

# LOCAL SYSTEMS WITH QUASI-UNIPOTENT MONODROMY AT INFINITY ARE DENSE\*

BY

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*Dedicated to Moshe Jarden*

ABSTRACT

We show that complex local systems with quasi-unipotent monodromy at infinity over a normal complex variety are Zariski dense in their moduli.

## 1. Introduction

Let  $G$  be a linear algebraic group over the complex numbers. In this short note we study  $G$ -representations of the topological fundamental group  $\pi := \pi_1(X(\mathbb{C}), x)$  of a normal complex variety  $X$  which are quasi-unipotent with respect to the monodromy at infinity. As  $\pi$  is finitely generated, the set of

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group homomorphisms (called  $G$ -representations)  $\rho: \pi \rightarrow G(\mathbb{C})$  is in a canonical way the set of complex points of an affine complex variety  $\text{Ch}_{G,\mathbb{C}}^\square(\pi)$ , the so-called framed character variety; see Section 3.

Our main result is motivated by the following conjecture about the density of representations of geometric origin. We fix an embedding of linear algebraic groups

$$G \xhookrightarrow{\iota} \text{GL}_{r,\mathbb{C}}.$$

We say that a  $G$ -representation  $\rho: \pi \rightarrow G(\mathbb{C})$  is of geometric origin if there is a smooth projective morphism  $f: Y \rightarrow U$ , where  $j: U \hookrightarrow X$  is a dense open subvariety such that the semi-simplification of the representation

$$\iota \circ \rho \circ j_*: \pi_1(U(\mathbb{C})) \rightarrow \text{GL}_{r,\mathbb{C}}$$

gives rise to a linear local system which is a direct summand of the local system  $\bigoplus_i R^i f_* \mathbb{C}$  on  $U$ . As linear systems of geometric origin are compatible with tensor products, direct sums and duals, [Mil17, Thm. 4.14] implies that the notion of  $\rho$  being of geometric origin does not depend on the choice of the embedding  $\iota$ .

**CONJECTURE 1.1 (Density):**<sup>1</sup> *The set of  $G$ -representations of geometric origin is Zariski dense in  $\text{Ch}_{G,\mathbb{C}}^\square(\pi)$ .*

This density conjecture is in accordance with the one stated in [EK19, Qu. 9.1] and can be generalized to special loci, see Conjecture 5.3. It is not difficult to show that Conjecture 1.1 holds for  $G$  abelian. Indeed, then  $G$  is a product of a unipotent group, a torus and a finite group, so one only has to study the case that  $G$  is a torus. However, the torus case follows from [EK19, Thm. 1.2].

A complex analytic analog of the density conjecture involving the Riemann–Hilbert correspondence is formulated in [BW20, Conj. 10.4.1].

Another way to formulate Conjecture 1.1 is to say that the image of

$$\text{Ch}_{G,\mathbb{C}}^\square(\pi) \rightarrow \text{Ch}_{\text{GL}_{r,\mathbb{C}}}(\pi)$$

contains a dense set of points corresponding to semi-simple representations  $\pi \rightarrow \text{GL}_{r,\mathbb{C}}(\mathbb{C})$  of geometric origin. Here

$$\text{Ch}_{\text{GL}_{r,\mathbb{C}}}(\pi) = \text{Ch}_{\text{GL}_{r,\mathbb{C}}}^\square(\pi) // \text{GL}_{r,\mathbb{C}}$$

is the character variety.

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<sup>1</sup> Aaron Landesman and Daniel Litt [LL22] just made available a preprint showing that there is a lower bound for the rank of geometric local systems with infinite monodromy on certain curves, and consequently the conjecture can not be true in this generality.

We say that  $\rho$  has **quasi-unipotent monodromy at infinity** if for one (equivalently for all, see Proposition 3.1) normal compactification  $X \subset \overline{X}$  the eigenvalues of  $\iota \circ \rho(T_D)$  are roots of unity. Here  $D \hookrightarrow \overline{X} \setminus X$  runs over the irreducible components which are of codimension one in  $\overline{X}$  and  $T_D$  is the canonical conjugacy class  $T_D \subset \pi$  corresponding to a “small loop around  $D$ ”; see Section 2.

**THEOREM 1.2 (Monodromy theorem):** *A  $G$ -representation  $\rho: \pi \rightarrow G(\mathbb{C})$  which is of geometric origin has quasi-unipotent monodromy at infinity.*

The monodromy theorem is due to Clemens and Landman; see [Gri70, Thm. 3.1]. Proofs which are based on the study of local systems were given by Brieskorn [Del70, III,2] and Grothendieck [SGA7.1, Thm 1.2]. The proof of our main result, Theorem 1.3 below, is motivated by Grothendieck’s proof of the monodromy theorem. In fact, in view of the monodromy theorem it can also be seen as a tiny bit of evidence for the density conjecture.

**THEOREM 1.3 (Theorem 3.2):** *The set of  $G$ -representations which have quasi-unipotent monodromy at infinity is Zariski dense in  $\text{Ch}_{G,\mathbb{C}}^{\square}(\pi)$ .*

After we lectured on Theorem 1.3, an alternative proof for  $G = \text{GL}_{r,\mathbb{C}}$  based on the Riemann–Hilbert correspondence and the Gelfond–Schneider theorem was given by B. Bakker and Y. Brunebarbe. Independently a similar density theorem involving the Riemann–Hilbert correspondence was obtained by Budur, Lerer and Wang [BLW21, Thm. 1.2].

Our proof of Theorem 1.3 is based on the action of an arithmetic Galois group on certain completions of the character variety. This action is induced by a comparison of the topological fundamental group with the étale fundamental group. On the monodromy at infinity the Galois action is given in terms of the cyclotomic character (see Lemma 2.1).

In Section 2 we recall some properties of fundamental groups. In Section 3 we introduce  $G$ -representations of the fundamental group of a variety with quasi-unipotent monodromy at infinity and formulate our main theorem. The proof of the main theorem is contained in Section 4. In the final Section 5 we explain how our proof can be applied more generally to certain special loci in  $\text{Ch}_{\text{GL}_{r,\mathbb{C}}}^{\square}(\pi)$  and how the density conjecture relates to the Fontaine–Mazur conjecture.

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**2. The monodromy at infinity**

THE SETUP. The proof of our main result relies on arithmetic Galois groups, so we introduce a setting in which we can later apply arithmetic arguments, even though the formulation of Theorem 3.2 is purely complex. Let  $F \subset \mathbb{C}$  be a finitely generated field. Let  $X_0$  be a normal, geometrically irreducible variety over  $F$  and let  $\overline{X}_0$  be a normal compactification of  $X_0$ . Let  $X \subset \overline{X}$  be the base change of these varieties to  $\mathbb{C}$ . We assume that there exists a rational point  $x_0 \in X_0(F)$  and let  $x$  be the associated complex point. We consider the following objects:

- $\overline{F}$  is the algebraic closure of  $F$  in  $\mathbb{C}$ ,  $\Gamma = \text{Gal}(\overline{F}/F)$  the Galois group of  $F$ .
- $\pi = \pi_1(X(\mathbb{C}), x)$  is the topological fundamental group of  $X(\mathbb{C})$  based at  $x$ .
- $\pi^{\text{ét}} = \pi_1^{\text{ét}}(X, x) \xrightarrow{\sim} \pi_1^{\text{ét}}(X_{0, \overline{F}}, x)$  is the geometric étale fundamental group of  $X$  based at  $x$ , which by the Riemann existence theorem [SGA1, Cor. XII.5.2] can be identified with the pro-finite completion of  $\pi$ .
- $\pi_1^{\text{ét}}(X_0, x)$  is the arithmetic fundamental group of  $X_0$  based at  $x$ .
- The conjugation action induced by the splitting of the homotopy exact sequence [SGA1, Thm. IX.6.1]

$$1 \longrightarrow \pi^{\text{ét}} \longrightarrow \pi_1^{\text{ét}}(X_0, x) \longrightarrow \Gamma \longrightarrow 1$$

given by the point  $\text{Spec}(F) \xrightarrow{x_0} X \rightarrow \text{Spec}(F)$  defines an action of  $\Gamma$  on  $\pi^{\text{ét}}$ .

- The family  $D_i \hookrightarrow \overline{X}_0$  ( $1 \leq i \leq s$ ) of irreducible components of  $\overline{X}_0 \setminus X_0$  which are of codimension one in  $\overline{X}_0$ . We assume that all  $D_i$  are geometrically irreducible. By abuse of notation we denote the base change of  $D_i$  to  $\mathbb{C}$  by the same symbol.

COMPLEX MONODROMY. To each  $D_i$  one associates a canonical conjugacy class  $T_i \subset \pi$  as follows. Consider the dense open subvariety

$$\overline{X}^\circ = \overline{X} \setminus (\overline{X}^{\text{sing}} \cup (\overline{X} \setminus X)^{\text{sing}})$$

of  $\overline{X}$ . Set  $D_i^\circ = D_i \cap \overline{X}^\circ$  and  $X^\circ = \overline{X}^\circ \cap X$ . Let us assume  $x \in X^\circ(\mathbb{C})$ . Then a “small loop” around  $D_i^\circ(\mathbb{C}) \hookrightarrow X^\circ(\mathbb{C})$  defines a canonical conjugation class  $T_i^\circ$  in  $\pi_1(X^\circ(\mathbb{C}), x)$ ; see [Kas81, 1.4]. We define  $T_i$  to be the image of  $T_i^\circ$  via the surjective homomorphism  $\pi_1(X^\circ(\mathbb{C}), x) \rightarrow \pi_1(X(\mathbb{C}), x)$ .

ÉTALE MONODROMY. We denote by  $T_i^{\text{ét}} \subset \pi^{\text{ét}}$  the conjugacy class induced by the image of  $T_i \subset \pi$  in  $\pi^{\text{ét}}$ . It can be described purely algebraically in terms of ramification theory; see [SGA7.2, XIV.1.1.10] for an exposition in the one-dimensional case. This implies the following well-known lemma; see also [EK22, Claim 7.1].

LEMMA 2.1: *For each  $1 \leq i \leq s$  the action of  $\gamma \in \Gamma$  on  $\pi^{\text{ét}}$  maps  $T_i^{\text{ét}}$  to  $(T_i^{\text{ét}})^{\chi(\gamma)}$ . Here  $\chi: \Gamma \rightarrow \widehat{\mathbb{Z}}^\times$  is the cyclotomic character.*

### 3. $G$ -representations and quasi-unipotent monodromy

QUASI-UNIPOTENT ELEMENTS AND  $G$ -REPRESENTATIONS. Let  $G/\mathbb{C}$  be a linear algebraic group. Recall that an element  $g \in G(\mathbb{C})$  is called quasi-unipotent, if for one (or equivalently for any) embedding of algebraic groups  $\iota: G \hookrightarrow \text{GL}_{r,\mathbb{C}}$  the eigenvalues of  $\iota(g)$  are roots of unity. Let  $\pi$  be a finitely generated group. A  **$G$ -representation** of  $\pi$  is a homomorphism  $\rho: \pi \rightarrow G(\mathbb{C})$ .

CHARACTER VARIETIES. Let  $R$  be a noetherian ring and let  $G$  be an affine group scheme of finite type  $R$ . There exists an affine scheme  $\text{Ch}_{G,R}^\square(\pi)$  of finite type over  $R$  such that for an  $R$ -algebra  $R'$  there is a functorial bijection

$$\text{Hom}(\pi, G(R')) \cong \text{Ch}_{G,R}^\square(\pi)(R').$$

The  $R$ -scheme  $\text{Ch}_{G,R}^\square(\pi)$  is called the **framed character variety**. If  $\pi$  has a presentation  $\langle w_1, \dots, w_\sigma \mid r_1, \dots, r_\tau \rangle$  then

$$(1) \quad \text{Ch}_{G,R}^\square(\pi) = \{g \in G^\sigma \mid r_1(g) = \dots = r_\tau(g) = 1\}.$$

QUASI-UNIPOTENT MONODROMY AT INFINITY. Let the notation be as in Section 2, in particular  $X$  is a normal complex variety,  $x \in X(\mathbb{C})$ ,  $\pi = \pi_1(X(\mathbb{C}), x)$  and  $X \subset \overline{X}$  is a normal compactification. Let  $G/\mathbb{C}$  be a linear algebraic group. We say that a  $G$ -representation  $\rho: \pi \rightarrow G(\mathbb{C})$  has a **quasi-unipotent monodromy at infinity** if for all  $1 \leq i \leq s$  the image of the monodromy  $\rho(T_i) \subset G(\mathbb{C})$  consists of quasi-unipotent elements. The following important theorem is shown in [Kas81, Thm. 3.1]. In fact Kashiwara’s result is about constructible sheaves and one can easily translate it into our setting of local systems.

PROPOSITION 3.1 (Kashiwara): *The property of  $\rho: \pi \rightarrow G(\mathbb{C})$  to have quasi-unipotent monodromy at infinity does not depend on the choice of the normal compactification  $\overline{X}$  of  $X$ .*

Our main theorem states:

THEOREM 3.2: *The set of representations  $\rho \in \text{Ch}_{G,\mathbb{C}}^\square(\pi)(\mathbb{C})$  with quasi-unipotent monodromy at infinity is Zariski dense in  $\text{Ch}_{G,\mathbb{C}}^\square(\pi)$ .*

In Theorem 5.2 we formulate a strengthening of Theorem 3.2 involving an arithmetic Galois action. Theorem 3.2 is shown in Section 4.

Remark 3.3: One can also show by the same technique that the set of representations  $\rho \in \text{Ch}_{G,\mathbb{C}}^\square(\pi)(\mathbb{C})$  with finite determinant and with quasi-unipotent monodromy at infinity is Zariski dense.

If  $G$  is reductive we can form the categorical quotient of  $\text{Ch}_{G,K}(\pi)$  with respect to the conjugation action of  $G$  to obtain the **character variety**

$$\text{Ch}_{G,\mathbb{C}}(\pi) = \text{Ch}_{G,\mathbb{C}}^\square(\pi) // G.$$

The points  $\text{Ch}_{G,K}(\pi)(\mathbb{C})$  correspond to isomorphism classes of completely reducible representations  $\rho: \pi \rightarrow G(\mathbb{C})$  [Sik12, Sec. 11]. Theorem 3.2 then immediately implies:

COROLLARY 3.4: *The set of isomorphism classes of completely reducible representations  $\pi \rightarrow G(\mathbb{C})$  with quasi-unipotent monodromy at infinity is Zariski dense in  $\text{Ch}_{G,\mathbb{C}}(\pi)$ .*

Example 3.5: For  $X = \mathbb{A}^1 \setminus (s \text{ points})$  the topological fundamental group  $\pi = \pi_1(X(\mathbb{C}), x)$  is a free group with  $s$  generators  $w_1, \dots, w_s$  (suitable loops around the  $s$  points based at a common point  $x \in X(\mathbb{C})$ ). The monodromy

at infinity for the canonical compactification  $X \subset \mathbb{P}^1$  consists of the conjugacy classes of  $w_1, \dots, w_s, (w_1 \cdots w_s)^{-1}$  which correspond to loops around the  $s$  points  $\mathbb{A}^1 \setminus X$  and the point  $\infty \in \mathbb{P}^1$ . In this case Theorem 3.2 says: The set of  $g = (g_1, \dots, g_s) \in G^s(\mathbb{C})$  such that  $g_1, \dots, g_s, g_1 \cdots g_s$  are quasi-unipotent is Zariski dense in  $G^s$ .

This example is related to [EK22, Thm. B] in the arithmetic situation.

**4. Proof of Theorem 3.2**

$\Gamma$ -ACTION AND  $\text{Ch}_G^\square(\pi)$ . We use the notation of Section 2, so  $X$  is the base change to  $\mathbb{C}$  of a variety  $X_0$  over a finitely generated field  $F \subset \mathbb{C}$ . Recall that  $G \hookrightarrow \text{GL}_{r,\mathbb{C}}$  is a linear algebraic group.

Let  $Q \subset \text{Ch}_{G,\mathbb{C}}^\square(\pi)$  be the Zariski closure of the set of quasi-unipotent representations  $\rho: \pi \rightarrow G(\mathbb{C})$ . We argue by contradiction and assume that  $Q \neq \text{Ch}_{G,\mathbb{C}}^\square(\pi)$ . In particular,  $\text{Ch}_{G,\mathbb{C}}^\square(\pi)$  is non-empty.

Choose a subring  $R \subset \mathbb{C}$  which is of finite type over  $\mathbb{Z}$ , such that  $G$  is induced by a group scheme  $\mathcal{G} \hookrightarrow \text{GL}_{r,R}$  over  $R$  and such that  $Q$  is induced by a closed subscheme  $\mathcal{Q}$  of  $\text{Ch}_{\mathcal{G},R}^\square(\pi)$ . Set  $\mathcal{W} = \text{Spec}(R)$  and let  $K \subset \mathbb{C}$  be the field of fractions of  $R$ .

For a scheme  $\mathcal{X}$  of finite type over  $R$ , let us denote by  $|\mathcal{X}|$  the set of closed points of  $\mathcal{X}$ . For  $x \in \mathcal{X}$  we let  $\mathcal{X}_x^\wedge$  be the local scheme  $\text{Spec}(\mathcal{O}_{\mathcal{X},x}^\wedge)$ , where  $\mathcal{O}_{\mathcal{X},x}^\wedge$  is the completed local ring.

The  $\Gamma$ -action on  $\pi^{\text{ét}}$  induces a continuous  $\Gamma$ -action on the discrete set of closed points  $|\text{Ch}_{\mathcal{G},R}^\square(\pi)|$ . Similarly, we get an induced  $\Gamma_x$ -action on  $\text{Ch}_{\mathcal{G},R}^\square(\pi)_x^\wedge$  for  $x \in |\text{Ch}_{\mathcal{G},R}^\square(\pi)|$  and for  $\Gamma_x \subset \Gamma$  the open stabilizer subgroup of  $x$ .

CHARACTERISTIC POLYNOMIAL OF MONODROMY. For each local monodromy at infinity  $T_i \subset \pi$ , choose  $g_i \in T_i$ . We have a morphism

$$\psi: \text{Ch}_{\mathcal{G},R}^\square \rightarrow \mathcal{N} = \prod_{i=1}^s (\mathbb{A}^{r-1} \times \mathbb{G}_m)$$

of affine schemes of finite type over  $R$  defined for each  $i = 1, \dots, s$  by the coefficients  $(\sigma_1(\rho(g_i)), \dots, \sigma_r(\rho(g_i))) \in \mathcal{N}(R')$  of the characteristic polynomials

$$\det(T \cdot \mathbb{I}_r - \rho(g_i)) = T^r - \sigma_1(\rho(g_i))T^{r-1} + \dots + (-1)^r \sigma_r(\rho(g_i))$$

of a  $G$ -representation  $\rho: \pi \rightarrow \mathcal{G}(R')$ , where  $R'$  is an  $R$ -algebra.

Furthermore, we have the finite flat morphism

$$\varphi: \mathcal{M} = (\mathbb{G}_m^r)^s \rightarrow \mathcal{N}$$

of affine schemes over  $R$  given by

$$\mathbb{G}_m^r \rightarrow \mathbb{A}^{r-1} \times \mathbb{G}_m, \quad (\mu_1, \dots, \mu_r) \mapsto (s_1(\mu_1, \dots, \mu_r), \dots, s_r(\mu_1, \dots, \mu_r)),$$

where  $s_i(\mu_1, \dots, \mu_r)$  is the  $i$ -th elementary symmetric function in the  $\mu_j$ .

The cyclotomic character  $\chi$  induces an action of  $\Gamma$  on  $|\mathcal{M}|$  and a compatible action on  $|\mathcal{N}|$  such that  $|\varphi|: |\mathcal{M}| \rightarrow |\mathcal{N}|$  is  $\Gamma$ -equivariant. For each point  $x \in |\mathcal{M}|$  the stabilizer  $\Gamma_x \subset \Gamma$  acts on  $\mathcal{M}_x^\wedge$  and on  $\mathcal{N}_x^\wedge$  such that  $\varphi_x^\wedge$  is  $\Gamma_x$ -equivariant.

CERTAIN CLOSED POINTS. Let  $\mathcal{T}$  be the reduced closure of the image of  $\psi$ . Let  $\mathcal{S}$  be  $\varphi^{-1}(\mathcal{T})_{\text{red}}$ . Note that the generic fibre  $\mathcal{S}_K$  of  $\mathcal{S}$  over  $\mathcal{W}$  is non-empty as  $\text{Ch}_{\mathcal{G}, \mathbb{C}}^\square(\pi)$  is non-empty, so the smooth locus  $\mathcal{S}^{\text{sm}}$  of  $\mathcal{S}$  over  $R$  is non-empty. By the generic flatness of  $\psi$  we can fix a closed point  $z \in \text{Ch}_{\mathcal{G}, R}^\square(\pi) \setminus \mathcal{Q}$  such that

- $\psi$  is flat at  $z$ ,
- $y = \psi(z) \in \varphi(\mathcal{S}^{\text{sm}})$ .

We also fix a closed point  $x \in \mathcal{S}^{\text{sm}} \cap \varphi^{-1}(y)$ . Let  $\Gamma'$  be the intersection of stabilizers  $\Gamma_x \cap \Gamma_z$ , which is thus open in  $\Gamma$ , and let  $w \in \mathcal{W} = \text{Spec}(R)$  be the image of the points  $x, y, z$ .

CLAIM 4.1: *The closed subscheme  $\mathcal{S}_x^\wedge \hookrightarrow \mathcal{M}_x^\wedge$  is  $\Gamma'$ -stable.*

*Proof.* As  $\psi$  is flat at the point  $z$  the closed subscheme  $\mathcal{T}_y^\wedge \hookrightarrow \mathcal{N}_y^\wedge$  is the schematic image of  $\psi_x^\wedge: \text{Ch}_{\mathcal{G}, R}^\square(\pi)_z^\wedge \rightarrow \mathcal{N}_y^\wedge$ . As the latter morphism is  $\Gamma'$ -equivariant, it follows that  $\mathcal{T}_y^\wedge$  is stabilized by  $\Gamma'$ . As  $\mathcal{S}_x^\wedge = \varphi^{-1}(\mathcal{T}_y^\wedge)_{\text{red}}$  we deduce that  $\mathcal{S}_x^\wedge$  is stabilized by  $\Gamma'$ . ■

DE JONG'S TRICK. For simplicity of notation we can assume that  $\Gamma = \Gamma'$ . Choose a normal integral ring  $A$  of finite type over  $\mathbb{Z}$  with field of fractions  $F$  such that the characteristic of the residue field  $k(x)$  of  $x$  is invertible in  $A$ . Then the action of  $\Gamma$  on  $\mathcal{M}_x^\wedge$  via the cyclotomic character  $\chi$  factors through  $\pi_1^{\text{ét}}(\text{Spec}(A))$  and for an  $\mathbb{F}_q$ -point  $a: \text{Spec}(\mathbb{F}_q) \rightarrow \text{Spec}(A)$  the associated Frobenius  $\text{Fr} = \text{Fr}_a \in \pi_1^{\text{ét}}(\text{Spec}(A))$ , which is well-defined up to conjugation, acts by multiplication by  $q$  on the group scheme  $\mathcal{M}$  and on  $\mathcal{M}_x^\wedge$ .



CLAIM 4.2 (De Jong’s trick): *The morphism of local schemes*

$$(\mathcal{S}_x^\wedge)^{\text{Fr}} \rightarrow \mathcal{W}_w^\wedge$$

*is finite, flat and surjective.*

*Proof.* The following argument is copied from [deJ01, 3.14]; see also [Dri01, Lem. 2.8], [EK19, Sec. 10] and [EK22, Sec. 8]. We can assume without loss of generality that  $k(x) = k(w)$ . By smoothness of  $\mathcal{S}/\mathcal{W}$  at  $x$

$$\mathcal{S}_x^\wedge \cong \text{Spec}(\mathcal{O}_{\mathcal{W},w}^\wedge \llbracket X_1, \dots, X_j \rrbracket).$$

Then

$$(\mathcal{S}_x^\wedge)^{\text{Fr}} \cong \text{Spec}(\mathcal{O}_{\mathcal{W},w}^\wedge \llbracket X_1, \dots, X_j \rrbracket / (1 - \text{Fr}(X_1), \dots, 1 - \text{Fr}(X_j)))$$

has fibre dimension zero over  $w$  as this fibre is a closed subscheme of the  $(q-1)$ -torsion subscheme of the torus  $\mathcal{M}_w$  over  $w$ . As in [deJ01, 3.14] basic commutative algebra shows that  $(\mathcal{S}_x^\wedge)^{\text{Fr}}$  is a local complete intersection, finite and flat over  $\mathcal{W}_w^\wedge$ . ■

CONCLUSION. By Claim 4.2 there exists a point  $\tilde{x} \in \mathcal{M}$  which is  $(q-1)$ -torsion, which maps to the generic point of  $\mathcal{W}$  and which specializes to  $x$ . In fact any point  $\tilde{x}$  in the image of the non-empty set  $(\mathcal{S}_x^\wedge)^{\text{Fr}}$  satisfies these properties. Then  $\tilde{y} = \varphi(\tilde{x})$  specializes to  $y$ . By going-down for flat morphisms there exists a point  $\tilde{z} \in \text{Ch}_{\mathcal{G},R}^\square(\pi)$  with  $\varphi(\tilde{z}) = \tilde{y}$  which specializes to  $z$ . By construction  $\tilde{z}$  corresponds to a representation of  $\pi$  which has quasi-unipotent monodromy at infinity, so  $\tilde{z} \in \mathcal{Q}$  and therefore  $z \in \mathcal{Q}$ . Contradiction!

### 5. Special loci

The aim of this section is to extend Conjecture 5.3 and Theorem 3.2 to certain subloci of the character variety  $\text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$ . We also relate our density conjectures to other classical conjectures. We use the notation of Sections 2, 3 and 4. Let  $\wp: \text{Ch}_{\text{GL}_r, \mathbb{C}}^\square(\pi) \rightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$  be the canonical quotient map. For a locally closed subscheme  $Z \hookrightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$ , we denote by  $\mathcal{Z} \hookrightarrow \text{Ch}_{\text{GL}_r, R}^\square(\pi)$  a suitable locally closed subscheme such that  $\wp(\mathcal{Z}_{\mathbb{C}}) \subset Z$  is dense. Here  $R \subset \mathbb{C}$  is a suitable subring of finite type over  $\mathbb{Z}$  as above.

*Definition 5.1:* A subscheme  $Z \hookrightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$  as above is **special** or **arithmetic** if there exist  $R$  and  $\mathcal{Z}$  as above such that for each closed point  $z \in \mathcal{Z}$ , there is an open subgroup of  $\Gamma$  which stabilizes the completion  $\mathcal{Z}_z^\wedge$ . A point in  $s \in \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)(\mathbb{C})$  is **special** or **arithmetic** if the subscheme  $Z = \{s\}$  is special.

**THEOREM 5.2:** *For a special subscheme  $Z \hookrightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$  the set of quasi-unipotent points in  $Z$  is Zariski dense in  $Z$ .*

*Proof.* The proof is analogous to that of Theorem 3.2. One just replaces  $\text{Ch}_{G, \mathbb{C}}^\square(\pi)$  by  $\wp^{-1}(Z)$  and  $Q$  by  $\wp^{-1}(Z) \cap Q$ . ■

Here is another natural density conjecture in this context:

**CONJECTURE 5.3 (Density):**<sup>2</sup> *Let  $Z \hookrightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$  be a special subscheme. Then the set of complex points of  $Z$  corresponding to representations of geometric origin  $\rho: \pi \rightarrow \text{GL}_r(\mathbb{C})$  is dense.*

*In particular, the special points are then dense on  $Z$ .*

Note that a point of  $\text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)(\mathbb{C})$  which corresponds to a representation of geometric origin is special by the comparison isomorphism between Betti cohomology and  $\ell$ -adic cohomology.

The following observations are easy to check.

*Remark 5.4:*

- (1) Conjecture 5.3  $\Rightarrow$  Conjecture 1.1. Indeed for  $G \xrightarrow{\ell} \text{GL}_{r, \mathbb{C}}$  given, the image of  $\text{Ch}_{G, \mathbb{C}}^\square(\pi) \rightarrow \text{Ch}_{\text{GL}_r, \mathbb{C}}(\pi)$  is constructible and we take  $Z$  to be a suitable dense subscheme in this image.
- (2) Conjecture 5.3 comprises the density conjecture formulated in [EK19, Qu. 9.1] (1), (2), except (3). For (1) and (2) this is by definition.
- (3) Conjecture 5.3 for  $\dim(Z) = 0$  implies Simpson’s “rigid  $\Rightarrow$  motivic” conjecture [Sim90, Conj. 4] as rigid representations are arithmetic [Sim92, Thm. 4].
- (4) Recent work of Petrov [Pet20] implies that the relative Fontaine–Mazur conjecture of Liu–Zhu [LZ17] implies Conjecture 5.3 for  $\dim(Z) = 0$ .

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<sup>2</sup> See the footnote to Conjecture 1.1.

In fact (4) is a direct consequence of [Pet20, Lem. 6.2] over a number field  $F$ . For a general finitely generated field  $F \subset \mathbb{C}$  one has to use a spreading argument similar to [Pet20, Prop. 6.1] in order to reduce to the number field case.

So (3) and (4) of Remark 5.4 together say that

relative F-M conj.  $\Rightarrow$  Simpson's "rigid  $\Rightarrow$  motivic" conj.

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