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Estimating Fock-state linear optics evolution using coherent states

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ABSTRACT

This paper presents two methods for simulating the interference of bosonic Fock states through linear interferometers using coherent states. The first method repeats the interferometer, injects coherent states in particular modes, and uses symmetric combinations of the outputs to reconstruct the state amplitudes of the Fock-state interference. The second method constructs a new interferometer that can be probed with coherent states on individual inputs to extract the required state amplitudes. The two approaches here show explicitly where the classical computational difficultly arises. In the first approach, the computational hardness is in the measurement post-processing, and in the second approach, it is within the construction of the required state evolution.

I. INTRODUCTION

There is often quite some confusion about the quantum nature of states in quantum optical experiments. This is undoubtedly due to the different paths of research that have been taken for quantum optics[1,2] and quantum information theory.[3] In quantum information theory, the focus is entirely on the ability to perform a task (or not), under specific complexity requirements, using classical or quantum resources.[4] In quantum optics, the desire is the ability to describe a state and subsequent measurement outcomes using classical probability distributions.

As this prescription is somewhat vague, there are a number of different constructions which meet these criteria. For example, in quantum information theory, we could describe a quantum computation using a particular choice of basis, complementary bases with special properties,[5,6] or more advanced "classical shadow" techniques.[7] In quantum optics, for example, one could use a P-function,[8,9] Q-function,[10] Wigner-function,[11] or a semi-classical technique.[12] If a quantum system can be described by these methods, then one declares the process "classical." However, these two definitions are definitely *not* equivalent.[13] One can construct an example (admittedly using mixed states) where a quantum computational speed-up is available with a system that would be classed as completely classical in quantum optics.[14,15]

Here, we show that a similar situation can exist with coherent states. Coherent states are a class of states which by quantum optics standards would be called classical. Coherent states, as a theoretical construct, are useful because they obey many desirable properties, particularly under passive linear optical evolution. Coherent states, in the world of optics, display no "quantum-interference" under linear optical transformations. Furthermore, the statistics of coherent states, under measurement of quanta of energy, obeys those statistics of random and independently generated quanta (i.e., a Possionan distribution). The action of linear optics evolution is such that single quanta creation operators evolve into other single quanta creation operators; hence, coherent states with linear optics is considered at most a "one particle" or "one photon" experiment. The state amplitudes of single particle states follow the wave amplitudes of a classical wave under the linear evolution, hence the conclusion that such experiments can only be classical. However, coherent states *are* quantum states and measurement in a Fock basis *is* a quantum measurement.

Here, we will argue that by careful construction of optical experiments, coherent state sources, linear optics, and photon detection can reconstruct the statistics of any multi-photon experiment. The caveat here is that the information required to perform this construction is necessarily computationally inefficient to extract from the description of the experiment being simulated. This is consistent with what is known about the complexity of multi-photon linear optics experiments.[16] An intriguing note of the constructions presented here is that a choice can be made as to where the most complex part lies. Either there is an exponential growth in the run-time of computing the linear optical network required to perform the simulation, or there is an exponential growth in the post-processing required to perform an analysis of the detection outcomes. Hopefully, the theory outlined in this paper can be of utility when considering the role of using coherent state sources in quantum optics experiments and the limitations on their uses.

The results in this paper will be presented as follows: In Sec. II, the standard approach of reconstructing single-photon interference using coherent states will be given and the expressions used later in the paper will be defined. Section III will consider the extension of the single-photon interference to multiple photon interference in two modes using coherent states and the extraction of the state amplitudes. Section IV will consider the generalization of the two-mode system to any number of modes. Section V will then present an alternative approach where the extraction of amplitudes is efficient, but the difficulty of the simulation is moved entirely into the construction of the interferometer. Section VI will discuss some implications of these methods and the effects of practical imperfections. Finally, in Sec. VII, the conclusions will be presented.

II. SINGLE-PHOTON INTERFERENCE

Any single-photon Fock state in m-modes can be written as

$$\sum_{i=1}^{m} c_i |\overbrace{\underbrace{0,0,\ldots,0,1}_{i},0,\ldots,0}^{m}\rangle = \sum_{i=1}^{m} c_i \hat{a}_i^\dagger |\overbrace{0,0,\ldots,0,0}^{m}\rangle, \tag{1}$$

with the single particle mode occupation number being represented by a single non-negative integer i and $\hat{a}_i^\dagger$ representing the creation operator on that mode. This has m complex parameters represented by the coefficients c_i. Requiring the state to be normalized makes the set of states into a projective linear space, which we will denote as V. One can think of this as a m-dimensional complex space with one restriction on the degrees of freedom.

The evolution of the photon state from Eq. (1) under a linear optical transformation $\hat{\mathcal{U}}$ is

$$\hat{\mathcal{U}} \sum_{i=1}^{m} c_i \hat{a}_i^\dagger |0,\ldots,0\rangle = \sum_{i=1}^{m} c_i \hat{\mathcal{U}} \hat{a}_i^\dagger \hat{\mathcal{U}}^\dagger \hat{\mathcal{U}} |0,\ldots,0\rangle \tag{2}$$

$$= \sum_{i=1}^{m} c_i \hat{\mathcal{U}} \hat{a}_i^\dagger \hat{\mathcal{U}}^\dagger |0,\ldots,0\rangle \tag{3}$$

$$= \sum_{i=1}^{m} c_i \sum_{j=1}^{m} U_{ij} \hat{a}_j^\dagger |0,\ldots,0\rangle \tag{4}$$

$$= \sum_{j=1}^{m} \left(\sum_{i=1}^{m} c_i U_{ij} \right) \hat{a}_j^\dagger |0,\ldots,0\rangle \tag{5}$$

$$= \sum_{j=1}^{m} d_j \hat{a}_j^\dagger |0,\ldots,0\rangle, \tag{6}$$

where the matrix U is defined by

$$\sum_{j=1}^{m} U_{ij} \hat{a}_j = \hat{\mathcal{U}} \hat{a}_i \hat{\mathcal{U}}^\dagger, \tag{7}$$

and the coefficients d_j are given by

$$d_j = \sum_{i=0}^{m} c_i U_{ij}. \tag{8}$$

The values d_j are the multiplication of the c_i coefficients from Eq. (1) as a vector with the matrix U_{ij}, which will be a unitary matrix from $U(m)$, the set of $m \times m$ complex matrices satisfying $U^\dagger U = UU^\dagger = I$. Therefore, for single-photon states, the linear mode transformation just directly applies the mode transformation matrix to the coefficients of the superposition.

Now, let us turn to the coherent state evolution. To distinguish coherent states from Fock basis states, the notation $|\alpha = c\rangle$ will represent a coherent state with amplitude c. That is, the action of the annihilation operator on these states is

$$\hat{a}|\alpha = c\rangle = c|\alpha = c\rangle, \tag{9}$$

with c taking on the value of any complex number. Consider now the m-mode coherent state,

$$\otimes_{i=1}^{m} |\alpha_i = c_i\rangle. \tag{10}$$

The evolution of this state through a linear optical transformation $\hat{\mathcal{U}}$ can be found by taking the ansatz that the output state will also be eigenstates of annihilation operators. To test this ansatz, consider the kth mode annihilation operator on the evolved coherent state input,

$$\hat{a}_k \hat{\mathcal{U}} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle = \hat{\mathcal{U}} \hat{\mathcal{U}}^\dagger \hat{a}_k \hat{\mathcal{U}} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle, \tag{11}$$

$$= \hat{\mathcal{U}} \sum_{j=1}^{m} \hat{a}_j U_{jk} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle \tag{12}$$

$$= \hat{\mathcal{U}} \sum_{j=1}^{m} c_j U_{jk} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle \tag{13}$$

$$= \left(\sum_{j=1}^{m} c_j U_{jk} \right) \hat{\mathcal{U}} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle \tag{14}$$

$$= d_k \hat{\mathcal{U}} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle. \tag{15}$$

Here, the same definitions for the matrix U in Eq. (7) and the vector d in Eq. (8) [note the use of the *inverse* unitary matrix going from Eq. (11) to (12)] are used. This is consistent with the ansatz and implies that the evolved state is also a multi-mode coherent state. This evolution can be written as

$$\hat{\mathcal{U}} \otimes_{i=1}^{m} |\alpha_i = c_i\rangle = \bigotimes_{k=1}^{m} |\alpha_k = d_k\rangle. \tag{16}$$

Here, it can be directly seen that the one-particle amplitudes are all encoded within the coherent state amplitudes.

III. TWO-MODE INTERFERENCE

When permitting states of more than one photon, there are many more possibilities. Before approaching the most general situation of N photons into m modes, first consider the case of N photons in $m = 2$ modes.

To depict the index set of allowed Fock basis states of m modes with N photons in total, we will use

$$\mathcal{N}_m^N = \left\{ (i_1, i_2, \ldots i_m) \in \mathbb{Z}_+^m \middle| \sum_{j=1}^{m} i_j = N \right\}. \tag{17}$$

So, initially, we will use $\mathcal{N}_2^N$, which consists of all ordered pairs of non-negative integers that sum to N.

An arbitrary two-mode Fock basis state with N photons can be written as

$$\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}\hat{a}_1^{\dagger i_1}\hat{a}_2^{\dagger i_2}|0,0\rangle. \tag{18}$$

Applying the Fock basis evolution $\hat{\mathcal{U}}$ on the Fock basis state of Eq. (18) gives

$$\begin{aligned}\hat{\mathcal{U}}&\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}\hat{a}_1^{\dagger i_1}\hat{a}_2^{\dagger i_2}|0,0\rangle\\
&=\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}\hat{\mathcal{U}}\hat{a}_1^{\dagger i_1}\hat{\mathcal{U}}^\dagger\hat{\mathcal{U}}\hat{a}_2^{\dagger i_2}\hat{\mathcal{U}}^\dagger\hat{\mathcal{U}}|0,0\rangle \qquad (19)\\
&=\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}(\hat{\mathcal{U}}\hat{a}_1^\dagger\hat{\mathcal{U}}^\dagger)^{i_1}(\hat{\mathcal{U}}\hat{a}_2^\dagger\hat{\mathcal{U}}^\dagger)^{i_1}|0,0\rangle \qquad (20)\\
&=\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}(U_{11}\hat{a}_1^\dagger+U_{12}\hat{a}_2^\dagger)^{i_1}\\
&\quad\times(U_{21}\hat{a}_1^\dagger+U_{22}\hat{a}_2^\dagger)^{i_2}|0,0\rangle \qquad (21)\\
&=\sum_{(i_1,i_2)\in\mathcal{N}_2^N}\frac{c_{(i_1,i_2)}}{\sqrt{i_1!i_2!}}\sum_{k_1=0}^{i_1}\sum_{k_2=0}^{i_2}\binom{i_1}{k_1}\binom{i_2}{k_2}\\
&\quad\times U_{11}^{k_1}U_{12}^{i_1-k_1}U_{21}^{k_2}U_{22}^{i_2-k_2}\hat{a}_1^{\dagger k_1+k_2}\hat{a}_2^{\dagger N-(k_1+k_2)}|0,0\rangle \qquad (22)\\
&=\sum_{(j_1,j_2)\in\mathcal{N}_2^N}\frac{d_{(j_1,j_2)}}{\sqrt{j_1!j_2!}}\hat{a}_1^{\dagger j_1}\hat{a}_2^{\dagger j_2}|0,0\rangle \qquad (23)\end{aligned}$$

with

$$d_{(j_1,j_2)}=\sum_{(i_1,i_2)\in\mathcal{N}^N}c_{(i_1,i_2)}\left[\sum_{\substack{k_1,k_2=0\\k_1+k_2=j_1}}^{i_1,i_2}\frac{\sqrt{j_1!j_2!}}{\sqrt{i_1!i_2!}}\times\binom{i_1}{k_1}\binom{i_2}{k_2}U_{11}^{k_1}U_{12}^{i_1-k_1}U_{21}^{k_2}U_{22}^{i_2-k_2}\right], \tag{24}$$

and $(j_1,j_2)\in\mathcal{N}_2^N$. This is clearly a more complex expression than that for the single-photon case, but the term in the square brackets can be thought of as a linear mapping of the c coefficients into the d coefficients. There are a number of observations that can be made on this expression. First, the action of $\hat{\mathcal{U}}$ does not change the form of the expression for a fixed N (i.e., this is a mapping of two-mode states of N photons onto two-mode states of N photons). Next, the expression is unchanged under interchange of U_{11} and U_{12} with U_{21} and U_{22}, respectively. Finally, if two different linear mappings were defined in this way using two matrices U, then their product will also be of this form. This can be seen by the fact that the $\hat{\mathcal{U}}$ operators representing the unitary matrices U will close to form a group. In this particular case, the term in the square brackets forms a *representation* of the group $U(2)$. It is for this reason that the binomial coefficients and square-roots of factorial terms appear in the expression.

Consider the case of Eq. (24), where $N=2$ with $c_{(1,1)}=1$ and all other c terms are zero. With all these restrictions, we find

$$d_{(1,1)}=U_{11}U_{22}+U_{12}U_{21}. \tag{25}$$

Here, the connection to a matrix permanent can be directly seen. This is also the main term responsible for the "Hong–Ou–Mandel dip,"[17] which can be achieved when $U_{11}=U_{22}=U_{12}=-U_{21}$. For the other components of this two-photon state, we have

$$d_{(2,0)}=\sqrt{2}U_{11}U_{21} \tag{26}$$

and

$$d_{(0,2)}=\sqrt{2}U_{12}U_{22}. \tag{27}$$

Now, we turn to a coherent state representation of this same two-mode interference effect. To begin, consider the single-photon interference pattern of Eq. (16). This expression essentially encodes the matrix elements of U into the coherent state amplitudes. In order to extract expressions involving two matrix elements, two amplitudes will be needed. So, for example, the amplitude of Eq. (26) can be extracted using the single-photon process of Eq. (16) using the output amplitude d_k of mode 1 when using the input amplitude for the two cases of $c_i=1$ of mode 1 with all other inputs being vacuum and $c_i=1$ for mode 2 with all other inputs being vacuum. These two amplitudes will result in U_{11} and U_{21}, which can then be multiplied together along with a factor of $\sqrt{2}$ to give $d_{(2,0)}$. If the c_i input terms are not 1, then the output amplitude d_k is scaled by this factor, which can be divided out before the multiplication. This same procedure will work for $d_{(0,2)}$, but using the output amplitudes of mode 2.

This is suggestive as it matches the Fock basis component of the input state with the input coherent states. There are two photons entering the interferometer in modes 1 and modes 2, and these are exactly the inputs that require coherent states. This would suggest that the pattern is for $d_{(1,1)}$ to require the estimate of an amplitude from the output of modes 1 and 2. However, here more work is needed to enforce the symmetry of the transformation. With the coherent state amplitude injected into the first mode, one can measure the output in either the first or second modes. Similarly, with amplitude injected into the second mode, either the first or the second mode amplitudes can be measured. In order to reconstruct the amplitude of Eq. (25), the symmetric combination of the two possible choices of how to measure the outputs must be taken. That is, for the case where a coherent state is injected into the first mode, if the output amplitude of the first mode is taken, then we have U_{11} and, hence, the amplitude for the case where the coherent state is injected into the second mode must take the amplitude of the output from the second mode giving U_{22}. The product of these two terms gives the first part of Eq. (25), and the case where the outputs are swapped is then added to give the final part of the expression.

This situation continues for the general two-mode expression in Eq. (24). For each photon, an experiment is performed with the injection of a coherent state amplitude into the mode representing the location of the single-photon input. That is, there will be N repetitions of the experiment. For each output location, the amplitudes are measured and the results are multiplied and symmetrically combined under addition. This will automatically incorporate the correct binomial terms, but the bosonic statistical factors involving the square-root of factorials must be multiplied into the final result. This extra factor will depend on the number of repetitions made in the input and the output modes.

IV. MULTI-MODE MULTI-PHOTON INTERFERENCE

We now move on to the general case with any number of modes.

A. Fock basis

An N photon state in m modes can be written as

$$\sum_{\mathbf{i}\in\mathcal{N}_m^N} \frac{c_{\mathbf{i}}}{\sqrt{\mathbf{i}!}} \hat{a}_1^{\dagger i_1} \hat{a}_2^{\dagger i_2} \ldots \hat{a}_m^{\dagger i_m} |0, 0, \ldots, 0\rangle, \tag{28}$$

where $\mathbf{i} = (i_1, i_2, \ldots, i_m)$ and $\mathbf{i}! = i_1! i_2! \cdots i_m!$. To compute the result of a linear optical operation $\hat{\mathcal{U}}$, the expansion of the annihilation operators proceeds as before. The resulting expression after this procedure will be a sum of multinomials in the annihilation operators with the same total degree of the original state. But as above the expression will be *linear* in the $c_{\mathbf{i}}$ values, it can be written down as an operator acting on these coefficients. The space defined by the Fock basis states in $\mathcal{N}^N$ has dimension $\begin{pmatrix} m+N-1 \\ N \end{pmatrix}$. So for each m where $\hat{\mathcal{U}}$ represents the action of $U(m)$, we end up with a linear representation of this continuous group on the Fock basis with N particles in m modes.

The method for giving the explicit expression of each coefficient in this evolution is known,[16] and the most compact form being that of the matrix permanent formed from appropriately repeated rows and columns of the original matrix,

$$\frac{\sqrt{\mathbf{j}!}}{\sqrt{\mathbf{i}!}} \mathrm{Per}\left(A_{(\mathbf{i})}^{(\mathbf{j})}\right). \tag{29}$$

In this expression $A_{(\mathbf{i})}^{(\mathbf{j})}$ is a square matrix formed from U in a two-step process. First, a rectangular matrix is formed by taking i_1 copies of the first column of U, i_2 copies of the second column of U, and so on. From this rectangular matrix, the square matrix is formed, matrix j_1 copies of its first row, j_2 copies of the second row, and so on. This expression naturally accounts for the binomial coefficients through the symmetry of the matrix permanent and the combinatorics of the repeated rows and columns (see Ref. 18). The symmetry of the matrix permanent means that it is independent of the order in which the rows and columns appear. The resulting matrix can then be used to form the transformation from the $c_{\mathbf{i}}$ coefficients to the $d_{\mathbf{j}}$ coefficients in the same form as Eq. (24). The resulting expression for the coefficient in the sum for a N photon state will result in degree N combinations of the matrix elements of the original matrix U.

B. Coherent states

We will now give the generalization of the equivalent coherent state circuit as described in Sec. III on the two-mode case. We want to perform some reconstruction of the amplitude of the Fock basis process from a configuration given by $\mathbf{i}$ to $\mathbf{j}$. Consider the tensor product space of $m \times N$ coherent states with i_1 copies of a coherent state in mode 1, i_2 copies of a coherent state in mode 2,

$$\underbrace{\bigotimes_{k=1}^{m} |\alpha_k = \delta_{k1}\rangle \ldots \bigotimes_{k=1}^{m} |\alpha_k = \delta_{k1}\rangle}_{i_1} \times \underbrace{\bigotimes_{k=1}^{m} |\alpha_k = \delta_{k2}\rangle \ldots \bigotimes_{k=1}^{m} |\alpha_k = \delta_{k2}\rangle}_{i_2} \ldots \bigotimes_{k=1}^{m} |\alpha_k = \delta_{km}\rangle, \tag{30}$$

where δ_{km} is the Kronecker's delta symbol,

$$\delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases} \tag{31}$$

A linear optics operation acting on this space can be constructed using n copies of the single-photon action U on the blocks of m amplitudes as

$$U_{tot} = \bigoplus_{k=1}^{N} U, \tag{32}$$

where U is the m-mode linear network matrix. The direct sum is used to combine the network matrices U as they describe the linear interaction of modes [as defined in Eq. (7)]. Applying this linear transformation operation on the complete coherent state input state will give in each block of m-modes the single-photon amplitudes for each possible cases of where single photons could be input into the network. Using the same evolution from Eq. (16), the output after this particular evolution can be written as

$$\underbrace{\bigotimes_{k=1}^{m} |\alpha_k = U_{1k}\rangle \ldots \bigotimes_{k=1}^{m} |\alpha_k = U_{1k}\rangle}_{i_1} \times \underbrace{\bigotimes_{k=1}^{m} |\alpha_k = U_{2k}\rangle \ldots \bigotimes_{k=1}^{m} |\alpha_k = U_{2k}\rangle}_{i_2} \ldots \bigotimes_{k=1}^{m} |\alpha_k = U_{mk}\rangle. \tag{33}$$

This expression explicitly forms the repeated rows of the input unitary matrix as required to form the matrix $A_{(\mathbf{i})}^{(\mathbf{j})}$ above. The required columns for the combination $\mathbf{j}$ can be formed by measuring mode 1 j_1 times, mode 2 j_2 times, and multiplying the results, but remembering to take all possible symmetric combinations of the arrangement of these modes as was done in Sec. III. The complete amplitude is formed by multiplying by the factor $\frac{\sqrt{\mathbf{j}!}}{\sqrt{\mathbf{i}!}}$. This configuration is shown schematically in Fig. 1 for the case of only single-photon inputs (i.e., without repeats).

C. Three-photon three-mode example

As an example of the general case, consider a three-photon linear interferometer described by the unitary matrix,

$$\begin{pmatrix} U_{11} & U_{12} & U_{13} \\ U_{21} & U_{22} & U_{23} \\ U_{31} & U_{32} & U_{33} \end{pmatrix}. \tag{34}$$

Into this network, inject the Fock state $|2, 1, 0\rangle$. In this example, we will consider the case of the amplitude for the output in the Fock basis $|1, 1, 1\rangle$, where the three photons exit all in individual modes.

In the Fock basis, the amplitude for this output can be computed using the same above-mentioned polynomial expansion as follows:

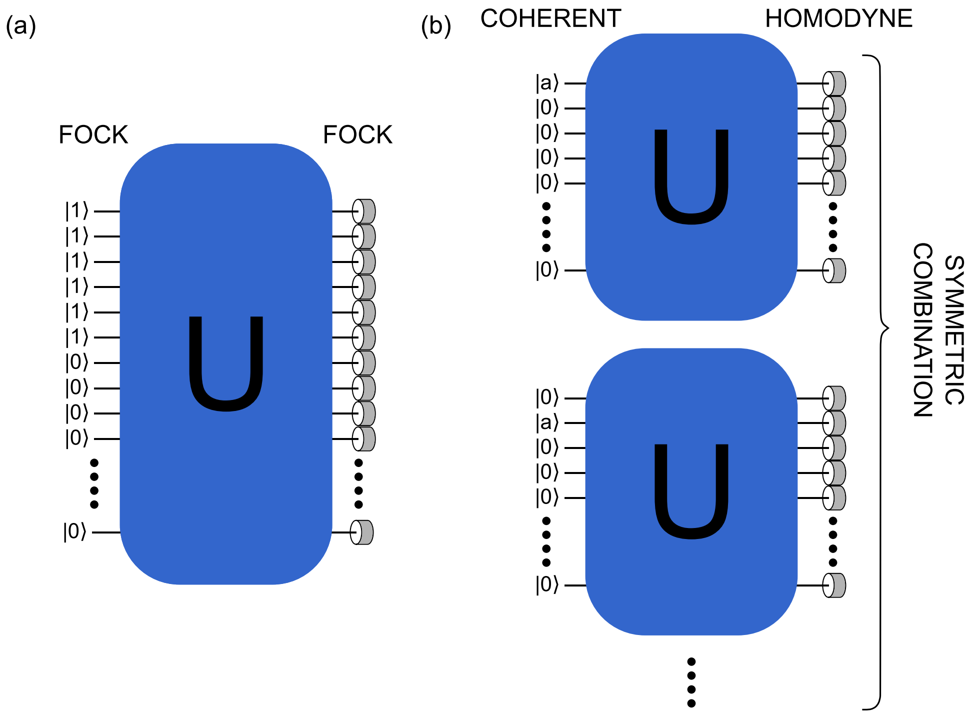

FIG. 1. Schematic of how amplitudes of a Fock basis state evolving under a linear network U can be derived from coherent states. (a) The configuration considered in this schematic is for single-photon inputs into some modes of a linear network U, and the output is measured in the Fock basis. (b) The method for extracting the amplitudes of the Fock basis process. For each input photon, the network U is repeated. A coherent state is then injected into each network for each photon individually. The output coherent state amplitude is then measured using homodyne detection. The Fock basis amplitude is encoded in the symmetric combination of the output modes of the corresponding modes of the Fock basis detection. Though this representation shows the case of individual photon inputs, the procedure works for the general case (see the text).

$$\begin{aligned}\hat{\mathcal{U}}|2,1,0\rangle &= \hat{\mathcal{U}}\frac{1}{\sqrt{2}}\hat{a}_1^2\hat{a}_2|0,0,0\rangle \\ &= \frac{1}{\sqrt{2}}(U_{11}\hat{a}_1 + U_{12}\hat{a}_2 + U_{13}\hat{a}_3)^2 \\ &\quad \times (U_{21}\hat{a}_1 + U_{22}\hat{a}_2 + U_{23}\hat{a}_3)|0,0,0\rangle \end{aligned} \tag{35}$$

$$\overset{|1,1,1\rangle}{\rightarrow} \sqrt{2}(U_{11}U_{12}U_{23} + U_{11}U_{13}U_{22} + U_{12}U_{13}U_{21}). \tag{36}$$

Using the coherent state construction above, with input amplitudes of

$$|\alpha=1\rangle|\alpha=0\rangle|\alpha=0\rangle|\alpha=1\rangle|\alpha=0\rangle|\alpha=0\rangle|\alpha=0\rangle|\alpha=1\rangle|\alpha=0\rangle, \tag{37}$$

under evolution of the block repeated interferometer, the output state is

$$\begin{aligned}&|\alpha=U_{11}\rangle|\alpha=U_{12}\rangle|\alpha=U_{13}\rangle|\alpha=U_{11}\rangle|\alpha=U_{12}\rangle \\ &\quad|\alpha=U_{13}\rangle|\alpha=U_{21}\rangle|\alpha=U_{22}\rangle|\alpha=U_{23}\rangle.\end{aligned} \tag{38}$$

Now, to extract the $|1,1,1\rangle$ Fock basis amplitude, a symmetric product of the amplitudes from the first, second, and third mode of each block is taken. There are six such combinations, which when added will give the expression

$$\begin{aligned}&U_{11}U_{12}U_{23} + U_{11}U_{13}U_{22} + U_{12}U_{13}U_{21} + U_{11}U_{12}U_{23} \\ &\quad + U_{11}U_{13}U_{22} + U_{12}U_{13}U_{21}.\end{aligned} \tag{39}$$

After multiplying this expression by the $\sqrt{\mathbf{j}!}/\sqrt{\mathbf{i}!}$, this becomes the same as the Fock basis amplitude in Eq. (36).

D. Efficiency of forming the amplitude

While this process of moving from a Fock basis interferometer to a coherent state is straight-forward, the ability to make these amplitude estimates must clearly be inefficient at scale. There is only one part of the process where this inefficiency lies. The process to lay out the interferometer repeatedly for the coherent state inputs is linear in the number of modes (m) and the number of photons (N). The estimation of the output amplitudes from the coherent states can be made efficiently, as there are no correlations between the modes to estimate. The only place where the inefficiency lies is in the combination of the output amplitudes to form the final Fock basis amplitude. In the worst case, there are $\binom{m}{N}$ amplitudes that need to be symmetrically combined.

This is distinct from the concept from quantum optics that the full effects of interference of photons cannot be reproduced in an experimental configuration using only classical light. This construction shows that the analogous interference effects *can* be recreated, but at the cost of a computationally inefficient process to generate the amplitudes. For fixed values of m and N, there are no variables to scale the complexity in, and this procedure will give a construction that will generate the required amplitudes.

V. EVOLUTION RECOMPUTATION METHOD

In the method presented in Sec. IV, the source of computational inefficiency all occurs within the measurement processing. Here, we show how one can move this complexity from the measurement and into the evolution of the coherent states. In this approach, the appropriate representation of the unitary matrix is encoded into a larger interferometer. The required amplitude can be directly measured on the output coherent state amplitude. However, the representation matrix elements need to be computed, which becomes the source of the computational inefficiency.

A. Hong–Ou–Mandel example

We will start with the familiar example of the Hong–Ou–Mandel two-mode two-photon interferometer.[17] We can write down the evolution of the annihilation operators from two modes that are of total degree 2,

$$\frac{1}{\sqrt{2}}\hat{a}_1^2 \rightarrow \frac{1}{\sqrt{2}}(U_{1,1}\hat{a}_1 + U_{1,2}\hat{a}_2)^2 \tag{40}$$

$$= U_{1,1}^2\frac{1}{\sqrt{2}}\hat{a}_1^2 + \sqrt{2}U_{1,1}U_{1,2}\hat{a}_1\hat{a}_2 + U_{1,2}^2\frac{1}{\sqrt{2}}\hat{a}_2^2, \tag{41}$$

$$\hat{a}_1\hat{a}_2 \rightarrow (U_{1,1}\hat{a}_1 + U_{1,2}\hat{a}_2)(U_{2,1}\hat{a}_1 + U_{2,2}\hat{a}_2) \tag{42}$$

$$= \sqrt{2}U_{1,1}U_{1,2}\frac{1}{\sqrt{2}}\hat{a}_1^2 + (U_{1,1}U_{2,2} + U_{1,2}U_{2,1})\hat{a}_1\hat{a}_2 + \sqrt{2}U_{1,2}U_{2,2}\frac{1}{\sqrt{2}}\hat{a}_2^2, \tag{43}$$

$$\frac{1}{\sqrt{2}}\hat{a}_2^2 \rightarrow \frac{1}{\sqrt{2}}(U_{2,1}\hat{a}_1 + U_{2,2}\hat{a}_2)^2 \tag{44}$$

$$= U_{2,1}^2\frac{1}{\sqrt{2}}\hat{a}_1^2 + \sqrt{2}U_{2,1}U_{2,2}\hat{a}_1\hat{a}_2 + \frac{1}{\sqrt{2}}U_{2,2}^2\hat{a}_2^2. \tag{45}$$

As everything is linear, we can write this in matrix form

$$\begin{pmatrix} U_{1,1}^2 & \sqrt{2}U_{1,1}U_{1,2} & U_{2,1}^2 \\ \sqrt{2}U_{1,1}U_{1,2} & (U_{1,1}U_{2,2} + U_{1,2}U_{2,1}) & \sqrt{2}U_{2,1}U_{2,2} \\ U_{1,2}^2 & \sqrt{2}U_{2,1}U_{2,2} & U_{2,2}^2 \end{pmatrix}, \tag{46}$$

where this matrix acts on the space of vectors formed from the coefficients of the terms $\hat{a}_1^2/\sqrt{2}$, $\hat{a}_1\hat{a}_2$, $\hat{a}_2^2/\sqrt{2}$. This matrix representation is also unitary, as it is a faithful representation of the underlying group. That is, if the inverse is the conjugate transpose of the group element, then it is also true of this representation. As this is a unitary matrix, we can think of it as forming another new linear network. This is the way in which we can make the coherent state isomorphism work.

If the two-photon two-mode coefficients within the Fock basis are $c_{1,1}$, $c_{1,2}$, and $c_{2,2}$, then a three-mode coherent state that represents this two-photon state can be written as

$$|\alpha_1 = c_{1,1}\rangle \otimes |\alpha_2 = c_{1,2}\rangle \otimes |\alpha_3 = c_{2,2}\rangle. \tag{47}$$

By using the 3×3 matrix from Eq. (46), we can recover the evolution of the two-photon states through this mapping.

B. Three-photon, two-mode example

Following the same rules, one can write out the case for three photons in two modes. The basis states are $\{|3,0\rangle, |2,1\rangle, |1,2\rangle, |0,3\rangle\}$, and hence, the unitary matrix will need to act on four-dimensional vectors. The result is

$$\begin{pmatrix} U_{1,1}^3 & \sqrt{3}U_{1,1}^2U_{1,2} & \sqrt{3}U_{1,1}U_{1,2}^2 & U_{1,2}^3 \\ \sqrt{3}U_{1,1}^2U_{2,1} & U_{1,1}(U_{1,1}U_{2,2} + 2U_{1,2}U_{2,1}) & U_{1,2}(2U_{1,1}U_{2,2} + U_{1,2}U_{2,1}) & \sqrt{3}U_{1,2}^2U_{2,2} \\ \sqrt{3}U_{1,1}U_{2,1}^2 & U_{2,1}(2U_{1,1}U_{2,2} + U_{1,2}U_{2,1}) & U_{2,2}(U_{1,1}U_{2,2} + 2U_{1,2}U_{2,1}) & \sqrt{3}U_{1,2}U_{2,2}^2 \\ U_{2,1}^3 & \sqrt{3}U_{2,1}^2U_{2,2} & \sqrt{3}U_{2,1}U_{2,2}^2 & U_{2,2}^3 \end{pmatrix}, \tag{48}$$

which again is a unitary matrix.

C. Two-photon, three-mode example

In the case of two-photons into three modes, the Fock space is spanned by a six-dimensional space. Here, we choose the basis

$$\{|2,0,0\rangle, |1,1,0\rangle, |1,0,1\rangle, |0,2,0\rangle, |0,1,1\rangle, |0,0,2\rangle\}. \tag{49}$$

The matrix for a linear optics network U as a transformation in this basis is

$$\begin{pmatrix} U_{1,1}^2 & \sqrt{2}U_{1,1}U_{1,2} & \sqrt{2}U_{1,1}U_{1,3} & U_{1,2}^2 & \sqrt{2}U_{1,2}U_{1,3} & U_{1,3}^2 \\ \sqrt{2}U_{1,1}U_{2,1} & (U_{1,1}U_{2,2} + U_{1,2}U_{2,1}) & (U_{1,1}U_{2,3} + U_{1,3}U_{2,1}) & \sqrt{2}U_{1,2}U_{2,2} & (U_{1,2}U_{2,3} + U_{1,3}U_{2,2}) & \sqrt{2}U_{1,3}U_{2,3} \\ \sqrt{2}U_{1,1}U_{3,1} & (U_{1,1}U_{3,2} + U_{1,2}U_{3,1}) & (U_{1,1}U_{3,3} + U_{1,3}U_{3,1}) & \sqrt{2}U_{1,2}U_{3,2} & (U_{1,2}U_{3,3} + U_{1,3}U_{3,2}) & \sqrt{2}U_{1,3}U_{3,3} \\ U_{2,1}^2 & \sqrt{2}U_{2,1}U_{2,2} & \sqrt{2}U_{2,1}U_{2,3} & U_{2,2}^2 & \sqrt{2}U_{2,2}U_{2,3} & U_{2,3}^2 \\ \sqrt{2}U_{2,1}U_{3,1} & (U_{2,1}U_{3,2} + U_{2,2}U_{3,1}) & (U_{2,1}U_{3,3} + U_{2,3}U_{3,1}) & \sqrt{2}U_{2,2}U_{3,2} & (U_{2,2}U_{3,3} + U_{2,3}U_{3,2}) & \sqrt{2}U_{2,3}U_{3,3} \\ U_{3,1}^2 & \sqrt{2}U_{3,1}U_{3,2} & \sqrt{2}U_{3,1}U_{3,3} & U_{3,2}^2 & \sqrt{2}U_{3,2}U_{3,3} & U_{3,3}^2 \end{pmatrix}. \tag{50}$$

D. General case

What is the general pattern for these matrices? Clearly, there is some kind of matrix permanents from matrices formed from repeated matrix elements of the smaller unitary. This motivates the definition of a generalized expression for matrix permanents. Before getting there, let us look at the matrix determinant.

This can be written using the totally antisymmetric symbol $\epsilon_{ijk\ldots}$ as

$$Det(A) = \sum_{ijk\ldots} \epsilon_{ijk\ldots} A_{i1} A_{j2} A_{k3} \ldots . \tag{51}$$

Note that this links the dimensionality of the matrix with the *rank* of ϵ. So the standard matrix permanent can be rewritten in this notation as

$$Per(A) = \sum_{ijk\ldots} \sigma_{ijk\ldots} A_{i1} A_{j2} A_{k3} \ldots, \tag{52}$$

where σ represents the totally *symmetric* tensor, that is, 1 if the indices are a permutation of 123… and zero otherwise. This directly suggests the generalized permanent expression as

$$Per^{abc\ldots}_{\alpha\beta\gamma\ldots}(A) = \sum_{ijk\ldots} \sigma^{abc\ldots}_{ijk\ldots} A_{i\alpha} A_{j\beta} A_{k\gamma} \ldots, \tag{53}$$

where $\sigma^{abc\ldots}_{ijk\ldots}$ is defined as being 1 if $ijk\ldots$ is a permutation of $abc\ldots$, and zero otherwise.

The matrix elements of the computed unitary matrix can be computed using this generalized permanent. First, define a function Δ, which takes the vector index notation from Eq. (28) into the index notation for the generalized permanent expression such that

$$\Delta((i_1, i_2, \ldots, i_m)) = \underbrace{11\cdots1}_{i_1}\underbrace{22\cdots2}_{i_2}\cdots\underbrace{mm\cdots m}_{i_m}. \tag{54}$$

Using this, the computed unitary matrix elements can be written as

$$U^{\mathrm{sym}} = \frac{\sqrt{\mathbf{j}!}}{\sqrt{\mathbf{i}!}} Per^{\Delta(\mathbf{j})}_{\Delta(\mathbf{i})}(U). \tag{55}$$

A schematic representation of this method is shown in Fig. 2.

As an example of this, consider the case of two photons in three modes ($m = 3$) above. The matrix element in the first column and second row is given by the indices $\mathbf{i} = (1,1,0)$ and $\mathbf{j} = (2,0,0)$, and with this expression, the value is

$$\frac{1}{\sqrt{2}} \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma^{12}_{ij} U_{i,1} U_{j,1} = \sqrt{2} U_{1,1} U_{1,2}, \tag{56}$$

which agrees with the value given. For another example, consider the three-photon two-mode case above and the entry in the third column and second row with indices $\mathbf{i} = (1,2,0)$ and $\mathbf{j} = (2,1,0)$. The entry is given by

$$\begin{aligned} &\frac{\sqrt{2}}{\sqrt{2}} \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} \sigma^{112}_{ijk} U_{i,1} U_{j,2} U_{k,2} \\ &= U_{1,1} U_{1,2} U_{2,2} + U_{1,1} U_{2,2} U_{1,2} + U_{2,1} U_{1,2} U_{1,2}, \end{aligned} \tag{57}$$

which is the same as the expression given in the matrix above.

VI. DISCUSSION

The methods presented in this paper give explicitly a construction which extends the notion that "a single-photon linear optics experiment can be recreated with coherent states" to that of multiple photon experiments. Even though this is possible, the key observation is that the methods presented are inefficient, in a computational sense, to implement. As the number of photons increases, there is some procedure that requires exponential resources to compute. In both cases

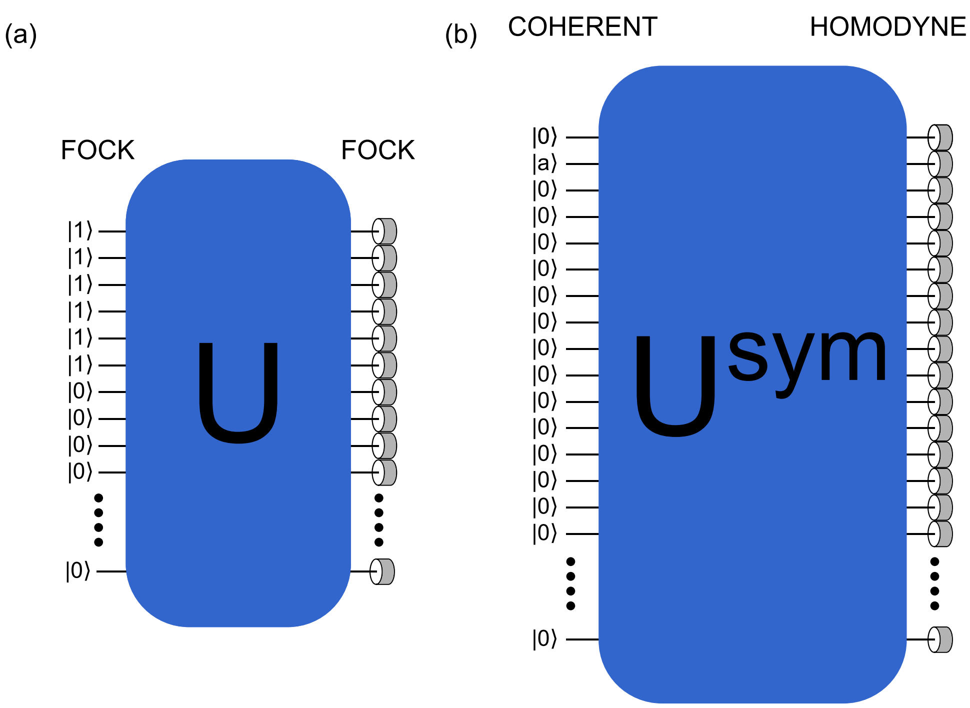

FIG. 2. Schematic example of the coherent state equivalent of Fock basis mixing under linear optics. (a) The network considered is single-photon states injected into a linear optics network given by the network matrix U. The output is measured in the Fock basis. (b) The equivalent configuration under the network recomputation method. A coherent state in injected into a single mode (here, shown in the second mode). The network is recomputed using the matrix of Eq. (55). The equivalent amplitude is extracted from homodyne measurements on the output. The representation shown is only for individual input photons, but the process works for multiple photons (see the text).

here, this is reflected by the presence of symmetric polynomials of amplitudes and quantities, which can be related to matrix permanents. In the context of what is known about the classical computational hardness of quantum systems, even under approximation, this situation is to be expected. Therefore, such constructions do not in any way supersede the need for the advancement of quantum technologies to generate and detect high number Fock states.

The approach considered here depends on the ability to utilize multiple copies, or the ability to construct linear networks. This implicitly requires very good knowledge of the network matrix parameters. If one were presented with a single instance of the network, then a computation of the required amplitudes could be performed after a tomographic characterization of the network.[19] In fact, for any one particular amplitude, only specific elements of the network matrix will be required. This work, therefore, complements the techniques for characterizing linear networks.

If the knowledge about the linear network is wrong or incomplete, then there will inevitably be an error associated with the estimated amplitudes in addition to any error associated with the estimation of complex amplitudes. This extra error will be (in the worst case) an additive error that accumulates for each term involved in the final estimation. Hence, one can expect $\binom{m}{N}$ accumulated error terms for the symmetric combinations of output amplitudes, and these errors will be multiplied by N for the unitary recomputation technique. It is interesting to note that the basis for the classical computational hardness of approximately sampling photons from linear optic devices is based on the hardness of approximating matrix permanents to within an additive error that scales as $\sqrt{N!}$. How and if this can be exploited here to find a regime of noises added to the linear network matrix where a classically efficient approximate estimate technique based on the observations made here is somewhat unclear.

This estimation technique does not seem to perform poorly in the presence of loss. If one can characterize the loss rates for all coherent state input–output combinations, then this loss rate can be merely corrected for by appropriately increasing an estimated coherent state amplitude. Alternatively, a proportionally higher coherent state could be injected into the network. This is not surprising, as the purity of coherent states under loss remains unchanged.

So, in what way is this result useful? The author of this paper believes that the answer to this question is in the development of new, larger scale, quantum technologies. It is an often used argument that an experiment can simply be performed by replacing single photons with coherent states. What this paper describes may indeed be true, but depending on the context, either some part of the quantum nature of the system is lost or the construction is particular and not scalable. This notion is not necessarily new, as it has been used as a conceptual basis in the past for determining what is and what is not an "interesting" quantum optics experiment. However, this has been previously been left somewhat ambiguous. This paper gives an explicit construction for which this notion can be compared to.

With all that being said, there is also the possibility that with the constructions given here, new applications could be identified. There may be an application which, with a specific construction, minimally passes some sense of a quantum/classical transition based on the computational difficulty. To the author, no such applications are immediately obvious, but such possibilities cannot be ruled out.

VII. CONCLUSION

In this paper, the standard approach of comparing single-photon experiments using linear optics with experiments involving coherent states with linear optics was presented. By extending the construction through repeated implementation of the network, particularly chosen coherent state inputs and post-processing of the output coherent state amplitudes, multiple photon experiments can have their amplitudes estimated in a similar way. As an alternative approach, a new linear optical network can be constructed for coherent state inputs, which mirrors the original Fock basis process. Coherent states can then be directly injected and amplitudes estimated. The cost of the first approach is that the output signals must be processed in a way that uses exponential computational time. The cost of the second approach is that computing the new network requires exponential computational resources. It is hoped that these approaches give a new perspective on the transition of experiments from what can be explained classically and what is a manifestly quantum phenomena.

DEDICATION

This work has been prepared in the memory of Jon Dowling. I first met Jon early in my Ph.D. studies, and his unique approach to life instantly appealed to me. Later on, in my Ph.D., our paths crossed again at a workshop held at Avoca Beach, about 100 km north of Sydney, Australia. One night at this workshop, Jon, Tim Ralph (my Ph.D. advisor), and I found ourselves at a bar across the road from the venue. After sufficient lubrication, a plan was hatched where I would spend a summer in Baton Rouge to work with the group Jon had formed since starting at Louisiana State University. The time I spent in Baton Rouge was an amazing time for me as I got to interact with a wide range of people from different backgrounds as well as solidify ideas in my head about fault-tolerant quantum computing with coherent states, which would eventually form the majority of my Ph.D. thesis. The members of the group were people with a diverse range of styles, and Jon was clearly the ideal catalyst to keep the group interested and engaged in science questions. In my case, we spent every lunchtime debating whether a path entangled single-photon state had any measurable entanglement. Jon's approach and anecdote telling kept us all interested in the scientific topics we were discussing. This approach was clearly successful, as Jon has been involved in an enormous quantity of successful research collaborations. Jon was a wholly enjoyable and amazing person to have known. I will remember him mostly through his unique laugh, a sound I will never forget.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Austin P. Lund: Conceptualization (equal); Investigation (equal); Methodology (equal); Writing – original draft (equal); Writing – review & editing (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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