Chapter 2

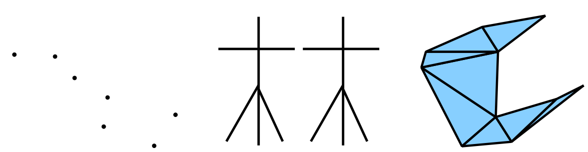
Preliminaries

This chapter gives basic definitions of the types of shapes and distance measures that we work with in this dissertation. An optional way to read this thesis for a reader who is familiar with the topic is to skip this chapter, and if necessary refer back to it later – maybe using the index of keywords at the end of this thesis – to look up a specific definition.

Throughout the whole thesis we consider shapes in d-dimensional real space $\mathbb{R}^d$, where the dimension $d \geq 2$ is a fixed constant.

2.1 Representation of Shapes

In general there are many ways to describe geometric shapes, which also varies according to the application and the field of research. In the most general setting a geometric shape can be modeled as composed of a finite number of basic objects (*sites*), such as



points, line segments, triangles.

A common way to represent shapes is to describe them as sets of pixels (or voxels). These can be seen as special sets of points, which are arranged in a grid fashion. However often a representation by higher-dimensional sites, such as line segments or triangles, can reduce the description complexity immensely. One can describe a shape either as a solid volume, or by its boundary. In two dimensions this yields the special case of closed curves representing the boundary of a shape. In general, curves are also a class of shapes which are worthwhile to consider since they appear in several applications. We will exploit the special property of curves that they are not only a set of points, but that they have a parameterization which describes an “order” of the points.

Let us now describe in detail the types of sites and shapes we consider in the sequel:

Definition 1 (Simplex) *A* k-simplex *in* $\mathbb{R}^d$*, for* $-1 \leq k \leq d$*, is the convex hull of* $k+1$ *affinely independent points in* $\mathbb{R}^d$*.* $k+1$ *points in* $\mathbb{R}^d$ *are* affinely independent *iff their affine hull has dimension* k*.*

There are several ways to define polyhedral sets. We consider them to be composed of simplices, and not of arbitrary polyhedra.

Definition 2 (Polyhedral Set) *A* polyhedral set *A of* complexity *n in* $\mathbb{R}^d$ *is a collection* $A = \{a_1, \ldots, a_n\}$ *of interior-disjoint simplices of dimension at most d.*

Note that since we assume throughout this thesis that the dimension d is constant, every k-simplex in $\mathbb{R}^d$ has $O(1)$ j-faces, $0 \leq j \leq k$, altogether. Hence, a polyhedral set A of complexity n has a description complexity proportional to n.

Definition 3 (Terrain / Polyhedral Terrain) *A* terrain *F in* $\mathbb{R}^d$ *is the graph* $F = \{(x, f(x)) \,|\, x \in D_f\}$ *of a function* $f : D_f \longrightarrow \mathbb{R}$, *with* $D_f \subseteq \mathbb{R}^{d-1}$. *F is a* polyhedral terrain *if* $D_f = \cup M_f$, *where* M_f *is a polyhedral subset of* $\mathbb{R}^{d-1}$, *and f is a linear function over each simplex of* M_f.

Hence, a polyhedral terrain is a polyhedral set with the property that, for each point in $\mathbb{R}^{d-1}$, which we consider embedded in $\mathbb{R}^d$ at $x_d = 0$, the line in x_d-direction intersects the terrain in at most one point only. Also note that our definition of terrains allows f to be any k-variate function for $0 \leq k \leq d-1$.

Definition 4 (Curve / Polygonal Curve) *A* curve in $\mathbb{R}^d$ *is a continuous mapping* $f : I \to \mathbb{R}^d$ *for a closed interval* $I \subset \mathbb{R}$. *We often use* $I = [0, 1]$.
A polygonal curve *P of complexity* $n \in \mathbb{N}$ *is a curve* $P : [0, n] \to \mathbb{R}^d$ *with* vertices $P_i := P(i)$ *for* $i \in \{0, 1, 2, ..., n\}$ *and* line segments $\overline{P}_j := P|_{[j,j+1]}$ *for* $j \in \{0, 1, ..., n-1\}$, *such that* $P(j + \lambda) = (1 - \lambda)P(j) + \lambda P(j + 1)$ *for all* $\lambda \in [0, 1]$.

Note that the curves we consider are not given as sets of points in $\mathbb{R}^d$ only, but a curve f is given as a *parameterization* $f : I \to \mathbb{R}^d$. Thus polygonal curves in our setting do not only consist of vertices and line segments, but they are also equipped with a fixed. We call a continuous mapping $\alpha : I' \to I$ a *reparameterization*, since for a curve $f : I \to \mathbb{R}^d$ it defines the reparameterized curve $f \circ \alpha : I' \to f(I)$ that has the same image of points $f(I) \subseteq \mathbb{R}^d$ as f but represents a different parameterization.

2.2 Distance Measures

There are many different ways of describing a shape, and there exists a variety of different distance measures between two shapes. We consider a small subset of distance measures which we define in the following. The Hausdorff and the directed Hausdorff distance are commonly used in practice since they are relatively easy to compute. Although the Fréchet distance is not as well known as the Hausdorff distance, it is a natural distance measure for curves which has a wide potential to being used in practice once there are practical algorithms for dealing with it. Now let us define the distance measures we consider:

Definition 5 (Metric) *Let* ρ *denote a* metric *in* $\mathbb{R}^d$. *For fixed* ρ *we implicitly consider the metric space* $(\mathbb{R}^d, \rho)$. *Let* $A, B \subseteq \mathbb{R}^d$, *and let* $x \in \mathbb{R}^d$ *be a point. We use the notation* $\rho(x, B) := \rho(B, x) := \inf_{y \in B} \rho(x, y)$ *and* $\rho(A, B) := \inf_{x \in A} \inf_{y \in B} \rho(x, y)$. *Note that for compact A and B the infima are realized at points of A and B.*

Examples of metrics are metrics induced by norms, such as the Euclidean L_2-norm $||.||$, the L_1-norm or the L_∞-norm. Throughout this thesis we denote by ρ an underlying metric.

Definition 6 (Polyhedral Metric) *Let P be a d-dimensional convex polytope in $\mathbb{R}^d$ which is centrally symmetric with respect to the origin. For two points $x, y \in \mathbb{R}^d$ the distance induced by P is*

$$d_P(x,y) := \min\{\sigma \in \mathbb{R}_0^+ \mid y \in x + \sigma P\}.$$

d_P is the (convex) polyhedral metric induced by P*.*

If the sum of the complexities of all faces of P is constant, d_P is called a polyhedral metric of constant description complexity*.*

$\mathbb{R}^d$ is a normed space with unit ball P. Thus d_P is a metric.

Definition 7 (Hausdorff distance) *Let ρ be a metric in $\mathbb{R}^d$, and let $A, B \subseteq \mathbb{R}^d$ be two compact sets with respect to ρ. Then the* directed Hausdorff distance $\vec{\delta}_{\mathrm{H}}(A,B)$ *is defined as*

$$\vec{\delta}_{\mathrm{H}}(A,B) := \max_{x\in A}\min_{y\in B}\rho(x,y) = \max_{x\in A}\rho(x,B). \tag{2.1}$$

If A and B are polyhedral sets in $\mathbb{R}^d$, then

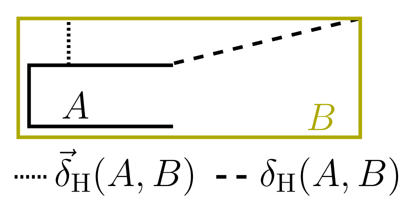
$$\vec{\delta}_{\mathrm{H}}(A,B) := \vec{\delta}_{\mathrm{H}}(\cup A, \cup B) = \max_{x\in a\in A}\min_{y\in b\in B}\rho(x,y) = \max_{x\in a\in A}\min_{b\in B}\rho(x,b). \tag{2.2}$$

For two curves $f, g : I \to \mathbb{R}^d$ the directed Hausdorff distance is defined on the images of the curves

$$\vec{\delta}_{\mathrm{H}}(f,g) := \vec{\delta}_{\mathrm{H}}(f(I), g(I)). \tag{2.3}$$

The (undirected) Hausdorff distance $\delta_{\mathrm{H}}(A,B)$ *is defined as*

$$\delta_{\mathrm{H}}(A,B) := \max\{\vec{\delta}_{\mathrm{H}}(A,B), \vec{\delta}_{\mathrm{H}}(B,A)\}. \tag{2.4}$$



The Hausdorff distance is a natural way to extend a metric ρ to the class of compact sets. Usually the shapes we consider will be compact, so that we consider δ_{H} as a natural *shape metric*. Note that δ_{H} is indeed a metric, however $\vec{\delta}_{\mathrm{H}}$ is not, since it is not symmetric. Nonetheless the directed Hausdorff distance is often used in partial matching applications, where the task is to find a subset of the shape B that resembles the shape A the most.

For terrains the following similarity notion naturally derives from the monotonicity property.

Definition 8 (Perpendicular Distance) *The* perpendicular distance *(also sometimes called* vertical distance*,* uniform metric*, or* Chebyshev metric*) for two terrains $F = \{(x, f(x)) | x \in D_f\}$, $G = \{(x, g(x)) | x \in D_g\}$ in $\mathbb{R}^d$ over the same domains $D_f = D_g$ is defined as*

$$\delta_{\perp}(F,G) := \sup_{x\in D_f} |f(x) - g(x)| \,. \tag{2.5}$$

Since f and g are functions, the perpendicular distance is the standard L_∞ Minkowski-metric for functions.

In contrast to the Hausdorff distance we can exploit the existence of a parameterization when the shapes we are concerned with are curves. For curves a natural distance measure is the Fréchet distance.

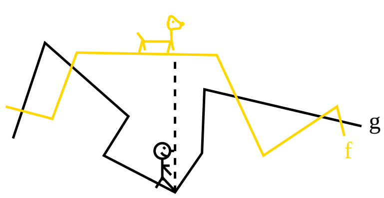
Definition 9 (Fréchet Distance) *Let $f : I = [l_I, r_I] \to \mathbb{R}^d$, $g : J = [l_J, r_J] \to \mathbb{R}^d$ be two curves, and let ρ be a metric in $\mathbb{R}^d$. Then the* Fréchet distance $\delta_{\mathrm{F}}(f, g)$ *is defined as*

$$\delta_{\mathrm{F}}(f,g) := \inf_{\substack{\alpha:[0,1]\to I \\ \beta:[0,1]\to J}} \max_{t\in[0,1]} \rho(f(\alpha(t)), g(\beta(t))).$$

where α and β range over all continuous and non-decreasing reparameterizations with $\alpha(0) = l_I$, $\alpha(1) = r_I$, $\beta(0) = l_J$, $\beta(1) = r_J$. If we drop the condition on the reparameterizations to be non-decreasing we obtain the weak Fréchet distance $\tilde{\delta}_{\mathrm{F}}(f, g)$.

We call two reparameterizations α^*, β^* which fulfill the constraints above witnessing reparameterizations *for $\delta_{\mathrm{F}}(f, g)$ iff $\delta_{\mathrm{F}}(f, g) = \max_{t\in[0,1]} \rho(f(\alpha^*(t)), g(\beta^*(t)))$.*

Both the Fréchet distance and the weak Fréchet distance are *pseudo-metrics*, see [10, 44], which means that they have almost all properties of a metric with the exception that two objects with distance 0 may be distinct. As an example consider $f, g : [0, 1] \to \mathbb{R}^2$ with $f(x) = (x, x)$ and $g(x) = (x^2, x^2)$: Clearly $\delta_{\mathrm{F}}(f, g) = \tilde{\delta}_{\mathrm{F}}(f, g) = 0$, however $f \neq g$.



For the Fréchet distance we will consider ρ to be the metric induced by the Euclidean norm $||.||$. A popular illustration of the Fréchetdistance is the following: Suppose a man is walking his dog, the man is walking on the one curve and the dog on the other. Both are allowed to control their speed but they are not allowed to go backwards. Then the Fréchet distance of the curves is the minimal length of a leash that is necessary for both to walk the curves from beginning to end.

For two curves f, g in $\mathbb{R}^d$ we can consider their Hausdorff distance, their Fréchet distance and their weak Fréchet distance. All three distance measures assign points of $f(I)$ and $g(J)$ according to certain rules to each other, and the respective distance measure is the maximum distance between any such point pair. The Hausdorff distance has the most freedom in picking for each point $x \in f(I)$ the closest point $y \in g(J)$. The weak Fréchet distance on the other hand already imposes restrictions on the choice of point pairs by forcing them to be continuously assigned according to continuous reparameterizations. The Fréchet distance imposes even more restrictions by requiring the continuous reparameterizations to be non-decreasing. Thus we can make the following observation.

Observation 1 *Let f, g be two curves in $\mathbb{R}^d$. Then*

$$\delta_{\mathrm{H}}(f,g) \leq \tilde{\delta}_{\mathrm{F}}(f,g) \leq \delta_{\mathrm{F}}(f,g) \; .$$

The standard example of how much those distance measures can differ is given in Figure 2.1. There two polygonal curves P and Q zigzag, one horizontally, and the other vertically, such that the Hausdorff distance is relatively small, but the Fréchet distance very large.

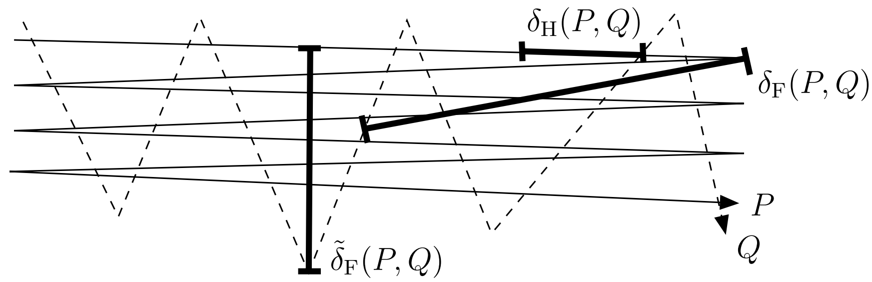


Figure 2.1: Example of two polygonal curves P and Q with large Fréchet distance but small Hausdorff distance.

2.3 Miscellaneous

In this subsection we gather a few notations and definitions that we will need lateron. Minkowski sums and semi-algebraic sets are important notions which we need to describe configuration spaces.

Definition 10 (Minkowski Sum) *For two sets $A, B \subseteq \mathbb{R}^d$*

$$A \oplus B := \{a + b \mid a \in A,\ b \in B\}$$

is called the Minkowski sum *(or* vector sum*) of A and B.*

Definition 11 (Algebraic Set / Semi-Algebraic Set / Algebraic Surface) *Let $f_1, \ldots, f_k \in \mathbb{R}[X_1, \ldots, X_d]$ be polynomials in d variables with coefficients in $\mathbb{R}$. Then*

$$\{x \in \mathbb{R}^d \mid \bigwedge_{i \in [1,\ldots,k]} f_i(x) = 0\} \tag{2.6}$$

is an algebraic set*. If $k = 1$ then it is also called an* algebraic surface*. Let $f_{i,j} \in \mathbb{R}[X_1, \ldots, X_d]$ for $i = 1, \ldots, k$ and $j = 1, \ldots, l_i$. And let $\leq_{i,j} \in \{\leq, =\}$. Then*

$$\{x \in \mathbb{R}^d \mid \bigwedge_{i \in [1,\ldots,k]} \bigvee_{j \in [1,\ldots,l_i]} f_{i,j}(x) \leq_{i,j} 0\} \tag{2.7}$$

is a semi-algebraic set*. If exactly one of the $\leq_{i,j}$ is a "$=$" then the semi-algebraic set is called a* semi-algebraic surface patch*.*

See [31] for properties and results on algebraic and semi-algebraic sets. See [8] for the definition and properties of algebraic surfaces and surface patches.
Some further miscellaneous notations we use in the sequel are the following:

- For a given metric ρ let
  - $\mathbf{B}^d_\varepsilon(y) := \{x \in \mathbb{R}^d \mid \rho(x, y) \leq \varepsilon\}$ be the d-dimensional *closed ε-ball* centered at $y \in \mathbb{R}^d$, and let $\mathbf{B}^d_\varepsilon := \mathbf{B}^d_\varepsilon(0, \ldots, 0)$ be the closed ε-ball at the origin.
  - $\mathring{\mathbf{B}}^d_\varepsilon(y) := \{x \in \mathbb{R}^d \mid \rho(x, y) < \varepsilon\}$ be the d-dimensional *open ε-ball* centered at $y \in \mathbb{R}^d$, and let $\mathring{\mathbf{B}}^d_\varepsilon := \mathring{\mathbf{B}}^d_\varepsilon(0, \ldots, 0)$ be the open ε-ball at the origin.

– $S_\varepsilon^{d-1}(y) := \mathbf{B}_\varepsilon^d(y) \setminus \mathring{\mathbf{B}}_\varepsilon^d(y)$ be the $(d-1)$-dimensional ε-*sphere* centered at $y \in \mathbb{R}^d$, and let $S_\varepsilon^{d-1} := \mathbf{B}_\varepsilon^d \setminus \mathring{\mathbf{B}}_\varepsilon^d$ be the ε-sphere at the origin.

- For a set $A \subseteq \mathbb{R}^d$ we denote with ∂A its boundary and with $\overline{A}$ its complement.
- The convex hull of a collection of points $p_1, \ldots, p_n \in \mathbb{R}^d$ is denoted by $CH(p_1, \ldots, p_n)$.
- With O_δ we denote the variant of the O-notation which subsumes factors n^δ, and in which the constant of proportionality depends on δ, for any $\delta > 0$. For example $O(n^{2+\delta})$ for any $\delta > 0$ is $O_\delta(n^2)$.
- We identify $\mathbb{R}^d$ with its vector space, and denote the i-th unit vector in the standard orthonormal basis with $\vec{e}_i$. Also we often identify points in $\mathbb{R}^d$ with their vectors.