linear, we can assume without loss of generality that $\tilde{H}(r_0) = -1$. We now want to prove that $\tilde{H}$ remains negative on $(r_0, r_{12}]$. Suppose there exists some smallest $\bar{r} \in (r_0, r_{12}]$ such that $\tilde{H}(\bar{r}) = 0$. Then $\tilde{H}$ is negative for $r \in (r_0, \bar{r})$. Note that $\tilde{A}_{21}$ and $\tilde{A}_{22}$ are positive on the same interval. Therefore $\tilde{H}_-'(r_0) = -\tilde{A}_{21}(r_0)$ is negative and since

$$\tilde{H}_-' = \tilde{A}_{21}\tilde{H} + \tilde{A}_{22}\tilde{H}_- \leq \tilde{A}_{22}\tilde{H}_- \tag{204}$$

holds, $\tilde{H}_-$ is negative for all $r \in (r_0, \bar{r}]$. This in turn implies that

$$\tilde{H}' = \tilde{A}_{11}\tilde{H} + \tilde{A}_{12}\tilde{H}_- \tag{205}$$

is positive, since $\tilde{A}_{11}$ and $\tilde{A}_{12}$ are both negative. Therefore $\tilde{H}(r) > -1$ for all $r \in (r_0, \bar{r})$.

Since $r \leq r_{12} \leq \sqrt{\mu}^{-1}$ holds, $\mu^2 r^3 \leq \sqrt{\mu}$ we can bound $\tilde{A}_{21}$ from above by

$$\tilde{A}_{21} = \mu^2 \frac{r - 2 + \mu^2 r^3}{(r-1)(1+\mu^2 r^3)} \leq C\mu^2. \tag{206}$$

Combining this with $\tilde{H}'_- = \tilde{A}_{21}\tilde{H} + \tilde{A}_{22}\tilde{H}_- \geq -\tilde{A}_{21}$ leads to

$$\tilde{H}_-(r) \geq -C\mu^2 (r - r_0) \geq -C\mu^{\frac{3}{2}} \tag{207}$$

for all $r \in (r_0, \bar{r})$. Combining this with the differential equation for $\tilde{H}'$ we arrive at

$$\tilde{H}' \leq \tilde{A}_{11}\tilde{H} - C\mu^{\frac{3}{2}}\tilde{A}_{12}. \tag{208}$$

As in the proof of the previous lemma, we can integrate this inequality from r_0 to $\bar{r}$ to get

$$\tilde{H}(\bar{r}) \leq e^{\alpha}\left(\tilde{H}(r_0) - C\mu^{\frac{3}{2}}\int_{r_0}^{\bar{r}} \tilde{A}_{12} e^{-\alpha} dr\right) \tag{209}$$

where

$$\alpha(r) = \int_{r_0}^{r} \tilde{A}_{11}. \tag{210}$$

$\tilde{A}_{11}$ is negative and can be bounded from below by

$$\begin{aligned} \tilde{A}_{11} &= -\frac{2(r-1) + \mu^2 r^3 (r+1)}{2r(r-1)(1+\mu^2 r^3)} \\ &= -\frac{1}{r}\left[\frac{1}{1+\mu^2 r^3} + \frac{\mu^2 r^3 (r+1)}{2(r-1)(1+\mu^2 r^3)}\right] \\ &\geq -\frac{3}{2}\frac{1}{r} \end{aligned} \tag{211}$$

for sufficiently small μ, since $\mu^2 r^3 \leq \sqrt{\mu}$. Consequently, $\exp(-\alpha)$ can be bounded from above by

$$\exp(-\alpha(r)) \leq \left(\frac{r}{r_0}\right)^{\frac{3}{2}} \leq \mu^{-\frac{3}{4}}. \tag{212}$$

Using $\tilde{A}_{12} \geq -C$ for $r \in (r_{21}, r_{12})$ we can combine this with equation (209) to

$$\tilde{H}(\bar{r}) \leq e^{\alpha}\left(\tilde{H}(r_0) + C\mu^{\frac{3}{4}}(\bar{r} - r_0)\right). \tag{213}$$

Since $\bar{r} \leq \sqrt{\mu}^{-1}$ and using $\tilde{H}(r_0) = -1$ this implies

$$\tilde{H}(\bar{r}) \leq e^{\alpha}\left(-1 + C\mu^{\frac{1}{4}}\right) < 0 \tag{214}$$

for sufficiently small μ, which contradicts $\tilde{H}(\bar{r}) = 0$. Thus, the zero $\bar{r}$ does not exist and $\tilde{H}(r)$ is negative for all $r \in (r_0, r_{12}]$. As argued before in this proof, this also implies, that $\tilde{H}_-$ is negative on the same interval. □

Lemma 3.16. *For $2\Omega = \mu$ and μ sufficiently small, the solution $u = u_0^+$ satisfies $a_\infty < 0$.*

Proof. From the previous lemma follows that at r_{12} both $\tilde{H}$ and $\tilde{H}_-$ are negative. For $r > r_{12}$, both $\tilde{A}_{12}$ and $\tilde{A}_{21}$ are positive. This implies that both $\tilde{H}$ and $\tilde{H}_-$ are negative for all $r > r_{12}$. Assume that this was not the case. Then there exists a smallest $r_0 > r_{12}$ such that either $\tilde{H}$, $\tilde{H}_-$ or both are zero. The latter can be easily excluded, since if they were both zero somewhere, they would have to vanish everywhere. Assume that $\tilde{H}(r_0) = 0$. Then using the differential equation this implies that

$$\tilde{H}'(r_0) = \tilde{A}_{12}\tilde{H}_-(r_0) \tag{215}$$

is negative. By continuity this also holds on some open interval of r_0. However, this contradicts $H(r_0) = 0$ since $\tilde{H}(r_0 - \epsilon)$ is negative for some small ϵ. A analogous argument shows that $\tilde{H}_-$ cannot become zero, either. Consequently, both $\tilde{H}$ and $\tilde{H}_-$ are negative for all $r > r_{12}$.

As already discussed in the proof of lemma 3.10, $a_\infty = 0$ implies that $\tilde{H}$ and $\tilde{H}_-$ have opposite sign for large r. Similarly, a positive a_∞ implies that both $\tilde{H}$ and $\tilde{H}_-$ are positive for large r. Consequently, a_∞ is negative. □

This result together with lemma 3.10 establishes the following theorem.

Theorem 3.17. *There exists a closed subset of $\{(\mu, \Omega) : \mu > 0$ and $\Omega > 0\}$ such that (45) has a unique one parameter family of solutions which are regular at both infinity and the horizon.*

Proof. From lemma 3.10 and lemma 3.16 follows the existence of two points where the regular solution u_0^+ has positive and negative coefficient a_∞, respectively. By continuity of Φ it follows that both of these points have an open neighborhood where those inequalities are satisfied. Then, the continuity of Φ in μ and Ω implies the existence of a closed subset of the space of parameters where u_0^+ satisfies $a_\infty = 0$. □

The previous theorem establishes the existence of regular solutions. In particular it shows that there exists some closed subset of the parameter space admitting regular solutions. However, several features of figure 3 are not covered by this result. One important feature is the existence of some critical μ_c such that there are no regular solutions for $\mu > \mu_c$. The quantity μ is a frequency in the separation of variables ansatz (38). Since the compact extra dimension is periodically identified, this implies that this instability only occurs for black strings which are longer than a certain threshold length $L_c = \frac{2\pi}{\mu_c}$. This shows that perturbations of the form (38) are not admissible for $L < L_c$ and suggests that short black strings are stable. In the following the existence of a critical μ_c will be proved.

Lemma 3.18. *There exists some positive constant μ_c such that the solution u_0^+ has positive coefficient a_∞ for all $\mu > \mu_c$.*

Proof. Let $F = -\left(\mu^2 + 2\Omega\right)\tilde{H} - \left(\frac{1}{2} - \Omega\right)\tilde{H}_-$. For the solution u_0^+ this quantity satisfies $\lim_{r\to 1} F = 0$, as can be seen from the asymptotic behavior at the horizon. Furthermore, F satisfies

$$\begin{aligned} F' = & -\left[\left(\mu^2 + 2\Omega\right)\tilde{A}_{11} + \left(\frac{1}{2} - \Omega\right)\tilde{A}_{21}\right]\tilde{H} \\ & -\left[\left(\mu^2 + 2\Omega\right)\tilde{A}_{12} + \left(\frac{1}{2} - \Omega\right)\tilde{A}_{22}\right]\tilde{H}_- \end{aligned} \tag{216}$$

and for $F = 0$ this simplifies to

$$F'|_{F=0} = \frac{\tilde{H}_- G}{r\left(1 + \mu^2 r^3\right)}, \tag{217}$$

where G is given by

$$G = G_0 + \mu^2 G_2 + \mu^4 G_4 - r^5 \mu^6. \tag{218}$$

The quantities G_i are polynomials in r and Ω that do not depend on μ,

$$\begin{aligned} G_0 =& -4\Omega^2 + 2\Omega - 4\Omega^3 r^2 (r+1) - 4\Omega^4 r^2 \left(r^2 - 1\right)(r+1) \\ G_2 =& 1 + \frac{r}{2} + \Omega^2 r \left(2 - 4r^2 - 4r^4\right) - 4\Omega^3 r^4 (r+1) \\ & + \Omega\left(r(r+1)(r-2) - 2\right) \\ G_4 =& -\Omega^2 r^4 (1+r) + \frac{1}{4} r^2 (3r-1) - 2\Omega\left(2r^2 + 1\right) r^3. \end{aligned} \tag{219}$$

By dropping most of the negative terms in the G_i, it follows that

$$\begin{aligned} G &\leq 2\Omega + \mu^2 \left(1 + \frac{r}{2} + \Omega \left(r\left(r+1\right)\left(r-2\right) - 2\right)\right) \\ &\quad + \mu^4 \left(\frac{1}{4} r^2 \left(3r-1\right) - 2\Omega \left(2r^2+1\right) r^3\right) - r^5 \mu^6 \\ &= 2\Omega \left(1-\mu^2\right) + \mu^2 \left(1 + \frac{r}{2} - \Omega \left(r^2 + 2r\right)\right) \\ &\quad + \mu^4 \left(\frac{1}{4} r^2 \left(3r-1\right) - 4\Omega r^5\right) + r^3 \Omega \left(\mu^2 - 2\mu^4\right) - \mu^6 r^5. \end{aligned} \tag{220}$$

If μ is large enough, all terms on the r.h.s. of the above inequality which contain Ω are negative. Furthermore the highest power of r is $-\mu^6 r^5$ and therefore, assuming that μ is large enough, G is negative for positive Ω and $r > 1$. Consequently, if $\tilde{H}_-$ is positive and F vanishes, then F' is negative. Hence, if F and $\tilde{H}_-$ are both negative, F cannot change sign. Furthermore, using the asymptotic behavior (94) of u_0^+ at the horizon, it follows that F behaves like

$$F = F_0 \left(r-1\right)^{\Omega} + F_1 \left(r-1\right)^{\Omega+1} + \mathcal{O}\left(\left(r-1\right)^{\Omega+2}\right) \tag{221}$$

where $F_0 = 0$ and

$$\begin{aligned} F_1 = &-\frac{1}{2\Omega^2 + \Omega} \left(8\Omega^5 + 4\Omega^4 \left(\mu^2 - 1\right) + \Omega^3 \left(11 + 2\mu^2\right)\right. \\ &\left. + \Omega^2 \left(10 + 5\mu^2 + 2\mu^4\right) + \Omega \left(\frac{13}{2}\mu^2 - 3\right) + \frac{\mu^4}{2} - 2\mu^2\right). \end{aligned} \tag{222}$$

This shows that F is negative for r close to 1, μ sufficiently big and all positive Ω. Close to the horizon the regular solution u_0^+ satisfies

$$u_0^+ = a_0^+ \left(r-1\right)^{\Omega} + a_1^+ \left(r-1\right)^{\Omega+1} + \mathcal{O}\left(\left(r-1\right)^{\Omega+2}\right). \tag{223}$$

Consequently, using the values of a_0^+ and a_1^+, it follows that $\tilde{H}_-$ is positive close to $r = 1$. In order to show that F is negative for all r it remains to show that $\tilde{H}_-$ is always positive. Assume there exists some r_0 such that $\tilde{H}_- \left(r_0\right)$ vanishes and that $\tilde{H}_-(r)$ is positive for all $r < r_0$. As shown before, this implies that F is negative for all $r < r_0$ and consequently that $F\left(r_0\right) \leq 0$. Using the definition of F this implies that $\tilde{H}\left(r_0\right)$ is nonnegative. The case $\tilde{H}\left(r_0\right) = 0$ is excluded, since u^+ is not the trivial solution. We therefore assume that $\tilde{H}\left(r_0\right)$ is positive. Using the differential equation and the fact that for μ sufficiently large, $\tilde{A}_{21}$ is positive, it follows that $\tilde{H}_-'\left(r_0\right)$ is positive.

Since $\tilde{H}'_-$ is continuous, this also holds for all $r \in (r_0 - \epsilon, r_0]$ for some ϵ, which by integrating leads to a contradiction to $\tilde{H}_-(r_0) = 0$. Consequently, the zero r_0 cannot exist and therefore $\tilde{H}_-$ is positive and F is negative for all $r > 1$.

We will now show that this implies that a_∞ is positive. Assume that a_∞ is negative. Using the asymptotic behavior at infinity (138) and equation (139) implies that for sufficiently large r the solution has positive $\tilde{H}_-$, which is a contradiction. Assume that a_∞ vanishes. Then b_∞ is either positive or negative. If b_∞ is negative, $\tilde{H}_-$ would be negative for sufficiently large r, which is not the case. If b_∞ is positive, the solution satisfies

$$\begin{aligned} F = {} & e^{-r\sqrt{\Omega^2+\mu^2}} r^{\gamma_-} \, b_\infty \\ & \cdot \left[\left(\mu^2 + 2\Omega\right) \sqrt{\Omega^2+\mu^2} - \left(\frac{1}{2} - \Omega\right) \mu^2 + \mathcal{O}\left(r^{-1}\right) \right] \end{aligned} \tag{224}$$

for sufficiently large r. For μ sufficiently big this implies that F must be positive for large r, which is a contradiction, too. □

The above lemma implies that for $\mu > \mu_c$ no regular solutions exist.

Theorem 3.19. *There exists some positive constant μ_c such that for $\mu > \mu_c$ equations (53) do not admit solutions which are regular both at the horizon and infinity.*

Remark. *When considering strings with compact extra dimensions, this also proves the existence of the corresponding threshold length $L_c = \frac{2\pi}{\mu_c}$.*

There are some other aspects of figure 3 that can be proven. The following result and its proof are essentially identical to an argument that can be found in [18]. There — under the assumption that $\Omega > 1$ — it is shown that the equations do not admit any regular solutions. While the numerical results suggest that this is true, the assumption $\Omega > 1$ is not sufficient for the argument to be correct. We therefore present the same argument here, albeit with a corrected assumption. Apart from that, the proof is virtually identical to the one given in [18].

Lemma 3.20. *If $\Omega > \sqrt{\frac{5}{4}}$ holds, the solution u_0^+ has positive a_∞.*

Proof. Under the assumptions, the second order equation (48) is equivalent to the first order system. We can write equation (48) as

$$A(\Omega, \mu, r) \frac{\mathrm{d}^2 H}{\mathrm{d} r^2} + B(\Omega, \mu, r) \frac{\mathrm{d} H}{\mathrm{d} r} + C(\Omega, \mu, r) H = 0. \tag{225}$$

It is evident, that given the assumption $\Omega > \sqrt{\frac{5}{4}}$, A is negative for all $\mu > 0$ and all $r > 1$. Similarly, C is positive everywhere. The asymptotic behavior of the quantity H corresponding to the regular solution u_0^+ is

$$H = \left(\Omega - \frac{1}{2}\right)(r-1)^{\Omega-1} + \mathcal{O}\left((r-1)^{\Omega}\right) \tag{226}$$

at the horizon. Consequently, on some interval $(1, r_0)$ H and H' are both positive. We will now prove that $a_\infty \leq 0$ will lead to a contradiction. In both cases, i.e. $a_\infty = 0$ and $a_\infty < 0$, H must have a maximum for some $R > r_0$, since H either converges to zero or H diverges to $-\infty$ for $r \to \infty$. Consequently, $H'(R) = 0$ and $H''(R) \leq 0$. Since C is positive, $H''(R) = 0$ implies $H(R) = 0$, which is a contradiction, since $u_0^+ \neq 0$. If $H''(R) < 0$, the second order equation for H implies

$$H(R)\, C < 0 \tag{227}$$

which is a contradiction to $H(R) > 0$. This implies that $a_\infty > 0$. □

4 Conclusion

The aim of this thesis has been to improve the understanding of the instability of uniform black strings. Since the discovery of this instability, one crucial missing step had been a complete proof of linear instability. We were able to fill this gap and replace the numerical existence result of Gregory and Laflamme by a rigorous mathematical proof. One last missing step towards proving linear instability is to show that solutions to the reduced system of Gregory and Laflamme correspond to solutions of the linearized Einstein equations. Although this is expected to be true, it has not been fully answered yet, and therefore remains an important question in this context.

There are several directions in which the results of this thesis could be extended. The numerical investigations done by R. Gregory and R. Laflamme [19] can be applied to *charged* uniform strings in any dimension greater than four. The results in this thesis are limited to uncharged strings in five dimensions. However, as has already been observed by Gregory and Laflamme, the qualitative behavior of the linear instability does not change when considering charges and higher dimensions. We therefore expect that the results presented here could be extended to cover charged strings in all dimensions.

Another very important, albeit ambitous direction for future research is the question of nonlinear instability. Given a better understanding of the mapping properties of the constraint equations it might be possible to establish the linearization stability of Einstein's equations near the uniform black string solution (see chapter 2.4). Using the existence of the unstable linear solutions proven here, this could be used to prove the existence of the nonuniform string branch. This would constitute an important starting point for better understanding of the evolution of nonuniform black strings (see chapter 2.9).

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