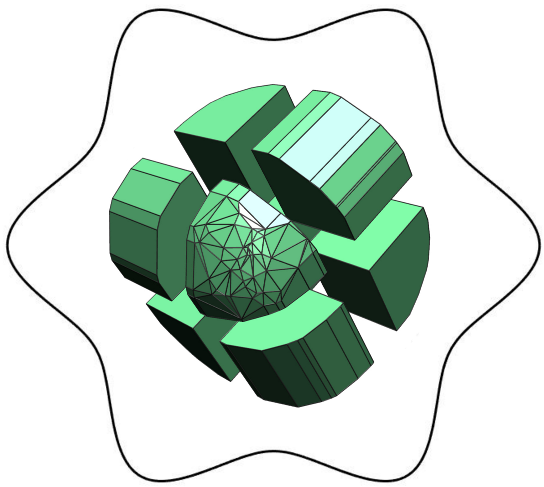
Fachbereich Mathematik und Informatik
der Freien Universität Berlin

Double posets and real invariant varieties

Two interactions between combinatorics and geometry



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Introduction

Mathematics is often most intriguing when we leave the boarders of classical disciplines and instead connect different areas. Such correspondences will help us to shed new light on mathematical questions, as they allow the translation of problems and techniques between different mathematical languages.

The topic of this thesis is, simply said, located in *combinatorics*, *algebra* and *discrete and real algebraic geometry*. The thesis consists of two parts. In Part I, we study polytopes associated to posets and double posets. These polytopes are very combinatorial in nature and much of their geometry can be described in terms of the combinatorics of the underlying (double) poset. Our main objects of study in Part II are real varieties invariant under the action of a finite reflection group and, in particular, relations to the discrete geometry of the associated reflection arrangement. In the following, we will give a concise overview over both topics. Rigorous definitions will be given later on.

A finite *partially ordered set* (or *poset*, for short) is a finite set P together with a reflexive, transitive and anti-symmetric relation $\preceq$. The notion of partial order pervades all of mathematics and the enumerative and algebraic combinatorics of posets is underlying in computations in virtually all areas. Stanley [73] studied two convex polytopes for every poset P which, in quite different ways, geometrically capture combinatorial properties of P. These "poset polytopes" are very natural objects that appear in a variety of contexts in combinatorics and beyond (see for instance [2, 26, 55, 71, 80]).

The *order polytope* $\mathcal{O}(P)$ is the set of all order-preserving functions into the interval $[0,1]$. That is, all functions $f : P \to \mathbb{R}$ such that

$$0 \ \leq \ f(a) \ \leq \ f(b) \ \leq \ 1$$

for all $a, b \in P$ with $a \preceq b$. Many properties of P are encoded in the geometry of $\mathcal{O}(P)$. Facets of $\mathcal{O}(P)$ correspond to *cover relations* and *maximal* and *minimal* elements in P and the vertices of $\mathcal{O}(P)$ are in bijection with *order ideals* in P. More generally, faces of arbitrary dimension relate to *quotients* of P, that is, posets obtained by consecutively contracting cover relations in P. In particular, faces of order polytopes are again order polytopes. Stanley [73] moreover describes a canonical unimodular triangulation of $\mathcal{O}(P)$ whose simplices arise from chains of order ideals. Maximal simplices correspond to *linear extensions* of P, which also yields a simple combinatorial formula for the volume of $\mathcal{O}(P)$. This bridge between geometry and combinatorics can, for example, be used to show that computing volume is *hard* (cf. [11]) and, conversely, geometric inequalities can be used on partially ordered sets (see [56, 73]). Generalizing the volume formula, even the *Ehrhart polynomial* $\mathrm{Ehr}_{\mathcal{O}(P)}(n) = |n\mathcal{O}(P) \cap \mathbb{Z}^P|$ can be described in terms of P: Up to a shift, it coincides with the *order polynomial* $\Omega_P(n)$ of P (see [74, Sec. 3.15.2]).

The *chain polytope* $\mathcal{C}(P)$ is the collection of functions $g : P \to \mathbb{R}$ such that

$$0 \ \leq \ g(a_1) + g(a_2) + \cdots + g(a_k) \ \leq \ 1$$

for all *chains* $a_1 \prec a_2 \prec \cdots \prec a_k$ in P. In contrast to the order polytope, $\mathcal{C}(P)$ does not determine P. In fact, $\mathcal{C}(P)$ only depends on the *comparability graph* of P and bears strong relations to *stable set polytopes* of perfect graphs. Order and chain polytopes are closely related. Stanley defined a piecewise linear homeomorphism Φ_P between $\mathcal{O}(P)$ and $\mathcal{C}(P)$ whose domains of linearity are exactly the simplices of the canonical triangulation. This *transfer map* takes the canonical triangulation of $\mathcal{O}(P)$ to a triangulation of $\mathcal{C}(P)$ and, since Φ_P is lattice-preserving, it follows that $\mathrm{Ehr}_{\mathcal{O}(P)}(n) = \mathrm{Ehr}_{\mathcal{C}(P)}(n)$. In particular, $\mathrm{vol}(\mathcal{O}(P)) = \mathrm{vol}(\mathcal{C}(P))$, which, on the combinatorial side, shows that the number of linear extensions of P only depends on its comparability graph.

A *double poset* $\mathbf{P}$ is a triple consisting of a finite ground set P and two partial order relations $\preceq_+$ and $\preceq_-$ on P. We will write $P_+ = (P, \preceq_+)$ and $P_- = (P, \preceq_-)$ to refer to the two underlying posets. Double posets were introduced by Malvenuto and Reutenauer [60], generalizing Stanley's labelled posets [72]. The combinatorial study of general double posets gained momentum in recent years with a focus on algebraic aspects (see for example [24, 25]). Our goal is to build a bridge to geometry by introducing "two double poset polytopes" that, like the chain- and the order polytope, geometrically reflect the combinatorial properties of double posets and, in particular, the interaction between the two partial orders. To any double poset $\mathbf{P}$, we associate the *double order polytope*

$$\mathbb{T}\mathcal{O}(\mathbf{P}) \ := \ \mathrm{conv}\big\{(2\mathcal{O}(P_+) \times \{1\}) \cup (-2\mathcal{O}(P_-) \times \{-1\})\big\}.$$

In other words, we embed $2\mathcal{O}(P_+)$ and $-2\mathcal{O}(P_-)$ at heights 1 and -1 in $\mathbb{R}^P \times \mathbb{R}$, respectively, and take their convex hull. Analogously, the *double chain polytope* is

$$\mathbb{T}\mathcal{C}(\mathbf{P}) \ := \ \mathrm{conv}\big\{(2\mathcal{C}(P_+) \times \{1\}) \cup (-2\mathcal{C}(P_-) \times \{-1\})\big\}.$$

Both polytopes are full-dimensional and the vertices of $\mathbb{T}\mathcal{O}(\mathbf{P})$ (resp. $\mathbb{T}\mathcal{C}(\mathbf{P})$) are in bijection with filters (resp. antichains) in P_+ and P_-. Analogous to the case of ordinary poset polytopes, we aim for a combinatorial description of faces, triangulations, volume and Ehrhart polynomials of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and $\mathbb{T}\mathcal{C}(\mathbf{P})$. Also, it is natural to ask whether Stanley's transfer map extends to the case of double posets (see Figure 1).

Our main results are the following. After treating basics regarding posets and double posets in Section 1.1 and order polytopes in Section 1.2, Section 1.3 is devoted to studying the facet structure of double order polytopes. We show that facets of $\mathbb{T}\mathcal{O}(\mathbf{P})$ relate to *alternating chains* and *alternating cycles* in the underlying double poset $\mathbf{P}$. For the important case of *compatible* double posets, that is, double posets where P_+ and P_- have a common linear extension, this yields a complete description of the facets of $\mathbb{T}\mathcal{O}(\mathbf{P})$ in terms of the combinatorics of $\mathbf{P}$. Furthermore, we determine which of these polytopes are 2-*level*, a geometric property which arises, for instance, in the context of centrally symmetric polytopes, optimization and statistics [36, 37,

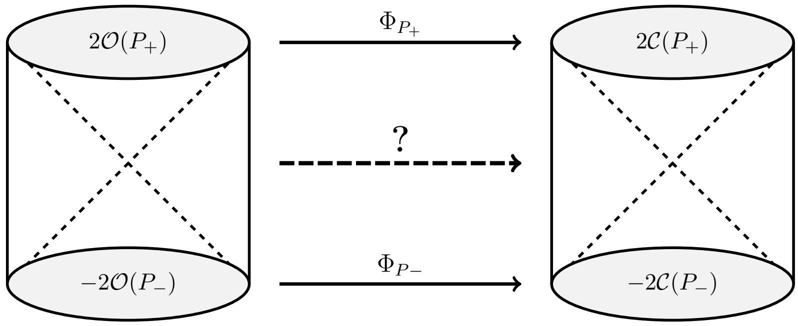

Figure 1: Stanley's transfer map relates the grey top and bottom facets. Does this extend to the whole double poset polytopes?

41, 68, 69, 78]. Moreover, using polars we prove a surprising connection between certain double order polytopes and *valuation polytopes*, another class of polytopes associated to distributive lattices, that was introduced by Geissinger [32].

Chapter 2 is again devoted to the study of double order polytopes, but we take a more algebraic perspective. Every integral polytope $\mathcal{P}$ has an associated *toric ideal* $I_\mathcal{P}$, an algebraic object that captures much of the geometry of $\mathcal{P}$. For example, Sturmfels' correspondence [77] relates *initial ideals* of $I_\mathcal{P}$ to *regular triangulations* of $\mathcal{P}$. We recall some of these well-known basics in Section 2.1. In Section 2.2, we apply this machinery to double order polytopes. The toric ideals corresponding to ordinary order polytopes are closely related to *Hibi rings*, which have been introduced in the context of algebras with straightening laws. Analogously, for the case of double order polytopes, we introduce *double Hibi rings*. For compatible $\mathbf{P}$, we describe a canonical Gröbner basis of the toric ideals of $\mathbb{T}\mathcal{O}(\mathbf{P})$, which can be determined directly from the combinatorics of $\mathbf{P}$. On the geometric side, this yields a regular, flag and unimodular triangulation of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and, in particular, a simple formula for its volume. We also obtain a description of the complete face lattice of $\mathbb{T}\mathcal{O}(\mathbf{P})$ in terms of the underlying *Birkhoff lattices* of P_+ and P_-.

In Chapter 3, we finally turn to the study of double chain polytopes. First note that the ordinary chain polytope $\mathcal{C}(P) \subset \mathbb{R}^P_{\geq 0}$ has the property that for any $f, g \in \mathbb{R}^P$

$$g \in \mathcal{C}(P) \text{ and } 0 \leq f(p) \leq g(p) \text{ for all } p \in P \text{ implies } f \in \mathcal{C}(P).$$

Polytopes with this property are called *anti-blocking polytopes* and were introduced by Fulkerson [30] in the context of combinatorial optimization. It turns out that much of the theory for double chain polytopes can be developed in this more general setting. Basics and relations between chain polytopes, *stable set polytopes* and anti-blocking polytopes are treated in Section 3.1. In Section 3.2, we study Cayley sums of the form

$$\mathcal{P}_1 \boxminus \mathcal{P}_2 := \operatorname{conv}(\mathcal{P}_1 \times \{1\} \cup -\mathcal{P}_2 \times \{-1\}) \subset \mathbb{R}^n \times \mathbb{R}$$

for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$. This subsumes double chain polytopes as well as *Hansen polytopes* [41]. We give an explicit description of the polar of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ in terms of $\mathcal{P}_1$ and $\mathcal{P}_2$, which specializes to a combinatorial description of the facets of $\mathbb{T}\mathcal{C}(\mathbf{P})$ for arbitrary double posets $\mathbf{P}$. For some cases, this yields a connection between the number of facets of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and $\mathbb{T}\mathcal{C}(\mathbf{P})$, which relates to work by Hibi and Li [47] for ordinary poset polytopes. Moreover, we classify the 2-level polytopes among the Cayley sums $\mathcal{P}_1 \boxminus \mathcal{P}_2$, which are precisely Hansen polytopes associated to perfect graphs. Section 3.3 is devoted to subdivisions and triangulations. We describe a canonical subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$, which can in particular be used to lift two triangulations of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, to a triangulation of $\mathcal{P}_1 \boxminus \mathcal{P}_2$. Returning to the case of double chain polytopes, for an arbitrary double poset $\mathbf{P}$, we are able to extend Stanley's triangulation of the bottom and top facets $\mathcal{C}(P_+)$ and $\mathcal{C}(P_-)$ to a triangulation of $\mathbb{T}\mathcal{C}(\mathbf{P})$. This triangulation is regular, flag and unimodular. On the algebraic side, it yields a squarefree quadratic Gröbner basis of the associated toric ideal. Moreover, we obtain a simple volume formula for $\mathcal{P}_1 \boxminus \mathcal{P}_2$, which in the special case of double chain polytopes only depends on the combinatorics of $\mathbf{P}$. Whenever $\mathbf{P}$ is compatible, it follows that the triangulations of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and $\mathbb{T}\mathcal{C}(\mathbf{P})$ are combinatorially equivalent and hence $\mathbb{T}\mathcal{O}(\mathbf{P})$ and $\mathbb{T}\mathcal{C}(\mathbf{P})$ have the same volume. A reason for this is given in Section 3.4: For compatible $\mathbf{P}$, we solve the problem from Figure 1 and define a lattice-preserving piecewise linear homeomorphism between $\mathbb{T}\mathcal{C}(\mathbf{P})$ and $\mathbb{T}\mathcal{O}(\mathbf{P})$ which takes the triangulation of $\mathbb{T}\mathcal{C}(\mathbf{P})$ to the triangulation of $\mathbb{T}\mathcal{O}(\mathbf{P})$ from Section 2.2. This, in particular, implies that $\mathbb{T}\mathcal{C}(\mathbf{P})$ and $\mathbb{T}\mathcal{O}(\mathbf{P})$ even have the same *Ehrhart polynomial*. For this purpose it seems easier to work with double chain polytopes and, more generally, we give an explicit description of the Ehrhart quasi-polynomial of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ for rational anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$. Yet again, this specializes to a combinatorial formula for the Ehrhart polynomial of $\mathbb{T}\mathcal{C}(\mathbf{P})$, which also holds true for $\mathbb{T}\mathcal{O}(\mathbf{P})$ whenever $\mathbf{P}$ is compatible.

The main geometric objects in Part II are the following. A *real variety* $X \subseteq \mathbb{R}^n$ is the set of real points simultaneously satisfying a system of polynomial equations with real coefficients, that is,

$$X \ = \ \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m) \ := \ \{\mathbf{p} \in \mathbb{R}^n : f_1(\mathbf{p}) = f_2(\mathbf{p}) = \cdots = f_m(\mathbf{p}) = 0\},$$

for some $f_1, \dots, f_m \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_n]$. A fundamental problem of real algebraic geometry is to certify when a real variety is nonempty (see for example [5]). Timofte [79] studied the special case of real varieties invariant under the action of the symmetric group $\mathfrak{S}_n$, in other words, varieties that are invariant under permuting coordinates. Every $\mathfrak{S}_n$-invariant variety can be defined in terms of *symmetric polynomials*, that is, polynomials $f \in \mathbb{R}[\mathbf{x}]$ such that $f(x_1, x_2, \dots, x_n) = f(x_{\tau(1)}, x_{\tau(2)}, \dots, x_{\tau(n)})$ for every $\tau \in \mathfrak{S}_n$. Important examples of symmetric polynomials are the *elementary symmetric polynomials*

$$e_k(\mathbf{x}) \ := \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_k}$$

for $k \in [n] := \{1, \dots, n\}$. In fact, by the fundamental theorem of symmetric polynomials, every symmetric f can uniquely be written as a polynomial in $e_1, \dots, e_n$. Phrased differently, we have $f(\mathbf{x}) = F(e_1(\mathbf{x}), \dots, e_n(\mathbf{x}))$ for some $F \in \mathbb{R}[y_1, \dots, y_n]$. We call f *k-sparse* if F only depends on the variables $y_1, \dots, y_k$ and an $\mathfrak{S}_n$-invariant variety X is called *k-sparse* if $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ for k-sparse symmetric polynomials $f_1, \dots, f_m$.

Theorem 1 ([79]). *Let $X \subseteq \mathbb{R}^n$ be a nonempty k-sparse $\mathfrak{S}_n$-invariant variety. Then X contains a point $\mathbf{p} = (p_1, \ldots, p_n)$ with at most k distinct coordinates, that is, we have $|\{p_1, \ldots, p_n\}| \leq k$.*

This result can be interpreted in a more geometric way. If we regard $\mathfrak{S}_n$ as a group of orthogonal transformations on $\mathbb{R}^n$, it is generated by the reflections fixing pointwise the hyperplanes $H_{ij} := \{\mathbf{p} : p_i = p_j\}$ for $1 \leq i < j \leq n$. Consider the associated collection of hyperplanes $\mathcal{H} := \{H_{ij} : 1 \leq i < j \leq n\}$. For $0 \leq k \leq n$, we write $\mathcal{H}_k$ for the set of points which lie on $n-k$ linearly independent hyperplanes in $\mathcal{H}$. Timofte's theorem now states that for $k \in [n]$, every nonempty k-sparse variety X intersects $\mathcal{H}_k$. Of course, if X is defined using all elementary symmetric polynomials, then X may or may not intersect the hyperplane arrangement (see Figure 2). Such a viewpoint can be taken in a more general setting.

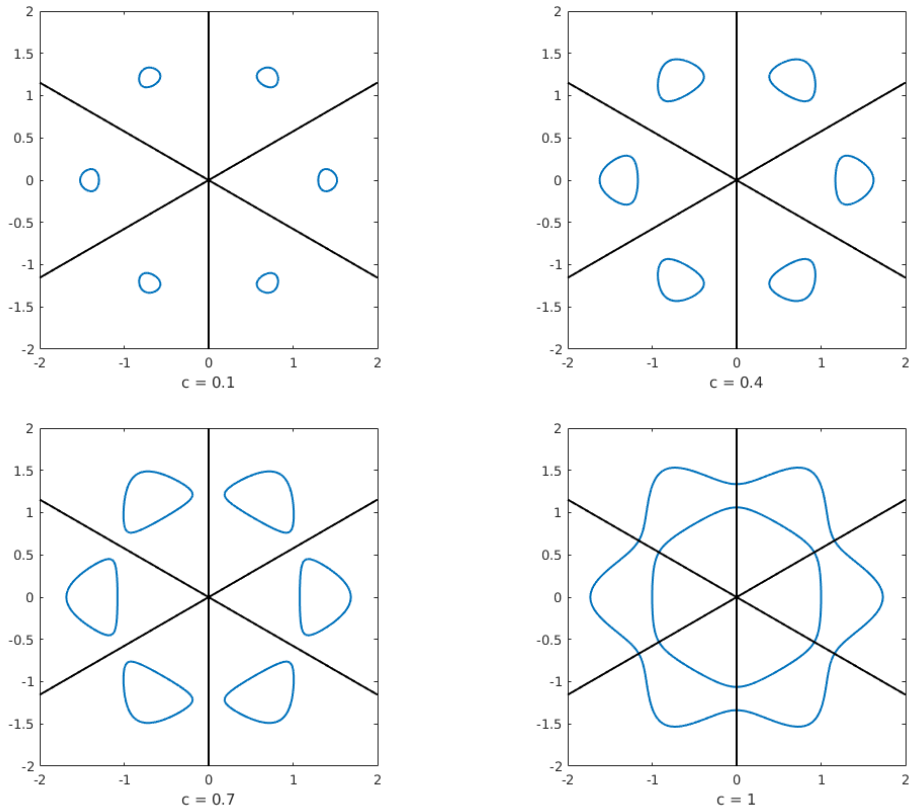

Figure 2: The symmetric zero set of $f = (x_1^2 + x_2^2 + x_3^2 - 2)^2 + (x_1^3 + x_2^3 + x_3^3)^2 - c$ for four values of c and the reflection arrangement of $\mathfrak{S}_3$ acting essentially on the subspace $\{\mathbf{p} : p_1 + p_2 + p_3 = 0\} \subset \mathbb{R}^3$.

A *(real) reflection group* G is a group of orthogonal transformations of $\mathbb{R}^n$ that is generated by reflections. Its associated *reflection arrangement* is

$$\mathcal{H} \;=\; \mathcal{H}(G) \;:=\; \{H = \ker g : g \in G \text{ reflection}\}$$

and the *flats* of $\mathcal{H}$ are the linear subspaces of $\mathbb{R}^n$ which arise as intersections of hyperplanes in $\mathcal{H}$. Analogous to the case of $\mathfrak{S}_n$, for $0 \leq k \leq n$, we denote by

$\mathcal{H}_k(G)$ (or simply $\mathcal{H}_k$) the union of all flats of dimension k and we set $\mathcal{H}_n := \mathbb{R}^n$. We call G *essential* if G does not fix a nontrivial linear subspace or, equivalently, if $\mathcal{H}_0 = \{0\}$. Reflection groups naturally occur in connection with Lie groups and Lie algebras and are well-studied from the perspective of geometry, algebra, and combinatorics [10, 31, 53]. A complete classification of reflection groups can be given in terms of *Dynkin diagrams* (see [53]). There are four infinite families of irreducible reflection groups, $\mathfrak{S}_n \cong A_{n-1}, B_n, D_n$ and $I_2(m)$, and six exceptional groups, H_3, H_4, F_4, E_6, E_7, and E_8.

The linear action of G on $\mathbb{R}^n$ induces an action on $\mathbb{R}[\mathbf{x}] = \mathbb{R}[x_1, \dots, x_n]$ by $g \cdot f(\mathbf{x}) := f(g^{-1} \cdot \mathbf{x})$. Chevalley's Theorem [53, Ch. 3.5] states that the ring $\mathbb{R}[\mathbf{x}]^G$ of polynomials invariant under G is generated by algebraically independent homogeneous polynomials $\pi_1, \pi_2, \dots, \pi_n \in \mathbb{R}[x]$. The collection $\pi_1, \dots, \pi_n$ is called a set of *basic invariants* for G. The basic invariants are not unique, but their degrees $d_i(G) := \deg \pi_i$ are. Throughout this thesis, we will assume that the basic invariants are labelled such that $d_1 \leq d_2 \leq \cdots \leq d_n$. In accord with $\mathfrak{S}_n$-invariant varieties, we call a G-invariant variety $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ *k-sparse* if $f_1, \dots, f_m$ can be chosen in $\mathbb{R}[\pi_1, \dots, \pi_k]$ for some choice of basic invariants $\pi_1, \dots, \pi_n$.

In the light of the above, it is natural to ask whether Theorem 1 extends to other reflection groups. Our main result is the following.

Theorem 2. *Let G be a reflection group of type $I_2(m), A_{n-1}, B_n, D_n, H_3$, or F_4 and X a nonempty G-invariant real variety. If X is k-sparse, then $X \cap \mathcal{H}_k(G) \neq \emptyset$.*

We also provide computational evidence that Theorem 2 should hold true for the group H_4 (see Section 6.1.2) and thus we conjecture that it extends to all essential and irreducible reflection groups. For arbitrary reflection groups, we can prove the following result, which in particular proves the case $k = n-1$ of Theorem 2 for all essential reflection groups.

Theorem 3. *Let G be an essential reflection group acting on $\mathbb{R}^n$ and assume that $X = \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m)$ is G-invariant and nonempty. If there exists some $j \in [n]$ such that $f_1, \dots, f_m \in \mathbb{R}[\pi_i : i \neq j]$, then $X \cap \mathcal{H}_{n-1}(G) \neq \emptyset$.*

Independent from our work, Acevedo and Velasco [1] addressed the related question of certifying non-negativity of G-invariant polynomials. Let us note that all our results also apply to *semialgebraic sets* instead of real varieties and, in particular, the main result in [1] is a consequence of Theorem 3. For details, see Section 6.1.1.

Part II is structured as follows. Chapter 4 serves as an extended introduction. In Section 4.1, we recall some concepts and theory on reflection groups, such as root systems, Dynkin diagrams and the classification of irreducible reflection groups. Section 4.2 treats their invariant theory and the geometry of invariant varieties. We recall Timofte's theorem for symmetric polynomials and give a more detailed overview of our results.

In Chapter 5, we prove Theorem 2 for the infinite families A_n, B_n and D_n. The proof for A_n and B_n presented in Section 5.1 uses a strengthened version of a result by Steinberg [76], which does not hold true for all reflection groups. However, in Section 5.2 we are still able to give a proof for D_n, which turns out to be considerably more involved than the previous cases.

Instead of studying reflection groups case by case, we take a general approach in Chapter 6. In Section 6.1, we introduce orbit spaces. We study their geometry and boundary structure and prove a result that directly implies Theorem 3. As a consequence, Theorem 2 follows for all groups of rank at most 3. Additionally, we provide a proof for the group F_4 and we give computational evidence that Theorem 2 should also hold true for H_4. In Section 6.2, we turn to studying k-sparse varieties for $k < n - 1$. Under a mild extra assumption on the defining polynomials, we prove a Timofte-type result for arbitrary reflection groups. However, the dimension of the strata that are intersected is in general not best possible (as can be seen for groups of type D_n) and difficult to compute. We provide upper bounds using the combinatorics of parabolic subgroups. Finally, using perturbation techniques, we obtain an alternative proof of Theorem 2 for the groups A_n and B_n.

Reflection groups are closely related to semi-simple Lie algebras and in Section 6.3, we translate Theorem 2 into an analogous result for varieties invariant under the action of the Lie groups SL_n and SO_n.

So far we have only worked over the real numbers $\mathbb{R}$. However, precisely the same questions can be asked for complex reflection groups. Note that neither real results have implications for the complex case nor vice versa and even for the simplest case of the symmetric group it is not known whether Timofte's theorem holds. As a first step into the complex world, we prove in Section 6.4 that for the large class of *well-generated* complex reflection groups, every nonempty $(n-1)$-sparse variety meets the reflection arrangement.

Part I is based on joint work with Tom Chappell and Raman Sanyal and will appear in [14]. Apart from Section 6.2.3 and Section 6.4, the results of Part II will be published in [29], which is joint work with Cordian Riener and Raman Sanyal.

Contents

Part I

Double poset polytopes

Chapter 1

Double order polytopes

A prominent class of polytopes associated to posets are Stanley's order polytopes [73]. In this chapter we will introduce and study *double* order polytopes arising from *double* posets, a notion introduced by Malvenuto and Reutenauer [60]. Double order polytopes are very combinatorial in flavor and much of their geometry, such as their volume, faces and triangulations, can be described in terms of the underlying double poset.

Section 1.1 deals with the basic combinatorial concepts that will be needed later on: Posets and double posets. In Section 1.2, we move on to the geometry of order polytopes and recall some of their properties discovered by Stanley [73]. Our main objects of interest, double order polytopes, will be studied in Section 1.3.

1.1 Posets and double posets

In this section, we introduce posets and double posets, the combinatorial objects which will be essential for all what comes later. As a reference for background and additional information on posets we recommend the first volume of Stanley's "Enumerative Combinatorics" [74]. Double posets were first studied by Malvenuto and Reutenauer [60] in the context of combinatorial Hopf algebras, generalizing Stanley's labelled posets [72]. The combinatorial study of general double posets gained momentum in recent years with a focus on algebraic aspects; see, for example, [24, 25].

1.1.1 Partially ordered sets

A **partially ordered set** (or **poset**) is a pair $(P, \preceq)$, where P is a set and $\preceq$ is a binary relation on P such that for all $a, b, c \in P$ we have

(i) $a \preceq a$ (reflexivity),

(ii) $a \preceq b$ and $b \preceq c$ imply $a \preceq c$ (transitivity) and

(iii) $a \preceq b$ and $b \preceq a$ imply $a = b$ (anti-symmetry).

We will sometimes abuse notation and write P instead of $(P, \preceq)$. Since we are exclusively working with finite posets, we will tacitly assume $|P| < \infty$ throughout. We denote the natural numbers $\{0, 1, 2, \dots\}$ by $\mathbb{N}$ and we write $\mathbb{N}_{>0}$ for the set of strictly positive natural numbers. Three very natural families of posets are the following.

Example 1.1. Let $n \in \mathbb{N}_{>0}$.

(1) The finite subset $[n] := \{1, 2, \dots, n\} \subset \mathbb{N}_{>0}$ together with the the usual order $\leq$ on $\mathbb{N}$ forms a poset, called the n-**chain**, which we will denote by C_n.

(2) The same ground set $[n]$ with no relations between distinct elements is called the n-**antichain** and will be denoted by A_n.

(3) The n-**comb** is the poset Comb_n on the ground set $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ such that $a_i \preceq a_j$ for all $i, j \in [n]$ with $i \leq j$ and $b_i \preceq a_i$ for $i \in [n]$.

Two elements a, b of a poset P are **comparable** if $a \preceq b$ or $b \preceq a$. Otherwise, a and b are **incomparable**. For instance, for $n \in \mathbb{N}_{>0}$, every two elements in the chain C_n are comparable, whereas every two elements in the antichain A_n are incomparable. If $a \preceq b$ and $a \neq b$ we write $a \prec b$. A relation of the form $a \prec b$ is called **cover relation** if it is minimal in the sense that there is no $c \in P$ such that $a \prec c \prec b$. In this case we write $a \prec\!\cdot\, b$. For $a, b \in P$ with $a \preceq b$ the **closed interval** between a and b is

$$[a, b] \;:=\; \{p \in P : a \preceq p \preceq b\}.$$

An element $a \in P$ is called **minimal** (resp. **maximal**) if there is no $b \in P$ with $b \prec a$ (resp. $a \prec b$). Moreover, a is called **least** (resp. **greatest**) element if for every $b \in P$ we have $a \preceq b$ (resp. $a \succeq b$). Of course, not every poset has a least and greatest element. However, for every poset P we can adjoin a least element $\widehat{0}$ and a greatest element $\widehat{1}$. The resulting poset with ground set $P \cup \{\widehat{0}, \widehat{1}\}$ will be denoted by $\widehat{P}$.

For two posets $(P, \preceq_P)$ and $(Q, \preceq_Q)$, a map $f : P \to Q$ is called **order-preserving** if for all $a, b \in P$

$$a \preceq_P b \quad \text{implies} \quad f(a) \preceq_Q f(b).$$

If f is bijective and its inverse map is also order-preserving, then f is an **isomorphism**. The posets P and Q are called **isomorphic** if there exists an isomorphism $f : P \to Q$. A **linear extension** of a poset P with $|P| = n$ is an order-preserving bijection $f : P \to \mathrm{C}_n$. A linear extension can be thought of as a labelling of the elements of P with $1, 2, \dots, n$ which respects the order relations in P.

The **Hasse diagram** of a poset P is the graph on the elements of P such that a and b form an edge if and only if $a \prec\!\cdot\, b$ is a cover relation. Usually the direction of the edges is indicated by drawing b above a. Figures 1.1, 1.2 and 1.3 show the Hasse diagrams of the posets from Example 1.1.

Figure 1.1: The chain C_4.

Figure 1.2: The antichain A_4.

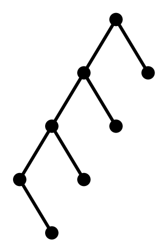

Figure 1.3: Comb_8.

In addition to the Hasse diagram, there is a second canonical way of associating a graph to a poset. The **comparability graph** $G(P)$ of a poset $(P, \preceq)$ is the undirected graph with vertex set P and edge set $\{xy : x \prec y \text{ or } y \prec x\}$. The poset P is called **connected** if the graph $G(P)$ is connected. Note that it is usually not possible to recover P from its comparability graph $G(P)$. For example, for any poset $(P, \preceq)$ consider its **opposite poset** $P^{\text{op}} = (P, \preceq_{\text{op}})$, where $a \preceq_{\text{op}} b$ in P^{op} if and only if $b \preceq a$ in P. Two elements are comparable in P if and only if they are comparable in P^{op} and hence $G(P) = G(P^{\text{op}})$.

A poset $(Q, \preceq_Q)$ is called a **subposet** of $(P, \preceq_P)$ if $Q \subseteq P$ and $a \preceq_Q b$ implies $a \preceq_P b$. If also the converse holds, that is, for $a, b \in Q$ with $a \preceq_P b$ we have $a \preceq_Q b$, then the subposet $(Q, \preceq_Q)$ is called **induced**. Every subset $Q \subseteq P$ determines a unique induced subposet, which we will denote by $P|_Q$. A subset $Q \subseteq P$ with $|Q| = k$ is a **chain** (resp. **antichain**) in P if the induced subposet $P|_Q$ is isomorphic to C_k (resp. A_k). A chain C in P is called **maximal** if it is maximal with respect to inclusion, that is, if there is no chain C' in P with $C \subset C'$ and $|C'| > |C|$.

A poset $(P, \preceq)$ is called a **lattice** if for every $a, b \in P$ there exist unique greatest lower and least upper bounds, that is there exist elements $a \wedge b$ and $a \vee b$ in P such that for any $c \in P$

$$\begin{aligned} &a \wedge b \preceq a, b \text{ and } c \preceq a, b \text{ implies } c \preceq a \wedge b \text{ and} \\ &a \vee b \succeq a, b \text{ and } c \succeq a, b \text{ implies } c \succeq a \vee b. \end{aligned}$$

For $a, b \in P$, the elements $a \wedge b$ and $a \vee b$ are called **meet** and **join** of a and b, respectively. Note that every finite lattice has a least element and a greatest element which are obtained by taking the meet or the join of all elements in the lattice.

Every poset comes with the following associated lattice. A subset $\mathsf{J} \subseteq P$ is an **order filter** or simply a **filter** if it is up-closed, that is if $a \prec b$ and $a \in \mathsf{J}$ implies $b \in \mathsf{J}$. Observe that there is a one-to-one correspondence between filters and antichains in P. For every filter $\mathsf{J} \subseteq P$ the set $\min(\mathsf{J})$ of minimal elements in J is an antichain. Conversely, every antichain $\mathrm{A} \subseteq P$ **generates** a filter $\langle \mathrm{A} \rangle := \{b \in P : b \succeq a \text{ for some } a \in \mathrm{A}\}$. The set $\mathcal{J}(P)$ consisting of all filters in P together with inclusion relations forms a lattice, called the **Birkhoff lattice** of P. In this case, for two filters J, K, their meet and join are simply the filters $\mathsf{J} \cap \mathsf{K}$ and $\mathsf{J} \cup \mathsf{K}$, respectively. The Birkhoff lattice $\mathcal{J}(P)$ is **distributive**, that is, for filters $\mathsf{J}, \mathsf{K}, \mathsf{L} \in \mathcal{J}(P)$ the distributivity laws

$$\begin{aligned} \mathsf{J} \cup (\mathsf{K} \cap \mathsf{L}) &= (\mathsf{J} \cap \mathsf{K}) \cup (\mathsf{J} \cap \mathsf{L}) \text{ and} \\ \mathsf{J} \cap (\mathsf{K} \cup \mathsf{L}) &= (\mathsf{J} \cup \mathsf{K}) \cap (\mathsf{J} \cup \mathsf{L}) \end{aligned}$$

are satisfied. Note that it is easy to reconstruct the poset P from $\mathcal{J}(P)$. A filter $\mathsf{J} \in \mathcal{J}(P)$ is called **meet-irreducible** if $\mathsf{J} \neq P$ and one cannot write $\mathsf{J} = \mathsf{K} \cap \mathsf{L}$ for two filters $\mathsf{K}, \mathsf{L} \neq \mathsf{J}$. It is straightforward to observe that P is isomorphic to the induced subposet of $\mathcal{J}(P)$ consisting of all meet-irreducible filters. In fact, this holds more generally: Birkhoff's theorem [74, Thm. 3.4.1] states that every finite distributive lattice $\mathcal{J}$ is isomorphic to $\mathcal{J}(P)$, where P denotes the induced subposet of meet-irreducible elements in $\mathcal{J}$.

Example 1.2. We denote by P_X the 5-element poset with Hasse diagram the shape of an "X" depicted in Figure 1.4. This poset has four distinct linear extensions and its Birkhoff lattice $\mathcal{J}(P_X)$ in Figure 1.5 consists of 8 filters.

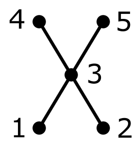

Figure 1.4: The poset P_X. The labels correspond to a linear extension.

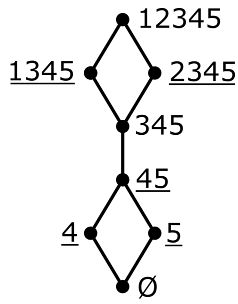

Figure 1.5: The Birkhoff lattice $\mathcal{J}(P_X)$. Meet-irreducible elements are underlined.

Example 1.3 (Dimension-2 posets)**.** For a poset $(P, \preceq)$ the smallest number d such that there exist linear extensions $f_1, \dots, f_d$ of P satisfying

$$a \preceq b \text{ if and only if } f_i(a) \leq f_i(b) \text{ for all } i \in [d]$$

for all $a, b \in P$ is called the **order dimension** of P. This notion was introduced by Dushnik and Miller [20], who also observed that posets of order dimension 2 can be characterized as follows. They are precisely posets of the form $([n], \prec_\pi)$, where $n \in \mathbb{N}_{>0}$ and $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ is a sequence of distinct integers and $i \preceq_\pi j$ if and only if $i \leq j$ and $\pi_i \leq \pi_j$. Note that we may of course assume $\pi_i \in [n]$ for all $i \in [n]$, but later on it will be more convenient to allow arbitrary integers. A chain in $P_\pi = ([n], \preceq_\pi)$ is a sequence $i_1 < i_2 < \cdots < i_k$ with $\pi_{i_1} < \pi_{i_2} < \cdots < \pi_{i_k}$. That is, chains in P_π are in bijection to **increasing subsequences** of π. Conversely, one checks that antichains (and hence filters) in P_π are in bijection to **decreasing subsequences** of π.

There is yet another characterization of posets of order dimension 2 due to Baker, Fishburn and Roberts [3]. For a poset P, consider the **incomparability graph** $\overline{G(P)}$ with vertex set P and edges between incomparable elements. Then P has order dimension 2 if and only if $\overline{G(P)}$ is again a comparability graph, that is if there exists a poset P' such that $\overline{G(P)} = G(P')$. In this case, such a P' can easily be constructed explicitly. For a poset of the form P_π with $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ we have $\overline{G(P_\pi)} = G(P_{-\pi})$, where $-\pi = (-\pi_1, -\pi_2, \dots, -\pi_n)$.

1.1.2 Double posets and alternating chains

The combinatorial objects we are primarily interested in are the following. A **double poset** is a triple $\mathbf{P} = (P, \preceq_+, \preceq_-)$ consisting of a finite ground set P and two partial order relations $\preceq_+$ and $\preceq_-$ on P. For $\sigma \in \{\pm\}$ we will write P_σ (or sometimes also $(\mathbf{P})_\sigma$) for the underlying ordinary posets $(P, \preceq_\sigma)$. The following classes of double posets will serve as running examples throughout the rest of this work.

Example 1.4. (1) Every poset $(P, \preceq)$ trivially gives rise to an **induced** double poset $\mathbf{P} = (P, \preceq, \preceq)$. We will often denote induced double posets by bold versions of the original poset: For example, $\mathbf{C}_n$, $\mathbf{A}_n$ and $\mathbf{Comb}_n$ are the double posets induced by C_n, A_n and Comb_n, respectively, for $n \in \mathbb{N}_{>0}$.

(2) For $n \in \mathbb{N}_{>0}$ we consider the double poset $\mathbf{Alt}_n$ with ground set $\{a_1, a_2, \ldots, a_n\}$ and relations

$$a_i \prec_+ a_{i+1} \quad \text{for } i \text{ odd and}$$
$$a_i \prec_- a_{i+1} \quad \text{for } i \text{ even.}$$

(3) For $n \in \mathbb{N}_{>0}$ we denote by $\mathbf{AC}_n$ the double poset consisting of a chain and an antichain, that is $(\mathbf{AC}_n)_+ = \mathrm{C}_n$ and $(\mathbf{AC}_n)_- = \mathrm{A}_n$.

Inspired by the previous example, we define the following. An **alternating chain** C of a double poset $\mathbf{P} = (P, \preceq_\pm)$ is a sequence of distinct elements in $\widehat{P} = P \cup \{\widehat{0}, \widehat{1}\}$ of the form

$$\widehat{0} \;=\; p_0 \;\prec_\sigma\; p_1 \;\prec_{-\sigma}\; p_2 \;\prec_\sigma\; \cdots \;\prec_{\pm\sigma}\; p_k \;=\; \widehat{1},$$

where $\sigma \in \{\pm\}$ and, to get the most out of our notational convention, we set

$$-\sigma \;:=\; \begin{cases} - & \text{if } \sigma = + \\ + & \text{if } \sigma = -. \end{cases}$$

. We always require alternating chains to start in $\widehat{0}$ and end in $\widehat{1}$, since this will be convenient later on. Note that it is possible that a sequence of elements $p_1, p_2, \ldots, p_k$ gives rise to two alternating chains, one starting with $\prec_+$ and one starting with $\prec_-$. For every double poset there exist two **trivial** alternating chains $\widehat{0} \prec_+ \widehat{1}$ and $\widehat{0} \prec_- \widehat{1}$. An **alternating cycle** C of $\mathbf{P}$ is a sequence of the form

$$p_0 \;\prec_\sigma\; p_1 \;\prec_{-\sigma}\; p_2 \;\prec_\sigma\; \cdots \;\prec_{-\sigma}\; p_{2k} \;=\; p_0,$$

with $k > 0$, $\sigma \in \{\pm\}$ and $p_i \neq p_j$ for $0 \leq i < j < 2k$. In fact, every alternating cycle of length $2k$ yields k alternating cycles starting with $\prec_+$ and k alternating starting with $\prec_-$.

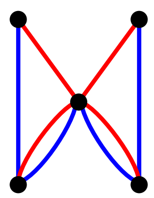

Figure 1.6: The 'XW'-double poset $\mathbf{P}_{XW}$. The red and blue lines are the Hasse diagram of P_+ and P_-, respectively.

Example 1.5. Consider the double poset $\mathbf{P}_{XW}$ from Figure 1.6. That is, $\mathbf{P}_{XW}$ is the double poset such that $(\mathbf{P}_{XW})_+ = P_X$ and $(\mathbf{P}_{XW})_-$ is the poset with Hasse diagram "W". There are no alternating cycles in $\mathbf{P}_{XW}$. The 28 alternating chains are shown in Figure 1.7.

We call a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ **compatible** if $(P, \preceq_+)$ and $(P, \preceq_-)$ have a common linear extension. All posets in Examples 1.4 and 1.5 are easily seen to be compatible. However it is easy to construct non-compatible examples. For instance, any double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ which contains two elements a, b such that $a \prec_+ b$ and $b \prec_- a$ is not compatible. However, if no such elements exist, this does not guarantee compatibility, as the following example shows.

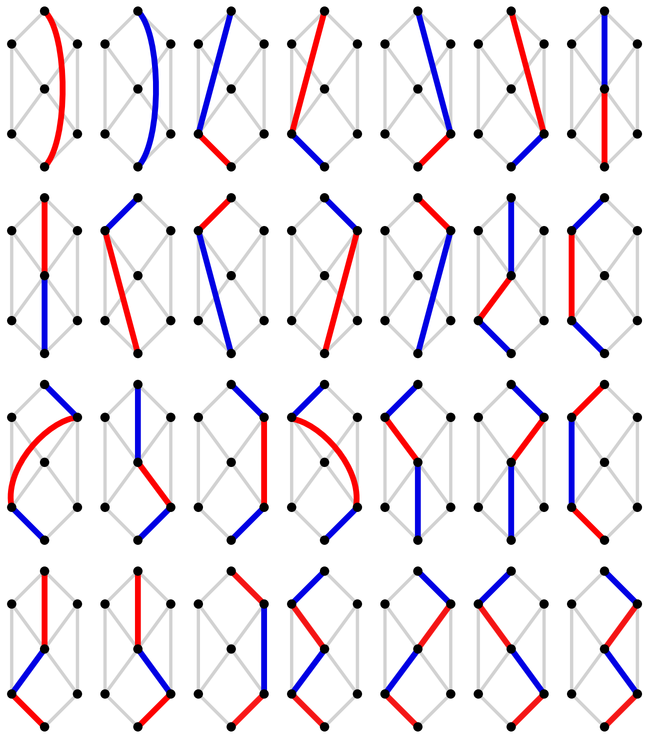

Figure 1.7: The 28 alternating chains in $\widehat{\mathbf{P}}_{XW}$.

Example 1.6. For $n \in \mathbb{N}$, we denote by $\mathbf{Cyc}_{2n}$ the double poset on the ground set $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ with relations $a_i \prec_+ b_i$ and $b_i \prec_- a_{i+1}$ for $i \in [n]$, where we set $a_{n+1} := a_1$. It is easily seen that $\mathbf{Cyc}_{2n}$ cannot be compatible. Indeed, for any linear extension f, the element $p = f^{-1}(1)$ must be minimal in both $(\mathbf{Cyc}_{2n})_+$ and $(\mathbf{Cyc}_{2n})_-$. However, such an element does not exist.

In fact, compatible double posets can be characterized as follows.

Proposition 1.7. *A double poset is compatible if and only if it does not contain any alternating cycles.*

Proof. Adapting the argument from Example 1.6 we see that compatible double posets cannot contain alternating cycles. For the converse, we observe that whenever a double poset $\mathbf{P} = (P, \preceq_+ \preceq_-)$ does not contain alternating cycles, then there exists an element $p \in P$ that is minimal in both P_+ and P_-. We start defining a linear extension by $f(p) := 1$. The double poset $(P \setminus \{p\}, \preceq_+, \preceq_-)$ does not contain alternating cycles and an inductive argument finishes the proof. $\square$

Following [60], we call a double poset $\mathbf{P}$ **special** if $\preceq_-$ is a total order. At the other extreme, we call $\mathbf{P}$ **anti-special** if $(P, \preceq_-)$ is an anti-chain. A **plane poset**, as defined in [24] is a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ such that distinct $a, b \in P$ are $\prec_+$-comparable if and only if they are *not* $\prec_-$-comparable. For two posets $(P_1, \preceq^1)$ and $(P_2, \preceq^2)$, one classically defines the **disjoint union** $\preceq_\uplus$ and the **ordinal sum** $\preceq_\oplus$ as the posets on $P_1 \uplus P_2$ as follows. For $a, b \in P_1 \uplus P_2$ set $a \preceq_\uplus b$ if $a, b \in P_i$ and $a \preceq^i b$ for some $i \in \{1, 2\}$. For the ordinal sum, $\preceq_\oplus$ restricts to $\preceq^1$ and $\preceq^2$

on P_1 and P_2, respectively, and $p_1 \prec_\oplus p_2$ for all $p_1 \in P_1$ and $p_2 \in P_2$. Malvenuto and Reutenauer [60] define the **composition** of two double posets $(P_1, \prec^1_\pm)$ and $(P_2, \prec^2_\pm)$ as the double poset $(P, \preceq_\pm)$ such that $(P, \preceq_+) = (P_1, \prec^1_+) \uplus (P_2, \prec^2_+)$ and $(P, \preceq_-) = (P_1, \prec^1_-) \oplus (P_2, \prec^2_-)$. The following is easily seen; for plane posets with the help of [24, Prop. 11].

Proposition 1.8. *Anti-special and plane posets are compatible. Moreover, the composition of two compatible double posets is a compatible double poset.*

Plane posets have a simple combinatorial description that relates to the posets of order dimension 2 from Example 1.3.

Example 1.9 (Plane posets)**.** Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. We may assume that $P = \{p_1, \dots, p_n\}$ are labelled such that $p_i \prec_\sigma p_j$ for $\sigma = +$ or $= -$ implies $i < j$. By [25, Prop. 15], $\mathbf{P}$ is a plane poset, if and only if there is a sequence of distinct numbers $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ such that for $p_i, p_j \in P$

$$\begin{aligned} p_i \prec_+ p_j \quad &\iff \quad i < j \text{ and } \pi_i < \pi_j \text{ and} \\ p_i \prec_- p_j \quad &\iff \quad i < j \text{ and } \pi_i > \pi_j. \end{aligned}$$

This is to say, P_+ is canonically isomorphic to the order-dimension-2 poset $([n], \preceq_\pi)$ and P_- is canonically isomorphic to $([n], \preceq_{-\pi})$. Note that this also follows from the characterization of dimension-2 posets in terms of comparability graphs from Example 1.3. The non-trivial alternating chains in $\mathbf{P}$ are in bijection to **alternating sequences**. That is, sequences $i_1 < i_2 < i_3 < \cdots < i_k$ such that $k \geq 1$ and $\pi_{i_1} < \pi_{i_2} > \pi_{i_3} < \pi_{i_4} > \cdots$.

1.2 Order polytopes

This section introduces order polytopes and at the same time serves as an introduction to basic concepts from the theory of polytopes. The discrete-geometric results we provide here and further background information on polyhedra can be found in Ziegler's "Lectures on Polytopes" [81]. For additional information on triangulations, we also refer the reader to [17]. Most of the results on the combinatorics and geometry of order polytopes we collect here can be found in [73]. Order polytopes are very natural objects that appear in a variety of contexts in combinatorics and beyond; see for instance [2, 26, 34, 55, 71, 80].

1.2.1 Polytopes and order polytopes

Before talking about order polytopes, we recall some basic definitions and results. A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called **convex** if for all $\mathbf{a}, \mathbf{b} \in \mathcal{P}$, any convex combination $\lambda\mathbf{a} + (1-\lambda)\mathbf{b}$ with $0 \leq \lambda \leq 1$ lies in $\mathcal{P}$. Moreover, $\mathcal{P}$ is a **(convex) cone** if for $\mathbf{a}, \mathbf{b} \in \mathcal{P}$ and $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$, the point $\lambda_1\mathbf{a} + \lambda_2\mathbf{b}$ lies in $\mathcal{P}$. The **convex hull** $\operatorname{conv}(\mathcal{P})$ and the **conical hull** $\operatorname{cone}(\mathcal{P})$ of a set $\mathcal{P} \subseteq \mathbb{R}^n$ are the inclusionwise smallest convex set containing $\mathcal{P}$ and the smallest cone containing $\mathcal{P}$, respectively. Moreover, the **affine hull** $\operatorname{aff}(\mathcal{P})$ of $\mathcal{P}$ is the smallest affine subspace of $\mathbb{R}^n$ that contains $\mathcal{P}$. Explicitly, we have

$$\begin{aligned}
\operatorname{conv}(\mathcal{P}) &= \Big\{ \sum_{i=1}^{k} \lambda_i \mathbf{a}_i : k \geq 1, \mathbf{a}_i \in \mathcal{P}, \lambda_i \in \mathbb{R}_{\geq 0} \text{ for } i \in [k], \sum_{i=1}^{k} \lambda_i = 1 \Big\}, \\
\operatorname{cone}(\mathcal{P}) &= \Big\{ \sum_{i=1}^{k} \lambda_i \mathbf{a}_i : k \geq 1, \mathbf{a}_i \in \mathcal{P}, \lambda_i \in \mathbb{R}_{\geq 0} \text{ for } i \in [k] \Big\} \text{ and} \\
\operatorname{aff}(\mathcal{P}) &= \Big\{ \sum_{i=1}^{k} \lambda_i \mathbf{a}_i : k \geq 1, \mathbf{a}_i \in \mathcal{P}, \lambda_i \in \mathbb{R} \text{ for } i \in [k], \sum_{i=1}^{k} \lambda_i = 1 \Big\}.
\end{aligned}$$

A **polyhedron** in $\mathbb{R}^n$ is the intersection of finitely many closed **halfspaces**, that is, sets of the form $\{\mathbf{x} \in \mathbb{R}^n : \ell(\mathbf{x}) \leq c\}$ for some linear functional $\ell : \mathbb{R}^n \to \mathbb{R}$ and some $c \in \mathbb{R}$. In other words, a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ is of the form

$$\mathcal{P} \;=\; \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} \leq \mathbf{b}\}$$

for some $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. If $\mathbf{b} = \mathbf{0}$, then $\mathcal{P}$ is a **polyhedral cone**. A bounded polyhedron is called a **polytope**. The **dimension** of a polyhedron $\mathcal{P}$ is the dimension of its affine hull $\operatorname{aff}(\mathcal{P})$. The following result is fundamental for the study of polytopes and polyhedral cones.

Theorem 1.10 ([81, Thm.1.1 and Thm.1.3])**.** (1) *For every finite set $V \subset \mathbb{R}^n$, the convex hull* $\operatorname{conv}(V)$ *is a polytope. Conversely, every polytope is the convex hull of finitely many points.*

(2) *For every finite set $V \subset \mathbb{R}^n$, the conical hull* $\operatorname{cone}(V)$ *is a polyhedral cone. Conversely, every polyhedral cone is the conical hull of finitely many points.*

Two polyhedra $\mathcal{P}_1 \subset \mathbb{R}^n$ and $\mathcal{P}_2 \subset \mathbb{R}^m$ are called **linearly isomorphic** (resp. **affinely isomorphic**) if there exists a linear (resp. affine) map $F : \mathbb{R}^n \to \mathbb{R}^m$ which restricts to a bijection between $\mathcal{P}_1$ and $\mathcal{P}_2$.

We will illustrate the above for the very combinatorial class of order polytopes. For a finite poset P we will write $\mathbb{R}^P$ for real vector space of functions from P to $\mathbb{R}$.

Definition 1.11. Let P be a finite poset. The **order polytope** $\mathcal{O}(P) \subset \mathbb{R}^P$ of P is set of all order preserving functions into the interval $[0, 1]$.

Order polytopes are always full-dimensional, that is, $\dim(\mathcal{O}(P)) = |P|$. The definition directly yields a representation of order polytopes as bounded intersections of halfspaces: A function $f \in \mathbb{R}^P$ is contained in $\mathcal{O}(P)$ if and only if

$$\begin{array}{rcll}
f(a) - f(b) & \leq & 0 & \text{for all cover relations } a \prec\!\cdot\, b \text{ in } P, \\
f(a) & \geq & 0 & \text{for all minimal elements } a \in P \text{ and} \\
f(a) & \leq & 1 & \text{for all maximal elements } a \in P.
\end{array}$$

Alternatively, they can be regarded as the convex hull a finite set as follows. For a set $\mathsf{J} \subseteq P$ we define the characteristic function $\mathbf{1}_{\mathsf{J}} \in \mathbb{R}^P$ by

$$\mathbf{1}_{\mathsf{J}}(a) = \begin{cases} 1 & \text{if } a \in F \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 1.12 ([73, Cor.1.3])**.** *Let P be a finite poset. Then*

$$\mathcal{O}(P) \;=\; \operatorname{conv}(\mathbf{1}_{\mathsf{J}} : \mathsf{J} \subseteq P \textit{ filter}).$$

Moreover, these points are in convex position, that is, for every filter $\mathsf{J} \subseteq P$ we have

$$\mathbf{1}_{\mathsf{J}} \notin \operatorname{conv}(\mathbf{1}'_{\mathsf{J}} : \mathsf{J} \neq \mathsf{J}' \subseteq P \textit{ filter}).$$

Example 1.13. Chains and antichains give rise to two very natural classes of polytopes. For $n \in \mathbb{N}_{>0}$, the order polytope $\mathcal{O}(\mathrm{C}_n)$ is an n-dimensional **simplex**, that is, the convex hull of the $n+1$ affinely independent points corresponding to filters in C_n. At the other extreme, $\mathcal{O}(\mathrm{A}_n)$ is the n**-cube** with vertex set $\{0,1\}^n$.

1.2.2 Faces

In the following, we denote by $(\mathbb{R}^n)^*$ the **dual space** consisting of all linear functionals on $\mathbb{R}^n$. Whenever convenient we identify $(\mathbb{R}^n)^*$ with $\mathbb{R}^n$: A linear functional $\ell \in (\mathbb{R}^n)^*$ yields a vector in $\mathbb{R}^n$ consisting of images of the elements of a given basis. Conversely, any $\mathbf{u} \in \mathbb{R}^n$ yields a functional $\ell_{\mathbf{u}}(\mathbf{p}) := \langle \mathbf{u}, \mathbf{p} \rangle$, where $\langle \mathbf{u}, \mathbf{p} \rangle$ denotes the standard inner product on $\mathbb{R}^n$. For a polyhedron $\mathcal{P} \subseteq \mathbb{R}^n$ and a linear functional $\ell \in (\mathbb{R}^n)^*$ we define

$$\mathcal{P}^\ell \;:=\; \{\mathbf{p} \in \mathcal{P} : \ell(\mathbf{p}) \geq \ell(\mathbf{q}) \text{ for all } \mathbf{q} \in \mathcal{P}\}.$$

A subset $F \subseteq \mathcal{P}$ is called **face** of $\mathcal{P}$ if either $\mathcal{P} = \emptyset$ or $F = \mathcal{P}^\ell$ for some $\ell \in (\mathbb{R}^n)^*$. A face $F \subseteq \mathcal{P}$ with $F \neq \mathcal{P}$ is called **proper**. The subset of $\mathcal{P}$ consisting of all points that are not contained in any proper face is called the **relative interior** of $\mathcal{P}$ and will be denoted by $\operatorname{relint}(\mathcal{P})$. Faces are again polyhedra and hence for a face F its dimension $\dim(F)$ is the dimension of the affine hull $\operatorname{aff}(F)$. **Facets** are faces of dimension $\dim(\mathcal{P}) - 1$. Zero-dimensional faces are called **vertices** and one-dimensional vertices are called **edges**. For a polyhedron $\mathcal{P}$, we denote its set of vertices by $V(\mathcal{P})$. The number of faces in each dimension is collected in the **face vector** or f**-vector**

$$f(\mathcal{P}) \;:=\; \big(f_0(\mathcal{P}), f_1(\mathcal{P}), \ldots, f_{\dim(\mathcal{P})-1}(\mathcal{P})\big),$$

where f_i denotes the number of i-dimensional faces of $\mathcal{P}$ for $0 \leq i \leq \dim(\mathcal{P}) - 1$. The set of all faces of a polytope $\mathcal{P}$ together with the inclusion relation forms a poset, which is called the **face lattice** of $\mathcal{P}$. Note that this poset is indeed a lattice and the meet and join of two faces $F, F' \subseteq \mathcal{P}$ are $F \cap F'$ and the inclusionwise smallest face containing $F \cup F'$, respectively. If two polytopes have isomorphic face lattices, they are called **combinatorially equivalent**. The **normal cone** of a face F of $\mathcal{P}$ is defined as

$$N_{\mathcal{P}}(F) \;=\; \{\ell \in (\mathbb{R}^n)^* : F \subseteq \mathcal{P}^\ell\}.$$

We can regard $N_{\mathcal{P}}(F)$ as a polyhedral cone with relative interior

$$\operatorname{relint}(N_{\mathcal{P}}(F)) \;=\; \{\ell \in (\mathbb{R}^n)^* : F = \mathcal{P}^\ell\}.$$

In the rest of this chapter, we will describe the facial structure of order polytopes. To this end, let $(P, \preceq)$ be a finite poset. For an order relation $a \prec b$ in P, we define a linear form $\ell_{a,b} : \mathbb{R}^P \to \mathbb{R}$ by

$$\ell_{a,b}(f) \;:=\; f(a) - f(b)$$

for any $f \in \mathbb{R}^P$. Moreover, for $a \in P$, we define $\ell_{a,\widehat{1}}(f) := f(a)$ and $\ell_{\widehat{0},a}(f) := -f(a)$. With this notation, $f \in \mathbb{R}^P$ is contained in $\mathcal{O}(P)$ if and only if

$$\begin{aligned} \ell_{a,b}(f) \;&\leq\; 0 \quad \text{for all cover relations } a \prec\cdot\, b \text{ in } P,\\ \ell_{\widehat{0},b}(f) \;&\leq\; 0 \quad \text{for all minimal elements } b \in P \text{ and}\\ \ell_{a,\widehat{1}}(f) \;&\leq\; 1 \quad \text{for all maximal elements } a \in P. \end{aligned} \tag{1.1}$$

None of the above equations is redundant and hence each inequality corresponds to a facet of $\mathcal{O}(P)$. In particular, the number of facets of $\mathcal{O}(P)$ equals the number of cover relations in $\widehat{P}$. More generally, faces of arbitrary dimension can be interpreted in a very combinatorial way: For a face $F \subseteq \mathcal{O}(P)$, define an equivalence relation on $\widehat{P}$ by

$$a \sim b \text{ if and only if } f(a) = f(b) \text{ for all } f \in F,$$

where we set $f(\widehat{0}) := 0$ and $f(\widehat{1}) := 1$ for all $f \in F$. Denote by $\mathcal{B}(F) = \{B_1, \ldots, B_m\}$ the set of equivalence classes, which form a partition of $\widehat{P}$. Such a partition is called a **face partition**. Since any face $F \subseteq \mathcal{O}(P)$ is the intersection of some facets, it follows from (1.1) that distinct faces yield distinct face partitions.

Stanley gave the following combinatorial characterization of the partitions of $\widehat{P}$ which occur as face partitions. A partition $\mathcal{B} = \{B_1, \ldots, B_m\}$ of $\widehat{P}$ is called **connected** if for $1 \leq i \leq m$ the block B_i is connected as an induced subposet of $\widehat{P}$. Moreover, $\mathcal{B}$ is called **compatible** if the binary relation $\preceq_{\mathcal{B}}$ on $\mathcal{B}$ defined by setting $B_i \preceq_{\mathcal{B}} B_j$ if $a \preceq b$ for some $a \in B_i$ and $b \in B_j$ is a partial order. Note that $\mathcal{B}$ is compatible if and only if the poset $(\mathcal{B}, \preceq_{\mathcal{B}})$ can be obtained from $\widehat{P}$ by consecutively contracting cover relations.

Proposition 1.14 ([73, Thm. 1.2])**.** *Let P be a finite poset. A partition $\mathcal{B}$ of $\widehat{P}$ is a face partition if and only if it is connected and compatible.*

Remark 1.15. It is easy to see that faces of order polytopes are again order polytopes. More precisely, for a poset P any face $F \subseteq \mathcal{O}(P)$ with associated face partition $\mathcal{B}(F)$ is affinely equivalent to the order polytope of the poset $(\mathcal{B}(F)\backslash\{B_{\widehat{1}}, B_{\widehat{0}}\}, \preceq_{\mathcal{B}(F)})$, where we denote by $B_{\widehat{1}}$ and $B_{\widehat{0}}$ the blocks containing $\widehat{1}$ and $\widehat{0}$, respectively. In particular, the dimension of a face F equals $|\mathcal{B}(F)| - 2$.

To determine a face partition $\mathcal{B}(F)$ it is of course sufficient to remember the non-singleton blocks and we define the **reduced face partition** of F as $\mathcal{B}^\circ(F) = \{B_i \in \mathcal{B}(F) : |B_i| > 1\}$. The following result follows directly from (1.1).

Proposition 1.16. *Let P be a finite poset and $F \subseteq \mathcal{O}(P)$ a nonempty face with reduced face partition $\mathcal{B}^\circ = \{B_1, \ldots, B_k\}$. Then*

$$N_{\mathcal{O}(P)}(F) \;=\; \operatorname{cone}\{\ell_{a,b} : a, b \in \widehat{P}, a \prec b, [a,b] \subseteq B_i \text{ for some } i \in [k]\}.$$

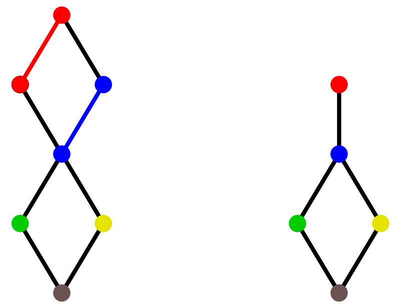

Figure 1.8: A face partition $\mathcal{B}$ of $\widehat{P_X}$ with blocks indicated by different colors and the block poset $(\mathcal{B}, \preceq_{\mathcal{B}})$, which is obtained from $\widehat{P_X}$ by contracting the red and blue cover relations, respectively. This partition corresponds to a three-dimensional face of $\mathcal{O}(P_X)$.

Later, we want to identify linear functionals with their vectors of coefficients and thus for $\ell \in (\mathbb{R}^P)^*$ we write

$$\ell(f) \;=\; \sum_{a \in P} \ell_a f(a).$$

We note the following simple but very useful consequence of Proposition 1.16.

Corollary 1.17. *Let P be a finite poset and let $F \subseteq \mathcal{O}(P)$ be a nonempty face with reduced face partition $\mathcal{B}^\circ(F) = \{B_1, \ldots, B_k\}$. Then for every $\ell \in \operatorname{relint} N_{\mathcal{O}(P)}(F)$ and $p \in P$ the following hold.*
(1) If $p \in \min(B_i)$ for some $i \in [k]$, then $\ell_p > 0$.
(2) If $p \in \max(B_i)$ for some $i \in [k]$, then $\ell_p < 0$.
(3) If $p \notin \bigcup_{i \in [k]} B_i$, then $\ell_p = 0$.

Proof. Let $p \in \min(B_i)$. For every $\ell \in N_{\mathcal{O}(P)}(F)$ we have $\ell_p \geq 0$ by Proposition1.16. If $\ell_p = 0$ holds, then ℓ is contained in the face $N_{\mathcal{O}(P)}(F) \cap \{\ell \in \mathbb{R}^P : \ell_p = 0\}$. Hence $\ell \notin \operatorname{relint} N_{\mathcal{O}(P)}(F)$, which proves (1). The proof of (2) is similar and (3) is immediate. $\square$

By Proposition 1.12 we have

$$V(\mathcal{O}(P)) \;=\; \{\mathbf{1}_{\mathsf{J}} : \mathsf{J} \in \mathcal{J}(P)\}$$

and hence we can identify the vertices of $\mathcal{O}(P)$ with filters in P. For a filter $\mathsf{J} \subseteq P$, we write $\widehat{\mathsf{J}} := \mathsf{J} \cup \{\widehat{1}\}$ for the filter induced in $\widehat{P}$. For a given face $F \subseteq \mathcal{O}(P)$ the following proposition, which follows immediately from (1.1), describes the filters that give rise to vertices of F.

Proposition 1.18. *Let $F \subseteq \mathcal{O}(P)$ be a face with reduced face partition $\mathcal{B}^\circ = \{B_1, \ldots, B_k\}$ and let $\mathsf{J} \subseteq P$ be a filter. Then $\mathbf{1}_{\mathsf{J}} \in F$ if and only if for all $i = 1, \ldots, k$*

$$\widehat{\mathsf{J}} \cap B_i \;=\; \emptyset \quad \text{or} \quad \widehat{\mathsf{J}} \cap B_i \;=\; B_i.$$

That is, $\mathbf{1}_{\mathsf{J}}$ belongs to F if and only if for all $i \in [k]$, the filter $\widehat{\mathsf{J}}$ does not separate any two elements in B_i.

Example 1.19. Consider the face $F \subset \mathcal{O}(P_X)$ arising from the face partition in Figure 1.8. If we use the labelling from Figure 1.4, the vertices of F correspond to the five filters $\big\{\{4\}, \{3,4,5\}, \{1,3,4,5\}, \{2,3,4,5\}, \{1,2,3,4,5\}\big\}$. The face F is 3-dimensional and affinely isomorphic to a pyramid over a square.

1.2.3 A canonical triangulation

Before triangulating order polytopes, we need some definitions. A **polytopal complex** is a finite nonempty collection $\mathcal{K}$ of polytopes in $\mathbb{R}^n$ such that
(i) for $\mathcal{P} \in \mathcal{K}$ and for every face $F \subseteq \mathcal{P}$ we have $F \in \mathcal{K}$ and
(ii) for $\mathcal{P}, \mathcal{Q} \in \mathcal{K}$, their intersection $\mathcal{P} \cap \mathcal{Q}$ is a face of each.
The elements of $\mathcal{K}$ are called **cells**. The dimension $\dim(\mathcal{K})$ of the polytopal complex is the largest dimension of a face in $\mathcal{K}$. As in the case of polytopes, zero-dimensional faces are called **vertices** and one-dimensional cells are **edges**. A cell is called **maximal** if it is not contained in a larger cell. To recover $\mathcal{K}$ it is sufficient to keep track of the maximal cells.

If all elements of $\mathcal{K}$ are simplices, then $\mathcal{K}$ is called a **geometric simplicial complex**. If we only want to record the face lattice of a geometric simplicial complex, we can use the following notion. An **abstract simplicial complex** on a finite ground set V is a collection Δ of subsets of V such that $X \in \Delta$ and $Y \subset X$ implies $Y \in \Delta$. Every geometric simplicial complex $\mathcal{K}$ with vertex set V has an underlying abstract simplicial complex Δ on the ground set V which for every $M \subseteq V$ satisfies

$$\operatorname{conv}(\mathbf{v} : \mathbf{v} \in M) \in \mathcal{K} \text{ if and only if } M \in \Delta.$$

Conversely, every abstract simplicial complex can be realized as a geometric simplicial complex, for example by choosing any set of affinely independent vertices.

Finally, for a polytope $\mathcal{P} \subset \mathbb{R}^n$, a polytopal complex $\mathcal{K}$ in $\mathbb{R}^n$ is called a **subdivision** of $\mathcal{P}$ if the union of all $\mathcal{Q} \in \mathcal{K}$ equals $\mathcal{P}$. If all elements of $\mathcal{K}$ are simplices, $\mathcal{K}$ is a **triangulation** of $\mathcal{P}$.

We are now ready to put the above concepts to good use and describe Stanley's triangulation of the order polytope. For a chain C of filters in $\mathcal{J}(P)$ of the form $\mathsf{J}_0 \subset \mathsf{J}_1 \subset \cdots \subset \mathsf{J}_k$ we define

$$F(C) \ := \ \operatorname{conv}(\{\mathbf{1}_{\mathsf{J}_0}, \mathbf{1}_{\mathsf{J}_1}, \mathbf{1}_{\mathsf{J}_2}, \ldots, \mathbf{1}_{\mathsf{J}_k}\}).$$

Note that $F(C)$ is a simplex of dimension k with $V(F(C)) \subseteq V(\mathcal{O}(P))$. Using this notation, it is easy to describe a canonical triangulation of order polytopes.

Theorem 1.20 ([73, Sec.5])**.** *Let P be a poset. Then the set of simplices $\{F(C) : C \subseteq \mathcal{J}(P) \text{ chain}\}$ forms a triangulation of $\mathcal{O}(P)$.*

The abstract simplicial complex underlying the triangulation in Theorem 1.20 consists of all chains in $\mathcal{J}(P)$. This complex is called the **order complex** of $\mathcal{J}(P)$ and is denoted by $\Delta(\mathcal{J}(P)) := \{C \subseteq \mathcal{J}(P) : C \text{ chain}\}$. It is worth noting that the maximal cells in $\Delta(\mathcal{J}(P))$ are in one-to-one correspondence with linear extensions of P. Indeed for any maximal chain of the form $\emptyset = \mathsf{J}_0 \subset \mathsf{J}_1 \subset \cdots \subset \mathsf{J}_n = P$ we define $p_i := \mathsf{J}_i \setminus \mathsf{J}_{i-1}$ for $i \in [n]$. With this notation at hand, the function $f : P \to [n]$ with

$f(p_i) := n - i$ is a linear extension of P. It is straightforward to check that every linear extension arises this way. In the sequel, we will denote the number of linear extensions of P by $\mathrm{e}(P)$.

Example 1.21. Consider the antichain A_3 with ground set $\{1, 2, 3\}$. The maximal chains in the Birkhoff lattice $\mathcal{J}(\mathrm{A}_3)$ are

$$\begin{aligned}
&\emptyset \subset \{1\} \subset \{1, 2\} \subset \{1, 2, 3\},\\
&\emptyset \subset \{1\} \subset \{1, 3\} \subset \{1, 2, 3\},\\
&\emptyset \subset \{2\} \subset \{1, 2\} \subset \{1, 2, 3\},\\
&\emptyset \subset \{2\} \subset \{2, 3\} \subset \{1, 2, 3\},\\
&\emptyset \subset \{3\} \subset \{1, 3\} \subset \{1, 2, 3\} \text{ and}\\
&\emptyset \subset \{3\} \subset \{2, 3\} \subset \{1, 2, 3\}.
\end{aligned}$$

Therefore, the triangulation of $\mathcal{O}(\mathrm{A}_3)$ obtained from Theorem 1.20 consists of six maximal simplices, which are shown in Figure 1.9.

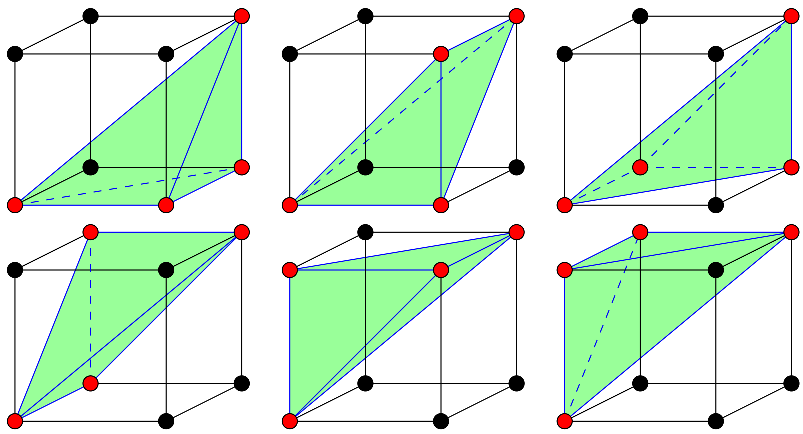

Figure 1.9: The six maximal simplices in the canonical triangulation of $\mathcal{O}(\mathrm{A}_3)$.

Example 1.22. More generally, for $n \in \mathbb{N}_{>0}$, the Birkhoff lattice $\mathcal{J}(\mathrm{A}_n)$ contains $n!$ maximal chains and hence the canonical triangulation of the n-cube $\mathcal{O}(\mathrm{A}_n)$ consists of $n!$ maximal simplices. This triangulation is particularly interesting since it is universal in the following sense: For every poset P with ground set $[n]$, chains in $\mathcal{J}(P)$ are also chains in $\mathcal{J}(\mathrm{A}_n)$ and hence every simplex in the canonical triangulation of $\mathcal{O}(P)$ is a simplex in the triangulation of $\mathcal{O}(\mathrm{A}_n)$. These triangulations are also known as Freudenthal triangulations, see for example [21, 27].

In the following, for a full-dimensional polytope $\mathcal{P} \subset \mathbb{R}^n$ we will denote by $\mathrm{vol}(\mathcal{P})$ its Euclidean volume in $\mathbb{R}^n$. Using Cayley-Menger determinants (see, for instance, [75]), it is easily seen that $\mathrm{vol}(F(C)) = \frac{1}{n!}$ for any maximal chain $C \in \Delta(\mathcal{J}(P))$. Hence, the above considerations yields a simple combinatorial formula for the volume of order polytopes.

Corollary 1.23. *Let P be a poset with $|P| = n$. Then*

$$\mathrm{vol}(\mathcal{O}(P)) \;=\; \frac{\mathrm{e}(P)}{n!}.$$

This result is interesting from the perspective of computational complexity: By the above, for a poset P, computing the volume of $\mathcal{O}(P)$ is equivalent to computing the number of linear extensions of P. For more information regarding the complexity of counting linear extensions and related problems, see [11].

1.3 Double order polytopes

This section focusses on the study of double order polytopes from a purely geometric viewpoint. The main results are the following. In Theorem 1.29, we give a description of the facets of double order polytopes arising from compatible double posets in terms of alternating chains. Theorem 1.37 determines which of these polytopes are 2-level, a geometric property that plays an important role in, for example, the study of centrally-symmetric polytopes [68, 41], polynomial optimization [36, 37], statistics [78], and combinatorial optimization [69]. Using polars, in Theorem 1.38 we explore a surprising connection to Geissinger's valuation polytopes, which were introduced in [32].

1.3.1 Facets and alternating chains

For two polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$, their **Minkowski sum** $\mathcal{P}_1 + \mathcal{P}_2$ and their **Cayley sum** $\mathcal{P}_1 \boxplus \mathcal{P}_2$ are defined as

$$\begin{aligned} \mathcal{P}_1 + \mathcal{P}_2 &:= \{\mathbf{p}_1 + \mathbf{p}_2 : \mathbf{p}_1 \in \mathcal{P}_1, \mathbf{p}_2 \in \mathcal{P}_2\} \subset \mathbb{R}^n \text{ and} \\ \mathcal{P}_1 \boxplus \mathcal{P}_2 &:= \operatorname{conv}(\mathcal{P}_1 \times \{1\} \cup \mathcal{P}_2 \times \{-1\}) \subset \mathbb{R}^n \times \mathbb{R}. \end{aligned}$$

We abbreviate $\mathcal{P}_1 - \mathcal{P}_2 := \mathcal{P}_1 + (-\mathcal{P}_2)$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2 := \mathcal{P}_1 \boxplus -\mathcal{P}_2$. For a single polytope $\mathcal{P} \subset \mathbb{R}^n$, we call $\operatorname{prism}(\mathcal{P}) := \mathcal{P} \boxplus \mathcal{P}$ the **prism** over $\mathcal{P}$ and $\operatorname{tprism}(\mathcal{P}) := \mathcal{P} \boxminus \mathcal{P}$ the **twisted prism** over $\mathcal{P}$. Minkowski and Cayley sums are related via

$$(\mathcal{P}_1 \boxplus \mathcal{P}_2) \cap (\mathbb{R}^n \times \{0\}) = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2) \times \{0\}. \tag{1.2}$$

We are now ready to define our main objects of study.

Definition 1.24. Let $\mathbf{P} = (P, \preceq_\pm)$ be a double poset. The **double order polytope** of $\mathbf{P}$ is

$$\mathbb{T}\mathcal{O}(\mathbf{P}) := 2\mathcal{O}(P_+) \boxminus 2\mathcal{O}(P_-) = \operatorname{conv}\big((2\mathcal{O}(P_+) \times \{1\}) \cup (-2\mathcal{O}(P_-) \times \{-1\})\big).$$

The double order polytope $\mathbb{T}\mathcal{O}(\mathbf{P})$ is a $(|P|+1)$-dimensional polytope in $\mathbb{R}^P \times \mathbb{R}$ with coordinates (f, t). Its vertices are exactly $(2\mathbf{1}_{\mathsf{J}_+}, 1), (-2\mathbf{1}_{\mathsf{J}_-}, -1)$ for filters $\mathsf{J}_+ \subseteq P_+$ and $\mathsf{J}_- \subseteq P_-$, respectively. Associated to $\mathbb{T}\mathcal{O}(\mathbf{P})$ is the **reduced** double order polytope

$$\mathbb{D}\mathcal{O}(\mathbf{P}) := \mathcal{O}(P_+) - \mathcal{O}(P_-),$$

which is obtained from $\mathbb{T}\mathcal{O}(\mathbf{P})$ by intersecting with the hyperplane $\{(f, t) : t = 0\}$.

Remark 1.25. There are several ways of embedding $\mathbb{T}\mathcal{O}(\mathbf{P})$ into $\mathbb{R}^{n+1}$, each having advantages and disadvantages. The definition we chose will be convenient for the study of polars in Section 1.3.3. For some purposes, for instance the study of unimodular triangulations in Section 3.3 and Ehrhart theory in Section 3.4.2 it is more convenient to think of $\mathbb{T}\mathcal{O}(\mathbf{P})$ as a **0/1-polytope**, that is with all vertex coordinates in $\{0,1\}$. Explicitly, $\mathbb{T}\mathcal{O}(\mathbf{P})$ is affinely equivalent to the 0/1-polytope

$$\operatorname{conv}\big(\{(\mathcal{O}(P_+)\times\{1\})\cup((\mathbf{1}-\mathcal{O}(P_-))\times\{0\})\}\big).$$

We will now investigate the facet structure of double order polytopes. By construction, $2\mathcal{O}(P_+)\times\{1\}$ and $-2\mathcal{O}(P_-)\times\{-1\}$ are the facets obtained by maximizing the linear function $\pm L_\emptyset(f,t) := \pm t$ over $\mathbb{T}\mathcal{O}(\mathbf{P})$. We call the remaining facets **vertical**, as they are of the form $2F_+ \boxminus 2F_-$, where $F_\sigma \subset \mathcal{O}(P_\sigma)$ are certain nonempty proper faces for $\sigma \in \{\pm\}$. The vertical facets of $\mathbb{T}\mathcal{O}(\mathbf{P})$ are in bijection with the facets of the reduced double order polytope $\mathbb{D}\mathcal{O}(\mathbf{P}) = \mathcal{O}(P_+) - \mathcal{O}(P_-)$.

Observe that any face $F = \mathbb{D}\mathcal{O}(\mathbf{P})^\ell$ with $\ell \in \mathbb{R}^P$ is of the form $F = F_+ - F_-$ where $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$. Moreover,

$$\dim(F) \;=\; |P| - \dim\big(\operatorname{relint} N_{\mathcal{O}(P_+)}(F_+) \cap \operatorname{relint} -N_{\mathcal{O}(P_-)}(F_-)\big).$$

In particular, F is a facet if and only if the linear function ℓ is unique up to scaling, that is

$$\operatorname{relint} N_{\mathcal{O}(P_+)}(F_+) \;\cap\; \operatorname{relint} -N_{\mathcal{O}(P_-)}(F_-) \;=\; \mathbb{R}_{>0}\cdot\ell\,. \tag{1.3}$$

We will call a linear function ℓ **rigid** if it satisfies (1.3) for a pair of faces (F_+,F_-). Our next goal is to give an explicit description of all rigid linear functions for $\mathbb{D}\mathcal{O}(\mathbf{P})$ which then yields a characterization of vertical facets of $\mathbb{T}\mathcal{O}(\mathbf{P})$.

For an alternating chain C of the form $\widehat{0} = p_0 \prec_\sigma \cdots \prec_{\pm\sigma} p_k = \widehat{1}$ with $\sigma \in \{\pm\}$ we define an associated linear function ℓ_C by

$$\ell_C(f) \;:=\; \sigma\sum_{i=1}^{k-1}(-1)^i f(p_i).$$

Here, we severely abuse notation and interpret σ as ±1. Note that $\ell_C \equiv 0$ if $k=1$ and we call C a **proper** alternating chain if $k>1$. Analogously, for an alternating cycle C of the form $p_0 \;\prec_\sigma\; p_1 \;\prec_{-\sigma}\; p_2 \;\prec_\sigma \cdots \;\prec_{-\sigma}\; p_{2k} = \; p_0$ we define

$$\ell_C(f) \;:=\; \sigma\sum_{i=0}^{2k-1}(-1)^i f(p_i).$$

Proposition 1.26. *Let $\mathbf{P} = (P,\preceq_\pm)$ be a double poset. If ℓ is a rigid linear function for $\mathbb{D}\mathcal{O}(\mathbf{P})$, then $\ell = \mu\ell_C$ for some alternating chain or alternating cycle C and $\mu > 0$.*

Proof. Let $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$ be the two faces for which (1.3) holds and let $\mathcal{B}_\pm = \{B_{\pm1}, B_{\pm2},\dots\}$ be the corresponding reduced face partitions. We define a directed bipartite graph $G = (V_+ \cup V_-, E)$ with nodes $V_+ := \{p\in P : \ell_p > 0\}$ and $V_- := \{p \in P : \ell_p < 0\}$. If $\widehat{1}$ is contained in some block in $\mathcal{B}_+$, then we add a corresponding node $\widehat{1}_+$ to V_-. Consistently, we add a node $\widehat{1}_-$ to V_+ if $\widehat{1}$ it occurs in a part of $\mathcal{B}_-$. Similarly we add $\widehat{0}_+$ to V_+ and $\widehat{0}_-$ to V_- if they appear in $\mathcal{B}_+$ and $\mathcal{B}_-$, respectively. Note that here we regard $\widehat{0}_-$ and $\widehat{0}_+$, as well as $\widehat{1}_-$ and $\widehat{1}_+$, as distinct nodes. By Corollary 1.17, we have ensured that $\max(B_{+i}) \subseteq V_-$ and $\max(B_{-i}) \subseteq V_+$ for all i.

For $u \in V_+$ and $v \in V_-$, we add the directed edge $uv \in E$ if $u \prec_+ v$ and $[u,v]_{P_+} \subseteq B_{+i}$ for some i. Similarly, we add the directed edge $vu \in E$ if $v \prec_- u$ and $[v,u]_{P_-} \subseteq B_{-i}$ for some i. We claim that every node u except for maybe the special nodes $\widehat{0}_\pm, \widehat{1}_\pm$ has an incoming and an outgoing edge. For example, if $u \in V_+$, then $\ell_u > 0$. By Corollary 1.17(iii), there is an i such that $u \in B_{+i}$ and by (ii), u is not a maximal element in B_{+i}. Thus, there is some $v \in \max(B_{+i})$ with $u \prec_+ v$ and uv is an edge. It follows that every longest path either yields an alternating cycle or a proper alternating chain.

For an alternating cycle $C = (p_0 \prec_+ \cdots \prec_- p_{2l})$, we observe that

$$\begin{aligned} \ell_C &= \ell_{p_0,p_1} + \ell_{p_2,p_3} + \cdots + \ell_{p_{2l-2},p_{2l-1}} \text{ and} \\ -\ell_C &= \ell_{p_1,p_2} + \ell_{p_3,p_4} + \cdots + \ell_{p_{2l-1},p_{2l}}. \end{aligned}$$

Since for every j, $[p_{2j}, p_{2j+1}]_{P_+}$ is contained in some part of $\mathcal{B}_+$, we conclude that $\ell_C \in N_{P_+}(F_+)$ by Proposition 1.16. Similarly, for all j, $[p_{2j-1}, p_{2j}]_{P_-}$ is contained in some part of $\mathcal{B}_-$, and hence $-\ell_C \in N_{P_-}(F_-)$. Assuming that ℓ is rigid then shows that $\ell = \mu\ell_C$ for some $\mu > 0$.

If G does not contain a cycle, then let $C = (p_0, p_1, \ldots, p_k)$ be a longest path in G. In particular $p_0 = \widehat{0}_\pm$ and $p_k = \widehat{1}_\pm$. The same reasoning applies and shows that $\ell_C \in N_{P_+}(F_+) \cap -N_{P_-}(F_-)$ and hence $\ell = \mu\ell_C$ for some $\mu > 0$. □

In general, not every alternating chain or cycle gives rise to a rigid linear function. For example, let $(P, \preceq)$ be a poset that is not the antichain and define the non-compatible double poset $\mathbf{P} = (P, \preceq, \preceq^{\mathrm{op}})$. In this case, $\mathbb{DO}(\mathbf{P})$ is, up to translation, the polytope $2\mathcal{O}(P, \preceq)$, whose facets correspond to cover relations in P by (1.1). Hence, every rigid ℓ is of the form $\ell = \mu\ell_{p,q}$ for cycles $p \prec_+ q \prec_- p$ where $p \prec\!\cdot\, q$ is a cover relation in P.

In the following, we will focus on the case of compatible double posets. We first observe that they can be characterized as follows.

Proposition 1.27. *Let* $\mathbf{P}$ *be a double poset. Then* $\mathbb{TO}(\mathbf{P})$ *contains the origin in its interior if and only if* $\mathbf{P}$ *is compatible.*

Proof. If $\mathbf{P} = (P, \preceq_\pm)$ is compatible, let f be a common linear extension of P_+ and P_- with associated maximal chain of filters $C \subseteq \mathcal{J}(P_+) \cap \mathcal{J}(P_-)$. Then $\mathbb{TO}(\mathbf{P})$ contains the full-dimensional Cayley sum $2F(C) \boxminus -2F(C)$, whose interior clearly contains the origin.

If $\mathbf{P}$ is not compatible, then by Proposition 1.7 it contains an alternating cycle C of the form $p_0 \prec_+ p_1 \prec_- p_2 \prec_+ \cdots \prec_- p_{2k} = p_0$. For a filter $\mathsf{J}_+ \subseteq P_+$ we have that $p_{2i} \in \mathsf{J}_+$ implies $p_{2i+1} \in \mathsf{J}_+$ for $0 \le i < k$. Hence $\ell_C(\mathbf{1}_{\mathsf{J}_+}) \le 0$ for every filter $\mathsf{J}_+ \subseteq P_+$ and therefore $\ell_C \le 0$ on $\mathcal{O}(P_+)$. Analogously, we obtain that $\ell_C \le 0$ on $-\mathcal{O}(P_-)$. Hence $\mathbb{TO}(\mathbf{P})$ is contained in the negative halfspace of $H = \{(f,t) : \ell_C(f) \le 0\}$. This finishes the proof, since $\mathbf{0} \in H$. □

Compatibility assures that in an alternating chain $p_i \prec_\sigma p_j$ implies $i < j$ for $\sigma \in \{\pm\}$. This also shows the following.

Lemma 1.28. *Let* $\mathbf{P} = (P, \preceq_\pm)$ *be a compatible double poset. If* $a_i \prec_\sigma a_{i+1} \prec_{-\sigma} \cdots \prec_{-\tau} a_j \prec_\tau a_{j+1}$ *is part of an alternating chain with* $\sigma, \tau \in \{\pm\}$ *and* $i < j$ *then there is no* $b \in P$ *such that* $a_i \prec_\sigma b \prec_\sigma a_{i+1}$ *and* $a_j \prec_\tau b \prec_\tau a_{j+1}$.

For compatible double posets, the following result gives a complete characterization of facets of double order polytopes.

Theorem 1.29. *Let* $\mathbf{P}$ *a compatible double poset. A linear function* ℓ *is rigid for* $\mathbb{DO}(\mathbf{P})$ *if and only if* $\ell \in \mathbb{R}_{>0}\ell_C$ *for some proper alternating chain* C*. In particular, the facets of* $\mathbb{TO}(\mathbf{P})$ *are in bijection with alternating chains.*

Proof. The facets $2\mathcal{O}(P_+) \times \{1\}$ and $-2\mathcal{O}(P_-) \times \{-1\}$ of $\mathbb{TO}(\mathbf{P})$ correspond to the improper alternating chains $\widehat{0} \prec_\sigma \widehat{1}$ for $\sigma \in \{\pm\}$. By Proposition 1.26 it remains to show that for any proper alternating chain C the function ℓ_C is rigid. We only consider the case that C is an alternating chain of the form

$$\widehat{0} \;=\; p_0 \;\prec_+\; p_1 \;\prec_-\; p_2 \;\prec_+\; \cdots \;\prec_+\; p_{2k-1} \;\prec_-\; p_{2k} \;\prec_+\; p_{2k+1} \;=\; \widehat{1}.$$

The other cases can be treated analogously. Let $F_+ = \mathcal{O}(P_+)^{\ell_C}$ and and $F_- = \mathcal{O}(P_-)^{-\ell_C}$ be the corresponding faces with reduced face partitions $\mathcal{B}_\pm$. Define $O = \{p_1, p_3, \ldots, p_{2k-1}\}$ and $E = \{p_2, p_4, \ldots, p_{2k}\}$. Then for any set $A \subseteq P$, we observe that $\ell_C(\mathbf{1}_A) = |E \cap A| - |O \cap A|$. If J is a filter of P_+, then $p_{2i} \in \mathsf{J}$ implies $p_{2i+1} \in \mathsf{J}$ and hence $\ell_C(\mathbf{1}_\mathsf{J}) \le 1$ and thus $\mathbf{1}_\mathsf{J} \in F_+$ if and only if J does not separate p_{2j} and p_{2j+1} for $1 \le j \le k$. Likewise, a filter $\mathsf{J} \subseteq P_-$ is contained in F_- if and only if J does not separate p_{2j-1} and p_{2j} for $1 \le j \le k$. Lemma 1.28 implies that

$$\begin{aligned} \mathcal{B}_+ &= \{[p_0, p_1]_{P_+}, [p_2, p_3]_{P_+}, \ldots, [p_{2k}, p_{2k+1}]_{P_+}\} \text{ and} \\ \mathcal{B}_- &= \{[p_1, p_2]_{P_-}, [p_3, p_4]_{P_-}, \ldots, [p_{2k-1}, p_{2k}]_{P_-}\}. \end{aligned}$$

To show that ℓ_C is rigid, pick a linear function $\ell(\varphi) = \sum_{p \in P} \ell_p \varphi(p)$ with $F_+ = \mathcal{O}(P_+)^\ell$ and $F_- = \mathcal{O}(P_-)^{-\ell}$. Since the elements in E and O are exactly the minima and maxima of the blocks in $\mathcal{B}_+$, it follows from Corollary 1.17 that $\ell_p > 0$ if $p \in E$, $\ell_p < 0$ for $p \in O$. By Lemma 1.28, it follows that if $q \in (p_i, p_{i+1})_{P_+}$, then q is not contained in a block of the reduced face partition $\mathcal{B}_-$ and vice versa. By Corollary 1.17(iii), it follows that $\ell_p = 0$ for $p \notin E \cup O$. Finally, $\ell_{p_i} + \ell_{p_{i+1}} = 0$ for all $1 \le i \le 2k$ by Proposition 1.16 and therefore $\ell = \mu \ell_C$ for some $\mu > 0$, which finishes the proof. □

Example 1.30. Let $n \in \mathbb{N}_{>0}$.

(1) Alternating chains in $\mathbf{C}_n$ can be identified with $\{-,+\}^{n+1}$ and hence, by Theorem 1.29, $\mathbb{TO}(\mathbf{C})$ has 2^{n+1} facets, which are all simplices. More explicitly, $\mathbb{TO}(\mathbf{C})$ is linearly isomorphic to the **cross polytope** $C^{\triangle}_{n+1} := \operatorname{conv}\{\pm \mathbf{e}_1, \ldots, \pm \mathbf{e}_{n+1}\}$ of dimension $n+1$. Here, for $i \in [n+1]$ we denote by $\mathbf{e}_i$ the ith standard basis vector in $\mathbb{R}^{n+1}$.
(2) All proper alternating chains in the double antichain $\mathbf{A}_n$ with ground set $[n]$ are of the form $\widehat{0} \prec_\sigma a \prec_{-\sigma} \widehat{1}$ for $a \in \mathbf{A}_n$ and $\sigma \in \{\pm\}$ and $\mathbb{TO}(\mathbf{A}_n) = [-2, +2]^{n+1}$ is isomorphic to the cube.
(3) It is easily seen that Comb_n (see Figure 1.3) has $2^{n+1} - 1$ filters and $3 \cdot 2^n - 2$ chains. Hence, $\mathbb{TO}(C_n, \preceq, \preceq)$ has $2^{n+2} - 2$ vertices and $3 \cdot 2^{n+1} - 4$ facets.
(4) Since the double poset $\mathbf{Alt}_n$ from Example 1.4 is compatible, the number of facets of $\mathbb{TO}(\mathbf{Alt}_n)$ equals the number of alternating chains which is easily computed to be $\binom{n+3}{2} + 1$.

(5) Consider the double poset $\mathbf{AC}_n$ from Example 1.4. Any proper alternating chain is either of the form $\widehat{0} \prec_\sigma a \prec_{-\sigma} \widehat{1}$ for $\sigma = \pm$ or of the form $\widehat{0} \prec_- a \prec_+ b \prec_- \widehat{1}$ with $a, b \in [n]$. Thus, $\mathbb{TO}(\mathbf{AC}_n)$ is a $(n+1)$-dimensional polytope with 2^n+n+1 vertices and $\binom{n}{2} + 2n + 2$ facets.

Example 1.31. Consider the double poset $\mathbf{P}_{XW}$ from Example 1.5, whose Hasse diagrams are given in Figure 1.6. The six-dimensional double order polytope $\mathbb{TO}(\mathbf{P}_{XW})$ has 28 facets, with corresponding alternating chains shown in Figure 1.7. More generally, using `sage` [18] we compute the f-vector

$$f(\mathbb{TO}(\mathbf{P}_{XW})) \;=\; (21, 112, 247, 263, 135, 28).$$

Example 1.32 (Dimension-2 posets and plane posets)**.** Let $\pi = (\pi_1, \pi_2, \dots, \pi_n)$ be an ordered sequence of distinct numbers with $n \in \mathbb{N}_{>0}$.

(1) Consider the associated dimension-2 poset $P_\pi = ([n], \preceq_\pi)$. We have seen in Example 1.3 that filters and chains in P_π correspond to decreasing and increasing sequences in π, respectively. Thus, if $\mathbf{P}_\pi$ denotes the double poset induced by P_π, it follows that vertices and facets of $\mathbb{TO}(\mathbf{P}_\pi)$ are in two-to-one correspondence with decreasing and increasing sequences, respectively.

(2) Let $\mathbf{P}$ be the plane poset associated to π, that is, the double poset with $(\mathbf{P})_+ = P_\pi$ and $(\mathbf{P})_- = P_{-\pi}$. By Example 1.9 and Theorem 1.29, the vertical facets of $\mathbb{TO}(\mathbf{P})$ are in bijection to alternating sequences in π, whereas the vertices are in bijection to increasing and decreasing sequences of π.

As a consequence of the proof of Theorem 1.29 we can determine a facet-defining inequality description of double order polytopes. For an alternating chain C, let us write $\operatorname{sgn}(C) = \tau \in \{\pm\}$ if the last relation in C is $\prec_\tau$.

Corollary 1.33. *Let $\mathbf{P} = (P, \preceq_\pm)$ be a compatible double poset. Then $\mathbb{TO}(\mathbf{P})$ is the set of points $(f, t) \in \mathbb{R}^P \times \mathbb{R}$ such that*

$$L_C(f, t) \;:=\; \ell_C(f) - \operatorname{sgn}(C)\, t \;\le\; 1$$

for all alternating chains C of $(P, \preceq_\pm)$.

Proof. Note that 0 is contained in the interior of $\mathbb{TO}(P_+, P_-)$ by Proposition 1.27. Hence by Theorem 1.29 every facet-defining halfspace of $\mathbb{TO}(P_+, P_-)$ is of the form $\{(\varphi, t) : L(\varphi, t) = \mu\ell_C + \beta t \le 1\}$ for some alternating chain C and $\mu, \beta \in \mathbb{R}$ with $\mu > 0$. If C is an alternating chain with $\operatorname{sgn}(C) = +$, then the maximal value of ℓ_C over $2\mathcal{O}(P_+)$ is 2 and 0 over $-2\mathcal{O}(P_-)$. The values are exchanged for $\operatorname{sgn}(C) = -$. It then follows that $\alpha = 1$ and $\beta = -\operatorname{sgn}(C)$. □

Let us remark that the number of facets of a given double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ can be computed by the transfer-matrix method (see [74, Sec. 4.7]). Define matrices $\eta^+, \eta^- \in \mathbb{R}^{\widehat{P} \times \widehat{P}}$ by

$$\eta^\sigma_{a,b} \;:=\; \begin{cases} 1 & \text{if } a \prec_\sigma b \\ 0 & \text{otherwise} \end{cases}$$

for $a, b \in \widehat{P}$ and $\sigma = \pm$.

Corollary 1.34. *Let* $\mathbf{P} = (P, \preceq_+, \preceq_-)$ *be a compatible double poset. Then the number of facets of* $\mathbb{TO}(\mathbf{P})$ *is given by*

$$\left[(\mathrm{Id} - \eta^+\eta^-)^{-1}(\mathrm{Id} + \eta^+) + (\mathrm{Id} - \eta^-\eta^+)^{-1}(\mathrm{Id} + \eta^-)\right]_{\widehat{0},\widehat{1}}.$$

Proof. By [74, Thm. 4.7.1], the entry $(\eta^+\eta^-)^k_{\widehat{0},\widehat{1}}$ equals the number of alternating chains of $\mathbf{P}$ on $k+1$ elements starting with $\prec_+$ and ending with $\prec_-$. Since $\mathbf{P}$ is compatible, this implies that the matrix $\eta^+\eta^-$ is nilpotent. It follows that $\left[(\mathrm{Id} - \eta^+\eta^-)^{-1}(\mathrm{Id} + \eta^+)\right]_{\widehat{0},\widehat{1}}$ equals the number of alternating chains starting with $\prec_+$. The analogous argument for $\eta^-\eta^+$ finishes the proof. $\square$

1.3.2 2-levelness

A polytope $\mathcal{P} \subset \mathbb{R}^n$ is called **2-level** if for any facet-defining hyperplane H there is a $t \in \mathbb{R}^n$ such that the two parallel hyperplanes H and $t + H$ contain all vertices of $\mathcal{P}$.

Proposition 1.35. *For any poset* P*, the order polytope* $\mathcal{O}(P)$ *is* 2*-level.*

Proof. For every cover relation $a \prec b$ in $\widehat{P}$ the functional $\ell_{a,b}$ takes only two distinct values on the vertex set $V(\mathcal{O}(P)) = \{\mathbf{1}_\mathsf{J} : \mathsf{J} \in \mathcal{J}(P)\}$. By (1.1), this finishes the proof. $\square$

Note that it is in general *not* true that $\mathcal{P}_1 \boxplus \mathcal{P}_2$ is 2-level if $\mathcal{P}_1$ and $\mathcal{P}_2$ are. Counterexamples are for instance the polytopes $\Delta_{6,2} \boxplus \Delta_{6,4}$ or $\mathrm{tprism}(\Delta_{6,2}) = \Delta_{6,2} \boxminus \Delta_{6,2}$, where

$$\Delta_{n,k} \ := \ \{\mathbf{p} \in [0,1]^n : p_1 + p_2 + \cdots + p_n = k\}$$

is the (n,k)**-hypersimplex** (see [37]). In the light of the above, it is only natural to ask for which double posets $\mathbf{P}$ the double order polytope $\mathbb{TO}(\mathbf{P})$ is 2-level and, in fact, this was the starting point for our investigations. In the following, we will give a complete answer for the case of compatible double posets.

In [38] a double poset $(P, \preceq_+, \preceq_-)$ is called **tertispecial** if a and b are $\prec_-$-comparable whenever $a \prec_+ b$ is a cover relation for $a, b \in P$. The next result gives a necessary condition for a double poset to yield a 2-level polytope.

Proposition 1.36. *Let* $\mathbf{P} = (P, \preceq_\pm)$ *be a double poset. If* $\mathbb{TO}(\mathbf{P})$ *is* 2*-level, then* $\mathbf{P}$ *as well as* $(P, \preceq_-, \preceq_+)$ *are tertispecial.*

Proof. Let $\sigma \in \{\pm\}$ and let $a \prec_\sigma b$ be a cover relation. The linear function $\ell_{a,b}$ is facet defining for $\mathcal{O}(P_\sigma)$ and hence yields a facet for $\mathbb{TO}(\mathbf{P})$. If a, b are not comparable in $P_{-\sigma}$, then the filters $\emptyset, \{c \in P : c \succeq_{-\sigma} a\}$ and $\{c \in P : c \succeq_{-\sigma} b\}$ take three distinct values on $\ell_{a,b}$. $\square$

In the compatible case, we are now ready to prove the following.

Theorem 1.37. *Let* $\mathbf{P} = (P, \preceq_\pm)$ *be a compatible double poset. Then* $\mathbb{TO}(\mathbf{P})$ *is* 2*-level if and only if* $\preceq_+ = \preceq_- = \preceq$*. In this case, the number of facets of* $\mathbb{TO}(\mathbf{P})$ *is twice the number of chains in* $(P, \preceq)$*.*

Proof. If $\preceq_+ = \preceq_- = \preceq$, then every chain $C \subseteq \widehat{P}$ gives rise to two alternating chains C_+ and C_- with signs $+$ and $-$, respectively and every alternating chain arises this way. Note that every filter $\mathsf{J} \subseteq P$ separates precisely one relation in C. Therefore $\ell_{C_+}(\mathbf{1}_\mathsf{J}) \in \{0,1\}$ and $\ell_{C_+}(-\mathbf{1}_\mathsf{J}) \in \{0,-1\}$ for all $\mathsf{J} \in \mathcal{J}(P)$, which implies that L_{C_+} attains only the two values 1 and -1 on the vertices of $\mathbb{T}\mathcal{O}(\mathbf{P})$. Since $L_{C_-} = -L_{C_+}$, Corollary 1.33 implies that $\mathbb{T}\mathcal{O}(\mathbf{P})$ is 2-level.

The converse follows from Proposition 1.36 by noting that if both $(P, \preceq_+, \preceq_-)$ and $(P, \preceq_-, \preceq_+)$ are compatible and tertispecial then $\preceq_+ = \preceq_-$. $\square$

Note that since faces of 2-level polytopes are again 2-level, Theorem 1.37 yields a second proof of Proposition 1.35.

1.3.3 Polars and valuation polytopes

We will now connect double order polytopes to another class of polytopes, whose underlying combinatorial objects are the following. A real-valued **valuation** on a finite distributive lattice $(\mathcal{J}, \vee, \wedge)$ is a function $h : \mathcal{J} \to \mathbb{R}$ such that for any $a, b \in \mathcal{J}$,

$$h(a \vee b) \;=\; h(a) + h(b) - h(a \wedge b) \tag{1.4}$$

and $h(\widehat{0}) = 0$, where $\widehat{0}$ denotes the least element in $\mathcal{J}$. Geissinger [32] studied the **valuation polytope**

$$\mathrm{Val}(\mathcal{J}) \;:=\; \{h : \mathcal{J} \to [0,1] : h \text{ valuation}\} \subset \mathbb{R}^{\mathcal{J}}$$

and conjectured that its vertices are exactly the valuations with values in $\{0,1\}$. This was shown by Dobbertin [19]. Not much is known about the valuation polytope and Stanley's '*5-*'-Exercise [74, Ex. 4.61(h)] challenges the reader to find interesting combinatorial properties of $\mathrm{Val}(\mathcal{J})$.

By Birkhoff's theorem we may assume $\mathcal{J} = \mathcal{J}(P)$, that is, $\mathcal{J}$ is the lattice of filters of some poset P. In particular, for every valuation $h : \mathcal{J}(P) \to \mathbb{R}$ there is a unique $h_0 : P \to \mathbb{R}$ such that

$$h(\mathsf{J}) \;=\; \sum_{a \in \mathsf{J}} h_0(a),$$

for every filter $\mathsf{J} \in \mathcal{J}(P)$. Hence, $\mathrm{Val}(\mathcal{J})$ is linearly isomorphic to the $|P|$-dimensional polytope

$$\mathrm{Val}_0(P) \;:=\; \{h_0 : P \to \mathbb{R} : 0 \le h(\mathsf{J}) \le 1 \text{ for all filters } \mathsf{J} \subseteq P\}.$$

To connect the above to double order polytopes, we denote by

$$S^{\triangle} \;:=\; \{\ell \in (\mathbb{R}^n)^* : \ell(s) \le 1 \text{ for all } s \in S\} \tag{1.5}$$

the **polar** of a convex set $S \subset \mathbb{R}^n$. Note that $S^{\triangle}$ is bounded if and only if 0 is contained in the interior of S. For additional information we refer to [81, Sec.2.3]. Polarity relates order polytopes and valuation polytopes in the following way.

Theorem 1.38. *For any finite poset $(P, \preceq)$ with induced double poset $\mathbf{P} = (P, \preceq, \preceq)$ we have*

$$\mathbb{T}\mathcal{O}(\mathbf{P})^{\triangle} \;\cong\; \mathrm{tprism}(-\mathrm{Val}_0(P)).$$

Proof. A chain $C = \{a_0 \prec a_1 \prec \cdots \prec a_k\}$ in P yields two alternating chains C_+ and C_- in $\widehat{P}$ of signs $+$ and $-$, respectively. Define

$$\ell'_C(f) \;:=\; \ell_{C_+} \;=\; \sum_{i=0}^{k} (-1)^{k-i} f(a_i)$$

and $L'_C(f,t) := L_{C_+}(f,t) = \ell'_C(f) - t$. Note that $-L'_C = L_{C_-}$ and therefore Corollary 1.33 yields

$$\mathbb{T}\mathcal{O}(\mathbf{P})^{\triangle} \;=\; \operatorname{conv}(\pm L'_C(f,t) : C \subseteq P \text{ chain}).$$

It is shown in Dobbertin [19, Theorem B] that

$$\mathrm{Val}_0(P) \;=\; \operatorname{conv}\left(\ell'_C : C \subseteq P \text{ chain}\right),$$

from which the claim follows. □

A polytope $\mathcal{P} \subset \mathbb{R}^n$ is called **centrally symmetric** if $\mathcal{P} = -\mathcal{P}$, that is for every point $\mathbf{p} \in \mathcal{P}$, its negative $-\mathbf{p}$ also lies in $\mathcal{P}$. Clearly, every centrally symmetric polytope contains the origin in its relative interior. Taking polars does in general not preserve 2-levelness. However, it does if we restrict to the class of centrally symmetric polytopes.

Proposition 1.39. *Let $\mathcal{P}$ be a full-dimensional centrally symmetric 2-level polytope. Then $\mathcal{P}^{\triangle}$ is also centrally symmetric and 2-level.*

Proof. It follows from (1.5) that the polar of any centrally symmetric polytope is again centrally symmetric. To prove 2-levelness of $\mathcal{P}^{\triangle}$, by duality it suffices to show that for every vertex $\mathbf{v} \in \mathcal{P}$ every facet $F \subset \mathcal{P}$ contains either $\mathbf{v}$ or $-\mathbf{v}$. If we assume the contrary, that is $F \cap \{\mathbf{v}, -\mathbf{v}\} = \emptyset$, then also $-F \cap \{\mathbf{v}, -\mathbf{v}\} = \emptyset$ since $\mathcal{P}$ is centrally symmetric. This contradicts the 2-levelness of $\mathcal{P}$. □

As a direct consequence, we note the following.

Corollary 1.40. *Let P be a finite poset, then* $\operatorname{tprism}(\mathrm{Val}_0(P))$ *is 2-level.*

Proof. Since $\mathbb{T}\mathcal{O}(\mathbf{P})$ is centrally-symmetric and, by Theorem 1.37, 2-level, the claim follows from Theorem 1.38 and Proposition 1.39. □

We can make the connection to valuations more transparent by considering valuations with values in $[-1,1]$. In this case, for a distributive lattice $\mathcal{J} = \mathcal{J}(P)$ the associated polytope

$$\mathrm{Val}(\mathcal{J}) \;:=\; \{h : \mathcal{J} \to [-1,1] : h \text{ valuation}\}$$

is centrally symmetric and linearly isomorphic to

$$\mathrm{Val}_0^{\pm}(P) \;:=\; \{h_0 : P \to \mathbb{R} : -1 \le h(\mathsf{J}) \le 1 \text{ for all filters } \mathsf{J} \subseteq P\} \;=\; (\mathcal{O}(P) \cup -\mathcal{O}(P))^{\triangle}.$$

Now, the convex hull of $\mathcal{O}(P) \cup -\mathcal{O}(P)$ is exactly the image of $\frac{1}{2}\mathbb{T}\mathcal{O}(\mathbf{P})$ under the projection $\pi : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P$ with $\pi(f,t) = f$. Denote by $\pi^* : (\mathbb{R}^P)^* \to (\mathbb{R}^P \times \mathbb{R})^*$ the dual map defined by $\pi^*(\ell)(f,t) := \ell(f)$. We have

$$\mathrm{Val}_0^{\pm}(P) \;\cong\; \pi(\tfrac{1}{2}\mathbb{T}\mathcal{O}(\mathbf{P}))^{\triangle} \;\cong\; (2\mathbb{T}\mathcal{O}(\mathbf{P})^{\triangle}) \cap \operatorname{im}(\pi^*) \;\cong\; \operatorname{tprism}(-2\mathrm{Val}_0(P)) \cap (\mathbb{R}^P \times \{0\}),$$

where the last step follows from Theorem 1.38. We can interpret the last expression as a Minkowski sum by (1.2), which yields the following.

Corollary 1.41. *For any poset P*

$$\mathrm{Val}_0^{\pm}(P) \;=\; \mathrm{Val}_0(P) - \mathrm{Val}_0(P).$$

Finally, we want to relate our results to work by Hibi, Matsuda, Ohsugi, Tsuchiya and Shibata ([48], [49], [50], [51]). A polytope $\mathcal{P} \subseteq \mathbb{R}^d$ is called **integral** if all its vertices have integer coordinates. Moreover, an integral polytope $\mathcal{P}$ is called **reflexive** if its polar $\mathcal{P}^{\triangle}$ is also integral. Reflexive polytopes are of particular interest, since they occur in connection with Calabi-Yau varieties and mirror symmetry (see [6]). For two polytopes $\mathcal{P}, \mathcal{Q} \subset \mathbb{R}^n$, write $\Gamma(\mathcal{P}, \mathcal{Q}) := \mathrm{conv}(\mathcal{P} \cup -\mathcal{Q})$. Thus, $\Gamma(\mathcal{P}, \mathcal{Q})$ is the projection of $\mathcal{P} \boxminus \mathcal{Q}$ onto the first n coordinates. Hibi and Matsuda [48] used Gröbner basis techniques to show that the polytopes $\Gamma(\mathcal{O}(P_+), \mathcal{O}(P_-))$ are reflexive whenever P_+ and P_- are posets on the same ground set that possess a common linear extension, i.e. whenever the double poset $\mathbf{P} := (P_+, P_-)$ is compatible. Our next observation in particular recovers a special case of their results.

Corollary 1.42. *For any poset P,*

$$\Gamma(\mathcal{O}(P), \mathcal{O}(P)) \;=\; (\mathrm{Val}_0(P) - \mathrm{Val}_0(P))^{\triangle}.$$

In particular, $\Gamma(\mathcal{O}(P), \mathcal{O}(P))$ is reflexive.

Chapter 2

Double Hibi rings

In this chapter, we study the toric ideals associated to double order polytopes. This will allow for geometric insights that are not obvious from a purely geometric perspective. Section 2.1 develops some necessary theory. We introduce Gröbner bases and toric ideals and explain a connection between triangulations and initial ideals which was discovered by Sturmfels [77]. In Section 2.2, after briefly treating toric ideals arising from ordinary order polytopes, which were first studied by Hibi [46], we compute a Gröbner basis for the toric ideals of double order polytopes. Finally, we use this Gröbner basis to describe triangulations and the complete facial structure of double order polytopes.

2.1 Toric ideals and polytopes

Every integral polytope has an associated toric ideal, an algebraic object which reflects the geometric properties of the polytope. Toric ideals connect concepts from discrete geometry, algebra and algebraic geometry (see for instance [16, 62, 77]) and can be used to study the geometry of lattice polytopes with the help of algebra and vice versa. In fact, discrete geometric questions are sometimes more accessible by taking the detour via algebra. In this section, we prepare for applying this strategy to double order polytopes: We introduce Gröbner bases and toric ideals and state a result that connects regular triangulations and initial ideals, which is nowadays known as Sturmfels' correspondence (see [77]).

2.1.1 Gröbner bases

We recall some basics from the theory of Gröbner bases. For additional information, the reader is referred to [23]. Let $\mathbf{x} := (x_1, \dots, x_n)$ be a list of variables with $n \in \mathbb{N}_{>0}$. A **monomial** is an expression of the form $\mathbf{x}^{\mathbf{a}} := x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ for some $\mathbf{a} \in \mathbb{N}^n$. The **support** of a monomial $\mathbf{x}^{\mathbf{a}}$ is the set $\operatorname{supp}(\mathbf{x}^{\mathbf{a}}) := \{i \in [n] : a_i \neq 0\}$ and its **degree** is $\deg(\mathbf{x}^{\mathbf{a}}) := a_1 + a_2 + \cdots + a_n$. A monomial is called **square-free** if $a_i \leq 1$ for all $i \in [n]$. For a fixed field $\mathbb{K}$, a finite $\mathbb{K}$-linear combination of monomials is called **polynomial** with coefficients in $\mathbb{K}$. For a polynomial f its **degree** $\deg(f)$ is the highest degree of a monomial occurring in f. If all terms of f have the same degree, f is called **homogeneous**. If $\deg(f) = 2$, we call f **quadratic**. A polynomial of the form $f = c\mathbf{x}^{\mathbf{a}} + d\mathbf{x}^{\mathbf{b}}$ with $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$ and $c, d \in \mathbb{K}$ is called a **binomial**.

The $\mathbb{K}$-vector space of polynomials in the variables $x_1, \ldots, x_n$ is called **polynomial ring** and will be denoted by $\mathbb{K}[\mathbf{x}] = \mathbb{K}[x_1, \ldots, x_n]$. An additive subgroup of $\mathbb{K}[\mathbf{x}]$ is called **ideal** if for any $f \in I$ and $g \in \mathbb{K}[\mathbf{x}]$ we have $fg \in I$. A set $G \subseteq \mathbb{K}[\mathbf{x}]$ **generates** I if I is the inclusionwise smallest ideal containing G. An ideal I is called a **monomial ideal** (resp. **binomial ideal**) if it can be generated by monomials (resp. binomials). Monomial ideals are called **square-free** if they are generated by square-free monomials and ideals which have a set of generators consisting of homogeneous polynomials are themselves called **homogeneous**. An ideal I is called **radical** if $f^k \in I$ for some $f \in \mathbb{K}[\mathbf{x}]$ and $k \geq 0$ implies $f \in I$. Moreover, I is called **prime** if whenever $fg \in I$ for some $f, g \in \mathbb{K}[\mathbf{x}]$ we have $f \in I$ or $g \in I$.

In the following, we will mainly consider that case $\mathbb{K} = \mathbb{C}$ and, unless stated differently, all polynomials have complex coefficients. Let M denote the set of monomials in the complex polynomial ring $\mathbb{C}[\mathbf{x}]$ and let $\preceq$ be a total order on M, that is, an ordering such that any two monomials are comparable. We call $\preceq$ a **monomial order** or **term order** on $\mathbb{C}[\mathbf{x}]$ if

(i) $1 \preceq u$ for all $u \in M$ and

(ii) if $u \preceq v$ and $w \in M$ then $uw \preceq uv$.

Some of the most common term orders are the following.

Example 2.1. (1) The **lexicographic order** $\preceq_{lex}$ is defined by $\mathbf{x}^{\mathbf{a}} \prec_{lex} \mathbf{x}^{\mathbf{b}}$ if and only if the left-most nonzero entry of $\mathbf{a} - \mathbf{b}$ is negative. We obtain the **degree lexicographic order** $\preceq_{dlex}$ by setting $\mathbf{x}^{\mathbf{a}} \prec_{dlex} \mathbf{x}^{\mathbf{b}}$ if either $\deg(\mathbf{x}^{\mathbf{a}}) < \deg(\mathbf{x}^{\mathbf{b}})$ or $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and $\mathbf{x}^{\mathbf{a}} \prec_{lex} \mathbf{x}^{\mathbf{b}}$.

(2) The **reverse lexicographic order** $\preceq_{rev}$ satisfies $\mathbf{x}^{\mathbf{a}} \prec_{rev} \mathbf{x}^{\mathbf{b}}$ if and only if the right-most nonzero entry of $\mathbf{a} - \mathbf{b}$ is positive. Analogously, the **degree reverse lexicographic order** $\preceq_{drev}$ is the monomial order with $\mathbf{x}^{\mathbf{a}} \prec_{drev} \mathbf{x}^{\mathbf{b}}$ if either $\deg(\mathbf{x}^{\mathbf{a}}) < \deg(\mathbf{x}^{\mathbf{b}})$ or $\deg(\mathbf{x}^{\mathbf{a}}) = \deg(\mathbf{x}^{\mathbf{b}})$ and $\mathbf{x}^{\mathbf{a}} \prec_{rev} \mathbf{x}^{\mathbf{b}}$.

(3) Any weight vector $\boldsymbol{\omega} \in \mathbb{R}^n_{\geq 0}$ whose entries are linearly independent over $\mathbb{Q}$ defines a monomial order $\preceq_{\boldsymbol{\omega}}$ called **weight order** by setting $\mathbf{x}^{\mathbf{a}} \prec_{\boldsymbol{\omega}} \mathbf{x}^{\mathbf{b}}$ whenever $\boldsymbol{\omega} \cdot \mathbf{a} < \boldsymbol{\omega} \cdot \mathbf{b}$.

Note that for the (reverse) lexicographic order we can think of x_1 as the largest variable, x_2 as the second largest variable and so on. More generally, for a given total ordering of the variables we can define an associated (reverse) lexicographic order by replacing "left-most" (resp. "right-most") by "largest" (resp. "smallest") with respect to the variable ordering.

For a polynomial $f \in \mathbb{C}[\mathbf{x}]$, the term which contains the largest monomial with respect to $\preceq$ among all monomials occurring in f is called the **initial term** or **leading term** of f and is denoted by $\mathrm{in}_{\preceq}(f)$. All other terms are called **trailing terms**. The non-zero constant coefficient of $\mathrm{in}_{\preceq}(f)$ is called the **leading coefficient** of f. For an ideal $I \subseteq \mathbb{C}[\mathbf{x}]$ the ideal $\mathrm{in}_{\preceq}(I)$ generated by $\{\mathrm{in}_{\preceq}(f) : f \in I\}$ is called the **initial ideal** of I with respect to $\preceq$. A subset $G \subseteq I$ is called a **Gröbner basis** of I with respect to $\preceq$ if $\{\mathrm{in}_{\preceq}(g) : g \in G\}$ generates $\mathrm{in}_{\preceq}(I)$. A Gröbner basis G is called **reduced** if

(i) all $g \in G$ have leading coefficient 1 and

(ii) for $g, g' \in G$ no term of g' is divisible by $\mathrm{in}_{\preceq}(g)$.

Give a term order $\preceq$, every ideal has a unique reduced Gröbner basis, which can be computed using Buchberger's algorithm (see [13]). Gröbner bases yield algorithms for many classical problems in commutative algebra and play a key role in computational algebra (see for instance [23]).

For an ideal I, we say that a term order $\preceq$ is **represented** by a weight order $\preceq_{\boldsymbol{\omega}}$ with $\boldsymbol{\omega} \in \mathbb{R}^n$ if $\mathrm{in}_{\preceq}(I) = \mathrm{in}_{\preceq_{\boldsymbol{\omega}}}(I)$. The following result shows that is is often sufficient to study weight orders.

Proposition 2.2 ([77, Prop.1.11])**.** *Let $I \subseteq \mathbb{C}[\mathbf{x}]$ be an ideal and $\preceq$ a term order. Then there exists a weight $\boldsymbol{\omega} \in \mathbb{R}^n_{\geq 0}$ such that $\preceq_{\boldsymbol{\omega}}$ represents $\preceq$ for I.*

2.1.2 Toric ideals

We provide a short introduction to toric ideals. For a more detailed treatment, we refer the reader to [62]. In the literature, the word *lattice* unfortunately has two different meanings. In addition to the poset-theoretic context it is used as follows. A set $\Lambda \subset \mathbb{R}^n$ is called a **lattice** if Λ is an abelian subgroup of $\mathbb{R}^n$ generated by k linearly independent vectors for some $k \leq n$. The number k is called the **rank** of Λ and will be denoted by $\mathrm{rk}(\Lambda)$. For a finite set $\mathcal{A} \subseteq \Lambda$, we denote by $\mathbb{Z}\mathcal{A}$ the set of all integral linear combinations of elements in $\mathcal{A}$. Phrased differently, $\mathbb{Z}\mathcal{A}$ is the inclusionwise smallest lattice in Λ that contains $\mathcal{A}$. If $\mathcal{A} \subset \Lambda$ with $|\mathcal{A}| = \mathrm{rk}(\Lambda)$ and $\mathbb{Z}\mathcal{A} = \Lambda$, then we call $\mathcal{A}$ **basis** of Λ. The Euclidean volume of the parallelotope spanned by a basis $A \subset \Lambda$ is called the **determinant** of Λ and will be denoted by $|\Lambda|$. Note that the volume does not depend on the choice of basis.

A subset $\Lambda \subset \mathbb{R}^n$ is called an **affine lattice** if there exists $\mathbf{v} \in \mathbb{R}^n$ such that $\Lambda + \mathbf{v}$ is a lattice. For a lattice $\Lambda \subset \mathbb{R}^n$ and an affine subspace $U \subseteq \mathbb{R}^n$ of dimension k such that $\Lambda \cap U \neq \emptyset$, the affine lattice $\Lambda \cap U \subset U \cong \mathbb{R}^k$ is called the **induced lattice** of Λ in U. Note that for every set $\mathcal{A} \subseteq \mathbb{Z}^n$, there exists a unique smallest affine lattice $\Lambda \subset \mathbb{Z}^n$ containing $\mathcal{A}$. We will call Λ the affine lattice **generated** by $\mathcal{A}$. Explicitly, Λ can be constructed as follows. Consider the homogenized point set $\tilde{\mathcal{A}} = \mathcal{A} \times \{1\} \subseteq \mathbb{Z}^{n+1}$. Then Λ is the induced lattice $\mathbb{Z}\tilde{\mathcal{A}} \cap \mathrm{aff}(\mathcal{A}) \subseteq \mathbb{Z}^n$.

Of course, the most natural example of a lattice is $\mathbb{Z}^n$ itself. For $\mathcal{A} \subseteq \mathbb{Z}^n$ we denote by $\mathbb{N}\mathcal{A}$ the **affine semigroup** generated by $\mathcal{A}$, that is, the set of $\mathbb{N}$-linear combinations of elements of $\mathcal{A}$. The affine semigroup $\mathbb{N}\mathcal{A}$ is called **normal** if every lattice point in the cone $C := \mathrm{cone}(\mathcal{A})$ lies in $\mathbb{N}\mathcal{A}$, that is, if $C \cap \mathbb{Z}\mathcal{A} = \mathbb{N}\mathcal{A}$.

For a set of lattice points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{Z}^n$, consider the semigroup homomorphism

$$\varphi : \mathbb{N}^m \to \mathbb{Z}^n, \mathbf{u} \mapsto u_1\mathbf{a}_1 + u_2\mathbf{a}_2 + \cdots + u_m\mathbf{a}_m$$

for $\mathbf{u} = (u_1, \ldots, u_m) \in \mathbb{N}^m$. Note that the image of φ is the affine semigroup $\mathbb{N}\mathcal{A}$. On the algebraic level, φ induces a ring homomorphism between the polynomial ring $\mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, \ldots, x_m]$ and the Laurent ring $\mathbb{C}[\mathbf{y}, \mathbf{y}^{-1}] = \mathbb{C}[y_1, y_1^{-1}, \ldots, y_n, y_n^{-1}]$ defined on generators as

$$\overline{\varphi} : \mathbb{C}[\mathbf{x}] \to \mathbb{C}[\mathbf{y}, \mathbf{y}^{-1}], x_i \mapsto \mathbf{y}^{\mathbf{a}_i}.$$

The ideal $I_{\mathcal{A}} = \ker(\overline{\varphi}) := \{f \in \mathbb{C}[\mathbf{x}] : \overline{\varphi}(f) = 0\}$ is called the **toric ideal** of $\mathcal{A}$. The image $\mathrm{im}(\overline{\varphi})$ is the **semigroup ring** associated to $\mathbb{N}\mathcal{A}$. Toric ideals have the following properties.

Proposition 2.3 ([77, Ch. 4])**.** *Let $\mathcal{A} \subset \mathbb{Z}^n$ with $|\mathcal{A}| = m$.*

(1) *The ideal $I_{\mathcal{A}}$ is a prime ideal generated by the binomials*

$$\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \text{ for } \mathbf{u}, \mathbf{v} \in \mathbb{N}^m \text{ satisfying } \varphi(\mathbf{u}) \;=\; \varphi(\mathbf{v}).$$

(2) *$I_{\mathcal{A}}$ is homogeneous if and only if $\mathcal{A}$ lies in a hyperplane that does not contain the origin.*

In the following, we will also call a set $\mathcal{A} \subset \mathbb{Z}^n$ **homogeneous** if it gives rise to a homogeneous toric ideal. A particularly interesting class of homogenous toric ideals arises from polytopes: To an integral polytope $\mathcal{P} \subset \mathbb{R}^n$ we associate the toric ideal $I_{\mathcal{P}} := I_{V(\mathcal{P})\times\{1\}}$, where $V(\mathcal{P}) \times \{1\} = \{(\mathbf{v}, 1) : \mathbf{v} \in V(\mathcal{P})\} \subset \mathbb{R}^{n+1}$ is the vertex set of $\mathcal{P}$ embedded at height 1 into $\mathbb{R}^{n+1}$.

Remark 2.4. Note that for an integral polytope $\mathcal{P} \subseteq \mathbb{R}^n$, there are several different ways of associating toric ideals. The most common way is to consider the ideal $I_{\mathcal{A}}$ where $\mathcal{A} = \mathcal{P} \cap \mathbb{Z}^n$ (see, for instance, [16, Sec. 2.3]). However, for our purpose it is more useful to only consider vertices instead of all integral points in $\mathcal{P}$.

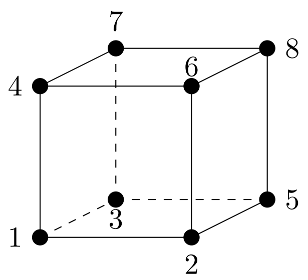

Figure 2.1: The cube $\mathcal{O}(\mathrm{A}_3)$.

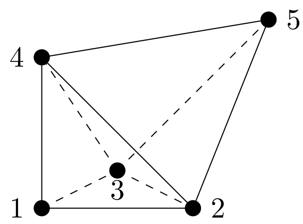

Figure 2.2: The bipyramid over a triangle, realized with vertices on the 3-cube.

Example 2.5. Consider the 3-cube $\mathcal{O}(\mathrm{A}_3)$ with vertices labelled as in Figure 2.1. We identify its vertices with the ordered variables $x_1 < \cdots < x_8$ and consider the induced monomial orders $\preceq_{rev}$ and $\preceq_{lex}$ (cf. Example 2.1). Then the reduced Gröbner bases of $I_{\mathcal{O}(\mathrm{A}_3)}$ with respect to $\preceq_{rev}$ and $\preceq_{lex}$ each consist of nine square-free quadratic binomials, given respectively in the columns

$$\begin{array}{lcl}
\underline{x_2x_3} - x_1x_5 & & \underline{x_1x_5} - x_2x_3 \\
\underline{x_2x_4} - x_1x_6 & & \underline{x_1x_6} - x_2x_4 \\
\underline{x_2x_7} - x_1x_8 & & \underline{x_1x_7} - x_3x_4 \\
\underline{x_3x_4} - x_1x_7 & \text{and} & \underline{x_1x_8} - x_4x_5 \\
\underline{x_3x_6} - x_1x_8 & & \underline{x_2x_7} - x_4x_5 \\
\underline{x_4x_5} - x_1x_8 & & \underline{x_2x_8} - x_5x_6 \\
\underline{x_5x_6} - x_2x_8 & & \underline{x_3x_6} - x_4x_5 \\
\underline{x_5x_7} - x_3x_8 & & \underline{x_3x_8} - x_5x_7 \\
\underline{x_6x_7} - x_4x_8 & & \underline{x_4x_8} - x_6x_7.
\end{array}$$

Here, the underlined term of every binomial denotes its leading term. We will use this notation throughout this work.

Example 2.6. Let $\mathcal{P} = \operatorname{conv}(\{(0,0,0),(1,0,0),(0,1,0),(0,0,1),(1,1,1)\})$ be a bipyramid over a triangle with vertex labels as in Figure 2.2. The reduced Gröbner basis with respect to the corresponding reverse lexicographic order $\preceq_{rev}$ is given by the single cubic $\underline{x_2x_3x_4} - x_1^2x_5$. The lexicographic order $\preceq_{lex}$ yields the same binomial, but with the other term as leading term, that is $\underline{x_1^2x_5} - x_2x_3x_4$.

The following result relates the facial structure of integral polytopes to algebraic properties of the associated toric ideals. We include a proof, since we could not find this exact statement in the literature.

Lemma 2.7. *Let $\mathcal{A} \subset \mathbb{Z}^n$ be homogeneous with associated integral polytope $\mathcal{P} = \operatorname{conv}(\mathcal{A})$. Let $M \subseteq \mathcal{A}$. Then $\operatorname{conv}(M)$ is a face of $\mathcal{P}$ satisfying $\operatorname{conv}(M) \cap \mathcal{A} = M$ if and only if*

$$f(\mathbf{1}_M) \;=\; 0 \text{ for all } f \in I_{\mathcal{A}},$$

where $\mathbf{1}_M \in \mathbb{C}^{\mathcal{A}}$ denotes the characteristic vector of M.

Proof. Let $M \subseteq \mathcal{A}$. Observe that since $\mathcal{A}$ is homogeneous we have $f(\mathbf{1}_M) = 0$ for all $f \in I_{\mathcal{A}}$ if and only for every $k \in \mathbb{N}_{>0}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_k, \mathbf{b}_1, \ldots, \mathbf{b}_k \in \mathcal{A}$ with

$$\mathbf{a}_1 + \cdots + \mathbf{a}_k \;=\; \mathbf{b}_1 + \cdots + \mathbf{b}_k$$

it holds that

$$\mathbf{a}_1, \ldots, \mathbf{a}_k \in M \text{ if and only if } \mathbf{b}_1, \ldots, \mathbf{b}_k \in M.$$

Again using the homogeneity of $\mathcal{A}$, this is equivalent to the condition that for $\mathbf{a}, \mathbf{b} \in \mathbb{N}\mathcal{A}$ we have that

$$\mathbf{a} + \mathbf{b} \in \mathbb{N}M \text{ implies } \mathbf{a}, \mathbf{b} \in \mathbb{N}M.$$

Using the language of [62, Ch. 7.2], this condition means that the affine semigroup $\mathbb{N}M$ is a face of $\mathbb{N}\mathcal{A}$, which is by [62, Lem.7.12] equivalent to $\operatorname{conv}(M)$ being a face of $\mathcal{P}$ and $\operatorname{conv}(M) \cap \mathcal{A} = M$. □

2.1.3 Regular triangulations and initial complexes

For a finite set $\mathcal{A} \subset \mathbb{Z}^n$, a polytopal complex $\mathcal{K}$ in $\mathbb{R}^n$ is called a **subdivision** (resp. **triangulation**) of $\mathcal{A}$ if it is a subdivision (resp. triangulation) of $\operatorname{conv}(\mathcal{A})$ (cf. Section 1.2.3) satisfying $V(\mathcal{P}) \subseteq \mathcal{A}$ for every $\mathcal{P} \in \mathcal{K}$. An important class of subdivisions and triangulations can be constructed as follows. For a set of points $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\} \subset \mathbb{R}^n$ and a vector $\boldsymbol{\omega} \in \mathbb{R}^m$, consider the lifted configuration $\mathcal{A}_{\boldsymbol{\omega}} = \{(\mathbf{a}_1, \omega_1), \ldots, (\mathbf{a}_m, \omega_m)\} \subset \mathbb{R}^{n+1}$. A **lower face** of the polytope $\mathcal{P}_{\boldsymbol{\omega}} := \operatorname{conv}(\mathcal{A}_{\boldsymbol{\omega}})$ is a face that is "visible from below." More precisely, a face is lower if it has an outer normal vector with negative last coordinate. Let $\Delta_{\boldsymbol{\omega}}$ be the polyhedral complex consisting of the projection of all lower faces of $\mathcal{P}_{\boldsymbol{\omega}}$ onto the first n coordinates. A subdivision or triangulation of $\mathcal{A}$ (resp. of the polytope $\mathcal{P} := \operatorname{conv}(\mathcal{A})$) is called **regular** if it is of the form $\Delta_{\boldsymbol{\omega}}$ for some height function $\boldsymbol{\omega}$.

One can also construct subdivisions and triangulations taking a more algebraic approach. Let $\preceq$ be a monomial order on $\mathbb{C}[\mathbf{x}] := \mathbb{C}[x_1, \ldots, x_m]$ and let $I \subseteq \mathbb{C}[\mathbf{x}]$ be an ideal. The **initial complex** $\Delta_{\preceq}(I)$ of I with respect to $\preceq$ is the abstract simplicial complex consisting of all $X \subseteq [m]$ such that there is no $f \in I$ whose initial monomial $\operatorname{in}_{\preceq}(f)$ has support X. In the case when I is toric, the abstract simplicial complex $\Delta_{\preceq}(I)$ comes with the following canonical geometric realization. For $\mathcal{A} = \{\mathbf{a}_1, \dots, \mathbf{a}_m\} \subset \mathbb{Z}^n$, we can identify the ground set of $\Delta_{\preceq}(I_{\mathcal{A}})$ with the set $\mathcal{A}$. Moreover, for every $X \in \Delta_{\preceq}(I_{\mathcal{A}})$, the points in $X \subseteq \mathcal{A}$ are affinely independent. Indeed, any affine dependence relation between the points in X yields a binomial in $I_{\mathcal{A}}$ of the form $u - v$ for binomials u, v satisfying $\operatorname{supp}(u) \cup \operatorname{supp}(v) \subseteq X$. Hence, we obtain a way of regarding $\Delta_{\preceq}(I_{\mathcal{A}})$ as a geometric simplicial complex with vertex set $\mathcal{A}$. In the following we will abuse notation and also denote this geometric simplicial complex by $\Delta_{\preceq}(I_{\mathcal{A}})$.

The following result, due to Sturmfels, elegantly connects regular triangulations and initial complexes.

Theorem 2.8 ([77, Thm.8.3]). *Let $\mathcal{A} \subset \mathbb{Z}^n$ be homogeneous with $|\mathcal{A}| = m$. Then the initial complexes of $I_{\mathcal{A}}$ are exactly the regular triangulations of $\mathcal{A}$. More precisely, let $\preceq$ be a monomial order with $\boldsymbol{\omega} \in \mathbb{R}^m$ such that $in_{\preceq_{\boldsymbol{\omega}}}(I_{\mathcal{A}}) = in_{\preceq}(I_{\mathcal{A}})$. Then*

$$\Delta_{\preceq}(I_{\mathcal{A}}) = \Delta_{\boldsymbol{\omega}}.$$

This theorem has some interesting implications, for which we need several more definitions. For a fixed lattice $\Lambda \subset \mathbb{R}^n$ we call a polytope $\mathcal{P} \subset \mathbb{R}^n$ a **lattice polytope** if $V(\mathcal{P}) \subseteq \Lambda$. If $\mathcal{P}$ is a full-dimensional lattice polytope, the **normalized volume** of $\mathcal{P}$ with respect to Λ is defined as

$$\operatorname{Vol}(\mathcal{P}) = \frac{n!}{|\Lambda|} \operatorname{vol}(\mathcal{P}),$$

where $|\Lambda|$ denotes the determinant of Λ. A triangulation Δ of a finite set $\mathcal{A} \subset \Lambda$ (or the polytope $\mathcal{P} := \operatorname{conv}(\mathcal{A})$) is called **unimodular** if every maximal simplex $F \in \Delta$ has normalized volume 1 with respect to Λ (see, for instance, [17, Ch. 9.3] or [77, Ch. 8]). In particular, if $\mathcal{P}$ has a unimodular triangulation with respect to Λ, then its normalized volume equals the number of simplices in the triangulation. Unless the lattice is specified explicitly, we will always assume that Λ is the affine lattice generated by $\mathcal{A}$.

A subdivision Δ of a finite set $\mathcal{A} \subset \mathbb{Z}^n$ is called **flag** if every minimal nonface has dimension 1. Phrased differently, this means that whenever $M \subseteq \mathcal{A}$ such that $\operatorname{conv}(M) \notin \Delta$ and $\operatorname{conv}(M') \in \Delta$ for all proper subsets $M' \subset M$ we have $|M| = 2$. The following results interpret the above geometric properties of triangulations as algebraic properties of the corresponding initial ideals.

Corollary 2.9 ([77, Cor.8.9]). *Let $\mathcal{A} \subset \mathbb{Z}^n$ with $|\mathcal{A}| = m$ be homogeneous and let $\preceq$ be a monomial order on $\mathbb{C}[x_1, \dots, x_m]$.*

(1) *The initial ideal $in_{\preceq}(I_{\mathcal{A}})$ is square-free if and only if the corresponding triangulation $\Delta_{\preceq}(I_{\mathcal{A}})$ is unimodular (with respect to the affine lattice generated by $\mathcal{A}$).*
(2) *The radical of $in_{\preceq}(I_{\mathcal{A}})$ is generated by quadratic monomials if and only if the corresponding triangulation $\Delta_{\preceq}(I_{\mathcal{A}})$ is flag.*

In the sequel, we will be mainly concerned with the case when $\mathcal{A}$ is the homogenized vertex set of an integral polytope. In this case, the regular triangulations corresponding to the monomial orders $\preceq_{lex}$ and $\preceq_{rev}$ are called **placing triangulations** and **pulling triangulations**, respectively (see [77, Ch. 8]). Both triangulations have a recursive geometric description. Let $\mathcal{P} \subset \mathbb{R}^n$ be a polytope with vertex set $\mathcal{A} = V(\mathcal{P}) = \{\mathbf{a}_1, \ldots, \mathbf{a}_m\}$. Note that for any subset $S \subseteq \mathcal{A}$, the ordering $\mathbf{a}_1 < \mathbf{a}_2 < \cdots < \mathbf{a}_m$ induces a total order on S and hence we may speak of placing and pulling triangulations of $\operatorname{conv}(S)$.

To construct the placing triangulation, we recursively consider a placing triangulation Δ' of $\operatorname{conv}(\mathcal{A} \setminus \{\mathbf{a}_1\})$. We call a face $F \in \Delta'$ **visible** from $\mathbf{a}_1$ if it has a supporting hyperplane that separates $\mathbf{a}_1$ from all other vertices $\mathcal{A} \setminus (F \cup \{\mathbf{a}_1\})$. To complete Δ' to a triangulation of $\mathcal{P}$, for every $F \in \Delta'$ visible from $\mathbf{a}_1$, we add the simplex $\{\operatorname{conv}(F \cup \{\mathbf{a}_1\})\}$ to finally obtain the placing triangulation $\Delta_{\prec_{lex}}(I_{\mathcal{P}})$.

For the pulling triangulation $\Delta_{\prec_{rev}}(I_{\mathcal{P}})$, we start with the vertex $\mathbf{a}_1$ and subdivide the polytope into pyramids of the form $\operatorname{conv}(F \cup \{\mathbf{a}_1\})$, where F is any facet of $\mathcal{P}$ that does not contain $\mathbf{a}_1$. Recursively, we consider a pulling triangulation of F, which yields a triangulation of the pyramids $\operatorname{conv}(F \cup \{\mathbf{a}_1\})$. Since the pulling order for every facet comes from the same global variable ordering, these triangulations are compatible on boundaries and together form the pulling triangulation of $\mathcal{P}$. For further details on placing and pulling triangulations, see for example [17, Sec. 4.3].

Example 2.10. Consider the 3-cube $\mathcal{O}(\mathrm{A}_3)$ from Example 2.5. It is straightforward to observe that the Gröbner basis with respect to the reverse lexicographic order $\preceq_{rev}$ corresponds to the canonical triangulation with six maximal simplices from Example 1.21. The placing triangulation arising from the lexicographic order $\preceq_{lex}$ also consists of six maximal simplices. In fact, up to an affine transformation, the two triangulations are the same. Both triangulations are square-free and unimodular by Example 2.6 and Corollary 2.9.

In general, pulling and placing triangulations do not have to be affinely isomorphic. The following example shows that even the number of simplices can differ.

Example 2.11. Consider the triangular bipyramid from Example 2.6. The pulling triangulation corresponding to the reverse lexicographic order $\preceq_{rev}$ consisting of the three simplices given in Figure 2.4. By Corollary 2.9, this triangulation is unimodular, but not flag. On the other hand, the lexicographic order $\preceq_{lex}$ yields the placing triangulation with two simplices depicted in Figure 2.3, which is flag, but not unimodular.

2.2 Double Hibi rings

The main goal of this section is to study toric ideals arising from double order polytopes and use them to better understand the underlying geometry. To begin with, we illustrate this interplay between geometry and algebra for the case of ordinary order polytopes. We recall a Gröbner basis of the toric ideals of order polytopes which was found by Hibi [46]. Using the machinery developed in Section 2.1, we recover the triangulation in Theorem 1.20. Moreover, we obtain a simple proof of a description of the complete face lattice of order polytopes, which was first stated in [80].

In the considerably more involved case of double posets, we explicitly compute a Gröbner basis in Theorem 2.17 whenever the underlying double poset is compatible. On the geometric side, this yields a regular triangulation of double order polytopes given in Corollary 2.20 and a simple formula for their volume in Corollary 2.21.

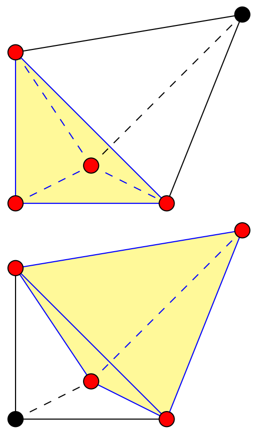

Figure 2.3: A placing triangulation with two simplices.

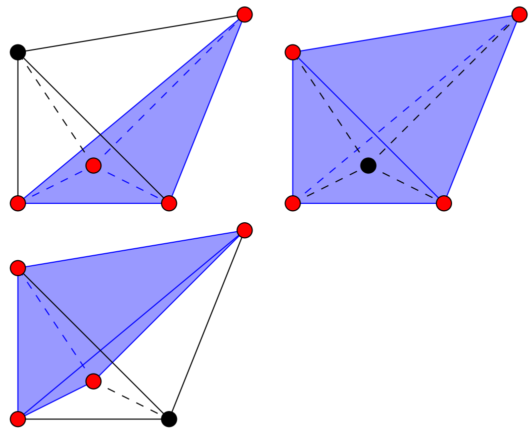

Figure 2.4: A pulling triangulation with three simplices.

Moreover, analogously to the case of order polytopes, in Theorem 2.22 we obtain a description of their whole face lattice in terms of the two underlying Birkhoff lattices. This turns out to be particularly useful for studying low-dimensional faces and in Corollary 2.24 we give a characterization of the edges of double order polytopes.

2.2.1 Hibi rings

Toric ideals of order polytopes were first studied by Hibi [46] in the context of algebras with straightening laws. They are also interesting from the perspective of classical algebraic geometry: For certain posets, they define projective toric varieties which deform to Grassmannians and flag varieties (see for instance [34] or [62, Sec. 14.3]). The projective varieties associated to order polytopes are, as the polytopes themselves, very combinatorial in nature and many geometric properties and invariants, such as the singular locus, the divisor class group and the Picard group can be read off directly from the poset (see [28],[43],[80]).

Let $(P, \preceq)$ be a finite poset with Birkhoff lattice $\mathcal{J}(P)$. Consider two polynomials rings $S = \mathbb{C}[t, s_a : a \in P]$ and $R = \mathbb{C}[x_\mathsf{J} : \mathsf{J} \in \mathcal{J}(P)]$. Recall from Section 2.1.2 that the toric ideal $I_{\mathcal{O}(P)}$ arises as the kernel of the homomorphism $\varphi : R \to S$ defined on generators by

$$\varphi(x_\mathsf{J}) \ = t\,\mathbf{s}^\mathsf{J} \text{ where } \ \mathbf{s}^\mathsf{J} \ := \prod_{a \in \mathsf{J}} s_a.$$

The graded semigroup ring $\mathbb{C}[\mathcal{O}(P)] := \operatorname{im}(\varphi) = \mathbb{C}[t\mathbf{s}^\mathsf{J} : \mathsf{J} \in \mathcal{J}(P)] \cong S/I_{\mathcal{O}(P)}$ is called the **Hibi ring** associated to P. Hibi showed in [46] that $\mathbb{C}[\mathcal{O}(P)]$ is a normal Cohen-Macaulay domain of dimension $|P|+1$ and that $\mathbb{C}[\mathcal{O}(P)]$ is Gorenstein if and only if P is a graded poset, that is, all maximal chains have the same length. For algebraic background and in particular a detailed treatment of toric Gorenstein rings from both algebraic and discrete-geometric viewpoints, see [12].

We briefly note the following: The algebraic definition of Gorenstein rings relates to the definition of reflexive polytopes from Section 1.3.3 as follows. An integral polytope $\mathcal{P} \subset \mathbb{R}^n$ is called **normal** if the affine semigroup $\mathcal{S}$ generated by the homogenized vertex set $V(\mathcal{P}) \times \{1\} \subset \mathbb{Z}^{n+1}$ is normal. An important class of normal polytopes is formed by polytopes that have a unimodular triangulation. If P is normal, the quotient algebra R/I_P is Gorenstein if and only if there exist $k \in \mathbb{Z}_{>0}$ and $\mathbf{x} \in \mathbb{Z}^n$ such that the polytope $k\mathcal{P} - \mathbf{x}$ is reflexive. For our purpose, we might as well take this as the definition of Gorenstein rings. On the level of normal semigroups, the Gorenstein property becomes the following. We denote by $\operatorname{int}(\mathcal{S}) := \mathcal{S} \cap \operatorname{relint}(\operatorname{cone}(\mathcal{S}))$ the **interior** of $\mathcal{S}$. The ring R/I_P is Gorenstein if and only if $\operatorname{int}(\mathcal{S}) = \mathbf{x} + \mathcal{S}$ for some point $\mathbf{x} \in \mathcal{S}$. Such a normal semigroup is also called **Gorenstein**.

Remark 2.12. Hibi's proof that the toric ring associated to order polytopes is Gorenstein if and only if the underlying poset is graded is phrased in algebraic terms. Translated to the language of cones and affine semigroups, the outline of the proof is as follows: An element $\mathbf{x} \in \operatorname{int}(\mathcal{S})$ which *cannot* be written as $\mathbf{x} = \mathbf{y} + \mathbf{s}$ with $\mathbf{y} \in \operatorname{int}(\mathcal{S})$ and $\mathbf{0} \neq \mathbf{s} \in \mathcal{S}$ corresponds to a *minimal* strictly order preserving map $\sigma : \widehat{P} \to \mathbb{N}$, in the sense that there is no strictly order-preserving map $\sigma' : \widehat{P} \to \mathbb{N}$ such that $\mathbf{0} \neq \sigma - \sigma'$ is order-preserving. There is only one such map if and only if the poset is graded. In this case, the unique generator corresponds to the rank function in $\widehat{P}$. For details, see [46].

Hibi elegantly described a reduced Gröbner basis of $I_{\mathcal{O}(P)}$ in terms of $\mathcal{J}(P)$. As before, the underlined term of a polynomial always denotes its leading term.

Theorem 2.13 ([45, Thm. 10.1.3])**.** *Let $(P, \preceq)$ be a finite poset. Fix a total order $\leq$ on the variables $\{x_{\mathsf{J}} : \mathsf{J} \in \mathcal{J}(P)\}$ such that $x_{\mathsf{J}} \leq x_{\mathsf{J}'}$ whenever $\mathsf{J} \subseteq \mathsf{J}'$ and denote by $\preceq_{rev}$ the induced reverse lexicographic order. Then the collection*

$$\underline{x_{\mathsf{J}}\, x_{\mathsf{J}'}} \;-\; x_{\mathsf{J}\cap\mathsf{J}'}\, x_{\mathsf{J}\cup\mathsf{J}'} \qquad \text{with } \mathsf{J}, \mathsf{J}' \in \mathcal{J}(P) \text{ incomparable} \tag{2.1}$$

is a reduced Gröbner basis of $I_{\mathcal{O}(P)}$ with respect to $\preceq_{rev}$.

The binomials in (2.1) are sometimes called **Hibi relations**. The Gröbner basis of $I_{\mathcal{O}(P)}$ in Theorem 2.13 is the algebraic counterpart to the triangulation of $\mathcal{O}(P)$ described in Theorem 1.20. Indeed, via Sturmfels' correspondence (Theorem 2.8), the Gröbner basis yields a regular triangulation of $\mathcal{O}(P)$ whose minimal non-faces are of the form $\operatorname{conv}(\mathbf{1}_{\mathsf{J}}, \mathbf{1}_{\mathsf{J}'})$ for incomparable filters $\mathsf{J}, \mathsf{J}' \in \mathcal{J}(P)$. Thus, the faces correspond to chains in $\mathcal{J}(P)$ and we recover precisely the canonical triangulation from Theorem 1.20. Moreover, we can use Theorem 2.8 to transfer algebraic properties of the Gröbner basis to geometric properties of the triangulation. The following result follows directly from Theorem 2.13 and Corollary 2.9.

Corollary 2.14. *The triangulation in Theorem 1.20 is a regular, flag and unimodular pulling triangulation.*

In Section 1.2.2, we have studied the complete facial structure of $\mathcal{O}(P)$ by thinking of faces as intersections of facets. In the following, we will give a different perspective by asking for which subsets $M \subseteq V(\mathcal{O}(P))$ the convex hull $\operatorname{conv}(M)$ is a face. For $\mathcal{L} \subseteq \mathcal{J}(P)$ of filters we shall write $F(\mathcal{L}) := \operatorname{conv}(\mathbf{1}_{\mathsf{J}} : \mathsf{J} \in \mathcal{L})$ in the following. An induced subposet $\mathcal{L} \subseteq \mathcal{J}(P)$ is an **embedded sublattice** if for any two filters $\mathsf{J}, \mathsf{J}' \in \mathcal{J}(P)$

$$\mathsf{J} \cup \mathsf{J}', \mathsf{J} \cap \mathsf{J}' \in \mathcal{L} \quad \text{if and only if} \quad \mathsf{J}, \mathsf{J}' \in \mathcal{L}. \tag{2.2}$$

For an embedded sublattice $\mathcal{L}$ we define $l(\mathcal{L}) := |C|$, where $C \subseteq \mathcal{L}$ is a longest chain in $\mathcal{L}$. Note that $l(\mathcal{L}) - 1$ equals the number of join-irreducible elements in the lattice $\mathcal{L}$. Embedded sublattices give an alternative way to characterize faces of $\mathcal{O}(P)$.

Proposition 2.15 ([80, Thm 1.1(f)]). *Let P be a poset and $\mathcal{L} \subseteq \mathcal{J}(P)$ a collection of filters. Then $F(\mathcal{L})$ is a face of $\mathcal{O}(P)$ if and only if $\mathcal{L}$ is an embedded sublattice. In this case,* $\dim F(\mathcal{L}) = l(\mathcal{L}) - 1$.

Proof. By Lemma 2.7, $F(\mathcal{L})$ is a face of $\mathcal{O}(P)$ if and only if for every $f \in I_{\mathcal{O}(P)}$ we have $f(\mathbf{1}_{\mathcal{L}}) = 0$. By Theorem 2.13, this means that $\mathbf{1}_{\mathcal{L}}$ has to satisfy the Hibi relations in (2.1). But this is clearly equivalent to $\mathcal{L}$ being an embedded sublattice. By Theorem 1.20, we have $\dim F(\mathcal{L}) = \dim F(C)$, where $C \subseteq \mathcal{L}$ is a longest chain. $\square$

This description is particularly useful for faces of low-dimension. For example it is now easy to describe the edges of $\mathcal{O}(P)$. The following result follows easily from Proposition 2.15 and the relations in (2.2).

Corollary 2.16. *Let P be a poset and $\mathsf{J}, \mathsf{J}' \in \mathcal{J}(P)$ distinct. Then* $\operatorname{conv}(\{\mathbf{1}_{\mathsf{J}}, \mathbf{1}_{\mathsf{J}'}\})$ *is an edge of $\mathcal{O}(P)$ if and only if J and J' are comparable (say $\mathsf{J} \subset \mathsf{J}'$) and their difference $\mathsf{J}' \setminus \mathsf{J}$ is a connected poset.*

Finally, note that for a face $F = F(\mathcal{L})$ it is easy to describe the associated face partition $\mathcal{B}(F)$ (cf. Section 1.2.2): Two poset elements $a, b \in \widehat{P}$ belong to the same block if and only if for every $\mathsf{J} \in \mathcal{L}$ we have $a \in \mathsf{J} \cup \{\widehat{1}\}$ if and only if $b \in \mathsf{J} \cup \{\widehat{1}\}$.

2.2.2 A Gröbner basis for double Hibi rings

For a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$, consider the Laurent ring $\hat{S} := \mathbb{C}[t_-, t_+, s_a, s_a^{-1} : a \in P]$ and the polynomial ring $\hat{R} := \mathbb{C}[x_{\mathsf{J}_+}, x_{\mathsf{J}_-} : \mathsf{J}_+ \in \mathcal{J}(P_+), \mathsf{J}_- \in \mathcal{J}(P_-)]$. On generators we define the map $\hat{\varphi} : \hat{R} \to \hat{S}$ by

$$\hat{\varphi}(x_{\mathsf{J}_+}) \;:=\; t_+ \, \mathbf{s}^{\mathsf{J}_+} = t_+ \prod_{a \in \mathsf{J}_+} s_a \quad \text{and} \quad \hat{\varphi}(x_{\mathsf{J}_-}) \;:=\; t_- \, (\mathbf{s}^{\mathsf{J}_-})^{-1} = t_- \prod_{a \in \mathsf{J}_-} s_a^{-1}.$$

The toric ideal associated with $\mathbb{T}\mathcal{O}(\mathbf{P})$ is $I_{\mathbb{T}\mathcal{O}(\mathbf{P})} = \ker \hat{\varphi}$. Analogously to Section 2.2.1 we call the subalgebra $\mathbb{C}[\mathbb{T}\mathcal{O}(\mathbf{P})] = \operatorname{im}(\hat{\varphi}) \subseteq \hat{S}$ the **double Hibi ring** of $\mathbf{P}$. We can regard double Hibi rings as quotients by identifying $\mathbb{C}[\mathbb{T}\mathcal{O}(\mathbf{P})] \cong \hat{R}/I_{\mathbb{T}\mathcal{O}(\mathbf{P})}$. The double Hibi ring $\mathbb{C}[\mathbb{T}\mathcal{O}(\mathbf{P})]$ is a graded semigroup ring of dimension $|P| + 1$. Moreover, whenever $\mathbf{P}$ is compatible, the double order polytope $\mathbb{T}\mathcal{O}(\mathbf{P})$ is reflexive by Corollary 1.33 and it follows that $\mathbb{C}[\mathbb{T}\mathcal{O}(\mathbf{P})]$ is Gorenstein. Note that the rings $\mathbb{C}[\mathbb{T}\mathcal{O}(\mathbf{P})]$ as well as the affine semigroup rings associated to the double chain polytopes $\mathbb{T}\mathcal{C}(\mathbf{P})$ as treated in Section 3.3.2 were also considered by Hibi and Tsuchiya [52].

Theorem 2.17. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Fix a total order $\leq$ on the variables $\{x_{\mathsf{J}_\sigma} : \sigma \in \{\pm\}, \mathsf{J}_\sigma \in \mathcal{J}(P_\sigma)\}$ such that*

(i) *$x_{\mathsf{J}_\sigma} < x_{\mathsf{J}'_\sigma}$ for any filters $\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ with $\sigma \in \{\pm\}$ and $\mathsf{J}_\sigma \subset \mathsf{J}'_\sigma$, and*
(ii) *$x_{\mathsf{J}_+} < x_{\mathsf{J}_-}$ for any filters $\mathsf{J}_+ \in \mathcal{J}(P_+)$ and $\mathsf{J}_- \in \mathcal{J}(P_-)$.*

Denote by $\preceq_{rev}$ the induced reverse lexicographic monomial order. Then a Gröbner basis for $\mathrm{I}_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ is given by the binomials

$$\underline{x_{\mathsf{J}_\sigma} x_{\mathsf{J}'_\sigma}} \; - \; x_{\mathsf{J}_\sigma \cup \mathsf{J}'_\sigma} x_{\mathsf{J}_\sigma \cap \mathsf{J}'_\sigma} \tag{2.3}$$

for incomparable filters $\mathsf{J}_\sigma, \mathsf{J}'_\sigma \in \mathcal{J}(P_\sigma)$ and $\sigma \in \{\pm\}$, and

$$\underline{x_{\mathsf{J}_+} x_{\mathsf{J}_-}} \; - \; x_{\mathsf{J}_+ \setminus A} x_{\mathsf{J}_- \setminus A} \tag{2.4}$$

for filters $\mathsf{J}_+ \in \mathcal{J}(P_+), \mathsf{J}_- \in \mathcal{J}(P_-)$ such that $A := \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) \neq \emptyset$.

It is clear that the binomials of the form (2.3) and (2.4) are contained in $\mathrm{I}_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ and hence it suffices to show that their leading terms generate $\mathrm{in}_{\preceq_{rev}}(I_{\mathbb{T}\mathcal{O}(\mathbf{P})})$. For this, let us take a closer look at the combinatorics of $\hat{\varphi}$. Let $\mathcal{G}$ be the collection of binomials given in (2.3) and (2.4) and let $f = \underline{m_1} - m_2$ be a binomial of the form

$$\underline{x_{\mathsf{J}_{+1}} x_{\mathsf{J}_{+2}} \cdots x_{\mathsf{J}_{+k_+}} \cdot x_{\mathsf{J}_{-1}} x_{\mathsf{J}_{-2}} \cdots x_{\mathsf{J}_{-k_-}}} - x_{\mathsf{J}'_{+1}} x_{\mathsf{J}'_{+2}} \cdots x_{\mathsf{J}'_{+k_+}} \cdot x_{\mathsf{J}'_{-1}} x_{\mathsf{J}'_{-2}} \cdots x_{\mathsf{J}'_{-k_-}}, \tag{2.5}$$

for filters $\mathsf{J}_{+1}, \ldots, \mathsf{J}_{+k_+}, \mathsf{J}'_{+1}, \ldots, \mathsf{J}'_{+k_+} \in \mathcal{J}(P_+)$ and $\mathsf{J}_{-1}, \ldots, \mathsf{J}_{-k_-}, \mathsf{J}'_{-1}, \ldots, \mathsf{J}'_{-k_-} \in \mathcal{J}(P_-)$. It suffices to show that the initial term m_1 lies in the the ideal generated by the initial terms of the binomials in $\mathcal{G}$. By reducing f by the binomials in (2.3), we can view f as a quadruple

$$\begin{array}{ll} \mathsf{J}_{+1} \subset \mathsf{J}_{+2} \subset \cdots \subset \mathsf{J}_{+k_+} & \quad \mathsf{J}_{-1} \subset \mathsf{J}_{+2} \subset \cdots \subset \mathsf{J}_{-k_-} \\ \mathsf{J}'_{+1} \subset \mathsf{J}'_{+2} \subset \cdots \subset \mathsf{J}'_{+k_+} & \quad \mathsf{J}'_{-1} \subset \mathsf{J}'_{+2} \subset \cdots \subset \mathsf{J}'_{-k_-}. \end{array} \tag{2.6}$$

Looking individually at the degree of the variables s_a for $a \in P$, it follows from the definition of $\hat{\varphi}$ that such a quadruple defines a binomial in $I_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ if and only if for every $q \in P$ we have

$$\max\{r : q \notin \mathsf{J}_{+r}\} - \max\{s : q \notin \mathsf{J}_{-s}\} \; = \; \max\{r : q \notin \mathsf{J}'_{+r}\} - \max\{s : q \notin \mathsf{J}'_{-s}\} \tag{2.7}$$

and we note the following implication.

Lemma 2.18. *Let the collection of filters in* (2.6) *correspond to a binomial $f \in I_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ and let $q \in P$. Then there is some $1 \leq i \leq k_+$ such that $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$ if and only if there is some $1 \leq j \leq k_-$ such that $q \in \mathsf{J}_{-j} \setminus \mathsf{J}'_{-j}$.*

Proof. If $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$, then $\max\{r : q \notin \mathsf{J}_{+r}\} < i$ and $\max\{r : q \notin \mathsf{J}'_{+r}\} \geq i$ and (2.7) implies that $q \in \mathsf{J}_{-j} \setminus \mathsf{J}'_{-j}$ for some j. The other direction is identical. □

We call $q \in P$ **moving** if it satisfies one of the two equivalent conditions of Lemma 2.18.

Proof of Theorem 2.17. Let $f = \underline{m_1} - m_2 \in \mathrm{I}_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ be a binomial represented by a collection of filters given in (2.6). If $k_- = 0$ or $k_+ = 0$, then the Hibi relations (2.3) for P_- or P_+ together with Theorem 2.13 yields the result. Thus, we assume that $k_-, k_+ > 0$ and we need to show that there are filters J_{+i} and J_{-j} such that $\min(\mathsf{J}_{+i}) \cap \min(\mathsf{J}_{-j}) \neq \emptyset$.

We claim that there is at least one moving element. First observe that $\mathsf{J}_{+1} \not\subseteq \mathsf{J}'_{+1}$ and hence $\mathsf{J}_{+1} \setminus \mathsf{J}'_{+1} \neq \emptyset$. Indeed, otherwise, $x_{\mathsf{J}_{+1}} < x_{\mathsf{J}'_{+1}}$ and the reverse lexicographic term order $\preceq_{rev}$ would not select m_1 as the leading term of f. Among all moving elements, choose q to be minimal with respect to $\preceq_+$ and $\preceq_-$. Since $\mathbf{P}$ is a compatible double poset, such a q exists. But then, if $q \in \mathsf{J}_{+i} \setminus \mathsf{J}'_{+i}$, then $q \in \min(\mathsf{J}_{+i})$. The same holds true for J_{-j} and this shows that m_1 is divisible by the leading term of a binomial of type (2.4). □

Remark 2.19. Reformulated in the language of double posets, Hibi, Matsuda, and Tsuchiya [48, 50, 51] computed related Gröbner bases of the toric ideals associated with the polytopes $\Gamma(\mathcal{O}(P_+), \mathcal{O}(P_-))$ (in the compatible case), $\Gamma(\mathcal{C}(P_+), \mathcal{C}(P_-))$, and $\Gamma(\mathcal{O}(P_+), \mathcal{C}(P_-))$ for a double poset $\mathbf{P}$. See the paragraph before Corollary 1.42 for notation.

2.2.3 Triangulations of double order polytopes

Theorem 2.17 together with Theorem 2.8 yields a canonical triangulation of compatible double order polytopes. Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. In the following we write $\mathbb{T}\mathcal{J}(\mathbf{P}) := \mathcal{J}(P_+) \uplus \mathcal{J}(P_-)$. The elements of $\mathbb{T}\mathcal{J}(\mathbf{P})$ correspond to vertices of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and hence $\mathbb{T}\mathcal{J}(\mathbf{P})$ may be regarded as a *double Birkhoff lattice*, a generalization of Birkhoff lattices to the case of double posets. However, we will not employ this terminology, since the poset $\mathbb{T}\mathcal{J}(\mathbf{P})$ does not carry a lattice structure. Let $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ be a chain of filters in P_σ for $\sigma \in \{\pm\}$. The pair of chains $C = C_+ \uplus C_- \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ is **non-interfering** if $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \emptyset$ for any $\mathsf{J}_+ \in C_+$ and $\mathsf{J}_- \in C_-$. Consider the simplicial complex

$$\Delta(\mathbb{T}\mathcal{J}(\mathbf{P})) \;=\; \{C_+ \uplus C_- : C_+ \in \Delta(\mathcal{J}(P_+)), C_- \in \Delta(\mathcal{J}(P_-))\}.$$

The subcomplex

$$\Delta^{\mathrm{ni}}(\mathbf{P}) \;:=\; \{C : C = C_+ \uplus C_- \in \Delta(\mathbb{T}\mathcal{J}(\mathbf{P})),\, C \text{ non-interfering}\}$$

is called the **non-interfering complex** of $\mathbf{P}$. For a non-interfering pair of chains $C = C_+ \uplus C_- \in \Delta^{\mathrm{ni}}(\mathbf{P})$ we define $\overline{F}(C) := 2F(C_+) \boxminus 2F(C_-)$.

Corollary 2.20. *Let $\mathbf{P}$ be a compatible double poset. The collection of simplices*

$$\{\overline{F}(C) : C \in \Delta^{\mathrm{ni}}(\mathbf{P})\}$$

forms a regular, unimodular, flag triangulation of $\mathbb{T}\mathcal{O}(\mathbf{P})$ with underlying simplicial complex $\Delta^{\mathrm{ni}}(\mathbf{P})$.

Proof. A monomial does not occur as a leading term of the Gröbner basis in Theorem 2.17 if and only it is of the form

$$\prod_{\mathsf{J}_+ \in C_+} x_{\mathsf{J}_+} \prod_{\mathsf{J}_- \in C_-} x_{\mathsf{J}_-}$$

for some non-interfering pair $C = C_+ \uplus C_- \in \Delta^{\mathrm{ni}}(\mathbf{P})$. Theorem 2.8 and Corollary 2.9 complete the proof. □

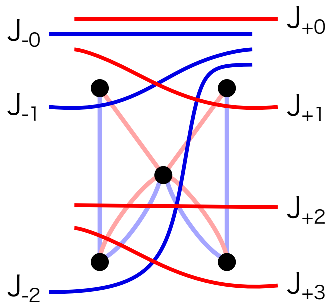

Figure 2.5: A maximal non-interfering set of filters in $\mathbf{P}_{XW}$. A red or blue curve denotes the filter consisting of all elements above the curve.

Note that associating the order complex $\Delta(\mathcal{J}(P))$ to a poset P is very natural and can be motivated, for example, from an algebraic-combinatorial approach to the order polynomial (cf. [8]). It would be very interesting to know if the association of $\Delta^{\mathrm{ni}}(\mathbf{P})$ to $\mathbf{P}$ is equally natural from a purely combinatorial perspective.

For a compatible double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$ counting the number of simplices in the above triangulation yields a simple combinatorial formula for the normalized volume of $\mathbb{T}\mathcal{O}(\mathbf{P})$. In the following we denote by $\mathrm{Vol}(\mathbb{T}\mathcal{O}(\mathbf{P}))$ the normalized volume with respect to $\Lambda = 2\mathbb{Z}^P \times (2\mathbb{Z}+1)$, the affine lattice generated by the vertices of $\mathbb{T}\mathcal{O}(\mathbf{P})$. In particular, if $|P| = n$, every full-dimensional unimodular simplex has ordinary Euclidean volume $\frac{2^{n+1}}{(n+1)!}$. Recall from Section 1.1 that for a poset $(\mathcal{P}, \preceq)$ we write $\mathrm{e}(P)$ for the number of linear extensions of P and for $J \subseteq \mathcal{P}$, we denote by $\mathcal{P}|_J$ the subposet induced by J.

Corollary 2.21. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Then*

$$\mathrm{Vol}(\mathbb{T}\mathcal{O}(\mathbf{P})) \;=\; \sum_{J \subseteq P} \mathrm{e}(P_+|_J) \cdot \mathrm{e}(P_-|_{J^c})$$

Proof. We will count maximal simplices in the triangulation from Corollary 2.20. Let $C = C_+ \uplus C_-$ be the non-interfering pair corresponding to a maximal simplex. We have $|C| = |P| + 2$ and hence every element of P must occur as a minimal element of some filter in C. More precisely, we have $P = M_+ \uplus M_-$, where for $\sigma \in \{\pm\}$ we define

$$M_\sigma := \{p \in P : p \in \min(\mathsf{J}_\sigma) \text{ for some } \mathsf{J}_\sigma \in \mathcal{J}(P_\sigma)\}.$$

For $\sigma \in \{\pm\}$, the chain C_+ induces a maximal chain of filters in the induced subposet $P_\sigma|_{M_\sigma}$ and hence corresponds to a linear extension of $P_\sigma|_{M_\sigma}$. Conversely, every partition $P = M_+ \uplus M_-$ together with a pair of linear extensions of $P_\sigma|_{M_\sigma}$ for $\sigma \in \{\pm\}$ uniquely determines a maximal pair of non-interfering chains. $\square$

2.2.4 Faces of double order polytopes

In Proposition 2.15, we have used toric ideals and Gröbner bases to describe faces of order polytopes. Now we will generalize this description to the case of double order polytopes. Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. For any subset $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ we will denote by $\mathcal{L}_+$ and $\mathcal{L}_-$ the sets $\mathcal{L} \cap \mathcal{J}(P_+)$ and $\mathcal{L} \cap \mathcal{J}(P_-)$, respectively. Moreover, we shall write $\overline{F}(\mathcal{L}) := 2F(\mathcal{L}_+) \boxminus 2F(\mathcal{L}_-) \subseteq \mathbb{T}\mathcal{O}(\mathbf{P})$. More explicitly,

$$\overline{F}(\mathcal{L}) \;=\; \operatorname{conv}\left(\{(2\mathbf{1}_{\mathsf{J}_+}, +1) : \mathsf{J}_+ \in \mathcal{L}_+\} \cup \{(-2\mathbf{1}_{\mathsf{J}_-}, -1) : \mathsf{J}_- \in \mathcal{L}_-\}\right). \qquad (2.8)$$

Clearly, every face $F \subseteq \mathbb{T}\mathcal{O}(P)$ is of the form $F = \overline{F}(\mathcal{L})$ for some $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ and the filters in $\mathcal{L}$ correspond to vertices of $\mathbb{T}\mathcal{O}(\mathbf{P})$ that lie in F. The following is a complete characterization of faces of $\mathbb{T}\mathcal{O}(\mathbf{P})$ in terms of their vertex sets in the case when $\mathbf{P}$ is compatible. For $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$, we denote by $cl(\mathcal{L})$ the largest number $|C| = |C_+| + |C_-|$ such that $C \subseteq \mathcal{L}$ is a pair of non-interfering chains.

Theorem 2.22. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset and $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$. Then $\overline{F}(\mathcal{L})$ is a face of $\mathbb{T}\mathcal{O}(\mathbf{P})$ if and only if*
(i) *$\mathcal{L}_+ \subseteq \mathcal{J}(P_+)$ and $\mathcal{L}_- \subseteq \mathcal{J}(P_-)$ are embedded sublattices and*
(ii) *for all filters $\mathsf{J}_+ \subseteq \mathsf{J}'_+ \in \mathcal{J}(P_+)$ and $\mathsf{J}_- \subseteq \mathsf{J}'_- \in \mathcal{J}(P_-)$ with*

$$\mathsf{J}'_+ \setminus \mathsf{J}_+ \;=\; \mathsf{J}'_- \setminus \mathsf{J}_-$$

it holds that $\{\mathsf{J}_+, \mathsf{J}_-\} \subseteq \mathcal{L}$ if and only if $\{\mathsf{J}'_+, \mathsf{J}'_-\} \subseteq \mathcal{L}$.
Moreover, in this case $\dim \overline{F}(\mathcal{L}) = cl(\mathcal{L}) - 1$.

We call a pair $\mathcal{L} = \mathcal{L}_+ \uplus \mathcal{L}_- \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ of embedded sublattices **cooperating** if it satisfies condition (ii) of Theorem 2.22 above. We may also rephrase condition (ii) as follows.

Lemma 2.23. *Let $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ such that $\mathcal{L}_+$ and $\mathcal{L}_-$ are embedded sublattices. Then $\mathcal{L}$ is cooperating if and only if only if for any two filters $\mathsf{J}_- \in \mathcal{L}_-, \mathsf{J}_+ \in \mathcal{L}_+$ we have*
(i) *for $A \subseteq \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-)$ we have $\mathsf{J}_- \setminus A \in \mathcal{L}_-$ and $\mathsf{J}_+ \setminus A \in \mathcal{L}_+$, and*
(ii) *for $B \subseteq \max(P_+ \setminus \mathsf{J}_+) \cap \max(P_- \setminus \mathsf{J}_-)$ we have $\mathsf{J}_- \cup B \in \mathcal{L}_-$ and $\mathsf{J}_+ \cup B \in \mathcal{L}_+$.*

Proof. First assume that $\mathcal{L}$ is cooperating. Let $B \subseteq \max(P_+ \setminus \mathsf{J}_+) \cap \max(P_- \setminus \mathsf{J}_-)$ and define filters $\mathsf{J}'_\sigma = \mathsf{J}_\sigma \cup B$ for $\sigma \in \{\pm\}$. Clearly, $\mathsf{J}'_+ \setminus \mathsf{J}_+ = \mathsf{J}'_- \setminus \mathsf{J}_- = B$ and hence $\mathsf{J}'_+, \mathsf{J}'_- \in \mathcal{L}$ since $\mathcal{L}$ is cooperating. The same argument applies to $A \subseteq \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-)$.

For the converse direction, let $\mathsf{J}_\sigma \subseteq \mathsf{J}'_\sigma$ for $\sigma \in \{\pm\}$ such that $\mathsf{J}'_+ \setminus \mathsf{J}_+ = \mathsf{J}'_- \setminus \mathsf{J}_- =: D$. If we assume $\mathsf{J}'_+, \mathsf{J}'_- \in \mathcal{L}$ then $A := \min(D) \subseteq \min(\mathsf{J}'_+) \cap \min(\mathsf{J}'_-)$ and by (i), $\mathsf{J}'_\sigma \setminus A \in \mathcal{L}_\sigma$ for $\sigma \in \{\pm\}$. Now induction on $|D|$ yields $\mathsf{J}_+, \mathsf{J}_- \in \mathcal{L}$. The other direction is similar. $\square$

Proof of Theorem 2.22. Let $\mathcal{L} \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$. It follows from Lemma 2.7 and Theorem 2.17 that $\overline{F}(\mathcal{L}) \subseteq \mathbb{T}\mathcal{O}(\mathbf{P})$ is a face if and only if $f(\mathbf{1}_\mathcal{L}) = 0$ for every f in the Gröbner basis given in (2.3) and (2.4). The Hibi relations in (2.3) ensure that $\mathcal{L}_+$ and $\mathcal{L}_-$ are embedded sublattices (cf. Proposition 2.15) and the conditions imposed by (2.4) are equivalent to those of Lemma 2.23. The last assertion follows directly from Corollary 2.20. $\square$

For a poset P, we have seen in Corollary 2.16 that edges of $\mathcal{O}(P)$ are of the form $\operatorname{conv}(\{\mathbf{1}_\mathsf{J}, \mathbf{1}_{\mathsf{J}'}\})$ where $\mathsf{J} \subseteq \mathsf{J}'$ are filters of P such that $\mathsf{J}' \setminus \mathsf{J}$ is a connected poset. Of course, this description captures all the *horizontal* edges of $\mathbb{T}\mathcal{O}(\mathbf{P})$. The upcoming characterization of *vertical* edges follows directly from Theorem 2.22 and Lemma 2.23, but we also supply a direct proof.

Corollary 2.24. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset and let $\mathsf{J}_+ \subseteq P_+$ and $\mathsf{J}_- \subseteq P_-$ be filters. Then $(2\mathbf{1}_{\mathsf{J}_+}, +1)$ and $(-2\mathbf{1}_{\mathsf{J}_-}, -1)$ are the endpoints of a vertical edge of $\mathbb{T}\mathcal{O}(\mathbf{P})$ if and only if $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is a vertex of $\mathbb{D}\mathcal{O}(\mathbf{P})$. This is the case if and only if*

$$\begin{aligned} \min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) &= \emptyset \textit{ and} \\ \max(P_+ \setminus \mathsf{J}_+) \cap \max(P_- \setminus \mathsf{J}_-) &= \emptyset. \end{aligned}$$

Proof. The first equivalence follows from the fact that

$$\mathbb{T}\mathcal{O}(\mathbf{P}) \cap \{(\varphi, t) : t = 0\} \;=\; (\mathcal{O}(P_+) - \mathcal{O}(P_-)) \times \{0\}$$

and $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is the midpoint between $(2\mathbf{1}_{\mathsf{J}_+}, +1)$ and $(-2\mathbf{1}_{\mathsf{J}_-}, -1)$.

Before we come to the second claim, let us note that the face partition of a vertex $\mathbf{1}_{\mathsf{J}}$ for a poset $(P, \preceq)$ is given by $\{\mathsf{J}, P \setminus \mathsf{J}\}$. Thus, if $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ is a vertex of $\mathbb{D}\mathcal{O}(\mathbf{P})$, then there is a linear function $\ell(f) = \sum_{a \in P} \ell_a f(a)$ such that $\mathcal{O}(P_+)^{\ell} = \{\mathbf{1}_{\mathsf{J}_+}\}$ and $\mathcal{O}(P_-)^{-\ell} = \{\mathbf{1}_{\mathsf{J}_-}\}$. Corollary 1.17 then yields that $\ell_a > 0$ for each $a \in \min(\mathsf{J}_+)$ and $\ell_a < 0$ for $a \in \min(\mathsf{J}_-)$. The same reasoning applies to $\max(P_+ \setminus \mathsf{J}_+)$ and $\max(P_- \setminus \mathsf{J}_-)$ and shows necessity.

Let $b \in \min(\mathsf{J}_+)$. If $b \notin \mathsf{J}_-$, then the linear function $\ell(f) := f(b)$ is maximized over $\mathcal{O}(P_+)$ at every filter that contains b and over $-\mathcal{O}(P_-)$ at every filter that does not contain b. If $b \in \mathsf{J}_-$, then, by assumption, $b \notin \min(\mathsf{J}_-)$ and there is some $p_2 \in \min(\mathsf{J}_-)$ with $p_2 \prec_- b$. Now, if $p_2 \in \mathsf{J}_+$, then there is $p_3 \in \min(\mathsf{J}_+)$ with $p_3 \prec_+ p_2$ and so on. Compatibility now assures us that we get a descending alternating chain of the form

$$\hat{1} \;\succ_+\; b =: p_1 \;\succ_-\; p_2 \;\succ_+\; p_3 \;\succ_-\; \cdots \;\succ_\sigma\; p_k \;\succ_{-\sigma}\; a \;\succ_\sigma\; \hat{0}$$

where $p_2, p_4, p_6, \ldots \in \min(\mathsf{J}_-) \cap \mathsf{J}_+$ and $p_1, p_3, p_5, \ldots \in \min(\mathsf{J}_+) \cap \mathsf{J}_-$ and $a \in \min(\mathsf{J}_{-\sigma}) \setminus \mathsf{J}_\sigma$. Consider the associated linear function

$$\ell(f) \;=\; f(p_0) - f(p_1) + f(p_2) - \cdots + (-1)^k f(p_k) + (-1)^{k+1} f(a)$$

for $f \in \mathbb{R}^P$. We claim that $\ell(\mathbf{1}_{\mathsf{J}'_+}) \leq 1$ for each filter $\mathsf{J}'_+ \subseteq P_+$ and with equality if $b \in \mathsf{J}'_+$. Indeed, if $p_{2i+1} \in \mathsf{J}'_+$, then $p_{2i} \in \mathsf{J}'_+$ for all $i \geq 1$. Conversely, $\ell(-\mathbf{1}_{\mathsf{J}'_-}) \leq 0 = \ell(-\mathbf{1}_{\mathsf{J}_-})$ for each filter $\mathsf{J}'_- \subseteq P_-$. This follows from the fact that $p_{2i} \in \mathsf{J}'_-$ implies $p_{2i-1} \in \mathsf{J}'_-$ for each $i \geq 1$.

For $a \in \max(P_+ \setminus \mathsf{J}_+)$ the situation is similar and we search for $b \in \max(P_- \setminus \mathsf{J}_-)$ with $a \prec_- b$ in the case that $a \notin \mathsf{J}_-$. This yields a linear function $\ell \in -N_{P_-}(\mathbf{1}_{\mathsf{J}_-})$ that is maximized over $\mathcal{O}(P_+)$ at filters $\mathbf{1}_{\mathsf{J}'_+}$ with $a \notin \mathsf{J}'_+$. Summing these linear functions for $b \in \min(\mathsf{J}_+)$ and $a \in \max(P_+ \setminus \mathsf{J}_+)$ yields a linear function ℓ^+ with $\mathcal{O}(P_+)^{\ell^+} = \{\mathbf{1}_{\mathsf{J}_+}\}$ and $\mathbf{1}_{\mathsf{J}_-} \in \mathcal{O}(P_-)^{-\ell^+}$.

Of course, the same reasoning applies to J_- instead of J_+ and it follows that $\ell^+ - \ell^-$ is uniquely maximized at $\mathbf{1}_{\mathsf{J}_+} - \mathbf{1}_{\mathsf{J}_-}$ over $\mathbb{D}\mathcal{O}(\mathbf{P}) = \mathcal{O}(P_+) - \mathcal{O}(P_-)$. □

We close this section by noting that in the case of ordinary order polytopes, most problems are rather straightforward to approach from both the geometric and the algebraic side. However, we have seen that for double order polytopes things get more involved and for many purposes (for example the triangulation in Corollary 2.20) it seems easier to take the algebraic perspective. However, this approach leaves some questions open; for example, it would be interesting to find a combinatorial algorithm which for a given point $\mathbf{p} \in \mathbb{T}\mathcal{O}(\mathbf{P})$ finds a simplex of the triangulation that contains $\mathbf{p}$.

Chapter 3

Double chain polytopes and anti-blocking polytopes

In this chapter, we study double chain polytopes, a second class of polytopes associated to double posets. As the double order polytopes in Chapter 1, they are very combinatorial in flavor and, again, our goal is to study their geometry in terms of the combinatorics of the underlying poset. It turns out that most of the theory can be developed in the more general setting of Fulkerson's anti-blocking polytopes [30] and their Cayley sums.

Section 3.1 introduces chain polytopes and anti-blocking polytopes, as well as stable set polytopes, a class of anti-blocking 0/1-polytopes associated to graphs which generalizes chain polytopes. In Section 3.2, we define double chain polytopes and more generally, we consider Cayley sums of anti-blocking polytopes, which also subsume Hansen polytopes [41]. We give an explicit description of the facets of these Cayley sums and in particular of double chain polytopes. Cayley sums of anti-blocking polytopes have a very natural subdivision, which we describe in Section 3.3. In the case of double chain polytopes, we refine this subdivision to a canonical triangulation, which, in particular, yields explicit combinatorial formulas for their volume. Finally, in Section 3.4, we establish a connection between double order polytopes and double chain polytopes by defining a piecewise-linear homeomorphism between the two polytopes whenever the underlying double poset is compatible. Eventually, we put this connection to good use: We compute the Ehrhart polynomial of double chain polytopes and use the transfer map to also obtain the Ehrhart polynomial of compatible double order polytopes.

3.1 Anti-blocking polytopes

The second class of poset polytopes introduced by Stanley [73] are chain polytopes. Before we turn to the study of their Cayley sums, we collect some basics on chain polytopes, as well as two larger families: Independent set polytopes associated to graphs (see for example [41, 58]) and, even more generally, anti-blocking polytopes, which were introduced by Fulkerson [30] in connection with min-max-relations in combinatorial optimization.

3.1.1 Chain polytopes and stable set polytopes

Definition 3.1. Let P be a poset. The **chain polytope** $\mathcal{C}(P)$ is the collection of functions $g : P \to \mathbb{R}_{\geq 0}$ such that

$$g(a_1) + g(a_2) + \cdots + g(a_k) \ \leq \ 1 \tag{3.1}$$

for all chains $a_1 \prec a_2 \prec \cdots \prec a_k$ in P.

Stanley [73] showed that such an inequality is facet-defining if and only if the corresponding chain is maximal and that the vertices of $\mathcal{C}(P)$ are the characteristic vectors of antichains, that is

$$V(\mathcal{C}(P)) \ = \ \{\mathbf{1}_A : A \subseteq P \text{ antichain}\}. \tag{3.2}$$

Since filters and antichains in P are in one-to-one correspondence, so are the vertices of $\mathcal{O}(P)$ and $\mathcal{C}(P)$. More generally, in Section 3.4.1, we will see that $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are related by a piecewise linear homeomorphism. In contrast to the order polytope, $\mathcal{C}(P)$ does not uniquely determine P. In fact, $\mathcal{C}(P)$ is determined by the comparability graph $G(P)$ defined in Section 1.1, and any two posets with the same comparability graph have the same chain polytope.

For a graph $G = (V, E)$, a **stable set** (resp. **clique**) is a subset $S \subseteq V$ satisfying $\binom{S}{2} \cap E = \emptyset$ (resp. $\binom{C}{2} \subseteq E$). We write $\mathbf{1}_S \in \{0,1\}^V$ for the characteristic vector of any $S \subseteq V$.

Definition 3.2. The **stable set polytope** of a graph $G = (V, E)$ is

$$\mathcal{P}_G \ := \ \operatorname{conv}(\mathbf{1}_S : S \subseteq V \text{ stable set}) \ \subset \ \mathbb{R}^V.$$

For the following well-studied graphs, the stable set polytope has a particularly simple inequality description. A graph G is called **perfect** if for every induced subgraph of G the chromatic number equals the size of a largest clique. The strong perfect graph theorem [15] states that G is perfect if and only if neither G nor its complement graph contain induced odd cycles of length 5 or more. The class of perfect graphs has been well-studied and is also particularly interesting in the context of stable set polytopes: Lovász [58] gave yet another characterization of perfect graphs in terms of their stable set polytopes. For a vector $\mathbf{x} \in \mathbb{R}^V$ and a subset $J \subseteq V$, we write $\mathbf{x}(J) = \sum_{j \in J} x_j$.

Theorem 3.3 ([58])**.** *A graph $G = (V, E)$ is perfect if and only if*

$$\mathcal{P}_G \ = \ \{\mathbf{x} \in \mathbb{R}^V : \mathbf{x} \geq 0, \mathbf{x}(C) \leq 1 \textit{ for all cliques } C \subseteq V\}.$$

Note that antichains (resp. chains) in a poset P are precisely stable sets (resp. cliques) in the comparability graph $G(P)$. Hence, every chain polytope is a stable set polytope. Explicitly,

$$\begin{aligned}\mathcal{P}_{G(P)} \ = \ \mathcal{C}(P) \ &= \ \{\mathbf{x} \in \mathbb{R}^P : \mathbf{x} \geq 0, \mathbf{x}(C) \leq 1 \text{ for all chains } C \subseteq P\} \\ &= \ \{\mathbf{x} \in \mathbb{R}^P : \mathbf{x} \geq 0, \mathbf{x}(C) \leq 1 \text{ for all cliques } C \text{ in } G(P)\}.\end{aligned}$$

In particular, Theorem 3.3 implies that comparability graphs are perfect.

3.1.2 Anti-blocking polytopes

Definition 3.4. A nonempty polytope $\mathcal{P} \subset \mathbb{R}^n_{\geq 0}$ is called **anti-blocking** if

$$\mathbf{q} \in \mathcal{P} \ \text{ and } \ \mathbf{0} \leq \mathbf{p} \leq \mathbf{q} \ \text{ implies } \ \mathbf{p} \in \mathcal{P}, \tag{3.3}$$

where $\mathbf{0} \leq \mathbf{p} \leq \mathbf{q}$ refers to componentwise order in $\mathbb{R}^n$.

It is obvious from (3.1) that chain polytopes are anti-blocking polytopes. More generally, also stable set polytopes of arbitrary graphs are anti-blocking. The following fundamental results on anti-blocking polytopes can be found in [69, Sect. 9.3]. For any $\mathbf{c}_1, \ldots, \mathbf{c}_r \in \mathbb{R}^n_{\geq 0}$ the polytope

$$\{\mathbf{c}_1, \ldots, \mathbf{c}_r\}^{\downarrow} \ := \ \mathbb{R}^n_{\geq 0} \cap (\operatorname{conv}(\mathbf{c}_1, \ldots, \mathbf{c}_r) - \mathbb{R}^n_{\geq 0})$$

has the anti-blocking property. Conversely, if $\mathcal{P} \subset \mathbb{R}^n$ is an anti-blocking polytope, then there always exist $\mathbf{c}_1, \ldots, \mathbf{c}_r \in \mathbb{R}^n_{\geq 0}$ such that $\mathcal{P} = \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}^{\downarrow}$. The unique minimal such set, denoted by $V^{\downarrow}(\mathcal{P})$, is given by the maxima of the vertex set of $\mathcal{P}$ with respect to the partial order $\leq$. Moreover, there is a dual minimal collection $\mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{R}^n_{\geq 0}$ such that

$$\mathcal{P} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for all } i = 1, \ldots, s\}. \tag{3.4}$$

For an arbitrary polytope $\mathcal{Q} \subseteq \mathbb{R}^n_{\geq 0}$, we consider an **associated** anti-blocking polytope

$$A(\mathcal{Q}) \ := \ \{\mathbf{d} \in \mathbb{R}^n_{\geq 0} : \langle \mathbf{d}, \mathbf{x} \rangle \leq 1 \text{ for all } \mathbf{x} \in \mathcal{Q}\}.$$

Anti-blocking polytopes contain the origin on their boundary, hence the ordinary polar is not bounded. This problem can be resolved by replacing polars by associated anti-blocking polytopes. Note that they relate to the ordinary polar since $A(\mathcal{Q}) = \mathcal{Q}^{\triangle} \cap \mathbb{R}^n_{\geq 0}$. The following is the structure theorem for anti-blocking polytopes akin to the bipolar theorem for convex bodies.

Theorem 3.5 ([69, Thm. 9.4])**.** *Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional anti-blocking polytope with*

$$\mathcal{P} \ = \ \{\mathbf{c}_1, \ldots, \mathbf{c}_r\}^{\downarrow} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \textit{ for all } i = 1, \ldots, s\}$$

for some $\mathbf{c}_1, \ldots, \mathbf{c}_r, \mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{R}^n_{\geq 0}$. Then

$$A(\mathcal{P}) \ = \ \{\mathbf{d}_1, \ldots, \mathbf{d}_s\}^{\downarrow} \ = \ \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{c}_i, \mathbf{x} \rangle \leq 1 \textit{ for all } i = 1, \ldots, r\}.$$

In particular, $A(A(\mathcal{P})) = \mathcal{P}$.

For independent set polytopes or perfect graphs, the associated anti-blocking polytopes have an easy combinatorial description. For a graph $G = (V, E)$ its **complement graph** is $\overline{G} := (V, \binom{V}{2} \setminus E)$. Note that the complement graph of a perfect graph is again perfect, which was first proven by Lovász [57]. Alternatively, this fact also follows from the strong perfect graph theorem [15].

Corollary 3.6. *Let G be a perfect graph. Then*

$$A(\mathcal{P}_G) \ = \ \mathcal{P}_{\overline{G}}.$$

Proof. The cliques of G are precisely the stable sets in $\overline{G}$. Since G is perfect, Theorem 3.3 and Theorem 3.5 finish the proof. □

For the special case of chain polytopes, this implies the following.

Corollary 3.7. *Let P be a poset. Then*

$$A(\mathcal{C}(P)) \;=\; \operatorname{conv}(\mathbf{1}_C : C \subseteq P \textit{ chain}).$$

Proof. We have

$$A(\mathcal{C}(P)) \;=\; A(\mathcal{P}_{G(\mathcal{P})}) \;=\; \mathcal{P}_{\overline{G(\mathcal{P})}}.$$

The claim follows since independent sets in $\overline{G(\mathcal{P})}$ are precisely chains in P. □

Remark 3.8. Recall that an inequality in (3.1) is facet-defining for $\mathcal{C}(P)$ if and only if it corresponds to a *maximal* chain in P and hence, by Theorem 3.5, we have $A(\mathcal{C}(P)) = \{\mathbf{1}_C : C \subseteq P \text{ maximal chain}\}^{\downarrow}$. However the vertices of the latter are of the form $\mathbf{1}_C$ for *arbitrary* chains in P.

3.2 Cayley sums of anti-blocking polytopes

In this section, we finally begin to study the double chain polytope $\mathbb{TC}(\mathbf{P})$ associated to a double poset $\mathbf{P}$, analogously to the double order polytope $\mathbb{TO}(\mathbf{P})$. Moreover, we will take a look at a closely related polytope, the Hansen polytope $\mathcal{H}(G)$ arising from a graph G (see [41]). Again, for our purpose we can work in the more general framework of anti-blocking polytopes, that is we will look at the polytopes $\mathcal{P}_1 \boxminus \mathcal{P}_2$ as well as their sections $\mathcal{P}_1 - \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n_{\geq 0}$.

The main results of this section are the following. We completely determine the facets of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ in terms of $\mathcal{P}_1, \mathcal{P}_2$ in Theorem 3.11. This result specializes to a combinatorial description of the facets of $\mathbb{TC}(\mathbf{P})$ for arbitrary double posets $\mathbf{P}$, which is given in Theorem 3.14. These results have several interesting implications. Corollary 3.15 exhibits a new class of reflexive polytopes arising from double graphs. In Corollary 3.16, we observe the curious fact that for induced double posets, the number of facets of the associated double order and double chain polytopes agrees and we conjecture a relation between their f-vectors, analogously to a conjecture by Hibi and Li [47] for the case of ordinary poset polytopes. Hansen [41] showed that the polytope $\mathcal{H}(G)$ is 2-level whenever G is perfect. In Corollary 3.20, we give a simple alternative proof of this result. Finally, in Theorem 3.24, we prove that Hansen polytopes of perfect graphs are the only 2-level polytopes of the form $\mathcal{P}_1 \boxminus \mathcal{P}_2$ with anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$.

3.2.1 Cayley sums and polars

Minkowski sums and Cayley sums of anti-blocking polytopes are of particular interest to us since they generalize the following families of polytopes.

(I) Our main objects of study in this chapter are the following.

Definition 3.9. Let $\mathbf{P} = (P, \preceq_\pm)$ be a double poset. The **double chain polytope** of $\mathbf{P}$ is

$$\mathbb{TC}(\mathbf{P}) \;:=\; 2\mathcal{C}(P_+) \boxminus 2\mathcal{C}(P_-) \;=\; \operatorname{conv}\big((2\mathcal{C}(P_+) \times \{1\}) \cup (-2\mathcal{C}(P_-) \times \{-1\})\big).$$

Sometimes it will be more convenient to look at the **reduced** double chain polytope $\mathbb{DC}(\mathbf{P}) := \mathcal{C}(P_+) - \mathcal{C}(P_-)$. Similar to double order polytopes, we will see that many geometric properties of double chain polytopes, such as their face structure, triangulations and volume, can be described in terms of the underlying double poset.

(II) More generally, we can look at Cayley sums of independent set polytopes: A **double graph** is a triple $\mathbf{G} = (V, E_+, E_-)$ consisting of a node set V with two sets of edges $E_+, E_- \subseteq \binom{V}{2}$. Again, we write $G_+ = (V, E_+)$ and $G_- = (V, E_-)$ to denote the two underlying ordinary graphs. The results of the preceding sections prompt the definition of the **double stable set polytope**

$$\mathcal{P}_{\mathbf{G}} \;:=\; 2\mathcal{P}_{G_+} \boxminus 2\mathcal{P}_{G_-}$$

associated to a double graph $\mathbf{G}$. Note that a double poset $\mathbf{P} = (P, \preceq_\pm)$ gives rise to a double graph $\mathbf{G}(\mathbf{P}) = (G(P_+), G(P_-))$ and the double chain polytope of $\mathbf{P}$ is simply $\mathbb{TC}(\mathbf{P}) = \mathcal{P}_{\mathbf{G}(\mathbf{P})}$.

(III) The case when $G_+ = G_-$ has been well-studied. For a graph G, Hansen [41] studied the Cayley sum $\mathcal{H}(G) := 2\mathcal{P}_G \boxminus 2\mathcal{P}_G$, which are nowadays called **Hansen polytopes**. If G is perfect, then Hansen showed that the polar $\mathcal{H}(G)^\triangle$ is linearly isomorphic to $\mathcal{H}(\overline{G})$ where $\overline{G}$ is the complement graph of G. He moreover showed that $\mathcal{H}(G)$ is 2-level if and only if G is perfect. We will generalize these results in Corollary 3.13 and Theorem 3.24, respectively.

In the sequel, we will consider Cayley sums of the form

$$\mathcal{P}_1 \boxminus \mathcal{P}_2 \;=\; \operatorname{conv}(\mathcal{P}_1 \times \{1\} \cup (-\mathcal{P}_2) \times \{-1\}),$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are anti-blocking polytopes in $\mathbb{R}^n$. Before we come to our first result regarding Cayley- and Minkowski-sums of anti-blocking polytopes, we note the following fact.

Proposition 3.10. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional anti-blocking polytopes. Then the vertices of* $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ *are exactly* $(V(\mathcal{P}_1) \cup V(-\mathcal{P}_2)) \setminus \{\mathbf{0}\}$.

Proof. It suffices to show that every $\mathbf{v} \in V(\mathcal{P}_1) \setminus \{\mathbf{0}\}$ is a vertex of $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$. Let $\mathbf{c} \in \mathbb{R}^n \cong (\mathbb{R}^n)^*$ such that $\mathcal{P}_1^{\mathbf{c}} = \{\mathbf{v}\}$. Since $\mathbf{v} \neq \mathbf{0}$, by (3.4) there is some $\mathbf{d} \in \mathbb{R}^n_{\geq 0} \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{d}, \mathbf{u}_1 \rangle \leq 1$ for all $\mathbf{u}_1 \in \mathcal{P}_1$ and $\langle \mathbf{d}, \mathbf{v} \rangle = 1$. Hence, for any $\mu \geq 0$, $\mathcal{P}_1^{\mathbf{c}+\mu\mathbf{d}} = \{\mathbf{v}\}$. Now, $\langle \mathbf{d}, -\mathbf{u}_2 \rangle \leq 0$ for all $\mathbf{u}_2 \in P_2$. In particular, for $\mu > 0$ sufficiently large,

$$\langle \mathbf{c} + \mu\mathbf{d}, -\mathbf{u}_2 \rangle \;\leq\; \langle \mathbf{c}, -\mathbf{u}_2 \rangle \;<\; \langle \mathbf{c}, \mathbf{v} \rangle + \mu \;=\; \langle \mathbf{c} + \mu\mathbf{d}, \mathbf{v} \rangle,$$

which shows that $\mathbf{v}$ uniquely maximizes $\langle \mathbf{c} + \mu\mathbf{d}, \mathbf{u} \rangle$ over $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$. $\square$

For $\mathbf{d} \in \mathbb{R}^n_{\geq 0}$ and $I \subseteq [n]$, we write $\mathbf{d}^{[I]}$ for the vector with

$$(\mathbf{d}^{[I]})_j \;=\; \begin{cases} d_j & \text{for } j \in I \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 3.11. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then*

$$(\mathcal{P}_1 - \mathcal{P}_2)^\triangle \;=\; \operatorname{conv}(A(\mathcal{P}_1) \cup -A(\mathcal{P}_2)).$$

Moreover,

$$(2\mathcal{P}_1 \boxminus 2\mathcal{P}_2)^\triangle \;=\; -A(\mathcal{P}_2) \boxminus -A(\mathcal{P}_1).$$

Proof. Let us denote the right-hand side of the first equation by $\mathcal{Q}$. Note that $\langle \mathbf{u}_1, -\mathbf{v}_2 \rangle \leq 0$ for $\mathbf{u}_1 \in A(P_1)$ and $\mathbf{v}_2 \in P_2$. This shows that $\langle \mathbf{u}_1, \mathbf{v} \rangle \leq 1$ for all $\mathbf{v} \in P_1 - P_2$. By symmetry, this yields $\mathcal{Q} \subseteq (\mathcal{P}_1 - \mathcal{P}_2)^\triangle$.

For the converse, we observe that every vertex of $\mathcal{Q}$ is of the form $\mathbf{d}^{[I]}$ with $\mathbf{d} \in V^\downarrow(A(\mathcal{P}_1)) \cup -V^\downarrow(A(\mathcal{P}_2))$ and $I \subseteq [n]$. It follows that $\mathbf{z} \in \mathcal{Q}^\triangle$ if and only if $\langle \mathbf{z}, \mathbf{d}^{[I]} \rangle \leq 1$ for all $\mathbf{d} \in V^\downarrow(A(\mathcal{P}_1)) \cup -V^\downarrow(A(\mathcal{P}_2))$ and all $I \subseteq [n]$. For $\mathbf{z} \in \mathcal{Q}^\triangle$, write $\mathbf{z} = \mathbf{z}^1 - \mathbf{z}^2$ with $\mathbf{z}^1, \mathbf{z}^2 \geq 0$ and $\operatorname{supp}(\mathbf{z}^1) \cap \operatorname{supp}(\mathbf{z}^2) = \emptyset$, where for any $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{R}^n$ we set $\operatorname{supp}(\mathbf{z}) := \{i : z_i \neq 0\}$. We claim that $\mathbf{z}^i \in \mathcal{P}_i$ for $i = 1, 2$. Indeed, let $I = \operatorname{supp}(\mathbf{z}^1)$. Then for any $\mathbf{d} \in V^\downarrow(\mathcal{P}_1)$ we have

$$\langle \mathbf{d}, \mathbf{z}^1 \rangle \;=\; \langle \mathbf{d}^{[I]}, \mathbf{z} \rangle \;\leq\; 1$$

and hence $\mathbf{z}^1 \in \mathcal{P}_1$. Applying the same argument to $\mathbf{z}^2$ shows that $\mathbf{z} \in \mathcal{P}_1 - \mathcal{P}_2$ and hence $(\mathcal{P}_1 - \mathcal{P}_2)^\triangle \subseteq \mathcal{Q}$.

For the second claim, note that any linear function on $\mathbb{R}^n \times \mathbb{R}$ that maximizes on a *vertical* facet of $2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$ is of the form $\alpha_{\mathbf{d}} \langle \mathbf{d}, \mathbf{x} \rangle + \delta_{\mathbf{d}} t$ for $\mathbf{d}$ a vertex of $(\mathcal{P}_1 - \mathcal{P}_2)^\triangle$ and some $\alpha_{\mathbf{d}}, \delta_{\mathbf{d}} \in \mathbb{R}$ with $\alpha_{\mathbf{d}} > 0$. By the first claim and Proposition 3.10, it follows that $\mathbf{d} \in (V(A(\mathcal{P}_1)) \cup V(-A(\mathcal{P}_2))) \setminus \{\mathbf{0}\}$.

If $\mathbf{d} \in V(A(\mathcal{P}_1)) \setminus \{\mathbf{0}\}$, then $\langle \mathbf{d}, \mathbf{u}_1 \rangle \leq 1$ is tight for $\mathbf{u}_1 \in \mathcal{P}_1$ whereas $\langle \mathbf{d}, -\mathbf{u}_2 \rangle \leq 0$ is tight for $-\mathbf{u}_2 \in -\mathcal{P}_2$. Hence,

$$\langle \mathbf{d}, \mathbf{x} \rangle - t \;\leq\; 1$$

is the corresponding facet-defining halfspace. Similarly, if $-\mathbf{d} \in -V(A(\mathcal{P}_1)) \setminus \{\mathbf{0}\}$, then

$$\langle -\mathbf{d}, x \rangle + t \;\leq\; 1$$

is facet-defining. Together with the two horizontal facets $\langle \mathbf{0}, \mathbf{x} \rangle \pm t \leq 1$ this yields an inequality description of $(-A(\mathcal{P}_2) \boxminus -A(\mathcal{P}_1))^\triangle$, which proves the second claim. $\square$

Theorem 3.11 together with Theorem 3.5 has a nice implication that was used in [68] in connection with Hansen polytopes.

Corollary 3.12. *For any full-dimensional anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$, the polytope $\mathcal{P} \boxminus A(\mathcal{P})$ is linearly isomorphic to its polar $(\mathcal{P} \boxminus A(\mathcal{P}))^\triangle$. In particular, we have that $\mathcal{P} \boxminus A(\mathcal{P})$ is self-dual, that is, combinatorially isomorphic to its polar.*

3.2.2 Facets of Double chain and Hansen polytopes

Theorem 3.11 yields a simple combinatorial description for the polars of double stable set polytopes. For a double graph $\mathbf{G}$, define the **complement graph** as $\overline{\mathbf{G}} = (V, E_+^c, E_-^c)$, that is, the double graph consisting of the two ordinary complement graphs.

Corollary 3.13. *Let* $\mathbf{G}$ *be a perfect double graph. Then* $\mathcal{P}_{\mathbf{G}}^{\triangle}$ *is linearly isomorphic to* $\mathcal{P}_{\overline{\mathbf{G}}}$.

Proof. By Theorem 3.11 and Corollary 3.6 we have

$$\mathcal{P}_{\mathbf{G}}^{\triangle} = (2\mathcal{P}_{G_+} \boxminus 2\mathcal{P}_{G_-})^{\triangle} \;=\; -A(\mathcal{P}_{G_-}) \boxminus -A(\mathcal{P}_{G_+}) \;=\; -\mathcal{P}_{\overline{G}_-} \boxminus -\mathcal{P}_{\overline{G}_+} \;\cong\; \mathcal{P}_{\overline{\mathbf{G}}}. \quad \square$$

Theorem 3.11 directly gives an explicit facet description of the double chain polytope. Note that compatibility is not required.

Theorem 3.14. *Let* $\mathbf{P}$ *be a double poset and* $\mathbb{TC}(\mathbf{P})$ *it double chain polytope. Then* $(g, t) \in \mathbb{R}^P \times \mathbb{R}$ *is contained in* $\mathbb{TC}(\mathbf{P})$ *if and only if*

$$\sum_{a \in C_+} g(a) - t \;\leq\; 1 \quad \text{and} \quad \sum_{a \in C_-} -g(a) + t \;\leq\; 1,$$

where $C_+ \subseteq P_+$ *and* $C_- \subseteq P_-$ *ranges over all chains.*

Proof. By Theorem 3.11, we have

$$\mathbb{TC}(\mathbf{P})^{\triangle} \;=\; (2\mathcal{C}(P_+) \boxminus 2\mathcal{C}(P_-))^{\triangle} \;=\; -A(\mathcal{C}(P_-)) \boxminus -A(\mathcal{C}(P_+)).$$

Hence, facets of $\mathbb{TC}(\mathbf{P})$ correspond to vertices of $-A(\mathcal{C}(P_-)) \boxminus -A(\mathcal{C}(P_+))$ and Theorem 3.5 together with (3.1) finishes the proof. $\square$

Recall that an integral polytope is called reflexive (cf. Corollary 1.42) if its polar is also integral. Our above results immediately imply the following.

Corollary 3.15. *Let* $\mathbf{G} = (G, E_+, E_-)$ *be a perfect double graph. Then the polytopes* $\mathcal{P}_{\mathbf{G}} = 2\mathcal{P}_{G_+} \boxminus 2\mathcal{P}_{G_-}$ *and* $\mathcal{P}_{G_+} - \mathcal{P}_{G_-}$ *are reflexive. In particular all (reduced) double chain polytopes are reflexive.*

For the usual order- and chain polytope, Hibi and Li [47] showed that $\mathcal{O}(P)$ has at most as many facets as $\mathcal{C}(P)$ and equality holds if and only if P does not contain the 5-element poset P_X from Example 1.2. This is different in the case of double poset polytopes.

Corollary 3.16. *Let* P *be a poset with induced double poset* $\mathbf{P}$. *Then* $\mathbb{TO}(\mathbf{P})$ *and* $\mathbb{TC}(\mathbf{P})$ *have the same number of facets.*

Proof. Every chain $C \subseteq P$ yields two distinct alternating chains in $\mathbf{P}$. Conversely, every alternating chain arises this way. $\square$

However, it is in general not true that $\mathbb{TO}(\mathbf{P})$ is combinatorially isomorphic to $\mathbb{TC}(\mathbf{P})$, as the following example shows.

Example 3.17. Let $\mathbf{P}_X$ be the double poset induced by P_X (cf. Example 1.2). Then, using `sage` [18] it can be checked that the face vectors of $\mathbb{TO}(\mathbf{P}_X)$ and $\mathbb{TC}(\mathbf{P}_X)$ are

$$\begin{aligned} f(\mathbb{TO}(\mathbf{P}_X)) &= (16, 88, 204, 240, 144, 36) \text{ and} \\ f(\mathbb{TC}(\mathbf{P}_X)) &= (16, 88, 222, 276, 162, 36). \end{aligned}$$

Hibi and Li [47] conjectured that $f(\mathcal{O}(P)) \leq f(\mathcal{C}(P))$ componentwise. Computations suggest that the same relation should hold for the double poset polytopes of induced double posets.

Conjecture 3.18. *Let $\mathbf{P} = (P, \preceq, \preceq)$ be a double poset induced by a poset $(P, \preceq)$. Then*

$$f_i(\mathbb{TO}(\mathbf{P})) \;\leq\; f_i(\mathbb{TC}(\mathbf{P}))$$

for $0 \leq i \leq |P|$.

An extension of the conjecture to general compatible double posets fails, as the following examples show.

Example 3.19. (1) Let $\mathbf{Alt}_n$ be the alternating chain from Example 1.4. It follows from Theorem 3.14 that the number of facets of $\mathbb{TC}(\mathbf{Alt}_n)$ is $3n+1$. Since $\mathbf{Alt}_n$ is compatible, then by Theorem 1.29 the number of facets of $\mathbb{TO}(\mathbf{Alt}_n)$ equals the number of alternating chains, which is easily computed to be $\binom{n+2}{2} + 1$. Thus, for $n \geq 4$, the alternating chains $\mathbf{Alt}_n$ fail Conjecture 3.18 for the number of facets. For $n = 4$, we used `sage` [18] to explicitly compute

$$\begin{aligned} f(\mathbb{TO}(\mathbf{Alt}_4)) &= (21, 70, 95, 60, 16) \quad \text{and} \\ f(\mathbb{TC}(\mathbf{Alt}_4)) &= (21, 67, 86, 51, 13). \end{aligned}$$

(2) Recall from Example 1.30 that the double order polytope $\mathbb{TO}(\mathbf{AC}_n)$ has $\binom{n}{2} + 2n + 2$ facets. On the other hand it follows easily from Theorem 3.14 that $\mathbb{TC}(\mathbf{AC}_n)$ has $2^n + n + 1$ facets. For $n = 3$, we explicitly have

$$\begin{aligned} f(\mathbb{TO}(\mathbf{AC}_3)) &= (12, 30, 29, 11) \quad \text{and} \\ f(\mathbb{TC}(\mathbf{AC}_3)) &= (12, 31, 31, 12). \end{aligned}$$

(3) For the double poset $\mathbf{P}_{XW}$ from Example 1.5, the polytopes $\mathbb{TO}(\mathbf{P}_{XW})$ and $\mathbb{TC}(\mathbf{P}_{XW})$ have the same number of facets, but they are not combinatorially equivalent:

$$\begin{aligned} f(\mathbb{TO}(\mathbf{P}_{XW})) &= (21, 112, 247, 263, 135, 28) \text{ and} \\ f(\mathbb{TC}(\mathbf{P}_{XW})) &= (21, 114, 254, 271, 138, 28). \end{aligned}$$

Theorem 3.11 also recovers one of the main results of Hansen [41].

Corollary 3.20 ([41, Thm. 4 and Thm. 6])**.** *Let G be a perfect graph. Then $\mathcal{H}(G)$ is 2-level and $\mathcal{H}(G)^\triangle$ is linearly isomorphic to $\mathcal{H}(\overline{G})$.*

Proof. A vertex of $\mathcal{H}(G)^\triangle$ is of the form $\mathbf{d} = \pm(-\mathbf{1}_C, 1)$ for some clique C of G. Thus, for any vertex $\mathbf{v} = \pm(2\mathbf{1}_S, 1) \in \mathcal{H}(G)$, where S is a stable set of G, we compute $\langle \mathbf{d}, \mathbf{v} \rangle = \pm(1 - 2|S \cap C|) = \pm 1$. The second claim follows from Corollary 3.13. $\square$

Example 3.21 (Double chain polytopes of dimension-two posets)**.** Following Example 1.3, let $\pi_+, \pi_- \in \mathbb{Z}^n$ be two integer sequences with associated posets P_{π_+} and P_{π_-} of order dimension two. Consider the double posets $\mathbf{P} = (P_{\pi_+}, P_{\pi_-})$ and $-\mathbf{P} = (P_{-\pi_-}, P_{-\pi_+})$. We have

$$\overline{\mathbf{G}(\mathbf{P})} \;=\; (\overline{G(P_{\pi_-})}, \overline{G(P_{\pi_+})}) \;=\; (G(P_{-\pi_-}), G(P_{-\pi_+})) \;=\; \mathbf{G}(-\mathbf{P})$$

and hence

$$\mathbb{T}\mathcal{C}(\mathbf{P})^{\triangle} \;\cong\; \mathbb{T}\mathcal{C}(-\mathbf{P})$$

by Corollary 3.13. Note that a similar statement does not hold true for double order polytopes. That is, it is not necessarily true that $\mathbb{T}\mathcal{O}(\mathbf{P})^{\triangle} \cong \mathbb{T}\mathcal{O}(-\mathbf{P})$, as can be checked for the dimension-two double poset $\mathbf{P}_X$ (cf. Example 3.17).

Example 3.22 (Double chain polytopes of plane posets)**.** Let $\mathbf{P}$ be a plane double poset. By the last example and Example 1.9, the double chain polytope $\mathbb{T}\mathcal{C}(\mathbf{P})$ is linearly isomorphic to its polar $\mathbb{T}\mathcal{C}(\mathbf{P})^{\triangle}$.

Among the 2-level polytopes, independence polytopes of perfect graphs play a distinguished role. The following observation, due to Samuel Fiorini (personal communication), characterizes 2-level anti-blocking polytopes.

Proposition 3.23. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional anti-blocking polytope. Then $\mathcal{P}$ is* 2*-level if and only if $\mathcal{P}$ is linearly isomorphic to $\mathcal{P}_G$ for some perfect graph G.*

Proof. The origin is a vertex of $\mathcal{P}$ and, since $\mathcal{P}$ is full-dimensional and anti-blocking, its neighbors are $\alpha_1\mathbf{e}_1, \dots, \alpha_1\mathbf{e}_n$ for some $\alpha_i > 0$. After a linear transformation, we can assume that $\alpha_1 = \cdots = \alpha_n = 1$. Since $\mathcal{P}$ is 2-level, it follows that $\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n_{\geq 0} : \langle \mathbf{d}_i, \mathbf{x}\rangle \leq 1 \text{ for } i = 1, \dots, s\}$ with $\mathbf{d}_i \in \{0,1\}^n$ for all $i = 1, \dots, s$. Let $G = ([n], E)$ be the minimal graph with cliques $\operatorname{supp}(\mathbf{d}_i)$ for all $i = 1, \dots, s$. That is, $E = \bigcup_i \binom{\operatorname{supp}(\mathbf{d}_i)}{2}$. We have $\mathcal{P}_G \subseteq \mathcal{P}$. Conversely, again by 2-levelness, any vertex of $\mathcal{P}$ is of the form $\mathbf{1}_S$ for some $S \subseteq [n]$ and $\langle \mathbf{d}_i, \mathbf{1}_S\rangle = |\operatorname{supp}(\mathbf{d}_i) \cap S| \leq 1$ shows that $\mathcal{P} \subseteq \mathcal{P}_G$. Finally, it follows from Theorem 3.3 that G is perfect. □

Proposition 3.23 can be used to characterize 2-level polytopes among Cayley sums of anti-blocking polytopes.

Theorem 3.24. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then $\mathcal{P} = \mathcal{P}_1 \boxminus \mathcal{P}_2$ is* 2*-level if and only if $\mathcal{P}$ is affinely isomorphic to $\mathcal{H}(G)$ for some perfect graph G.*

Proof. Sufficiency is Hansen's result (Corollary 3.20). For necessity, observe that $\mathcal{P}_1$ and $\mathcal{P}_2$ are faces and hence have to be 2-level. By the proof of Proposition 3.23, we may assume that $\mathcal{P}_1 = \mathcal{P}_{G_1}$ for some perfect graph G_1 and $\mathcal{P}_2 = A\mathcal{P}_{G_2}$ for some perfect G_2 and a diagonal matrix $A \in \mathbb{R}^{n\times n}$ with diagonal entries $a_i > 0$ for $i \in [n]$. We will proceed in two steps: We first prove that A must be the identity matrix and then show that $G_1 = G_2$.

For every $i \in [n]$, the inequality $x_i \geq 0$ is facet-defining for $\mathcal{P}_1$. Hence, it induces a facet-defining inequality for the Cayley sum $\mathcal{P}$, which must be of the form

$$\ell_i(\mathbf{x}) \;:=\; -b_i x_i + t \;\leq\; 1$$

for some $b_i > 0$, where t denotes the last coordinate in $\mathbb{R}^{n+1}$. Observe that ℓ_i takes the values 1 and $1 - b_i$ on the vertices $\{\mathbf{0}, \mathbf{e}_i\} \times \{1\}$ of the face $\mathcal{P}_1 \times \{1\}$. On the other hand, on $\{\mathbf{0}, -a_i\mathbf{e}_i\} \times \{-1\} \subset -\mathcal{P}_2 \times \{-1\}$, the values are -1 and $-1 + a_ib_i$. Now 2-levelness implies $a_i = 1$ and $b_i = 2$.

It now follows from Theorem 3.11 and Corollary 3.6 that the facet-defining inequalities for $\mathcal{P}$ are

$$\begin{aligned} 2\mathbf{1}_{C_1}(\mathbf{x}) - t &\leq 1 \text{ and} \\ -2\mathbf{1}_{C_2}(\mathbf{x}) + t &\leq 1, \end{aligned}$$

where C_1 and C_2 are cliques in G_1 and G_2, respectively. By 2-levelness each of these linear functions takes the values -1 and 1 on the vertices of P. This implies that every clique in G_1 must be a clique in G_2 and conversely. Hence $G_1 = G_2$. □

3.3 Subdivisions and triangulations

In this section, we are concerned with subdivisions and triangulations of the polytopes from the previous section. For anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$, there is a canonical subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$, which we describe in Lemma 3.25. Moreover, in Theorem 3.26, we prove that if $\mathcal{P}_1$ and $\mathcal{P}_2$ have regular, unimodular, or flag triangulations, then so has the Cayley sum $\mathcal{P}_1 \boxminus \mathcal{P}_2$. In Corollary 3.29, we apply this to the case of double chain polytopes: Stanley's canonical triangulation of chain polytopes [73] lifts to a regular flag and unimodular triangulation of double chain polytopes. In Theorem 3.31 we use Sturmfels' correspondence to translate this triangulation into the language of algebra and we obtain a Gröbner basis of the toric ideals associated to double chain polytopes.

The subdivision from Lemma 3.25 implies a formula for the volume of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$, given in Theorem 3.32, which in the case of double chain polytopes only depends on the combinatorics of the underlying double poset $\mathbf{P}$ (Corollary 3.33). Curiously, in the case when $\mathbf{P}$ is compatible, the volumes of $\mathbb{T}\mathcal{O}(\mathbf{P})$ and $\mathbb{T}\mathcal{C}(\mathbf{P})$ agree and, in fact, the two canonical triangulations even have the same underlying simplicial complex. This is not a coincidence, as we will see in Section 3.4.

3.3.1 Subdividing Cayley sums

We now turn to the canonical subdivisions of $\mathcal{P}_1 - \mathcal{P}_2$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$. A subdivision of $\mathcal{P} = \mathcal{P}_1 - \mathcal{P}_2$ with maximal cells $\mathcal{Q}^1, \ldots, \mathcal{Q}^m \subseteq \mathcal{P}$ is called **mixed** if each $\mathcal{Q}^i$ is of the form $\mathcal{Q}_1^i - \mathcal{Q}_2^i$ where $\mathcal{Q}_j^i$ is a vertex-induced subpolytope of $\mathcal{P}_j$, that is $V(\mathcal{Q}_j^i) \subseteq V(\mathcal{P}_j)$ for $j = 1, 2$. Finally, a mixed subdivision is **exact** if $\dim \mathcal{Q}^i = \dim \mathcal{Q}_1^i + \dim \mathcal{Q}_2^i$. That is, $\mathcal{Q}^i$ is linearly isomorphic to the Cartesian product $\mathcal{Q}_1^i \times \mathcal{Q}_2^i$. The Cayley trick [17, Thm 9.2.18] states that mixed subdivisions of the Minkowski sum $\mathcal{P}_1 - \mathcal{P}_2$ are in bijection with subdivisions of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ without new vertices, where a cell $\mathcal{Q}_1^i - \mathcal{Q}_2^i$ corresponds to $\mathcal{Q}_1^i \boxminus \mathcal{Q}_2^i$. Moreover, an exact mixed subdivision of $\mathcal{P}_1 - \mathcal{P}_2$ yields a subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ into **joins**: The cells $\mathcal{Q}_1^i \boxminus \mathcal{Q}_2^i$ are obtained by embedding $\mathcal{Q}_1^i$ and $\mathcal{Q}_2^i$ in skew affine subspaces and taking their convex hull. In this case we will also write $\mathcal{Q}_1^i * \mathcal{Q}_2^i$ for $\mathcal{Q}_1^i \boxminus \mathcal{Q}_2^i$.

For a full-dimensional anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$, every index set $J \subseteq [n]$ defines a distinct face $\mathcal{P}|_J := \{x \in \mathcal{P} : x_j = 0 \text{ for } j \notin J\}$. This is an anti-blocking polytope of dimension $|J|$.

Lemma 3.25. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then $\mathcal{P}_1 - \mathcal{P}_2$ has a regular exact mixed subdivision with maximal cells $\mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$ for all $J \subseteq [n]$. In particular, $\mathcal{P}_1 \boxminus \mathcal{P}_2$ has a regular subdivision whose maximal cells are the joins $\mathcal{P}_1|_J * \mathcal{P}_2|_{J^c}$ for all $J \subseteq [n]$.*

We call the subdivisions of Lemma 3.25 the **canonical subdivisions** of $\mathcal{P}_1 - \mathcal{P}_2$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2$, respectively.

Proof. By the Cayley trick [17, Thm 9.2.18], it is suffices to prove only the first claim. The subdivision of $\mathcal{P}_1 - \mathcal{P}_2$ is very easy to describe: Let us first note that the polytopes $\mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$ for $J \subseteq [n]$ only meet in faces. Hence, we only need to verify that they cover $\mathcal{P}_1 - \mathcal{P}_2$. It suffices to show that for any point $\mathbf{x} \in \mathcal{P}_1 - \mathcal{P}_2$ with $x_i \neq 0$ for all i, there is a $J \subseteq [n]$ with $\mathbf{x} \in \mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$. Let $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n_{\geq 0}$ with $\mathbf{x} = \mathbf{x}_1 - \mathbf{x}_2$ and $\operatorname{supp}(\mathbf{x}_1) \cap \operatorname{supp}(\mathbf{x}_2) = \emptyset$. We claim that $\mathbf{x}_i \in \mathcal{P}_i$ for $i = 1, 2$. Indeed, if $\mathbf{x} = \mathbf{y}_1 - \mathbf{y}_2$ for some $\mathbf{y}_i \in \mathcal{P}_i$, then $0 \leq \mathbf{x}_i \leq \mathbf{y}_i$ and $\mathbf{x}_i \in \mathcal{P}_i$ by (3.3). In particular, $\mathbf{x}_1 \in \mathcal{P}_1|_J$ and $\mathbf{x}_2 \in \mathcal{P}_2|_{J^c}$ and therefore $\mathbf{x} \in \mathcal{P}_1|_J - \mathcal{P}_2|_{J^c}$.

To show regularity, we pick the following height function $\omega : V(\mathcal{P}) \to \mathbb{R}$. Any vertex of $\mathcal{P}$ is of the form $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ with $\mathbf{v}_1 \in \mathcal{P}_1$ and $\mathbf{v}_2 \in \mathcal{P}_2$ and we set

$$\omega(\mathbf{v}) \ := \ \langle \mathbf{1}, \mathbf{v}_1 \rangle + \langle \mathbf{1}, \mathbf{v}_2 \rangle,$$

that is we take the sum of all coordinates of $\mathbf{v}_1$ and $\mathbf{v}_2$. We denote by $\overline{\mathcal{P}} = \operatorname{conv}\{(\mathbf{v}, \omega(\mathbf{v})) : \mathbf{v} \in V(\mathcal{P})\}$ the lifted polytope in $\mathbb{R}^{n+1}$ and for $J_1, J_2 \subseteq [n]$ with $J_1 \cap J_2 = \emptyset$ we define

$$\ell_{J_1,J_2} \ = \ (\mathbf{1}_{J_1} - \mathbf{1}_{J_2}, -1) \in \mathbb{R}^{n+1}.$$

For any vertex $\overline{\mathbf{v}} = (\mathbf{v}_1 - \mathbf{v}_2, \omega(\mathbf{v}_1, \mathbf{v}_2))$ of $\overline{\mathcal{P}}$ with $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n_{\geq 0}$ we have

$$\langle \ell_{J_1,J_2}, \overline{\mathbf{v}} \rangle \ = \ \langle \mathbf{1}_{J_1} - \mathbf{1}_{J_2}, \mathbf{v}_1 - \mathbf{v}_2 \rangle - \omega(\mathbf{v}_1 - \mathbf{v}_2) \ \leq \ 0,$$

where equality holds if and only if $\operatorname{supp}(\mathbf{v}_1) \subseteq J_1$ and $\operatorname{supp}(\mathbf{v}_2) \subseteq J_2$. Thus, the subdivision is regular, which finishes the proof. □

For the following result, recall the definition of regular, flag and unimodular triangulations from Section 2.1.3.

Theorem 3.26. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes with subdivisions $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. For $J \subseteq [n]$, let $\mathcal{S}_i|_J := \{S \cap \mathcal{P}_i|_J : S \in \mathcal{S}_i\}$ be the restriction of $\mathcal{S}_i$ to $\mathcal{P}_i|_J$ for $i = 1, 2$. Moreover, for $J_1, J_2 \subseteq [n]$ disjoint let $\mathcal{S}_1|_{J_1} * \mathcal{S}_2|_{J_2} := \{\sigma_1 * \sigma_2 : \sigma_i \in \mathcal{S}_i|_{J_i}\}$. Then*

$$\mathcal{S} \ := \ \bigcup_{J_1 \cap J_2 = \emptyset} \mathcal{S}_1|_{J_1} * \mathcal{S}_2|_{J_2}$$

is a subdivision of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ and the following hold:

(i) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are regular, then $\mathcal{S}$ is regular.*

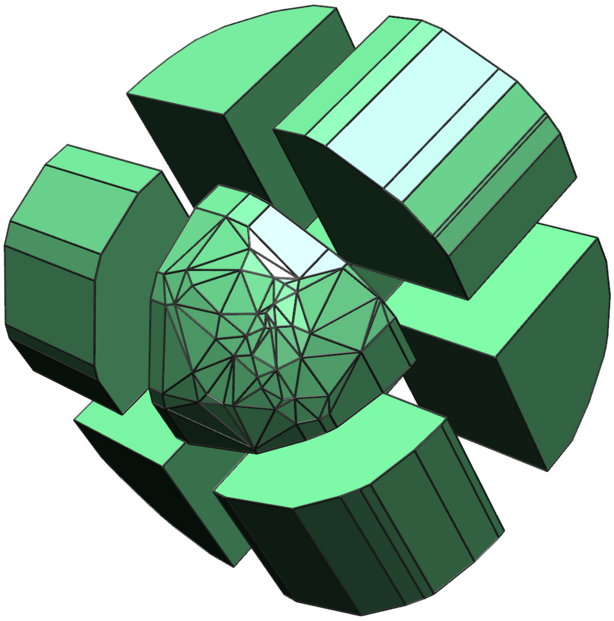

Figure 3.1: The canonical subdivision of $\mathcal{P}_1 - \mathcal{P}_2$ for two (random) anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^3_{\geq 0}$.

(ii) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are flag, then $\mathcal{S}$ is flag.*
(iii) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are triangulations, then $\mathcal{S}$ is a triangulation.*
(iv) *If $\mathcal{S}_1$ and $\mathcal{S}_2$ are unimodular triangulations with respect to a lattice $\Lambda \subset \mathbb{R}^n$, then $\mathcal{S}$ is a unimodular triangulation with respect to the affine lattice $\Lambda \times (2\mathbb{Z} + 1)$.*

Proof. For the first claim, observe that $\mathcal{S}_i|_J$ is a subdivision of the face $\mathcal{P}_i|_J$. By [17, Thm 4.2.7], $\mathcal{S}_1|_J * \mathcal{S}_2|_{J^c}$ is a subdivision of $\mathcal{P}_1|_J * \mathcal{P}_2|_{J^c}$. Hence, $\mathcal{S}$ is a refinement of the canonical subdivision of Lemma 3.25.

If $\mathcal{S}_i$ is a regular subdivision of $\mathcal{P}_i$, then there are weights $\omega_i : V(\mathcal{P}_i) \to \mathbb{R}$ for $i \in \{1, 2\}$. By adding a constant weight to every vertex if necessary, we can assume that $\omega_1(\mathbf{v}_1) > 0$ and $\omega_2(\mathbf{v}_2) < 0$ for all $\mathbf{v}_1 \in V(\mathcal{P}_1)$ and $\mathbf{v}_2 \in V(\mathcal{P}_2)$. Again using the Cayley trick, it is straightforward to verify that $\omega : V(\mathcal{P}_1 \boxminus \mathcal{P}_2) \to \mathbb{R}$ given by $\omega(\mathbf{v}_1, +1) := \omega_1(\mathbf{v}_1)$ and $\omega(\mathbf{v}_2, -1) := \omega_2(\mathbf{v}_2)$ induces $\mathcal{S}$.

For claim (ii), first note that for $J_1, J_2 \subseteq [n]$ disjoint, the subdivision $\mathcal{S}_1|_{J_1} * \mathcal{S}_2|_{J_2}$ is flag if $\mathcal{S}_1$ and $\mathcal{S}_2$ are. To show that $\mathcal{S}$ is flag, let $\sigma = \sigma_1 \boxminus \sigma_2$ be a minimal non-face. Since $\mathcal{S}_1$ and $\mathcal{S}_2$ are flag, it suffices to consider the case when $\sigma_i \neq \emptyset$ for $i \in \{1, 2\}$. By minimality we have $\sigma_1 \in \mathcal{S}_1$ and $\sigma_2 \in \mathcal{S}_2$. We may also assume that σ is not contained in any of the flag joins $\mathcal{S}_1|_{J_1} * \mathcal{S}_2|_{J_2}$ for $J_1, J_2 \in [n]$ with $J_1 \cap J_2 = \emptyset$. Thus, there are vertices $v_i \in \sigma_i$ for $i \in \{1, 2\}$ such that $\operatorname{supp}(v_1) \cap \operatorname{supp}(v_2) \neq \emptyset$. But then $\{v_1, v_2\}$ is already a non-face and the claim follows.

Finally, the claims (iii) and (iv) simply follows from the fact that the join of two (unimodular) simplices is a (unimodular) simplex. $\square$

The theorem has some immediate consequences.

Corollary 3.27. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional anti-blocking polytopes each having a unimodular triangulation. Then $\mathcal{P}_1 - \mathcal{P}_2$ and $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ both have unimodular triangulations.*

Proof. By Theorem 3.26 and the Cayley trick, $\mathcal{P}_1 - \mathcal{P}_2$ has a mixed subdivision into Cartesian products of unimodular simplices. Products of unimodular simplices are 2-level and, for example by [78, Thm. 2.4], have unimodular triangulations. The polytope $\operatorname{conv}(\mathcal{P}_1 \cup -\mathcal{P}_2)$ inherits a triangulation from the upper or lower hull of $\mathcal{P}_1 \boxminus \mathcal{P}_2$, which has a unimodular triangulation by Theorem 3.26. $\square$

We close this section with some immediate implications about double stable set polytopes.

Corollary 3.28. *Let* $\mathbf{G}$ *be a perfect double graph. Then* $\mathcal{P}_{\mathbf{G}}$, $\mathcal{P}_{G_+} - \mathcal{P}_{G_-}$, *and* $\operatorname{conv}(\mathcal{P}_{G_+} \cup -\mathcal{P}_{G_-})$ *have regular unimodular triangulations.*

Proof. By Theorem 3.3, both polytopes $\mathcal{P}_{G_+}$ and $\mathcal{P}_{G_-}$ are 2-level and by [78, Thm. 2.4] have unimodular triangulations. The results now follows from Theorem 3.26 and the previous corollary. $\square$

3.3.2 Triangulations of double chain polytopes

Before turning to the case of double chain polytopes, we review some results on ordinary chain polytopes. Stanley [73] elegantly transferred the canonical triangulation of $\mathcal{O}(P)$ to $\mathcal{C}(P)$ in the following sense. Define the **transfer map** $\Phi_P : \mathcal{O}(P) \to \mathcal{C}(P)$ by

$$(\Phi_P f)(b) \;:=\; \min\{f(b) - f(a) : a \prec b\}, \tag{3.5}$$

for $f \in \mathcal{O}(P)$ and $b \in P$. This is a **piecewise linear** map which is linear on the full-dimensional simplices of the triangulation in Theorem 1.20. In particular, $\Phi_P(\mathbf{1}_{\mathsf{J}}) = \mathbf{1}_{\min(\mathsf{J})}$ for any filter $\mathsf{J} \subseteq P$, which shows that Φ_P maps $\mathcal{O}(P)$ into $\mathcal{C}(P)$. To show that Φ_P is a PL homeomorphism of the two polytopes, Stanley explicitly defines an inverse $\Psi_P : \mathcal{C}(P) \to \mathcal{O}(P)$ by

$$(\Psi_P g)(b) \;:=\; \max\{g(a_0) + \cdots + g(a_{k-1}) + g(a_k) : a_0 \prec \cdots \prec a_{k-1} \prec a_k \preceq b\}, \tag{3.6}$$

for any $g \in \mathcal{C}(P)$. Note that our definition of Ψ_P differs from that in [73] in that we do *not* require that the chain has to end in b. This will be important later. It can be checked that Ψ_P is an inverse to Φ_P. Hence, the simplices

$$H(C) \;:=\; \operatorname{conv}(\mathbf{1}_{\min(\mathsf{J}_0)}, \ldots, \mathbf{1}_{\min(\mathsf{J}_k)}) \quad \text{for} \quad C = \{\mathsf{J}_0 \subset \cdots \subset \mathsf{J}_k\} \in \Delta(\mathcal{J}(P))$$

constitute a flag triangulation of $\mathcal{C}(P)$, which also turns out to be unimodular. The underlying simplicial complex is $\Delta(\mathcal{J}(P))$ and hence, the canonical triangulations of $\mathcal{O}(P)$ and $\mathcal{C}(P)$ are combinatorially equivalent. In particular, $\mathcal{O}(P)$ and $\mathcal{C}(P)$ have the same volume and the number of maximal simplices in the triangulation of $\mathcal{C}(P)$ equals $\mathrm{e}(P)$. This also shows that the number of linear extensions of a poset only depends on its comparability graph.

In the case of ordinary poset polytopes the triangulation of $\mathcal{O}(P)$ seems somewhat more natural than the one of $\mathcal{C}(P)$. Curiously, for a double poset $\mathbf{P}$, from a geometric perspective it is more straightforward to triangulate $\mathbb{TC}(\mathbf{P})$ instead of $\mathbb{TO}(\mathbf{P})$, since a triangulation can be obtained by refining the canonical subdivision from Lemma 3.25. Recall from Section 2.2.3 that a pair of chains $C = C_+ \uplus C_-$ with $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ is non-interfering if $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \emptyset$ for any $\mathsf{J}_\sigma \in C_\sigma$ for $\sigma \in \{\pm\}$.

Corollary 3.29. *Let* $\mathbf{P} = (P, \preceq_+, \preceq_-)$ *be a double poset. Then a triangulation of* $\mathbb{T}\mathcal{C}(P_1, P_2)$ *is given as follows: The* $(k-1)$*-dimensional simplices are in bijection to non-interfering pairs of chains* $C = C_+ \uplus C_- \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$ *with* $|C| = |C_+| + |C_-| = k$. *Explicitly,* C *corresponds to the simplex* $\overline{H}(C) := 2H(C_+) \boxminus 2H(C_-)$. *Moreover, the triangulation is regular, unimodular, and flag.*

Proof. The canonical triangulation of $\mathcal{C}(P_\sigma)$ is regular, unimodular, and flag for $\sigma \in \{\pm\}$. As described above, its $(l_\sigma - 1)$-simplices are in bijection to chains $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ of length $|C_\sigma| = l_\sigma$. By Theorem 3.26 applied to $\mathbb{T}\mathcal{C}(\mathbf{P}) = 2\mathcal{C}(P_+) \boxminus 2\mathcal{C}(P_-)$ it follows that a regular, unimodular and flag triangulation is given by the joins $2H(C_+) \boxminus 2H(C_-)$ for all chains $C_\sigma \subseteq \mathcal{J}(P_\sigma)$ such that $H(C_+)$ and $H(C_-)$ lie in complementary coordinate subspaces. This, however, is exactly the case when $\min(\mathsf{J}_+) \cap \min(\mathsf{J}_-) = \emptyset$ for all $\mathsf{J}_\sigma \in C_\sigma$ for $\sigma \in \{\pm\}$. □

As a simplicial complex, the triangulation in Corollary 3.29 is the non-interfering complex introduced in Section 2.2.3, which implies the following.

Corollary 3.30. *Let* $\mathbf{P}$ *be a compatible double poset. Then the triangulations of* $\mathbb{T}\mathcal{O}(\mathbf{P})$ *and* $\mathbb{T}\mathcal{C}(\mathbf{P})$ *given in Corollary 2.20 and Corollary 3.29, respectively, are combinatorially equivalent.*

In Section 2.2.3, we have used Gröbner bases to obtain triangulations of double order polytopes. Now we will use Sturmfels' correspondence in the other direction to find Gröbner bases in the case of double chain polytopes. For a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$, the toric ideal $I_{\mathbb{T}\mathcal{C}(\mathbf{P})}$ is contained in the polynomial ring $\mathbb{C}[x_{A_+}, x_{A_-}]$, where A_σ ranges over all anti-chains in P_σ for $\sigma \in \{\pm\}$. To describe a Gröbner basis for $I_{\mathbb{T}\mathcal{C}(\mathbf{P})}$, we introduce the following notation. For $\sigma \in \{\pm\}$ and two antichains $A, A' \subseteq P_\sigma$ define

$$\begin{aligned} A \sqcup A' &:= \min(A \cup A') \text{ and} \\ A \sqcap A' &:= (A \cap A') \cup \big((A \cup A') \setminus \min(A \cup A')\big). \end{aligned}$$

Theorem 3.31. *Let* $\mathbf{P}$ *be a double poset. Fix a total order* $\leq$ *on the variables* $\{x_{A_\sigma} : \sigma \in \{\pm\}, A_\sigma \subseteq P_\sigma \text{ antichain}\}$ *such that*
(i) $x_{A_\sigma} < x_{A'_\sigma}$ *for* $\sigma \in \{\pm\}$ *whenever* $\langle A_\sigma \rangle \subset \langle A'_\sigma \rangle$ *and*
(ii) $x_{A_+} < x_{A_-}$ *for any* $A_+ \subseteq P_+$ *and* $A_- \subseteq P_-$.
Denote by $\preceq_{rev}$ *the induced reverse lexicographic monomial order. A Gröbner basis for* $I_{\mathbb{T}\mathcal{C}(\mathbf{P})}$ *with respect to* $\preceq_{rev}$ *is given by the binomials*

$$\underline{x_A\, x_{A'}} \;-\; x_{A \sqcup A'}\, x_{A \sqcap A'} \qquad \langle A \rangle_\sigma, \langle A' \rangle_\sigma \in \mathcal{J}(P_\sigma) \text{ incomparable}$$

for antichains $A, A' \subset P_\sigma$ *for* $\sigma \in \{\pm\}$ *and*

$$\underline{x_{A_+}\, x_{A_-}} \;-\; x_{A_+ \setminus A_-} x_{A_- \setminus A_+} \qquad \text{for antichains } A_\sigma \subseteq P_\sigma.$$

Proof. Firstly, it is easy to check that the above binomials lie in $I_{\mathbb{T}\mathcal{C}(\mathbf{P})}$ and that the underlined terms are indeed the leading terms. For the latter we use that for $\sigma \in \{\pm\}$ and antichains $A, A' \subseteq P_\sigma$ we have $A \sqcap A' \subseteq \langle A \rangle \cap \langle A' \rangle$. The triangulation in Corollary 3.29 is unimodular and hence, by Corollary 2.9, corresponds to a Gröbner basis G of $I_{\mathbb{TC}(\mathbf{P})}$ with respect to some monomial order $\preceq$ such that the monomials $\{\mathrm{in}_\preceq(g) : g \in G\}$ are precisely the leading terms of the above binomials. Hence, since G is a Gröbner basis, we have $\mathrm{in}_\preceq(I_{\mathbb{TC}(\mathbf{P})}) \subseteq \mathrm{in}_{\preceq_{rev}}(I_{\mathbb{TC}(\mathbf{P})})$. But this implies $\mathrm{in}_\preceq(I_{\mathbb{TC}(\mathbf{P})}) = \mathrm{in}_{\preceq_{rev}}(I_{\mathbb{TC}(\mathbf{P})})$, which finishes the proof. □

Note that Theorem 3.31 in particular implies that the triangulation in Corollary 3.29 is the pulling triangulation arising from any ordering of the vertices of $\mathbb{TC}(\mathbf{P})$ which satisfies (i) and (ii).

3.3.3 Volume formulas

The canonical subdivision from Lemma 3.25 directly implies formulas for the (normalized) volume of $\mathcal{P}_1 - \mathcal{P}_2$ and $\mathcal{P}_1 \boxminus \mathcal{P}_2$ in terms of the volumes of the anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$.

Theorem 3.32. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be full-dimensional anti-blocking polytopes. Then*

$$\begin{aligned} \mathrm{vol}(\mathcal{P}_1 - \mathcal{P}_2) &= \sum_{J \subseteq [n]} \mathrm{vol}(\mathcal{P}_1|_J)\,\mathrm{vol}(\mathcal{P}_2|_{J^c}) \text{ and} \\ \mathrm{vol}(\mathcal{P}_1 \boxminus \mathcal{P}_2) &= \frac{2}{n+1} \sum_{J \subseteq [n]} \frac{1}{\binom{n}{|J|}} \mathrm{vol}(\mathcal{P}_1|_J)\,\mathrm{vol}(\mathcal{P}_2|_{J^c}). \end{aligned}$$

If $\mathcal{P}_1$ and $\mathcal{P}_2$ have vertices on a lattice $\Lambda \subset \mathbb{R}^n$, then then the normalized volume of $\mathcal{P}_1 - \mathcal{P}_2$ with respect to Λ is

$$\mathrm{Vol}(\mathcal{P}_1 - \mathcal{P}_2) = \sum_{J \subseteq [n]} \binom{n}{|J|} \mathrm{Vol}(\mathcal{P}_1|_J)\,\mathrm{Vol}(\mathcal{P}_2|_{J^c})$$

and the normalized volume of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ with respect to the affine lattice $\Lambda \times (2\mathbb{Z}+1)$ is

$$\mathrm{Vol}(\mathcal{P}_1 \boxminus \mathcal{P}_2) = \sum_{J \subseteq [n]} \mathrm{Vol}(\mathcal{P}_1|_J)\,\mathrm{Vol}(\mathcal{P}_2|_{J^c}).$$

Proof. The first claim follows directly from Lemma 3.25. For the second claim observe that for $J \subseteq [n]$ we have

$$\begin{aligned} \mathrm{vol}(\mathcal{P}_1|_J * \mathcal{P}_2|_{J^c}) &= 2 \int_0^1 \mathrm{vol}\big((1-t)\mathcal{P}_1|_J + t\mathcal{P}_2|_{J^c}\big)\mathrm{d}t \\ &= 2\Big(\int_0^1 (1-t)^{|J|} t^{|J^c|}\mathrm{d}t\Big)\,\mathrm{vol}(\mathcal{P}_1|_J)\,\mathrm{vol}(\mathcal{P}_2|_{J^c}) \\ &= \frac{2}{(n+1)\binom{n}{|J|}} \mathrm{vol}(\mathcal{P}_1|_J)\,\mathrm{vol}(\mathcal{P}_2|_{J^c}), \end{aligned}$$

which proves the claim.

The remaining assertions follow from the first part using the definition of normalized volume from Section 2.1.3. Alternatively, one can count simplices in unimodular triangulations, for $\mathcal{P}_1 - \mathcal{P}_2$ with the help of [17, Sec.6.2.2]. □

We will now turn to the case of double chain polytopes. Note that for a poset $(P, \preceq)$ and $J \subseteq P$, the face $\mathcal{C}(P)|_J$ is just the chain polytope associated to the induced poset $\mathcal{P}|_J = (J, \preceq)$. Recall that for a double poset $\mathbf{P}$ we regard $\mathbb{DC}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ as lattice polytopes with respect to the affine lattices generated by the vertices. Explicitly, these are $\mathbb{Z}^P$ and the affine lattice $2\mathbb{Z}^P \times (2\mathbb{Z}+1)$, respectively, and the normalized volumes in the following result are taken with respect to these lattices.

Corollary 3.33. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. Then*

$$\begin{aligned}\mathrm{Vol}(\mathbb{DC}(\mathbf{P})) &= \sum_{J \subseteq P} \binom{|P|}{|J|} \mathrm{e}(P_+|_J) \cdot \mathrm{e}(P_-|_{J^c}) \text{ and} \\ \mathrm{Vol}(\mathbb{TC}(\mathbf{P})) &= \sum_{J \subseteq P} \mathrm{e}(P_+|_J) \cdot \mathrm{e}(P_-|_{J^c})\end{aligned}$$

Proof. The results follow directly from Corollary 3.32, using that for $\sigma \in \{\pm\}$ we have $\mathrm{Vol}(\mathcal{C}(P_\sigma|_J)) = \mathrm{e}(P_\sigma|_J)$. □

Combining Corollary 3.33 and Corollary 2.21 we immediately obtain the following, which alternatively also follows from Corollary 3.30.

Corollary 3.34. *Let $\mathbf{P}$ be a compatible double poset. Then $\mathbb{TO}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ have the same (normalized) volume.*

The formulas of Corollary 3.33 are particularly simple when $\mathbf{P}$ is special or anti-special. We illustrate these cases at some simple examples.

Example 3.35. For the compatible 'XW'-double poset $\mathbf{P}_{XW}$ from Example 1.5 we have

$$\begin{aligned}\mathrm{Vol}(\mathbb{TO}(\mathbf{P}_{XW})) = \mathrm{Vol}(\mathbb{TC}(\mathbf{P}_{XW})) = 128 \text{ and} \\ \mathrm{Vol}(\mathbb{DO}(\mathbf{P}_{XW})) = \mathrm{Vol}(\mathbb{DC}(\mathbf{P}_{XW})) = 880.\end{aligned}$$

Example 3.36. Let $n \in \mathbb{N}_{>0}$.

(1) Consider the double poset $\mathbf{C}_n$ induced by the n-chain (cf. Example 1.4). Then $\mathrm{Vol}(\mathbb{TC}(\mathbf{C}_n)) = 2^n$ and it follows from Vandermonde's identity that

$$\mathrm{Vol}(\mathbb{DC}(\mathbf{P})) = \sum_{i=0}^{n} \binom{n}{i}^2 = \binom{2n}{n}.$$

(2) For the double antichain $\mathbf{A}_n$ we have that $\mathbb{TC}(\mathbf{A}_n)$ is isomorphic to the cube $[0,1]^{n+1}$ and its normalized volume is

$$\mathrm{Vol}(\mathbb{TC}(\mathbf{A}_n)) = \sum_{i=0}^{n} \binom{n}{i} i!(n-i)! = (n+1)!.$$

Likewise, $\mathbb{DC}(\mathbf{A}_n)$ is isomorphic to $[-1,1]^n$, which can be decomposed into 2^n unit cubes. Consequently, its normalized volume is

$$\mathrm{Vol}(\mathbb{DC}(\mathbf{A}_n)) = \sum_{i=0}^{n} \binom{n}{i}^2 i!(n-i)! = 2^n n!.$$

(3) For the double poset $\mathbf{AC}_n$ from Example 1.4, we compute

$$\mathrm{Vol}(\mathbb{TC}(\mathbf{AC}_n)) \;=\; \sum_{i=0}^{n} \frac{n!}{i!},$$

which is the number of choices of ordered subsets of an n-set. Moreover,

$$\mathrm{Vol}(\mathbb{DC}(\mathbf{AC}_n)) \;=\; \sum_{i=0}^{n} \binom{n}{i}^2 i!$$

is the number of partial permutation matrices, i.e. 0/1-matrices of size n with at most one nonzero entry per row and column. Indeed, such a matrix is uniquely identified by an i-by-i permutation matrix and a choice of i rows and i columns in which it is embedded.

(4) For Comb_n (cf. Example 1.1), the number of linear extensions is

$$e(\mathrm{Comb}_n) = (2n-1)!! := (2n-1)(2n-3)(2n-5)\cdots .$$

For the induced double poset $\mathbf{Comb}_n$, an induction argument shows that

$$\mathrm{Vol}(\mathbb{TC}(\mathbf{Comb}_n)) \;=\; 4^n\, n!.$$

It would be nice to have a bijective proof of this equality.

Remark 3.37. Let $(P, \preceq)$ be a poset and $\mathbf{P} = (P, \preceq, \preceq)$ the induced double poset. By Corollary 3.34, the polytopes $\mathbb{TO}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ have the same normalized volume. Since both polytopes are 2-level, it follows from [78, Thm. 2.4] that the number of maximal simplices in any pulling triangulation of $\mathbb{TO}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ coincides. From Theorem 1.38, we know that $\mathbb{TO}(\mathbf{P})^\triangle$ is linearly isomorphic the twisted prism over the valuation polytope associated to P. On the other hand, we know from Corollary 3.20 that $\mathbb{TC}(\mathbf{P})^\triangle$ is linearly isomorphic to the Hansen polytope $\mathcal{H}(\overline{G(P)})$. Moreover, $\mathbb{TO}(\mathbf{P})^\triangle$ and $\mathbb{TC}(\mathbf{P})^\triangle$ are both 2-level by Proposition 1.39 and it is enticing to conjecture that their normalized volumes also agree. Unfortunately, this is not the case. For the double poset $\mathbf{P}_X$, we used `sage` [18] to compute that any pulling triangulation of $\mathbb{TC}(\mathbf{P}_X)^\triangle$ has 324 simplices, whereas for $\mathbb{TO}(\mathbf{P}_X)^\triangle$ pulling triangulations have 320 simplices.

3.4 Transferring triangulations and Ehrhart polynomials

For a given poset P, Stanley's transfer map from Section 3.3.2 is a close connection between $\mathcal{O}(P)$ and $\mathcal{C}(P)$. In the following, we will extend this map to the case of double poset polytopes of a compatible double poset $\mathbf{P}$. This will in particular give a satisfying reason for the so far rather surprising fact that, for compatible $\mathbf{P}$, the canonical triangulations of $\mathbb{TO}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ have the same underlying simplicial complex.

The main results are the following. In Theorem 3.39, we describe the lattice-preserving transfer map between $\mathbb{TO}(\mathbf{P})$ and $\mathbb{TC}(\mathbf{P})$ for compatible $\mathbf{P}$. Corollary 3.40 relates their two canonical triangulations. As another application, we compute

Ehrhart polynomials: After recalling some basics, Proposition 3.44 uses the transfer map to show that order and chain polytopes have the same Ehrhart polynomial. In this case, it turns out to be easier to work with double chain polytopes. More generally, Theorem 3.47 gives an explicit description of the Ehrhart polynomial of $\mathcal{P}_1 \boxminus \mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$ for anti-blocking polytopes $\mathcal{P}_1, \mathcal{P}_2$ under a mild extra assumption. This in particular implies a formula for Ehrhart polynomials of double chain polytopes, given in Corollary 3.48 and this formula also holds true for compatible double order polytopes.

3.4.1 A transfer map for double poset polytopes

Recall from Section 3.3.2 that for a given poset P, the inverse transfer map is given by $\Psi_P : \mathbb{R}^P \to \mathbb{R}^P$ with

$$(\Psi_P g)(b) \; = \; \max\{0, g(a_0)+\cdots+g(a_{k-1})+g(a_k) : a_0 \prec \cdots \prec a_{k-1} \prec a_k \preceq b\}, \quad (3.7)$$

for any $g \in \mathbb{R}^P$. Note that we have slightly altered the definition since will now regard Ψ_P as a map on the whole of $\mathbb{R}^P$ rather than $\mathcal{C}(P)$. Given a double poset $\mathbf{P} = (P, \preceq_+, \preceq_-)$, we define a piecewise linear map $\mathbb{D}\Psi_{\mathbf{P}} : \mathbb{R}^P \to \mathbb{R}^P$ by

$$\mathbb{D}\Psi_{\mathbf{P}}(g) \; := \; \Psi_{P_+}(g) \; - \; \Psi_{P_-}(-g), \quad (3.8)$$

for $g \in \mathbb{R}^P$. Observe that whenever $g(p) < 0$ for some $p \in P$, then p is never contained in any chain that attains the maximum in (3.7). Hence, if we write $g = g^+ - g^-$, where $g^+, g^- \in \mathbb{R}^P_{\geq 0}$ with disjoint supports, then $\Psi_{P_+}(g) = \Psi_{P_+}(g^+)$ and $\Psi_{P_-}(-g) = \Psi_{P_-}(g^-)$. Thus, for any $g \in \mathbb{R}^P$ we have

$$\mathbb{D}\Psi_{\mathbf{P}}(g) \; = \; \Psi_{P_+}(g^+) - \Psi_{P_-}(g^-). \quad (3.9)$$

Lemma 3.38. *Let $\mathbf{P} = (\mathcal{P}, \preceq_+, \preceq_-)$ be compatible. Then $\mathbb{D}\Psi_{\mathbf{P}} : \mathbb{R}^P \to \mathbb{R}^P$ is a piecewise linear homeomorphism satisfying*
(1) $\mathbb{D}\Psi_{\mathbf{P}}(\mathbb{Z}^P) = \mathbb{Z}^P$ *and*
(2) $\mathbb{D}\Psi_{\mathbf{P}}(\lambda g) = \lambda \cdot \mathbb{D}\Psi_{\mathbf{P}}(g)$ *for* $g \in \mathbb{R}^P, \lambda \geq 0$.

Proof. It follows directly from (3.8) and (3.7) that $\mathbb{D}\Psi_{\mathbf{P}}$ is piecewise linear. To show that $\mathbb{D}\Psi_{\mathbf{P}}$ is an isomorphism, we explicitly construct for $f \in \mathbb{R}^P$ a $g \in \mathbb{R}^P$ such that $\mathbb{D}\Psi_{\mathbf{P}}(g) = f$. Since $\mathbf{P}$ is compatible, we can assume that $P = \{a_1, \ldots, a_n\}$ such that $a_i \prec_+ a_j$ or $a_i \prec_- a_j$ implies $i < j$.

It follows from (3.8) that $\mathbb{D}\Psi_{\mathbf{P}}(g')(a_1) = g'(a_1)$ for any $g' \in \mathbb{R}^P$ and hence, we can set $g(a_1) := f(a_1)$. Now assume that g is already defined on $D_k := \{a_1, \ldots, a_k\}$ for some k. For $g' \in \mathbb{R}^P$ observe that

$$\Psi_{P_+}(g')(a_{k+1}) \; = \; \max(g'(a_{k+1}), 0) + r$$

where $r = 0$ or $r = \Psi_{P_+}(g')(a_i)$ for some $i \leq k$. Analogously,

$$\Psi_{P_-}(-g')(a_{k+1}) \; = \; \max(-g'(a_{k+1}), 0) + s$$

where $s = 0$ or $s = \Psi_{P_-}(-g')(a_j)$ for some $j \leq k$. Thus, we set

$$g(a_{k+1}) \; := \; f(a_{k+1}) - r + s$$

This uniquely determines g by induction on k. To prove that $\mathbb{D}\Psi_{\mathbf{P}}$ is lattice-preserving, observe that by (3.8) we have $\mathbb{D}\Psi_{\mathbf{P}}(\mathbb{Z}^P) \subseteq \mathbb{Z}^P$. Moreover, if $f = \mathbb{D}\Psi_{\mathbf{P}}(g)$ with $f \in \mathbb{Z}^P$ then the above construction shows that $g \in \mathbb{Z}^P$. Hence, we have $\mathbb{Z}^P \subseteq \mathbb{D}\Psi_{\mathbf{P}}(\mathbb{Z}^P)$. Finally, the last assertion is an immediate consequence of (3.7) and (3.8).

□

Theorem 3.39. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a compatible double poset. Then the map $\mathbb{T}\Psi_{\mathbf{P}} : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P \times \mathbb{R}$ defined by*

$$\mathbb{T}\Psi_{\mathbf{P}}(g, t) = (\mathbb{D}\Psi_{\mathbf{P}}(g), t)$$

induces a lattice-preserving and piecewise linear homeomorphism between $\mathbb{T}\mathcal{C}(\mathbf{P})$ and $\mathbb{T}\mathcal{O}(\mathbf{P})$ whose domains of linearity are the simplices $\overline{H}(C)$ for non-interfering pairs $C = C_+ \uplus C_- \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$.

Proof. It follows directly from Lemma 3.38 that $\mathbb{T}\Psi_{\mathbf{P}} : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P \times \mathbb{R}$ is a piecewise linear and lattice-preserving homeomorphism. Moreover, it follows from (3.8) that for an antichain $A \subseteq P_\sigma$ with $\sigma \in \{\pm\}$ we have

$$\mathbb{T}\Psi_{\mathbf{P}}\big(\sigma(2\mathbf{1}_A, 1)\big) \;=\; \sigma(2\mathbf{1}_{\mathsf{J}}, 1),$$

where $\mathsf{J} := \langle A \rangle \subseteq P_\sigma$ is the filter generated by A. Here, as in Section 1.3, we abuse notation and interpret σ as $+1$ or -1. Hence $\mathbb{T}\Psi_{\mathbf{P}}$ induces a bijection between the vertices of $\mathbb{T}\mathcal{C}(\mathbf{P})$ and $\mathbb{T}\mathcal{O}(\mathbf{P})$. By Corollary 2.20 and Corollary 3.29, to finish the proof it suffices to show that $\mathbb{T}\Psi_{\mathbf{P}}$ is linear on the simplices $\overline{H}(C)$ for non-interfering $C = C_+ \uplus C_- \subseteq \mathbb{T}\mathcal{J}(\mathbf{P})$.

Observe that $\mathbb{D}\Psi_{\mathbf{P}}$ is linear on $\operatorname{cone}(H(C_+) - H(C_-))$: Indeed, since C_+ and C_- are non-interfering, we have $\operatorname{cone}(H(C_+) - H(C_-)) \cong \operatorname{cone}(H(C_+)) \times - \operatorname{cone}(H(C_-))$ and $\mathbb{D}\Psi_{\mathbf{P}}$ is the product of the two linear maps $\Psi_{P_+}|_{\operatorname{cone}(H(C_+))}$ and $-\Psi_{P_-}|_{\operatorname{cone}(H(C_-))}$. This finishes the proof, since $\pi(\overline{H}(C)) \subset \operatorname{cone}(H(C_+) - H(C_-))$, where the map $\pi : \mathbb{R}^P \times \mathbb{R} \to \mathbb{R}^P$ is the projection onto the coordinates in P. □

This immediately implies the following.

Corollary 3.40. *Let $\mathbf{P}$ be a compatible double poset. Then the homeomorphism $\mathbb{T}\Psi_{\mathbf{P}}$ maps the triangulation of $\mathbb{T}\mathcal{C}(\mathbf{P})$ from Corollary 3.29 to the triangulation of $\mathbb{T}\mathcal{O}(\mathbf{P})$ from Corollary 2.20.*

It is easy to check that Theorem 3.39 does not extend to the case of non-compatible double posets by considering the following example.

Example 3.41. Consider the double poset $\mathbf{P} = ([2], \leq, \geq)$, that is, P_+ is the 2-chain $\{1, 2\}$ and P_- is the opposite poset. Then

$$\mathcal{C}(P_+) = \mathcal{C}(P_-) = \{x \in \mathbb{R}^2 : x \geq 0, x_1 + x_2 \leq 1\}$$

and $\mathbb{T}\mathcal{C}(\mathbf{P})$ is a three-dimensional octahedron with volume $\frac{16}{3}$. Any triangulation of the octahedron has at least four simplices. In contrast, $\mathcal{O}(P_-) = \mathbf{1} - \mathcal{O}(P_+)$ and hence $\mathbb{T}\mathcal{O}(\mathbf{P})$ is linearly isomorphic to a prism over a triangle and has volume 4. Any triangulation of the prism has exactly 3 tetrahedra.

We have seen that for any double poset $\mathbf{P}$, the polytope $\mathbb{TC}(\mathbf{P})$ depends only on the double graph $\mathbf{G}(\mathbf{P})$. In particular, there can be different compatible double posets, such that their associated double order polytopes have the same image under the transfer map. Conversely, it is natural to ask whether for any arbitrary double poset $\mathbf{P}$, the double chain polytope $\mathbb{TC}(\mathbf{P})$ can always be realized as the double chain polytope associated to a *compatible* double poset, phrased differently, whether there is always a compatible double poset $\mathbf{P}'$ on the same ground set such that $\mathbf{G}(\mathbf{P}) = \mathbf{G}(\mathbf{P}')$. Although, this seems to be true for small posets, it fails in general, as the following example shows.

Example 3.42. Let $n \geq 7$ and consider the set $P = \{(i,j) \in \mathbb{Z}^2 : 1 \leq i,j \leq n\}$. Let $P_+ := (P, \preceq_+)$ be the poset with the natural product order on $\mathbb{Z}^2$, that is $(i,j) \preceq_+ (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. As can be seen by drawing Hasse diagrams, for every poset $P'_+ = (P, \preceq'_+)$ with $G(P_+) = G(P'_+)$ the *interior* elements $\{(i,j) \in P : 2 \leq i,j \leq n-1\}$ cannot be minimal elements in P'_+. Since $n \geq 7$, more than half of the elements of P are interior. Hence, there is a permutation $\pi : P \to P$ such that for every $p \in P$ either p or $\pi(p)$ is an interior element. Now we define $P_- := (P, \preceq_-)$ where $p \preceq_- q$ if and only if $\pi(p) \preceq_+ \pi(q)$ for all $p,q \in P$. Define $\mathbf{P} := (P, \preceq_+, \preceq_-)$. By the above, for any double poset $\mathbf{P}' = (P, \preceq_+, \preceq_-)$ satisfying $\mathbf{G}(\mathbf{P}) = \mathbf{G}(\mathbf{P}')$ there cannot exist an element $p \in P$ that is minimal in both P'_+ and P'_-. Hence $\mathbf{P}'$ cannot be compatible.

3.4.2 Ehrhart polynomials

We briefly recall the basics of Ehrhart theory. For more, see, for example, [7, 8]. For a polytope $\mathcal{P} \subset \mathbb{R}^n$ we define the **Ehrhart function** $\mathrm{Ehr}_{\mathcal{P}} : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ by

$$\mathrm{Ehr}_{\mathcal{P}}(k) := |k\mathcal{P} \cap \mathbb{Z}^n|.$$

Ehrhart [22] showed that if $\mathcal{P}$ is integral and of dimension $d \leq n$, then $\mathrm{Ehr}_{\mathcal{P}}$ agrees with a polynomial of degree d. We will identify $\mathrm{Ehr}_{\mathcal{P}}(k)$ with this polynomial, called the **Ehrhart polynomial** of $\mathcal{P}$. The constant coefficient of $\mathrm{Ehr}_{\mathcal{P}}$ is always 1 and if $\mathcal{P}$ is full-dimensional, then the leading coefficient is its volume $\mathrm{vol}(\mathcal{P})$.

More generally, if $\mathcal{P} \subset \mathbb{R}^n$ is a d-dimensional **rational** polytope, that is $V(\mathcal{P}) \subset \mathbb{Q}^n$, then the function $\mathrm{Ehr}_{\mathcal{P}}(k) := |k\mathcal{P} \cap \mathbb{Z}^n|$ agrees with a quasi-polynomial of degree d. In other words, we have

$$\mathrm{Ehr}_{\mathcal{P}}(k) \;=\; c_d(k)k^d + c_{d-1}(k)k^{d-1} + \cdots + c_0(k),$$

where c_i is a periodic function with integral period for $1 \leq i \leq d$. We will need the following fundamental result of Ehrhart theory.

Theorem 3.43 (Ehrhart–Macdonald reciprocity, [59])**.** *Let $\mathcal{P} \subset \mathbb{R}^n$ be a rational polytope of dimension d, then*

$$(-1)^d \mathrm{Ehr}_{\mathcal{P}}(-k) \;=\; |\,\mathrm{relint}(k\mathcal{P}) \cap \mathbb{Z}^n|$$

for all $k \geq 0$.

In the following we will compute the Ehrhart polynomial of double poset polytopes. If $\mathbf{P}$ is compatible, we can use Section 3.4.1 to obtain the following.

Proposition 3.44. *Let* $\mathbf{P}$ *be a compatible double poset. Then* $\mathbb{T}\mathcal{O}(\mathbf{P})$ *and* $\mathbb{T}\mathcal{C}(\mathbf{P})$ *have the same Ehrhart polynomial.*

Proof. It follows directly for Theorem 3.39 together with Lemma 3.38 that for $k \geq 0$ a lattice point $\mathbf{z} \in \mathbb{Z}^P \times \mathbb{Z}$ lies in $k\mathbb{T}\mathcal{C}(\mathbf{P})$ if and only if $\mathbb{T}\Psi_{\mathbf{P}}(\mathbf{z})$ is contained in $k\mathbb{T}\mathcal{O}(\mathbf{P})$. This finishes the proof since $\mathbb{T}\Psi_{\mathbf{P}}(\mathbb{Z}^P \times \mathbb{Z}) = \mathbb{Z}^P \times \mathbb{Z}$ again by Lemma 3.38. □

Let us briefly return to the case of ordinary poset polytopes. Let P be a finite poset. For the Ehrhart polynomial $\mathrm{Ehr}_{\mathcal{O}(P)}(n)$ of $\mathcal{O}(P)$ it suffices to interpret the lattice points in $n\mathcal{O}(P)$ for $n > 0$. Every point in $n\mathcal{O}(P) \cap \mathbb{Z}^P$ is an order preserving map from P into the interval $[0, n]$ with values in $\mathbb{Z}$. Thus, we may interpret lattice points in $n\mathcal{O}(P) \cap \mathbb{Z}^P$ as order preserving maps between P and the $(n+1)$-chain C_{n+1}. Moreover, points in $\mathrm{relint}(n\mathcal{O}(P)) \cap \mathbb{Z}^P$ correspond to **strictly order preserving** maps $f : P \to \mathrm{C}_{n-1}$, that is, $f(a) < f(b)$ whenever $a \prec b$. Counting order preserving maps is classical [74, Sect. 3.15]: the **order polynomial** $\Omega_P(n)$ of P counts the number of order preserving maps into n-chains. The **strict order polynomial** $\Omega_P^\circ(n)$ counts the number of strictly order preserving maps $f : P \to \mathrm{C}_n$. Now Proposition 3.44, together with Theorem 3.43, yield the following.

Corollary 3.45 ([73, Thm. 4.1])**.** *Let* P *be a finite poset. Then for every* $k > 0$

$$\begin{aligned} \Omega_P(k+1) \;&=\; \mathrm{Ehr}_{\mathcal{O}(P)}(k) \;=\; \mathrm{Ehr}_{\mathcal{C}(P)}(k) \text{ and} \\ (-1)^{|P|}\Omega_P^\circ(k-1) \;&=\; \mathrm{Ehr}_{\mathcal{O}(P)}(-k) \;=\; \mathrm{Ehr}_{\mathcal{C}(P)}(-k). \end{aligned}$$

Note that in the above, we computed the Ehrhart polynomial of $\mathcal{O}(P)$ and transferred it to $\mathcal{C}(P)$. Curiously, for double poset polytopes, it is more natural to go the other way: We will give an explicit description of $\mathrm{Ehr}_{\mathbb{T}\mathcal{C}(\mathbf{P})}$ for any double poset $\mathbf{P}$ and use the transfer map to obtain $\mathrm{Ehr}_{\mathbb{T}\mathcal{O}(\mathbf{P})}$ for compatible $\mathbf{P}$. In fact, instead of computing Ehrhart polynomials for double chain polytopes, we will work in the more general setting of rational anti-blocking polytopes. Note that if $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ are rational polytopes, then so are $2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$ and $\mathcal{P}_1 - \mathcal{P}_2$. We call an anti-blocking polytope $\mathcal{P} \subset \mathbb{R}^n$ **dual integral** if $A(\mathcal{P})$ is an integral polytope. By Theorem 3.5, this means that there are $\mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{Z}^n_{\geq 0}$ such that

$$\mathcal{P} \;=\; \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for } i = 1, \ldots, s\}.$$

Examples of dual integral anti-blocking polytopes are for example stable set polytopes of perfect graphs by Theorem 3.3. In particular, chain polytopes are dual integral. For a bounded set $S \subset \mathbb{R}^n$, let us write $E(S) := |S \cap \mathbb{Z}^n|$.

Theorem 3.46. *Let* $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ *be two full-dimensional rational anti-blocking polytopes and assume that* $\mathcal{P}_1$ *is dual integral. Then for any* $a, b \in \mathbb{Z}_{>0}$

$$E(a\mathcal{P}_1 - b\mathcal{P}_2) \;=\; \sum_{J \subseteq [n]} E(\mathrm{relint}((a+1)\mathcal{P}_1)) \cdot E(b\mathcal{P}_2).$$

Proof. It follows from Lemma 3.25 that for any $a, b \in \mathbb{Z}_{>0}$,

$$a\mathcal{P}_1 - b\mathcal{P}_2 \;=\; \bigcup_{J \subseteq [n]} (a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}).$$

For $J \subseteq [n]$, the cell $a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}$ is contained in the orthant $\mathbb{R}^J_{\geq 0} \times \mathbb{R}^{J^c}_{\leq 0}$. It is easy to see that

$$\mathbb{Z}^n \;=\; \biguplus_{J \subseteq [n]} \mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}$$

is a partition and for each $J \subseteq [n]$

$$\begin{aligned}(a\mathcal{P}_1 - b\mathcal{P}_2) \cap (\mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}) &= (a\mathcal{P}_1|_J - b\mathcal{P}_2|_{J^c}) \cap (\mathbb{Z}^J_{>0} \times \mathbb{Z}^{J^c}_{\leq 0}) \\ &= (a\mathcal{P}_1|_J \cap \mathbb{Z}^J_{>0}) - (b\mathcal{P}_2|_{J^c} \cap \mathbb{Z}^{J^c}).\end{aligned}$$

If $\mathcal{P}_1$ is dual integral, then $\mathcal{P}_1|_J$ is dual integral. Thus, for a fixed J, there are $\mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{Z}^J_{\geq 0}$ such that

$$\begin{aligned}a\mathcal{P}_1|_J \cap \mathbb{Z}^J_{>0} &= \{\mathbf{x} \in \mathbb{Z}^J : \mathbf{x} > 0, \langle \mathbf{d}_i, \mathbf{x}\rangle \leq a\} \\ &= \{\mathbf{x} \in \mathbb{Z}^J : \mathbf{x} > 0, \langle \mathbf{d}_i, \mathbf{x}\rangle < a+1\} \\ &= \operatorname{relint}((a+1)\mathcal{P}_1|_J) \cap \mathbb{Z}^J.\end{aligned}$$

This proves the result. □

By Theorem 3.43, we can interpret the above in terms of Ehrhart polynomials.

Theorem 3.47. *Let $\mathcal{P}_1, \mathcal{P}_2 \subset \mathbb{R}^n$ be two full-dimensional rational anti-blocking polytopes such that $\mathcal{P}_1$ is dual integral. Then*

$$\mathrm{Ehr}_{\mathcal{P}_1 - \mathcal{P}_2}(k) \;=\; \sum_{J \subseteq [n]} (-1)^{|J|} \mathrm{Ehr}_{\mathcal{P}_1|_J}(-k-1) \cdot \mathrm{Ehr}_{\mathcal{P}_2|_{J^c}}(k).$$

Moreover, for $\mathcal{P} := 2\mathcal{P}_1 \boxminus 2\mathcal{P}_2$ we have

$$\mathrm{Ehr}_{\mathcal{P}}(k) \;=\; \sum_{J \subseteq [n]} (-1)^{|J|} \sum_{t=-k}^{k} \mathrm{Ehr}_{\mathcal{P}_1|_J}(t-k-1) \cdot \mathrm{Ehr}_{\mathcal{P}_2|^c_J}(k+t).$$

Proof. The first part follows directly from Theorem 3.46. For the second assertion, observe that for $k > 0$,

$$k\mathcal{P} \;=\; \operatorname{conv}(2k\mathcal{P}_1 \times \{k\} \cup -2k\mathcal{P}_2 \times \{-k\}).$$

In particular, if $(\mathbf{p}, t)$ is a lattice point in $k\mathcal{P}$, then $-k \leq t \leq k$. For fixed t,

$$\{\mathbf{p} \in \mathbb{Z}^n : (\mathbf{p}, t) \in k\mathcal{P}\} \;=\; ((k-t)\mathcal{P}_1 - (k+t)\mathcal{P}_2) \cap \mathbb{Z}^n.$$

Theorems 3.46 and 3.43 then complete the proof. □

For the case of double poset polytopes, Theorem 3.47 together with Corollary 3.45 immediately yields the following, which generalizes Corollary 3.33.

Corollary 3.48. *Let $\mathbf{P} = (P, \preceq_+, \preceq_-)$ be a double poset. Then*

$$\mathrm{Ehr}_{\mathbb{DC}(\mathbf{P})}(k) \;=\; \sum_{J \subseteq P} \Omega^\circ_{P_+|_J}(k) \cdot \Omega_{P_-|_{J^c}}(k+1) \;\; \textit{and}$$

$$\mathrm{Ehr}_{\mathbb{TC}(\mathbf{P})}(k) \;=\; \sum_{J \subseteq P} \Big(\sum_{t=-k}^{k} \Omega^\circ_{P_+|_J}(k-t) \cdot \Omega_{P_-|_{J^c}}(k+t+1) \Big).$$

Remark 3.49. Note that in the above, the underlying lattice is $\mathbb{Z}^P$. This is slightly unnatural, since the triangulations of $\mathbb{TC}(\mathbf{P})$ described in Section 3.3.2 are unimodular with respect to the affine lattice $\Lambda = \mathbb{Z}^P \times (2\mathbb{Z}+1)$. However, the Ehrhart polynomial of $\mathbb{TC}(\mathbf{P})$ with respect to Λ can be obtained by substituting $\frac{1}{2}k$ for k in the above formula. This works analogously for $\mathbb{DC}(\mathbf{P})$ and the underlying lattice $\Lambda = 2\mathbb{Z}^P$.

We close with some thoughts on the class of dual integral anti-blocking polytopes. We have seen that independent set polytopes of perfect graphs are dual integral. In fact, these are the only dual integral anti-blocking polytopes that are also *integral*.

Proposition 3.50. *Let $\mathcal{P} \subset \mathbb{R}^n$ be a full-dimensional dual integral anti-blocking polytope. If the vertices of $\mathcal{P}$ lie in $\mathbb{Z}^n$, then $\mathcal{P} = \mathcal{P}_G$ for some perfect graph G.*

Proof. Let $\mathcal{P}$ be given by

$$\mathcal{P} \; = \; \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq 0, \langle \mathbf{d}_i, \mathbf{x} \rangle \leq 1 \text{ for } i = 1, \ldots, s\}$$

for some $\mathbf{d}_1, \ldots, \mathbf{d}_s \in \mathbb{Z}^n_{\geq 0}$. Since $\mathcal{P}$ is full-dimensional and integral, it follows that $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathcal{P}$ and for any $1 \leq j \leq s$ we compute

$$0 \;\leq\; \langle \mathbf{d}_j, \mathbf{e}_i \rangle \;\leq\; 1$$

for all i and since the $\mathbf{d}_j$ are integer vectors, it follows that $\mathbf{d}_j = \mathbf{1}_{C_j}$ for some $C_j \subseteq [n]$. Consequently, the vertices of $\mathcal{P}$ are in $\{0,1\}^n$ and $\mathcal{P}$ is 2-level. By Proposition 3.23, $\mathcal{P} = \mathcal{P}_G$ for some perfect graph G. $\square$

Of course this severely limits the applicability of Theorem 3.47 to integral polytopes. On the other hand, the class of stable set polytopes is already interesting by itself. For example, there is no known combinatorial description of their Ehrhart polynomial or even their volume.

Part II

Reflection groups, arrangements and real varieties

Chapter 4

Reflection groups and invariant varieties

This chapter recalls basics regarding real reflection groups, their combinatorics and their invariant theory, and introduces the problem we will be concerned with in the sequel. In Section 4.1, we treat reflection groups, their associated reflection arrangements and root systems and we specifically look at certain infinite families of reflection groups that will play an important role later on. For the case of symmetric groups, Timofte [79] proved an interesting structural result, which connects invariant varieties, reflection arrangements and the associated invariant rings. In Section 4.2 we recall some well-known invariant theory of real finite reflection groups. This yields a more systematic viewpoint on Timofte's result and it is only natural to ask for a more general version for arbitrary reflection groups, which will be our main objective throughout. Our main results are given in Theorem 4.7 and Theorem 4.8.

4.1 Reflection groups

Reflection groups appear in many different areas of mathematics. They connect to Lie groups and Lie algebras and are well-studied from the perspective of geometry, algebra, and combinatorics [10, 31, 40, 53]. In the following, we collect some essential basics, which can, for instance, be found in [53].

Let V be a finite-dimensional real vector space endowed with an inner product $\langle \cdot, \cdot \rangle$. The **orthogonal group** $\mathrm{O}(V)$ consists of all invertible linear transformations on V that preserve the inner product. In other words,

$$\mathrm{O}(V) \ = \ \{s : V \to V \text{ linear} : \langle s(\mathbf{p}), s(\mathbf{q}) \rangle = \langle \mathbf{p}, \mathbf{q} \rangle \text{ for all } \mathbf{p}, \mathbf{q} \in V\}.$$

A transformation $s \in \mathrm{O}(V)$ is called a **reflection** if it sends some nonzero $\boldsymbol{\alpha} \in V$ to its negative and fixes pointwise the hyperplane $H_{\boldsymbol{\alpha}} := \boldsymbol{\alpha}^{\perp} = \{\mathbf{p} \in V : \langle \boldsymbol{\alpha}, \mathbf{p} \rangle = 0\}$. Explicitly, a reflection is of the form

$$s_{\boldsymbol{\alpha}}(\mathbf{p}) \ := \ \mathbf{p} - 2\frac{\langle \mathbf{p}, \boldsymbol{\alpha} \rangle}{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle}\boldsymbol{\alpha}$$

for some nonzero $\boldsymbol{\alpha} \in V$ and every $\mathbf{p} \in V$. A finite subgroup G of $\mathrm{O}(V)$ which is generated by reflections is called a **(real) finite reflection group** acting on V. The reflection group G is **irreducible** if it is not the product of two nontrivial reflection groups. Associated to G is its **reflection arrangement**

$$\mathcal{H} \;=\; \mathcal{H}(G) \;:=\; \{H_{\boldsymbol{\alpha}} : s_{\boldsymbol{\alpha}} \in G \text{ reflection}\}.$$

The **flats** of $\mathcal{H}$ are the linear subspaces arising as intersections of hyperplanes in $\mathcal{H}$. The arrangement of linear hyperplanes stratifies V with **strata** given by

$$\mathcal{H}_i \;=\; \mathcal{H}_i(G) \;:=\; \{\mathbf{p} \in V : \mathbf{p} \text{ is contained in a flat of dimension } i\}$$

for $0 \leq i \leq n-1$ and we set $\mathcal{H}_n := V$. We call G **essential** if G does not fix a nontrivial linear subspace or, equivalently, if $\mathcal{H}_0 = \{0\}$. For every reflection group G acting on V, there exists a unique subspace $W \subseteq V$ such that G restricts to an essential reflection group acting on W and the dimension of W is called the **rank** of G, denoted by $\mathrm{rk}(G)$.

Example 4.1. For $m \geq 3$, consider the **regular** m**-gon** $\mathcal{P}_m \subset \mathbb{R}^2$ with vertices

$$\big(\sin\left(\tfrac{2k\pi}{m}\right), \cos\left(\tfrac{2k\pi}{m}\right)\big) \text{ for } k \in [m] = \{1, \ldots, m\}.$$

The **dihedral group** of order $2m$ is the symmetry group $I_2(m) := \{s \in \mathrm{O}(\mathbb{R}^2) : s(\mathcal{P}_m) = \mathcal{P}_m\}$. It contains m rotations and m reflections. In fact, the latter generate $I_2(m)$, which makes it an essential irreducible reflection group of rank 2. Figure 4.1 illustrates the cases $m = 3$ and $m = 4$.

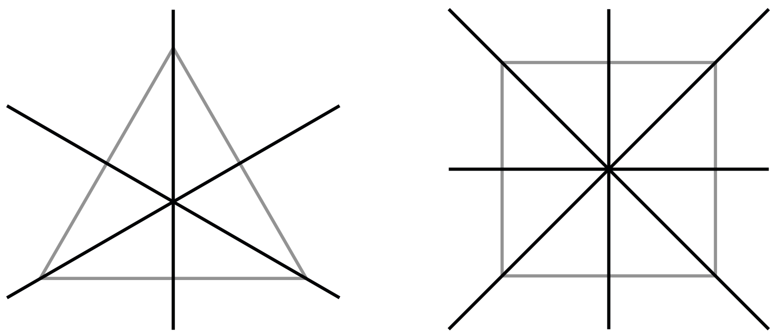

Figure 4.1: The dihedral groups $I_2(3)$ and $I_2(4)$.

Example 4.2. We record three more infinite families of irreducible reflection groups. Let $n \in \mathbb{N}_{>0}$.

A_{n-1}: We denote by $\mathfrak{S}_n$ the **symmetric group** consisting of all $n!$ permutations of the set $[n]$. The symmetric group $\mathfrak{S}_n$ acts on $\mathbb{R}^n$ by permuting coordinates, that is, for $\tau \in \mathfrak{S}_n$ and $\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n$ we have

$$\tau \cdot \mathbf{p} = (p_{\tau^{-1}(1)}, p_{\tau^{-1}(2)}, \ldots, p_{\tau^{-1}(n)}).$$

The symmetric group is generated by transpositions, which in geometric terms correspond to reflections in the hyperplanes

$$H_{ij} \;:=\; \{\mathbf{p} = (p_1, \ldots, p_n) \in \mathbb{R}^n : p_i = p_j\} \text{ for } 1 \leq i < j \leq n,$$

and hence we can think of $\mathfrak{S}_n$ as a reflection group on $\mathbb{R}^n$. The stratum $\mathcal{H}_k(\mathfrak{S}_n)$ consists of points $\mathbf{p}$ with at most k distinct coordinates, that is

$$\mathcal{H}_k(\mathfrak{S}_n) \;=\; \{\mathbf{p} = (p_1, \dots, p_n) \in \mathbb{R}^n : |\{p_1, \dots, p_n\}| \leq k\}.$$

This action is not essential, as the line $\mathbb{R}\mathbf{1}$ is fixed and hence we have $\mathrm{rk}(\mathfrak{S}_n) = n-1$. If we restrict $\mathfrak{S}_n$ to the subspace $\mathbf{1}^\perp = \{\mathbf{p} \in \mathbb{R}^n : p_1 + \cdots + p_n = 0\}$, we obtain the essential reflection group of type A_{n-1}. Note that $A_2 = I_2(3)$ and the associated reflection arrangement is given in Figure 4.1.

B_n: The group B_n acting on $\mathbb{R}^n$ consists of all *signed* permutations, that is, in addition to permuting entries of a point $\mathbf{p}$, the sign of every individual entry can be changed. Hence, B_n is of order $2^n n!$ and the associated reflection arrangement consists of the hyperplanes $\{p_i = \pm p_j\}$ for $1 \leq i < j \leq n$ together with the coordinate hyperplanes $\{p_i = 0\}$ for $i \in [n]$. A point $\mathbf{p}$ lies in $\mathcal{H}_k$ if and only if $(|p_1|, \dots, |p_n|)$ has at most k distinct nonzero coordinates. Note that $B_2 = I_2(4)$, see Figure 4.1.

D_n: The reflection group D_n is a subgroup of index 2 in B_n. It is the semidirect product of $\mathfrak{S}_n$ with the subgroup of B_n which consists of sign changes involving an *even* number of signs. The reflection arrangement of D_n consists of the hyperplanes $\{p_i = \pm p_j\}$ for $1 \leq i < j \leq n$. The k-stratum of D_n is a bit more involved to describe: Denote by M the set of all $\mathbf{p} \in \mathbb{R}^n$ with exactly one zero coordinate. Then

$$\mathcal{H}_k(D_n) \;=\; (\mathcal{H}_k(B_n) \setminus M) \cup (\mathcal{H}_{k-1}(B_n) \cap M). \tag{4.1}$$

It is often convenient to look at reflection groups from a different perspective. A **root system** is a finite set $\Phi \subset V$ such that

(i) $\Phi \cap \mathbb{R}\boldsymbol{\alpha} = \{\boldsymbol{\alpha}, -\boldsymbol{\alpha}\}$ for every $\boldsymbol{\alpha} \in \Phi$ and
(ii) $s_{\boldsymbol{\alpha}}(\Phi) = \Phi$ for all $\boldsymbol{\alpha} \in \Phi$.

The elements of Φ are called **roots**. Every root system gives rise to a finite reflection group with generators $\{s_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Phi\}$. Conversely, every finite reflection group arises this way: For a reflection group G the associated hyperplane arrangement is of the form $\mathcal{H}(G) = \{H_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in R\}$ for some $R \subset V$ and we may assume that all $\boldsymbol{\alpha} \in R$ are of unit length. Now a suitable root system is given by $\Phi := R \cup -R$. Note that the length of the roots is not uniquely determined and it is only necessary that any two roots in the same G-orbit have the same length.

Given a root system $\Phi \subset V$, a **simple system** is a set of roots $\Delta \subset \Phi$ such that

(i) The roots in Δ form a basis of the $\mathbb{R}$-span of Φ in V and
(ii) every root in Φ can be written as a linear combination of elements of Δ such that all coefficients have the same sign.

Every root system contains a simple system. For every root system $\Phi \subset V$ with associated reflection group G and simple system $\Delta \subseteq \Phi$, we have $\mathrm{rk}(G) = |\Delta|$ and the reflections $\{s_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Delta\}$ generate G. Moreover, any two simple systems $\Delta, \Delta' \subseteq \Phi$ are conjugate under G, that is, there exists $g \in G$ such that $g\Delta = \Delta'$. For details, see [53, Ch. 1].

Consider a fixed reflection group G with associated root system Φ and a simple system $\Delta \subseteq \Phi$. For two roots $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \Delta$, we write $m(\boldsymbol{\alpha}, \boldsymbol{\beta})$ for the order of the group element $s_{\boldsymbol{\alpha}} s_{\boldsymbol{\beta}}$ in G. It is convenient to record this data in a labelled graph D, called a **Dynkin diagram**, whose vertex set is Δ and two vertices $\boldsymbol{\alpha}, \boldsymbol{\beta}$ are connected by an edge if and only if $m(\boldsymbol{\alpha}, \boldsymbol{\beta}) \geq 3$. In this case, we label the edge by $m(\boldsymbol{\alpha}, \boldsymbol{\beta})$.

Dynkin diagrams do not depend on the choice of the underlying simple system, that is, two different simple systems yield isomorphic labelled graphs. Moreover, any two non-isomorphic essential reflection groups give rise to different Dynkin diagrams. If a Dynkin diagram D is not connected, say $D = D_1 \uplus D_2$, this yields a representation of the associated reflection group G as a product $G = G_1 \times G_2$, such that D_1 and D_2 are the Dynkin diagrams of G_1 and G_2, respectively. In particular, connected Dynkin diagrams correspond to irreducible reflection groups.

A complete classification of reflection groups can be given in terms of their Dynkin diagrams (see [53, Ch. 2]). There are four infinite families of irreducible reflection groups $\mathfrak{S}_n \cong A_{n-1}, B_n, D_n, I_2(m)$ (cf. Examples 4.1 and 4.2) and six exceptional reflection groups H_3, H_4, F_4, E_6, E_7, and E_8, where the index always denotes the rank. The corresponding Dynkin diagrams are given in Figure 4.2. Note that for any question regarding reflection groups, this leaves us with two strategies: Either, we approach the problem using general properties of reflection groups or we use the classification and tackle the problem case by case.

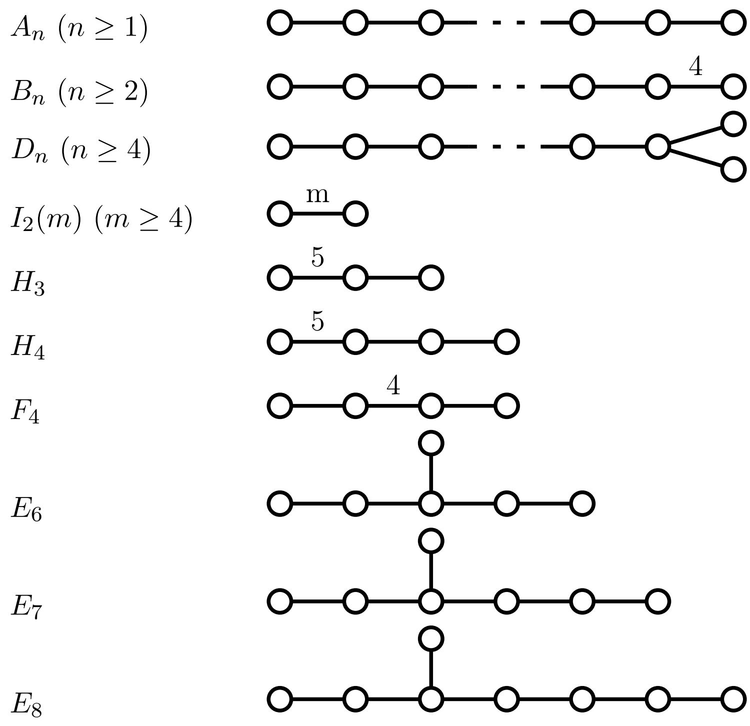

Figure 4.2: The Dynkin diagrams of irreducible real reflection groups. For simplicity, the edge label 3 is omitted.

4.2 Invariant polynomials and real varieties

Timofte [79] related the degree of a symmetric polynomial with real coefficients to the existence of real roots with only few distinct coordinates. In this section, we look at this result from a more geometric viewpoint and regard it as a connection between symmetric polynomials, invariant varieties and the $\mathfrak{S}_n$-arrangement. Chevalley's theorem [53, Thm. 3.5] allows us to take this viewpoint for arbitrary reflection groups, and in the rest of this thesis we prove a more general version for irreducible real reflection groups, stated in Theorem 4.7.

A **real variety** $X \subseteq \mathbb{R}^n$ is the set of real points simultaneously satisfying a system of polynomial equations with real coefficients, that is,

$$X \;=\; \mathrm{V}_{\mathbb{R}}(f_1,\dots,f_m) \;:=\; \{\mathbf{p}\in\mathbb{R}^n : f_1(\mathbf{p}) = f_2(\mathbf{p}) = \cdots = f_m(\mathbf{p}) = 0\},$$

for some $f_1,\dots,f_m \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1,\dots,x_n]$. In contrast to working over an algebraically closed field, the question if $X \neq \emptyset$ is considerably more difficult to answer, both theoretically and in practice; see [5]. Timofte [79] studied real varieties invariant under the action of the symmetric group $\mathfrak{S}_n$, and proved an interesting structural result, with a simplified proof given by Riener [66]. An $\mathfrak{S}_n$-invariant variety can be defined in terms of **symmetric** polynomials, that is, polynomials $f \in \mathbb{R}[\mathbf{x}]$ such that $f(x_{\tau^{-1}(1)},\dots,x_{\tau^{-1}(n)}) = f(x_1,\dots,x_n)$ for all permutations $\tau\in\mathfrak{S}_n$. Recall that the fundamental theorem of symmetric polynomials states that a polynomial f is symmetric if and only if f is a polynomial in the **elementary symmetric polynomials** $e_1,\dots,e_n$, where

$$e_k(\mathbf{x}) \;:=\; \sum_{1\le i_1<\cdots<i_k\le n} x_{i_1}\cdots x_{i_k}$$

for $k\in[n]$. We say that a symmetric polynomial f is k**-sparse** if $f\in\mathbb{R}[e_1,\dots,e_k]$, that is, if f is a polynomial in $e_1,\dots,e_k$. On the geometric side, we call a $\mathfrak{S}_n$-invariant variety X k**-sparse** if $X = \mathrm{V}_{\mathbb{R}}(f_1,\dots,f_m)$ for k-sparse symmetric polynomials $f_1,\dots,f_m$.

Theorem 4.3 ([79]). *Let $X\subseteq\mathbb{R}^n$ be a nonempty $\mathfrak{S}_n$-invariant real variety. If X is k-sparse, then there is a point $\mathbf{p}\in X$ with at most k distinct coordinates, that is, $X\cap\mathcal{H}_k(\mathfrak{S}_n)\neq\emptyset$.*

Clearly, the dimension of the stratum $\mathcal{H}_k$ does not depend on the number of variables and hence the result is particularly powerful when the number of variables is large. Also note that the sparsity of a symmetric polynomial $f\in\mathbb{R}[\mathbf{x}]$ relates to its degree as follows. If $\deg(f)\le k$, then f is k-sparse, since the elementary symmetric polynomials are algebraically independent. On the other hand, there exist k-sparse polynomials of arbitrarily high degree, the simplest examples being powers of e_1. In fact, Timofte's original result is phrased in terms of degrees instead of sparsity and is therefore known as *degree principle* for symmetric polynomials.

In the following, we aim for a generalization of Theorem 4.3 to other reflection groups. To begin with, we recall some facts from invariant theory of finite reflection groups. For more details, see [53, Ch. 3]. For any finite group G acting on $V\cong\mathbb{R}^n$, there is an induced action of G on the coordinate ring $\mathbb{R}[V]\cong\mathbb{R}[\mathbf{x}]$ given by $g\cdot f(\mathbf{x}) := f(g^{-1}\cdot\mathbf{x})$. We write $\mathbb{R}[V]^G := \{f\in\mathbb{R}[V] : g\cdot f = f \text{ for all } g\in G\}$ for the subring consisting of G-invariant polynomials. Chevalley's theorem states that in the case of reflection groups, the associated invariant ring has a particularly simple description.

Theorem 4.4 ([53, Thm. 3.5]). *Let G be a finite real reflection group acting on $V\cong\mathbb{R}^n$.*

(1) *The invariant ring* $\mathbb{R}[V]^G$ *is generated by* n *algebraically independent homogeneous polynomials.*
(2) *Let* $\pi_1, \ldots, \pi_n$ *and* $\pi'_1, \ldots, \pi'_n$ *be two choices of algebraically independent homogeneous generators of* $\mathbb{R}[V]^G$. *Then, after renumbering the generators if necessary, we have* $\deg(\pi_i) = \deg(\pi'_i)$ *for all* $i \in [n]$.

Any collection $\pi_1, \ldots, \pi_n$ of homogeneous algebraically independent generators of $\mathbb{R}[V]^G$ is called a set of **basic invariants** for G. Their degrees $d_i := \deg(\pi_i)$ for $i \in [n]$ are called the **degrees** of G. The degrees of all irreducible reflection groups are listed in Table 4.1 below. From now on, we will always assume that basic invariants $\pi_1, \ldots, \pi_n$ are labelled such that their degrees are non-decreasing, that is, $d_1 \leq d_2 \leq \cdots \leq d_n$.

Group	Degrees
A_{n-1}	$2, 3, \ldots, n-1$
B_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, \ldots, 2n-2, n$
$I_2(m)$	$2, m$
H_3	$2, 6, 10$
H_4	$2, 12, 20, 30$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

Table 4.1: The degrees of irreducible reflection groups.

Example 4.5. Let $n \in \mathbb{N}_{>0}$.
(1) For the symmetric group $\mathfrak{S}_n$, we have seen that basic invariants are given by the elementary symmetric polynomials $e_1, \ldots, e_n$. An alternative set of basic invariants are the **power sums**

$$s_k(\mathbf{x}) \; := \; x_1^k + x_2^k + \cdots + x_n^k,$$

for $k \in [n]$. The linear form $e_1(\mathbf{x}) = s_1(\mathbf{x}) = x_1 + \cdots + x_n$ corresponds to the invariant linear subspace $\mathbb{R}\mathbf{1}$ and omitting this invariant yields basic invariants for A_{n-1}.
(2) Given basic invariants $\pi_1, \ldots, \pi_n$ for $\mathfrak{S}_n$, we immediately obtain basic invariants for B_n defined by $\pi'_k(\mathbf{x}) := \pi_k(x_1^2, \ldots, x_n^2)$ for $k \in [n]$. In particular, the even power sums $s_2, s_4, \ldots, s_{2n}$ are basic invariants for B_n.
(3) The invariant that distinguishes D_n from B_n is given by $e_n(\mathbf{x}) = x_1 x_2 \cdots x_n$. A set of basic invariants for D_n are $\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x})$ with

$$\pi_k(\mathbf{x}) \; := \; \begin{cases} s_{2k}(\mathbf{x}) & \text{for } 1 \leq k \leq \lfloor \frac{n}{2} \rfloor, \\ e_n(\mathbf{x}) & \text{for } k = \lfloor \frac{n}{2} \rfloor + 1, \text{ and} \\ s_{2k-2}(\mathbf{x}) & \text{for } \lfloor \frac{n}{2} \rfloor + 1 < k \leq n. \end{cases} \tag{4.2}$$

Note that the action of G on V is essential if and only if $d_1 = 2$ and the number of linear basic invariants equals the dimension of the fixed point space of G. Whenever G is in addition irreducible, there is only one invariant of degree 2, which must be a scalar multiple of $p_2(\mathbf{x}) = \|\mathbf{x}\|^2 := \langle \mathbf{x}, \mathbf{x} \rangle$.

For $0 \leq k \leq n$, we call a G-invariant variety $X = \mathrm{V}_{\mathbb{R}}(f_1, \ldots, f_m)$ k**-sparse** if there exist basic invariants $\pi_1, \ldots, \pi_n$ such that $f_1, \ldots, f_m$ lie in $\mathbb{R}[\pi_1, \ldots, \pi_k]$. Keep in mind that, analogously to the case of symmetric polynomials, any invariant polynomial $f \in \mathbb{R}[V]^G$ with $\deg(f) < d_{k+1}$ is contained in $\mathbb{R}[\pi_1, \ldots, \pi_k]$ and, hence, all our results imply *degree principles*, where instead of working with sparse polynomials we restrict their degree.

Remark 4.6. Comparing with the previous definition of k-sparsity for symmetric polynomials, it may seem that we have slightly altered the definition, since now we allow arbitrary basic invariants instead of only considering elementary symmetric polynomials. However, we will see in Lemma 5.1 that for the symmetric group it does not matter which basic invariants we choose.

Our main result is the following.

Theorem 4.7. *Let G be a reflection group of type $I_2(m), A_{n-1}, B_n, D_n, H_3$, or F_4 and X a nonempty G-invariant real variety. If X is k-sparse, then $X \cap \mathcal{H}_k(G) \neq \emptyset$.*

The reason why we only consider essential groups is simply that if there exists an l-dimensional subspace $W \subseteq V$ which is fixed pointwise by G, then we have $\mathcal{H}_0 = \cdots = \mathcal{H}_{l-1} = \emptyset$. If the action of G is not essential, the theorem remains true for all k such that $\mathcal{H}_k \neq \emptyset$.

Using that the first basic invariant is always a scalar multiple of $s_2(\mathbf{x}) = \|\mathbf{x}\|^2$, Theorem 4.7 is trivially true for reflection groups of rank at most 2. The infinite families A_{n-1}, B_n, and D_n are treated in Chapter 5. Timofte's original proof and its simplification given in [66] use properties of the symmetric group that are not shared by all reflection groups (such as D_n) and we highlight this difference in Example 5.3 and Remark 5.4. In Section 6.1, we prove the following general result for invariant real varieties that implies the case $k = n - 1$ of Theorem 4.7.

Theorem 4.8. *Let G be an essential reflection group of rank n and consider a nonempty G-invariant variety $X = \mathrm{V}_{\mathbb{R}}(f_1, \ldots, f_m)$. If there is $j \in [n]$ such that $f_1, \ldots, f_m \in \mathbb{R}[\pi_i : i \neq j]$, then $X \cap \mathcal{H}_{n-1}(G) \neq \emptyset$.*

In particular, this result yields Theorem 4.7 for all reflection groups of rank ≤ 3. The group F_4 is treated in Section 6.1 and we provide computational evidence that Theorem 4.7 also holds for H_4. This supports the following conjecture.

Conjecture 4.9. *Let G be an irreducible essential reflection group. Then any nonempty and k-sparse G-invariant real variety X intersects $\mathcal{H}_k(G)$.*

In Section 6.2, we prove a weaker form of Conjecture 4.9 under an extra assumption on the defining polynomials of X and we obtain upper bounds on the dimension of the stratum that meets X in terms of the combinatorics of G.

Independently from our work, Acevedo and Velasco [1] considered the related problem of certifying nonnegativity of G-invariant homogeneous polynomials. They show that low-degree forms (where the exact degree depends on the group) are nonnegative if and only if they are nonnegative on $\mathcal{H}_{n-1}(G)$. Questions of nonnegativity of polynomials $f \in \mathbb{R}[V]^G$ are subsumed by our results. A **basic semialgebraic set** is of the form

$$S \;=\; S(f_1,\ldots,f_m) \;=\; \{\mathbf{p} \in V : f_1(\mathbf{p}),\ldots,f_m(\mathbf{p}) \geq 0\}.$$

for $f_1,\ldots,f_m \in \mathbb{R}[V]$. A general semialgebraic set is a Boolean combination of basic semialgebraic sets. Generalizing the case of real varieties, let us call a G-invariant semialgebraic set $S \subseteq V$ **k-sparse** if S can be defined in terms of inequalities with polynomials in $\mathbb{R}[\pi_1,\ldots,\pi_k]$.

Proposition 4.10. *Let G be a reflection group for which Conjecture 4.9 holds. Let $S \subseteq V$ be a k-sparse semialgebraic set and let $f \in \mathbb{R}[\pi_1,\ldots,\pi_k]$. Then f is nonnegative/positive on S if and only if f is nonnegative/positive on $\mathcal{H}_k(G) \cap S$.*

Proof. We only treat non-negativity, since the case of positivity is similar. If S is k-sparse, then the G-invariant variety

$$X_k(\mathbf{q}) \;:=\; \{\mathbf{p} \in V : \pi_i(\mathbf{p}) = \pi_i(\mathbf{q}) \text{ for all } i \in [k]\} \tag{4.3}$$

is contained in S for any $\mathbf{q} \in S$. Assume that there is a point $\mathbf{q} \in S$ with $f(\mathbf{q}) < 0$. By assumption $f = F(\pi_1,\ldots,\pi_k)$ for some $F \in \mathbb{R}[y_1,\ldots,y_k]$. Hence, f is negative (and constant) on $X_k(\mathbf{q}) \subseteq S$. By construction $X_k(\mathbf{q})$ is k-sparse and, since G satisfies Conjecture 4.9, $X_k(\mathbf{q}) \cap \mathcal{H}_k(G) \neq \emptyset$. □

The proof of Proposition 4.10 makes use of a key observation: It suffices to consider invariant varieties of the form (4.3) as any k-sparse variety X contains $X_k(\mathbf{q})$ for all $\mathbf{q} \in X$. We call $X_k(\mathbf{q})$ a **principal** k-sparse variety. Lastly, let us emphasize again that for now we will work with *real* varieties exclusively. In particular, set-theoretically, every real variety $X = \mathrm{V}_{\mathbb{R}}(f_1,\ldots,f_m)$ is the set of solutions to the equation $f(\mathbf{x}) = 0$ for $f = f_1^2 + f_2^2 + \cdots + f_m^2$.

Chapter 5

Groups of type A_n, B_n and D_n

In this chapter, we prove Theorem 4.7 for the three infinite classes A_n, B_n, and D_n. The proof for A_n and B_n presented in Section 5.1 uses a strengthening of a result by Steinberg [76] given in Corollary 5.2, which does not hold true for type D_n. Thus, the case of D_n, treated in Section 5.2, turns out to be considerably more involved.

5.1 Symmetric polynomials and Vandermonde determinants

We start with some observations regarding k-sparsity. For a reflection group G acting on $\mathbb{R}^n$ with basic invariants $\pi_1, \ldots, \pi_n$, the **Jacobian** of $\pi_1, \ldots, \pi_k$ at a point $\mathbf{p} \in V$ is the $(k \times n)$-matrix of partial derivatives

$$\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k) \;:=\; \begin{pmatrix} \partial_{x_1}\pi_1(\mathbf{p}) & \partial_{x_2}\pi_1(\mathbf{p}) & \cdots & \partial_{x_n}\pi_1(\mathbf{p}) \\ \partial_{x_1}\pi_2(\mathbf{p}) & \partial_{x_2}\pi_2(\mathbf{p}) & \cdots & \partial_{x_n}\pi_2(\mathbf{p}) \\ \vdots & \vdots & & \vdots \\ \partial_{x_1}\pi_k(\mathbf{p}) & \partial_{x_2}\pi_k(\mathbf{p}) & \cdots & \partial_{x_n}\pi_k(\mathbf{p}) \end{pmatrix}$$

and the rows of $\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k)$ are the **gradients** $\nabla\pi_1(\mathbf{p}), \ldots, \nabla\pi_k(\mathbf{p})$. The next result, which in particular shows that k-sparsity is independent of the choice of basic invariants in most cases, will be useful later on.

Lemma 5.1. *Fix a reflection group G acting on V. Let $\pi_1, \ldots, \pi_n$ and $\pi'_1, \ldots, \pi'_n$ be two sets of basic invariants and let $k \in [n]$ such that the degrees satisfy $d_{k+1} > d_k$. Then $\mathbb{R}[\pi_1, \ldots, \pi_k] = \mathbb{R}[\pi'_1, \ldots, \pi'_k]$ and $\mathrm{rk}\,\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k) = \mathrm{rk}\,\mathrm{Jac}_{\mathbf{p}}(\pi'_1, \ldots, \pi'_k)$ for all $\mathbf{p} \in V$.*

Proof. For every $1 \le i \le k$, $\pi_i = F_i(\pi'_1, \ldots, \pi'_n)$ for some polynomial $F_i(y_1, \ldots, y_n)$. Homogeneity and algebraic independence imply that $F_i \in \mathbb{R}[y_1, \ldots, y_k]$. This shows the inclusion $\mathbb{R}[\pi_1, \ldots, \pi_k] \subseteq \mathbb{R}[\pi'_1, \ldots, \pi'_k]$. Note that

$$\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_k) \;=\; \mathrm{Jac}_{\pi'(\mathbf{p})}(F_1, \ldots, F_k) \cdot \mathrm{Jac}_{\mathbf{p}}(\pi'_1, \ldots, \pi'_k)$$

for every $\mathbf{p} \in V$. The same argument applied to π'_i now proves the first claim and shows that $\mathrm{Jac}_{\pi'(\mathbf{p})}(F_1, \ldots, F_k)$ has full rank and this proves the second claim. □

We start with the verification of Theorem 4.7 for $\mathfrak{S}_n$, which is exactly Theorem 4.3. The proofs for B_n and D_n will rely on the arguments for A_{n-1}.

Proof of Theorem 4.7 for $A_{n-1} \cong \mathfrak{S}_n$. In the case of $\mathfrak{S}_n$, the degrees $d_1, \ldots, d_n$ are pairwise distinct and, hence, Lemma 5.1 implies that k-sparsity does not depend on the choice of basic invariants. In the following, we choose to work with the power sums $s_1, \ldots, s_n$. It suffices to prove that for any $\mathbf{p}_0 \in \mathbb{R}^n$ and $2 \le k \le n-1$, the principal k-sparse variety

$$X_k(\mathbf{p}_0) \;=\; \{\mathbf{p} \in \mathbb{R}^n : s_i(\mathbf{p}) = s_i(\mathbf{p}_0) \text{ for } i \in [k]\}$$

meets the stratum $\mathcal{H}_k$. Since $s_2(\mathbf{p}) = \|\mathbf{p}\|^2$, we conclude that $X_k(\mathbf{p}_0)$ is compact. Hence, the continuous function s_{k+1} attains its maximum over $X_k(\mathbf{p}_0)$ in a point $\mathbf{q}$. By the inverse function theorem [39, Ch. 1.3], at this point, the Jacobian $J = \mathrm{Jac}_{\mathbf{q}}(s_1, \ldots, s_{k+1})$ has rank $< k+1$. We claim that this condition is equivalent to $\mathbf{q} \in \mathcal{H}_k$. Indeed, up to scaling columns, J is given by

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ q_1 & q_2 & \cdots & q_n \\ \vdots & & & \vdots \\ q_1^k & q_2^k & \cdots & q_n^k \end{pmatrix},$$

where $\mathbf{q} = (q_1, \ldots, q_n)$. We have $\mathrm{rk}(J) < k+1$ if and only if all its $(k+1)$-minors vanish. But the latter are Vandermonde determinants, which yield that for any $I \subset [n]$ with $|I| = k+1$ we have

$$\prod_{i,j \in I, i<j} (q_i - q_j) \;=\; 0.$$

Now, using the description of k-strata for $\mathfrak{S}_n$ from Example 4.2 finishes the proof. □

We proceed to the reflection groups of type B_n.

Proof of Theorem 4.7 for B_n. By Lemma 5.1 and the fact that the degrees $d_i(B_n)$ are all distinct, we may assume that $\pi_i(\mathbf{x}) = s_{2i}(\mathbf{x})$ for all $i \in [n]$. Moreover, we can assume that X is a principal k-sparse variety, that is,

$$X \;=\; X_k(\mathbf{p}) \;=\; \{\mathbf{q} \in \mathbb{R}^n : s_{2i}(\mathbf{q}) = s_{2i}(\mathbf{p}) \text{ for all } i \in [k]\}.$$

Since $X_k(\mathbf{p}) = X_k(\mathbf{q})$ for all $\mathbf{q} \in X_k(\mathbf{p})$, we can assume that $\mathbf{p} = (p_1, \ldots, p_r, 0, \ldots, 0) \in X$ with the property that $p_1 \cdots p_r \neq 0$ and r is minimal.

If $r = n$, then X does not meet any of the coordinate hyperplanes $\{x_i = 0\}$. Let $\mathbf{q} \in X$ be an extreme point of π_{k+1} over X. At this point, the Jacobian $J = \mathrm{Jac}_{\mathbf{q}}(\pi_1, \ldots, \pi_{k+1})$ does not have full rank and hence every maximal minor of

$$J \;=\; \begin{pmatrix} q_1 & q_2 & \cdots & q_n \\ \vdots & & & \vdots \\ q_1^{2k-1} & q_2^{2k-1} & \cdots & q_n^{2k-1} \end{pmatrix}$$

vanishes. Since $q_i \neq 0$ for all $i \in [n]$, the Vandermonde formula implies that $(q_1^2, q_2^2, \ldots, q_n^2)$ has at most k distinct coordinates, which yields the claim.

If $r < n$, we can restrict X to the linear subspace $U = \{\mathbf{x} \in \mathbb{R}^n : x_{r+1} = \cdots = x_n = 0\} \cong \mathbb{R}^r$. The set $X' := X \cap U \subseteq \mathbb{R}^r$ is nonempty and, in particular, a k-sparse B_r-invariant variety that stays away from the coordinate hyperplanes in $\mathbb{R}^r$. By the previous case, there is a point $\mathbf{q}' \in X'$ such that $(|q'_1|, \ldots, |q'_r|)$ has at most k distinct coordinates. By construction, $\mathbf{q} = (\mathbf{q}', \mathbf{0}) \in X \cap \mathcal{H}_k(B_n)$, which finishes the proof. □

The key to the proof of Theorem 4.7 for A_{n-1} and B_n is the strong connection between the strata $\mathcal{H}_k$ and the ranks of the Jacobians $\mathrm{Jac}(\pi_1, \ldots, \pi_{k+1})$.

Corollary 5.2. *Let $G \in \{\mathfrak{S}_n, B_n\}$ and $\pi_1, \ldots, \pi_n$ a set of basic invariants for G. Then a point $\mathbf{p} \in V$ lies in $\mathcal{H}_k(G)$ for $0 \leq k \leq n-1$ if and only if $\mathrm{Jac}_{\mathbf{p}}(\pi_1, \ldots, \pi_{k+1})$ has rank at most k.*

It is tempting to believe that such a statement holds true for all reflection groups and, indeed, necessity follows from a well-known result of Steinberg [76]. However, the following example shows that Corollary 5.2 does not hold true in general.

Example 5.3. Consider the group $G = D_5$ acting on $\mathbb{R}^5$ and the point $\mathbf{p} = (1,1,1,1,0)$, which is contained in the three linearly independent D_5-hyperplanes $\{x_1 = x_2\}$, $\{x_2 = x_3\}$ and $\{x_3 = x_4\}$. By (4.1) it follows that $\mathbf{p} \in \mathcal{H}_2(D_5) \setminus \mathcal{H}_1(D_5)$. On the other hand, for any choice of basic invariants $\pi_1, \ldots, \pi_5$ the gradients $\nabla_{\mathbf{p}}\pi_1, \nabla_{\mathbf{p}}\pi_2$ are linearly dependent. Indeed, for $\pi_1 = \|x\|^2 = x_1^2 + \cdots + x_5^2$ and $\pi_2 = x_1^4 + \cdots + x_5^4$, this is easy to check and this extends to all choices of basic invariants using Lemma 5.1. Hence, the Jacobian $\mathrm{Jac}_{\mathbf{p}}(\pi_1, \pi_2)$ has rank 1, but $\mathbf{p}$ is not contained in $\mathcal{H}_1(D_5)$.

Remark 5.4. Example 5.3 also serves as a counterexample to generalizations of Corollary 5.2 to all finite reflection groups claimed in [4, Statement 3.3] and [33, Lemma 1'] (without a proof). Moreover, in the language of Acevedo and Velasco [1, Definition 7], it is the first example of a reflection group not satisfying the *minor factorization condition.*

5.2 Groups of type D_n

In the following, we turn to reflection groups of type D_n. We have seen that Corollary 5.2 does not extend to this case. However, the claim of Theorem 4.7 remains true.

Proof of Theorem 4.7 for D_n. Let $\pi_1, \ldots, \pi_n$ be a choice of basic invariants for D_n and let $X = X_k(\mathbf{q}) \subseteq \mathbb{R}^n$ for some $\mathbf{q} \in \mathbb{R}^n$ and $1 \leq k < n$. If n is odd or if $k \neq \lfloor \frac{n}{2} \rfloor$, then $d_{k+1} > d_k$ and, by Lemma 5.1, we can assume that the basic invariants are given by (4.2). If n is even and $k = \frac{n}{2}$, then $\pi_k(\mathbf{x}) = \alpha s_n(\mathbf{x}) + \beta e_n(\mathbf{x})$, for some $\beta \neq 0$. We can also assume that $\mathbf{q} = (q_1, \ldots, q_l, 0, \ldots, 0)$ with $q_1 \cdots q_l \neq 0$ and l is maximal among all points in $X_k(\mathbf{q})$. We distinguish two cases.

Case $l < n$: In this case, $e_n(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$ and $X' := X_k(\mathbf{q}) \cap \{\mathbf{x} : x_n = 0\}$ is nonempty. If $k \geq \lfloor \frac{n}{2} \rfloor + 1$, then we can identify

$$X' = \{\mathbf{x}' \in \mathbb{R}^{n-1} : s_{2i}(\mathbf{x}', 0) = s_{2i}(\mathbf{q}) \text{ for } i = 1, \ldots, k-1\}.$$

Hence X' is a real variety in $\mathbb{R}^{n-1}$ invariant under the action of B_{n-1} and X' is $(k-1)$-sparse with respect to B_{n-1}. By Theorem 4.7 for B_{n-1}, $X' \cap \mathcal{H}_{k-1}(B_{n-1}) \neq \emptyset$. The claim now follows the description of $\mathcal{H}_k(D_n)$ given in (4.1).

If $k < \frac{n}{2}$, consider the Jacobian of $\pi_1 = s_2, \ldots, \pi_k = s_{2k}$ and the $(l+1)$-th elementary symmetric polynomial $e_{l+1}(\mathbf{x})$ at $\mathbf{q}$. This is given by

$$J = \begin{pmatrix} q_1 & q_2 & \cdots & q_l & 0 & 0 & \cdots & 0 \\ q_1^3 & q_2^3 & \cdots & q_l^3 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ q_1^{2k-1} & q_2^{2k-1} & \cdots & q_l^{2k-1} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & q_1 \cdots q_l & 0 & \cdots & 0 \end{pmatrix}. \tag{5.1}$$

We observe that the $(l+1)$-th elementary symmetric function $e_{l+1}(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$ and hence the gradients of $\pi_1, \ldots, \pi_k$ and e_{l+1} are linearly dependent on $X_k(\mathbf{q})$. In particular, the Jacobian J has rank $\leq k$. Since $q_1 \cdots q_l \neq 0$, the Vandermonde minors imply

$$\prod_{i,j \in I, i<j} (q_i^2 - q_j^2) = 0$$

for any $I \subseteq \{1, \ldots, l\}$ with $|I| = k$. This shows that $\mathbf{q} \in \mathcal{H}_{k-1}(B_n) \subseteq \mathcal{H}_k(D_n)$.

If $k = \frac{n}{2}$, then n is even and X is defined in terms of the power sums $s_2, \ldots, s_{n-2}$ and the special invariant $\pi_k = \alpha s_n + \beta e_n$. However, since $e_n(\mathbf{x})$ is identically zero on $X_k(\mathbf{q})$, the variety X is in fact cut out by the power sums $s_2, \ldots, s_n$ and the above argument remains valid.

Case $l = n$: If $k < \frac{n}{2}$, set $f := e_n$. If $k \geq \lfloor \frac{n}{2} \rfloor + 1$, the function e_n takes a constant nonzero value on X and instead we set $f := s_{2k}$. For the special case that n is even and $k = \frac{n}{2}$, we set $f = e_n$ if $\alpha \neq 0$ and $f = s_n$ otherwise. Let $\mathbf{r} = (r_1, \ldots, r_n) \in X_k(\mathbf{q})$ be a maximizer of $|f(\mathbf{x})|$, which exists since X is compact. It is easy to observe that we always have $r_1 \cdots r_n \neq 0$. Moreover, in all cases, up to row and column operations, the Jacobian $J = \mathrm{Jac}_{\mathbf{q}}(s_2, s_4, \ldots, s_{2k}, f)$ is of the form

$$J = \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ r_1^3 & r_2^3 & \cdots & r_n^3 \\ \vdots & \vdots & & \vdots \\ r_1^{2k-1} & r_2^{2k-1} & \cdots & r_n^{2k-1} \\ \widehat{r_1} r_2 \cdots r_n & r_1 \widehat{r_2} \cdots r_n & \cdots & r_1 r_2 \cdots \widehat{r_n} \end{pmatrix}, \tag{5.2}$$

where $\widehat{r_i}$ is to be omitted from the product. Multiplying the i-th column by r_i and dividing the last row by $r_1 \cdots r_n$, we get a Vandermonde matrix of rank $\leq k$. Thus, all the $(k+1)$-minors vanish which implies that $(|r_1|, \ldots, |r_n|)$ has at most k distinct entries. Since all entries are nonzero, it follows from (4.1) that $\mathbf{r} \in \mathcal{H}_k(D_n)$. $\square$

The proof actually gives stronger implications for the B_n-case.

Corollary 5.5. *Let $1 \leq k \leq n-2$. Then every nonempty B_n-invariant, k-sparse variety X meets $\mathcal{H}_k(D_n)$.*

Proof. For $X = X_k(\mathbf{q})$, we can assume that $\mathbf{q} = (q_1, \ldots, q_l, 0, \ldots, 0)$ with $q_i \neq 0$ for $1 \leq i \leq l$ and l maximal. If $l \leq k$, then $\mathbf{q} \in \mathcal{H}_k(D_n)$ and we are done. So assume $k < l \leq n$. We distinguish two cases: If $l \leq n-1$, let $f = e_{l+1}$ and $\mathbf{r} \in X_k(\mathbf{q})$ arbitrary. If $l = n$, let $f = e_n$ and $\mathbf{r} \in X_k(\mathbf{q})$ a maximizer of $|f|$. The corresponding Jacobians (5.1) and (5.2) for $s_2, \ldots, s_{2k}, f$ at $\mathbf{r}$ yield the claim. $\square$

Chapter 6

General reflection groups

In Chapter 5, we have studied reflection groups case by case. In this chapter, we take a different approach and try to develop strategies that do not use the classification and work with arbitrary reflection groups. Section 6.1 introduces orbit spaces, which allow us to prove a stronger version of Theorem 4.8. The case of higher codimension, that is $k < n-1$, is treated in Section 6.2, where we prove a weaker version of Conjecture 4.9 under a mild extra assumption on the underlying invariant polynomials. Our results also relate to varieties invariant under the action of certain Lie groups, which is illustrated in Section 6.3. Finally, in Section 6.4, we prove a first result for the related case of varieties invariant under the action of *complex* reflection groups.

6.1 Orbit spaces

In this section, we introduce orbit spaces, an important tool which in Proposition 6.1 gives yet another viewpoint on Conjecture 4.9. Taking this perspective, in Theorem 6.5 we prove a result which in particular implies Conjecture 4.9 for $k = n-1$. This also settles the case of groups of rank at most 3. The group F_4 is treated separately and we moreover give computational evidence that the conjecture should also hold for H_4.

6.1.1 Orbit spaces and reflection arrangements

As before, let G be a reflection group acting on V. The reflection arrangement $\mathcal{H} = \mathcal{H}(G)$ decomposes V into relatively open polyhedral cones. The closure σ of any full-dimensional cone in this decomposition serves as a **fundamental domain**: For every $\mathbf{p} \in V$, the orbit $G\mathbf{p}$ meets σ in a unique point. On the other hand, the basic invariants define an **orbit map** $\pi : V \to \mathbb{R}^n$ given by $\pi(\mathbf{x}) = (\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x}))$. The basic invariants separate orbits, that is, $\pi(\mathbf{p}) = \pi(\mathbf{q})$ if and only if $\mathbf{q} \in G\mathbf{p}$ for all $\mathbf{p}, \mathbf{q} \in V$. The image $\mathcal{S} := \pi(V)$ is homeomorphic to V/G and, by abuse of terminology, we call $\mathcal{S}$ the **real orbit space**. Since π is an algebraic map, $\mathcal{S}$ is semialgebraic, with an explicit inequality description given in [65]. Restricted to σ the map $\pi|_\sigma : \sigma \to \mathcal{S}$ is a homeomorphism. Moreover,

$$\pi^{-1}(\partial\mathcal{S}) \;=\; \mathcal{H}_{n-1}(G), \tag{6.1}$$

where $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$. Observe that the orbit space $\mathcal{S}$ is not uniquely determined by G, but depends on the basic invariants we choose.

In terms of the orbit map, Conjecture 4.9 can be put in a more general context. For $J \subseteq [n]$, let us write $\pi_J(\mathbf{x}) = (\pi_i(\mathbf{x}) : i \in J)$. For given J, we can ask for the smallest $0 \leq t \leq n$ such that $\pi_J(V) = \pi_J(\mathcal{H}_t)$.

Proposition 6.1. *Let G be an irreducible and essential reflection group. Then Conjecture 4.9 is true for G if and only if for $J = [k]$ we have*

$$\pi_J(V) = \pi_J(\mathcal{H}_k).$$

Proof. For $\mathbf{q} \in V$, we have $X_k(\mathbf{q}) = \pi_J^{-1}(\pi_J(\mathbf{q}))$. Hence, $X_k(\mathbf{q}) \cap \mathcal{H}_k \neq \emptyset$ for $\mathbf{q} \in V$ if and only if there is some $\mathbf{p} \in \mathcal{H}_k$ such that $\pi_J(\mathbf{q}) = \pi_J(\mathbf{p})$. □

A generalization of Theorem 4.3 to *J-sparse* symmetric polynomials $f \in \mathbb{R}[\pi_i : i \in J]$ was considered in [67]. The correspondence given in Proposition 6.1 also shows that the dimensions of strata in Conjecture 4.9 are best possible.

Proposition 6.2. *Let $J \subseteq [n]$ and $0 \leq t \leq n$ such that $\pi_J(V) = \pi_J(\mathcal{H}_t)$. Then $t \geq |J|$.*

Proof. The set $\pi_J(V)$ is the projection of the real orbit space $\mathcal{S}$ onto the coordinates indexed by J and hence is of full dimension $|J|$. By invariance of dimension, this implies that $t = \dim \mathcal{H}_t \geq |J|$. □

For the next result, recall that, by definition, $G \subset O(V)$ and hence $\|\mathbf{x}\|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$ is an invariant of G. In the following, we call a set $\mathcal{S} \subseteq V$ **line-free** if every nonempty affine subspace $L \subseteq V$ satisfying $L \subseteq \mathcal{S}$ is a point.

Lemma 6.3. *Let G be a finite reflection group acting on V and let $\pi_1, \ldots, \pi_n$ be a choice of basic invariants such that $\pi_i(\mathbf{x}) = \|\mathbf{x}\|^2$ for some i. Then the orbit space $\mathcal{S} = \pi(V)$ is line-free.*

Proof. Since $\pi_i(\mathbf{x}) = \|\mathbf{x}\|^2 \geq 0$ for all $\mathbf{x} \in V$, the linear function $\ell(\mathbf{y}) = y_i$ is nonnegative on $\mathcal{S} \subset \mathbb{R}^n$. Hence, if $L \subseteq \mathcal{S}$ is an affine subspace, then ℓ is constant on L. Let $\sigma \subseteq V$ be a fundamental domain for G. Then $L = \mathcal{S} \cap L$ is homeomorphic to $\hat{L} := \{\mathbf{p} \in \sigma : \|\mathbf{p}\|^2 = c\}$ for some $c \geq 0$. This implies that L is compact, which finishes the proof. □

Example 6.4. For the essential reflection group B_3 and the power sums as basic invariants, Figure 6.1 shows the semialgebraic compact slice $K := \mathcal{S} \cap \{y_1 = 1\}$ and its defining algebraic curves of degree up to 9, obtained using the description in [65]. Any fundamental domain for B_3 is a full-dimensional cone with three generating rays. Hence, K is homeomorphic to a triangle. Moreover, since the basic invariants are homogeneous and of even degree, every other slice of the form $\mathcal{S} \cap \{y_1 = c\}$ with $c > 0$ can be obtained from K by simply rescaling the two coordinate axes.

Theorem 6.5. *Let G be an essential reflection group with a choice of basic invariants $\pi_1, \ldots, \pi_n$. Let $f \in \mathbb{R}[V]^G$ be an invariant polynomial such that f is at most linear in π_k for some k. Then $\mathrm{V}_\mathbb{R}(f) \neq \emptyset$ if and only if $\mathrm{V}_\mathbb{R}(f) \cap \mathcal{H}_{n-1} \neq \emptyset$.*

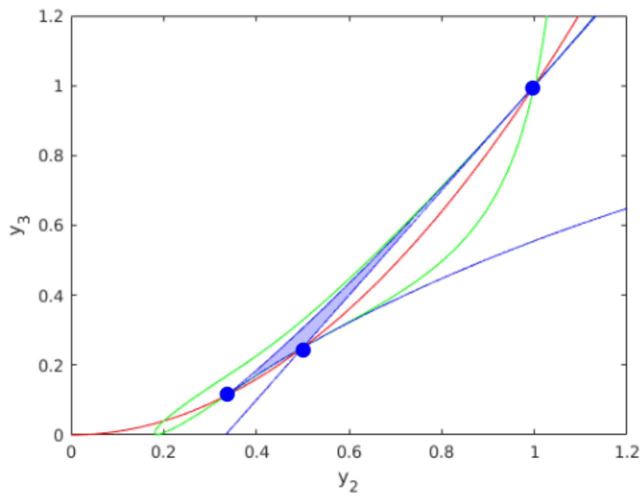

Figure 6.1: A compact slice of the B_3-orbit space at $y_1 = 1$, given in the blue region.

Proof. Without loss of generality, we can assume that $f(0) < 0$. Since the arrangement $\mathcal{H}_{n-1}$ is path connected, it suffices to show that there is a point $\mathbf{p}_+ \in \mathcal{H}_{n-1}(G)$ with $f(\mathbf{p}_+) \geq 0$.

We can assume that $\pi_1(\mathbf{x}) = \|\mathbf{x}\|^2$. Indeed, since G is essential, all basic invariants have degree at least 2 and $\|\mathbf{x}\|^2$ is a linear combination of the degree 2 basic invariants. Let $\mathbf{p} \in V_{\mathbb{R}}(f)$ and define $K = \{\mathbf{q} \in V : \pi_1(\mathbf{q}) = \pi_1(\mathbf{p})\}$, the sphere centered at the origin that contains $\mathbf{p}$. The function f attains its maximum over K in a nonempty closed set $M \subseteq K$. We claim that $M \cap \mathcal{H}_{n-1} \neq \emptyset$. Let $\mathbf{p}_0$ be a point in M.

We may pass to the real orbit space $\mathcal{S} = \pi(V)$ associated to G and $\pi_1, \ldots, \pi_n$ and consider the compact set $\overline{K} := \pi(K) = \{\mathbf{y} \in \mathcal{S} : y_1 = \pi_1(\mathbf{p})\}$. We can write $f = F(\pi_1, \ldots, \pi_n)$ for some $F \in \mathbb{R}[y_1, \ldots, y_n]$. In this setting, our assumption states that F is at most linear in y_k. If $\mathbf{p}_0 \in V \setminus \mathcal{H}_{n-1}$, then, by (6.1), $\overline{\mathbf{p}}_0 := \pi(\mathbf{p}_0)$ is in the interior of $\mathcal{S}$ and hence in the relative interior of $\overline{K}$. Let $L = \{\overline{\mathbf{p}}_0 + t\mathbf{e}_k : t \in \mathbb{R}\}$ be the affine line through $\overline{\mathbf{p}}_0$ in direction of the kth standard basis vector $\mathbf{e}_k$. Restricted to L, the polynomial F has degree at most 1. By Lemma 6.3 and our choice of $\overline{K}$, the line L meets $\partial\overline{K}$ in two points $\overline{\mathbf{p}}_-, \overline{\mathbf{p}}_+$ and $F(\overline{\mathbf{p}}_-) \leq F(\overline{\mathbf{p}}_0) \leq F(\overline{\mathbf{p}}_+)$. This implies that $\pi^{-1}(\overline{\mathbf{p}}_+) \subseteq M$ and, since $\partial\overline{K} \subseteq \partial\mathcal{S}$, equation (6.1) shows that $\pi^{-1}(\overline{\mathbf{p}}_+) \subseteq \mathcal{H}_{n-1}$. $\square$

The assumption in Theorem 6.5 that G is essential is indeed necessary, as the following example shows.

Example 6.6. Let $G = B_n$ act on $V = \mathbb{R}^n \times \mathbb{R}$ by fixing the last coordinate. A set of basic invariants is given by $\pi_1(\mathbf{x}, x_{n+1}) = x_{n+1}$ and $\pi_i(\mathbf{x}, x_{n+1}) = s_{2i-2}(\mathbf{x})$ for $i = 2, \ldots, n+1$. Pick $\mathbf{p} \in \mathbb{R}^n$ with all coordinates positive and distinct. The variety

$$X \quad = \quad \{(\mathbf{x}, x_{n+1}) \in V : s_{2i}(\mathbf{x}) = s_{2i}(\mathbf{p}) \text{ for } i = 1, \ldots, n\} \tag{6.2}$$

is defined over $\mathbb{R}[\pi_2, \ldots, \pi_{n+1}]$, but is a collection of affine lines that does not meet the reflection arrangement.

As a consequence of Theorem 6.5, we immediately obtain Theorem 4.8 from Section 4.2.

Proof of Theorem 4.8. Let $f_1, \dots, f_m \in \mathbb{R}[\pi_i : i \neq j]$ for some $j \in [n]$. Then

$$X \ := \ \mathrm{V}_{\mathbb{R}}(f_1, \dots, f_m) \ = \ \mathrm{V}_{\mathbb{R}}(f),$$

where $f := f_1^2 + \cdots , f_m^2$. Hence, Theorem 6.5 applies. $\square$

We give two further applications.

Corollary 6.7. *Let G be an essential reflection group and let $J \subset [n]$ with $|J| = n-1$. For polynomials $f, f_1, \dots, f_m \in \mathbb{R}[\pi_i : i \in J]$, the following hold:*

(i) *The semialgebraic set $\mathrm{S} \ := \ \{\mathbf{p} : f_1(\mathbf{p}) \geq 0, \dots, f_m(\mathbf{p}) \geq 0\}$ is nonempty if and only if $\mathrm{S} \cap \mathcal{H}_{n-1}(G) \neq \emptyset$.*

(ii) *We have $f(\mathbf{q}) \geq 0$ for all $\mathbf{q} \in \mathrm{S}$ if and only if $f(\mathbf{q}) \geq 0$ for all $\mathbf{q} \in \mathrm{S} \cap \mathcal{H}_{n-1}(G)$.*

Proof. For $\mathbf{q} \in \mathrm{S}$, it suffices to prove the claim for

$$X \ := \ \{\mathbf{p} \in V : \pi_j(\mathbf{p}) = \pi_j(\mathbf{q}) \text{ for } j \in J\} \ \subseteq \ S.$$

Claim (i) now follows from Theorem 6.5. As for (ii), assume that $\mathbf{q} \in \mathrm{S} \setminus \mathcal{H}_{n-1}$ and $f(\mathbf{q}) < 0$. Then the same argument applied to $X \cap \{\mathbf{p} : f(\mathbf{p}) = f(\mathbf{q})\}$ finishes the proof. $\square$

If $f \in \mathbb{R}[V]^G$ has degree $\deg(f) < 2\deg(\pi_n)$, then the algebraic independence of the basic invariants implies that Theorem 6.5 can be applied to prove the following corollary. Under the assumption that f is homogeneous, the second part of the corollary recovers the main result of Acevedo and Velasco [1].

Corollary 6.8. *Let $f \in \mathbb{R}[V]^G$ with $\deg(f) < 2d_n(G) = 2\deg(\pi_n)$. Then $\mathrm{V}_{\mathbb{R}}(f) \neq \emptyset$ if and only if $\mathrm{V}_{\mathbb{R}}(f) \cap \mathcal{H}_{n-1} \neq \emptyset$. In particular, $f \geq 0$ on V if and only if $f \geq 0$ on $\mathcal{H}_{n-1}$.*

The bound on the degree is tight: For a point $\mathbf{p} \in V \setminus \mathcal{H}_{n-1}(G)$, the set of solutions to

$$f(\mathbf{x}) \ := \ \sum_{i=1}^{n} (\pi_i(\mathbf{x}) - \pi_i(\mathbf{p}))^2 \ = \ 0$$

is exactly $G\mathbf{p}$, which does not meet $\mathcal{H}_{n-1}(G)$. The defining polynomial $f(\mathbf{x})$ is of degree exactly $2\deg(\pi_n)$.

6.1.2 Some exceptional types

Theorem 6.5 also allows us to prove Theorem 4.7 for groups of small rank.

Proof of Theorem 4.7 for $\mathrm{rk}(G) \leq 3$. For $k = \mathrm{rk}(G)$, there is nothing to prove. For $k = 1$, we observe that $X_1(\mathbf{p})$ is the sphere through $\mathbf{p}$, which meets the arrangement $\mathcal{H}_1(G)$ of lines through the origin. Thus, the only nontrivial case is $\mathrm{rk}(G) = 3$ and $k = \mathrm{rk}(G) - 1 = 2$. This is covered by Theorem 4.8. $\square$

Let G be an essential reflection group of rank ≥ 4. Since G acts on V by orthogonal transformations, we have that $\pi_1(\mathbf{x}) = \|\mathbf{x}\|^2$ and $X_k(\mathbf{p})$ is a subvariety of a sphere centered at the origin. Since the basic invariants are homogeneous, we may assume that $\pi_1(\mathbf{p}) = 1$ and hence $X_k(\mathbf{p}) \subseteq S^{n-1} = \{\mathbf{x} \in V : \|\mathbf{x}\| = 1\}$. To prove Theorem 4.7 for $k = 2$ we can proceed as follows. Let $\delta_{\min}$ and $\delta_{\max}$ be the minimum and maximum of π_2 over S^{n-1}. Then it suffices to find points $\mathbf{p}_{\min}, \mathbf{p}_{\max} \in \mathcal{H}_2(G) \cap S^{n-1}$ with $\pi_2(\mathbf{p}_{\min}) = \delta_{\min}$ and $\pi_2(\mathbf{p}_{\max}) = \delta_{\max}$. Indeed, since $\mathcal{H}_2(G)$ is connected (for $\mathrm{rk}(G) \geq 3$), this shows that $\pi_J(V) = \pi_J(\mathcal{H}_2(G))$ for $J = \{1, 2\}$, which, by Proposition 6.1, then proves the claim. For the group F_4, we can implement this strategy.

Proof of Theorem 4.7 for F_4. Since F_4 is of rank 4, we only need to consider the case $k = 2$ and can use the strategy outlined above. Let $\delta_{\min}$ and $\delta_{\max}$ be the minimum and maximum of π_2 over S^3. An explicit description of π_2 for F_4 is

$$\pi_2(\mathbf{x}) \;=\; \sum_{1 \leq i < j \leq 4} (x_i + x_j)^6 + (x_i - x_j)^6;$$

see, for example, Mehta [61] or [54, Table 5]. The points $\mathbf{p} = (1, 0, 0, 0)$ and $\mathbf{p}' = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)$ are contained in $\mathcal{H}_1(F_4) \subseteq \mathcal{H}_2(F_4)$ and takes values $\pi_2(\mathbf{p}) = 1$ and $\pi_2(\mathbf{p}') = \frac{3}{2}$. We claim, that these values are exactly $\delta_{\min}$ and $\delta_{\max}$, respectively.

Note that $\pi_2(\mathbf{x}) = g(x_1^2, x_2^2, x_3^2, x_4^2)$ for

$$g(\mathbf{y}) \;=\; 5 s_1(\mathbf{y}) \cdot s_2(\mathbf{y}) - 4 s_3(\mathbf{y}).$$

Let $\Delta_3 = \{\mathbf{x} \in \mathbb{R}^4 : x_1, \ldots, x_4 \geq 0, x_1 + \cdots + x_4 = 1\}$ be the standard 3-simplex. We have that $\rho(S^3) = \Delta_3$ where $\rho(x_1, \ldots, x_4) := (x_1^2, \ldots, x_4^2)$. Hence,

$$\delta_{\max} \;=\; \max\{g(\mathbf{p}) : \mathbf{p} \in \Delta_3\} \quad \text{and} \quad \delta_{\min} \;=\; \min\{g(\mathbf{p}) : \mathbf{p} \in \Delta_3\}.$$

Now, D_4 is a subgroup of F_4 and $\pi_2 \in \mathbb{R}[s_2, s_4, s_6]$ and does not depend on $e_4(\mathbf{x})$. By Theorem 4.7 for D_4, the varieties $S^3 \cap \{\pi_2(\mathbf{x}) = \delta_{\min}\}$ and $S^3 \cap \{\pi_2(\mathbf{x}) = \delta_{\max}\}$ both meet $\mathcal{H}_3(D_4)$. Hence, it suffices to minimize or maximize $g(\mathbf{x})$ over

$$\Delta_3 \cap \{\mathbf{x} \in \mathbb{R}^4 : x_1 = x_2\}.$$

This leaves us with the (standard) task to maximize and minimize a bivariate polynomial $g'(s,t)$ of degree 3 over a triangle. In the plane, the polynomial has 3 critical points with values $1, \frac{11}{9}, \frac{11}{9}$. On the boundary, the extreme values are attained at the points given above. $\square$

For the rank-4 reflection group H_4, the invariant $\pi_2(\mathbf{x})$ is a polynomial of degree 12 in four variables; see, for example, [54, Table 6]. Since $(B_1)^4$ is a reflection subgroup of F_4, π_2 is a polynomial in the squares $x_1^2, \ldots, x_4^2$ and, following the argument in the proof above, we are left with minimizing and maximizing a degree-6 polynomial $g(\mathbf{x})$ over the simplex Δ_3. However, finding the critical points is not easy and an extra computational challenge is the fact that $g(\mathbf{x})$ is a polynomial with coefficients in $\mathbb{Q}(\sqrt{5})$. `GloptiPoly` [44] *numerically* computes $\delta_{\min} = -\frac{5}{16}$ and $\delta_{\max} = 1$. These values are attained at $\mathbf{p}_{\min} = \frac{1}{\sqrt{2}}(1, 1, 0, 0)$ and $\mathbf{p}_{\max} = (1, 0, 0, 0)$, respectively, and both points lie in $\mathcal{H}_2((B_1)^4) \subseteq \mathcal{H}_2(H_4)$. This is strong evidence for the validity of Conjecture 4.9 for H_4 but, of course, not a rigorous proof.

6.2 A general approach

We turn to the case when $k < n-1$ for arbitrary reflection groups. In Section 6.2.1, we first use a naive inductive approach, which in most cases gives bounds that are far from optimal, but nevertheless, yield interesting results for the exceptional types in Proposition 6.9. In Theorem 6.11 we prove a weaker version of Conjecture 4.9 under a mild extra assumption on the underlying invariant polynomials. The bounds on dimension of the strata in Theorem 6.11 are not easy to compute explicitly and we give upper bounds using the combinatorics of parabolic subgroups in Section 6.2.2. Finally, in Section 6.2.3, we apply perturbation techniques, which in particular yield a second proof of Conjecture 4.9 for types A_n and B_n.

6.2.1 Strata of higher codimension

We have seen in the previous section that every nonempty $(n-1)$-sparse variety meets the hyperplane arrangement $\mathcal{H}_{n-1}$. In this section, we want to extend this result to k-sparse varieties for $k < n-1$. This case is considerably more difficult but we can make good use of the techniques and ideas developed in Section 6.1.

Let G be an essential finite reflection group acting on $V \cong \mathbb{R}^n$. Consider a G-invariant k-sparse variety X with $k < n$. If X is nonempty, then Theorem 6.5 yields that for some reflection hyperplane $H \in \mathcal{H}$ the variety $X' := X \cap H$ is nonempty. An inductive argument could now replace G by some other reflection subgroup $G' \subseteq G$ that fixes H. If X' remains sparse with respect to G' we can again apply Theorem 6.5 to obtain a point $\mathbf{p} \in \mathcal{H}_{n-2}(G') \subseteq \mathcal{H}_{n-2}(G)$. However, the results obtained using this strategy are far from optimal. We will briefly illustrate this for $G = \mathfrak{S}_n$: Let X be a nonempty k-sparse $\mathfrak{S}_n$-invariant variety for $k < n$. The largest subgroup of $\mathfrak{S}_n$ that fixes a given reflection hyperplane H is of the form $G' \cong \mathfrak{S}_{n-2} \times \mathfrak{S}_2$. Hence, Theorem 6.5 only applies for $X' = X \cap H$ and G' if $k = d_k(G) < d_n(G') = n-2$, in other words if the original variety X is $(k-3)$-sparse. Inductively, this yields that every nonempty k-sparse $\mathfrak{S}_n$-invariant variety meets $\mathcal{H}_l$ where $l = \lfloor \frac{n+k}{2} \rfloor$. However, applying the above method to the exceptional types gives nontrivial bounds.

Proposition 6.9. *Let $6 \le n \le 8$. Then every nonempty 2-sparse E_n-invariant variety intersects $\mathcal{H}_{n-2}(E_n)$.*

Proof. We exemplify the argument for the case $n = 8$. Let X be a nonempty 2-sparse E_8-invariant variety. By Theorem 6.5 we find a point $\mathbf{p} \in X \cap \mathcal{H}_7(E_8)$. The orbit of $\mathbf{p}$ meets every hyperplane in $\mathcal{H}(E_8)$ (see [53, Sect. 2.10]) and hence we may assume that $\mathbf{p}$ lies on the hyperplane $H = \{\mathbf{x} \in \mathbb{R}^8 : x_1 = x_2\}$. Consider the subgroup $G' \cong D_6 \subset E_8$ acting essentially on the coordinates $x_3, \ldots, x_8$. Since $d_6(D_6) = 10 > 8 = d_2(E_8)$, we can apply Theorem 6.5 to finish the proof. □

By restricting the class of invariant polynomials, we obtain better bounds than those in Proposition 6.9. In the following, a point $\mathbf{p}$ is called **G-general** if it does not lie on any reflection hyperplane of G, and hence $|G\mathbf{p}| = |G|$.

Definition 6.10. For a positive integer d, let $\delta_G(d)$ be the largest number ℓ such that for every $p \in \mathcal{H}_{\ell+1}$ there is a reflection subgroup $G' \subseteq G$ such that p is G'-general and $2d_n(G') > d$. Moreover, we define $\sigma_G(k) := \delta_G(2d_k(G))$. That is, $\sigma_G(k)$

is the largest $0 \leq \ell \leq d$ such that for every $p \in \mathcal{H}_{\ell+1}$, there is a reflection subgroup $G' \subseteq G$ such that p is G'-general and $d_n(G') > d_k(G)$.

We call an invariant polynomial $f \in \mathbb{R}[V]^G$ G**-finite** if either $V_{\mathbb{R}}(f) = \emptyset$ or if there is a point $\mathbf{p} \in V_{\mathbb{R}}(f)$ such that f has finitely many extreme points restricted to the sphere $K = \{\mathbf{q} \in V : \|\mathbf{q}\| = \|\mathbf{p}\|\}$. We want to think of G-finiteness as some sort of genericity assumption. Indeed in the non-invariant setting, it is shown in [63] that a *generic* polynomial $f \in \mathbb{R}[\mathbf{x}]$ has only finitely many extreme points on K.

Theorem 6.11. *Let $f \in \mathbb{R}[V]^G$ be a G-finite polynomial and $X = \mathrm{V}_{\mathbb{R}}(f)$. If $f \in \mathbb{R}[\pi_1, \ldots, \pi_k]$, then*

$$X \neq \emptyset \quad \text{if and only if} \quad X \cap \mathcal{H}_{\sigma_G(k)} \neq \emptyset.$$

If $d = \deg(f)$, then

$$X \neq \emptyset \quad \text{if and only if} \quad X \cap \mathcal{H}_{\delta_G(d)} \neq \emptyset.$$

Proof. We only give a proof for the second result. The proof of the first is analogous. Suppose $\mathrm{V}_{\mathbb{R}}(f) \neq \emptyset$. We may assume that $f(0) \leq 0$ and, since $\mathcal{H}_{\delta_G(d)}$ is connected, it suffices to show that there is some point $\mathbf{p}_+ \in \mathcal{H}_{\delta_G(d)}$ with $f(\mathbf{p}_+) \geq 0$.

By assumption, there is a zero $\mathbf{p}_0 \in \mathrm{V}_{\mathbb{R}}(f)$ such that f has only finitely many extreme points restricted to $K = \{\mathbf{q} : \|\mathbf{q}\| = \|\mathbf{p}_0\|\}$. Let $\mathbf{p}_+ \in K$ be a point maximizing f over K and hence $f(\mathbf{p}_+) \geq f(\mathbf{p}_0) \geq 0$. We claim that $\mathbf{p}_+ \in \mathcal{H}_{\delta_G(d)}$. Otherwise, there is a reflection subgroup $G' \subset G$ such that $\mathbf{p}_+$ is G'-general and $2d_n(G') > d$. Let $\pi'_1, \ldots, \pi'_n$ be a choice of basic invariants of G' and, without loss of generality, $\pi'_1(\mathbf{x}) = \|\mathbf{x}\|^2$. Thus, $\overline{\mathbf{p}}_+ = \pi'(\mathbf{p}_+)$ is in the interior of $\mathcal{S} = \pi'(V)$. We can write $f = F(\pi'_1, \ldots, \pi_n)$ for some $F \in \mathbb{R}[y_1, \ldots, y_n]$. On the level of orbit spaces, our assumption states that restricted to $\overline{K} = \pi(L) = \{\mathbf{y} \in \mathcal{S} : y_1 = \pi_1(\mathbf{p}_+)\}$, the polynomial F has only finitely many extreme points. However, F is linear in y_n and thus $\overline{\mathbf{p}}_+$ is a maximum only if $\overline{\mathbf{p}}_+ \in \partial\overline{K} \subseteq \partial\mathcal{S}$. This is a contradiction. □

6.2.2 Bounds from parabolic subgroups

Given a reflection group G, a subgroup $W \subseteq G$ is called **parabolic**, if there exists a simple system Δ for G and a subset $\Delta' \subseteq \Delta$ such that W is generated by the reflections $\{s_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \Delta'\}$ (cf. [53, Sec. 1.10]). The numbers $\delta_G(d)$ and $\sigma_G(k)$ defined in Section 6.2.1 are difficult to compute in general, but in this section we compute upper bounds coming from the combinatorics of parabolic subgroups.

Lemma 6.12. *Fix a finite irreducible reflection group G with Dynkin diagram D. Let $D' \subset D$ be a subdiagram obtained by removing a node from D and let $H \in \mathcal{H}(G)$ be a reflection hyperplane. Then there is a parabolic subgroup $W \subset G$ with Dynkin diagram D' and H is not a reflection hyperplane of W.*

Proof. Let W be a parabolic subgroup with Dynkin diagram D'. Since every parabolic subgroup with Dynkin diagram D' is conjugate to W, it suffices to show that there is a $g \in G$ such that $gH \notin \mathcal{H}(W)$. In terms of the roots $\Phi(G)$ of G, this is equivalent to showing that for every $\alpha \in \Phi(G)$, there is a $g \in G$ such that $g\alpha \notin \Phi(W)$. Since G acts transitively on the roots in every orbit (see [53]) and $\Phi(W) \subsetneq \Phi(G)$

for any proper parabolic subgroup, this yields the claim for all types except F_4, B_n and $I_2(2m)$ for $m > 1$. For F_4, the possible proper parabolic subgroups are B_3 and $A_1 \times A_2$ and the result follows by inspection. For $I_2(2m)$, there are two orbits with each $2m \geq 4$ roots whereas the only nontrivial proper parabolic subgroup is A_1. For B_n, this follows from counting the number of elements in each of the two orbits. □

The lemma yields the following result about finite reflection groups that might be interesting in its own right.

Proposition 6.13. *Let G be a finite irreducible reflection group with Dynkin diagram D acting on a real vector space V. For $k \geq 1$, let $\mathbf{p} \in \mathcal{H}_k \setminus \mathcal{H}_{k-1}$ and $D' \subset D$ be a connected subdiagram on k nodes. Then there is a parabolic subgroup $W \subset G$ with Dynkin diagram D' such that $\mathbf{p}$ is W-general.*

Proof. We argue by induction on $s = \dim V - k$. For $s = 0$, $\mathbf{p} \in V \setminus \mathcal{H}_{\dim V - 1}$ and $\mathbf{p}$ is by definition G-general. Otherwise, let $D_1 \subset D$ be a subdiagram obtained by removing a leaf such that $D' \subseteq D_1$ and let H_1 be a reflection hyperplane of G containing $\mathbf{p}$. We may use Lemma 6.12 to obtain a parabolic subgroup W_1 with Dynkin diagram D_1 and not containing H_1 as a reflection hyperplane. In particular, $\mathbf{p}$ is contained in precisely $s-1$ linearly independent reflection hyperplanes of W_1. By induction, there is a parabolic subgroup $W \subseteq W_1$ with Dynkin diagram D' for which $\mathbf{p}$ is W-general. In particular, W is a parabolic subgroup of G which concludes the proof. □

For a choice of a simple system $\Delta \subseteq \Phi(G)$, let us write W_I for the standard parabolic subgroup generated by $I \subseteq \Delta$. We define

$$\tilde{\delta}_G(d) \ := \ \min\{|I| - 1 : I \subseteq \Delta, 2d_n(W_I) > d\},$$

and, analogously, we define $\tilde{\sigma}_G(k) := \tilde{\delta}_G(2d_k(G))$. Proposition 6.13 implies the following bound on δ_G.

Corollary 6.14. $\delta_G(d) \leq \tilde{\delta}_G(d)$, *for all* $d \geq 0$.

The clear advantage is a simple way to compute upper bounds on $\delta_G(d)$ from the knowledge of (standard) parabolic subgroups of reflection groups; cf. [53]. The explicit values are given in Table 6.1. However, not every reflection subgroup is parabolic (e.g. $I_2(m) \subset I_2(2m)$). Nevertheless, we conjecture that $\delta_G(d)$ is attained at a parabolic subgroup.

Conjecture 6.15. *For any finite reflection group G*

$$\delta_G(d) \ = \ \tilde{\delta}_G(d)$$

for all d.

G	d	$\tilde{\delta}_G(d)$	W	$d_n(W)$	k	$\tilde{\sigma}_G(k)$
$A_{n-1}/\mathfrak{S}_n$	$0 — 2n-1$	$\lfloor d/2 \rfloor$	$A_{\lfloor d/2 \rfloor}$	$\lfloor d/2 \rfloor + 1$	$1 — n$	k
B_n	$0 — 4n-1$	$\lfloor d/4 \rfloor$	$B_{\lfloor d/4 \rfloor+1}$	$2(\lfloor d/4 \rfloor + 1)$	$0 — n$	k
D_n	$0 — 4n-5$	$\lfloor d/4 \rfloor + 1$	$D_{\lfloor d/4 \rfloor+2}$	$2(\lfloor d/4 \rfloor + 1)$	$0 — \lfloor \frac{n}{2} \rfloor$	$k+1$
					$\lfloor \frac{n}{2} \rfloor + 1 — n$	k
$I_2(m)$	$1 — 2m-1$	1	$I_2(m)$	m	1	1
E_6	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	3
	8 — 11	3	D_4	6	3	4
	12 — 15	4	D_5	8	4	5
	16 — 23	5	E_6	12	5	5
E_7	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	4
	8 — 11	3	D_4	6	3	5
	12 — 15	4	D_5	8	4	5
	16 — 23	5	E_6	12	5	6
	24 — 35	6	E_7	18	6	6
E_8	1 — 5	1	A_2	3	1	1
	6 — 7	2	A_3	4	2	5
	8 — 11	3	D_4	6	3	6
	12 — 15	4	D_5	8	4	6
	16 — 23	5	E_6	12	5	7
	24 — 35	6	E_7	18	6	7
	36 — 59	7	E_8	30	7	7
F_4	1 — 7	1	B_2	4	1	1
	8 — 11	2	B_3	6	2	3
	12 — 23	3	F_4	12	3	3
H_3	1 — 9	1	$I_2(5)$	5	1	1
	10 — 19	2	H_3	10	2	2
H_4	1 — 9	1	$I_2(m)$	5	1	1
	10 — 19	2	H_3	10	2	3
	20 — 59	3	H_4	30	3	3

Table 6.1: Computation for $\tilde{\delta}_G(d)$ and $\tilde{\sigma}_G(k)$. $a — b$ refers to the range $a, a+1, ..., b$. The column W gives the parabolic subgroup that attains $\tilde{\delta}_G$.

6.2.3 Perturbation techniques

We will now extend Theorem 6.11 to polynomials that are limits of G-finite polynomials. For $d \geq 0$, we write $\mathbb{R}[V]_{\leq d}$ for the vector space of polynomials of degree at most d. For convenience, we will call an invariant polynomial $f \in \mathbb{R}[V]^G$ **G-sparse** if $f \in \mathbb{R}[\pi_1, \ldots, \pi_k]$ for some choice basic invariants $\pi_1, \ldots, \pi_n$.

Proposition 6.16. *Let G be a finite reflection group acting on V and $f \in \mathbb{R}[V]^G$ such that $X = \mathrm{V}_{\mathbb{R}}(f) \neq \emptyset$. Let $(f_n)_{n\geq 0} \in \mathbb{R}[V]^G_{\leq d}$ be a sequence of polynomials such that $f_n \overset{n\to\infty}{\longrightarrow} f$ and assume that for some $\mathbf{p} \in X$, every f_n has only finitely many extreme points on the sphere of radius $\|\mathbf{p}\|$. Then $X \cap \mathcal{H}_{\delta_G(d)} \neq \emptyset$. If all f_n are k-sparse, then $X \cap \mathcal{H}_{\sigma_G(k)} \neq \emptyset$.*

Proof. We can assume $f(0) < 0$ and let m be the maximum of f over $K := \{\mathbf{q} : \|\mathbf{q}\| = \|\mathbf{p}\|\}$. By assumption, $m \geq 0 = f(\mathbf{p})$ and every maximizer $\mathbf{q}_n \in K$ of f_n over K is contained in $\mathcal{H}_{\delta_G(d)}$ for all $n \geq 0$ by the proof of Theorem 6.11. Choosing a convergent subsequence of $(\mathbf{q}_n)_{n\geq 0}$, there is a point $\mathbf{q} \in \mathcal{H}_{\delta_G(d)}$ such that $f(\mathbf{q}) \geq 0$. Using that $\mathcal{H}_{\delta_G(d)}$ is path connected completes the proof. The argument if f_n is k-sparse is analogous. □

In the following, we want to establish some criteria on when we are in the above situation. To do this, we will work over complex projective space. The basic results and notation from complex algebraic geometry we use here are taken from [42]. We identify the complexification of V with $\mathbb{C}^n$ that is embedded in the complex projective space $\mathbb{CP}^n$ as the affine chart $\{\mathbf{p} = [p_0 : \cdots : p_n] \in \mathbb{CP}^n : p_0 \neq 0\}$, where by $[p_0 : \cdots : p_n]$ we denote homogeneous coordinate in $\mathbb{CP}^n$. We write $K \subset \mathbb{C}^n$ for the unit sphere and $\tilde{K} = \{\mathbf{p} \in \mathbb{CP}^n : p_0^2 = p_1^2 + \cdots + p_n^2\}$ for its projectivization. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ let

$$\mathrm{Crit}(f) \ := \ \{\mathbf{p} \in K : \mathrm{rk}(\nabla f(\mathbf{p})), \mathbf{p}) \leq 1\}$$

be the complex critical locus of f over K. Whenever $f \in \mathbb{R}[V]$ and $\mathrm{Crit}(f)$ is finite, then f has only finitely many extreme points on the unit sphere in V and we can apply the machinery of G-finite polynomials.

As a next step we will homogenize all polynomials to apply results on complex projective varieties. For a polynomial $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $d \geq \deg(f)$, we write

$$f^{(d)} \ := \ x_0^d f(\tfrac{x_1}{x_0}, \tfrac{x_2}{x_0}, \ldots, \tfrac{x_n}{x_0}).$$

Thus $f^{(d)}$ is homogeneous of degree d. For $d = \deg(f)$ we simply write $\tilde{f}$. This operation, in particular, takes $\mathbb{C}[x_1, \ldots, x_n]_{\leq d}$ bijectively to $\mathbb{C}[x_0, \ldots, x_n]_d$, the vector space of d-forms. For $f \in \mathbb{C}[x_1, \ldots, x_n]_{\leq d}$, we define

$$\widetilde{\mathrm{Crit}}_d(f) \ := \ \Big\{[p_0 : p_1 : \cdots : p_n] \in \tilde{K} : \mathrm{rk}\begin{pmatrix} p_1 & \partial_{x_1} f^{(d)}(p_0, \ldots, p_n) \\ \vdots & \vdots \\ p_n & \partial_{x_n} f^{(d)}(p_0, \ldots, p_n) \end{pmatrix} \leq 1\Big\}. \quad (6.3)$$

This is indeed a projective variety since the rank condition is equivalent to the vanishing of the 2-minors. For $d = \deg f$, this is exactly the projectivization of $\mathrm{Crit}(f)$.

By Bézout's theorem ([42, Thm. 18.3]), if $\widetilde{\mathrm{Crit}}_d(f)$ is of positive dimension, then $\widetilde{\mathrm{Crit}}_d(f) \cap H \neq \emptyset$ for any hyperplane $H \subset \mathbb{C}\mathrm{P}^n$. For fixed H, define the variety $Y^{(d)} \subset \mathbb{C}[x_1, \ldots, x_n]_{\leq d}$ as the projection of the incidence variety

$$\left\{(f, p) \in \mathbb{C}[x_1, \ldots, x_n]_{\leq d} \times H : p \in \widetilde{\mathrm{Crit}}_d(f)\right\}$$

onto the first factor. Since $H \cong \mathbb{C}\mathrm{P}^{n-1}$, this projection is Zariski-closed in the affine space $\mathbb{C}[x_1, \ldots, x_n]_{\leq d}$ by [42, Thm. 3.12]. For our purpose, the above implies the following.

Proposition 6.17. *Let G be a finite reflection group acting on V and $d > 0$.*

(i) *Assume that there is $f_0 \in \mathbb{R}[V]^G_{\leq d}$ such that $\widetilde{\mathrm{Crit}}_d(f_0)$ is 0-dimensional. Then for any $f \in \mathbb{R}[V]^G_{\leq d}$*

$$\mathrm{V}_{\mathbb{R}}(f) \neq \emptyset \iff \mathrm{V}_{\mathbb{R}}(f) \cap \mathcal{H}_{\delta_G(d)} \neq \emptyset.$$

(ii) *Assume that there is a k-sparse $f_0 \in \mathbb{R}[V]^G_{\leq d}$ such that $\widetilde{\mathrm{Crit}}_d(f_0)$ is 0-dimensional. Then for any k-sparse $f \in \mathbb{R}[V]^G_{\leq d}$*

$$\mathrm{V}_{\mathbb{R}}(f) \neq \emptyset \iff \mathrm{V}_{\mathbb{R}}(f) \cap \mathcal{H}_{\sigma_G(k)} \neq \emptyset.$$

Proof. By assumption, there is a hyperplane $H \subset \mathbb{C}\mathrm{P}^n$ for which $f_0 \notin Y^{(d)}$. Therefore, $Z := Y^{(d)} \cap \mathbb{R}[V]^G_{\leq d}$ is nowhere dense in $\mathbb{R}[V]^G_{\leq d}$ and every polynomial outside Z has only finitely many critical points on K. Let $f \in \mathbb{R}[V]^G_{\leq d}$ such that $\mathrm{V}_{\mathbb{R}}(f) \neq \emptyset$. We may assume that $\mathrm{V}_{\mathbb{R}}(f) \cap K \neq \emptyset$ by replacing $f(\mathbf{x})$ by $f(\|\mathbf{p}\|\mathbf{x})$ for any $\mathbf{p} \in \mathrm{V}_{\mathbb{R}}(f)$. Since Z is nowhere dense f is a limit of polynomials with only finitely many extreme points on K and (i) follows from Proposition 6.16. The proof of (ii) is analogous. □

In particular, this yields a second proof of Theorem 4.7 for types A_n and B_n.

Alternative proof of Theorem 4.7 for A_n and B_n. For A_n and B_n, a set of basic invariants can be chosen from among the power sums and, by Lemma 5.1, it suffices to consider these particular basic invariants. Using (6.3), it is straightforward to verify that for any $d \geq 1$, the projective variety $\widetilde{\mathrm{Crit}}_d(s_d)$ consists of only finitely many points, which, with the help of Table 6.1, finishes the proof. □

A similar strategy can be applied for the case of D_{2n}. However, for the exceptional groups we were so far unable to find invariants of arbitrarily high degree such that the associated projective critical locus is 0-dimensional.

6.3 Lie groups

In this section, we extend some of our results to polynomials invariant under the action of a Lie group. More precisely, we consider the case of a real simple Lie group G with the adjoint action on its Lie algebra $\mathfrak{g}$. For basic information on Lie groups and Lie algebras, we refer the reader to [31].

We illustrate our results for the case $G = \mathrm{SL}_n$. Its Lie algebra $\mathfrak{sl}_n$ is the vector space of real n-by-n matrices of trace 0. The adjoint action of SL_n on $\mathfrak{sl}_n$ is by conjugation: $g \in \mathrm{SL}_n$ acts on $A \in \mathfrak{sl}_n$ by $g \cdot A := gAg^{-1}$. The following description of its ring of invariants is well-known. We briefly recall the standard proof which immediately suggests a connection to our treatment of reflection groups. For $k \geq 1$ and $A \in \mathfrak{sl}_n$ we define $s_k(A) := \mathrm{tr}(A^k)$.

Theorem 6.18 ([35, Ch. 12.5.3]). *For $G = \mathrm{SL}_n$ we have*

$$\mathbb{R}[\mathfrak{sl}_n]^G \;=\; \mathbb{R}[s_2, \ldots, s_n].$$

Moreover, the generators $s_2, \ldots, s_n$ are algebraically independent.

Proof. We write $D \subset \mathfrak{sl}_n$ for the set of diagonalizable matrices and we denote by $\lambda(A) = (\lambda_1, \ldots, \lambda_n)$ the eigenvalues of $A \in D$. Then for any $A \in D$

$$s_k(A) \;=\; s_k(\lambda(A)) \;=\; \lambda_1(A)^k + \lambda_2(A)^k + \cdots + \lambda_n(A)^k$$

and $s_2, \ldots, s_n$ are simply the power sums restricted to the linear subspace $\Delta \subset D$ of diagonal matrices. This shows that $s_2, \ldots, s_n$ are algebraically independent. Now for a polynomial $f(\mathbf{X}) \in \mathbb{R}[\mathfrak{sl}_n]$ invariant under the action of SL_n, the restriction to $\Delta \cong \mathbb{R}^{n-1}$ is a polynomial $f(\mathbf{x})$ that is invariant under A_{n-1}. Hence $f(\mathbf{x}) = F(s_2(\mathbf{x}), \ldots, s_n(\mathbf{x}))$ for some $F \in \mathbb{R}[y_2, \ldots, y_n]$. The polynomial $\tilde{f}(\mathbf{X}) = F(s_2(\mathbf{X}), \ldots, s_n(\mathbf{X}))$ is invariant under SL_n and agrees with f on D. Since D contains a nonempty open set, $f = \tilde{f}$ as required. $\square$

For the special orthogonal group SO_n, its Lie algebra $\mathfrak{so}_n \subset \mathfrak{sl}_n$ is the vector space of skew-symmetric n-by-n matrices on which SO_n acts by conjugation. If $n = 2k+1$, then the corresponding Weyl group is B_k and D_k if $n = 2k$. Hence $\mathbb{R}[\mathfrak{so}_{2k+1}]^{\mathrm{SO}_{2k+1}}$ is generated by $s_2(\mathbf{X}), s_4(\mathbf{X}), \ldots, s_{2k}(\mathbf{X})$. For $n = 2k$, a minimal generating set is given by $s_2(\mathbf{X}), s_4(\mathbf{X})$, ..., $s_{2n-2}(\mathbf{X})$ and the Pfaffian $\mathrm{pf}(\mathbf{X}) := \sqrt{\det \mathbf{X}}$.

Analogous to the case of reflection groups, we call a G-invariant variety $X \subseteq \mathfrak{g}$ ***k*-sparse** if $X = \mathrm{V}_{\mathbb{R}}(f_1, \ldots, f_m)$ for some $f_1, \ldots, f_m \in \mathbb{R}[\pi_1, \ldots, \pi_k]$ The **discriminant locus** $\mathcal{D} \subset \mathfrak{g}$ of the Lie group G are the points with nontrivial stabilizer. This is exactly the orbit of the corresponding reflection arrangement $\mathcal{H}_{n-1} \subset \mathfrak{g}$ under G. Together with Theorem 6.18 and the result by Steinberg [76], this is a real G-invariant hypersurface. This yields a stratification of $\mathfrak{g}$ by defining $\mathcal{D}_i$ to be the orbit of $\mathcal{H}_i$, which corresponds to the points for which the discriminant vanishes up to order $n-i$. Hence, the results from the previous sections generalize to Lie groups. Theorem 4.7 yields the following.

Theorem 6.19. *Let $G \in \{\mathrm{SL}_n, \mathrm{SO}_n : n \in \mathbb{N}_{>0}\}$ and let $X \subseteq \mathfrak{g}$ be G-invariant and k-sparse. If X is nonempty, it intersects $\mathcal{D}_k$.*

For suitable Lie groups G, Theorem 6.19 gives a first relation between real varieties invariant under the action of G and the discriminant locus and Conjecture 4.9 is reasonable for this setting. It would be very interesting to explore this connection further.

6.4 Outlook: The complex case

In this final section, we will change the underlying field and work over the complex numbers $\mathbb{C}$. Let V be a finite-dimensional complex vector space and denote by $\mathrm{U}(V)$

the group of unitary transformations. A **complex reflection** on V is a nontrivial unitary transformation which fixes a complex hyperplane pointwise. Analogously to the real case, a finite subgroup of $\mathrm{U}(V)$ is called **complex reflection group** if it is generated by complex reflections. As before, we denote by $\mathcal{H} = \mathcal{H}(G)$ the **reflection arrangement**, that is the set of all complex hyperplanes which are fixed by some reflection in G. Again, this arrangement induces a stratification of V with strata $\mathcal{H}_i(G)$ being the union of all flats of dimension i in $\mathcal{H}$.

Note that many features of complex reflection groups differ from the real case. For any hyperplane $H \in \mathcal{H}$, the orthogonal space $H^\perp$ is isomorphic to $\mathbb{C}$. Hence, a complex reflection is not necessary of order two, but can for instance also rotate the complex plane $H^\perp$. Also, since complex hyperplanes can be thought of subspaces of real codimension two, $\mathcal{H}_{n-1}$ does not divide V into distinct connected regions.

Every linear transformation of $\mathbb{R}^n$ gives rise to a corresponding transformation on the complex space $\mathbb{C}^n$. In particular, every real reflection group G acting on $\mathbb{R}^n$ trivially yields a corresponding **complexified real reflection group**. Of course, not every complex reflection group is of this form. A complete classification of irreducible complex reflection groups was given by Shephard and Todd [70]: There exists a three-parameter infinite family $G(m,p,n)$ and, additionally, 34 exceptional groups. They moreover prove that Chevalley's theorem (Theorem 4.4) also holds in the complex case. That is, for every complex reflection group G acting on $\mathbb{C}^n$, there exist n algebraically independent **basic invariants** $\pi_1, \ldots, \pi_n$ such that $\mathbb{C}[\mathbf{x}]^G = \mathbb{C}[\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x})]$ an their degrees are uniquely determined up to permutation. We can therefore also define k**-sparse** varieties as in the real case and the following question is only natural.

Question 6.20. *Let G be an essential irreducible complex reflection group acting on V and let $X \subseteq V$ be a nonempty k-sparse variety. Is it always true that $X \cap \mathcal{H}_k \neq \emptyset$?*

Note that in the real case, we could, for convenience always assume that the varieties were defined by only a single polynomial. Over the complex numbers this is not the case. For any polynomial $f \in \mathbb{C}[\mathbf{x}]$, its zero set $V(f) \subseteq V$ is a variety of complex codimension 1. In particular, it follows easily that if G is a reflection group whose arrangement $\mathcal{H}(G)$ consists of N hyperplanes, then for any, not necessarily invariant non-constant polynomial $f \in \mathbb{C}[\mathbf{x}]$ of degree at most $N-1$, we have $V(f) \cap \mathcal{H}_{n-1} \neq \emptyset$.

For a choice of basic invariants $\pi_1, \ldots, \pi_n$, we consider again the **orbit map** $\pi : V \to \mathbb{R}^n$ defined by $\pi(\mathbf{x}) := (\pi_1(\mathbf{x}), \ldots, \pi_1(\mathbf{x}))$ (cf. Section 6.1.1). In contrast to the real case, the fundamental theorem of algebra yields that π is surjective. The image of $\mathcal{H}_{n-1}$ under π is a complex hypersurface, which is easy to describe: For every reflection hyperplane $H \in \mathcal{H}$, consider a linear functional ℓ_H with kernel H and denote by e_H the order of the cyclic subgroup fixing H. For example, in the case of complexified real reflection groups, we have $e_H = 2$ for all H. The polynomial

$$\tilde{\Delta}(\mathbf{x}) \;:=\; \prod_{H \in \mathcal{H}} \ell_H(\mathbf{x})^{e_H}$$

is G-invariant and hence of the form

$$\tilde{\Delta}(\mathbf{x}) \;=\; \Delta(\pi_1(\mathbf{x}), \ldots, \pi_n(\mathbf{x}))$$

for a unique polynomial Δ, called the **discriminant** of G with respect to $\pi_1, \ldots, \pi_n$. The **discriminant locus** $V(\Delta) = \pi(\mathcal{H}_{n-1})$ inherits a stratification from $\mathcal{H}$, with algebraic strata $\mathcal{S}_i := \pi(\mathcal{H}_i)$, whose defining equations are explicitly given in [64]. The following shows that, in the complex case, Question 6.20 can be translated into a simple question regarding the geometry of the discriminant locus and its stratification.

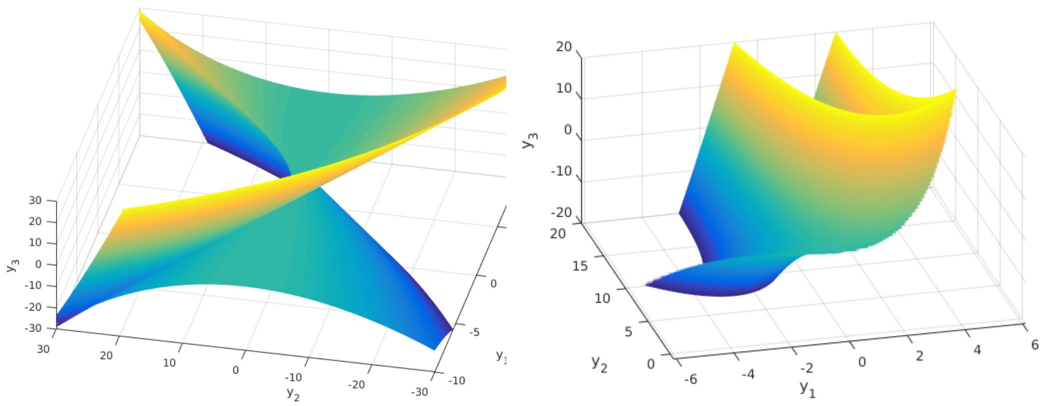

Figure 6.2: The real part of the discriminant locus for the non-essential group $\mathfrak{S}_3$ with respect to elementary symmetric polynomials and power sums, respectively.

Lemma 6.21. *Every non-empty k-sparse variety meets $\mathcal{H}_k$ if and only if for every $\mathbf{p} \in \mathbb{C}^n$, the affine space $L_k(\mathbf{q}) := \{\mathbf{q} \in \mathbb{C}^n : q_i = p_i \text{ for } i \in [k]\}$ intersects $\mathcal{S}_k$.*

Proof. It suffices to consider principal invariant varieties of the form

$$X_k(\mathbf{p}) \;=\; \{\mathbf{q} \in V : \pi_i(\mathbf{q}) = \pi_i(\mathbf{p}) \text{ for } i \in [k]\}$$

for $\mathbf{p} \in V$. But we have $\pi(X_k(\mathbf{p})) = L_k(\pi(\mathbf{p}))$, which finishes the proof. $\square$

Using a deep result on discriminants given in [9], we can give a partial answer to Question 6.20 for the case $k = n-1$. A complex reflection group G is **well-generated** if it can be generated by $\mathrm{rk}(G)$ reflections. By the considerations in Section 4.1, complexified real groups are well-generated.

Theorem 6.22. *Let G be a well-generated, essential and irreducible complex reflection group of rank $n \geq 1$. Then every non-empty $(n-1)$-sparse G-invariant variety intersects $\mathcal{H}_{n-1}$.*

Proof. By [9, Thm 2.4], for any choice $\pi_1, \ldots, \pi_n$, the discriminant $\Delta \in \mathbb{C}[y_1, \ldots, y_n]$ of G is monic in the last variable, that is, up to multiplication by a nonzero constant we have

$$\Delta \;=\; y_n^n + g_{n-1} y_n^{n-1} + \cdots + g_1 y_n + g_0,$$

where $g_i \in \mathbb{C}[y_1, \ldots, y_{n-1}]$ for $0 \leq i \leq n-1$. Hence, for every $\mathbf{q} \in \mathbb{C}^n$, the restriction $\Delta|_{L_{n-1}(\mathbf{q})}$ is a univariate non-constant polynomial, which, by the fundamental theorem of algebra, yields a root $\mathbf{p} \in V(\Delta) \cap L_{n-1}(\mathbf{q})$. Lemma 6.21 finishes the proof. $\square$

Theorem 6.22 is only a first step and for most cases, Question 6.20 remains open. Surprisingly, even for the symmetric group, the answer is not known and all existing proofs for the real case employ techniques which do not work over the complex numbers.

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Zusammenfassung

Richard Stanley [73] assoziierte 1986 zu einer gegebenen endlichen *partiell geordneten Menge* zwei geometrische Objekte, das *Ordnungs-* und das *Kettenpolytop*, deren Geometrie die Kombinatorik der zugrunde liegenden partiellen Ordnung widerspiegelt. Im ersten Teil dieser Dissertation wird eine ähnliche Theorie für *Doppel-Posets*, für endliche Mengen mit zwei Ordnungsstrukturen (nach Malvenuto und Reutenauer [60]), entwickelt. Wir assoziieren zu jedem Doppel-Poset $\mathbf{P}$ ein *Doppel-Ordnungspolytop* $\mathbb{T}\mathcal{O}(\mathbf{P})$ und ein *Doppel-Kettenpolytop* $\mathbb{T}\mathcal{C}(\mathbf{P})$.

Kapitel 1 behandelt Doppel-Ordnungspolytope. Wir zeigen, dass im Fall von *kompatiblen* Doppel-Posets die Facetten von $\mathbb{T}\mathcal{O}(\mathbf{P})$ genau *alternierenden Ketten* in $\mathbf{P}$ entsprechen. Des Weiteren charakterisieren wir die 2-*level-Polytope* der Form $\mathbb{T}\mathcal{O}(\mathbf{P})$ und wir etablieren eine Verbindung zu Geissingers *Bewertungs-Polytopen.* In Kapitel 2 betrachten wir die *torischen Ideale* von $\mathbb{T}\mathcal{O}(\mathbf{P})$. Für kompatible Doppel-Posets finden wir eine quadratische *Gröbnerbasis* und eine entsprechende *unimodulare reguläre Triangulierung* von $\mathbb{T}\mathcal{O}(\mathbf{P})$, sowie eine Beschreibung der kompletten Seitenflächen-Struktur. Kapitel 3 behandelt $\mathbb{T}\mathcal{C}(\mathbf{P})$. Wir arbeiten erst in der größeren Klasse von *Cayley-Summen* von *Anti-blocking-Polytopen* und beschreiben die Facetten und eine kanonische Unterteilung. Für den Spezialfall von $\mathbb{T}\mathcal{C}(\mathbf{P})$ erhalten wir eine unimodulare Triangulierung, eine kombinatorische Interpretation des Volumens und, allgemeiner, des *Ehrhart-Polynoms.* Für kompatibles $\mathbf{P}$ definieren wir eine *Transfer-Abbildung*, die die Triangulierungen und Ehrhart-Polynome von $\mathbb{T}\mathcal{O}(\mathbf{P})$ und $\mathbb{T}\mathcal{C}(\mathbf{P})$ verbindet.

Die zentralen Objekte im zweiten Teil sind *reelle Varietäten*, die invariant unter der Operation einer *endlichen reellen Spiegelungsgruppe* sind. Für den Spezialfall der symmetrischen Gruppe besagt Timoftes Grad-Prinzip [79], dass jede nicht-leere Varietät, die mithilfe der ersten k elementarsymmetrischen Polynome definiert werden kann, einen k-dimensionalen Unterraum des dazugehörigen Hyperebenen-Arrangements schneidet. Unser Ziel ist, dieses Ergebnis auf beliebige Spiegelungsgruppen zu verallgemeinern.

In Kapitel 5 behandeln wir die unendlichen Familien A_n, B_n und D_n. Wir zeigen in jedem der Fälle, dass jede nicht-leere Varietät, die von den ersten k nach Grad geordneten *basic invariants* definiert wird, einen k-dimensionalen Unterraum des assoziierten Arrangements schneidet und stellen die Vermutung auf, dass dies für alle irreduziblen Spiegelungsgruppen gilt. In Kapitel 6 beweisen wir die Vermutung im Fall $k = n-1$ für alle irreduziblen Spiegelungsgruppen und für beliebige k für H_3 und F_4. Zudem zeigen wir eine abgeschwächte Version der Vermutung. Wir stellen auch eine Verbindung zu *Lie-Gruppen* und deren invarianten Varietäten her und beweisen ein erstes Ergebnis für *komplexe* Spiegelungsgruppen.

Selbstständigkeitserklärung

Gemäß §7 (4) der Promotionsordnung des Fachbereichs Mathematik und Informatik der Freien Universität Berlin versichere ich hiermit, dass ich alle Hilfsmittel und Hilfen angegeben und auf dieser Grundlage die Arbeit selbstständig verfasst habe. Des Weiteren versichere ich, dass ich diese Arbeit nicht schon einmal zu einem früheren Promotionsverfahren eingereicht habe.

Berlin, den

Tobias Friedl