Assembly and norm maps via genuine equivariant homotopy theory

Dissertation

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Alexander Müller

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Erstgutachter: Prof. Dr. Holger Reich
Zweitgutachter: Prof. Dr. Marco Varisco
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Alexander Müller
Berlin, der 14. 10. 2023.

Abstract. We give a new, conceptual proof and sharp generalization of a Theorem by Cary Malkiwiech [Mal17] about how the assembly map of the algebraic K-theory of a group ring (spectrum) with respect to a finite group G admits a dual coassembly map, such that the composition of assembly and coassembly is the well-studied norm map of $K(R)$.

Using the equivariant perspective on assembly of [DL98] and the precise understanding of the ∞-category of genuine G-spectra that the theory of spectral G-Mackey functors of [Bar17] affords, we show the above theorem by contemplating various universal properties, and that it holds for any additive functor $\mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ instead of K-theory.

Let's thank some people, no?
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Contents

1. Introduction

The algebraic K-theory of a ring or ring spectrum R is a powerful invariant, but notoriously inaccessible to computations. Fixing a finite group G, one might attempt to recover information about the K-theory of the group ring $K(R[G])$ from knowledge of both $K(R)$ and the group homology of G, and indeed there is an *assembly map*

$$H_*(\mathrm{B}G; K(R)) \to K_*(R[G]),$$

coming from a map of spectra

$$\mathrm{B}G_+ \otimes K(R) \to K(R[G]).$$

As we will explain in more detail in Chapter 4, this map occurs as the universal left approximation of Waldhausen A-theory

$$A\colon \mathcal{S} \to \mathsf{Sp}, \quad x \mapsto K(\mathsf{Perf}R[\Omega x])$$

by a colimit-preserving functor.

Unfortunately, this map is generally far from an isomorphism (or even an injection on homotopy groups). Segue - another map that is not an isomorphism is the *norm map*

$$\mathfrak{nm}\colon X_{hG} \to X^{hG}$$

associated to any $X \in \mathsf{Sp}^{\mathrm{B}G}$. However, the norm map becomes an isomorphism after localizing with respect to any Morava theory $K(n)$ (prime p implicit), e.g. rationally (see [HS96] or, for a generalized and modern account [HL13]).

In 2017, Cary Malkiewich published the following theorem, which at least $K(n)$-locally implies that the assembly map is a split-injection:[1]

Theorem 1.1 (Malkiewich, [Mal17])**.** *If G is a finite group and R a ring (or ring spectrum), the composite*

$$\begin{tikzcd}
\mathrm{B}G_+ \otimes K(R) \arrow[r, "\text{assembly}"] & K(R[G]) \arrow[d, "\text{cartan}"'] & \\
& G^R(R[G]) \arrow[r, "\text{coassembly}"] & \mathrm{map}(BG_+, K(R))
\end{tikzcd}$$

is equivalent to the norm map

$$\mathfrak{nm}\colon K(R)_{hG} \to K(R)^{hG}$$

associated to $K(R)$ with the trivial G-action.

Here, $G^R(R[G]) \coloneqq K(\mathrm{Fun}(\mathrm{B}G, \mathsf{Perf}R))$ is the K-theory of perfect R-modules with G-action, also called *Swan-theory.* The difference between perfect R-modules with G-action and perfect $R[G]$-modules is captured by the fact that in general,

$$\mathrm{Fun}(\mathrm{B}G, \mathsf{Mod}R)^\omega \not\simeq \mathrm{Fun}(\mathrm{B}G, \mathsf{Mod}R^\omega)$$

However, if G is finite, there is a comparison map going from left to right, and the K-theory of this map is exactly the cartan map $K(R[G]) \to G^R(R[G])$ appearing in Theorem 1.1. The coassembly map $G^R(R[G]) \to G^R(R)^{hG} = \mathrm{map}(BG_+, K(R))$ then stems from the observation that Swan-theory assembles into a *contravariant* functor

$$\mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}, \quad x \mapsto K(\mathrm{Fun}(x, \mathsf{Perf}R)).$$

By an exactly dual procedure to that of assembly, this admits a canonical right approximation by the limit preserving functor

$$\mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}, \quad x \mapsto \mathrm{map}(x, K(R))$$

[1] We refer to [Mal17, Ch. 2] for a more careful statement of the relation to $K(n)$-localization.

and we evaluate at $\mathrm{B}G$ to obtain said coassembly map. Note how the cartan map is defined only pointwise, at the spaces $\mathrm{B}G$ for G a finite group. There is no obvious naturality statement possible, since the functors connected by it are of different variance.

Malkiewich arduously proves Theorem 1.1 using explicit simplicial model categories of parametrized spectra, and a geometric description of the norm map $X_{hG} \to X^{hG}$ associated to any spectrum with G-action. In particular, at no point do *genuine* G-spectra come up.

The project of this thesis is to understand Theorem 1.1 through the lens of modern ∞-category theory and genuine equivariant homotopy theorem. One way to summarize our results might be the following sharp generalization. Here and in the following, $\mathrm{Cat}_{\mathrm{perf}}$ denotes the ∞-category of small, stable, and idempotent-complete ∞-categories, which we take as the source of K-theory.

Main Theorem 1. *Let R be an $\mathbb{E}_1$-ring, G a finite group and*

$$E\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$$

an additive functor. The classical assembly map

$$\mathrm{B}G_+ \otimes E(\mathsf{Perf}R) \xrightarrow{\alpha} E(\mathsf{Perf}R[G])$$

associated to[2]

$$\mathcal{S} \to \mathsf{Sp}, \quad x \mapsto E(x \otimes \mathsf{Perf}R)$$

factorizes the norm map *associated to the spectrum $E(\mathsf{Perf}R)$, equipped with the trivial G-action. That is, there is a natural* coassembly map

$$E(\mathsf{Perf}R[G]) \xrightarrow{\gamma} \mathrm{map}(\mathrm{B}G_+, E(\mathsf{Perf}R))$$

such that the composition

$$\begin{tikzcd}
E(\mathsf{Perf}R)_{hG} \arrow[d, "\sim"] & & \\
\mathrm{B}G_+ \otimes E(\mathsf{Perf}R) \arrow[r, "\alpha"] & E(\mathsf{Perf}R[G]) \arrow[r, "\gamma"] & \mathrm{map}(\mathrm{B}G_+, E(\mathsf{Perf}R)) \arrow[d, "\sim"] \\
& & E(\mathsf{Perf}R)^{hG}
\end{tikzcd}$$

is equivalent to the norm map.

For $E = K\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ we immediately recover Theorem 1.1 (modulo the fact that the norm map factorizes over $G^R(R[G])$, but we will recover that too, below). But now, no feature of K-theory is present in the proof except for it being an additive functor, so the result holds for e.g. non-connective K-theory, topological Hochschild Homology THH, topological cyclic Homology TC etc. It is also natural in transformations of these functors such as trace maps, the pre-eminent tool to extract results about K-theory from other, more computable invariants. For example, for the cyclotomic trace $\mathtt{tr}\colon K \to TC$ (see e.g. [LRRV17], [NS18]), we

[2]Here, to connect this to Waldhausen A-theory, we implicitly use the not-quite-trivial identification $\mathrm{B}G_+ \otimes \mathsf{Perf}R \simeq \mathsf{Perf}R[G]$. Owing considerably to [CMNN20], we prove this as our Theorem 11.3.

obtain the commutative diagram

$$\begin{tikzcd}
K(R)_{hG} \arrow[r, "\alpha"] \arrow[d, "\mathrm{tr}"] \arrow[rr, bend left, "\mathrm{nm}"] & K(R[G]) \arrow[r, "\gamma"] \arrow[d, "\mathrm{tr}"] & K(R)^{hG} \arrow[d, "\mathrm{tr}"] \\
TC(R)_{hG} \arrow[r, "\alpha"] \arrow[rr, bend right, "\mathrm{nm}"'] & TC(R[G]) \arrow[r, "\gamma"] & TC(R)^{hG}.
\end{tikzcd}$$

Our proof of 1 relies on connections between the norm map and assembly through equivariant homotopy theory. We give a quick tour through the relevant ideas here. Let Or_G be the orbit-category of the finite group G, i.e. the category formed by the G-sets G/H and equivariant maps. Recall that any G-spectrum X not only admits a 'naive' underlying spectrum with G-action $uX \in \mathrm{Fun}(\mathrm{B}G, \mathsf{Sp})$, but also an underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum:A functor $\overleftarrow{u}X : \mathsf{Or}_G^{\mathrm{op}} \to \mathsf{Sp}$), given by

$$G/H \mapsto X^H.$$

The functoriality is expressed in the formula $X^H \simeq \mathrm{map}_{\mathsf{Sp}_G}(G/H_+, X)$. Dually, X also restricts to an Or_G-spectrum (that is, a functor $\vec{u}X : \mathsf{Or}_G \to \mathsf{Sp}$), given by the formula

$$G/H \mapsto X^H \simeq (G/H_+ \otimes X).$$

Also note that taking (homotopy) orbits determines a canonical functor

$$c\colon \mathsf{Or}_G \to \mathcal{S}, \quad G/H \mapsto \mathrm{B}H.$$

We shall prove the following two theorems in this thesis, of which Theorem 1 is then an immediate corollary.

Theorem A. *Let $F\colon \mathcal{S} \to \mathsf{Sp}$ be any functor satisfying the property that the composite*

$$\mathsf{Or}_G \xrightarrow{c} \mathcal{S} \xrightarrow{F} \mathsf{Sp}$$

is equivalent to the underlying Or_G-spectrum $\vec{u}X$ of some genuine G-spectrum X. Then we can associate to the $\mathsf{Or}_G^{\mathrm{op}}$-spectrum $\overleftarrow{u}X$ a coassembly map

$$\gamma\colon \overleftarrow{u}X(G/G) \to \overleftarrow{u}X(G/1)^{hG}$$

such that

$$\begin{tikzcd}
 & \overleftarrow{u}X(G/G) \arrow[r, "\gamma"] \arrow[d, "\sim"] & \overleftarrow{u}X(G/1)^{hG} \arrow[d, "\sim"] \\
F(*)_{hG} \arrow[r, "\alpha"] \arrow[rr, bend right, "\mathrm{nm}"'] & F(\mathrm{B}G) \arrow[r, dashed, "\gamma"] & F(*)^{hG}
\end{tikzcd}$$

commutes.

Theorem B. *For any additive $E\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$, the composite functor*

$$\begin{array}{ccccc}
\mathsf{Or}_G & \longrightarrow & \mathcal{S} & \longrightarrow & \mathsf{Sp} \\
 & & & & \\
G/H & \longmapsto & \mathrm{B}H & \longmapsto & E(\mathrm{B}H \otimes \mathsf{Perf}R)
\end{array}$$

is the underlying Or_G-spectrum of a genuine G-spectrum.

Before we comment on the method of proof, let us mention the strong symmetry that is at play here: There are immediate dual versions of both theorems.

Theorem A'. *Let $H\colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$ be any functor satisfying the property that the composite*

$$\mathsf{Or}_G^{\mathrm{op}} \xrightarrow{c} \mathcal{S}^{\mathrm{op}} \xrightarrow{H} \mathsf{Sp}$$

is equivalent to the underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum $\vec{u}Y$ of some genuine G-spectrum Y. Then we can associate to $\vec{u}Y$ an assembly map

$$\alpha\colon \vec{u}Y(G/1)_{hG} \to \vec{u}Y(G/G)$$

such that

$$\begin{tikzcd}
\vec{u}Y(G/1)_{hG} \arrow[r, "\alpha"] \arrow[d, "\sim"] & \vec{u}Y(G/G) \arrow[d, "\sim"] & \\
H(*)_{hG} \arrow[r, dashed, "\alpha"] \arrow[rr, bend right, "\mathrm{nm}"'] & H(\mathrm{B}G) \arrow[r, "\gamma"] & H(*)^{hG}.
\end{tikzcd}$$

commutes.

Here, the assembly map $H(*)_{hG} \to H(\mathrm{B}G)$ was not a priori present on the functor $\mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$, where in the case of Theorem A, it is the coassembly map that cannot in general be constructed on a functor $\mathcal{S} \to \mathsf{Sp}$.

Theorem B'. *For any additive $E\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$, the functor*

$$\begin{array}{ccccc}
\mathsf{Or}_G^{\mathrm{op}} & \longrightarrow & \mathcal{S}^{\mathrm{op}} & \longrightarrow & \mathsf{Sp} \\
G/H & \longmapsto & \mathrm{B}H & \longmapsto & E(\mathrm{Fun}(\mathrm{B}G, \mathsf{Perf}R))
\end{array}$$

is the underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum of a genuine G-spectrum.

This perspective also closes the loop on the role that Swan-theory plays: If $X \in \mathsf{Sp}_G$ is the G-spectrum associated to

$$x \mapsto K(x \otimes \mathsf{Perf}R)$$

by Theorem B (i.e. the one modeling assembly in K-theory), and $Y \in \mathsf{Sp}_G$ is the G-spectrum associated to

$$x \mapsto K(\mathrm{Fun}(x, \mathsf{Perf}R)),$$

by Theorem B' (i.e. the one modeling coassembly in Swan-theory), we will observe a map of G-spectra $X \to Y$ whose G-fixed points represent the cartan map, and which is an equivalence on underlying spectra with G-action. The situation is summarized in the commutative diagram

$$\begin{tikzcd}
X_{hG} \arrow[r] \arrow[d, "\sim"] \arrow[rr, bend left, "\mathrm{nm}"] & X^G \arrow[r] \arrow[d, "\sim"] & X^{hG} \arrow[d, "\sim"] \\
\mathrm{B}G_+ \otimes K(R) \arrow[r] \arrow[d, "\sim"] & K(R[G]) \arrow[r] \arrow[d, "\mathrm{cartan}"] & \mathrm{map}(BG_+, K(R)) \arrow[d, "\sim"] \\
\mathrm{B}G_+ \otimes G^R(R) \arrow[r] \arrow[d, "\sim"] & G^R(R[G]) \arrow[r] \arrow[d, "\sim"] & \mathrm{map}(BG_+, G^R(R)) \arrow[d, "\sim"] \\
Y_{hG} \arrow[r] \arrow[rr, bend right, "\mathrm{nm}"'] & Y^G \arrow[r] & Y^{hG}.
\end{tikzcd}$$

Here, all horizontal maps in the left column are assembly maps, all horizontal maps in the right column are coassembly maps, and all horizontal compositions are identified with the norm map on $K(R)$.

Our proofs of Theorems A and B rely on the precise control of the homotopy theory of G-spectra afforded by the theory of *spectral Mackey functors.* Recall that the ∞-category of G-spaces is equivalent to $\mathrm{Fun}(\mathsf{Or}_G^{\mathrm{op}}, \mathcal{S})$, or equivalently

$$\mathcal{S}_G \simeq \mathrm{Fun}^{\times}(\mathsf{Fin}_G^{\mathrm{op}}, \mathcal{S}).$$

This is known as *Elmendorf's theorem.* Here, the functor associated to a G-space X is given by

$$G/H \mapsto X^H \simeq \mathrm{Map}_{\mathcal{S}_G}(G/H, X).$$

A naive guess towards a comparable description of G-spectra might then be

$$\mathsf{Sp}_G \overset{?}{\simeq} \mathrm{Fun}^{\times}(\mathsf{Fin}_G^{\mathrm{op}}, \mathsf{Sp}).$$

However, for reasons visible to the geometry of equivariant suspension, the ∞-category Sp_G allows *transfer maps,* i.e. 'wrong way' maps $\mathbb{S}[G/H] \to \mathbb{S}[G/K]$ associated to equivariant maps $G/K \to G/H$. For example, the mapping space

$$\mathrm{Map}_{\mathsf{Sp}_G}(\mathbb{S}[G/G], \mathbb{S}[G/1])$$

is not contractible, but equivalent to $\Omega^\infty \mathbb{S}$. These wrong-way maps are not present in the purported description $\mathrm{Fun}^{\times}(\mathsf{Fin}_G^{\mathrm{op}}, \mathsf{Sp})$, but it turns out that accounting for them and their compatibility is exactly the missing ingredient.

Definition 1.2. Let $\mathrm{Span}(\mathsf{Fin}_G)$ denote the ∞-category with objects the finite G-sets, and morphisms $S \to T$ given by *spans*

$$\begin{tikzcd}
 & U \arrow[dl] \arrow[dr] & \\
S & & T,
\end{tikzcd}$$

composition given by forming pullbacks.[3]

Theorem 1.3 (Guillou-May [GM17a], Barwick [Bar17])**.** *There is an equivalence of ∞-categories*

$$\mathsf{Sp}_G \simeq \mathrm{Fun}^{\times}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathsf{Sp})$$

that sends a G-spectrum X to the diagram

$$G/H \mapsto X^H.$$

[3]See Chapter 6, which closely follows [Bar17], for a complete definition.

The target of this equivalence, the full subcategory of $\mathrm{Fun}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathsf{Sp})$ on the product-preserving functors, is called the ∞-category of *spectral Mackey functors* $\mathrm{Mack}_G(\mathsf{Sp})$. We give our perspective on Theorem 1.3 in Chapter 9, following a detailed investigation of the ∞-category $\mathrm{Mack}_G(\mathsf{Sp})$ in Chapter 8.

This lets us construct G-spectra in new ways: For example, given a product-preserving $\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}$-diagram X with values in some other preadditive ∞-category $\mathcal{C}$ and a product-preserving functor $F\colon \mathcal{C} \to \mathsf{Sp}$, we can form the composition

$$\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}} \xrightarrow{X} \mathcal{C} \xrightarrow{F} \mathsf{Sp}$$

as an object of $\mathrm{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Sp}_G$. The G-spectra proclaimed in Theorems B and B' will be of this kind, with intermediate coefficients $\mathcal{C} = \mathrm{Cat}_{\mathrm{perf}}$, the ∞-category of small stable idempotent-complete ∞-categories and exact functors. We will finish the proof of both Theorems in Chapter 12, after carefully establishing the necessary formal properties of $\mathrm{Cat}_{\mathrm{perf}}$ in Chapter 11.

The other ingredient to this project is the equivariant perspective on assembly laid out in [DL98]. To a functor

$$E\colon \mathsf{Or}_G \to \mathsf{Sp}$$

an assembly map $E(G/1)_{hG} \to E(G/G)$ is associated. Given $F\colon \mathcal{S} \to \mathsf{Sp}$, in Chapter 4 we confirm in our modern language the folklore result that this construction applied to

$$E\colon \mathsf{Or}_G \xrightarrow{c\colon G/H \mapsto \mathrm{B}H} \mathcal{S} \xrightarrow{\quad F \quad} \mathsf{Sp}$$

recovers the classical assembly map associated to F. Carefully passing from genuine G-spectra to Mackey functors and their underlying Or_G-spectra and back is then what facilitates the proofs of Theorems A and A', which we finish in Chapter 10.

2. Conventions, notations & recollections

2.1. **∞-categories.** This thesis is written in the language of ∞-categories, as laid out in the foundational works of Jacob Lurie [Lur09], [Lur17], and [Lur21]. With these works at our back, we can mostly work model-independent, that is we consider as given an ∞-category of ∞-categories and all the usual categorical notions like functor ∞-categories, colimits, limits, adjunctions, etc. Unless otherwise specified, these words shall always refer to their ∞-categorical definitions.

A classical category is of course a special case of an ∞-category (one with the property that all mapping spaces are discrete), and we will refer to these not as 'ordinary' categories, but 1-categories. We remark that all the above categorical notions coincide with their classical counterpart if the ∞-categories in which they are applied are 1-categories.

2.2. **A word on the size of the universe.** We follow the usual convention: There is a fixed universe of 'small' sets, and a small ∞-category is then a (levelwise) small simplicial set. Note that many of our ∞-categories then will not be small, such as the ∞-category of (small!) spaces $\mathcal{S}$ or the ∞-category of small ∞-categories Cat_∞, so that for example $\mathcal{S}$ is not an object of Cat_∞. However, one may enlarge the universe to obtain an ∞-category of 'large' ∞-categories $\widehat{\mathrm{Cat}_\infty}$ (of which $\mathcal{S}$ is then an object), and so on.

When no real size issues are in sight, we will not bother mentioning the 'small', e.g. when stating that an ∞-category admits all colimits indexed by a small ∞-category, we will usually just say it admits 'all colimits'.

2.3. **Commutative monoids and spectra.** We define the ∞-category of spectra as the limit (in $\widehat{\mathrm{Cat}_\infty}$) of

$$\cdots \to \mathcal{S}_\bullet \xrightarrow{\Omega} \mathcal{S}_\bullet \xrightarrow{\Omega} \mathcal{S}_\bullet \xrightarrow{\Omega} \mathcal{S}_\bullet,$$

i.e. a spectrum is given by a sequence of pointed spaces $\{E_n\}$ and equivalences $\Omega E_{n+1} \simeq E_n$. We think of E_0 as its underlying space, which comes equipped with the structure of chosen deloopings, ad infinitum.

This is the fundamental object of study in stable homotopy theory, the canonical *stabilization* of the ∞-category of spaces and closely related to the ∞-category of commutative monoids (where we of course mean the homotopical version, see Chapter 5 for details.).

To give some context, let us review a hierarchy of properties an ∞-category $\mathcal{C}$ might possess, giving increasingly more structure on its mapping spaces.

We call $\mathcal{C}$ *pointed* if it admits a *zero-object*, i.e. an object $0 \in \mathcal{C}$ that is both initial and terminal, i.e. all mapping spaces to and from 0 are contractible. This canonically makes all mapping spaces themselves pointed (at the map $x \to 0 \to y$), and one might think of $\mathcal{C}$ as enriched over pointed spaces.

We call an ∞-category $\mathcal{C}$ *preadditive* if it is pointed, admits finite coproducts and products and for any finite collection of objects $\{c_i\}$ the natural comparison map

$$\phi\colon \coprod c_i \to \prod c_i$$

given as

$$\phi_{ij} = \begin{cases} \mathrm{id} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

is an equivalence. This canonically promotes each mapping space to a commutative monoid (i.e. an $\mathbb{E}_\infty$-algebra in $\mathcal{S}$, see Chapter 5) via

$$f + g\colon x \xrightarrow{\triangle} x \times x \simeq x \amalg x \xrightarrow{f \amalg g} y \amalg y \xrightarrow{\nabla} y,$$

and one might think of $\mathcal{C}$ as enriched over commutative monoids.

We call an ∞-category $\mathcal{C}$ *additive* if it is preadditive and its mapping spaces are *grouplike*, i.e. for each $x, y \in \mathcal{C}$, we have that $\pi_0 \operatorname{Map}_{\mathcal{C}}(x, y)$ is not merely a discrete commutative monoid but an abelian group. Somewhat tautologically, one might think of $\mathcal{C}$ as enriched over grouplike commutative monoids.

We call an ∞-category *stable* if it is pointed and a square in $\mathcal{C}$ is a pushout if and only if it is a pullback. This certainly implies preadditivity, and with a little more thought also implies additivity ([Lur17, Lemma 1.1.2.9]). Further, the mapping spaces of $\mathcal{C}$ admit canonical deloopings by

$$\operatorname{Map}_{\mathcal{C}}(x, y) \simeq \Omega \operatorname{Map}_{\mathcal{C}}(x, \Sigma y),$$

allowing us to think of $\mathcal{C}$ as enriched over the ∞-category of spectra Sp. We shall consistently denote these mapping spectra by $\operatorname{map}_{\mathcal{C}}(x, y)$.

Notice that pointed, preadditive and stable ∞-categories are characterized by identifying increasingly more diagrams as both colimit and limit diagrams simultaneously. We point to [GGN15] and [CSY20, Ch. 5], for more thorough discussions of a certain poset of properties including the above.

We also remark that we have right adjoint 'forgetful' functors

$$\begin{tikzcd} \mathsf{Sp} \arrow[r] \arrow[rrrr, bend right, "\Omega^\infty"'] & \mathrm{CMon}^{gp} \arrow[r] & \mathrm{CMon} \arrow[r] & \mathcal{S}_\bullet \arrow[r] & \mathcal{S}. \end{tikzcd}$$

Here, the second term identifies with $\mathsf{Sp}_{\geq 0}$ via the classical *May Recognition Principle* (see [May72] or [Lur17, Thm. 5.2.6.26]): Given an $\mathbb{E}_1$-structure on a space x, one may form a pointed, connected delooping $Bx \in \mathcal{S}_{\bullet, \geq 1}$, and pass back via the loop-space functor (which canonically lands in $\mathcal{S}^{gp}_{\mathbb{E}_1}$, the ∞-category of grouplike $\mathbb{E}_1$-spaces). This gives an equivalence of categories

$$\mathcal{S}^{gp}_{\mathbb{E}_1} \simeq \mathcal{S}_{\bullet, \geq 1}$$

and repeated application of this principle then gives an equivalence

$$\mathrm{CMon}^{gp} \simeq \lim\big(\cdots \to \mathcal{S}_{\bullet, \geq 2} \xrightarrow{\Omega} \mathcal{S}_{\bullet, \geq 1} \xrightarrow{\Omega} \mathcal{S}_{\bullet, \geq 0} = \mathcal{S}_\bullet\big) = \mathsf{Sp}_{\geq 0}.$$

2.4. **Presentable ∞-categories.** Presentable ∞-categories have many convenient formal properties. Morally, they are ∞-categories that admit all small colimits (and thus are usually large), but are still controlled by a small subcategory - the prototypical example is the ∞-category of spaces $\mathcal{S}$, the ∞-category obtained from the terminal one by freely adjoining all (small) colimits.

Definition 2.1. An ∞-category is presentable exactly if it is an *accessible localization* of $\mathcal{P}(\mathcal{C})$ for some small $\mathcal{C}$.

Here, $\mathcal{P}(-)$ denotes the category of presheaves on $\mathcal{C}$, i.e. the ∞-category obtained by freely adjoining all colimits, and we refer to [Lur09, Sect. 5.4.2] for accessibility concerns. Here are three particularly pleasant consequences of presentability:

- Presentable ∞-categories are *cocomplete*, i.e. admit all (small) colimits [Lur09, Thm. 5.5.1.1], and *complete*, i.e. admit all (small) limits [Lur09, Cor. 5.5.2.4].
- The *adjoint functor theorem* [Lur09, Cor. 5.5.2.9]: A functor $F \colon \mathcal{C} \to \mathcal{D}$ between presentable ∞-categories admits a right adjoint if and only if F preserves all colimits. A functor $G \colon \mathcal{D} \to \mathcal{C}$ admits a left adjoint if and only if G preserves all limits and preserves filtered colimits.
- Presentable ∞-categories and left adjoint functors organize into the ∞-category $\mathrm{Pr_L}$ [Lur09, Def. 5.5.3.1]. This ∞-category then is symmetric monoidal with respect to the *Lurie tensor product* [Lur17, Ch. 4.8.1], a

fairly strict generalization of the tensor product of vector spaces: Given $\mathcal{C}, \mathcal{D} \in \mathrm{Pr_L}$ it is characterized by a functor

$$\mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$$

such that colimit-preserving functors $\mathcal{C} \otimes \mathcal{D} \to \mathcal{E}$ identify with functors $\mathcal{C} \times \mathcal{D} \to \mathcal{D}$ preserving colimits in either variable.

A commutative algebra object in $\mathrm{Pr_L}$ is then the same as a presentable ∞-category with symmetric monoidal structure that preserves colimits in either variable, and we refer to such an ∞-category as *presentably symmetric monoidal*, and collect these in the ∞-category $\mathrm{CAlg}(\mathrm{Pr}_L)$.

Further examples of presentable ∞-categories include the ∞-category of small ∞-categories Cat_∞ and the ∞-category of spectra Sp. If K is a small simplicial set and $\mathcal{C}$ is presentable, so is $\mathrm{Fun}(K, \mathcal{C})$, and so is the full subcategory of $\mathrm{Fun}(K, \mathcal{C})$ spanned by the colimit-preserving functors, and similarly for a whole host of other constructions one might want to perform.

2.5. **Equivariant homotopy theory.** Let us fix a finite group G. Given some convenient model category of G-spaces (such as G-CW-spaces), we may invert the equivariant equivalences (i.e. equivariant maps that induce weak equivalences of spaces after taking fixed points with respect to all subgroups of G) to obtain an ∞-category of G-spaces $\mathcal{S}_G$. Let Or_G be the full subcategory of $\mathcal{S}_G$ spanned by the orbits G/H. It is then the content of *Elmendorf's Theorem* that the functor

$$\mathcal{S}_G \to \mathrm{Fun}(\mathsf{Or}_G^{\mathrm{op}}, \mathcal{S}), \quad X \mapsto \left[G/H \mapsto \mathrm{Map}_{\mathcal{S}_G}(G/H, X)\right]$$

is an equivalence of ∞-categories, so we might as well take the right hand side as our definition. In view of [Lur09, Thm. 5.1.5.6], this says that G-spaces are obtained by freely forming colimits of orbits. As a category of presheaves, $\mathcal{S}_G$ is itself presentable, and we consider it symmetric monoidal (i.e. as an object of $\mathrm{CAlg}(\mathrm{Pr_L})$) under the cartesian symmetric monoidal structure.

The ∞-category of pointed G-spaces $\mathcal{S}_{G\bullet}$ is symmetric monoidal under the smash product of G-spaces (i.e. the essentially unique symmetric monoidal structure rendering the left adjoint of the forgetful functor $\mathcal{S}_{G\bullet} \to \mathcal{S}_G$ symmetric monoidal), and we define the ∞-category of genuine G-spectra Sp_G as the initial presentably symmetric monoidal ∞-category under $\mathcal{S}_{G\bullet}$ in which the representation spheres are inverted. That is, by definition, Sp_G is presentably symmetric monoidal and equipped with a symmetric monoidal left adjoint

$$\Sigma^G \colon \mathcal{S}_{G\bullet} \to \mathsf{Sp}_G$$

that sends the representation spheres in $\mathcal{S}_{G\bullet}$ to invertible objects of Sp_G.

This definition is powered by the results of Marco Robalo's thesis [Rob15]. In the appendix of [GM20], it is proven that this is indeed the ∞-category obtained from the model category of orthogonal spectra, which is the more classical way to define Sp_G, see e.g. [NS18] for a detailed account. As mentioned in the introduction, Sp_G is known to be equivalent to the very concrete ∞-category of *spectral G-Mackey functors*. We will explain the latter in Chapter 8, and carefully study the equivalence in Chapter 9.

3. Kan extensions

In this chapter, we give a brief account of the main features of the theory of *Kan extensions* (along fully faithful functors), collect some useful results and compute examples. Our exposition is based on [Lur21, Tag 02YP], and is heavily streamlined to our needs.

Definition 3.1. [Lur21, Def. 7.4.2.1] Given a functor $F\colon \mathcal{J} \to \mathcal{C}$, a full subcategory $\mathcal{I} \hookrightarrow \mathcal{J}$, and an object $X \in \mathcal{J}$ we say that F is *left Kan extended from* $\mathcal{I}$ *at* X if the composition

$$(\mathcal{I}_{/X})^{\triangleright} \to (\mathcal{J}_{/X})^{\triangleright} \xrightarrow{c} \mathcal{J} \xrightarrow{F} \mathcal{C}$$

is a colimit-diagram in $\mathcal{C}$.

Here, the *relative overcategory* $\mathcal{I}_{/X}$ is the pullback[4]

$$\begin{tikzcd} \mathcal{I}_{/X} \arrow[r] \arrow[d] & \mathcal{J}_{/X} \arrow[d] \\ \mathcal{I} \arrow[r] & \mathcal{J}, \end{tikzcd}$$

i.e. the ∞-category whose n-simplices are those $(n+1)$-simplices of $\mathcal{J}$ whose restriction to $[0,...,n]$ is contained in $\mathcal{I}$ and whose final vertex is X. We refer to App. B.1 for a discussion of absolute overcategories. The map $c\colon (\mathcal{J}_{/X})^{\triangleright} \to \mathcal{J}$ is the *slice contraction morphism*, which restricts on $\mathcal{J}_{/X}$ to the canonical right fibration $\mathcal{J}_{/X} \to \mathcal{J}$, sends the final object ∞ to $X \in \mathcal{J}$ and maps the (essentially) unique edge from an object $(j \to X)$ to ∞ to the edge $j \to X$ of $\mathcal{J}$. We refer to [Lur21, Constr. 4.3.5.12] for details.

Informally, this says that if F is left Kan extended from $\mathcal{I}$ at X, its value $F(X)$ is determined by its restriction $F|_{\mathcal{I}}$, namely as

$$F(X) \simeq \operatorname{colim}(\mathcal{I}_{/X} \to \mathcal{I} \xrightarrow{F} \mathcal{C}), \tag{3.2}$$

exhibited as such by the natural diagram under $\mathcal{I}_{/X}$.

Remark 3.3. We note that any functor $F\colon \mathcal{J} \to \mathcal{C}$ is automatically left Kan extended from $\mathcal{I}$ at i if the object i is in $\mathcal{I}$. This is so since the overcategory $\mathcal{I}_{/i}$ then admits a terminal object $i \xrightarrow{=} i$. The value of the relevant composition $(\mathcal{I}_{/i})^{\triangleright} \to \mathcal{C}$ at the conepoint agrees with its value at the terminal object of $\mathcal{I}_{/i}$ via the natural map, so it is a colimit-diagram and the condition is satisfied.

We say that functor $F\colon \mathcal{J} \to \mathcal{C}$ is left Kan extended from $\mathcal{I} \hookrightarrow \mathcal{J}$ if it is left Kan extended at X for all $X \in \mathcal{J}$, and we say $F\colon \mathcal{J} \to \mathcal{C}$ is a *left Kan extension of* $F_0\colon \mathcal{I} \to \mathcal{C}$ if F is, well, left Kan extended from $\mathcal{I}$ and there is an equivalence $\eta\colon F_0 \xrightarrow{\sim} F|_{\mathcal{I}}$. Note that by Remark 3.3, any left Kan extended functor is a left Kan extension of its restriction.

As a consequence of the essential uniqueness of colimiting diagrams, any two left Kan extensions of some $F_0\colon \mathcal{I} \to \mathcal{C}$ are naturally isomorphic, i.e. equivalent in the ∞-category $\operatorname{Fun}(\mathcal{J}, \mathcal{D})$.

If $\mathcal{C}$ has all (or at least all $\mathcal{I}_{/X}$-shaped) colimits, every functor $F_0\colon \mathcal{I} \to \mathcal{C}$ admits a left Kan extension $\mathfrak{l}F_0\colon \mathcal{J} \to \mathcal{C}$, and these assemble to a fully faithful functor

$$\mathfrak{l}\colon \operatorname{Fun}(\mathcal{I}, \mathcal{C}) \to \operatorname{Fun}(\mathcal{J}, \mathcal{C})$$

with essential image exactly the left Kan extended functors, see [Lur09, Prop. 4.3.2.15].

The colimit-formula for left Kan extensions not only determines the functor F from its restriction to $\mathcal{I}$, but importantly it also forces all natural transformations

[4] Taken equivalently in Cat_∞ or $s\mathrm{Set}$, see Remark B.1.

into any other functor $G\colon \mathcal{J} \to \mathcal{C}$ to be determined by their restriction to $\mathcal{I}$. More precisely:

The composition

$$\mathrm{map}_{\mathrm{Fun}(\mathcal{J},\mathcal{D})}(\mathfrak{l}F_0, G) \xrightarrow{\mathrm{res}} \mathrm{map}_{\mathrm{Fun}(\mathcal{I},\mathcal{D})}((\mathfrak{l}F_0)|_{\mathcal{I}}, G|_{\mathcal{I}}) \xrightarrow{\eta^*} \mathrm{map}_{\mathrm{Fun}(\mathcal{I},\mathcal{D})}(F_0, G|_{\mathcal{I}})$$

is an equivalence - so the functor $\mathfrak{l}$ provides a left adjoint to the restriction functor, with unit specified by the identification $\eta\colon F_0 \xrightarrow{\sim} F|_{\mathcal{I}}$ ([Lur09, Prop. 4.3.2.17]).

Everything just said dualizes, mutatis mutandis, to give the theory of *right Kan extensions:* A right Kan extension $F\colon \mathcal{J} \to \mathcal{C}$ of a functor $F_0\colon \mathcal{I} \to \mathcal{J}$ is uniquely determined by the requirement

$$F(X) \simeq \lim(\mathcal{I}_{X/} \to \mathcal{I} \to \mathcal{C})$$

via the natural diagram over $\mathcal{I}_{X/}$ with conepoint $F(X)$. Equivalently, a functor $F\colon \mathcal{J} \to \mathcal{C}$ is right Kan extended from $\mathcal{I} \to \mathcal{J}$ if the opposite functor $F^{\mathrm{op}}\colon \mathcal{I}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is left Kan extended from $\mathcal{I}^{\mathrm{op}} \to \mathcal{J}^{\mathrm{op}}$.

Right Kan extensions exist if the ∞-category $\mathcal{C}$ admits all limits of shape $\mathcal{I}_{X/}$, and assemble to a fully faithful functor

$$\mathfrak{r}\colon \mathrm{Fun}(\mathcal{I},\mathcal{C}) \to \mathrm{Fun}(\mathcal{J},\mathcal{C}),$$

which is right adjoint to the restriction functor.

3.1. **Examples.** We now compute some examples and obtain useful structural results.

Example 3.4. Let $\mathcal{C}$ be a cocomplete ∞-category, and let $F\colon * \to \mathcal{C}$ classify some object $F_0 \in \mathcal{C}$. The left Kan extension of F along $* \hookrightarrow \mathcal{S}$ at a given space X is then given by

$$\mathfrak{l}F(X) \simeq \mathrm{colim}(*_{/X} \to * \to \mathcal{C}),$$

i.e. by the colimit of shape $*_{/X}$ with constant value F_0. The pullback square

$$\begin{tikzcd} *_{/X} \arrow[r] \arrow[d] & \mathcal{S}_{/X} \arrow[d] \\ * \arrow[r, hook] & \mathcal{S} \end{tikzcd}$$

identifies $*_{/X}$ with $\mathrm{Map}_{\mathcal{S}}(*, X) \simeq X$. Thus, recalling that we may consider any cocomplete ∞-category as tensored over spaces via

$$X \otimes c := \operatorname*{colim}_X c,$$

we identify the left Kan extension of F along $* \hookrightarrow \mathcal{S}$ with the functor

$$\mathcal{S} \to \mathcal{C}, \quad X \mapsto X \otimes F_0.$$

Note that $\mathfrak{l}F$ preserves all colimits.

Dually, for $\mathcal{C}$ a complete ∞-category, the right Kan extension of F along $* \hookrightarrow \mathcal{S}^{\mathrm{op}}$ at a given space X is then given by

$$\mathfrak{r}F(X) \simeq \lim(*_{X/} \to * \to \mathcal{C}),$$

where the relative overcategory $*_{X/}$ is computed in $\mathcal{S}^{\mathrm{op}}$, so again is equivalent to X. Thus we have

$$\begin{aligned} \mathfrak{r}F(X) &\simeq \mathrm{Map}_{\mathcal{S}}(X, F_0) \quad \text{for } \mathcal{C} = \mathcal{S}, \\ \mathfrak{r}F(X) &\simeq \mathrm{Fun}(X, F_0) \quad \text{for } \mathcal{C} = \mathrm{Cat}_\infty, \\ \mathfrak{r}F(X) &\simeq \mathrm{map}_{\mathsf{Sp}}(X, F_0) \quad \text{for } \mathcal{C} = \mathsf{Sp}, \quad \text{see Remark B.3.} \end{aligned}$$

Remark 3.5. Maybe now is a good time to comment on an unfortunate almost-clash of notation: If $\mathcal{C} = \mathsf{Sp}$, the ∞-category of spectra, E any spectrum and X any space, it is customary to write $X_+ \otimes E$ for the colimit (in Sp) of the constant diagram of shape X with value E. Indeed, here $X_+ \otimes E$ is shorthand for $\Sigma^\infty_+ X \otimes E$ (note that $\otimes$ now denotes the smash product of spectra), and we have consistency since

$$\begin{aligned}\Sigma^\infty_+ X \otimes E &\simeq (\Sigma^\infty_+ \operatorname*{colim}_X *) \otimes E \\ &\simeq \operatorname*{colim}_X (\Sigma^\infty_+ * \otimes E) \\ &\simeq \operatorname*{colim}_X (\mathbb{S} \otimes E) \simeq \operatorname*{colim}_X E.\end{aligned}$$

In contrast, if the space X is already pointed, $X \otimes E$ is classically shorthand for $\Sigma^\infty X \otimes E$, for example in $S^n \otimes E \simeq \Sigma^n E$. The difference becomes clear if we are unsure whether to consider $*$ as an unpointed space or a pointed space: we would read $* \otimes E$ either as $\operatorname{colim}_* E = E$ or as $(\Sigma^\infty *) \otimes E = 0$, respectively. Thus, in the case of $\mathcal{C} = \mathsf{Sp}$ we shall stick to the classical $X_+ \otimes E$ for a colimit of shape X, to avoid confusion.

Example 3.6. If $*$ is the terminal object of $\mathcal{J}$, the left Kan extension of some functor $F_0 \colon \mathcal{I} \to \mathcal{C}$ to $\mathcal{J}$ at $*$ is given by

$$\mathfrak{l}F_0(*) \simeq \operatorname{colim}(\mathcal{I} \xrightarrow{F_0} \mathcal{C})$$

since $\mathcal{I}_{/X} \to \mathcal{I}$ is an equivalence if $*$ is terminal in $\mathcal{J}$. If F_0 is the restriction of some $F \colon \mathcal{J} \to \mathcal{C}$, the comparison map

$$\mathfrak{l}F_0 \to F$$

is then the natural map

$$\operatorname{colim}(\mathcal{I} \xrightarrow{F_0} \mathcal{C}) \to \operatorname{colim}(\mathcal{J} \xrightarrow{F} \mathcal{C}) \simeq F(*).$$

Proposition 3.7. [Gla17, Lem. 2.20] *Let $\mathcal{I} \hookrightarrow \mathcal{J}$ be an additive, fully-faithful inclusion of preadditive ∞-categories, and*

$$F \colon \mathcal{I} \to \mathcal{C}$$

an additive functor into a bicomplete, preadditive ∞-category $\mathcal{C}$. Then both Kan extensions

$$\mathfrak{l}F, \mathfrak{r}F \colon \mathcal{J} \to \mathcal{C}$$

are additive.

Proof. We prove the assertion for the left Kan extension $\mathfrak{l}F$, and begin by remarking that $\mathfrak{l}F$ certainly preserves 0-objects since both the inclusion $\mathcal{I} \hookrightarrow \mathcal{J}$ and F are assumed additive.

Now we show that for any objects $X, Y \in \mathcal{J}$, the functor

$$\mathcal{I}_{/X} \times \mathcal{I}_{/Y} \to \mathcal{I}_{/X \oplus Y}, \quad (i \to X, j \to Y) \mapsto (i \oplus j \to X \oplus Y)$$

is final, i.e. computing colimits is invariant under precomposition with this functor[5]. Indeed, it is the statement of [Lur21, Tag 02P3] that exhibiting a left adjoint suffices, which is given by

$$\mathcal{I}_{/X \oplus Y} \to \mathcal{I}_{/X} \times \mathcal{I}_{/Y}, \quad (i \to X \oplus Y) \mapsto (i \to X, i \to Y).$$

[5]Note that in [Lur21], Lurie calls this property of functors *right cofinal*, and historically there has been no real consensus. We go with the simple *final*.

Thus, we see

$$\begin{aligned}
\mathfrak{l}F(X \oplus Y) &\simeq \operatorname{colim}(\mathcal{I}_{/X\oplus Y} \to \mathcal{I} \to \mathcal{C}) \\
&\simeq \operatorname*{colim}_{\mathcal{I}_{/X}\times\mathcal{I}_{/Y}} F \oplus F \\
&\simeq \operatorname*{colim}_{i\in\mathcal{I}_{/X}} \left(\operatorname*{colim}_{j\in\mathcal{I}_{/Y}} F(i) \oplus \operatorname*{colim}_{j\in\mathcal{I}_{/Y}} F(j)\right) \\
&\simeq \operatorname*{colim}_{i\in\mathcal{I}_{/X}} \left(F(i) \oplus \operatorname*{colim}_{j\in\mathcal{I}_{/Y}} F(j)\right) \\
&\simeq \operatorname*{colim}_{i\in\mathcal{I}_{/X}} F(i) \oplus \operatorname*{colim}_{j\in\mathcal{I}_{/Y}} F(j) \\
&\simeq \mathfrak{l}F(X) \oplus \mathfrak{l}F(Y).
\end{aligned}$$

Here, we use in turn the general formula for left Kan extensions 3.2, the cofinality result above and F being additive, and colimits distributing over products of categories and sums of functors. In the third and second to last step, we used the fact that colimits of a constant diagram over any weakly contractible category are given by its value, by [Lur09, Cor. 4.4.4.10], and that the overcategories $\mathcal{I}_{/X}$ are weakly contractible (since $0 \to X$ specifies an initial object).

The proof of the additivity of right Kan extensions is exactly dual: Precomposing the functor

$$\mathcal{I}_{X/} \times \mathcal{I}_{Y/} \to \mathcal{I}_{X\oplus Y/}, \quad (X \to i, Y \to j) \mapsto (X \oplus Y \to i \oplus j)$$

is seen to preserve limits (as it admits a right adjoint), so the formula for right Kan extensions yields the result. □

We also need a slightly less structured version of this result. Let $\mathcal{C}$ be an ∞-category, and write $\mathcal{C}'$ for the ∞-category obtained by formally adjoining finite sums, i.e. the full subcategory of $\mathcal{P}(\mathcal{C})$ spanned by finite coproducts of representables. Here, the examples we care about are $\mathsf{Or}G' \simeq \mathsf{Fin}_G$ and $<G/1>' \simeq \mathsf{Free}_G$.

Proposition 3.8. *Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, the left Kan extension (if it exists) $\mathfrak{l}F\colon \mathcal{C}' \to \mathcal{D}$ preserves finite coproducts.*

Proof. For any objects a, b, c of $\mathcal{C}$, we have

$$\begin{aligned}
\operatorname{Map}_{\mathcal{C}'}(a, b \amalg c) &\simeq \operatorname{Map}_{\mathcal{P}(\mathcal{C})}(\operatorname{Map}_{\mathcal{C}}(-,a), \operatorname{Map}_{\mathcal{C}}(-,b) \amalg \operatorname{Map}_{\mathcal{C}}(-,c)) \\
&\simeq \operatorname{Map}_{\mathcal{C}}(a,b) \coprod \operatorname{Map}_{\mathcal{C}}(a,c)
\end{aligned}$$

by the Yoneda lemma, and thus an equivalence of relative overcategories

$$\mathcal{C}_{/b\amalg c} \simeq \mathcal{C}_{/b} \coprod \mathcal{C}_{/c}.$$

We then use the colimit formula for left Kan extensions to compute

$$\begin{aligned}
\mathfrak{l}F(a \amalg b) &\simeq \operatorname{colim}(\mathcal{C}_{/a\amalg b} \to \mathcal{C} \to \mathcal{D}) \\
&\simeq \operatorname{colim}(\mathcal{C}_{/a} \coprod \mathcal{C}_{/b} \to \mathcal{D}) \\
&\simeq F(a) \coprod F(b),
\end{aligned}$$

again using the fact that $\mathcal{C}_{/a}$ has a terminal object given by $a \xrightarrow{=} a$. □

We now consider Kan extensions of an object with an action by a finite group G to the orbit category Or_G. Recall that functors from the transport groupoid $G\textstyle\int * = \mathrm{B}G$ to some ∞-category $\mathcal{C}$ correspond to objects of $\mathcal{C}$ equipped with a 'left' action by G, $\mathsf{Or}G$ is the 1-category spanned by the set of objects $\{G/H\}_{H\leq G}$ and equivariant maps, and we have a natural identification $\mathrm{B}G^{\mathrm{op}} = <G/1>$. Given an object $X \in \mathcal{C}^{\mathrm{B}G}$, we shall write $\bar{X}$ for the composition

$$\mathrm{B}G^{\mathrm{op}} \xrightarrow{\sim} \mathrm{B}G \xrightarrow{X} \mathcal{C},$$

i.e. the associated object with 'right' G-action.

Proposition 3.9. *Given an object $X \in \mathcal{C}^{\mathrm{B}G}$, the left Kan extension $\mathfrak{l}\bar{X} \colon \mathsf{Or}G \to \mathcal{C}$ is characterized by $G/H \mapsto X_{hH}$.*

Proof. As Or_G is a 1-category, so is the relative overcategory $<G/1>_{/G/H}$. Writing out objects and morphisms, we identify it with the groupoid $(G\int G/H)^{\mathrm{op}} \simeq \mathrm{B}H^{\mathrm{op}}$ (compare also Lemma 4.9, which provides a more structured identification). Thus, by the colimit formula, we have

$$\begin{aligned} \mathfrak{l}\bar{X}(G/H) &\simeq \operatorname{colim}(\mathrm{B}H^{\mathrm{op}} \to \mathrm{B}G^{\mathrm{op}} \xrightarrow{\bar{X}} \mathcal{C}) \\ &\simeq X_{hH}. \end{aligned}$$

$\square$

Dually, right Kan extensions along $\mathrm{B}G \hookrightarrow \mathsf{Or}G^{\mathrm{op}}$ are characterized by

$$G/H \mapsto \lim_{\mathrm{B}H} X = X^{hH}.$$

We remark that Proposition 3.8 and Proposition 3.9 together fully describe left Kan extensions from $\mathrm{B}G$ to Fin_G, and right Kan extensions from $\mathrm{B}G$ to $\mathsf{Fin}_G^{\mathrm{op}}$.

4. Assembly and coassembly

The theory of assembly maps has a long history, going back to ideas by Frank Quinn, Jean-Louis Loday and Friedhelm Waldhausen, among others. The throughline is the attempt to approximate some (interesting, but difficult to compute) construction by another, less intractable construction in a canonical fashion. An early reference that formalizes this circle of ideas is the often cited 1991 note [WW95] by Michael Weiss and Bruce Williams, and (after translating it into the language of ∞-categories) we use their basic setup as our definition.

Recall that a functor $\mathcal{S} \to \mathsf{Sp}$ is called *strongly excisive* if it preserves coproducts and pushout squares, or equivalently all colimits, or equivalently is left Kan extended along $* \hookrightarrow \mathcal{S}$, or equivalently its homotopy groups form a generalized homology theory.

Using simplicial models for (homotopy) colimits, Weiss and Williams associate to any functor $F\colon \mathcal{S} \to \mathsf{Sp}$ a strongly excisive functor $F_\%\colon \mathcal{S} \to \mathsf{Sp}$, equipped with a natural transformation $F_\% \to F$ with the property $F_\%(*) \xrightarrow{\sim} F(*)$. We think of this as a more tractable approximation of F from the left: on homotopy groups, we read it as as a map

$$H_*(x; F(*)) \to \pi_* F(x),$$

so given knowledge of $\pi_* F(*)$, the left hand side might be computable using the usual spectral sequences.

Of course, in modern language, $F_\%$ has to be the functor mapping

$$x \mapsto \operatorname*{colim}_x F(*) \simeq x_+ \otimes F(*),$$

(i.e. the one left Kan extended from the value of F at the point) and the approximation is given by the natural comparison map

$$\operatorname*{colim}_x F(*) \longrightarrow F(\operatorname*{colim}_x *) \simeq F(x)$$

(i.e. the unit transformation associated to left Kan extensions), compare Example 3.4. In the relevant applications, one is interested in the approximation of F at the classifying spaces $\mathrm{B}G$, so we offer the following succinct definition.

Definition 4.1. Given a group G and a functor $F\colon \mathcal{S} \to \mathsf{Sp}$, the *Weiss-Williams-assembly map* α_{WW} associated to F with respect to G is the counit of the adjunction

$$\mathsf{Sp} \underset{\mathrm{res}_{*\hookrightarrow\mathcal{S}}}{\overset{\iota}{\rightleftarrows}} \mathsf{Sp}^{\mathcal{S}}$$

at F evaluated at $\mathrm{B}G$, i.e. the natural comparison map

$$\begin{tikzcd} \mathrm{B}G_+ \otimes F(*) \arrow[r, dashed, "\alpha_{WW}"] \arrow[d, "\sim"] & F(\mathrm{B}G) \\ \operatorname{colim}_{\mathrm{B}G} F(*) \arrow[r] & F(\operatorname{colim}_{\mathrm{B}G} *) \arrow[u, "\sim"'] \end{tikzcd}$$

Definition 4.2. Given any ring spectrum $R \in \mathsf{Sp}_{\mathbb{E}_1}$ and taking F to be the functor

$$A\colon \mathcal{S} \to \mathsf{Sp}; \quad x \mapsto K(x \otimes \mathsf{Perf} R),$$

the *(classical*, or sometimes *Loday) assembly map in K-theory of the group ring* $R[G]$ is the Weiss-Williams-assembly map associated to F. Indeed, we will later proof (see Theorem 11.3) that

$$\mathrm{B}G \otimes \mathsf{Perf} R \simeq \mathsf{Perf} R[G],$$

so this assembly map is a map of spectra

$$\mathrm{B}G_+ \otimes K(R) \to K(R[G]),$$

or, even more classically, on homotopy groups a map

$$H_*(\mathrm{B}G; K(R)) \to K_*(R[G]).$$

In the above definition, the K-theory functor may of course be replaced by any other meaningful invariant of stable ∞-categories (such as Topological Hochschild Homology THH or Topological Cyclic Homology TC) to obtain assembly maps for group rings in those theories.

4.1. **Coassembly.** Let us quickly dualize the preceding discussion: Given a functor $G\colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$, we may canonically and uniquely approximate it from the right by a cohomology theory. That is, there is a functor $G^{\%}\colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$ equipped with a natural transformation $G \to G^{\%}$ such that the map $G(*) \to G^{\%}(*)$ is an equivalence, and $G^{\%}$ takes colimits of spaces to limits of spectra (i.e. as a functor $\mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$, it preserves limits).

Of course, $G^{\%}$ then has to be the functor

$$x \mapsto \lim_{x} G(*) \simeq \mathrm{map}_{\mathsf{Sp}}(x, G(*)),$$

i.e. the right Kan extension of $G(*)$ considered as a functor $* \to \mathsf{Sp}$ along $* \to \mathcal{S}^{\mathrm{op}}$, see Example 3.4. The natural transformation $G \to G^{\%}$ is then given by the unit of the adjunction

$$\mathrm{Fun}(\mathcal{S}^{\mathrm{op}}, \mathsf{Sp}) \underset{\mathrm{r}}{\overset{\mathrm{res}}{\rightleftarrows}} \mathsf{Sp}.$$

Definition 4.3. The dual version of Definition 4.2 is given by considering the functor

$$\forall\colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}; \quad x \mapsto K(\mathrm{Fun}(x, \mathsf{Perf}R)).$$

This functor sends $\mathrm{B}G$ to the K-theory of perfect R-modules equipped with G-action, sometimes called the *Swan Theory* $G^R(R[G])$. The coassembly map is then a map of spectra

$$K(\mathsf{Perf}R^{\mathrm{B}G}) \to \mathrm{map}(\mathrm{B}G, K(R))$$

or on homotopy groups

$$G^R_*(R[G]) \to H^{-*}(\mathrm{B}G, K(R)).$$

4.2. **The equivariant perspective.** In their 1998 paper [DL98], James F. Davis and Wolfgang Lück establish a framework for assembly maps that allows more direct access to the equivariance involved. We restate it in ∞-categorical terms in the next definition.

Originally, the usefulness of this setup stems from the fact that it affords easy definitions of the *relative* assembly maps taking center stage in the Farrell-Jones conjecture and other isomorphism conjectures, see e.g. [RV18] and [Lüc20]. To us, it is very useful as a bridge between the world of classical assembly and that of genuine G-spectra, and thus a crucial change of perspective facilitating the proof of our main Theorem 1. In the language of [DL98], we will only consider assembly maps relative to the trivial family.

In the following, for simplicity of notation, we consider $\mathrm{B}G$ a full subcategory of Or_G via the usual

$$i\colon \mathsf{B}G \simeq \mathsf{B}G^{\mathrm{op}} =<G/1>\hookrightarrow \mathsf{Or}_G.$$

Definition 4.4. Given a group G and a functor $E\colon \mathsf{Or}_G \to \mathsf{Sp}$, the *Davis-Lück-assembly map* associated to E is the counit of the adjunction

$$\mathsf{Sp}^{\mathrm{B}G} \underset{\mathrm{res}_{\mathrm{B}G \hookrightarrow \mathsf{Or}_G}}{\overset{\mathfrak{l}}{\rightleftarrows}} \mathsf{Sp}^{\mathsf{Or}_G}$$

at E evaluated at G/G, i.e. the natural comparison map

$$E(G/1)_{hG} \xrightarrow{\alpha_{DL}} E(G/G).$$

Recall that we computed the relevant Kan extension in Proposition 3.9.

The Davis-Lück-assembly map generalizes the Weiss-Williams-assembly map, and the latter is recovered in the following way: Precomposition with

$$c\colon \mathsf{Or}_G \longrightarrow \mathrm{Fun}(\mathrm{B}G, \mathcal{S}) \xrightarrow{\mathrm{colim}} \mathcal{S}$$

(i.e. the functor that sends G/H to $\mathrm{B}H$) associates to any $F\colon \mathcal{S} \to \mathsf{Sp}$ a functor

$$E\colon \mathsf{Or}_G \to \mathcal{S} \xrightarrow{F} \mathsf{Sp},$$

and the Davis-Lück-assembly map of E is equivalent to the Weiss-Williams-assembly map of F, both taken with respect to G.

Remark 4.5. This well-known result appears to lack a full proof recorded in the literature. We close this gap by the following, mostly formal argument.

Definition 4.6. Consider the commutative square of ∞-categories

$$\begin{tikzcd} \mathrm{B}G \arrow[r, "p"] \arrow[d, "i"] & * \arrow[d, "j"] \\ \mathsf{Or}_G \arrow[r, "c"] & \mathcal{S}. \end{tikzcd}$$

and the square induced by applying $\mathrm{Fun}(-, \mathsf{Sp})$, i.e.

$$\begin{tikzcd} \mathsf{Sp}^{\mathrm{B}G} & \mathsf{Sp} \arrow[l, "p^*"'] \\ \mathsf{Sp}^{\mathsf{Or}_G} \arrow[u, "i^*"] & \mathsf{Sp}^{\mathcal{S}}. \arrow[l, "c^*"'] \arrow[u, "j^*"'] \end{tikzcd} \tag{4.7}$$

In the following, we write $i_!$ and $j_!$ for the left adjoints to i^* and j^* (i.e. the left Kan extensions along i and j), and e.g. η_i for the unit of the adjunction associated to i, and ε_i for the counit of the same adjunction. The Weiss-Williams-assembly map is then the counit $\varepsilon_j\colon j_!j^* \to \mathrm{id}$ at F evaluated at G/G, and the Davis-Lück-assembly map is the counit $\varepsilon_i\colon i_!i^* \to \mathrm{id}$ at E evaluated at $c(G/G) \simeq \mathrm{B}G$.

Recall from [Lur17, Def. 4.7.4.13] that the square 4.7 is called *left adjointable* if the canonical basechange transformation

$$\beta\colon \quad i_!p^* \xrightarrow{\eta_j} i_!p^*j^*j_! \xrightarrow{\sim} i_!i^*c^*j_! \xrightarrow{\varepsilon_i} c^*j_!$$

is an equivalence, i.e. if the two a priori distinct ways to form a functor $\mathsf{Or}_G \to \mathsf{Sp}$ from a given spectrum by forming vertical left adjoints in 4.7 are identified via the basechange transformation.

Theorem 4.8. *The square 4.7 is left adjointable, and the dashed composition (of natural transformations of functors* $\mathrm{Fun}(\mathcal{S}, \mathsf{Sp}) \to \mathrm{Fun}(\mathsf{Or}_G, \mathsf{Sp})$*)*

$$\begin{tikzcd} i_!i^*c^* \arrow[r, dashed] \arrow[d, "\sim"] & c^* \\ i_!p^*j^* \arrow[r, "\beta"] & c^*j_!j^* \arrow[u, "\varepsilon_j"'] \end{tikzcd}$$

identifies with the counit ε_i.

In particular, for any $F\colon \mathcal{S} \to \mathsf{Sp}$ and evaluating at the final object G/G, we obtain a canonical identification

$$\begin{tikzcd} i_!i^*c^*F\ (G/G) \simeq \mathrm{B}G_+ \otimes F(*) \arrow[r, "\alpha_{DL}"] \arrow[d, "\sim"] & F(BG) \\ c^*j_!j^*F\ (G/G) \simeq \mathrm{B}G_+ \otimes F(*) \arrow[ur, "\alpha_{WW}"'] & \end{tikzcd}$$

of the Davis-Lück-assembly map associated to $c^*F\colon \mathsf{Or}_G \to \mathcal{S} \to \mathsf{Sp}$ *and the Weiss-Williams assembly map associated to* F*.*

Proof. For the first statement, we first observe that, since left Kan extension (along a fully faithful functor) gives a fully faithful functor $\mathsf{Sp}^{\mathrm{B}G} \to \mathsf{Sp}^{\mathsf{Or}_G}$, the unit transformation η_j is always an equivalence. Thus, to recognize the basechange transformation

$$\beta\colon \quad i_!p^* \xrightarrow{\eta_j} i_!p^*j^*j_! \xrightarrow{\sim} i_!i^*c^*j_! \xrightarrow{\varepsilon_i} c^*j_!$$

as an equivalence, it remains to see that the counit

$$\varepsilon_i\colon i_!i^*c^*F \to c^*F$$

is an equivalence if F is in the image of $j_!$, i.e. is left Kan extended along $j\colon * \hookrightarrow \mathcal{S}$, i.e. that c^*F is itself left Kan extended along $\mathrm{B}G \hookrightarrow \mathsf{Or}_G$ if F was left Kan extended along $* \hookrightarrow \mathcal{S}$. By definition, we need to see that the induced contraction morphism

$$(<G/1>_{/G/H})^{\triangleright} \to \mathcal{S} \xrightarrow{F} \mathsf{Sp}$$

associated to c^*F is a colimiting cocone. It factorizes as

$$(<G/1>_{/G/H})^{\triangleright} \to (*_{/\mathrm{B}H})^{\triangleright} \to \mathcal{S} \xrightarrow{F} \mathsf{Sp}.$$

Here, the first functor is induced by c and an equivalence (by Lemma 4.9, proved right after), and the following composition is the contraction morphism associated to F and thus a colimiting cocone by the assumption that F is left Kan extended.

For the second statement, we consider the commutative diagram

$$\begin{tikzcd}
i_!i^*c^* \arrow[d, "\sim"] & i_!p^*j^* \arrow[r, "\sim"] & i_!i^*c^* \arrow[r, "\varepsilon_i"] & c^* \\
i_!p^*j^* \arrow[ur, "\sim"] \arrow[r, "\eta_j"] \arrow[rrr, bend right, "\beta"'] & i_!p^*j^*j_!j^* \arrow[u, "\varepsilon_j"] \arrow[r, "\sim"] & i_!i^*c^*j_!j^* \arrow[u, "\varepsilon_j"] \arrow[r, "\varepsilon_i"] & c^*j_!j^* \arrow[u, "\varepsilon_j"]
\end{tikzcd}$$

where commutativity of the left most triangle is the triangle identity, and the squares commute by naturality of counits.

The third statement is then confirmed by unraveling the definitions of the Weiss-Williams-assembly map as the counit ε_j, the Davis-Lück-assembly map as the counit ε_i, evaluated at G/G and $c(G/G) \simeq \mathrm{B}G$ respectively, and using the second part. □

We still owe the proof (and statement) of Lemma 4.9 to conclude the above proof of Theorem 4.8.

Lemma 4.9. *The functor of relative overcategories* $<G/1>_{/G/H} \to *_{/\mathrm{B}H}$ *induced by* $c\colon \mathsf{Or}_G \to \mathcal{S},\ G/H \mapsto \mathrm{B}H$ *is an equivalence.*

Proof. We first observe that $<G/1>_{/G/H}$ sits in the diagram

$$\begin{tikzcd}
<G/1>_{/G/H} \arrow[r] \arrow[d] & \mathsf{Or}_{G_{/G/H}} \arrow[r] \arrow[d] & \mathcal{S}^{\mathrm{B}G}_{/G/H} \arrow[d] \\
<G/1> \arrow[r, hook] & \mathsf{Or}_G \arrow[r, hook] & \mathcal{S}^{\mathrm{B}G}
\end{tikzcd}$$

where the left square is cartesian by definition and the right square is cartesian by inspection (i.e. it is a strict pullback in sSet, which suffices since the right hand map is a right fibration). Thus the outer rectangle is cartesian and we are reduced to showing that the transformation of cospans

$$
\begin{tikzcd}
{<G/1>} \arrow[dr, hook] \arrow[dd] & & \mathcal{S}^{\mathrm{B}G}_{/G/H} \arrow[dl] \arrow[dd] \\
& \mathcal{S}^{\mathrm{B}G} \arrow[dd] & \\
{<*>} \arrow[dr, hook] & & \mathcal{S}_{/\mathrm{B}H} \arrow[dl] \\
& \mathcal{S} &
\end{tikzcd}
\tag{4.10}
$$

induced by the colimit-functor induces an equivalence on pullbacks.

Recall that the Straightening/Unstraightening equivalence relative to a *space* X gives an identification $\mathrm{Fun}(X, \mathcal{S}) \simeq \mathcal{S}_{/X}$, under which the colimit-functor is identified with the usual right fibration $\mathcal{S}_{/X} \to \mathcal{S}$.[6]

Writing $G\int G/H$ for the G-transport groupoid of G/H, we observe that the canonical left fibration

$$G\int G/H \to G\int G/G$$

straightens to the G-set G/H. Now since $G\int G/H$ is equivalent to $\mathrm{B}H$ we may rewrite the top row of 4.10 to

$$<* \to \mathrm{B}G> \hookrightarrow \mathcal{S}_{/\mathrm{B}G} \leftarrow \mathcal{S}_{/\mathrm{B}H},$$

where for the term on the right, $\mathcal{S}^{\mathrm{B}G}_{/G/H}$, we used the identification

$$(\mathcal{S}_{/\mathrm{B}G})_{/(\mathrm{B}H \to \mathrm{B}G)} \simeq \mathcal{S}_{/\mathrm{B}H}$$

as in the proof of Proposition B.2, and under this identification the right vertical map in 4.10 is the identity on $\mathcal{S}_{/\mathrm{B}H}$.

Thus, we are considering the cube

$$
\begin{tikzcd}
P \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{S}_{/\mathrm{B}H} \arrow[dr, "="] \arrow[dd] & \\
& Q \arrow[rr] \arrow[dd] & & \mathcal{S}_{/\mathrm{B}H} \arrow[dd] \\
{<* \to \mathrm{B}G>} \arrow[rr] \arrow[dr] & & \mathcal{S}_{/\mathrm{B}G} \arrow[dr] & \\
& {<*>} \arrow[rr] & & \mathcal{S}
\end{tikzcd}
$$

where $P \to Q$ is the map of pullbacks we are interested in. The back and front faces are cartesian by definition. Since the bottom face is a pullback square, the composition of back and bottom face is cartesian, and thus so is the composition of top and front face. This implies that the top face is already cartesian, and thus the map $P \to Q$ is an equivalence, as a pullback of an equivalence. □

[6] This well-known result appears to lack a citeable reference. It holds since for X an ∞-groupoid, every map $Y \to X$ is a left fibration. That the colimit of $F\colon X \to \mathcal{S}$ is given by the total space of its unstraightening is e.g. [Lur21, Tag 02VF].

This concludes the proof of 4.8, and thus the identification of Weiss-Williams-assembly maps and Davis-Lück-assembly maps. In particular we note that the classical assembly map associated to the group ring $R[G]$,

$$\mathsf{B}G_+ \otimes K(R) \to K(R[G]),$$

identifies with the Davis-Lück assembly map associated to the functor

$$\mathsf{Or}_G \to \mathsf{Sp}, \quad G/H \mapsto K(\mathsf{B}H \otimes \mathsf{Perf}R).$$

Now, as above, the situation dualizes without resistance:

Definition 4.11. Given a group G and a functor $H\colon \mathsf{Or}_G^{\mathrm{op}} \to \mathsf{Sp}$, the *Davis-Lück-coassembly map* associated to G is the unit of the adjunction

$$\mathsf{Sp}^{\mathsf{Or}_G^{\mathrm{op}}} \underset{\mathrm{r}}{\overset{\mathrm{res}_{\mathsf{B}G \hookrightarrow \mathsf{Or}_G}}{\rightleftarrows}} \mathsf{Sp}^{\mathsf{B}G}$$

at E evaluated at G/G, i.e. the natural comparison map

$$E(G/G) \xrightarrow{\gamma_{DL}} E(G/1)^{hG}.$$

Viewing this as the left Kan extension (and its comparison map) of the opposite functor

$$\mathsf{Or}_G \to \mathsf{Sp}^{\mathrm{op}}$$

we have the following corollary of Theorem 4.8.

Proposition 4.12. *Given a functor $F\colon \mathcal{S}^{\mathrm{op}} \to \mathsf{Sp}$, the Weiss-Williams-coassembly map*

$$F(\mathsf{B}G) \to \mathrm{map}(\mathsf{B}G, F(*))$$

identifies with the Davis-Lück-coassembly map

$$F(*_{hG}) \to \mathrm{map}(*_{hG}, F(G/1))$$

associated to

$$\mathsf{Or}_G^{\mathrm{op}} \xrightarrow{c} \mathcal{S}^{\mathrm{op}} \xrightarrow{F} \mathsf{Sp}.$$

5. Commutative monoids in presentable ∞-categories

We collect notation and some well-known results about commutative monoids. The reason for our interest is Proposition 5.6, which states that for any presentable, preadditive ∞-category $\mathcal{E}$, the ∞-category of objects of $\mathcal{E}$ equipped with an action of the finite group G is equivalent to $\mathrm{Fun}^{\times}(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{E})$, where $\mathcal{L}(G)$ is the full subcategory of commutative monoids with G-action which are free on a finite set. We also describe this equivalence concretely, and proceed to use it in Chapter 12 to describe certain adjoints as Kan extensions.

Let $\mathcal{C}$ be an ∞-category admitting finite products. A *commutative monoid* in $\mathcal{C}$ is a functor A from the category of pointed finite sets $\mathtt{Fin}_*$ to $\mathcal{C}$ that satisfies the *Segal condition* (going back to Graeme Segal's seminal paper [Seg74]). The Segal condition states that for $i = 1, ..., n$ the characteristic maps $\rho_i \colon \boldsymbol{n}_+ \to \boldsymbol{1}_+$ together induce an equivalence

$$A(\boldsymbol{n}_+) \xrightarrow{\sim} \prod_n A(\boldsymbol{1}_+).$$

The ∞-category $\mathrm{CMon}(\mathcal{C})$ is then defined as the full subcategory of $\mathrm{Fun}(\mathtt{Fin}_*, \mathcal{C})$ of functors satisfying the Segal condition, and we abbreviate $\mathrm{CMon}(\mathcal{S})$ as CMon. The 'underlying object' or 'forgetful' functor

$$u\colon\ \mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$$

is given by evaluation at $\boldsymbol{1}_+$, and we think of A as equipping its underlying object with a 'homotopy coherently associative and commutative' multiplication via the map

$$uA \times uA \xleftarrow{\sim} A(\boldsymbol{2}_+) \xrightarrow{(\boldsymbol{2}\to\boldsymbol{1})_+} uA.$$

This notion coincides with that of an $\mathbb{E}_\infty$-algebra in $\mathcal{C}$ with respect to the cartesian symmetric monoidal structure as in [Lur17, Sect. 2.4.2].

Note that the Segal condition immediately implies that u is conservative, i.e. that a map of commutative monoids in $\mathcal{C}$ is an equivalence if and only if its underlying map in $\mathcal{C}$ is an equivalence.

Proposition 5.1. *The ∞-category* CMon *admits limits and sifted colimits, and the functor $u\colon$ CMon $\to \mathcal{S}$ preserves these.*

Proof. By [Lur09, Cor. 5.1.2.3], evaluation at an object preserves all limits and colimits as a functor from $\mathrm{Fun}(\mathtt{Fin}_*, \mathcal{S})$, so it remains to check that the inclusion $\mathrm{CMon} \hookrightarrow \mathrm{Fun}(\mathtt{Fin}_*, \mathcal{S})$ preserves limits and sifted colimits, i.e. that the pointwise limit or sifted colimit of functors satisfying the Segal condition also satisfies the Segal condition. But this is clear since limits and sifted colimits commute with finite products in the ∞-category of spaces, see [Lur09, Lemma 5.5.8.11 & Remark 5.5.8.12]. $\square$

The next result we cite from [GGN15].

Proposition 5.2. CMon *is a presentable, preadditive ∞-category.*

Presentability follows from expressing $\mathrm{CMon} \subseteq \mathrm{Fun}(\mathtt{Fin}_*, \mathcal{S})$ as the local objects with respect to a (small) set of maps, via some adjunction games. We refer to [GGN15, Prop. 4.1] for details. Preadditivity follows from the fact that coproducts in CMon are given by products in $\mathcal{S}$ by [Lur17, Cor. 3.2.4.7] (and so are products in CMon, by Proposition 5.1). We again refer to [GGN15, Prop. 2.3] for a more thorough discussion.

Proposition 5.1, Proposition 5.2, and the adjoint functor theorem [Lur09, Cor. 5.5.2.9] imply that $u\colon$ CMon $\to \mathcal{S}$ has a 'free' left adjoint $\mathcal{F}$, and then so does the forgetful functor $\mathrm{CMon}^{\mathrm{B}G} \to \mathrm{CMon} \to \mathcal{S}$. We denote the left adjoint of this composition by $\mathcal{F}_G$.

Next, for $\mathcal{C}$ presentable, we want to describe $\mathrm{CMon}(\mathcal{C})$ as the models of an algebraic theory (or Lawvere theory) in the sense of [Cra10] or [GGN15, App. B], that is, we identify an ∞-category $\mathbb{T}$ such that finite product-preserving functors $\mathbb{T} \to \mathcal{C}$ identify with commutative monoids in $\mathcal{C}$. We immediately cover the case of monoids equipped with an action by a group G.

Definition 5.3. $\mathcal{L}(G)$ denotes the full subcategory of $\mathrm{CMon}^{\mathrm{B}G}$ spanned by the essential image of

$$\mathsf{Fin} \hookrightarrow \mathcal{S} \xrightarrow{\mathcal{F}_G} \mathrm{CMon}^{\mathrm{B}G}.$$

We remark that $\mathcal{L}(G)$ is a full subcategory closed under finite coproducts of the preadditive ∞-category $\mathrm{CMon}^{\mathrm{B}G}$ and thus preadditive.

Now recall from [Lur09, Sect. 5.5.8] that for an ∞-category $\mathcal{C}$ admitting small coproducts, its *free cocompletion under sifted colimits* $\mathcal{P}_\Sigma(\mathcal{C})$ is given by

$$\begin{aligned} j\colon \mathcal{C} &\longrightarrow \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathcal{S}) \\ c &\longmapsto \mathrm{Map}_{\mathcal{C}}(-, c), \end{aligned}$$

characterized by the universal property that for every ∞-category $\mathcal{D}$ admitting sifted colimits, restriction along the Yoneda embedding induces an equivalence

$$\mathrm{Fun}_\Sigma(\mathcal{P}_\Sigma(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D}),$$

where $\mathrm{Fun}_\Sigma(\mathcal{C}, \mathcal{D})$ denotes the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors that preserve sifted colimits, see in particular [Lur09, Prop. 5.5.8.15 & Cor. 5.5.8.17].

Proposition 5.4. $\mathrm{CMon}^{\mathrm{B}G}$ *is the free cocompletion of* $\mathcal{L}(G)$ *under sifted colimits, i.e. the unique sifted functor* Φ *rendering the diagram*

$$\begin{tikzcd} \mathcal{L}(G) \arrow[d, "j"] \arrow[dr, hook] & \\ \mathcal{P}_\Sigma(\mathcal{L}(G)) \arrow[r, dashed, "\Phi"] & \mathrm{CMon}^{\mathrm{B}G} \end{tikzcd}$$

commutative is an equivalence.

Proof. Φ is fully faithful: By Proposition 5.1 (and the fact that $\mathrm{res}_{*\to \mathrm{B}G}\colon \mathcal{C}^{\mathrm{B}G} \to \mathcal{C}$ preserves all colimits), the forgetful functor

$$u\colon\ \mathrm{CMon}^{\mathrm{B}G} \to \mathcal{S}$$

preserves sifted colimits. Thus, using the fact that in $\mathcal{S}$, sifted colimits commute with finite products, we confirm that the corepresented functor $\mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(\mathcal{F}_G(x), -)$ preserves sifted colimits, for x a finite set:

$$\begin{aligned} \mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(\mathcal{F}_G(x), \operatorname*{colim}_S A_s) &\simeq \mathrm{Map}_{\mathcal{S}}(x, u \operatorname*{colim}_S A_s) \\ &\simeq \prod_x \operatorname*{colim}_S uA_s \\ &\simeq \operatorname*{colim}_S \prod_x uA_s \simeq \operatorname*{colim}_S \mathrm{Map}_{\mathcal{S}}(x, uA_s) \\ &\simeq \operatorname*{colim}_S \mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(\mathcal{F}_G(x), A_s). \end{aligned}$$

Since every object of $\mathcal{P}_\Sigma(\mathcal{L}(G))$ is equivalent to a sifted colimit of representables (see [Lur09, Cor. 5.1.5.8], whose proof adapts without change), we write arbitrary objects of $\mathcal{P}_\Sigma(\mathcal{L}(G))$ as

$$A \simeq \operatorname*{colim}_S ja_s, \quad B \simeq \operatorname*{colim}_R jb_r$$

for S, R sifted. We then factorize

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{L}(G))}(A,B) \arrow[r, "\Phi"] & \mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(\Phi A, \Phi B) \\
\lim_S \mathrm{colim}_R \mathrm{Map}_{\mathcal{L}(G)}(a_s, b_r) \arrow[r, "\sim"] \arrow[u, "\sim"] & \lim_S \mathrm{colim}_R \mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(a_s, b_r). \arrow[u, "\sim"']
\end{tikzcd}$$

Here, the left vertical equivalence is a consequence of the Yoneda lemma (and $\mathcal{P}_\Sigma(\mathcal{C})$ being a full subcategory closed under sifted colimits of $\mathcal{P}(\mathcal{C}) = \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$). The lower horizontal equivalence is the definition of $\mathcal{L}(G)$ as a full subcategory of $\mathrm{CMon}^{\mathrm{B}G}$, and the right vertical equivalence uses the above preservation of sifted colimits of $\mathrm{Map}_{\mathrm{CMon}^{\mathrm{B}G}}(a_s, -)$ and the universal property of Φ.

Φ is essentially surjective: We need to show that $\mathrm{CMon}^{\mathrm{B}G}$ is generated under sifted colimits by $\mathcal{L}(G)$. The monadic bar construction of the monad $u \circ \mathcal{F}_G$ associated to the adjunction expresses any $A \in \mathrm{CMon}^{\mathrm{B}G}$ as the geometric realization of a simplicial object (i.e. a sifted colimit) taking values in the essential image $\mathcal{F}_G(\mathcal{S})$ (see [Lur17, Sect. 4.7], in particular Proposition 4.7.3.14). But of course any space is a sifted colimit of finite sets (since infinite sets are filtered colimits of finite sets), so we conclude. □

Remark 5.5. Let us describe Φ concretely: The underlying space of ΦE for E a product preserving functor

$$\big[E \colon \mathcal{L}(G)^{\mathrm{op}} \to \mathcal{S}\big] \in \mathcal{P}_\Sigma(\mathcal{L}(G))$$

is equivalent to its evaluation at $\mathcal{F}_G(\mathbf{1})$.

This holds since $\mathrm{ev}(\mathcal{F}_G(\mathbf{1}))\colon \mathcal{P}_\Sigma(\mathcal{L}(G)) \to \mathcal{S}$ factorizes as

$$\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{S}) \hookrightarrow \mathrm{Fun}(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{ev}(\mathbf{1})} \mathcal{S}$$

both of which preserve sifted colimits by [Lur09, Prop. 5.5.8.10.(4)]. So does the functor $u\colon \mathrm{CMon}^{\mathrm{B}G} \to \mathcal{S}$, thus the functors

$$u\Phi,\ \mathrm{ev}(\mathcal{F}_G(1))\colon\ \mathcal{P}_\Sigma(\mathcal{L}(G)) \to \mathcal{S}$$

are equivalent if they agree on representable functors $E \simeq \mathrm{Map}_{\mathcal{L}(G)}(-, x)$. In this case, $u\Phi E \simeq ux$ by definition of Φ, and

$$\mathrm{ev}(\mathcal{F}_G(\mathbf{1}))E \simeq \mathrm{Map}_{\mathcal{L}(G)}(\mathcal{F}_G(\mathbf{1}), x) \simeq \mathrm{Map}_{\mathcal{S}}(\mathbf{1}, ux) \simeq ux$$

by the adjunction.

Further, let us describe the action maps on $u\Phi E \in \mathcal{S}^{\mathrm{B}G}$: For $E \simeq \mathrm{Map}_{\mathcal{L}(G)}(-, \mathcal{F}_G(\mathbf{1}))$, the action by $g \in G$ is exactly the action by g on $u\mathcal{F}_G(\mathbf{1})$, which under the equivalence

$$u\mathcal{F}_G(\mathbf{1}) \simeq \mathrm{Map}_{\mathcal{L}(G)}(\mathcal{F}_G(\mathbf{1}), \mathcal{F}_G(\mathbf{1}))$$

corresponds to the pushforward along the (equivariant!) map $\mathcal{F}_G(\mathbf{1}) \xrightarrow{\cdot g} \mathcal{F}_G(\mathbf{1})$, given on underlying commutative monoids (without action) as

$$\bigoplus_G \mathcal{F}(\mathbf{1}) \xrightarrow{\cdot g} \bigoplus_G \mathcal{F}(\mathbf{1}),$$

permuting the factors by right multiplication with g. By equivariance, the map on $\mathrm{Map}_{\mathcal{L}(G)}(\mathcal{F}_G(\mathbf{1}), \mathcal{F}_G(\mathbf{1}))$ induced by pushforward along $\cdot g$ on the target and the map induced by pullback along $\cdot g$ on the source agree. This description then carries over to arbitrary functors $E\colon \mathcal{L}(G)^{\mathrm{op}} \to \mathcal{S}$ by extending along finite and sifted colimits.

Proposition 5.6. *For $\mathcal{C}$ any presentable ∞-category, there is an equivalence of categories*
$$\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{C}) \simeq \mathrm{Fun}(\mathrm{B}G, \mathrm{CMon}(\mathcal{C})).$$
If in addition $\mathcal{C}$ is preadditive, we obtain an equivalence
$$\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{C}) \simeq \mathcal{C}^{\mathrm{B}G}.$$

Proof. For the Lurie tensor product of presentable ∞-categories, we have equivalences

$$\mathrm{Fun}(K, \mathcal{S}) \otimes \mathcal{C} \simeq \mathrm{Fun}(K, \mathcal{C}), \tag{5.7}$$
$$\mathrm{Fun}^\times(K, \mathcal{S}) \otimes \mathcal{C} \simeq \mathrm{Fun}^\times(K, \mathcal{C}), \tag{5.8}$$
$$\mathrm{CMon} \otimes \mathcal{C} \simeq \mathrm{CMon}(\mathcal{C}), \tag{5.9}$$

all of which are proved in essentially the same fashion: Use the the description $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^R(\mathcal{D}^{\mathrm{op}}, \mathcal{C})$ of [Lur17, Prop. 4.8.1.17] to identify both sides of each equivalence with the same full subcategory of $\mathrm{Fun}(K \times \mathcal{C}^{\mathrm{op}}, \mathcal{S})$, see also [GGN15, Thm. 4.6].

With these, we tensor the equivalence of Proposition 5.4 with $\mathcal{C}$ to obtain

$$\begin{tikzcd}
\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{S}) \otimes \mathcal{C} \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathrm{Fun}(\mathrm{B}G, \mathrm{CMon}) \otimes \mathcal{C} \arrow[d, "\sim"] \\
\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{C}) \arrow[r, dashed] & \mathrm{Fun}(\mathrm{B}G, \mathrm{CMon}(\mathcal{C})).
\end{tikzcd}$$

The second claim of the proposition is true since for $\mathcal{C}$ preadditive, the forgetful functor $u\colon \mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ is an equivalence, see [GGN15, Prop. 2.3(iv)]. Said informally: commutative monoids are $\mathbb{E}_\infty$-algebras with respect to the cartesian symmetric monoidal structure, which, by preadditivity, is equivalent to the cocartesian symmetric monoidal structure. But in the latter, there is one and only one coherent multiplication for any object, namely the fold map $c \coprod c \xrightarrow{\triangledown} c$. ☐

Remark 5.10. The description of Remark 5.5 extends naturally to this more general situation, i.e. the equivalence
$$\mathrm{Fun}^\times(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\sim} \mathcal{E}^{\mathrm{B}G}$$
for $\mathcal{E}$ preadditive sends a functor E to the object $E(\mathcal{F}_G(1))$ with action by g given by
$$E\big(\mathcal{F}_G(1) \xleftarrow{\cdot g} \mathcal{F}_G(1)\big).$$

In a preadditive ∞-category $\mathcal{E}$, the mapping spaces $\mathrm{Map}_{\mathcal{E}}(a, b)$ are canonically equipped with the structure of a commutative monoid, given on maps $f, g\colon a \to b$ by
$$f \oplus g\colon a \xrightarrow{\triangle} a \oplus a \xrightarrow{f \coprod g} b \oplus b \xrightarrow{\triangledown} b.$$
More formally, we claim that we have for $\mathcal{E}$ preadditive a canonical lift

$$\begin{tikzcd}
 & \mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathrm{CMon}) \arrow[d, "u"] \\
\mathcal{E} \arrow[ur, dashed] \arrow[r, "j"'] & \mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathcal{S})
\end{tikzcd}$$

where again j denotes the Yoneda embedding $c \mapsto \mathrm{Map}_{\mathcal{E}}(-, c)$. Since $\mathcal{P}_\Sigma(\mathcal{E})$ is generated by the representables under sifted colimits, this suggests (and is indeed implied by) the following.

Proposition 5.11. *For $\mathcal{E}$ preadditive, the forgetful functor*

$$\mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}_\Sigma(\mathcal{E})$$

is an equivalence.

Proof. Via the arguments explained in the proof of Proposition 5.6, we see

$$\begin{aligned}\mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathrm{CMon}) &\simeq \mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathcal{S}) \otimes \mathrm{CMon}\\ &\simeq \mathrm{CMon}(\mathrm{Fun}^\times(\mathcal{E}^{\mathrm{op}}, \mathcal{S})),\end{aligned}$$

so the map in question identifies with the forgetful functor

$$u\colon \mathrm{CMon}(\mathcal{P}_\Sigma(\mathcal{E})) \to \mathcal{P}_\Sigma(\mathcal{E}).$$

Thus, we are reduced to recognizing $\mathcal{P}_\Sigma(\mathcal{E})$ as preadditive, again as explained in the proof of Proposition 5.6, following [GGN15, Prop. 2.3(iv)]. For sifted colimits of representables, the relevant comparison map is an equivalence:

$$\begin{tikzcd}
(\mathrm{colim}_S\, ja_s) \coprod (\mathrm{colim}_R\, jb_r) \arrow[r] & (\mathrm{colim}_S\, ja_s) \times (\mathrm{colim}_R\, jb_r) \\
\mathrm{colim}_{S\times R}(ja_s \coprod jb_r) \arrow[r] \arrow[u, "\sim"] \arrow[d, "\sim"'] & \mathrm{colim}_{S\times R}(ja_s \times jb_r) \arrow[u, "\sim"] \\
\mathrm{colim}_{S\times R}(j(a_s \coprod b_r)) \arrow[r, "\sim"] & \mathrm{colim}_{S\times R}(j(a_s \times b_r)) \arrow[u, "\sim"]
\end{tikzcd}$$

Here, the upper left equivalence is commuting the colimit with the coproduct, combined with the fact that sifted ∞-categories are weakly contractible ([Lur09, Prop. 5.5.8.7]) and colimits of weakly contractible shape of constant functors agree with their value ([Lur09, Cor. 4.4.4.10]). The lower left equivalence is the fact that $j\colon \mathcal{E} \to \mathcal{P}_\Sigma(\mathcal{E})$ preserves finite coproducts by [Lur09, Proposition 5.5.8.10(2)]. The upper right equivalence then is commuting sifted colimits with finite products (which holds since the colimits are equivalently computed in the ∞-topos $\mathcal{P}(\mathcal{E})$, see [Lur09, Remark 5.5.8.12]), and the lower right equivalence is the Yoneda embedding preserving products. Finally, the lower horizontal equivalence is of course the assumption of $\mathcal{E}$ being preadditive.

Since $\mathcal{P}_\Sigma(\mathcal{E})$ is generated by the representables under sifted colimits, we conclude. □

6. The ∞-category of spans

Specializing the approach of [Bar17], we define an ∞-category of *spans* associated to any ∞-category $\mathcal{C}$ admitting pullbacks. Its objects will agree with those of $\mathcal{C}$, while a morphism $c \to d$ will be given by span in $\mathcal{C}$, i.e. a diagram

$$\begin{tikzcd} & e \arrow[dl] \arrow[dr] & \\ c & & d. \end{tikzcd}$$

Composition will correspond to the formation of pullbacks, so that a functor

$$\mathrm{Span}(\mathcal{C}) \to \mathcal{D}$$

encodes both a covariant and a contravariant functoriality in $\mathcal{C}$ with the same value on objects, and the following compatibility: If

$$\begin{tikzcd} \cdot \arrow[r, "v"] \arrow[d, "u"] & \cdot \arrow[d, "g"] \\ \cdot \arrow[r, "f"] & \cdot \end{tikzcd}$$

is a pullback square in $\mathcal{C}$, we have an equivalence

$$g^* \circ f_* \simeq v_* \circ u^*.$$

Of course, there is actually an infinite hierarchy of coherences hidden in the functor $\mathrm{Span}(\mathcal{C}) \to \mathcal{D}$, but one example where this intuition can be made precise is in the setting of adjoint functors and adjointable squares as in Definition 4.6. This is Clark Barwick's *Unfurling* machinery, see [Bar17, Ch. 11].
Let us begin by associating to a poset P its poset of *twisted arrows*

$$\mathsf{Tw}(P) := \{(p, q) \mid p \leq q\} \subset P \times P^{\mathrm{op}}.$$

Of course $\mathsf{Tw}\colon \mathsf{Poset} \to \mathsf{Poset}$ is easily seen to be a functor, and as usual we regard $\mathsf{Tw}(P)$ as an ∞-category without additional notation. We also remark that this definition extends without much hassle to general categories and even to ∞-categories (see [Bar17, Ch. 2]), but we won't have need for either. Note that in *loc.cit.,* the author refers to the opposite of our $\mathsf{Tw}(P)$ as the twisted arrow category. We follow the convention of e.g. [Lur21].

To illustrate, $\mathsf{Tw}([3])$ is the category

$$\begin{tikzcd}[column sep=small] & & & 03 \arrow[dl] \arrow[dr] & & & \\ & & 02 \arrow[dl] \arrow[dr] & & 13 \arrow[dl] \arrow[dr] & & \\ & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & & 23 \arrow[dl] \arrow[dr] & \\ 00 & & 11 & & 22 & & 33, \end{tikzcd}$$

and an ordered inclusion $[k] \overset{i}{\hookrightarrow} [n]$ picks exactly the subdiagram $\mathsf{Tw}([k]) \hookrightarrow \mathsf{Tw}([n])$ spanned by the vertices with both indices in the image of i.

Given an ∞-category $\mathcal{C}$, a functor $\mathsf{Tw}([n]) \to \mathcal{C}$ is called *cartesian* if every square

$$\begin{tikzcd} & pq \arrow[dl] \arrow[dr] & \\ pq' \arrow[dr] & & p'q \arrow[dl] \\ & p'q' & \end{tikzcd}$$

for $p \leq p' \leq q' \leq q$ is sent to a pullback in $\mathcal{C}$, and we let $\mathrm{Hom}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C})$ denote the *set* of cartesian functors.

Certainly, the association

$$[n] \mapsto \mathrm{Hom}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C})$$

defines a simplicial set, and it is a theorem of Clark Barwick that if $\mathcal{C}$ admits pullbacks, this indeed defines an ∞-category [Bar17, Ch. 3].

Definition 6.1. Given an ∞-category $\mathcal{C}$ which admits pullbacks, the *span-category of* $\mathcal{C}$, denoted $\mathrm{Span}(\mathcal{C})$, is the ∞-category given by the simplicial set

$$[n] \mapsto \mathrm{Hom}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C}).$$

Barwick refers to this as the *effective Burnside category* A^{eff}, but to the present author it appears more natural to call it the span-category of $\mathcal{C}$. We remark that a pullback-preserving functor $\mathcal{C} \to \mathcal{D}$ defines a functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$ in the obvious way. For concreteness, let us state that the objects of $\mathrm{Span}(\mathcal{C})$ are exactly the objects of $\mathcal{C}$, edges are spans $x \leftarrow z \to y$, and 2-simplices are cartesian diagrams of the form

$$\begin{tikzcd} & & 02 \arrow[dl] \arrow[dr] \arrow[dll, bend right] \arrow[drr, bend left] & & \\ & 01 \arrow[dl] \arrow[dr] & & 12 \arrow[dl] \arrow[dr] & \\ 00 & & 11 & & 22, \end{tikzcd}$$

with colors indicating face maps corresponding to

$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2. \end{tikzcd}$$

Remark 6.2. The mapping spaces in $\mathrm{Span}(\mathcal{C})$ are indeed given by what one would reasonably expect, namely by the ∞-groupoid of spans, or more precisely said

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x, y) \simeq (\mathcal{C}_{/\{x,y\}})^{\sim}.$$

This is an immediate consequence of the way Barwick constructs the span-category, so we should briefly comment on it. He defines a bisimplicial set, i.e. a functor

$$\Delta^{\mathrm{op}} \to \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathsf{Set})$$

by

$$[n] \mapsto (\underline{\mathrm{Hom}}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C}))^{\sim}$$

where $\underline{\mathrm{Hom}}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C})$, as usual, denotes the simplicial set (or ∞-category) of cartesian functors, a full subcategory of $\mathrm{Fun}(\mathsf{Tw}([n]), \mathcal{C})$. He then proceeds to prove that, if $\mathcal{C}$ admits pullbacks, this bisimplicial set is in fact a *complete Segal space*, a notion introduced by Charles Rezk in [Rez01]. Complete Segal spaces model homotopy theories in essentially the same way ∞-categories do, and a concrete Quillen equivalence to the model category of ∞-categories was established by Joyal and Tierney [JT07]. Constructing an equivalent ∞-category from a complete Segal space $\boldsymbol{X}$ is as simple as one could hope: It is the simplicial set given by the 0'th row of $\boldsymbol{X}$, i.e. the simplicial set whose n-simplices are the vertices of $\boldsymbol{X}_n$, see [JT07, Ch. 4].

Note that for the complete Segal space above, this gives exactly the simplicial set

$$\mathrm{Span}(\mathcal{C})\colon [n] \mapsto \mathrm{Hom}^{\mathrm{cart}}(\mathsf{Tw}([n]), \mathcal{C}).$$

Mapping spaces in complete Segal spaces are described in [Rez01, 5.1], as the (point-set = homotopy, see *loc.cit*) fibers of the map

$$\begin{tikzcd}
\boldsymbol{X}_1 \arrow[rr, "{d_1, d_0}"] \arrow[d, "="] & & \boldsymbol{X}_0 \times \boldsymbol{X}_0 \arrow[d, "="] \\
(\underline{\mathrm{Hom}}^{\mathrm{cart}}(\mathsf{Tw}([1]), \mathcal{C}))^{\sim} \arrow[rr, "{\mathrm{ev}_{00}, \mathrm{ev}_{11}}"] \arrow[d, "\cong"] & & (\underline{\mathrm{Hom}}^{\mathrm{cart}}(\mathsf{Tw}([0]), \mathcal{C}))^{\sim} \arrow[d, "\cong"] \\
\mathrm{Fun}(\Delta^0 \star (\Delta^0 \coprod \Delta^0), \mathcal{C})^{\sim} \arrow[rr] & & \mathcal{C}^{\sim} \times \mathcal{C}^{\sim}
\end{tikzcd}$$

where in the latter step we used that being cartesian is vacuous for functors out of

$$\mathsf{Tw}([1]) \cong \Delta^0 \star (\Delta^0 \coprod \Delta^0).$$

The fiber of

$$\mathrm{Fun}(\Delta^0 \star (\Delta^0 \coprod \Delta^0), \mathcal{C}) \to \mathcal{C} \times \mathcal{C}$$

at x, y identifies with $\mathcal{C}_{/\{x,y\}}$. Now recall that $\mathcal{C} \mapsto \mathcal{C}^{\sim}$ is right adjoint to the inclusion $\mathcal{S} \to \mathrm{Cat}_\infty$ and thus commutes with taking fibers and the formation of pullbacks, so we conclude

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x, y) \simeq (\mathcal{C}_{/\{x,y\}})^{\sim}.$$

Remark 6.3. The span-category comes equipped with natural functors

$$\begin{aligned}
-_*\colon \mathcal{C} &\longrightarrow \mathrm{Span}(\mathcal{C}), \\
-^*\colon \mathcal{C} &\longrightarrow \mathrm{Span}(\mathcal{C})^{\mathrm{op}}, \\
\tau\colon \mathrm{Span}(\mathcal{C}) &\xrightarrow{\sim} \mathrm{Span}(\mathcal{C})^{\mathrm{op}}.
\end{aligned}$$

The first two are defined, on simplices, by precomposition with the natural maps

$$\mathsf{Tw}([n]) \xrightarrow{p_1} [n], \quad \mathsf{Tw}([n]) \xrightarrow{p_2} [n]^{\mathrm{op}}$$

respectively, and the latter is induced by

$$\begin{tikzcd}
\mathsf{Tw}([n]^{\mathrm{op}}) \arrow[r, hook] \arrow[d, dashed, "\sim"] & {[n]^{\mathrm{op}} \times [n]} \arrow[d, "\mathrm{swap}"] \\
\mathsf{Tw}([n]) \arrow[r, hook] & {[n] \times [n]^{\mathrm{op}}}.
\end{tikzcd}$$

These act exactly as one would guess: E.g. for a morphism $f\colon x \to y$, we have

$$f_* = \quad x \xleftarrow{=} x \xrightarrow{f} y, \quad f^* = \quad y \xleftarrow{f} x \xrightarrow{=} x,$$

and $\tau(\sigma_*) = \sigma^*$ for any simplex σ of $\mathcal{C}$.

Under mild conditions on the category $\mathcal{C}$, the span-category $\mathrm{Span}(\mathcal{C})$ is preadditive.

Definition 6.4. An ∞-category $\mathcal{C}$ is *disjunctive* if in addition to admitting pullbacks it

- admits finite coproducts,
- which are disjoint, i.e.

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & X \coprod Y \end{tikzcd}$$

is a pullback square for any two objects X, Y in $\mathcal{C}$,
- and universal, i.e. given any morphism $T \to S$ and two objects X, Y over S, the square

$$\begin{tikzcd} X \times_S T \coprod Y \times_S T \arrow[r] \arrow[d] & X \coprod Y \arrow[d] \\ T \arrow[r] & S \end{tikzcd} \tag{6.5}$$

is a pullback square.

The categories we have in mind are the usual suspects, Fin_G and Free_G.

Proposition 6.6. [Bar17, Proposition 4.3]. *Given a disjunctive ∞-category $\mathcal{C}$, the span-category* $\mathrm{Span}(\mathcal{C})$ *is preadditive, with sums given by coproducts in $\mathcal{C}$.*

Proof. In *loc.cit.*, Clark Barwick shows that the functor $-_*\colon \mathcal{C} \to \mathrm{Span}(\mathcal{C})$ preserves finite coproducts if $\mathcal{C}$ is disjunctive. That these are exhibited as finite products by the natural maps is then a consequence of the duality-equivalence $\tau\colon \mathrm{Span}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Span}(\mathcal{C})^{\mathrm{op}}$ preserving objects. $\square$

We close this chapter with a technical lemma, used in Chapter 12 to compare certain Kan extensions. We write $\beta^*\alpha_*$ for a generic span $x \xleftarrow{\beta} z \xrightarrow{\alpha} y$.

Lemma 6.7. *Let $\mathcal{C}$ be an ∞-category admitting pullbacks, and $c \in \mathcal{C}$. Then*

$$\mathcal{C}_{/c} \to \mathrm{Span}(\mathcal{C})_{/c}, \quad \sigma \mapsto \sigma_*$$

(i) is fully faithful.
(ii) admits a left adjoint, given on objects as $e^*f_* \mapsto f_*$.

Proof of (i). We are looking to show that, for any two morphisms $x \xrightarrow{f} c$ and $y \xrightarrow{g} c$ of $\mathcal{C}$, the map

$$\mathrm{Map}_{\mathcal{C}_{/c}}(f, g) \longrightarrow \mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*)$$

is an equivalence. By proposition B.2, which describes the mapping spaces of overcategories as fibers of mapping spaces, this map is the left vertical map in the following diagram of horizontal fiber sequences:

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}_{/c}}(f,g) \arrow[r] \arrow[d, dashed] & \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r, "g\circ -"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(x,c) \arrow[d] \\ \mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*,g_*) \arrow[r] & \mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x,y) \arrow[r, "g_*\circ -"] & \mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x,c) \end{tikzcd} \tag{6.8}$$

with fibers taken at $f \in \mathrm{Map}_{\mathcal{C}}(x, c)$ and $f_* \in \mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x, c)$ respectively. In remark 6.2, we have identified the mapping spaces of the span-category as

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(x, y) \simeq (\mathcal{C}_{/\{x,y\}})^{\sim} = (\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim}.$$

Under this equivalence, composition with g_* corresponds to the map

$$(\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim} \xrightarrow{\mathrm{id} \times_{\mathrm{id}} (g\circ -)} (\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/c})^{\sim},$$

as seen from the cartesian diagram

$$\begin{tikzcd}[column sep=small]
 & & 02 \arrow[dl, "="'] \arrow[dr, "\beta"] & & \\
 & 01 \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & 12 \arrow[dl, "="'] \arrow[dr, "g"] & \\
00 & & 11 & & 22.
\end{tikzcd}$$

Again, since $\mathcal{C} \mapsto \mathcal{C}^\sim$ is right adjoint to the inclusion $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ (and preserves the terminal object), it commutes with taking fibers and the formation of pullbacks, and we may rewrite the lower fiber sequence in diagram 6.8 as

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*) \to (\mathcal{C}_{/x})^\sim \times_{\mathcal{C}^\sim} (\mathcal{C}_{/y})^\sim \to (\mathcal{C}_{/x})^\sim \times_{\mathcal{C}^\sim} (\mathcal{C}_{/c})^\sim.$$

Since the map with respect to which the fiber is computed is the identity on the first coordinate, projection to the second induces an equivalence on fibers, and since further the fiber (taken in Cat_∞) of

$$\mathcal{C}_{/y} \xrightarrow{g\circ -} \mathcal{C}_{/c}$$

is already in $\mathcal{S}$ (e.g. as a consequence of Proposition B.2), we can do without the $(-)^\sim$ to arrive at the diagram of fiber sequences

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{/c}}(f,g) \arrow[r] \arrow[d, dashed] & \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r, "g\circ -"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(x,c) \arrow[d] \\
\mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*) \arrow[r] & \mathcal{C}_{/y} \arrow[r, "g_*\circ -"] & \mathcal{C}_{/c}.
\end{tikzcd}$$

Since the right two vertical maps are the canonical inclusions (sending a map $x \to y$ to itself), we recognize the familiar fiber sequences defining (right) mapping spaces and extend vertically down to the commutative square of (horizontal and vertical) fiber sequences

$$\begin{tikzcd}
\mathrm{Map}_{\mathcal{C}_{/c}}(f,g) \arrow[r] \arrow[d, dashed] & \mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r, "g\circ -"] \arrow[d] & \mathrm{Map}_{\mathcal{C}}(x,c) \arrow[d] \\
\mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*) \arrow[r] \arrow[d] & \mathcal{C}_{/y} \arrow[r, "g_*\circ -"] \arrow[d] & \mathcal{C}_{/c} \arrow[d] \\
* \arrow[r] & \mathcal{C} \arrow[r, "="] & \mathcal{C}.
\end{tikzcd}$$

Thus the dashed map is an equivalence, and we conclude. □

Proof of (ii). The left adjoint $\mathrm{Span}(\mathcal{C})_{/c} \to \mathcal{C}_{/c}$ is, on simplices, induced by precomposition with

$$[n+1] \to \mathsf{Tw}([n+1]), \quad i \mapsto (i, n+1).$$

Note that this is natural in $[n]$, with respect to the natural identifications $[n] \star [0] = [n+1]$.

To check that this provides the left adjoint to the (fully faithful, by (i)) functor $\mathcal{C}_{/c} \to \mathrm{Span}(\mathcal{C})_{/c}$, we would like to see a natural equivalence

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*) \to \mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(e^* f_*, g_*),$$

for any

$$\begin{tikzcd} & x \arrow[dl, "e"'] \arrow[dr, "f"] & & & y \arrow[dr, "g"] & \\ x' & & c, & & & c. \end{tikzcd}$$

As above, the left hand side is the fiber of

$$(\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim} \xrightarrow{\mathrm{id} \times (g \circ -)} (\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/c})^{\sim}$$

at (id, f), and by an analogous computation, the right hand side is the fiber of

$$(\mathcal{C}_{/x'} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim} \xrightarrow{\mathrm{id} \times (g \circ -)} (\mathcal{C}_{/x'} \times_{\mathcal{C}} \mathcal{C}_{/c})^{\sim}$$

at (e, f). We obtain a map of fiber sequences

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_*, g_*) \arrow[r] \arrow[d, dashed] & (\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim} \arrow[r] \arrow[d] & (\mathcal{C}_{/x} \times_{\mathcal{C}} \mathcal{C}_{/c})^{\sim} \arrow[d] \\ \mathrm{Map}_{\mathrm{Span}(\mathcal{C})_{/c}}(f_* e^*, g_*) \arrow[r] & (\mathcal{C}_{/x'} \times_{\mathcal{C}} \mathcal{C}_{/y})^{\sim} \arrow[r] & (\mathcal{C}_{/x'} \times_{\mathcal{C}} \mathcal{C}_{/c})^{\sim} \end{tikzcd}$$

where the vertical maps are given by

$$(\alpha, \beta) \mapsto (e \circ \alpha, \beta).$$

In the same way as in the proof of (i), we argue that projection to the second coordinate induces an equivalence on fibers on either column, and thus conclude that the dashed map is an equivalence. $\square$

7. Mackey Functors

Let $\mathcal{E}$ be a preadditive ∞-category. For brevity, we will write Span_G for $\mathrm{Span}(\mathsf{Fin}_G)$. By Proposition 6.6, this is a preadditive ∞-category.

Definition 7.1. A *G-Mackey functor with coefficients in $\mathcal{E}$* is a sum-preserving functor

$$\mathcal{M}\colon \mathrm{Span}_G^{\mathrm{op}} \longrightarrow \mathcal{E}.$$

These assemble into an ∞-category $\mathrm{Mack}_G(\mathcal{E})$ as the full subcategory

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E}).$$

As the group G and the coefficient category will usually be clear from context, we will often refer to these as just *Mackey functors*. The evaluation at an object G/H of Span_G we will sometimes refer to as the *fixed points* of $\mathcal{M}$, alluding to Theorem 1.3.

Observation 7.2. Perhaps unsurprisingly, $\mathrm{Mack}_G(\mathcal{E})$ is closed under all limits and colimits (that exist in $\mathcal{E}$) as a full subcategory of $\mathrm{Fun}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E})$, since any colimit of finite-coproduct-preserving functors preserves finite coproducts, and vice-versa for limits. This implies that for any object x of $\mathrm{Span}(\mathsf{Fin}_G)$, evaluation at x preserves all limits and colimits of Mackey functors.

Since Span_G is equivalent to its opposite by the self-duality τ of Remark 6.3, choosing the domain category to be $\mathrm{Span}_G^{\mathrm{op}}$ instead of Span_G is purely notational convention. Our choice is motivated by considering $\mathrm{Mack}_G(\mathcal{E})$, for $\mathcal{E}$ presentable, as a tensor product of presentable ∞-categories: Since Span_G is small and admits finite coproducts, its completion under sifted colimits

$$\mathcal{P}_\Sigma(\mathrm{Span}_G) = \mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S})$$

is presentable by [Lur09, Prop. 5.5.8.10.(1)]. By the considerations in the proof of Proposition 5.6 (and since product-preserving functors between preadditive ∞-categories are precisely sum-preserving) we have

$$\mathrm{Mack}_G(\mathcal{E}) \simeq \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathcal{E},$$

a perspective we will exploit heavily.

Given an additive functor $F\colon \mathcal{E} \to \mathcal{D}$ between preadditive categories, we obtain a functor

$$\mathrm{Mack}_G(\mathcal{E}) \to \mathrm{Mack}_G(\mathcal{D})$$

induced by postcomposition. If $\mathcal{C}, \mathcal{D}$ are presentable and F preserves colimits (i.e. we are describing functoriality in the full subcategory of Pr_{L} on the preadditive ∞-categories), the above functor is equivalently given by

$$\mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathcal{E} \xrightarrow{\mathcal{P}_\Sigma(\mathrm{Span}_G)\otimes F} \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathcal{D}.$$

Definition 7.3. Given a Mackey functor $\mathcal{M}$, we extract an underlying Or_G-object by restriction, i.e. we define $\vec{u}\mathcal{M}\colon \mathsf{Or}_G \to \mathcal{E}$ as the composition

$$\mathsf{Or}_G \hookrightarrow \mathsf{Fin}_G \xrightarrow{-^*} \mathrm{Span}_G^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{E}.$$

Dually, we have an underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum defined as the composition

$$\overleftarrow{u}\mathcal{M}\colon \mathsf{Or}_G^{\mathrm{op}} \hookrightarrow \mathsf{Fin}_G^{\mathrm{op}} \xrightarrow{-_*} \mathrm{Span}_G^{\mathrm{op}} \xrightarrow{\mathcal{M}} \mathcal{E}.$$

Restricting one more time, we get an underlying object with G-action $u\mathcal{M}\colon \mathrm{B}G \to \mathcal{E}$ as the composition

$$\mathrm{B}G = \langle G/1 \rangle^{\mathrm{op}} \hookrightarrow \mathsf{Or}_G^{\mathrm{op}} \xrightarrow{\overleftarrow{u}\mathcal{M}} \mathcal{E}.$$

Remark 7.4. A priori, there is another way to extract an underlying object with G-action from a Mackey functor $\mathcal{M}$, along the intermediate step $\vec{u}\mathcal{M}$: Restriction along

$$\mathrm{B}G^{\mathrm{op}} = \langle G/1 \rangle \hookrightarrow \mathsf{Fin}_G \xrightarrow{-^*} \mathrm{Span}_G^{\mathrm{op}},$$

gives an object $\overline{u}\mathcal{M} \in \mathcal{E}^{\mathrm{B}G^{\mathrm{op}}}$. Precomposing with the equivalence $\mathrm{B}G \simeq \mathrm{B}G^{\mathrm{op}}$ then identifies this G-object with $u\mathcal{M}$, i.e. using the notation introduced just before Proposition 3.9, we have

$$\overline{u}\mathcal{M} \simeq \overline{u\mathcal{M}}.$$

We see this by displaying a natural homotopy between the morphisms g^* and g_*^{-1} of Span(Fin_G), namely the 2-simplex

$$\begin{tikzcd}
& & {} \arrow[dl, "="'] \arrow[dr, "g^{-1}"] & & \\
& {} \arrow[dl, "="'] \arrow[dr, "="] & & {} \arrow[dl, "g"'] \arrow[dr, "="] & \\
{} & & {} & & {}
\end{tikzcd}.$$

8. Spectral Mackey Functors

The most important coefficient category for Mackey functors is the (presentable, stable and thus additive) ∞-category of spectra Sp. Indeed, the whole motivation for developing this theory is the insight that $\mathrm{Mack}_G(\mathsf{Sp})$ models genuine stable equivariant homotopy theory, which originally goes back to a triple of papers by Bertrand Guillou and Peter May [GM17a] [GM17b] [GM20]. Analyzing the structure of the stable presentable ∞-category

$$\mathrm{Mack}_G(\mathsf{Sp}) = \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathsf{Sp}$$

directly, we prepare for a modern interpretation of this result in this chapter. Everything here is apparently well-known to experts, but we failed to find a published account for much of it. We want to acknowledge and thank Maxime Ramzi for many helpful conversations on this material.

First, we contemplate the relation $\mathrm{Mack}_G(\mathsf{Sp})$ holds to the ∞-category of G-spaces. We have an 'underlying G-space'-functor $\Omega^\infty_G \colon \mathrm{Mack}_G(\mathsf{Sp}) \to \mathcal{S}_G$ given as the composition

$$\begin{tikzcd}
\mathrm{Mack}_G(\mathsf{Sp}) \arrow[r] \arrow[d, "="] & \mathcal{P}_\Sigma(\mathrm{Span}_G) \arrow[r] \arrow[d, "="] & \mathcal{S}_G \arrow[d, "="] \\
\mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp}) \arrow[r] & \mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}) \arrow[r] & \mathrm{Fun}^\times(\mathsf{Fin}_G^{\mathrm{op}}, \mathcal{S}).
\end{tikzcd}$$

Here the first functor is given by postcomposition with $\Omega^\infty \colon \mathsf{Sp} \to \mathcal{S}$ and the second functor is given by precomposition with

$$-_* \colon \mathsf{Fin}_G^{\mathrm{op}} \to \mathrm{Span}_G^{\mathrm{op}}.$$

Thus, for any spectral Mackey functor $\mathcal{M}$, we have that its underlying G-space has fixed points at H the underlying space of its evaluation at G/H, i.e. that

$$(\Omega^\infty_G \mathcal{M})^H \simeq \Omega^\infty(\mathcal{M}(G/H)). \tag{8.1}$$

We claim that the two functors defining Ω^∞_G admit left adjoints, so we obtain an adjunction

$$\mathbb{S}_G \colon \mathcal{S}_G \rightleftarrows \mathrm{Mack}_G(\mathsf{Sp}) \colon \Omega^\infty_G$$

which we want to describe concretely.[7]

Let us first consider the second step. To show that $\mathcal{P}_\Sigma(\mathrm{Span}_G) \to \mathcal{S}_G$ admits a left adjoint, we check the following lemma.

Lemma 8.2. *Given ∞-categories $\mathcal{C}, \mathcal{D}$ which admit finite coproducts and a functor $\mathcal{C} \to \mathcal{D}$ preserving these, the restriction*

$$\mathrm{Fun}^\times(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{res}^\times} \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$$

admits a left adjoint, given by the covariant functoriality present on the construction $\mathrm{Fun}^\times(-^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}_\Sigma(-)$, i.e. the unique functor preserving sifted colimits that renders the diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d, "j"] & \mathcal{D} \arrow[d, "j"] \\
\mathcal{P}_\Sigma(\mathcal{C}) \arrow[r, dashed] & \mathcal{P}_\Sigma(\mathcal{D})
\end{tikzcd}$$

commutative.

[7] The classical notation in genuine G-spectra for this functor would be $T \mapsto \Sigma^\infty_G(T_+)$. In [Bar17], Barwick denotes it as $\boldsymbol{S}^T$, see also Remark 8.4.

Proof. For the analogous statement regarding the whole category of presheaves $\mathrm{Fun}(-^{\mathrm{op}}, \mathcal{S}) = \mathcal{P}(-)$, this is [Lur09, Prop. 5.2.3.6].

The restriction to the full subcategory $\mathcal{P}_\Sigma(\mathcal{C}) \hookrightarrow \mathcal{P}(\mathcal{C})$ (i.e. to *sifted* colimits of representables) of the unique colimit preserving functor $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ (which extends $\mathcal{C} \to \mathcal{D}$ along the Yoneda embedding) lands in the full subcategory of $\mathcal{P}(\mathcal{D})$ spanned by sifted colimits of representables, i.e. $\mathcal{P}_\Sigma(\mathcal{D})$, and this is of course the functor $\mathcal{P}_\Sigma(\mathcal{C}) \to \mathcal{P}_\Sigma(\mathcal{D})$ claimed to give the left adjoint.

On the other hand, since $\mathcal{C} \to \mathcal{D}$ preserves finite coproducts, the restriction functor restricts to a functor

$$\mathrm{Fun}^\times(\mathcal{D}^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{res}^\times} \mathrm{Fun}^\times(\mathcal{C}^{\mathrm{op}}, \mathcal{S}),$$

so the adjunction of [Lur09, Prop. 5.2.3.6] restricts to the claimed adjunction. □

This identifies the left adjoint to

$$\mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}) \xrightarrow{\mathrm{res}^\times} \mathrm{Fun}^\times(\mathsf{Fin}_G^{\mathrm{op}}, \mathcal{S})$$

as the unique functor $\mathcal{S}_G \to \mathcal{P}_\Sigma(\mathrm{Span}_G)$ that preserves sifted colimits and restricts to $\mathsf{Fin}_G \to \mathrm{Span}_G$.

Considering the first step, i.e. the functor

$$\mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp}) \xrightarrow{\Omega^\infty_*} \mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}),$$

we first observe that the left adjoint to Ω^∞, i.e. $\Sigma^\infty_+ \colon \mathcal{S} \to \mathsf{Sp}$, does not preserve products, so the desired left adjoint cannot be constructed 'pointwise'. However, we recall that $\Omega^\infty \colon \mathsf{Sp} \to \mathcal{S}$ factorizes as the composition of right adjoints

$$\mathsf{Sp} \xrightarrow{\tau_{\geq 0}} \mathsf{Sp}_{\geq 0} \simeq \mathrm{CMon}^{gp} \hookrightarrow \mathrm{CMon} \xrightarrow{u} \mathcal{S} \tag{8.3}$$

so the functor we are interested in factorizes as

$$\mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp}) \to \mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}).$$

We checked that the second functor here is an equivalence in Proposition 5.11.

Now the left adjoint to $\mathsf{Sp} \to \mathrm{CMon}$ is given by the composition of left adjoints of 8.3, that is in turn: group completion

$$-^{gp} \colon \mathrm{CMon} \to \mathrm{CMon}^{gp},$$

the equivalence

$$\mathrm{CMon}^{gp} \xrightarrow{\sim} \mathsf{Sp}_{\geq 0}$$

and the natural inclusion

$$\mathsf{Sp}_{\geq 0} \hookrightarrow \mathsf{Sp}.$$

We will also refer to this composition $\mathrm{CMon} \to \mathsf{Sp}$ as just group completion.

As left adjoints between preadditive categories, these preserve finite sums, and thus the desired left adjoint

$$\mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp})$$

is given pointwise, i.e. by postcomposition with the group completion functor $\mathrm{CMon} \to \mathsf{Sp}$.

We thus have constructed the desired left adjoint $\mathbb{S}_G$, sitting in the diagram

$$\begin{tikzcd}
& \mathsf{Fin}_G \arrow[r] \arrow[d, "j"] & \mathrm{Span}_G \arrow[d, "j"] \arrow[dr, dashed] & \\
\mathcal{S}_G = & \mathcal{P}_\Sigma(\mathsf{Fin}_G) \arrow[r] \arrow[rr, bend right, "\mathbb{S}_G"'] & \mathcal{P}_\Sigma(\mathrm{Span}_G) \arrow[r] & \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathsf{Sp} = \mathrm{Mack}_G(\mathsf{Sp}).
\end{tikzcd}$$

Remark 8.4. We will also write $\mathbb{S}_G$ for the dashed composition $\mathrm{Span}_G \to \mathrm{Mack}_G$, which explicitly models the wrong-way maps present on $\mathbb{S}_G[G/H]$. It is the restriction to Span_G of the left adjoint to

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp}) \xrightarrow{\Omega^\infty \circ -} \mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}).$$

It is exactly this left adjoint that Barwick in [Bar17] refers to as Mackey stabilization, denoted $T \mapsto \boldsymbol{S}^T$.

Remark 8.5. Let us be explicit about our notational choices: We have constructed a functor $\mathbb{S}_G$ with domain $\mathcal{S}_G$ (or Span_G) and valued in G-Mackey functors. We shall write square brackets for the evaluation of this functor, i.e. for a finite G-set T, we have $\mathbb{S}_G[T] \in \mathrm{Mack}_G(\mathsf{Sp})$. As this is itself a Mackey functor, we will again evaluate it on finite G-sets, writing standard brackets, i.e. $\mathbb{S}_G[T](S) \in \mathsf{Sp}$.

In the special case of $\mathbb{S}_G[G/G]$, we will simply write $\mathbb{S}_G$ for this Mackey functor.

Example 8.6. To summarize, let us compute $\mathbb{S}_G[T]$ for a finite G-set T, i.e. a representable in $\mathcal{S}_G = \mathrm{Fun}^{\times}(\mathsf{Fin}_G^{\mathrm{op}}, \mathcal{S})$. Since $\mathcal{S}_G$ is generated under sifted colimits by representables and $\mathbb{S}_G$ preserves these, this describes $\mathbb{S}_G$ completely.

The first left adjoint $\mathcal{P}_\Sigma(\mathsf{Fin}_G) \to \mathcal{P}_\Sigma(\mathrm{Span}_G)$ sends it to $\mathrm{Map}_{\mathrm{Span}_G}(-, T)$ by Lemma 8.2.

The second left adjoint, i.e. the inverse of

$$u\colon \mathrm{Fun}^{\oplus}(\mathrm{Span}_G^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S})$$

just remembers that $\mathrm{Map}_{\mathrm{Span}_G}(S,T)$ is canonically a commutative monoid $\mathrm{Hom}_{\mathrm{Span}_G}(S,T)$, since Span_G is preadditive.

The third left adjoint

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}_G^{\mathrm{op}}, \mathrm{CMon}) \to \mathrm{Fun}^{\oplus}(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp})$$

then group completes this commutative monoid at every $S \in \mathrm{Span}_G$ and understands it as a (connective) spectrum, i.e. we have

$$\begin{aligned} \mathbb{S}_G[T](S) &\simeq \mathrm{Hom}_{\mathrm{Span}_G}(S,T)^{gp} \\ &\simeq (\mathsf{Fin}_{G/\{S,T\}})^{\sim, gp} \in \mathsf{Sp}. \end{aligned}$$

Here, the second equivalence is Remark 6.2, and the addition (with respect to which we group complete) is easily seen to be given by the coproduct on the source, i.e. for $U \to S \times T$ and $V \to S \times T$, their sum in $(\mathsf{Fin}_{G/\{S,T\}})^{\sim}$ is given by $U \amalg V \to S \times T$, as a consequence of sums in Span_G given by coproducts.

More concretely, let us compute the fixed points and the underlying spectrum of the *equivariant sphere* $\mathbb{S}_G := \mathbb{S}_G[G/G]$: We have

$$\begin{aligned} \mathbb{S}_G(G/G) &\simeq \mathrm{Hom}_{\mathrm{Span}_G}(G/G, G/G)^{gp} \\ &\simeq (\mathsf{Fin}_{G/\{G/G, G/G\}})^{\sim, gp} \\ &\simeq \mathsf{Fin}_G^{\sim, gp}. \end{aligned}$$

The first equivalence is the above description of the functor $\mathbb{S}_G$. The second equivalence is Remark 6.2. The last equivalence follows from G/G being final in Fin_G. Since sums in Span_G are given by coproducts (i.e. disjoint unions) in Fin_G and the monoid structure on $\mathrm{Hom}_{\mathrm{Span}_G}(S,T)$ comes from the preadditivity of Span_G, the relevant monoid structure on $\mathsf{Fin}_G^{\sim}$ is the one given by disjoint union of G-sets.

On the other hand, we have

$$\begin{aligned} \mathbb{S}_G(G/1) &\simeq \mathrm{Hom}_{\mathrm{Span}_G}(G/1, G/G)^{gp} \\ &\simeq (\mathsf{Fin}_{G/\{G/1\}})^{\sim, gp} \\ &\simeq \mathsf{Fin}^{\sim, gp} \qquad\qquad \simeq \mathbb{S}. \end{aligned}$$

Here, in the second to last step we used the equivalence

$$\begin{tikzcd} \mathsf{Fin}_{G/\{G/1\}} \arrow[r] \arrow[rr, bend right, "\sim"'] & \mathsf{Fin}_G \arrow[r, "(-)_{/G}"] & \mathsf{Fin}, \end{tikzcd}$$

since a G-set equipped with a map into $G/1$ is necessarily free, and every orbit is assigned a basepoint by its identification with $G/1$. Note that this identifies the monoid structure induced from disjoint union of G-sets/sets. We also see that the G-action on $\mathbb{S}_G(G/1)$ (which is then given by functoriality in $G/1$ on $\mathsf{Fin}_{G/\{G/1\}}$) is trivial.

Remark 8.7. Taking the equivalence $\mathrm{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Sp}_G$ for granted, this recovers (modulo the multiplicative structure) the classical statement that the underlying spectrum of the equivariant sphere $\mathbb{S}_G$ is the sphere spectrum with trivial action, but surprisingly its genuine fixed points model (at π_0) the G-Burnside ring.

Elaborations on this argument then identify the Mackey functor (valued in Abelian groups) $\pi_0^?\mathbb{S}_G$ with the Mackey functor structure present on the ?-Burnside ring by induction/restriction. One may also recover the Tom-Dieck splitting from this perspective.

Thinking of the Mackey functors $\mathbb{S}_G[G/H]$ as representable Mackey functors, naturally valued in Sp, we expect the following refinement of the Yoneda lemma:

Proposition 8.8. *For $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$, we have an equivalence of spectra*

$$\mathcal{M}(G/H) \simeq \mathrm{map}_G(\mathbb{S}_G[G/H], \mathcal{M}),$$

natural in $G/H \in \mathrm{Span}_G$.

Here we write, for brevity, $\mathrm{map}_G(\mathcal{M}, \mathcal{N})$ for the mapping spectrum associated to $\mathrm{Mack}_G(\mathsf{Sp})$ being stable. Some preparatory recollections: Since $\mathrm{Mack}_G(\mathsf{Sp})$ is presentable, we have another way to construct the mapping spectrum: Sp is a presentably symmetric monoidal ∞-category (i.e. a commutative algebra object in $\mathrm{Pr_L}$ with respect to the Lurie tensor product), so $\mathrm{Mack}_G(\mathsf{Sp}) = \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathsf{Sp}$ is canonically tensored over Sp, classified by a functor

$$\mathrm{Mack}_G(\mathsf{Sp}) \times \mathsf{Sp} \to \mathrm{Mack}_G(\mathsf{Sp})$$

preserving colimits in either variable. Thus, fixing an $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$, we have a colimit-preserving functor

$$\mathsf{Sp} \xrightarrow{\mathcal{M}\otimes -} \mathrm{Mack}_G(\mathsf{Sp})$$

whose right adjoint we claim is given by the mapping spectrum $\mathrm{map}_G(\mathcal{M}, -)$.

Indeed, the left adjoint $\mathcal{M} \otimes -$ is characterized as the unique colimit-preserving functor $\mathsf{Sp} \to \mathrm{Mack}_G(\mathsf{Sp})$ sending the sphere spectrum $\mathbb{S}$ to $\mathcal{M}$, so we obtain a natural equivalence of mapping spectra

$$\mathrm{map}_G(\mathcal{M} \otimes A, \mathcal{N}) \simeq \mathrm{map}_{\mathsf{Sp}}(A, \mathrm{map}_G(\mathcal{M}, \mathcal{N}))$$

by observing that both sides agree on $A = \mathbb{S}$ and (in the variable A) transform colimits to limits, see Appendix B.2. Applying Ω^∞ to these mapping spectra gives the desired natural equivalence of mapping spaces characterizing the adjunction

$$\mathsf{Sp} \underset{\mathrm{map}_G(\mathcal{M},-)}{\overset{\mathcal{M}\otimes-}{\rightleftarrows}} \mathrm{Mack}_G(\mathsf{Sp}).$$

In the other variable, we get another adjunction, now for any fixed $A \in \mathsf{Sp}$:

$$\mathrm{Mack}_G(\mathsf{Sp}) \underset{F_A}{\overset{-\otimes A}{\rightleftarrows}} \mathrm{Mack}_G(\mathsf{Sp})$$

where we claim $(F_A\mathcal{M})(G/H) \simeq \mathrm{map}_{\mathsf{Sp}}(A, \mathcal{M}(G/H))$.

Indeed, for stable ∞-categories of the form $\mathrm{Fun}(K, \mathsf{Sp})$ the tensoring

$$\mathrm{Fun}(K, \mathsf{Sp}) \xrightarrow{-\otimes A} \mathrm{Fun}(K, \mathsf{Sp})$$

is computed pointwise, so its right adjoint is also computed pointwise, i.e. as the functor

$$\mathrm{Fun}(K, \mathsf{Sp}) \xleftarrow{F \mapsto \mathrm{map}_{\mathsf{Sp}}(A, F-)} \mathrm{Fun}(K, \mathsf{Sp}).$$

If K admits finite products, both these functors restrict to functors $\mathrm{Fun}^{\times}(K, \mathsf{Sp}) \to \mathrm{Fun}^{\times}(K, \mathsf{Sp})$, so specializing to $\mathrm{Mack}_G(\mathsf{Sp}) = \mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp})$ yields the claimed description of the right adjoint .

Proof of Proposition 8.8. Let us think about G/H as an object of Fin_G first. We have the following chain of equivalences of mapping spaces, for $A \in \mathsf{Sp}$:

$$\begin{aligned}
\mathrm{Map}_{\mathsf{Sp}}(A, \mathcal{M}(G/H)) &\simeq \mathrm{Map}_{\mathcal{S}_G}\big(G/H,\ \mathrm{Map}_{\mathsf{Sp}}(A, \mathcal{M}(-))\big) \\
&\simeq \mathrm{Map}_{\mathcal{S}_G}\big(G/H,\ \Omega^\infty(F_A\mathcal{M}(-))\big) \\
&\simeq \mathrm{Map}_{\mathcal{S}_G}\big(G/H,\ \Omega^\infty_G(F_A\mathcal{M})\big) \\
&\simeq \mathrm{Map}_G\big(\mathbb{S}_G[G/H],\ F_A\mathcal{M}\big) \\
&\simeq \mathrm{Map}_G\big(\mathbb{S}_G[G/H] \otimes A,\ \mathcal{M}\big) \\
&\simeq \mathrm{Map}_{\mathsf{Sp}}\big(A,\ \mathrm{map}_G(\mathbb{S}_G[G/H], \mathcal{M})\big)
\end{aligned}$$

The first equivalence is the Yoneda lemma for the representable functor $G/H \in \mathcal{S}_G$. The second is the given description of $F_A(\mathcal{M})$ and the compatibility of mapping spaces and mapping spectra. The third is the compatibility of taking fixed points and underlying spaces 8.1. The rest is applying the various adjunctions. Every step is natural in G/H, A, and $\mathcal{M}$, so the Yoneda lemma lets us conclude.

The proof as written confirms a natural equivalence $\mathcal{M}(-) \simeq \mathrm{map}_G(\mathbb{S}_G[-], \mathcal{M})$ of functors $\mathsf{Fin}_G^{\mathrm{op}} \to \mathsf{Sp}$, but notice how everything was purely formal - taking Remark 8.4 into account and replacing $\mathcal{S}_G = \mathcal{P}_\Sigma(\mathsf{Fin}_G)$ by $\mathcal{P}_\Sigma(\mathrm{Span}_G)$ where necessary, we obtain the desired natural equivalence of functors $\mathrm{Span}_G^{\mathrm{op}} \to \mathsf{Sp}$. □

Remark 8.9. Given a presentable stable ∞-category $\mathcal{C}$, recall that a set of objects $\{X_\alpha\}$ is called a set of generators if for every $Y \in \mathcal{C}$, Y is 0 if and only if $\mathrm{map}_{\mathcal{C}}(X_\alpha, Y) \simeq 0$ for all X_α. This nomenclature is justified by the fact that this property is equivalent to the property that the smallest stable subcategory that is closed under colimits and contains $\{X_\alpha\}$ is all of $\mathcal{C}$. This is proved e.g. in [MNN17, Lemma 7.6].

Corollary 8.10. *For $T \in \mathrm{Span}_G$, $\mathbb{S}_G[T]$ is a co mpact object of $\mathrm{Mack}_G(\mathsf{Sp})$, and $\{\mathbb{S}_G[T]\}_{T \in \mathrm{Span}_G}$ is a set of generators. Thus so is the set of orbits $\mathbb{S}_G[G/H]$.*

Proof. For compactness, we need to show that mapping out of $\mathbb{S}_G[T]$ preserves filtered colimits. For $\mathcal{M} \simeq \mathrm{colim}_S\, \mathcal{M}_s$ a filtered colimit in $\mathrm{Mack}_G(\mathsf{Sp})$, we have by the above Proposition 8.8

$$\begin{aligned}\mathrm{Map}_G(\mathbb{S}_G[T], \operatorname*{colim}_S \mathcal{M}_s) &\simeq \Omega^\infty((\operatorname*{colim}_S \mathcal{M}_s)(T)) \\ &\simeq \operatorname*{colim}_S \Omega^\infty \mathcal{M}_s(T) \\ &\simeq \operatorname*{colim}_S \mathrm{Map}_G(\mathbb{S}_G[T], \mathcal{M}_s),\end{aligned}$$

since evaluation preserves all colimits by Observation 7.2, and the functor

$$\Omega^\infty \colon \mathsf{Sp} \to \mathcal{S}$$

preserves filtered colimits.

To see that the $\mathbb{S}_G[T]$ generate $\mathrm{Mack}_G(\mathsf{Sp})$, we need to see that $\mathcal{M}$ is 0 if and only if $\mathrm{map}_G(\mathbb{S}_G[T], \mathcal{M})$ is 0 for all T. Certainly, $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$ is 0 if and only if it is 0 in $\mathrm{Fun}(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp})$ if and only if all its evaluations are 0.

Since every $\mathbb{S}_G[T]$ is a finite sum of orbits (as the functor $\mathbb{S}_G$ preserves coproducts), the second part is immediate. □

Remark 8.11. That the $\mathbb{S}_G[G/H]$ generate $\mathrm{Mack}_G(\mathsf{Sp})$ in the above sense implies that a natural transformation α of colimit preserving functors $\mathrm{Mack}_G(\mathsf{Sp}) \to \mathcal{D}$ for $\mathcal{D}$ stable and cocomplete is an equivalence if it is an equivalence at the $\mathbb{S}_G[G/H]$. This is so since the subcategory of objects on which α is an equivalence is stable, closed under colimits and contains a set of generators, so is all of $\mathrm{Mack}_G(\mathsf{Sp})$.

8.1. **Tensor products of Mackey functors.** Under mild conditions on $\mathcal{C}$, the ∞-category $\mathrm{Span}(\mathcal{C})$ carries a natural symmetric monoidal structure, which then (by an ∞-categorical version of Day convolution) leads to a symmetric monoidal structure on $\mathrm{Mack}_G(\mathcal{E})$. Of course, this will model classical constructions of the smash product of G-spectra. We give a very brief exposition focusing on the case $\mathcal{C} = \mathsf{Fin}_G$, and refer to [BGS20] for the whole story.

The first step is to observe that if $\mathcal{C}$ is a disjunctive ∞-category (see Definition 6.4) that admits a terminal object 1, the ∞-category $\mathrm{Span}(\mathcal{C})$ admits a symmetric monoidal structure given by the product in $\mathcal{C}$ (which makes 1 the tensor unit), see [BGS20, Ex. 2.13, Prop. 2.14]. We will write $c \times d$ for this tensor product, but remind the reader that by Proposition 6.6, this is not the categorical product in $\mathrm{Span}(\mathcal{C})$ (which is given by the coproduct in $\mathcal{C}$).

As a consequence of universality 6.5, the tensor product functor

$$- \times - \colon \mathrm{Span}(\mathcal{C}) \times \mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{C})$$

preserves direct sums in either variable.

In the above case, the functor

$$\tau \colon \mathrm{Span}(\mathcal{C}) \xrightarrow{\sim} \mathrm{Span}(\mathcal{C})^{\mathrm{op}}$$

of Remark 6.3 provides a functorial self-duality for every object c in $\mathrm{Span}(\mathcal{C})$: There is an *evaluation* morphism $c \times \tau c \to 1$ given by the diagram

$$\begin{tikzcd} & c \arrow[dl, "\Delta"'] \arrow[dr] & \\ c \times c & & 1 \end{tikzcd}$$

and, dually, a coevaluation morphism $1 \to \tau c \times c$ given by τ of the evaluation, i.e. the diagram

$$\begin{tikzcd} & c \arrow[dl] \arrow[dr, "\Delta"] & \\ 1 & & c \times c. \end{tikzcd}$$

These are easily confirmed (see [BGS20, 2.18]) to exhibit τc as a dual of c (with the standard 1-categorical definition, applied to the homotopy category of $\mathrm{Span}(\mathcal{C})$). In particular, we have an adjunction-equivalence of mapping spaces

$$\mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(\tau c \times d, e) \simeq \mathrm{Map}_{\mathrm{Span}(\mathcal{C})}(d, c \times e)$$

natural in c.

In [Gla16], Saul Glasman developed a general theory of Day convolutions for ∞-categories: He constructs a symmetric monoidal structure on $\mathrm{Fun}(K, \mathcal{C})$ given that K and $\mathcal{C}$ are symmetric monoidal ∞-categories with $\mathcal{C}$ admitting all colimits and the functors $c \otimes - \colon \mathcal{C} \to \mathcal{C}$ preserving these, generalizing Lurie's construction for $\mathcal{C} = \mathcal{S}$, see [Lur17, Rem. 4.8.1.13].

Just as 1-categorically, on objects, this convolution product of $F, G \in \mathrm{Fun}(K, \mathcal{C})$ is given as the dashed left Kan extension in the diagram

$$\begin{tikzcd} K \times K \arrow[r, "F \times G"] \arrow[d, "- \otimes -"'] & \mathcal{C} \times \mathcal{C} \arrow[r, "- \otimes -"] & \mathcal{C}. \\ K \arrow[urr, dashed] & & \end{tikzcd}$$

Remark 8.12. As summarized in [BGS20, before Prop. 1.6], it is the unique symmetric monoidal structure on $\mathrm{Fun}(K, \mathcal{C})$ that preserves colimits in either variable, and such that the functor

$$K^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Fun}(K, \mathcal{S}) \times \mathcal{C} \to \mathrm{Fun}(K, \mathcal{C})$$

admits a symmetric monoidal structure, where the first functor is the Yoneda embedding, and the second functor is

$$(F, c) \mapsto \big[k \mapsto F(k) \otimes c \simeq \operatorname*{colim}_{F(k)} c\big].$$

In particular, in the case $\mathcal{C} = \mathcal{S}$, the Yoneda embedding itself is a symmetric monoidal functor.

For $\mathcal{E}$ presentably symmetric monoidal and preadditive, we thus obtain a symmetric monoidal structure on $\mathrm{Fun}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E})$ given by Day-convolution. Unfortunately, it is not quite true that $\mathrm{Mack}_G(\mathcal{E}) = \mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E})$ is a symmetric monoidal subcategory, but using the general principles established in [Lur17, Sect. 2.2.1], one confirms [BGS20, Lemma 3.7] that there is a symmetric monoidal structure on $\mathrm{Mack}_G(\mathcal{E})$ given by Day convolution followed by 'Mackeyfication' i.e. the left adjoint to the inclusion

$$\mathrm{Fun}^{\times}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E}) \hookrightarrow \mathrm{Fun}(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{E}).$$

Certainly, this symmetric monoidal structure on $\mathrm{Mack}_G(\mathcal{E})$ preserves colimits in either variable, and indeed it makes $\mathrm{Mack}_G(\mathsf{Sp})$ a presentably symmetric monoidal ∞-category, i.e. a commutative algebra-object of $\mathrm{Pr_L}$ with respect to the Lurie tensor product [CMNN20, Constr. 2.7]. The analysis carried out in [BGS20, Ch. 4] shows that in the case $\mathcal{E} = \mathsf{Sp}$, the natural refinement of the Yoneda embedding

$$\mathbb{S}_G \colon \mathrm{Span}_G \to \mathrm{Mack}_G(\mathsf{Sp})$$

admits a canonical symmetric monoidal structure, which together with colimit-preservation uniquely characterizes the tensor product on $\mathrm{Mack}_G(\mathsf{Sp})$.

Thus, we immediately see that $\mathbb{S}_G[G/G]$ is the tensor unit of $\mathrm{Mack}_G(\mathsf{Sp})$ (since it is so in Span_G), and more generally that for $T \in \mathrm{Span}_G$, the Mackey functor $\mathbb{S}_G[T]$ is a dualizable object of $\mathrm{Mack}_G(\mathsf{Sp})$, with dual $\mathbb{S}_G[\tau T]$. Note that the object $\mathbb{S}_G[\tau T]$ is of course the same as $\mathbb{S}_G[T]$, but this way we may write the natural equivalence of functors $\mathrm{Span}_G \to \mathsf{Sp}$

$$\mathrm{map}_G(\mathbb{S}_G[\tau T] \otimes \mathcal{M}, \mathcal{N}) \simeq \mathrm{map}_G(\mathcal{M}, \mathbb{S}_G[T] \otimes \mathcal{N}).$$

Since $\mathrm{Mack}_G(\mathsf{Sp})$ is presentably symmetric monoidal, it has internal mapping objects characterized by adjunctions

$$\mathrm{Mack}_G(\mathsf{Sp}) \underset{F(\mathcal{M},-)}{\overset{-\otimes\mathcal{M}}{\rightleftarrows}} \mathrm{Mack}_G(\mathsf{Sp}). \tag{8.13}$$

For orbits $\mathbb{S}_G[T]$, we have described this right adjoint above. In general, we also have the following:

Corollary 8.14. *For any $\mathcal{M}, \mathcal{N}$ in $\mathrm{Mack}_G(\mathsf{Sp})$, we have*

$$F(\mathcal{M},\mathcal{N})(G/G) \simeq \mathrm{map}_G(\mathcal{M},\mathcal{N}).$$

This should recover the intuition that the fixed points of the mapping G-spectrum model the spectrum of equivariant maps.

Proof. We have

$$\begin{aligned} F(\mathcal{M},\mathcal{N})(G/G) &\simeq \mathrm{map}_G(\mathbb{S}_G, F(\mathcal{M},\mathcal{N})) \\ &\simeq \mathrm{map}_G(\mathcal{M},\mathcal{N}). \end{aligned}$$

where the first equivalence is an application of Proposition 8.8, and the second uses the adjunction 8.13 and the fact that $\mathbb{S}_G$ is the tensor unit of $\mathrm{Mack}_G(\mathsf{Sp})$. □

Remark 8.15. We have arrived at an alternative description for the underlying Or_G-spectrum $\vec{u}\mathcal{M}\colon \mathsf{Or}_G \to \mathsf{Sp}$ for a given spectral Mackey functor $\mathcal{M}$. With the definitions of 7.3, and the fact that $-^* = \tau \circ -_*$ as functors $\mathsf{Fin}_G \to \mathsf{Or}_G^{\mathrm{op}}$, we have

$$\begin{aligned} \vec{u}\mathcal{M}(-) &\simeq \mathrm{map}_G(\mathbb{S}_G[\tau-],\mathcal{M}) \\ &\simeq F(\mathbb{S}_G[\tau-],\mathcal{M})(G/G) \\ &\simeq (\mathbb{S}_G[-]\otimes\mathcal{M})(G/G). \end{aligned}$$

Similarly, the underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum $\overleftarrow{u}\mathcal{M}\colon \mathrm{Span}_G^{\mathrm{op}} \to \mathsf{Sp}$ is given by

$$\overleftarrow{u}\mathcal{M}(-) \simeq F(\mathbb{S}_G[-],\mathcal{M})(G/G).$$

Before we move on, we have one more structural property regarding the dualizable objects of $\mathrm{Mack}_G(\mathsf{Sp})$ to prove.

Proposition 8.16. $\mathrm{Mack}_G(\mathsf{Sp})$ *is* rigidly-compactly generated*, that is it has a set of compact generators, and the class of compact object coincides with that of the dualizable objects.*

Proof. In light of Proposition 8.10, the thing left to check is that compactness and dualizability coincide. Assume $\mathcal{M}$ is a dualizable object with dual $\mathcal{M}^\vee$ and $\mathrm{colim}_F\,\mathcal{N}_i$ some filtered colimit in $\mathrm{Mack}_G(\mathsf{Sp})$. Then we have the following equivalences of mapping spectra

$$\begin{aligned} \mathrm{map}_G(\mathcal{M}, \operatorname*{colim}_F \mathcal{N}_i) &\simeq F(\mathcal{M}, \operatorname*{colim}_F \mathcal{N}_i)(G/G) \\ &\simeq (\mathcal{M}^\vee \otimes \operatorname*{colim}_F \mathcal{N}_i)(G/G) \\ &\simeq \operatorname*{colim}_F(\mathcal{M}^\vee\otimes\mathcal{N}_i)(G/G) \\ &\simeq \operatorname*{colim}_F \mathrm{map}_G(\mathcal{M},\mathcal{N}_i), \end{aligned}$$

so applying the filtered-colimit preserving functor $\Omega^\infty\colon \mathsf{Sp} \to \mathcal{S}$ gives the equivalence characterizing compactness of $\mathcal{M}$.

The other direction is a little more involved. Writing from here on out M^d for the full subcategory spanned by the dualizable objects of $\mathrm{Mack}_G(\mathsf{Sp})$, let us first note some closure properties of M^d.

Lemma 8.17. *(1) M^d is stable.*

(2) Any retract of a dualizable object is dualizable.
(3) M^d is closed under finite colimits.

Proof. In a presentably symmetric monoidal stable ∞-category, if some x is dualizable with dual $x^\vee$, then so is Ωx with dual object $\Sigma(x^\vee)$ (and vice-versa), exhibited as such by the coevaluation map

$$1 \to x^\vee \otimes x \simeq \Sigma\Omega(x^\vee \otimes x) \simeq \Sigma(x^\vee) \otimes \Omega x.$$

This says that M^d is a stable subcategory.

To see closure under retracts, first note that since the definition of a dualizable object is that of a dualizable object in the homotopy-category, we import the characterization (of a dualizable object in a closed symmetric monoidal 1-category with internal function objects $F(x, y)$) that x is dualizable if and only if, for all y, the natural map

$$F(x, 1) \otimes y \to F(x, y) \tag{8.18}$$

is an equivalence. Since internal function objects of a presentably symmetric monoidal ∞-category are (contravariantly) functorial (see Appendix B.2), given a retract x of some dualizable X, the map 8.18 classifying dualizability of x is a retract of the corresponding map for X, i.e. we have a commutative square

$$\begin{tikzcd}
F(x,1) \otimes y \arrow[r] \arrow[d, shift right] & F(x,y) \arrow[d, shift right] \\
F(X,1) \otimes y \arrow[r, "\sim"] \arrow[u, shift right] & F(X,y). \arrow[u, shift right]
\end{tikzcd}$$

The upper horizontal map is then an equivalence and thus M^d is closed under formation of retracts in $\mathrm{Mack}_G(\mathsf{Sp})$.

We argue similarly for closure under finite colimits: In the stable (thus preadditive) setting, the functor $F(-, z)$ commutes with coproducts (see again Appendix B.2) and so does $- \otimes y$, so the characterizing map 8.18 of a finite coproduct of dualizables is a finite coproduct of equivalences. Similarly, a pushout square is transformed into a pullback square, and since by stability $- \otimes y$ commutes with the formation of pullbacks, so the characterizing map 8.18 of a pushout of dualizables is a pullback of equivalences. We have seen that the dualizables are closed under finite coproducts and pushouts, and thus all finite colimits. $\square$

To proceed, we quickly recall some notions from [Lur09, Ch. 5]: The Ind-construction of [Lur09, Sect. 5.3.5] associates to an ∞-category $\mathcal{C}$ an ∞-category $\mathrm{Ind}(\mathcal{C})$ admitting all filtered colimits, with a fully faithful functor $j\colon \mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ such that the functor

$$\mathrm{Fun}^F(\mathrm{Ind}(\mathcal{C}), \mathcal{D}) \xrightarrow{\sim} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

is an equivalence, where $\mathrm{Fun}^F(-,-)$ denotes filtered-colimit preserving functors and $\mathcal{D}$ is some ∞-category admitting filtered colimits [Lur09, 5.3.5.10].

The functor $\mathcal{C} \hookrightarrow \mathrm{Ind}(\mathcal{C})$ preserves those finite colimits that exist in $\mathcal{C}$ by [Lur09, Prop. 5.3.5.14], and we argue that if $\mathcal{C} \to \mathcal{D}$ preserves finite colimits, so does the induced functor $\mathrm{Ind}(\mathcal{C}) \to \mathcal{D}$. Indeed, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Ind}(\mathcal{C}) \arrow[r] \arrow[dr, dashed] & \mathrm{Ind}(\mathcal{D}) \\
\mathcal{C} \arrow[r] \arrow[u, hook, "j"] & \mathcal{D} \arrow[u, hook, "j"']
\end{tikzcd}$$

where three of the solid functors preserve finite colimits by assumption or the above. That the top horizontal functor also preserves finite colimits (even all colimits) then is a consequence of [Lur09, Prop. 5.3.5.13], and the conclusion that the dashed functor preserves finite colimits is immediate.

Above we proved that $M^d \hookrightarrow \mathrm{Mack}_G(\mathsf{Sp})$ preserves finite colimits, so the dashed filtered-colimit preserving functor

$$\begin{tikzcd} \mathrm{Ind}(M^d) \arrow[dr, dashed, "F"] & \\ M^d \arrow[u, hook, "j"] \arrow[r, hook] & \mathrm{Mack}_G(\mathsf{Sp}) \end{tikzcd}$$

also preserves finite colimits, and thus all colimits. We further claim it is an equivalence:

Since M^d is stable, so is $\mathrm{Ind}(M^d)$ by [Lur17, Prop. 1.1.3.6]. Then the essential image of F is stable, closed under all colimits and contains the set of generators $\{\mathbb{S}_G[G/H]\}$, so is all of $\mathrm{Mack}_G(\mathsf{Sp})$.

To see that its fully-faithful, first we see that for any $d \in M^d$, the functors

$$\mathrm{Ind}(M^d) \to \mathcal{S}, \qquad \mathrm{Map}_{\mathrm{Ind}(M^d)}(jd, -) \text{ and } \mathrm{Map}_{\mathrm{Mack}_G(\mathsf{Sp})}(Fjd, F-)$$

preserve filtered colimits (since d is compact in $\mathrm{Mack}_G(\mathsf{Sp})$ and jd is compact in $\mathrm{Ind}(M^d)$ by [Lur09, 5.3.5.5]). Thus the map

$$\mathrm{Map}_{\mathrm{Ind}(M^d)}(jd, x) \to \mathrm{Map}_{\mathrm{Mack}_G(\mathsf{Sp})}(Fjd, Fx)$$

is an equivalence if it is one for $x \in \operatorname{im} M^d$, where it is clear. Now the functors

$$\mathrm{Ind}(M^d) \to \mathcal{S}^{\mathrm{op}}, \qquad \mathrm{Map}_{\mathrm{Ind}(M^d)}(-, x) \text{ and } \mathrm{Map}_{\mathrm{Mack}_G(\mathsf{Sp})}(F-, Fx)$$

always preserve filtered colimits, so the map induced by F is an equivalence if it is so on the restriction to M^d, and that we have just proved. Thus F is fully faithful.

We have shown $F\colon \mathrm{Ind}(M^d) \to \mathrm{Mack}_G(\mathsf{Sp})$ is an equivalence, which we may then restrict to an equivalence of the full subcategories of compact objects.

Now recall that the functor $\mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ always factors through the full subcategory of compact objects $\mathrm{Ind}(\mathcal{C})^\omega$ by [Lur09, 5.3.5.5], and this exhibits $\mathrm{Ind}(\mathcal{C})^\omega$ as the *idempotent-completion* of $\mathcal{C}$ by [Lur09, Lemma 5.4.2.4].

An ∞-category is *idempotent-complete* if every idempotent corresponds to a retraction [Lur09, Sect. 4.4.5]. Since $\mathrm{Mack}_G(\mathsf{Sp})$ admits all colimits, it is idempotent-complete ([Lur09, Rem. 5.3.1.10]). Therefore M^d is also idempotent-complete as it is closed under retractions, so the natural functor $M^d \to \mathrm{Ind}(M^d)^\omega$ is an equivalence. We summarize in this diagram:

$$\begin{tikzcd} \mathrm{Ind}(M^d) \arrow[r, "\sim"] & \mathrm{Mack}_G(\mathsf{Sp}) \\ \mathrm{Ind}(M^d)^\omega \arrow[u, hook] \arrow[r, "\sim"] & \mathrm{Mack}_G(\mathsf{Sp})^\omega \arrow[u, hook] \\ M^d \arrow[u, "\sim"] \arrow[ur, hook] & \end{tikzcd}$$

We have shown the inclusion of the dualizable into the compact objects of $\mathrm{Mack}_G(\mathsf{Sp})$ to be an equivalence.

□

8.2. **Pointed suspension, geometric fixed points and restriction.** To prepare for the proof that $\mathrm{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Sp}_G$, we will need a couple more constructions present on $\mathrm{Mack}_G(\mathsf{Sp})$. These model the familiar functors for genuine G-spectra

$$\begin{aligned}\Sigma^\infty &\colon \mathcal{S}_{G\bullet} \to \mathsf{Sp}_G,\\ \mathrm{res}_H &\colon \mathsf{Sp}_G \to \mathsf{Sp}_H,\\ \Phi^G &\colon \mathsf{Sp}_G \to \mathsf{Sp}.\end{aligned}$$

This material is explained in [BH21] and [CMNN20], and we follow their exposition quite closely.

We would like all these constructions to be symmetric monoidal left adjoints, so we recall the formal setup from [BH21, Sect. 2.1], which gives an alternative description of the symmetric monoidal structure on Mack_G.

Given $\mathcal{C}$ admitting finite products (considered as a cartesian symmetric monoidal ∞-category), we write $\mathcal{C}_\bullet$ for the ∞-category of pointed objects, which admits a canonical symmetric monoidal structure such that the left adjoint $(-)_+\colon \mathcal{C} \to \mathcal{C}_\bullet$ promotes to a symmetric monoidal functor. We then write $\mathcal{C}_+$ for the full subcategory of $\mathcal{C}_\bullet$ spanned by the image of $(-)_+$, which is, by construction, a symmetric monoidal subcategory. For $\mathcal{C}$ symmetric monoidal, the sifted cocompletion $\mathcal{P}_\Sigma(\mathcal{C})$ is a presentably symmetric monoidal ∞-category, given that $\mathcal{C}$ admits finite coproducts which distribute over the tensor product in $\mathcal{C}$. Here $\mathcal{P}_\Sigma(\mathcal{C})$ is equipped with the Day-convolution structure, i.e. the presentably symmetric monoidal structure classified by the property that the Yoneda embedding is symmetric monoidal. In this situation, we have a symmetric monoidal equivalence

$$\mathcal{P}_\Sigma(\mathcal{C}_+) \simeq \mathcal{P}_\Sigma(\mathcal{C})_\bullet$$

by [BH21, Lem. 2.2]. We are of course interested in the case $\mathcal{C} = \mathsf{Fin}_G$, to arrive at a symmetric monoidal equivalence

$$\mathcal{S}_{G\bullet} \simeq \mathcal{P}_\Sigma(\mathsf{Fin}_{G+}).$$

Now consider the functor $\iota\colon \mathsf{Fin}_{G+} \to \mathrm{Span}(\mathsf{Fin}_G)$ which sends T_+ to T and a morphism $f\colon T_+ \to S_+$ to the span

$$\begin{tikzcd} & \{t \in T | f(t) \neq \bullet\} \arrow[dl, hook'] \arrow[dr] & \\ T & & S. \end{tikzcd}$$

As laid out in [BH21, p. 50] and our Lemma 8.2, applying $\mathcal{P}_\Sigma(-)$ to ι yields a symmetric monoidal left adjoint

$$\mathcal{S}_{G\bullet} \to \mathcal{P}_\Sigma(\mathrm{Span}_G).$$

Postcomposing with the (symmetric monoidal, left adjoint) stabilization functor

$$\mathcal{P}_\Sigma(\mathrm{Span}_G) \to \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathsf{Sp}$$

finally yields the symmetric monoidal left adjoint

$$\Sigma^M_G\colon \mathcal{S}_{G\bullet} \to \mathrm{Mack}_G(\mathsf{Sp}).$$

Its right adjoint is given by

$$\mathrm{Fun}^\oplus(\mathrm{Span}_G^{\mathrm{op}}, \mathsf{Sp}) \to \mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}_\bullet) \to \mathrm{Fun}^\times(\mathsf{Fin}_G^{\mathrm{op}}, \mathcal{S}_\bullet) \simeq \mathcal{S}_{G\bullet}$$

so we have a factorization of $\mathbb{S}_G$ into symmetric monoidal left adjoints:

$$\begin{tikzcd} \mathcal{S}_G \arrow[rr, "\mathbb{S}_G"] \arrow[dr, "(-)_+"'] & & \mathrm{Mack}_G(\mathsf{Sp}). \\ & \mathcal{S}_{G\bullet} \arrow[ur, "\Sigma^M_G"'] & \end{tikzcd}$$

We turn to the construction of a model for the geometric fixed points functor. Consider the functor $(-)^G\colon \mathsf{Fin}_G \to \mathsf{Fin}$. It commutes with pullbacks, so we have a functor $\mathrm{Span}(\mathsf{Fin}_G) \to \mathrm{Span}(\mathsf{Fin})$, which preserves finite coproducts so induces a left adjoint on sifted cocompletions $\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}_G)) \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}))$, see again Lemma 8.2. Since the original functor $(-)^G\colon \mathsf{Fin}_G \to \mathsf{Fin}$ preserves products, the induced functor on spans is symmetric monoidal, and so is

$$\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}_G)) \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})).$$

We tensor this functor with Sp to obtain the symmetric monoidal left adjoint

$$\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}_G)) \otimes \mathsf{Sp} \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \otimes \mathsf{Sp}.$$

Recall from e.g. [BH21, Prop. C.1] (or originally [Cra10, Sect. 5]) that

$$\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \simeq \mathrm{CMon},$$

and thus (by Proposition 5.11) we have a canonical (symmetric monoidal) equivalence

$$\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \otimes \mathsf{Sp} \xrightarrow{\sim} \mathsf{Sp},$$

so we finally obtain the desired functor

$$\Phi^G_M\colon \mathrm{Mack}_G(\mathsf{Sp}) \to \mathsf{Sp}.$$

In addition to being a symmetric monoidal left adjoint, it by construction sits in the commutative diagram of symmetric monoidal left adjoints

$$\begin{tikzcd}
\mathcal{S}_{G\bullet} \simeq \mathcal{P}_\Sigma(\mathsf{Fin}_{G+}) \arrow[r] \arrow[d, "(-)^G"] \arrow[rr, bend left, "\Sigma^M_G"] & \mathcal{P}_\Sigma(\mathrm{Span}_G) \arrow[r] \arrow[d] & \mathcal{P}_\Sigma(\mathrm{Span}_G) \otimes \mathsf{Sp} = \mathrm{Mack}_G(\mathsf{Sp}) \arrow[d, "\Phi^G_M"] \\
\mathcal{S}_\bullet = \mathcal{P}_\Sigma(\mathsf{Fin}_+) \arrow[r] \arrow[rr, bend right, "\Sigma^\infty"'] & \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \arrow[r] & \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \otimes \mathsf{Sp} \simeq \mathsf{Sp}.
\end{tikzcd}$$

Thus, this functor satisfies the three properties that usually classify the geometric fixed points functor on genuine G-spectra: It commutes with colimits, is symmetric monoidal, and computes the expected values on suspension G-spectra, namely

$$\Phi^G_M(\Sigma^M_G X) \simeq \Sigma^\infty X^G$$

for any pointed G-space X.

Example 8.19. This immediately determines the values of $\Phi^G_M \mathbb{S}_G[G/H]$ as $\Sigma^\infty * \simeq 0$ if H is a proper subgroup of G and $\Sigma^\infty S^0 \simeq \mathbb{S}$ if $H = G$.

Via the same procedure, we now produce symmetric monoidal left adjoint change-of-group functors

$$\mathrm{res}_H\colon \mathrm{Mack}_G(\mathsf{Sp}) \to \mathrm{Mack}_H(\mathsf{Sp})$$

associated to a subgroup H of G. This time, consider the forgetful functor res_H from finite G-sets to finite H-sets. It preserves pullbacks and finite coproducts and thus defines a symmetric monoidal left adjoint

$$\mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}_G)) \to \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin}_H)),$$

and we again obtain the symmetric monoidal left adjoint

$$\mathrm{res}_H\colon \mathrm{Mack}_G(\mathsf{Sp}) \to \mathrm{Mack}_H(\mathsf{Sp})$$

by tensoring with Sp. By displaying the analogous diagram as above, we see that it commutes with suspension in the sense that there is a natural equivalence of functors $\mathcal{S}_{G\bullet} \to \mathrm{Mack}_H(\mathsf{Sp})$

$$\mathrm{res}_H \circ \Sigma^M_G \simeq \Sigma^M_H \circ \mathrm{res}_H \,. \tag{8.20}$$

By naturality, we also have the expected compatibility of further restriction, i.e. for $K \hookrightarrow H \hookrightarrow G$ the diagram

$$\begin{tikzcd}
\mathrm{Mack}_G(\mathsf{Sp}) \arrow[r, "\mathrm{res}_H"] \arrow[dr, "\mathrm{res}_K"'] & \mathrm{Mack}_H(\mathsf{Sp}) \arrow[d, "\mathrm{res}_K"] \\
& \mathrm{Mack}_K(\mathsf{Sp})
\end{tikzcd}$$

commutes.

For a G-space X and subgroups $K \hookrightarrow H \hookrightarrow G$, we have $(\mathrm{res}_H X)^K = X^K$. We would like restriction of Mackey functors to act in the same way on fixed points, namely that

$$(\mathrm{res}_H \mathcal{M})(H/K) \simeq \mathcal{M}(G/K).$$

In clarifying this, we also recover the 'Wirthmüller Isomorphism' from classical equivariant stable homotopy theory.

First note that the forgetful functor $\mathrm{res}_H \colon \mathsf{Fin}_H \to \mathsf{Fin}_G$ has a left adjoint

$$\mathrm{ind}_H \colon \mathsf{Fin}_G \to \mathsf{Fin}_H, \qquad U \mapsto G \times_H U.$$

Both res_H and ind_H preserve pullbacks, so induce functors

$$\mathrm{Span}(\mathsf{Fin}_G) \underset{\mathrm{res}_H}{\overset{\mathrm{ind}_H}{\rightleftarrows}} \mathrm{Span}(\mathsf{Fin}_H).$$

which are not only adjoint in the original direction, but as a consequence of the self-duality of $\mathrm{Span}(\mathcal{C})$ also adjoint in the other direction [BH21, Cor. C.21]. We will refer to a pair of functors which are simultaneously each others left and right adjoint as a bi-adjunction.

Now recall from Lemma 8.2 that the right adjoint to $\mathcal{P}_\Sigma(\mathrm{res}_H)$ is given by pre-composition with $\mathrm{res}_H \colon \mathrm{Span}_G^{\mathrm{op}} \to \mathrm{Span}_H^{\mathrm{op}}$ as expressed in the diagram

$$\begin{tikzcd}
\mathcal{P}_\Sigma(\mathrm{Span}_G) \arrow[r, "\mathcal{P}_\Sigma(\mathrm{res}_H)"] \arrow[d, "="] & \mathcal{P}_\Sigma(\mathrm{Span}_H) \arrow[d, "="] \\
\mathrm{Fun}^\times(\mathrm{Span}_G^{\mathrm{op}}, \mathcal{S}) & \mathrm{Fun}^\times(\mathrm{Span}_H^{\mathrm{op}}, \mathcal{S}). \arrow[l, dashed, "\mathrm{res}_H^*"]
\end{tikzcd} \tag{8.21}$$

But of course it is a general principle that precomposition along a left adjoint is right adjoint to precomposition with its right adjoint, and vice-versa. Maybe more clearly: If

$$\mathcal{C} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{D}$$

is an adjunction of ∞-categories (with unit transformation $\mu \colon c \to RLc$) and $\mathcal{E}$ is a third ∞-category, we have an adjunction

$$\mathrm{Fun}(\mathcal{C}, \mathcal{E}) \underset{L^*}{\overset{R^*}{\rightleftarrows}} \mathrm{Fun}(\mathcal{D}, \mathcal{E})$$

witnessed as such by the unit transformation

$$(\alpha \colon \mathcal{C} \to \mathcal{E}) \to L^* R^* \alpha$$

given at c as $\alpha(c) \xrightarrow{\alpha(\mu)} \alpha(RLc) = L^* R^* \alpha(c)$.

Now if both L and R preserve products, the adjunction above restricts to an adjunction

$$\mathrm{Fun}^{\times}(\mathcal{C},\mathcal{E}) \underset{L^*}{\overset{R^*}{\rightleftarrows}} \mathrm{Fun}^{\times}(\mathcal{D},\mathcal{E}).$$

Applying this principle to the bi-adjunction

$$\mathrm{Span}(\mathsf{Fin}_G) \underset{\mathrm{res}_H}{\overset{\mathrm{ind}_H}{\leftrightarrows}} \mathrm{Span}(\mathsf{Fin}_H).$$

identifies the left adjoint $\mathcal{P}_\Sigma(\mathrm{res}_H)$ in the square 8.21 as given by precomposition with $\mathrm{ind}_H\colon \mathrm{Span}_H \to \mathrm{Span}_G$, and the other way around. Tensoring with Sp then gives the bi-adjunction[8]

$$\mathrm{Mack}_G(\mathsf{Sp}) \underset{\mathrm{res}_H}{\overset{\mathrm{ind}_H}{\leftrightarrows}} \mathrm{Mack}_H(\mathsf{Sp}).$$

where for an H-Mackey functor $\mathcal{N}$ we have

$$(\mathrm{ind}_H \mathcal{N})(G/K) \simeq \mathcal{N}(\mathrm{res}_H G/K)$$

and more importantly, for a G-Mackey functor $\mathcal{M}$ we have

$$(\mathrm{res}_H \mathcal{M})(H/K) \simeq \mathcal{M}(G \times_H H/K) = \mathcal{M}(G/K).$$

Applying the principle $\mathrm{res}_H X^K = X^K$ to geometric fixed points leads to the following definition.

Definition 8.22. For H a subgroup of G, define the symmetric monoidal left adjoints

$$\Phi^H_M\colon \mathrm{Mack}_G(\mathsf{Sp}) \xrightarrow{\mathrm{res}_H} \mathrm{Mack}_H(\mathsf{Sp}) \xrightarrow{\Phi^H_M} \mathsf{Sp}.$$

The next order of business will be to see that the family of functors $\{\Phi^H_M\}_{H\leq G}$ is *jointly conservative*, i.e. that $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$ is 0 if and only if $\Phi^H_M\mathcal{M} \simeq 0$ for all H.

8.3. **Joint conservativity of $\{\Phi^H\}$.** To prove that the family of symmetric monoidal left adjoints $\{\Phi^H_M\}_{H\leq G}$ is jointly conservative, we will need an explicit formula for $\Phi^G_M\mathcal{M}$. Again, this pretty closely follows Appendix A of [CMNN20].

We begin by constructing the *classifying space* $\mathsf{E}\mathcal{F}$ of a family of subgroups of G. Recall that a collection of subgroups $\mathcal{F}$ is called a family if its closed under taking subgroups and conjugation. Its classifying space $\mathsf{E}\mathcal{F}$ is then characterized by the property that

$$\mathsf{E}\mathcal{F}^H = \begin{cases} * & \text{if } H \in \mathcal{F} \\ \varnothing & \text{if } H \notin \mathcal{F}. \end{cases}$$

Since $\mathcal{S}_G = \mathrm{Fun}(\mathsf{Or}_G^{\mathrm{op}}, \mathcal{S})$ and there is a map $G/H \to G/K$ in Or_G if and only if H is subconjugate to K, it should be obvious that $\mathsf{E}\mathcal{F}$ exists (and uniquely so) if $\mathcal{F}$ is indeed a family. We also have the following description, where we write $\mathsf{Or}_G^{\mathcal{F}}$ for the full subcategory of Or_G spanned by the orbits G/H with $H \in \mathcal{F}$.

Proposition 8.23. *$\mathsf{E}\mathcal{F}$ is given as the colimit of the natural inclusion $\mathsf{Or}_G^{\mathcal{F}} \hookrightarrow \mathcal{S}_G$.*

Proof. Since evaluation commutes with taking colimits, we need to see that the colimit (in $\mathcal{S}$) of

$$\mathsf{Or}_G^{\mathcal{F}} \to \mathcal{S}, \quad G/H \mapsto (G/H)^K = \mathrm{Map}_{\mathsf{Or}_G}(G/K, G/H)$$

[8]The fact that these functors are adjoined both ways is classically called the 'Wirthmüller Isomorphism'.

is contractible for $K \in \mathcal{F}$ and empty otherwise. The latter is clear: A space $\mathrm{Map}_{\mathsf{Or}_G}(G/K, G/H)$ being non-empty implies K being subconjugate to H, so K would have to be in $\mathcal{F}$. Thus we are computing a colimit with constant value the initial object $\varnothing$.

For $K \in \mathcal{F}$, we are computing the colimit of the corepresentable functor

$$\mathrm{Map}_{\mathsf{Or}_G^{\mathcal{F}}}(G/K, -)\colon \mathsf{Or}_G^{\mathcal{F}} \to \mathcal{S}$$

and colimits over corepresentable functors are always contractible: The colimit is given by the geometric realization of the total space of an unstraightening by [Lur21, Cor. 7.3.6.5]. A corepresentable functor unstraightens to the left fibration

$$\mathcal{C}_{c/} \to \mathcal{C}$$

and $|\mathcal{C}_{c/}|$ is contractible since $c \xrightarrow{=} c$ is an initial object. □

In this section, we will be particularly interested in the classifying space $\mathsf{E}\mathcal{P}$ of the family of proper subgroups $\mathcal{P} = \{H < G\}$.

Definition 8.24. We define $\widetilde{\mathsf{E}\mathcal{P}}$ to be the cofiber (in $\mathcal{S}_{G\bullet}$) of the map

$$\mathsf{E}\mathcal{P}_+ \to S^0$$

that sends only the basepoint to the basepoint. Its fixed points at H are thus the cofiber (in $\mathcal{S}_\bullet$) of $\mathsf{E}\mathcal{P}_+^H \to S^0$, i.e.

$$\widetilde{\mathsf{E}\mathcal{P}}^H \simeq \begin{cases} * & \text{if } H \in \mathcal{P} \\ S^0 & \text{if } H = G. \end{cases}$$

Proposition 8.25. [CMNN20, Prop. A.8] *We have an equivalence*

$$(\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M})(G/G) \xrightarrow{\sim} \Phi_M^G \mathcal{M}.$$

natural in $\mathcal{M}$.

Proof. Under the Day-convolution symmetric monoidal structure on $\mathrm{Fun}(\mathrm{Mack}_G(\mathsf{Sp}), \mathsf{Sp})$, lax symmetric monoidal functors $\mathrm{Mack}_G(\mathsf{Sp}) \to \mathsf{Sp}$ correspond to algebra-objects ([Gla16, Prop. 2.12]), and the Sp-valued functor corepresented by the unit

$$\mathcal{M} \mapsto \mathrm{map}_G(\mathbb{S}_G, \mathcal{M}) \simeq \mathcal{M}(G/G)$$

becomes the unit of the symmetric monoidal structure on $\mathrm{Fun}(\mathrm{Mack}_G(\mathsf{Sp}), \mathsf{Sp})$, as a consequence of the characterization of Remark 8.12. Thus, since Φ_M^G is symmetric monoidal, there is a lax symmetric monoidal transformation

$$\mathcal{M}(G/G) \to \Phi_M^G \mathcal{M}.$$

Since $\Phi_M^G(\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M}) \simeq (\Sigma^\infty \widetilde{\mathsf{E}\mathcal{P}}^G) \otimes \Phi_M^G \mathcal{M} \simeq \Phi_M^G \mathcal{M}$, replacing $\mathcal{M}$ by $\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M}$ gives the desired transformation

$$\phi\colon (\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M})(G/G) \to \Phi_M^G \mathcal{M}.$$

Since both sides preserve colimits, by Remark 8.11 it is enough to check that is an equivalence on all orbits $\mathcal{M} = \Sigma_G^M(G/H_+)$.

We know the right hand side to be

$$\Phi_M^G \Sigma_G^M(G/H_+) \simeq \begin{cases} 0 & \text{if } H \in \mathcal{P} \\ \mathbb{S} & \text{if } H = G. \end{cases}$$

Since every map $0 \to 0$ is an equivalence and every algebra-map $\mathbb{S} \to \mathbb{S}$ is an equivalence, we are reduced to confirming that we have the same values on the left hand side. Since by Remark 8.15 we have

$$(\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}} \otimes \Sigma_G^M(G/H_+))(G/G) \simeq (\Sigma_G^M \widetilde{\mathsf{E}\mathcal{P}})(G/H),$$

it remains to see that

$$(\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}})(G/H) \simeq \begin{cases} 0 & \text{if } H \in \mathcal{P} \\ \mathbb{S} & \text{if } H = G. \end{cases}$$

The first bit is easy enough:

$$\begin{aligned}(\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}})(G/H) &\simeq (\mathrm{res}_H \Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}})(H/H) \\ &\simeq \Sigma^M_H (\mathrm{res}_H \widetilde{\mathsf{E}\mathcal{P}})(H/H) \\ &\simeq 0,\end{aligned}$$

since $\mathrm{res}_H \widetilde{\mathsf{E}\mathcal{P}}$ is contractible in $\mathcal{S}_{G\bullet}$ for $H \in \mathcal{P}$. The second statement is a little more involved.

Lemma 8.26. [CMNN20, Lemma A.3] *We have an equivalence*

$$\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}}(G/G) \simeq \mathbb{S}.$$

Proof. Since $\mathbb{S}_G$ and evaluation at G/G preserve cofiber sequences and colimits, $\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}}(G/G)$ is the cofiber of the evaluation at G/G of the map of Mackey functors

$$\operatorname*{colim}_{G/H \in \mathsf{Or}^{\mathcal{P}}_G} (\mathbb{S}_G[G/H]) \to \mathbb{S}_G[G/G]$$

induced by $G/H \to G/G$. Let us take as input that there is a cofiber sequence of commutative monoids

$$\operatorname*{colim}_{G/H \in \mathsf{Or}^{\mathcal{P}}_G} (\mathsf{Fin}_{G}{}_{/G/H})^{\sim} \to \mathsf{Fin}^{\sim}_G \to \mathsf{Fin}^{\sim}$$

where the first map is induced from $\mathsf{Fin}_{G}{}_{/G/H} \to \mathsf{Fin}_{G}{}_{/G/G} \simeq \mathsf{Fin}_G$. This is a consequence of [CMNN20, Lemma A.4]. Recalling from Example 8.6 that the fixed points of orbits in $\mathrm{Mack}_G(\mathsf{Sp})$ are given by

$$\mathbb{S}_G[G/H](G/G) \simeq \mathrm{map}_G(\mathbb{S}_G, \mathbb{S}_G[G/H]) \simeq (\mathsf{Fin}_{G}{}_{/G/H})^{\sim, gp},$$

we apply the left adjoints (between pointed ∞-categories, and thus cofiber preserving)

$$\mathrm{CMon} \xrightarrow{(-)^{gp}} \mathrm{CMon}^{gp} \simeq \mathsf{Sp}_{\geq 0} \hookrightarrow \mathsf{Sp},$$

and obtain the desired cofiber sequence of spectra

$$\operatorname*{colim}_{G/H \in \mathsf{Or}^{\mathcal{P}}_G} \mathbb{S}_G[G/H](G/G) \to \mathbb{S}_G(G/G) \to \mathbb{S}.$$

□

This also concludes the proof of Proposition 8.25, i.e. that

$$\Phi^G_M \mathcal{M} \simeq (\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M})(G/G).$$

□

Remark 8.27. Since the functors Σ^M_G, $- \otimes \mathcal{M}$ and evaluating at G/G all preserve cofiber sequences, we have a cofiber sequence

$$(\Sigma^M_G \operatorname*{colim}_{\mathsf{Or}^{\mathcal{P}}_G}(G/H_+)) \otimes \mathcal{M})(G/G) \to \mathcal{M}(G/G) \to (\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathcal{M})(G/G).$$

Commuting the colimit to the outside and by Proposition 8.25, we equivalently obtain the cofiber sequence

$$\operatorname*{colim}_{G/H \in \mathsf{Or}^{\mathcal{P}}_G} \mathcal{M}(G/H) \to \mathcal{M}(G/G) \to \Phi^G_M \mathcal{M}. \tag{8.28}$$

This is the 'isotropy separation sequence', expressing the categorical G-fixed points as a twisted sum of the fixed points with respect to proper subgroups and geometric fixed points. It also directly implies that if all fixed points with respect to proper subgroups vanish, the 'genuine' and the geometric fixed points with respect to G agree.

That the collection of geometric fixed points functors $\{\Phi^H_M\}$ is jointly conservative is now an easy consequence, and then so is the fact that the representation spheres are invertible objects of $\mathrm{Mack}_G(\mathsf{Sp})$.

Proposition 8.29. *The collection of geometric fixed points functors $\{\Phi^H_M\}$ is jointly conservative, i.e. $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$ is 0 if and only if $\Phi^H_M(\mathcal{M}) \simeq 0$ for all subgroups H.*

Proof. For $G = \mathbf{1}$ the geometric fixed points functor is defined as the equivalence

$$\mathrm{Mack}_{\mathbf{1}}(\mathsf{Sp}) = \mathcal{P}_\Sigma(\mathrm{Span}(\mathsf{Fin})) \otimes \mathsf{Sp} \simeq \mathsf{Sp},$$

so the result holds.

We now argue by induction on the order of the finite group G: By hypothesis, the vanishing of the geometric fixed points implies that $\mathrm{res}_H \mathcal{M} \simeq 0$ for all proper subgroups H. Thus, since $\mathcal{M}$ is 0 if and only if all its evaluations are $0 \in \mathsf{Sp}$ and $\mathcal{M}(G/H) \simeq \mathrm{res}_H \mathcal{M}(H/H)$, it only remains to see that $\mathcal{M}(G/G) \simeq 0$.

But now both the left hand term and the right hand term in the isotropy separation sequence 8.28

$$\operatorname*{colim}_{G/H \in \mathsf{Or}_G^{\mathcal{P}}} \mathcal{M}(G/H) \to \mathcal{M}(G/G) \to \Phi^G_M \mathcal{M}$$

vanish, so the result follows. □

Recall that for a given finite-dimensional real representation V of the group G (i.e. a homomorphism $G \to \mathrm{Aut}(V)$ for some finite dimensional vector space V), the pointed G-space S^V is the one-point compactification of the G-space V. We call these spaces *representation spheres.* They are always finite colimits of orbits $(G/H)_+ \in \mathcal{S}_{G\bullet}$. An object x of a symmetric monoidal ∞-category is *invertible* if there is an object x^{-1} such that $x \otimes x^{-1}$ is the tensor unit.

Corollary 8.30. [CMNN20, Prop. A.10] *The symmetric monoidal left adjoint $\Sigma^M_G : \mathcal{S}_{G\bullet} \to \mathrm{Mack}_G(\mathsf{Sp})$ sends all representation spheres to invertible objects.*

Proof. We have seen that the orbits $\Sigma^M_G(G/H_+)$ are dualizable and, in the proof of Proposition 8.16, that the dualizable objects in $\mathrm{Mack}_G(\mathsf{Sp})$ are closed under finite colimits. Thus any representation sphere $\Sigma^M_G S^V$ is dualizable. Call its dual S^{-V} and consider the coevaluation map

$$\mathbb{S}_G \to S^{-V} \otimes \Sigma^M_G S^V.$$

Applying the symmetric monoidal functor Φ^H_M sends this to the coevaluation map in Sp

$$\mathbb{S} \to D \otimes \Phi^H_M \Sigma^M_G S^V$$

where D is the dual of $\Phi^H_M \Sigma^M_G S^V$. We have

$$\Phi^H_M \Sigma^M_G S^V \simeq \Sigma^\infty (S^V)^H \simeq \Sigma^\infty S^{(V^H)}$$

which is a sphere of dimension $\dim V^H$ and thus invertible in Sp. Therefore, the coevaluation map in Sp is an equivalence, and by joint conservativity of the functors Φ^H_M, so is the coevaluation map in $\mathrm{Mack}_G(\mathsf{Sp})$. □

9. Mackey Functors and G-spectra

We can now prove Theorem 1.3, i.e. that there is an equivalence of symmetric monoidal ∞-categories
$$\mathsf{Sp}_G \simeq \mathrm{Mack}_G(\mathsf{Sp}).$$
Before giving the proof found in [CMNN20, Appendix A], with only minor changes and some clarifications, let us recap the history of this result.

In the before-times, there were symmetric monoidal model categories of genuine G-spectra: Orthogonal G-spectra and S_G-modules are prominent examples, which are both described in [MM02]. Using the model-theoretic version of the Schwede-Shipley theorem of [SS03] (which describes stable model categories with a set of compact generators as spectrally-enriched presheaves on the full subcategory of those compact generators) Guillou and May managed to establish a zig-zag of Quillen equivalences between orthogonal G-spectra and a certain model category of spectrally-enriched functors $\boldsymbol{B}_G \to \mathscr{S}$. The target here is the model category of symmetric spectra, enriched over itself, and the source is a somewhat ad-hoc definition of the spectral Burnside-category: It is obtained by first considering the 2-category $\mathrm{Span}(\mathsf{Fin}_G)$ and its mapping categories as symmetric monoidal under the coproduct. Applying some monoidal model for K-theory (in lieu of a group-completion functor) yields a spectrally-enriched category, and it is then essentially an equivariant version of the Barrat-Priddy-Quillen theorem that identifies $\boldsymbol{B}_G$ with the full subcategory spanned by the (suspension G-spectra of) orbits. The triple of papers [GM17a], [GM17b], [GM20] is testament to their hardships.

But the idea was clear: The equivariant stable homotopy category is equivalent to spectral presheaves on the full subcategory spanned by the orbits, and the (connective) spectrum of equivariant maps $\Sigma^\infty_G(G/H_+) \to \Sigma^\infty_G(G/K_+)$ is the group completion of the commutative monoid $\mathrm{Hom}_{\mathrm{Span}_G}(G/H, G/K)$. Barwick transported (and greatly generalized) these ideas then to the language of ∞-categories [Bar17]. Since group-completion is left adjoint to the inclusion and the ∞-category Sp is already additive, there is no more need to group-complete the mapping spaces of $\mathrm{Span}(\mathsf{Fin}_G)$, and we arrive at the definition
$$\mathrm{Mack}_G(\mathsf{Sp}) \coloneqq \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathsf{Sp}).$$
A direct proof that the ∞-category associated to orthogonal G-spectra is equivalent to the ∞-category of spectral G-Mackey functors then appears in [Nar16], using the heavy machinery of parametrized higher category theory.

Robalo's thesis [Rob15] establishes the technical background to characterize, given a presentably symmetric monoidal ∞-category $\mathcal{C}$ and an object c in $\mathcal{C}$, the initial (presentably symmetric monoidal) ∞-category under $\mathcal{C}$ such that c is sent to an invertible object. In [GM20, App. C], Gepner and Meier then provide a careful proof that the ∞-category associated to orthogonal G-spectra is indeed the initial object of $\mathrm{CAlg}(\mathrm{Pr_L})$ under $\mathcal{S}_{G\bullet}$ with inverted representation spheres, and in Section 2.5 we took this uinversal property as our definition of Sp_G.

As summarized in [CMNN20, Thm A.2], the essential insight to show that (the ∞-category associated to) orthogonal G-spectra satisfies said universal property is that, by [Rob15, Prop. 2.19, Cor. 2.22], the mapping spaces between (the image of) finite G-sets T_+, T'_+ in Sp_G are given by
$$\operatorname*{colim}_{V \in \mathcal{U}} \mathrm{Map}_{\mathcal{S}_{G\bullet}}(S^V \otimes T_+, S^V \otimes T'_+),$$
as they are in orthogonal G-spectra.

In the appendix of [CMNN20], it is shown that $\mathrm{Mack}_G(\mathsf{Sp})$ satisfies the same universal property as Sp_G, and it is this proof that we give here. From this perspective, the equivariant Barrat-Priddy-Quillen is a corollary of two constructions satisfying the same universal property [CMNN20, Cor. A.11], where in the Guillou-May version it is the essential ingredient of the proof. We also remark that we import some structural properties of Sp_G from orthogonal G-spectra, but believe that the relevant properties can be deduced, without too much trouble, from the universal property (and Robalo's description) of Sp_G, giving a proof without recourse to any model.

9.1. **The proof that** $\mathsf{Sp}_G \simeq \mathrm{Mack}_G(\mathsf{Sp})$**.** Recall from Section 2.5 that we defined Sp_G as the initial presentably symmetric monoidal ∞-category equipped with a symmetric monoidal functor $\Sigma^\infty_G \colon \mathcal{S}_{G\bullet} \to \mathsf{Sp}_G$ such that the representation spheres are send to invertible objects. Since $\Sigma^M_G \colon \mathcal{S}_{G\bullet} \to \mathrm{Mack}_G(\mathsf{Sp})$ inverts the representation spheres by Corollary 8.30, we obtain an essentially unique symmetric monoidal left adjoint

$$\begin{tikzcd} & \mathcal{S}_{G\bullet} \arrow[dl, "\Sigma^\infty_G"'] \arrow[dr, "\Sigma^M_G"] & \\ \mathsf{Sp}_G \arrow[rr, dashed, "L"] & & \mathrm{Mack}_G(\mathsf{Sp}). \end{tikzcd}$$

rendering the diagram commutative.

Proposition 9.1. [CMNN20, Thm. A.1] *The above functor* $L \colon \mathsf{Sp}_G \to \mathrm{Mack}_G(\mathsf{Sp})$ *is an equivalence.*

Let us be very explicit about what well-known facts about the ∞-category Sp_G we will use. In light of the identification in the appendix of [GM20], these will be derived from analysis of one of the equivalent models for Sp_G such as orthogonal G-spectra [HHR15] or S_G-modules [MM02], and match the structural results we proved for $\mathrm{Mack}_G(\mathsf{Sp})$ in the preceding sections.

- There are symmetric monoidal left adjoints $\mathrm{res}_H \colon \mathsf{Sp}_G \to \mathsf{Sp}_H$ [MNN17, after Rem. 5.12].
- There is a symmetric monoidal *geometric fixed points* functor [NS18, Def. II.2.5.(ii)] $\Phi^G \colon \mathsf{Sp}_G \to \mathsf{Sp}$ given by the formula [Sch15, Prop. 7.6]
$$\Phi^G X \simeq (\Sigma^\infty_G \widetilde{E\mathcal{P}} \otimes X)^G.$$
Setting
$$\Phi^H \colon \mathsf{Sp}_G \xrightarrow{\mathrm{res}_H} \mathsf{Sp}_H \xrightarrow{\Phi^H} \mathsf{Sp}$$
gives a jointly conservative family of functors $\{\Phi^H\}_{H \le G}$. If X is such that $\mathrm{res}_H X$ vanishes for all proper subgroups H, there is an equivalence $X^G \xrightarrow{\sim} \Phi^G X$. These last two facts follow from the isotropy separation sequence in the same way we derived them for Mackey functors.
- Sp_G is *rigidly-compactly generated*, i.e. there is a set of compact generators, and the dualizable and the compact objects agree. Given that the orbits $\Sigma^\infty_G(G/H_+) \in \mathsf{Sp}_G$ are compact generators ([GM20, Lemma C.2] and [MNN17, Ex. 5.11]) and dualizable [GM17a, Prop. 3.9], the proof of our Proposition 8.16 adapts word for word.

Proof. We first collect some preliminaries about $R\colon \mathrm{Mack}_G(\mathsf{Sp}) \to \mathsf{Sp}_G$, the right adjoint to L.

(i) The functor R, like L, also preserves all colimits: L is symmetric monoidal so preserves dualizable objects, which coincide with compact objects on either side. We give the easy proof that the right adjoint of a compact-preserving functor between presentable compactly generated stable ∞-categories preserves all colimits in Lemma 11.6.

(ii) We have the *projection formula*: The natural map

$$X \otimes R\mathcal{M} \xrightarrow{\sim} R(LX \otimes \mathcal{M}) \tag{9.2}$$

is an equivalence, for any $X \in \mathsf{Sp}_G$ and any $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$.

Indeed, by the above both sides are colimit-preserving functors of X, so since Sp_G is rigidly-compactly generated, it is enough to check on dualizable objects. But for dualizable $X \in \mathsf{Sp}_G$ with dual $X^\vee$, we compare mapping properties:

$$\begin{aligned}\mathrm{Map}_{\mathsf{Sp}_G}(Y, X \otimes R\mathcal{M}) &\simeq \mathrm{Map}_{\mathsf{Sp}_G}(Y \otimes X^\vee, R\mathcal{M})\\ &\simeq \mathrm{Map}_{\mathrm{Mack}_G}(LY \otimes LX^\vee, \mathcal{M})\\ &\simeq \mathrm{Map}_{\mathrm{Mack}_G}(LY, LX \otimes \mathcal{M})\\ &\simeq \mathrm{Map}_{\mathsf{Sp}_G}(Y, R(LX \otimes \mathcal{M})).\end{aligned}$$

(iii) The functor R is conservative: If $R\mathcal{M} \simeq 0$, we have

$$\begin{aligned}0 &\simeq \mathrm{Map}_{\mathsf{Sp}_G}(\Sigma^\infty_G(G/H_+), R\mathcal{M})\\ &\simeq \mathrm{Map}_{\mathrm{Mack}_G}(L\Sigma^\infty_G(G/H_+), \mathcal{M})\\ &\simeq \mathrm{Map}_{\mathrm{Mack}_G}(\Sigma^M_G(G/H_+), \mathcal{M})\\ &\simeq \mathcal{M}(G/H)\end{aligned}$$

for all $H \leq G$, so $\mathcal{M}$ is 0.

We have confirmed the hypotheses (i)-(iii) of Proposition 5.29 of [MNN17], and may reap its reward: There is an equivalence

$$\mathrm{Mack}_G(\mathsf{Sp}) \simeq \mathsf{Mod}_{\mathsf{Sp}_G}(R\mathbb{S}_G),$$

given by equipping $R\mathcal{M}$ with its canonical $R\mathbb{S}_G$-module structure. Here both the algebra-structure on $R\mathbb{S}_G$ and the module-structure on $R\mathcal{M}$ are derived from the fact that a right adjoint of a symmetric monoidal functor is lax symmetric monoidal. Thus, we may conclude that R is an equivalence if the forgetful functor $\mathsf{Mod}_{\mathsf{Sp}_G}(R\mathbb{S}_G) \to \mathsf{Sp}_G$ is an equivalence, which is the case if the unit map

$$\mathfrak{u}\colon 1_{\mathsf{Sp}_G} \to RL(1_{\mathsf{Sp}_G}) \simeq R\mathbb{S}_G \tag{9.3}$$

is an equivalence.

Certainly for the trivial group $G = \mathbf{1}$, we have $\mathrm{Mack}_{\mathbf{1}(\mathsf{Sp})} \simeq \mathsf{Sp}_{\mathbf{1}} \simeq \mathsf{Sp}$ in the desired fashion, and we may proceed by induction on the group order: We assume, for all proper subgroups H, that

$$\mathsf{Sp}_H \xrightarrow{L_H} \mathrm{Mack}_H(\mathsf{Sp})$$

is an equivalence. These sit in squares

$$\begin{array}{ccc}\mathsf{Sp}_G & \xrightarrow{L} & \mathrm{Mack}_G(\mathsf{Sp})\\ \big\downarrow{\scriptstyle \mathrm{res}_H} & & \big\downarrow{\scriptstyle \mathrm{res}_H}\\ \mathsf{Sp}_H & \xrightarrow{L_H} & \mathrm{Mack}_H(\mathsf{Sp})\end{array}$$

which commute since both compositions are symmetric monoidal left adjoint functors $\mathsf{Sp}_G \to \mathrm{Mack}_H$ that agree after precomposing with $\mathcal{S}_{G\bullet} \xrightarrow{\Sigma^\infty_G} \mathsf{Sp}_G$. Now from the induction hypothesis, it follows that the unit map 9.3 is an equivalence on H-fixed points for H a proper subgroup, i.e. that

$$\mathrm{map}_{\mathsf{Sp}_G}(G/H_+, 1_{\mathsf{Sp}_G}) \xrightarrow{\mathfrak{u}_*} \mathrm{map}_{\mathsf{Sp}_G}(G/H_+, RL1_{\mathsf{Sp}_G})$$

is an equivalence: The above factorizes as

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{Sp}_G}(G/H_+, 1_{\mathsf{Sp}_G}) \arrow[r, "u_*"] \arrow[d, "L"] & \mathrm{map}_{\mathsf{Sp}_G}(G/H_+, RL1_{\mathsf{Sp}_G}) \\
\mathrm{map}_{\mathrm{Mack}_G}(L(G/H_+), L1_{\mathsf{Sp}_G}), \arrow[ur, "\sim"] &
\end{tikzcd}$$

and the left vertical again factorizes as

$$\begin{tikzcd}
\mathrm{map}_{\mathsf{Sp}_G}(G/H_+, 1_{\mathsf{Sp}_G}) \arrow[r, "L"] \arrow[d, "\sim"] & \mathrm{map}_{\mathrm{Mack}_G}(G/H_+, L1_{\mathsf{Sp}_G}) \arrow[d, "\sim"] \\
\mathrm{map}_{\mathsf{Sp}_H}(\mathbb{S}_H, \mathrm{res}_H 1_{\mathsf{Sp}_G}) \arrow[r, "L_H", "\sim"'] & \mathrm{map}_{\mathrm{Mack}_H}(\mathbb{S}_H, \mathrm{res}_H 1_{\mathrm{Mack}_G})
\end{tikzcd}$$

It remains to see that

$$(1_{\mathsf{Sp}_G})^G \to (R\mathbb{S}_G)^G$$

is an equivalence, or (by the isotropy seperation sequence) equivalently that

$$\Phi^G 1_{\mathsf{Sp}_G} \to \Phi^G R\mathbb{S}_G$$

is. Since $\Phi^G \colon \mathsf{Sp}_G \to \mathsf{Sp}$ is symmetric monoidal, this is a map of algebras with source equivalent to $\mathbb{S}$, so it is enough to see that that there is some equivalence $\Phi^G R(\mathbb{S}_G) \simeq \mathbb{S}$.

For any $\mathcal{M} \in \mathrm{Mack}_G(\mathsf{Sp})$, we have

$$\begin{aligned}(R\mathcal{M})^G &:= \mathrm{map}_{\mathsf{Sp}_G}(\Sigma^\infty_G(G/G_+), R\mathcal{M}) \\ &\simeq \mathrm{map}_{\mathrm{Mack}_G}(\mathbb{S}_G, \mathcal{M}) \simeq \mathcal{M}(G/G).\end{aligned}$$

We also have the formula

$$\Phi^G X \simeq (\Sigma^\infty_G \widetilde{\mathsf{E}\mathcal{P}} \otimes X)^G$$

for geometric fixed points of genuine G-spectra, and with these we compute

$$\begin{aligned}\Phi^G R(\mathbb{S}_G) &\simeq (\Sigma^\infty_G \widetilde{\mathsf{E}\mathcal{P}} \otimes R(\mathbb{S}_G))^G \\ &\simeq R(L\Sigma^\infty_G \widetilde{\mathsf{E}\mathcal{P}} \otimes \mathbb{S}_G)^G \\ &\simeq R(\Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}})^G \\ &\simeq \Sigma^M_G \widetilde{\mathsf{E}\mathcal{P}}(G/G) \\ &\simeq \mathbb{S}\end{aligned}$$

where the second equivalence uses the projection formula 9.2 and the last uses Lemma 8.26. □

Remark 9.4. In the introduction, we introduced the underlying Or_G-spectrum of a genuine G-spectrum X as

$$G/H \mapsto (G/H_+ \otimes X)^G.$$

Given that the equivalence $\mathsf{Sp}_G \to \mathrm{Mack}_G(\mathsf{Sp})$ is symmetric monoidal, compatible with G-suspensions and sends G-fixed points to evaluation at G/G, and in light of Remark 8.15, this is equivalent to the underlying Or_G-spectrum of the associated Mackey functor.

For the same reason, the underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum of a genuine G-spectrum X, defined as

$$G/H \mapsto \mathrm{map}_{\mathsf{Sp}_G}(G/H_+, X) \simeq F(G/H_+, X)^G,$$

agrees with the underlying $\mathsf{Or}_G^{\mathrm{op}}$-spectrum of the associated Mackey functor.

10. (Co-)Assembly and the norm

In this chapter, we explain why for any (genuine) G-spectrum, the assembly and coassembly maps associated to their underlying Or_G and $\mathsf{Or}_G^{\mathrm{op}}$-spectrum compose to the norm map of the underlying spectrum with G-action.

First some background on the norm map: Given an abelian group A equipped with an action of a finite group G, there is a map

$$\mathfrak{nm}\colon A_G \to A^G, \quad [a] \mapsto \sum_{g \in G} g \cdot a$$

from the quotient group A_G to the fixed points A^G. This situation generalizes to the ∞-categorical setting: For G a finite group and $\mathcal{C}$ a preadditive ∞-category admitting colimits and limits indexed by finite groupoids, there is a natural *norm map*

$$\mathfrak{nm}\colon X_{hG} \to X^{hG}$$

associated to any $X \in \mathcal{C}^{\mathrm{B}G}$. The general construction of this map is performed in [Lur17, Setc. 6.1.6], and it is the content of [Lur17, Rem. 6.1.6.23] that the constructed map indeed generalizes the classical situation. Its cofiber is called the *Tate construction*.

However, we are not concerned with this generality, and refer to the much older [GM95]: For a genuine G-spectrum X, its *Tate cohomology spectrum* is constructed as the cofiber of the composition

$$\begin{tikzcd}
(\mathsf{E}G_+ \otimes X)^G \arrow[r, "①"] \arrow[dr, dashed, "\mathfrak{nm}"'] & X^G \simeq F(\mathbb{S}_G, X)^G \arrow[d, "②"] \\
 & F(\mathsf{E}G_+, X)^G.
\end{tikzcd}$$

Here, as usual, $\mathsf{E}G$ denotes the unique G-space whose fixed points with respect to any nontrivial subgroup are empty, but whose underlying space is contractible, i.e. the classifying space of the trivial family $\{1\}$, i.e. by Proposition 8.23

$$\mathsf{E}G \simeq \operatorname{colim}(\mathsf{Or}_G^{\{1\}} \hookrightarrow \mathcal{S}_G).$$

That the two constructions of the map $\mathfrak{nm}$ coincide appears to be another folklore result. We prove this identification in Appendix A, as it uses the results of Chapter 12. The following proposition together with Theorem 4.8 will easily imply Theorem A and its dual Theorem A'.

Proposition 10.1.

(i) The map

$$(\mathsf{E}G_+ \otimes X)^G \xrightarrow{①} X^G$$

is the Davis-Lück-assembly map 4.4 associated to $\vec{u}\mathcal{M}\colon \mathsf{Or}_G \to \mathsf{Sp}$, where $\mathcal{M}$ is the spectral Mackey functor associated to X by Theorem 9.1.

(ii) The map

$$X^G \xrightarrow{②} F(\mathsf{E}G_+, X)^G$$

is the Davis-Lück-coassembly map 4.11 associated to $\overleftarrow{u}\mathcal{M}\colon \mathsf{Or}_G^{\mathrm{op}} \to \mathsf{Sp}$, where $\mathcal{M}$ is the spectral Mackey functor associated to X by Theorem 9.1.

Proof. (i) Under the symmetric monoidal equivalence $\mathsf{Sp}_G \to \mathrm{Mack}_G(\mathsf{Sp})$, the map ① is sent to

$$(\Sigma^M_G \mathsf{E}G_+ \otimes \mathcal{M})(G/G) \to \mathcal{M}(G/G).$$

By colimit-preservation of the various functors involved and Proposition 8.23, this is equivalent to

$$\operatorname*{colim}_{G/H\in \mathsf{Or}_G^{\{1\}}} (\Sigma^M_G(G/H_+)\otimes \mathcal{M})(G/G) \to \mathcal{M}(G/G)$$

which is

$$\operatorname*{colim}_{G/H\in \mathsf{Or}_G^{\{1\}}} \vec{u}\mathcal{M} \to \vec{u}\mathcal{M}(G/G)$$

by Remark 8.15. But this is the comparison map of the left Kan extension from $\mathsf{Or}_G^{\{1\}}$ to the terminal object by Example 3.6

$$\big(\mathfrak{l}(\vec{u}\mathcal{M}|_{\mathsf{Or}_G^{\{1\}}}) \to \vec{u}\mathcal{M}\big)(G/G),$$

i.e. the Davis-Lück-assembly map 4.4 associated to $\vec{u}\mathcal{M}$.

(ii) Dually, the map ② is sent to

$$\mathcal{M}(G/G) \to F(\Sigma^M_G(\mathsf{E}G_+), \mathcal{M})(G/G).$$

Since the mapping G-Mackey functor transforms colimits into limits by Appendix B.2 and again in light of Remark 8.15, this is equivalent to the natural map

$$\overleftarrow{u}\mathcal{M}(G/G) \to \lim_{\mathrm{B}G} \overleftarrow{u}\mathcal{M},$$

i.e. the Davis-Lück-coassembly map 4.11 associated to $\vec{u}\mathcal{M}$. □

Combining the results of this chapter with Theorem 4.8 and Proposition 4.12, this concludes the proof of Theorems A and A'.

Remark 10.2. The largely formal observation

$$(\mathsf{E}G_+ \otimes X)^G \simeq uX_{hG}$$

obtained from commuting various colimits and the fact

$$(\mathbb{S}_G[G/1]\otimes X)^G \simeq uX$$

is classically known as the *Adams isomorphism*.

11. The ∞-category of small stable idempotent complete ∞-categories

We follow [BGT13] in defining $\mathrm{Cat}_{\mathrm{perf}}$ as the ∞-category of small stable idempotent-complete ∞-categories and exact functors. It enjoys many pleasant formal properties: It is presentable [BGT13, Cor. 4.25], and thus complete and cocomplete. It is not only a full subcategory of $\mathrm{Cat}_{\mathrm{ex}}$ (the ∞-category of small stable ∞-categories) but also a localization (i.e. the inclusion is right adjoint) by [BGT13, Lemma 2.20]. Combining this with [Lur17, Theorem 1.1.4.4] tells us that limits in $\mathrm{Cat}_{\mathrm{perf}}$ are computed underlying, i.e. as in Cat_∞. Characterizing the formation of various colimits in $\mathrm{Cat}_{\mathrm{perf}}$ will occupy the rest of this chapter.

11.1. **Preadditivity.** The first colimits we want to consider are those over finite sets, i.e. finite coproducts. It turns out that these identify with finite limits in the canonical way, rendering the ∞-category $\mathrm{Cat}_{\mathrm{perf}}$ *preadditive.* This will allow us to use $\mathrm{Cat}_{\mathrm{perf}}$ on the same footing as Sp as coefficients of *Mackey functors* in Chapter 12.

Recall that an ∞-category is preadditive if it admits direct sums, that is, it has a *zero object* 0 (i.e. an object that is both terminal and initial, necessarily unique up to contractible choice), admits finite products and coproducts, and these are identified by the natural map afforded by the zero object, i.e. the map

$$\phi\colon \coprod_{\mathcal{I}} c_i \to \prod_{\mathcal{I}} c_i$$

given by the maps $\phi_{ij}\colon c_i \longrightarrow c_j$ with $\phi_{ij} = 0$ for $i \neq j$, and $\phi_{ii} = \mathrm{Id}_{c_i}$ (the 'identity matrix').

The following largely formal proof is a retelling of Clark Barwick's version in the analogous setting of *Waldhausen* ∞-categories, see [Bar16, Prop. 4.11].

Proposition 11.1. *The ∞-category $\mathrm{Cat}_{\mathrm{perf}}$ is preadditive.*

Proof. The category consisting of one point and no nontrivial morphisms is of course small, stable, and idempotent-complete and thus provides a zero object of $\mathrm{Cat}_{\mathrm{perf}}$. Let $(\mathcal{C}_i)_{i\in\mathcal{I}}$ be a finite collection of objects in $\mathrm{Cat}_{\mathrm{perf}}$. Limits and thus products exist and are formed as in Cat_∞, so we have to show that $\mathcal{C} := \prod_{\mathcal{I}} \mathcal{C}_i$ is also the coproduct, exhibited as such by the natural maps $\phi_i\colon \mathcal{C}_i \longrightarrow \mathcal{C}$ (which are the identity on the i'th component and 0 on all others).

Recall that colimit-diagrams in any ∞-category may be recognized by their property to induce limit-diagrams on all mapping spaces, i.e. we need to show that, for any $\mathcal{D} \in \mathrm{Cat}_{\mathrm{perf}}$, the map

$$\mathrm{Map}_{\mathrm{Cat}_{\mathrm{perf}}}(\mathcal{C}, \mathcal{D}) \longrightarrow \prod_{\mathcal{I}} \mathrm{Map}_{\mathrm{Cat}_{\mathrm{perf}}}(\mathcal{C}_i, \mathcal{D})$$

induced by the ϕ_i is an equivalence of spaces. Just as the mapping space between ∞-categories $\mathcal{C}, \mathcal{D}$ in Cat_∞ is obtained from the ∞-category $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ by taking its maximal sub-groupoid, we may identify $\mathrm{Map}_{\mathrm{Cat}_{\mathrm{perf}}}(\mathcal{C}, \mathcal{D})$ with the maximal sub-groupoid in $\mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$, where the latter is the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ on the exact functors.

Thus, we are reduced to recognizing the functor

$$F\colon \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \prod_{\mathcal{I}} \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}_i, \mathcal{D})$$

as an equivalence of ∞-categories.

We do this by explicitly constructing an inverse: First, let

$$u\colon \prod_{\mathcal{I}} \mathrm{Fun}(\mathcal{C}_i, \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \prod_{\mathcal{I}} \mathcal{D})$$

be adjoint to

$$\mathcal{C} \times \prod_{\mathcal{I}} \mathrm{Fun}(\mathcal{C}_i, \mathcal{D}) \simeq \prod_{\mathcal{I}} (\mathcal{C}_i \times \mathrm{Fun}(\mathcal{C}_i, \mathcal{D})) \longrightarrow \prod_{\mathcal{I}} \mathcal{D}$$

where the last map is evaluation in each factor.

As any stable ∞-category admits finite sums, for any finite set $\mathcal{I}$, there is a functor $\mathrm{Fun}(\mathcal{I}, \mathcal{D}) \xrightarrow{\mathrm{colim}} \mathcal{D}$, and thus (since $\prod_{\mathcal{I}} \mathcal{D} \simeq \mathrm{Fun}(\mathcal{I}, \mathcal{D})$) we have a functor

$$e\colon \mathrm{Fun}(\mathcal{C}, \prod_{\mathcal{I}} \mathcal{D}) \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

sending $(f_i)_{i \in \mathcal{I}}$ to

$$c \mapsto \coprod_{\mathcal{I}} f_i(c).$$

The composition $e \circ u$ restricts to a functor

$$G\colon \prod_{\mathcal{I}} \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}_i, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

by the observation that a finite sum of exact functors is again exact.

Now we check that G is indeed an inverse to F: If we write $p_i\colon \mathcal{C} \to \mathcal{C}_i$ for the i'th projection, the composition

$$F \circ G\colon \prod_{\mathcal{I}} \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}_i, \mathcal{D}) \longrightarrow \prod_{\mathcal{I}} \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}_i, \mathcal{D})$$

sends a tuple of exact functors $(f_i\colon \mathcal{C}_i \to \mathcal{D})_{i \in \mathcal{I}}$ to the tuple

$$\Big(\coprod_{j \in \mathcal{I}} f_j \circ p_j \circ \phi_i \Big)_{i \in \mathcal{I}}$$

which is equivalent to the original tuple $(f_i)_{i \in \mathcal{I}}$ via the equivalences $p_j \circ \phi_i \simeq \mathrm{Id}_{\mathcal{C}_i}$ if $j = i$, $p_j \circ \phi_i \simeq 0$ in all other cases, and $X \coprod 0 \simeq X$ which holds in any pointed ∞-category.

In the other direction, the composition

$$G \circ F\colon \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D}) \longrightarrow \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$$

sends $f \in \mathrm{Fun}_{\mathrm{ex}}(\mathcal{C}, \mathcal{D})$ to the functor

$$\coprod_{\mathcal{I}} f \circ \phi_i \circ p_i.$$

By exactness of f, we are finished if we see $\coprod_{\mathcal{I}} \phi_i \circ p_i$ to be the identity functor on $\mathcal{C}$. This is true since coproducts of (exact) functors are of course computed pointwise, and again $X \coprod 0 \simeq X$.

$\square$

Remark 11.2. In this project, we care about functors $\mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ that preserve sums, and we simply call them *additive functors*. We remark that the notion of an *additive invariant* from [BGT13] asks for more: By definition, an additive invariant $\mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ preserves filtered colimits (a condition we will not need), and also preserves split-exact sequences. A sequence of functors

$$\mathcal{A} \to \mathcal{B} \to \mathcal{C}$$

in $\mathrm{Cat}_{\mathrm{perf}}$ is called *exact* if it is both a fiber sequence and cofiber sequence ([BGT13, Ch. 5], compare also [LT19, Def. 1.2]), and it is called *split-exact* if it is exact and the fully faithful functor $\mathcal{A} \to \mathcal{B}$ admits a right adjoint, and the functor $\mathcal{B} \to \mathcal{C}$ admits a fully faithful right adjoint. We easily confirm that a sum decomposition

$$\mathcal{A} \to \mathcal{A} \oplus \mathcal{C} \to \mathcal{C}$$

gives such a split-exact sequence, so indeed every additive invariant in the sense of [BGT13] is an additive functor (but not vice-versa).

These include connective K-theory, non-connective K-theory and topological Hochschild Homology, which are constructed as functors

$$K, \mathbf{K}, THH \colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$$

and proven to be additive invariants in sections 7, 9.1, 10.1 of [BGT13] respectively. Topological cyclic homology only fails to be an additive invariant because it does not preserve filtered colimits (it is, however, a limit of the additive invariants TC^n), so it also gives an additive functor $TC\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$, see [BGT13, Sec. 10.3].

11.2. **Homotopy orbits.** We now study the formation of homotopy orbits in $\mathrm{Cat}_{\mathrm{perf}}$. Unlike finite coproducts, these will not coincide with the corresponding limit, but we can still exploit a close relation to give a description. The reason for our interest is contained in the following statement, whose proof will occupy the rest of the section.

Theorem 11.3. *Let R be an $\mathbb{E}_1$-ring, G a finite group. We equip $\mathsf{Perf}R$, the category of compact R-modules, with the trivial action. Then*

$$\mathsf{Perf}R_{hG} := \operatorname*{colim}_{\mathrm{B}G} \mathsf{Perf}R \simeq \mathsf{Perf}R[G].$$

To calculate this colimit, we shall exploit the relation $\mathrm{Cat}_{\mathrm{perf}}$ holds to the ∞-category $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ (of stable presentable ∞-categories and left adjoint functors) via the functor of Ind-completion, to arrive at the following description of homotopy orbits in $\mathrm{Cat}_{\mathrm{perf}}$: The homotopy orbits of any $X \in \mathrm{Fun}(\mathrm{B}G, \mathrm{Cat}_{\mathrm{perf}})$ are given by the compact objects of the homotopy *fixed points* of $\mathrm{Ind} \circ X \in \mathrm{Fun}(\mathrm{B}G, \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}})$, and these homotopy fixed points may be computed as a limit in the ambient ∞-category of large ∞-categories $\widehat{\mathrm{Cat}_\infty}$, see [CMNN20, Ex. 2.17].

The crucial facts about Ind-completion are the following: The functor

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$$

(which freely adjoins filtered colimits to any small ∞-category, see [BGT13, Sect. 2.4] for a summary and [Lur09, Ch. 5.3] for a thorough discussion) factors as an equivalence onto the subcategory $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega}$ of *compactly generated* (stable, presentable) ∞-categories, and (left adjoint) functors that preserve compact objects. Its inverse is given by the functor that assigns to a stable presentable ∞-category its subcategory of compact objects, denoted $(-)^\omega$. We summarize this situation in this diagram:

$$\begin{tikzcd} \mathrm{Cat}_{\mathrm{perf}} \arrow[r, "\mathrm{Ind}"] \arrow[d, shift right, "\sim"'] & \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}} \\ \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega} \arrow[u, shift right, "-^\omega"'] \arrow[ur, hook] & \end{tikzcd}$$

This is rephrasing the contents of [BGT13, Lemma 2.20] and the surrounding paragraphs. We also collected some properties of the Ind-functor in the proof of Proposition 8.16. We shall prove the following theorem.

Theorem 11.4. *The inclusion $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega} \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ preserves colimits. As a corollary of the above, so does the functor $\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$.*

Remark 11.5. Our recipe to compute colimits in $\mathrm{Cat}_{\mathrm{perf}}$ via comparison to $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ is already explained in [CMNN20, Constr. 2.16]. However, the stated reason for Ind to commute with colimits is not quite correct - It is said that $(-)^\omega$ is right adjoint to Ind. Passage to compact objects only defines a functor on the subcategory of functors preserving compact objects (for obvious reasons). Even with Theorem 11.4 proved by other means, the adjoint functor theorem cannot provide a right adjoint for Ind, since $\mathrm{Pr}^{\mathrm{ex}}_{\mathrm{L}}$ is not locally small: Concretely, consider the ∞-category of compact spectra $\mathsf{Sp}^{\omega} \in \mathrm{Cat}_{\mathrm{perf}}$ (with $\mathrm{Ind}(\mathsf{Sp}^{\omega}) \simeq \mathsf{Sp}$), and presume the existence of an adjunction

$$\mathrm{Cat}_{\mathrm{perf}} \underset{R}{\overset{\mathrm{Ind}}{\rightleftarrows}} \mathrm{Pr}^{\mathrm{ex}}_{\mathrm{L}}.$$

We would obtain, for every stable presentable ∞-category $\mathcal{C}$ an equivalence of mapping spaces

$$\begin{aligned}(R\mathcal{C})^{\sim} &\simeq \mathrm{Fun}_{\mathrm{ex}}(\mathsf{Sp}^{\omega}, R\mathcal{C})^{\sim} \\ &\simeq \mathrm{Map}_{\mathrm{Cat}_{\mathrm{perf}}}(\mathsf{Sp}^{\omega}, R\mathcal{C}) \\ &\simeq \mathrm{Map}_{\mathrm{Pr}^{\mathrm{ex}}_{\mathrm{L}}}(\mathsf{Sp}, \mathcal{C}) \\ &\simeq \mathrm{Fun}_{\mathrm{ex}}(\mathsf{Sp}, \mathcal{C})^{\sim} \\ &\simeq \mathcal{C}^{\sim}\end{aligned}$$

where the term on the far left is the core of a small ∞-category (and thus itself small), and the term on the right is not small in general, e.g. for $\mathcal{C} = \mathsf{Sp}$. Note how restricting to only those functors in $\mathrm{Pr}^{\mathrm{ex}}_{\mathrm{L}}$ that preserve compacts immediately alleviates this contradiction.

While certainly known to experts, to the authors knowledge no proof of Ind preserving colimits has been recorded in the literature. We first check some easy preliminaries.

Lemma 11.6. *Let*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be an adjunction of presentable ∞-categories, F being the left adjoint.

(i) If G preserves filtered colimits, F preserves compact objects.
(ii) If F preserves compact objects and $\mathcal{C}$ is compactly generated, G preserves filtered colimits.
(iii) If $\mathcal{C}$ and $\mathcal{D}$ are stable, and G preserves filtered colimits, then G also admits a right adjoint.

Proof. (i) For c a compact object of $\mathcal{C}$ and $\{d_i\}$ any filtered diagram in $\mathcal{D}$, we see the natural equivalence

$$\begin{aligned}\mathrm{Map}_{\mathcal{D}}(Fc, \operatorname{colim} d_i) &\simeq \mathrm{Map}_{\mathcal{C}}(c, G \operatorname{colim} d_i) \\ &\simeq \mathrm{Map}_{\mathcal{C}}(c, \operatorname{colim} Gd_i) \\ &\simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}(c, Gd_i) \\ &\simeq \operatorname{colim} \mathrm{Map}_{\mathcal{D}}(Fc, d_i)\end{aligned}$$

by in turn applying the adjunction, G preserving filtered colimits, c being compact, and the adjunction again.

(ii) Similarly, for c compact, $\{d_i\}$ any filtered diagram and F compact-preserving, we obtain

$$\mathrm{Map}_{\mathcal{C}}(c, G \operatorname{colim} d_i) \simeq \mathrm{Map}_{\mathcal{C}}(c, \operatorname{colim} Gd_i)$$

so the two colimit-preserving functors

$$\mathrm{Map}_{\mathcal{C}}(-, G \operatorname{colim} d_i), \mathrm{Map}_{\mathcal{C}}(-, \operatorname{colim} Gd_i) \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{S}$$

agree on compacts, so they agree everywhere as $\mathcal{C}$ is assumed compactly generated.

The Yoneda lemma then supplies the natural equivalence

$$\operatorname{colim} Gd_i \simeq G \operatorname{colim} d_i.$$

(iii) In the stable situation, G being right adjoint implies it preserves all pullback squares and thus pushout squares, as well as all finite products and thus finite coproducts. A functor preserving finite coproducts and pushouts preserves all finite colimits, so if it in addition preserves filtered colimits, it preserves all colimits.

The adjoint functor theorem then supplies its right adjoint.

□

For full subcategories of ∞-categories, it is easy to see that the inclusion preserves colimits if and only if the full subcategory is closed under the formation of colimits. For non-full inclusions of subcategories (such as $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega} \to \mathrm{Pr}_{\mathrm{L}}$), the following is the relevant analogue.

Lemma 11.7. *Let $\mathcal{C}_0 \to \mathcal{C}$ be the inclusion of a subcategory, $\mathcal{C}$ cocomplete. It preserves colimits if for any diagram*

$$\begin{tikzcd} K \arrow[r, "F_0"] \arrow[rr, bend right, "F"'] & \mathcal{C}_0 \arrow[r] & \mathcal{C}, \end{tikzcd}$$

any colimiting cocone $\bar{F}\colon K^{\triangleright} \to \mathcal{C}$ factors through $\mathcal{C}_0$, and in addition for any cocone $E\colon K^{\triangleright} \to \mathcal{C}_0$ extending F_0, the map $\bar{F}(\triangleright) \to E(\triangleright)$ (induced by the universal property of $\bar{F}$ being a colimit in $\mathcal{C}$) is already in $\mathcal{C}_0$.

Proof. The point here is that for a subcategory inclusion of ∞-categories $\mathcal{C}_0 \to \mathcal{C}$, the induced map

$$\mathrm{Map}_{\mathcal{C}_0}(c,d) \to \mathrm{Map}_{\mathcal{C}}(c,d)$$

is an inclusion of components, for any two objects $c, d \in \mathcal{C}_0$. This lets us confirm that the factorization of $\bar{F}$ through $\mathcal{C}_0$ is a colimiting cocone in $\mathcal{C}_0$, so uniqueness of colimits gives the result.

Indeed, we can directly compare the components hit by the horizontal arrows in

$$\begin{tikzcd} \mathrm{Map}_{\mathcal{C}_0}(\bar{F}(\triangleright), c) \arrow[r, hook] & \mathrm{Map}_{\mathcal{C}}(\bar{F}(\triangleright), c) \arrow[d, "\sim"] \\ \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C}_0)}(F_0, c) \arrow[r, hook] & \mathrm{Map}_{\mathrm{Fun}(K,\mathcal{C})}(F, c) \end{tikzcd}$$

to deduce the desired natural equivalence of the terms on the left. □

Now we are in shape to verify Theorem 11.4, by verifying the conditions of the above lemma. Recall that $\mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$ is the ∞-category of presentable stable ∞-categories and *right adjoint* functors, and we have a canonical equivalence $(\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$ which is the identity on objects and sends any functor to its right adjoint, see [Lur09, Cor. 5.5.3.4].

Proof of 11.4. We consider a diagram

$$\begin{tikzcd} K \arrow[r] \arrow[rr, bend right, "A"'] & \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega} \arrow[r] & \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}} \end{tikzcd}$$

and check the conditions of Lemma 11.7. By the above, we can always form the diagram of right adjoints

$$K^{\mathrm{op}} \xrightarrow{\overleftarrow{A}} \mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}},$$

and any limiting cone ${}^{\triangleleft}(K^{\mathrm{op}}) \to \mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$ of $\overleftarrow{A}$ yields a colimiting cocone for A by passing back to left adjoints.

Limits in Pr_{R} (and thus in $\mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$) exist and are formed in the ambient (huge) ∞-category of large ∞-categories $\widehat{\mathrm{Cat}_\infty}$, by [Lur09, 5.5.3.18]. In our situation of a diagram of compactly generated categories and compact-preserving left adjoints, all maps in the diagram of right adjoints $\overleftarrow{A}$ now preserve colimits by Lemma 11.6 (ii) and (iii), so it is simultaneously a diagram $K^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$.

But limits in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ are also formed underlying by [Lur09, Cor. 5.5.3.13], so the limit of $\overleftarrow{A}$ taken in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ and in $\mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$ agree, and thus all the natural projection maps from the limit are simultaneously left and right adjoints! Thus by Lemma 11.6 (i), their left adjoints preserve compacts, and these are exactly the natural maps to the colimit in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$.

To factor the colimiting cocone formed in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ through $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex},\omega}$, it only remains to see that $A_\infty := \operatorname{colim} A$ is compactly generated. For this, consider A_c the full subcategory of A_∞ generated under filtered colimits by the essential images of the compact objects in the $\{A_k\}_{k\in K}$. These are all compact since we above observed that the natural maps $A_k \to A_\infty$ preserve compacts. We shall now prove that $A_c \hookrightarrow A_\infty$ is an equivalence, where the former is clearly compactly generated.

As the A_k are compactly generated, the maps $A_k^\omega \to A_c$ extend uniquely to filtered-colimit-preserving functors $A_k \to A_c$, and these (again, by comparing on compacts) factor the natural maps $A_k \to A_\infty$, as expressed in the square

$$\begin{tikzcd} A_k \arrow[r] \arrow[dr, dashed] & A_\infty \\ A_k^\omega \arrow[u, hook] \arrow[r] & A_c. \arrow[u, hook] \end{tikzcd}$$

Thus, we obtain a a factorization of the colimiting cocone $A \to A_\infty$ through A_c. Note that there is no homotopy coherence to worry about, as A_c is defined to be a full subcategory of A_∞, so factoring each of the natural maps suffices.

This factorization of the universal cocone and the universal property of the colimit give a retraction

$$\begin{tikzcd} A_\infty \arrow[r, dashed, "r"] \arrow[rr, bend right, "\mathrm{id}"'] & A_c \arrow[r, hook] & A_\infty. \end{tikzcd}$$

The composition $A_c \hookrightarrow A_\infty \xrightarrow{r} A_c$ is equivalent to the identity since postcomposing with the inclusion $A_c \hookrightarrow A_\infty$ yields that very inclusion, and fully faithful inclusions are monomorphisms. Thus $A_c \simeq A_\infty$, so A_∞ is compactly generated.

The final condition to verify is that for a cocone E under A, the induced map $A_\infty \to E(\triangleright)$ preserves compacts. But this is again clear by 11.6 (i), since its right adjoint preserves filtered colimits, since said right adjoint is the map induced by the limiting cone of right adjoints, which is simultaneously a limiting cone in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$. This concludes the proof that

$$\mathrm{Ind}\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$$

commutes with colimits. □

Thus we can compute any colimit in $\mathrm{Cat}_{\mathrm{perf}}$ by pushing forward along the Ind-functor, computing the colimit in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$, and then passing back to compact objects, as expressed by the following chain of equivalences.

$$\begin{aligned} \operatorname*{colim}_K A_i &\simeq (\mathrm{Ind} \operatorname*{colim}_K A_i)^\omega \\ &\simeq (\operatorname*{colim}_K \mathrm{Ind}\, A_i)^\omega. \end{aligned}$$

General colimits in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ are still difficult to compute, but those indexed over spaces are actually computed as *limits* in $\widehat{\mathrm{Cat}_\infty}$, as we explain now.

In case of a diagram $F\colon K \to \mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$ we may again form the opposite diagram $\overleftarrow{F}\colon K^{\mathrm{op}} \to \mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$ and compute its limit to compute the colimit of F. Limits in Pr_{R} (and thus in $\mathrm{Pr}_{\mathrm{R}}^{\mathrm{ex}}$) are formed underlying, so if $p\colon X \to K$ is a cartesian fibration classified by F^{op}, the ∞-category of its *cartesian sections* is a model for $\lim F^{\mathrm{op}}$, i.e. $\operatorname{colim} F$.

We know that $p\colon X \to K$ is automatically also a cocartesian fibration, classified by the diagram of left adjoints to $\overleftarrow{F}$ - that is, (up to equivalence) exactly F. Using the fact that the inclusion $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}} \to \widehat{\mathrm{Cat}_\infty}$ also preserves limits, by [Lur09, Cor. 5.5.3.13]), we see the limit of F to be the ∞-category of *co*cartesian sections of p. If the diagram category K is an ∞-groupoid, being cartesian or cocartesian is automatic for any section, so indeed we have canonical identifications (in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$)

$$\operatorname*{colim}_K F \simeq \lim_K F,$$

of which $\mathcal{C}_{hG} \simeq \mathcal{C}^{hG}$ is of course a special case, for any presentable ∞-category with G-action. We thus arrive at the description of homotopy orbits in $\mathrm{Cat}_{\mathrm{perf}}$ promised above: They identify with the category of compact objects in the homotopy fixed points of $\operatorname{Ind}\mathcal{C}$, taken in $\mathrm{Pr}_{\mathrm{L}}^{\mathrm{ex}}$:

$$\mathcal{C}_{hG} \simeq ((\operatorname{Ind}\mathcal{C})^{hG})^{\omega}.$$

Applying the Ind-functor to $\mathsf{Perf}R$ of course yields $\mathsf{Mod}(R)$, so by the above description we are looking to identify the ∞-category of compact objects in $\mathsf{Mod}(R)^{hG}$, homotopy fixed points taken with respect to the trivial action. The trivial action classifies the fibration $p_2\colon \mathsf{Mod}(R) \times \mathrm{B}G \to \mathrm{B}G$ so we see its sections to be

$$\mathrm{Fun}_{/\mathrm{B}G}(\mathrm{B}G, \mathsf{Mod}(R) \times \mathrm{B}G) \simeq \mathrm{Fun}(\mathrm{B}G, \mathsf{Mod}(R))$$

Thus we may conclude $\mathsf{Perf}(R)_{hG} \simeq \mathsf{Perf}(R[G])$ upon confirming

$$\mathrm{Fun}(\mathrm{B}G, \mathsf{Mod}(R)) \simeq \mathsf{Mod}(R[G]).$$

The left hand side is equivalent to $\mathrm{Fun}(\mathrm{B}G, \mathsf{Sp}) \otimes \mathsf{Mod}(R)$, since for any presentable ∞-categories $\mathcal{C}, \mathcal{D}$ the equivalence

$$\mathrm{Fun}(K, \mathcal{C} \otimes \mathcal{D}) \simeq \mathrm{Fun}(K, \mathcal{C}) \otimes \mathcal{D}$$

holds, by identifying both sides with the same full subcategory of $\mathrm{Fun}(\mathcal{D}^{\mathrm{op}} \times K, \mathcal{C})$.

The right hand side is equivalent to $\mathsf{Mod}(\mathbb{S}[G]) \otimes \mathsf{Mod}(R)$, since for any $\mathbb{E}_1$-ring spectra R and S, the equivalence

$$\mathsf{Mod}(R) \otimes \mathsf{Mod}(S) \simeq \mathsf{Mod}(R \otimes S)$$

holds as a consequence of [Lur17, Thm. 4.8.5.16].

This reduces us to the case $R \simeq \mathbb{S}$, i.e. the claim

$$\mathrm{Fun}(\mathrm{B}G, \mathsf{Sp}) \simeq \mathsf{Mod}(\mathbb{S}[G]).$$

The crucial theorem in this identification is the *Schwede-Shipley recognition principle,* which we cite in its ∞-categorical form from [Lur17, Theorem 7.1.2.1], see also Remark 8.9.

Theorem 11.8 (Schwede-Shipley)**.** *Let $\mathcal{C}$ be a stable, presentable ∞-category.*

Assume there is a compact object $c \in \mathcal{C}$ which generates $\mathcal{C}$ in the sense that $\mathrm{map}_{\mathcal{C}}(c, d) \simeq 0$ implies $d \simeq 0$. Then $\mathcal{C}$ is equivalent to the ∞-category of right *modules over the endomorphism ring spectrum of c.*

We confirm that $\mathbb{S}[G] \in \mathsf{Sp}^{\mathrm{B}G}$ satisfies the assumptions of Theorem 11.8. Note that it is the image of the compact generator $\mathbb{S} \in \mathsf{Sp}$ under the 'free/forgetful' adjunction

$$\mathsf{Sp} \underset{u}{\overset{-\otimes G_+}{\rightleftarrows}} \mathsf{Sp}^{\mathrm{B}G}.$$

We refer to Appendix B.4 for a neat justification of this description of the left adjoint to the forgetful functor $\mathsf{Sp}^{\mathrm{B}G} \to \mathsf{Sp}$.

Compactness: Given a filtered colimit $\operatorname{colim} X_i$, we have

$$\begin{aligned}\operatorname{map}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G], \operatorname{colim} X_i) &\simeq \operatorname{map}_{\mathsf{Sp}}(\mathbb{S}, \operatorname{colim} uX_i)\\ &\simeq \operatorname{colim} \operatorname{map}_{\mathsf{Sp}}(\mathbb{S}, uX_i) \simeq \operatorname{colim} \operatorname{map}_{\mathsf{Sp}}(\mathbb{S}[G], X_i)\end{aligned}$$

where we also used that the right adjoint u commutes with colimits, since it itself admits a right adjoint.

Generation:

If $\operatorname{map}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G], X) \simeq 0$, then so is $\operatorname{map}_{\mathsf{Sp}}(\mathbb{S}, uX) \simeq uX$, so $X \simeq 0$.

Thus one is done if one believes the equivalence of ring spectra

$$\operatorname{end}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G]) \simeq \mathbb{S}[G]$$

and is not particularly careful about the distinction between left and right modules. For the skeptical, we offer a little more explanation.

We want to avoid wading too deeply into the higher algebra of endomorphism objects as laid out in [Lur17, Sect. 4.7.1] and take a pragmatic approach. The pertinent facts are:

- For any ∞-category $\mathcal{C}$ and any object $x \in \mathcal{C}$, the mapping space $\operatorname{Map}_{\mathcal{C}}(x, x)$ can be endowed with an $\mathbb{E}_1$-structure corresponding to the composition of maps. We write $\operatorname{End}_{\mathcal{C}}(x) \in \mathcal{S}_{\mathbb{E}_1}$ for this object.
- If in addition $\mathcal{C}$ is stable, the mapping spectrum $\operatorname{map}_{\mathcal{C}}(x, x)$ promotes compatibly to an endomorphism ring spectrum $\operatorname{end}_{\mathcal{C}}(x) \in \mathsf{Sp}_{\mathbb{E}_1}$. Compatible here means $\Omega^\infty \operatorname{end}_{\mathcal{C}}(x) \simeq \operatorname{End}_{\mathcal{C}}(x, x)$.
- Every $\mathbb{E}_1$-object X has an opposite X^{op}, obtained by reversing the order of multiplication. (Note that Lurie writes X^{rev} for this object.)
- Right modules over a ring spectrum $R \in \mathsf{Sp}_{\mathbb{E}_1}$ correspond to left modules over R^{op}.
- For group ring spectra $R[G]$, we have $R[G]^{\mathrm{op}} \simeq R^{\mathrm{op}}[G^{\mathrm{op}}]$, so in particular $\mathbb{S}[G]^{\mathrm{op}} \simeq \mathbb{S}[G^{\mathrm{op}}]$.

Thus (with our convention of $\mathsf{Mod}(-)$ denoting *left* modules), Theorem 11.8 provides an equivalence $\operatorname{Fun}(\mathrm{B}G, \mathsf{Sp}) \simeq \mathsf{Mod}(\operatorname{end}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G])^{\mathrm{op}})$, so it remains to see an equivalence

$$\operatorname{end}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G])^{\mathrm{op}} \simeq \mathbb{S}[G].$$

By construction of the category $\mathrm{B}G$, we have $\operatorname{End}_{\mathrm{B}G}(*) \simeq G$ (as $\mathbb{E}_1$-objects in spaces), and the corepresented functor $\operatorname{Map}_{\mathrm{B}G}(*, -)\colon \mathrm{B}G \to \mathcal{S}$ models G as a space with left G-action. The (Co-)Yoneda equivalence

$$\operatorname{Map}_{\mathcal{S}^{\mathrm{B}G}}(\operatorname{Map}_{\mathrm{B}G}(*, -), \operatorname{Map}_{\mathrm{B}G}(*, -)) \;\simeq\; \operatorname{Map}_{\mathrm{B}G}(*, *)$$

reverses order of composition, so promotes to an equivalence of $\mathbb{E}_1$-spaces

$$\operatorname{End}_{\mathcal{S}^{\mathrm{B}G}}(\operatorname{Map}_{\mathrm{B}G}(*, -)) \simeq \operatorname{End}_{\mathrm{B}G}(*)^{\mathrm{op}}.$$

Thus we see the chain of equivalences of $\mathbb{E}_1$-spectra

$$\begin{aligned}\operatorname{end}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[G]) &\simeq \operatorname{end}_{\mathsf{Sp}^{\mathrm{B}G}}(\mathbb{S}[\operatorname{Map}_{\mathrm{B}G}(*, -)])\\ &\simeq \mathbb{S}[\operatorname{End}_{\mathrm{B}G}(*)^{\mathrm{op}}] \simeq \mathbb{S}[G^{\mathrm{op}}]\\ &\simeq \mathbb{S}[G]^{\mathrm{op}}\end{aligned}$$

by the spectral enhancement of the Yoneda lemma of Proposition B.4.

This concludes the proof of Theorem 11.3.

12. 'Free' and 'Borel' Mackey functors

For what follows, fix a presentable and preadditive ∞-category $\mathcal{E}$, such as $\mathrm{Cat}_{\mathrm{perf}}$ or Sp. We want to specify two fully faithful functors

$$\mathcal{E}^{\mathrm{B}G} \to \mathrm{Mack}_G(\mathcal{E}),$$

whose essential images in Mack_G we will to refer to as the *free* and the *Borel* Mackey functors.

Of course, neither the existence nor the fully faithfulness of these adjoints in the ∞-categorical setting are new, see e.g. [NS18, Thm. II.2.7] or [CMNN20, Constr. 2.9 and 2.15], who arrive at similar results with different methods.

To this end, consider the 'underlying object with G-action'-functor of Definition 7.3

$$u\colon \mathrm{Mack}_G(\mathcal{E}) \to \mathcal{E}^{\mathrm{B}G}$$

given by restriction along

$$\mathrm{B}G \to \mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}.$$

It factorizes as

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathcal{E}) \to \mathrm{Fun}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathcal{E}) \to \mathcal{E}^{\mathrm{B}G},$$

where the first inclusion preserves all limits and colimits by Observation 7.2, and the second functor always does. Thus, by the adjoint functor theorem, u has both a left adjoint and a right adjoint (denoted $(-)_{\mathrm{free}}$ and $(-)_{\mathrm{Bor}}$ respectively) and it will be our goal to show that these are fully faithful. It will also become clear that for $\mathcal{E} = \mathsf{Sp}$, they recover the classical subcategories of *free G-spectra* (i.e. those with $X \otimes EG_+ \simeq X$) and of *Borel complete G-spectra* (i.e. those with $X \simeq F(EG_+, X)$).

The crucial insight is the following result of Saul Glasman, for which we provide an alternative proof. Let Free_G denote the category of finite free G-sets and equivariant maps.

Lemma 12.1. [Gla17, Thm. A.1]. *For any $\mathcal{E}$ presentable and preadditive, restriction along $\mathrm{B}G = <G/1>^{\mathrm{op}} \to \mathrm{Span}(\mathsf{Free}_G)^{\mathrm{op}}$ induces an equivalence*

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathsf{Free}_G)^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\sim} \mathrm{Fun}(\mathrm{B}G, \mathcal{E}).$$

In other words, $\mathrm{Span}(\mathsf{Free}_G)$ *is the free presentable preadditive ∞-category on* $\mathrm{B}G$.

Equipped with this result, the conclusion that the adjoints to

$$u\colon \mathrm{Mack}_G(\mathcal{E}) \to \mathcal{E}^{\mathrm{B}G}$$

are fully faithful is largely formal. Note that Free_G is disjunctive, and

$$\mathrm{Span}(\mathsf{Free}_G) \hookrightarrow \mathrm{Span}(\mathsf{Fin}_G)$$

is the inclusion of a full (since free G-sets only receive maps from free G-sets) subcategory closed under sums. The functor $u\colon \mathrm{Mack}_G(\mathcal{E}) \to \mathcal{E}^{\mathrm{B}G}$ thus factorizes as

$$\mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathsf{Fin}_G)^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\mathrm{res}^{\oplus}} \mathrm{Fun}^{\oplus}(\mathrm{Span}(\mathsf{Free}_G)^{\mathrm{op}}, \mathcal{E}) \xrightarrow{\sim} \mathcal{E}^{\mathrm{B}G}$$

and we are reduced to showing that the adjoints of $\mathrm{res}^{\oplus}$ are fully faithful. But these are given by Kan extending along a fully faithful inclusion, by Proposition 3.7, and hence themselves fully faithful..

As the proof of Lemma 12.1 in [Gla17, App. A] remains somewhat opaque to the author, let us offer an alternative. We heartily thank Maxime Ramzi for help with this argument.

Proof of Lemma 12.1. We recall the situation of Proposition 5.6: Writing $\mathcal{L}(G)$ for the full subcategory of $\mathrm{CMon}^{\mathrm{B}G}$ spanned by those free on a finite set (Definition 5.3), the functor

$$\begin{aligned} \mathrm{Fun}^{\times}(\mathcal{L}(G)^{\mathrm{op}}, \mathcal{E}) &\to \mathcal{E}^{\mathrm{B}G} \\ E &\mapsto E(\mathcal{F}_G(\mathbf{1})) \end{aligned}$$

is an equivalence. To prove Lemma 12.1, it thus suffices to exhibit an equivalence

$$\Psi\colon \mathcal{L}(G)^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Span}(\mathsf{Free}_G),$$

that sends $\mathcal{F}_G(\mathbf{1})$ to $G/1$, to ensure the proposed description of the equivalence $\mathrm{Fun}^{\times}(\mathrm{Span}(\mathsf{Free}_G)^{\mathrm{op}}, \mathcal{E}) \simeq \mathcal{E}^{\mathrm{B}G}$.

Since $\mathrm{Span}(\mathsf{Free}_G)$ is preadditive, we may use Proposition 5.6 (with $\mathcal{E} = \mathrm{Span}(\mathsf{Free}_G)$) to construct Ψ, i.e. by exhibiting $G/1$ as an object of $\mathrm{Span}(\mathsf{Free}_G)^{\mathrm{B}G}$.

Of course, the natural choice here is the inclusion

$$\mathsf{B}G = <G/1>^{op} \hookrightarrow \mathrm{Span}(\mathsf{Free}_G),$$

i.e. we consider $G/1$ with left action by the spans

$$\begin{tikzcd} & G/1 \arrow[dl, "\cdot g"'] \arrow[dr, "="] & \\ G/1 & & G/1. \end{tikzcd} \tag{12.2}$$

It remains to see that the functor Ψ is essentially surjective and fully faithful. By construction, Ψ preserves finite products. Since every object of $\mathcal{L}(G)^{\mathrm{op}}$ is of the form $\mathcal{F}_G(\boldsymbol{n}) \simeq \oplus_n \mathcal{F}_G(\mathbf{1})$ and every object of $\mathrm{Span}(\mathsf{Free}_G)$ is of the form $\oplus_n G/1$, essential surjectivity follows immediately from $E(\mathcal{F}_G(\mathbf{1})) = G/1$.

To check fully faithfulness, it is then sufficient to see that the map

$$\mathrm{Map}_{\mathcal{L}(G)^{\mathrm{op}}}(\mathcal{F}_G(\mathbf{1}), \mathcal{F}_G(\mathbf{1})) \xrightarrow{\Psi} \mathrm{Map}_{\mathrm{Span}(\mathsf{Free}_G)}(G/1, G/1) \tag{12.3}$$

is an equivalence. As both source and target of Ψ are preadditive categories and Ψ preserves products, this map naturally lifts to a map of commutative monoids, and we shall conclude by identifying both sides with the free monoid on the set G, $\mathcal{F}(G)$, such that generators are sent to generators.

Let us first consider the target of 12.3. By Proposition 6.2, it is given as the maximal subgroupoid contained in the overcategory $\mathsf{Free}_G{}_{/\{G/1, G/1\}}$. As Free_G is a 1-category, this space is modeled by the groupoid of isomorphisms of spans in the 1-categorical sense ([Lur21, Tag 0183]). The isomorphism of spans

$$\begin{tikzcd} & G/1 \arrow[dl, "\cdot y"'] \arrow[dr, "\cdot x"] \arrow[dd, "\cdot x"] & \\ G/1 & & G/1 \\ & G/1 \arrow[ul, "\cdot x^{-1} y"] \arrow[ur, "="'] & \end{tikzcd}$$

shows that the set of spans

$$\begin{tikzcd} & \coprod_{i \in \boldsymbol{n}} G/1 \arrow[dl, "(\cdot g_i)_i"'] \arrow[dr, "(\mathrm{id})_i"] & \\ G/1 & & G/1 \end{tikzcd} \tag{12.4}$$

are a set of representants for the isomorphism classes of $\mathsf{Free}_{G/\{G/1,G/1\}}$. Note that the monoid structure is given by forming the coproduct.

Recall that the commutative monoid $\mathcal{F}(G)$ is modeled by the symmetric monoidal 1-groupoid $\prod_G \mathsf{Fin}^{\sim}$. This follows from Graeme Segal's classical result that $\mathsf{Fin}^{\sim}$ is the free commutative monoid on a point [Seg74, Prop. 3.5], since $\mathcal{F}\colon \mathcal{S} \to \mathrm{CMon}$ preserves coproducts, and coproducts in CMon are given by products, which are formed underlying.

Now we give an explicit equivalence of symmetric monoidal 1-categories

$$(\mathsf{Free}_{G/\{G/1,G/1\}})^{\sim} \xrightarrow{\ \sim\ } \prod_G \mathsf{Fin}^{\sim},$$

which in turn specifies an equivalence of the corresponding objects in $\mathrm{CMon}(\mathcal{S})$. We work on skeletons for simplicity.

On objects, we send the span 12.4, denoted s, to the G-tuple $(\boldsymbol{k}_g)_{g\in G}$ with g'th component the finite set with cardinality the number of occurrences of g in $(g_i)_i$. An automorphism of s may only reorder components, to ensure commutativity of the right triangle, and can send the i'th component to the j'th if and only if $g_i = g_j$, to ensure commutativity of the left triangle. Thus, we have an obviously functorial bijection

$$\mathrm{Aut}_{\mathsf{Free}_{G/\{G/1,G/1\}}}(s) \longrightarrow \prod_G \mathrm{Sym}_{\boldsymbol{k}_g} = \mathrm{Aut}_{\prod_G \mathsf{Fin}}((\boldsymbol{k}_g)_{g\in G}).$$

Note that the generators of the monoid $\mathcal{F}(G) \simeq \prod_G \mathsf{Fin}^{\sim}$ are given by the G-tuples of finite sets of the form $(\mathbf{0},\ldots,\mathbf{1},\ldots,\mathbf{0})$, which under the given equivalence correspond to the action maps 12.2.

On the source, the result follows formally, from pushing around adjunctions. $\mathrm{Map}_{\mathcal{L}(G)^{\mathrm{op}}}(\mathcal{F}_G(\mathbf{1}),\mathcal{F}_G(\mathbf{1}))$ is (monoidally) equivalent to the underlying commutative monoid (without G-action) of $\mathcal{F}_G(\mathbf{1})$, which in turn we identify with $\mathcal{F}(G)$ since the left adjoint to the forgetful functor $\mathrm{CMon}^{\mathrm{B}G} \to \mathrm{CMon}$ is given by $A \mapsto \oplus_G A$ and $\mathcal{F}$ commutes with coproducts.

Under the equivalence $u\mathcal{F}(G) \simeq \mathrm{Map}_{\mathcal{L}(G)^{\mathrm{op}}}(\mathcal{F}_G(\mathbf{1}),\mathcal{F}_G(\mathbf{1}))$, the g'th generator, i.e. the map $\{g\} \hookrightarrow G \to u\mathcal{F}(G)$ corresponds to $\mathcal{F}(\mathbf{1}) \to \oplus_G \mathcal{F}(\mathbf{1})$, inclusion of the g'th summand, corresponds to the map $\mathcal{F}_G(\mathbf{1}) \xrightarrow{\cdot g} \mathcal{F}_G(\mathbf{1})$. By Remark 5.10, the functor E sends this map to the action-by-g-map on $G/1$, which we above identified with the g'th generator on the target. Thus, the map

$$\mathrm{Map}_{\mathcal{L}(G)^{\mathrm{op}}}(\mathcal{F}_G(\mathbf{1}),\mathcal{F}_G(\mathbf{1})) \xrightarrow{E} \mathrm{Map}_{\mathrm{Span}(\mathsf{Free}_G)}(G/1,G/1)$$

is an equivalence. $\square$

Traditionally, we might have defined a free G-spectrum as satisfying $X^H \simeq X_{hH}$ for all subgroups H, and a Borel complete G-spectrum as satisfying $X^H \simeq X^{hH}$. This characterization holds for Mackey functors, by another observation of Saul Glasman [Gla17, Lemma 2.28].

We say a full subcategory $\mathcal{C} \hookrightarrow \mathcal{D}$ is *upwardly closed* if the existence of a morphism $x \to c$ for $c \in \mathcal{C}$ implies $x \in \mathcal{C}$. The relevant example is of course $\mathsf{Free}_G \hookrightarrow \mathsf{Fin}_G$.

Proposition 12.5. *Let $\mathcal{C} \hookrightarrow \mathcal{D}$ be a fully faithful inclusion of an upwardly closed subcategory which is closed under pullbacks. Consider a functor $F\colon \mathrm{Span}(\mathcal{C}) \to \mathcal{E}$ and form its left Kan extension $\mathfrak{l}F$ along $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$. Then its restriction* *to $\mathcal{D}$ is itself left Kan extended from $\mathcal{C}$.*

$$
\begin{tikzcd}
\mathcal{C} \arrow[r] \arrow[d] & \mathrm{Span}(\mathcal{C}) \arrow[r, "F"] \arrow[d] & \mathcal{E} \\
\mathcal{D} \arrow[r] & \mathrm{Span}(\mathcal{D}) \arrow[ur, dashed, bend right, "\iota F"'] &
\end{tikzcd}
$$

Let us summarize the rewards of Proposition 12.5 before proving it.

Remark 12.6. Starting with an object $X \in \mathcal{E}^{\mathrm{B}G}$, we take F to be the sum-preserving functor $\mathrm{Span}(\mathsf{Free}_G) \to \mathcal{E}$ supplied by Lemma 12.1. Its left Kan extension to $\mathrm{Span}(\mathsf{Fin}_G)$ is then the Mackey functor X_{free}. The above theorem says that the restriction of X_{free} to Or_G is again Kan extended from $<G/1>$, i.e. its values are the homotopy orbits of X (in light of Propositions 3.8 and 3.9). Put differently, this procedure promotes the canonical diagram

$$\mathsf{Or}_G \to \mathcal{E}, \quad G/H \mapsto X_{hH}$$

induced by X to a (free) Mackey functor.

Proof of Theorem B. By the above, there is a free Mackey functor $\mathsf{Perf}R_{\mathrm{free}}$ valued in $\mathrm{Cat}_{\mathrm{perf}}$ obtained from $\mathsf{Perf}R \in \mathrm{Cat}_{\mathrm{perf}}^{\mathrm{B}G}$ by left Kan extension. Its underlying functor

$$\vec{u}(\mathsf{Perf}R_{\mathrm{free}})\colon \mathsf{Or}_G \to \mathrm{Cat}_{\mathrm{perf}}$$

is also left Kan extended, i.e. takes values

$$G/H \mapsto (\mathrm{B}H \otimes \mathsf{Perf}R) \simeq \mathsf{Perf}(R[H]).$$

For any additive functor $E\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ (such as K-theory), the composition

$$\mathrm{Span}_G^{\mathrm{op}} \xrightarrow{\mathsf{Perf}R_{\mathrm{free}}} \mathrm{Cat}_{\mathrm{perf}} \xrightarrow{E} \mathsf{Sp}$$

is then a spectral Mackey functor, i.e. a genuine G-spectrum with prescribed underlying Or_G-spectrum

$$G/H \mapsto E(\mathsf{Perf}R[H]).$$

□

We also note that the whole situation dualizes without resistance, proving Theorem B'.

Proof of Theorem B'. By the dual version of Proposition 12.5 above, there is a Borel Mackey functor $\mathsf{Perf}R_{\mathrm{Bor}}$ valued in $\mathrm{Cat}_{\mathrm{perf}}$ obtained from $\mathsf{Perf}R \in \mathrm{Cat}_{\mathrm{perf}}^{\mathrm{B}G}$ by right Kan extension. Its underlying functor

$$\overleftarrow{u}(\mathsf{Perf}R_{\mathrm{Bor}})\colon \mathsf{Or}_G^{\mathrm{op}} \to \mathrm{Cat}_{\mathrm{perf}}$$

is also right Kan extended, i.e. takes values

$$G/H \mapsto (\lim_{\mathrm{B}H} \mathsf{Perf}R) \simeq \mathrm{Fun}(\mathrm{B}H, \mathsf{Perf}(R)).$$

For any additive functor $E\colon \mathrm{Cat}_{\mathrm{perf}} \to \mathsf{Sp}$ (such as K-theory), the composition

$$\mathrm{Span}_G^{\mathrm{op}} \xrightarrow{\mathsf{Perf}R_{\mathrm{Bor}}} \mathrm{Cat}_{\mathrm{perf}} \xrightarrow{E} \mathsf{Sp}$$

is then a spectral Mackey functor, i.e. a genuine G-spectrum with prescribed underlying Or_G-spectrum

$$G/H \mapsto E(\mathrm{Fun}(\mathrm{B}H, \mathsf{Perf}(R)).$$

□

Proof of Proposition 12.5. Let us first remark that if $\mathcal{C}$ is upwardly closed in $\mathcal{D}$, the functor $\mathrm{Span}(\mathcal{C}) \to \mathrm{Span}(\mathcal{D})$ is fully faithful. In Lemma 6.7, we showed that the induced functor of overcategories $\mathcal{D}_{/d} \to \mathrm{Span}(\mathcal{D})_{/d}$ is also fully faithful, and so are the vertical functors in the square

$$\begin{array}{ccc} \mathcal{C}_{/d} & \longrightarrow & \mathrm{Span}(\mathcal{C})_{/d} \\ \big\downarrow & & \big\downarrow \\ \mathcal{D}_{/d} & \longrightarrow & \mathrm{Span}(\mathcal{D})_{/d}. \end{array}$$

The left adjoint to the bottom horizontal map we constructed in Lemma 6.7 clearly restricts to a functor $\mathrm{Span}(\mathcal{C})_{/d} \to \mathcal{C}_{/d}$, so the top horizontal map also has a left adjoint, and is thus final by [Lur21, Tag 02P3].[9]

Now the left Kan extension $\mathfrak{l}F\colon \mathrm{Span}(\mathcal{D}) \to \mathcal{E}$ is characterized by the property that the natural contraction morphism

$$(\mathrm{Span}(\mathcal{C})_{/d})^{\triangleright} \to \mathcal{E}$$

is a colimiting cocone, and thus (by finality of $\mathcal{C}_{/d} \to \mathrm{Span}(\mathcal{C})_{/d}$) so is

$$(\mathcal{C}_{/d})^{\triangleright} \to (\mathrm{Span}(\mathcal{C})_{/d})^{\triangleright} \to \mathcal{E},$$

which is exactly the condition that identifies the composition $\mathcal{D} \to \mathrm{Span}(\mathcal{D}) \to \mathcal{E}$ to be left Kan extended from $\mathcal{C}$. $\square$

[9]Note that Lurie refers to the property that we call final as right cofinal

Appendix A. Identification of norm maps

We explain why for X a genuine G-spectrum, the composition

$$\Phi_X \colon (\mathsf{E}G_+ \otimes X)^G \to X^G \to F(\mathsf{E}G_+, X)^G$$

identifies with the norm map

$$\mathfrak{nm}_X \colon X_{hG} \to X^{hG}.$$

Crucially, the norm map $\mathfrak{nm}$ has a universal property: Theorem B of [Kle02] characterizes it as the terminal left approximation of the functor

$$(-)^{hG} \colon \mathsf{Sp}^{\mathrm{B}G} \to \mathsf{Sp}$$

by a colimit-preserving functor. Identifying $(\mathsf{E}G_+ \otimes X)^G$ with X_{hG} and $F(\mathsf{E}G_+, X)^G$ with X^{hG} as we did in the proof of Proposition 10.1, we produce a canonical dashed comparison map

$$\begin{tikzcd}
(\mathsf{E}G_+ \otimes X)^G \arrow[r] \arrow[d, dashed, "\alpha"] & X^G \arrow[r] & F(\mathsf{E}G_+, X)^G \simeq X^{hG} \\
X_{hG} \arrow[urr, bend right, "\mathfrak{nm}"'] & &
\end{tikzcd}$$

and we would like to see it an equivalence.

First we note that, as in the proof of Proposition 10.1, source and target of Φ_X depend only on the underlying spectrum with G-action of X, so replacing X along $(uX)_{\mathrm{free}} \to X$ gives equivalent maps Φ. Thus we are reduced to considering free X. Recall the free-forgetful adjunction from Chapter 12, which identifies $\mathsf{Sp}^{\mathrm{B}G}$ with the full subcategory $(\mathsf{Sp}_G)_{\mathrm{free}}$ of free G-spectra:

$$\begin{tikzcd}
\mathsf{Sp}^{\mathrm{B}G} \arrow[r, "\sim"] \arrow[rr, bend left, "(-)_{\mathrm{free}}"] & (\mathsf{Sp}_G)_{\mathrm{free}}. \arrow[r, hook] & \mathsf{Sp}_G \arrow[ll, bend left, "u"]
\end{tikzcd}$$

Since $\mathbb{S}[G]$ is a generator of $\mathsf{Sp}^{\mathrm{B}G}$ in the sense of Remark 8.9, $\mathbb{S}[G]_{\mathrm{free}}$ generates the (colimit-closed) subcategory $(\mathsf{Sp}_G)_{\mathrm{free}}$, and it thus suffices to see that α is an equivalence on the object

$$X = \mathbb{S}[G]_{\mathrm{free}}.$$

First note that $\mathbb{S}[G]_{\mathrm{free}}$ coincides with $\mathbb{S}_G[G/1]$: In the commutative diagram of 'free' left adjoints

$$\begin{tikzcd}
\mathcal{S} \arrow[r] \arrow[d] & \mathcal{S}^{\mathrm{B}G} \arrow[r] \arrow[d] & \mathcal{S}_G \arrow[d, "\mathbb{S}_G"] \\
\mathsf{Sp} \arrow[r] & \mathsf{Sp}^{\mathrm{B}G} \arrow[r] & \mathsf{Sp}_G,
\end{tikzcd}$$

$\mathbb{S}[G]_{\mathrm{free}}$ is the image of $*$ along the lower horizontal, while $\mathbb{S}_G[G/1]$ is the image of $*$ along the upper horizontal, see also App. B.4.

Now the result will follow if we see that $\Phi_{\mathbb{S}_G[G/1]}$ and $\mathfrak{nm}_{\mathbb{S}[G]}$ are equivalences. The latter statement is [Lur17, Ex. 6.1.6.26].

As summarized in Remark 12.6, the map $(\mathsf{E}G_+ \otimes X)^G \to X^G$ is an equivalence for free X. By the techniques of Example 8.6, for $X = \mathbb{S}_G[G/1]$ the map

$$X^G \to F(\mathsf{E}G_+, X)^G \tag{A.1}$$

identifies with

$$\begin{tikzcd}
\mathbb{S}_G[G/1](G/G) \arrow[r] \arrow[d, "\sim"] & \lim_{BG} \mathbb{S}_G[G/1](-) \arrow[d, "\sim"] \\
(\mathsf{Fin}_{G/\{G/1\}})^{\sim,gp} \arrow[r] & \lim_{BG} (\mathsf{Fin}_{G/\{G/1,G/1\}})^{\sim,gp},
\end{tikzcd}$$

where the functor $\mathsf{Fin}_{G/\{G/1\}} \to \mathsf{Fin}_{G/\{G/1,G/1\}}$ is given by

$$\big(U \xrightarrow{f} G/1\big) \mapsto \big(G/1 \times U \xrightarrow{(\mathrm{id},f)} G/1 \times G/1\big),$$

as can be seen by computing the composition of spans

$$\begin{tikzcd}
 & G/1 \arrow[dl] \arrow[dr] & & U \arrow[dl] \arrow[dr] & \\
G/1 & & G/G & & G/1.
\end{tikzcd}$$

We have identified $\mathsf{Fin}_{G/\{G/1\}}$ with Fin at the very end of Example 8.6, and we have identified $\mathsf{Fin}_{G/\{G/1,G/1\}}$ with $\prod_G \mathsf{Fin}$ in the proof of Lemma 12.1. Under these identifications, we calculate the map in question to be given by the diagonal

$$\begin{aligned}
\mathsf{Fin} &\xrightarrow{\triangle} \prod_G \mathsf{Fin} \\
S &\longmapsto (S)_{g\in G}.
\end{aligned}$$

With $(\mathsf{Fin})^{\sim,gp} \simeq \mathbb{S}$ and $(\prod_G \mathsf{Fin})^{\sim,gp} \simeq \bigoplus_G \mathbb{S}$, we have finally identified the map A.1 with

$$\mathbb{S} \xrightarrow{\triangle} (\bigoplus_G \mathbb{S})^{hG}$$

which is an equivalence, and we conclude.

Appendix B. ∞-categorical generalities

This appendix is a collection of observations on ∞-categories that were deemed to interrupt the flow of exposition too much. Everything here is likely written down somewhere else, or at least well-known to experts. We will be very concrete and detailed sometimes, but also not shy away from blackboxing big parts of the theory.

B.1. **Overcategories & undercategories.** An important tool in the theory of ∞-categories is the formation of *overcategories* $\mathcal{C}_{/p}$ for $\mathcal{C}$ an ∞-category and $p\colon K \to \mathcal{C}$ a diagram in $\mathcal{C}$, as well as the dual notion of *undercategories* $\mathcal{C}_{p/}$. We define these as explicit simplicial sets as follows.

Recall the *join* $K \star S$ of two simplicial sets K, S of [Lur09, Def. 1.2.8.1]. It is a functor

$$K \star -\colon \mathsf{sSet} \to \mathsf{sSet}_{K/}$$

and likewise in the other variable, characterized by admitting canonical isomorphisms $\Delta^{i-1} \star \Delta^{j-1} \simeq \Delta^{i+j-1}$ and preserving colimits in either variable. Geometrically, we picture it as constructed from $K \coprod S$ by adjoining an edge $k \to s$ for every pair of vertices $(k, s) \in K_0 \times S_0$, a corresponding 2-simplex for every pair $(e, s) \in K_1 \times S_0$ as well as $(k, e) \in K_0 \times S_1$ and so forth.

Now, given a map of simplicial sets $p\colon K \to \mathcal{C}$, we define the simplicial set $\mathcal{C}_{/p}$ by the requirement

$$\mathrm{Hom}(X, \mathcal{C}_{/p}) = \mathrm{Hom}_{K/}(X \star K, \mathcal{C}),$$

and dually the simplicial set $\mathcal{C}_{p/}$ by the requirement

$$\mathrm{Hom}(X, \mathcal{C}_{p/}) = \mathrm{Hom}_{K/}(K \star X, \mathcal{C}).$$

Specializing to $X = \Delta^n$, this concretely describes the set n-simplices of $\mathcal{C}_{/p}$ as the set of maps $\Delta^n \star K \to \mathcal{C}$ restricting to p on K. Further specializing to $K = \Delta^0$ (and identifying p with its value $c \in \mathcal{C}$), we note that the n-simplices of $\mathcal{C}_{/c}$ are exactly the $(n+1)$-simplices of $\mathcal{C}$ with final vertex c.

We have obvious projection maps $\mathcal{C}_{/p} \to \mathcal{C}$ and $\mathcal{C}_{p/} \to \mathcal{C}$, given on n-simplices by restricting along $\Delta^n \hookrightarrow \Delta^n \star K$ and $\Delta^n \hookrightarrow K \star \Delta^n$ respectively.

It is a fundamental theorem of André Joyal [Joy02] that if $\mathcal{C}$ is an ∞-category, these projection maps are right and left fibrations respectively, and as a consequence, $\mathcal{C}_{/p}$ and $\mathcal{C}_{p/}$ are ∞-categories. Specializing to $K = \Delta^0$ again, these straighten to functors

$$\mathcal{C}^{\mathrm{op}} \to \mathcal{S} \qquad \text{and} \qquad \mathcal{C} \to \mathcal{S}$$

which give functorial models for the mapping spaces $\mathrm{map}_{\mathcal{C}}(-, x)$ and $\mathrm{map}_{\mathcal{C}}(x, -)$ respectively. In particular, mapping spaces in an ∞-category $\mathcal{C}$ sit in a natural fiber sequence

$$\begin{array}{ccccc} \mathrm{map}_{\mathcal{C}}(x, y) & \longrightarrow & \mathcal{C}_{/y} & \longrightarrow & \mathcal{C}, \\ & & & & \\ (x \to y) & \longmapsto & (x \to y) & \longmapsto & x, \end{array}$$

of which we will make use often.

Remark B.1. Here, the fiber taken in the 1-category sSet models the fiber taken in the ∞-category Cat_∞, since the relevant map is a right fibration and thus a categorical fibration. We of course prefer to view this as a fiber sequence in Cat_∞, with its cone point happily contained in the full subcategory $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$.

As an application, the mapping spaces of $\mathcal{C}_{/c}$ may be described as fibers of mapping spaces of $\mathcal{C}$ by the exact formula one might guess from 1-category theory:

Proposition B.2. *Given maps $f\colon x \to c$ and $g\colon y \to c$ in $\mathcal{C}$, consider the map*

$$\mathrm{Map}_{\mathcal{C}}(x,y) \xrightarrow{g\circ -} Map_{\mathcal{C}}(x,c).$$

Then $\mathrm{Map}_{\mathcal{C}_{/c}}(f,g)$ identifies with the fiber (in $\mathcal{S}$) of $\mathrm{Map}_{\mathcal{C}}(x,y) \to \mathrm{Map}_{\mathcal{C}}(x,c)$, taken at f.

Proof. In the following, all fiber constructions are considered in Cat_∞.

Above, we defined $\mathrm{Map}_{\mathcal{C}_{/c}}(f,g)$ as the fiber of $(\mathcal{C}_{/c})_{/g} \to \mathcal{C}_{/c}$ at f. We want to identify the domain of this map with $\mathcal{C}_{/y}$. Unraveling the definition, we see an honest isomorphism of simplicial sets

$$(\mathcal{C}_{/c})_{/g} \cong \mathcal{C}_{/g}.$$

Now the map $\mathcal{C}_{/g} \to \mathcal{C}_{/y}$ induced by $\Delta^0 \overset{0}{\hookrightarrow} \Delta^1$ is an equivalence of ∞-categories by [Lan21, Cor. 1.4.24(2)], so $\mathrm{Map}_{\mathcal{C}_{/c}}(f,g)$ is the fiber of $\mathcal{C}_{/y} \to \mathcal{C}_{/c}$ at f, with the map given by composing with g (see *loc.cit.* for a detailed construction of the map $\mathcal{C}_{/y} \to \mathcal{C}_{/c}$). Thus, starting with the diagram of the two lower horizontal fiber sequences, we take vertical fibers and see the square

$$\begin{tikzcd}
F \arrow[r] \arrow[d] & \mathrm{Map}_{\mathcal{C}_{/c}}(f,g) \arrow[r] \arrow[d] & * \arrow[d] \\
\mathrm{Map}_{\mathcal{C}}(x,y) \arrow[r] \arrow[d, "g\circ -"] & \mathcal{C}_{/y} \arrow[r] \arrow[d, "g\circ -"] & \mathcal{C} \arrow[d, "="] \\
\mathrm{Map}_{\mathcal{C}}(x,c) \arrow[r] & \mathcal{C}_{/c} \arrow[r] & \mathcal{C}.
\end{tikzcd}$$

Since the top row of vertical fibers then also forms a fiber sequence, the map $F \to \mathrm{Map}_{\mathcal{C}_{/c}}(f,g)$ is an equivalence of ∞-categories.

The inclusion of the full subcategory $\mathcal{S} \hookrightarrow \mathrm{Cat}_\infty$ is a right adjoint and preserves the terminal object $*$, so preserves fiber sequences, so we may consider this a fiber sequence in $\mathcal{S}$. ☐

B.2. **Internal Function-objects.** Consider a presentably symmetric monoidal ∞-category $\mathcal{C}$, i.e. a commutative algebra-object in Pr_{L} with respect to the Lurie tensor product [Lur17, Ch. 4.8]. The symmetric monoidal structure is classified by a functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{-\otimes -} \mathcal{C}$$

preserving colimits in either variable. Fixing the first variable, we have an adjunction

$$\mathcal{C} \underset{F(x,-)}{\overset{-\otimes x}{\rightleftarrows}} \mathcal{C}$$

by the adjoint functor theorem [Lur09, Prop. 5.5.2.9].

One should hope that the construction $(x,y) \mapsto F(x,y)$ promotes to a functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C},$$

and that this functor has the expected mapping properties on limits and colimits, i.e. that it preserves limits in the second variable and transforms colimits (in $\mathcal{C}$, not $\mathcal{C}^{\mathrm{op}}$) into limits in the first variable. We will construct the functor and prove the specified mapping properties.

Indeed, if such a functor existed, postcomposing it with the Yoneda-embedding $j\colon \mathcal{C} \to \mathcal{P}(\mathcal{C})$ would give a functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{P}(\mathcal{C}), \qquad (x,y) \mapsto \mathrm{Map}_{\mathcal{C}}(-, F(x,y)).$$

Now by the defining adjunction, we have a natural equivalence of spaces

$$\mathrm{Map}_{\mathcal{C}}(-, F(x,y)) \simeq \mathrm{Map}_{\mathcal{C}}(- \otimes x, y)$$

and the latter construction is evidently functorial $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{P}(\mathcal{C})$ as an adjoint construction to

$$\begin{aligned} \mathcal{C}^{\mathrm{op}} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{C} &\to \mathcal{S} \\ (z, x, y) &\mapsto \mathrm{Map}_{\mathcal{C}}(z \otimes x, y). \end{aligned}$$

The mere existence of internal function objects equivalently says that the above functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \xrightarrow{(x,y)\mapsto \mathrm{Map}_{\mathcal{C}}(-\otimes x, y)} \mathcal{P}(\mathcal{C})$$

takes values in the image of the Yoneda-embedding j. Since j is of course fully faithful, we may lift against it to obtain the factorization

$$\begin{tikzcd} & \mathcal{C} \arrow[d, hook, "j"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, dashed, "F(x,y)"] \arrow[r, "\mathrm{Map}_{\mathcal{C}}(-\otimes x,y)"'] & \mathcal{P}(\mathcal{C}), \end{tikzcd}$$

so we have constructed the desired functor.

Certainly the functor

$$\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{P}(\mathcal{C}), \qquad (x, y) \mapsto \mathrm{Map}_{\mathcal{C}}(- \otimes x, y)$$

preserves limits in the second variable and (since the tensor product commutes with colimits) transforms colimits in the first into limits. By [Lur09, Prop. 5.3.1.2], the Yoneda-embedding preserves all limits which exist in $\mathcal{C}$ (i.e. all limits) so we conclude the same for $F(x, y)$.

Remark B.3. As a simple application of this principle, we have that for a space X a constant limit of shape X with value E is given by $F(1_{\mathcal{C}} \otimes X, E)$:

$$\begin{aligned} \lim_X E &\simeq \lim_X F(1_{\mathcal{C}}, E) \\ &\simeq F(\operatorname*{colim}_X 1_{\mathcal{C}}, E) \\ &\simeq F(1_{\mathcal{C}} \otimes X, E). \end{aligned}$$

B.3. **The spectral Yoneda lemma.** We prove the following easy enhancement of the ∞-categorical Yoneda lemma.

Proposition B.4. *Let $\mathcal{D}$ be an ∞-category and $F\colon \mathcal{D} \to \mathsf{Sp}$ some functor. For any object d of $\mathcal{D}$, we have an equivalence of spectra*

$$\mathrm{map}_{\mathsf{Sp}^{\mathcal{D}}}\Big(\mathbb{S}[\,\mathrm{Map}_{\mathcal{D}}(d, -)\,],\ F\Big) \simeq F(d).$$

Proof. From the adjunction

$$\mathcal{S}^{\mathcal{D}} \underset{\Omega^\infty}{\overset{\mathbb{S}[-]}{\rightleftarrows}} \mathsf{Sp}^{\mathcal{D}}$$

and the ordinary (∞-categorical) Yoneda lemma, we readily derive the natural equivalence of underlying spaces

$$\mathrm{Map}_{\mathsf{Sp}^{\mathcal{D}}}\Big(\mathbb{S}[\,\mathrm{Map}_{\mathcal{D}}(d, -)\,],\ F\Big) \simeq \Omega^\infty F(d).$$

To promote this to an equivalence of spectra, observe that by [Lur17, Cor. 1.4.2.23] for $\mathcal{C}$ stable and any two *exact* functors $G, H\colon \mathcal{C} \to \mathsf{Sp}$, an equivalence of underlying spaces

$$\Omega^\infty G \simeq \Omega^\infty H$$

always promotes to one of G and H. Applying this in the situation $\mathcal{C} = \mathsf{Sp}^{\mathcal{D}}$,

$$G(?) = \mathrm{map}_{\mathsf{Sp}^{\mathcal{D}}}\Big(\mathbb{S}[\,\mathrm{Map}_{\mathcal{D}}(d,-)\,],\ ?\Big),$$

and $H = \mathrm{ev}_d$ gives the claimed equivalence. □

B.4. **Free G-objects via Kan extensions.** Given an ∞-category $\mathcal{C}$, we have a forgetful functor from objects of $\mathcal{C}$ equipped with an action by a group G to $\mathcal{C}$,

$$u\colon \mathcal{C}^{\mathrm{B}G} \to \mathcal{C}$$

given by restriction along $* \to \mathrm{B}G$. For $\mathcal{C}$ presentable, u admits a left adjoint, given by the 'free G-object' on $x \in \mathcal{C}$. It certainly seems plausible that this should be $\coprod_G x$, and indeed it is.

In the body of the text we introduced Kan extensions only for fully faithful inclusions $\mathcal{I} \hookrightarrow \mathcal{J}$, but we take the liberty of assuring the concerned reader that left Kan extensions exist along arbitrary transformations of diagrams, and (given the required existence of colimits in $\mathcal{C}$) again furnish a left adjoint to the restriction functor. Further, the same formula holds: Given diagrams $\mathcal{I} \to \mathcal{J}$ and a functor $F_0\colon \mathcal{I} \to \mathcal{C}$, the left Kan extension F is characterized by

$$F(x) \simeq \mathrm{colim}(\mathcal{I}_{/x} \to \mathcal{I} \to \mathcal{C}).$$

Note that in this more general situation, it is no longer true that $F_0(i) \simeq F(i)$ for $i \in \mathcal{I}$.

In the situation of left Kan extending along $* \to \mathrm{B}G$, we thus want to compute

$$\mathrm{colim}(K \to * \xrightarrow{x} \mathcal{C}),$$

where K is the relevant relative overcategory, i.e. the pullback

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{B}G_{/*} \arrow[d] \\ * \arrow[r] & \mathrm{B}G. \end{tikzcd}$$

Of course, by the usual description of mapping spaces as fibers of this right fibration, the pullback K is

$$\mathrm{Map}_{\mathrm{B}G}(*,*) \simeq \Omega \mathrm{B}G \simeq G.^{10}$$

Thus, the free G-object on x is given by the colimit of the constant functor from G with value x, that is $\coprod_G x$. A further inspection shows that the action of G is indeed given by permuting the summands, as expected.

[10] Alternatively, one could of course argue that $\mathrm{B}G_{/*}$ is contractible.

Appendix C. Zusammenfassung in deutscher Sprache

Ausgangspunkt der vorliegenden Arbeit ist ein Theorem von Cary Malkiwiech [Mal17]. Es besagt dass die sogenannte *assembly map* in der algebraischen K-theorie eines Gruppenringes bezüglich einer endlichen Gruppe eine duale *coassembly map* zulässt, so dass die Komposition beider die *norm map* auf der K-Theorie des zugrundeliegenden Ringes formt. Während Malkiewich den Beweis mit geometrischen Methoden und einem genauen Studium konkreter Modellkategorien parametrisierter Spektren antritt, nutzen wir gänzlich andere Methoden. Die entscheidenden Einflüsse sind erstens die equivariante Perspektive auf die assembly map von John Davis und Wolfgang Lück [DL98], und zweitens der präzise Zugang zu genuin-equivarianter Homotopietheorie den die Theorie von *spektralen G-Mackey Funktoren* [GM17a],[Bar17] erlaubt. Zusätzlich arbeiten wir ausschließlich in der modernen Sprache von ∞-Kategorien, die es uns ermöglicht präzise Definitionen mittels universeller Eigenschaften zu formulieren.

Nach der Einführung in Kapitel 1 erinnern wir in Kapitel 2 an einige zugrundeliegende Konzepte der Homotopietheorie aus der Perspektive von ∞-Kategorien. In Kapitel 3 geben wir eine auf unsere Anwendungen fokussierte Übersicht über Kan-Erweiterungen entlang Inklusionen voller Unterkategorien, und berechnen konkrete Beispiele. In Kapitel 4 definieren wir die assembly maps, zunächst nach Weiß-Williams [WW95] und dann nach Davis-Lück [DL98]. Wir liefern einen konzeptuellen Beweis in moderner Sprache dass beide Konstruktionen, nach den nötigen Identifikationen, übereinstimmen.

In Kapitel 5 sammeln wir einige klassische oder zumindest bekannte Ergebnisse über kommutative Monoide (also $\mathbb{E}_\infty$-Algebren bezüglich der kartesisch-monoidalen Struktur), mal mit Wirkung einer Gruppe, mal mit Werten in einer präsentablen ∞-Kategorie.

In Kapitel 6 definieren wir, [Bar17] folgend, die *Spann-Kategorie* einer disjunktiven ∞-Kategorie $\mathcal{C}$. Wir beweisen ein technisches Lemma über das Verhältnis der Kommakategorien $\mathcal{C}_{/c}$ und $\mathrm{Span}(\mathcal{C})_{/c}$, welches uns erlauben wird gewisse Kan-Erweiterungen zu identifizieren.

In Kapitel 7 definieren wir die ∞-Kategorie $\mathrm{Mack}_G(\mathcal{E})$ der G-Mackey Funktoren mit Werten in einer präadditiven ∞-Kategorie $\mathcal{E}$, und in Kapitel 8 widmen wir uns dem Studium des Spezialfalles $\mathcal{E} = \mathsf{Sp}$, der ∞-Kategorie der Spektren. Wir bereiten alles nötige vor um in Kapitel 9 dann den Beweis nach [CMNN20] zu geben dass $\mathrm{Mack}_G(\mathsf{Sp})$ genau die ∞-Kategorie genuiner G-Spektren modelliert.

In Kapitel 10 identifizieren wir, die Definition der Norm-Abbildung für genuine G-Spektren aus [GM95] zugrundelegend, die Komposition aus assembly und coassembly mit der Norm-Abbildung für gewisse Funktoren $\mathcal{S} \to \mathsf{Sp}$. Entscheidend ist hier der Übergang zum Davis-Lück-Modell der (co-)assembly maps und dem Verständnis genuiner G-Spektren als Diagram-Kategorien.

Es folgt in Kapitel 11 ein Studium (der Bildung gewisser Kolimiten in) der ∞-Kategorie $\mathrm{Cat}_{\mathrm{perf}}$ der stabilen, idempotent-vollständigen ∞-Kategorien, welches uns dann in Kapitel 12 ermöglicht $\mathrm{Cat}_{\mathrm{perf}}$ als Koeffizienten-Kategorie $\mathcal{E}$ für Mackey Funktoren zu nutzen. Eine genaue Untersuchung der Relation zwischen 'naiven' G-Objekten (also $\mathrm{Fun}(\mathrm{B}G, \mathcal{E})$) und 'genuinen' G-Objekten (also $\mathrm{Mack}_G(\mathcal{E})$) erlaubt es dann nachzuweisen, dass Waldhausen A-Theorie (mit Koeffizienten in einem beliebigen $\mathbb{E}_1$-Ring Spektrum) die nötige Bedingung für die Identifikation aus Kapitel 10 erfüllt.

Im Anhang A geben wir die Identifikation der Greenlees-May-Konstruktion der Norm-Abbildung für genuine G-Spektren mit der universellen Eigenschaft der Norm-Abbildung des unterliegenden naiven G-Spektrums nach [Kle02], und in Anhang B sammeln wir einige kleine Ergebnisse über ∞-Kategorien.

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