Convex partitions of vector bundles and fibrewise configuration spaces

Dissertation zur Erlangung des Grades
eines Doktors der Naturwissenschaften (Dr. rer. nat.)
am Fachbereich Mathematik und Informatik
der Freien Universität Berlin

vorgelegt von

Tatiana Levinson

Berlin
2023

Betreuer und Erstgutachter:
Prof. Pavle Blagojević

Zweitgutachter:
Prof. Florian Frick

Tag der Disputation:
5. September 2023

There are two methods for solving problems:
by intense staring or by meticulous rewriting.
S. Shashkov

Summary

We begin this thesis with a discussion of problems from geometry and combinatorics, to which methods from equivariant algebraic topology have been successfully applied in the past.

The generalised Nandakumar & Ramana Rao problem (due to Karasev, Hubard & Aronov and Blagojević & Ziegler) asks whether given a full-dimensional compact convex body K in $\mathbb{R}^n$, $n-1$ continuous real functions on the space of all full-dimensional compact convex bodies in $\mathbb{R}^n$ and a natural number m, one can always find a partition of K into m convex pieces of equal volume such that the value of each function is equal on all the pieces of the partition.

Inspired by this problem and recent parameterised generalisations of mass partition types by several groups of researchers, we formulate a parameterised version of the Nandakumar & Ramana Rao problem, where we aim to equipart $j > n-1$ functions, but are allowed to choose a convex body K from some family parameterised by a vector bundle E over a CW-complex B.

After we make the notion of "parameterised by the vector bundle E" precise in Chapter II, we follow the strategy developed by Karasev, Hubard & Aronov to formulate a topological criterion for the existence of solutions to the parameterised Nandakumar & Ramana Rao problem. Due to the limitations of our topological methods, we restrict our attention to the case when m equals some prime p.

Chapter III contains a brief overview of various standard algebraic topology results that we use extensively in the later chapters.

In Chapter IV we extend the results of Jaworowski concerning Fadell-Husseini indices of sphere bundles, equipped with free fibrewise action of the cyclic group $\mathbb{Z}_p$, by considering the symmetric group $\mathfrak{S}_p$ on the place of $\mathbb{Z}_p$. Next, we compute the index of the fibrewise configuration space $\mathrm{Fconf}(p, E)$ of p distinct points with respect to $\mathfrak{S}_p$ in the case of vector bundle E of an odd rank. In the case when E has an even rank, we provide bounds on the index, showing that the upper bound is tight in some cases. Then we change the group that acts on E, and compute the index of the space $\mathrm{Fconf}(p, E)$ with respect to $\mathbb{Z}_p$-action in the special case when E admits two linearly independent nowhere zero sections.

In Chapter V we use these computations to find a partial solution to the parameterised Nandakumar & Ramana Rao problem. For any pair of a vector bundle E and a prime p, we describe a range of j such that the parameterised Nandakumar & Ramana Rao has a solution for the family of convex bodies parameterised by E, the desired number p of pieces in partition and a choice of j appropriately defined continuous functions. Finally, we apply these computations to the case of a tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over the Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$.

Zusammenfassung

Die Dissertation beginnt mit einer Diskussion einiger geometrischer und kombinatorischer Probleme, zu deren Studium sich Methoden der Äquivarianten Algebraischen Topologie in der Vergangenheit als geeignet erwiesen haben.

Das verallgemeinerte Nandakumar & Ramana Rao-Problem (nach Karasev, Hubard & Aronov, sowie Blagojević & Ziegler) besteht in der Frage, ob sich für einen gegebenen volldimensionalen, kompakten, konvexen Körper K im $\mathbb{R}^n$, eine Familie von $n-1$ stetigen reellen Funktionen auf dem Raum aller solchen Körper im $\mathbb{R}^n$, sowie eine gegebene natürliche Zahl m stets eine Partition von K in m konvexe Teilmengen gleichen Volumens finden lässt, derart, dass jede einzelne der Funktionen auf allen Teilen gleiche Werte annimmt.

Wir formulieren eine parametrisierte Version dieses Problems, die nach einer Gleichteilung bezüglich einer Familie von möglicherweise mehr als $n-1$ Funktionen fragt, allerdings erlaubt, den konvexen Körper K aus einer durch ein Vektorbündel E über einem CW-Komplex parametrisierten Familie zu wählen.

Nachdem wir im zweiten Kapitel den Begriff der „Parametrisierung durch ein Vektorbündel“ präzisieren, verfolgen wir die von Karasev, Hubard & Aronov entwickelte Strategie, topologische Kriterien für die Lösbarkeit des parametriserten Nandakumar & Ramana Rao-Problems in gewissen Fällen zu finden. Den Grenzen unserer topologischen Methoden ist es geschuldet, dass wir uns dabei auf den Fall beschränken, indem $m = p$ eine Primzahl ist.

Das dritte Kapitel der Arbeit gibt einen Überblick über verschiedene bekannte Ergebnisse der Algebraischen Topologie, die wir in den späteren Kapiteln benutzen werden.

Im vierten Kapitel erweitern wir Ergebnisse Jaworowskis über Fadell-Husseini-Indizes gewisser Sphärenbündeln, die mit einer faserweisen Wirkung der zyklischen Gruppe $\mathbb{Z}_p$ ausgestattet sind, wobei eine Wirkung der symmetrischen Gruppe $\mathfrak{S}_p$ an die Stelle der $\mathbb{Z}_p$-Wirkung tritt. Als Nächstes berechnen wir die Indizes (bzgl. $\mathfrak{S}_p$-Wirkung) des Faserweisen Konfigurationsraums $\operatorname{Fconf}(p, E)$ von p Punkten in einem Vektorbündel E ungeraden Rangs, geben für den Fall geraden Rangs (teils bestmögliche) Schranken an, und berechnen den Index (bzgl. $\mathbb{Z}_p$ oder $\mathfrak{S}_p$-Wirkung) im Spezialfall, dass E zwei linear unabhängige Schnitte zulässt.

Das fünfte Kapitel behandelt, wie die Ergebnisse unserer Berechnungen zu einer teilweisen Lösung des parametrisierten Nandakumar & Ramana Rao-Problems führen. Für jedes Paar, bestehend aus einem Vektorbündel E und einer Primzahl p beschreiben wir einen Bereich möglicher Werte von j, für die das parametrisierte Nandakumar & Ramana Rao-Problem bezüglich dem Tripel (E, p, j) eine Lösung besitzt. Schließlich wenden wir unsere Überlegungen auf den Spezialfall Tautologischer Bündel $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ über Grassmann-Mannigfaltigkeiten $\operatorname{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ an.

Acknowledgements

My heartfelt thanks go to:

- My advisor, **Pavle Blagojević**, for introducing me to the world of equivariant algebraic topology, generously sharing math and life experience, and for inexhaustible patience in cases when I decided not to follow his advice.
- My second reviewer, **Florian Frick**, for many valuable comments on my thesis, our math discussion and overall moral support in the role of Günter IV.
- **Michael Crabb**, for generously sharing his knowledge of equivariant topology and, in particular, Euler classes, which saved this thesis from the dead end and gave it a new direction.
- **Evgeniya Lagoda**, for moral support when dealing with spectral sequences and for bringing my attention to the construction of characteristic classes via Leray-Hirsch theorem, which ended up to be one of the key techniques used in this thesis.
- **Simona Boyadzhiyska**, **Sampada Kolhatkar**, **Georg Oberdieck**, **Vira Raichenko**, **Nikola Sadovek**, and **Manuel Staiger** for spending so much personal time helping me with proofreading. Special thanks to Manuel, whose translation of the Summary to German made it so much better that I needed to translate it back into English.
- The **Villa** community. I was very lucky to be a part of the most friendly research group I have ever heard of. I will miss the Villa, people, cakes, BBQs, Paellas, ice cream in the fridge and Blauregenblütenblätten in my tea. I am especially grateful to **Melanie Gessele**, for taking care of all of us, from making keys work to "Nikolaus" chocolate on our desks. As well as to **Sampada Kolhatkar** and **Manuel Staiger** for our Keller discussions about everything in the world.
- the **BMS**, which funded all my studies in Germany, and **BMS One-Stop office**, whose members were always there to help. Special thanks to **Annika Preuß-Vermeulen**, who seemed to have an answer ready just minutes after receiving an email with a question.
- **Matthias Beck**, for Villa Soft-Skills seminars (I wish I would pay more attention to the one about writing!) and for keeping Villa together even during the lockdown. Special thanks to **Kumi-the-dog**, for the readiness to share the positive mood, and, sometimes, even a ball!
- My teachers. This thesis marks the end of my "being a student" (for 21 years) and "studying math" (for 10 out of them). Especially, I would like to thank my middle school teacher, **Borisova Elena Petrovna**, for encouraging me to try harder problems; my high school teachers, **Dorichenko Sergei Alexsandrovich** and **Sergei Shashkov**, for admitting me to the school 179 and showing that math can be fun; my bachelor's advisor, **Leonid Rybnikov**, for teaching me the foundations of many things that appear in this thesis; **Vadim Vologodsky** for the only "Analysis on Manifolds" course I ever heard of, that included the h-cobordism theorem, which inspired me to study topology; my English

teacher, **Pavel Rakitin**, for managing to teach me English despite my complete disregard for grammar and our university not paying for his lessons.

- My many friends: **Anton Markov** for supporting me for so many of these years, sharing the academic life experience, and for the ultimate form of support – giving up his copy of Hatcher; my first friend in Berlin, **Vignesh Nadarajan**; **Evgeniya Lagoda**, **Vira Raichenko**, and **Sophie Rehberg** for being here and always ready to help and listen; **Nikola Sadovek** for being always interested in whatever math one wants to share; **Alexander Kuklev**, for our endless night discussions; **Anna Denisenko**, **Solomiya Mizyuk**, **Valeria Starichkova**, **Zhanna Webb**, **Daria Pchelina**, for sharing this journey with me.
- Last but not least, **my family**, for always supporting me in the decision to study math, despite the fact that I neither can explain what I do nor what I will do with that after I graduate.

Contents

Notation

General:

$\mathbb{Z}, \mathbb{R}, \mathbb{C}$	the integers, the real numbers, the complex numbers
$\mathbb{F}_p, \mathbb{F}_p^\times$	the field with p elements, the invertible elements of $\mathbb{F}_p$
d, n	dimensions of complex, respectively real vector spaces
S^d	a sphere of dimension d
p	an odd prime number
$(m, m') = 1$	m and m' are coprime
$\lfloor x \rfloor$	number x rounded down
$\smile$	cup product
$\simeq, \cong$	homotopy equivalence, homeomorphism

Chapter II:

bar	the map from $\mathcal{K}$ to $\mathbb{R}^n$, see p. 28
bar_E^m	the map from $\mathcal{K}(E)^m$ to E^m, see p. 30
$bar_E^m(U_i)$	the map from $\mathcal{K}(E)^m \restriction_{U_i}$ to $E^m \restriction_{U_i}$, see p. 30
bar_K	the map from EVP to $\mathrm{Conf}(m, \mathbb{R}^n)$, see p. 26
bar_s	the map from $EVP(s)$ to $\mathrm{Fconf}(m, E)$, see p. 31
d_H, d_s	the Hausdorff-, the symmetric difference metric see p. 22
EVP	the space of all equal volume partitions, see p. 25
$EVP(s)$	the fibrewise analogue of EVP, see p. 29
ev	the map from EVP to W_m^{n-1} or $S(W_m^{n-1})$, see p. 25
ev_E	the map from $EVP(s)$ to $W_m^{n-1} \times B$, see p. 29
$\mathcal{K}$	the space of all full-dim., comp., conv. bodies in $\mathbb{R}^n$ see p. 22
$\mathcal{K}(E)$	the fibrewise analogue of $\mathcal{K}$, see p. 22
v	the map from $\mathrm{Fconf}(m, E) \times \mathcal{K}$ to $\mathcal{K}^m$, see p. 27
v_K	the map from $\mathrm{Conf}(m, \mathbb{R}^n)$ to EVP, see p. 26
v_s	the map from $\mathrm{Fconf}(m, E)$ to $EVP(s)$, see p. 31
$v_E(U_i)$	the map from $\mathrm{Fconf}(m, E) \restriction_{U_i}$ to $\mathcal{K}^m \restriction_{U_i}$, see p. 32
$K \Delta L$	the symmetric difference between K and L
$\pi_{EMP(s)}$	the projection from $EVP(s)$ onto B

$\pi_{\mathcal{K}}$	the projection map from $\mathcal{K}(E)$ onto B

Other:

$\mathrm{Ann}(x)$	**non-standard use of notation**, see p. 69
a, b	the generators of $H^*(\mathrm{B}\mathfrak{S}_p)$, see p. 37
$\mathrm{B}G$	the universal classifying space for the group G
$\mathrm{Conf}(m, \mathbb{R}^n)$	the ordered config. space of m distinct points in $\mathbb{R}^n$
$c_i(E), c(E)$	ith Chern class of E, the total Chern class of E
$\check{\mathrm{c}}\mathrm{h}(E)$	see p. 49
${}_pc_j(E)$	a characteristic class, see p. 42
E, B, π	a vector bundle E, its base B, the projection map π
$\mathrm{E}G$	the universal cover of $\mathrm{B}G$
$E \otimes_{\mathbb{R}} \mathbb{C}$	the complexification of the bundle E
$E_i^{s,q}$	the position (s, q) on the ith page of spectral sequence
$E \restriction_U$	the restriction of vector bundle E onto open set U
$e(E), u_E$	the Euler class of E, the Thom class of E
e, t	the generators of $H^*(\mathrm{B}\mathbb{Z}_p)$, see p. 37
$\mathrm{Fconf}(m, E)$	the ordered fibrewise configuration space, see p. 24
$\mathrm{Fl}\, E$	the flag fibre bundle associated with the vector bundle E
f^*E	the pullback of a vector bundle E along the map f
G	an arbitrary group
$\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$	the Grassmannian of all linear subspaces $\mathbb{R}^n \subset \mathbb{R}^N$
$H^{\Pi}(B)$	the ring of infinite series of $H^*(B)$
$\mathcal{H}^*(X; \mathbb{F}_p)$	a local coefficients system
$\mathrm{Index}_B^G\, F$	the index of F over B with respect to G-action see p. 45
$i^{\perp}$	the involution between $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ and $\mathrm{Gr}(\mathbb{R}^{N-n}, \mathbb{R}^N)$
L	a representation representation of $\mathbb{Z}_p$ in $\mathbb{C}$, see p. 52
P^i, P	i-th Steenrod power, total Steenrod power
$p_i(E)$	ith Pontryagin class of E
$\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$	**non-standard use of notation**, see p. 8
$S(E)$	the sphere bundle associated with the vector bundle E
Sq^i	ith Steenrod square
W_pE	the orthogonal complement of $\Delta(E) \subset E^p$, see p. 47
W_p^j	the orthogonal complement of $\Delta(\mathbb{R}^j)$ in $(\mathbb{R}^j)^p$
$W_p^j\mathbb{C}$	the orthogonal complement of$\Delta(\mathbb{C}^j)$ in $(\mathbb{C}^j)^p$
$\underline{X}$	a trivial bundle with fibre X
$X \times_G \mathrm{E}G$	the Borel construction, see p. 44
$\mathbb{Z}_m$	the cyclic group of order m
$\gamma(\mathbb{R}^n, \mathbb{R}^N)$	the tautological bundle over $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$

$\bar{\gamma}\left(\mathbb{R}^n, \mathbb{R}^N\right)$	the pullback of $\gamma(\mathbb{R}^{N-n}, \mathbb{R}^n)$ along the involution $i^{\perp}$
$\Delta^*(E_1 \times \cdots \times E_k)$	the pullback of a product bundle, see p. 24
$\Delta^*(\pi_1 \times \cdots \times \pi_k)$	the projection from $\Delta^*(E_1 \times \ldots E_k)$ onto B
$\zeta_E, \theta_E, \varepsilon_E$	see p. 50
$\pi_1(X)$	the fundamental group of a space X
π_{Conf}	the projection map from $\mathrm{Fconf}(m, E)$ to B
$\mathfrak{S}_m$	the symmetric group on m elements
ς_E	see p. 50
0_B	the zero section of a bundle over B
$\perp$	the projection map from E^p to W_pE

Conventions. The base B of a vector (alternatively fibre) bundle E is always assumed to be a CW-complex.

We fix an odd prime p for the duration of the thesis. Unless specified otherwise we work over the field $\mathbb{F}_p$ and cohomology are taken with $\mathbb{F}_p$ coefficients. We write simply $H^*(B)$ for the cohomology ring $H^*(B;\mathbb{F}_p)$.

When working with the tensor product $H^*(B) \otimes H^*(\mathrm{B}G)$ we identify $H^*(B)$ with the subring $H^*(B) \otimes 1$ and $H^*(\mathrm{B}G)$ with the subring $1 \otimes H^*(\mathrm{B}G)$. Using this identification for an element $x \in H^*(B)$ and an element $y \in H^*(\mathrm{B}G)$ we write xy instead of $x \otimes y$.

Whenever an embedding $\mathbb{Z}_p \hookrightarrow \mathfrak{S}_p$ is mentioned, it is assumed to be the regular embedding, see [AM13, Example 2.7 on page 100] or a brief overview in Section 3.1.

By volume of a full-dimensional compact convex body $K \subsetneq \mathbb{R}^n$ we understand the Lebesgue measure $\mathcal{L}^n$ of the underlining set K.

Chapter 1

Introduction and motivation

As the title suggests, this thesis has two sides to it. Motivated by parameterised Nandakumar & Ramana Rao problem coming from discrete geometry, we look closer at fibrewise configuration spaces and their equivariant cohomology, which are interesting objects of their own. Returning to the geometric side, we use our topological results to find a partial solution to this problem.

1.1 Nandakumar & Ramana Rao problem, background

Nandakumar & Ramana Rao-type problems. The original Nandakumar & Ramana Rao problem [Nan06] asks the following:

Problem 1.1 (Nandakumar & Ramana Rao problem)**.** *Given a polygon P on the plane and a positive integer m, is there a partition of this polygon into m convex pieces $P_1, \dots, P_m$ with pairwise disjoint interiors, such that all of them are of the same area and perimeter?*

This problem was posted in Nandakumar's blog in 2006 along with a conjecture that the answer is positive for any value of m. A couple of years later, Nandakumar & Ramana Rao published a paper [NRR12], where they prove this conjecture for $m = 2$ using the intermediate value theorem and provide some arguments for $m = 2^k$, $k > 1$. The answer for the case $m = 3$ was soon proved positive by Bárány, Blagojević & Szűcs [BBS10] using heavy algebraic topology tools such as Borel constructions and their associated spectral sequences.

Soon afterwards, Problem 1.1 was generalised [KHA14, BZ14]:

Problem 1.2 (Generalised Nandakumar & Ramana Rao Problem)**.** *Given a full-dimensional compact convex body K in $\mathbb{R}^n$, a natural number m, and $n-1$ continuous functions $(f_1, \dots, f_{n-1})$ on the space of all full-dimensional compact convex bodies in $\mathbb{R}^n$, is it possible to find a partition of K into m convex pieces $K = K_1 \cup \dots \cup K_m$ such that for each $1 \leqslant j \leqslant n-1$ the equalities*

$$f_j(K_1) = \dots = f_j(K_m),$$

holds and all pieces in the partition have equal volume?

Generalising even further, one can consider an absolutely continuous probability measure μ on $\mathbb{R}^n$ in place of a full-dimensional compact convex body, and ask for a convex partition $(C_1, \dots, C_m)$ of $\mathbb{R}^n$, such that $\mu(C_1) = \dots = \mu(C_m)$. As we will see, all current approaches to Problem 1.2 work in exactly the same way for these two versions.

In the same paper [KHA14], Karasev, Hubard & Aronov used generalised Voronoi diagrams to show that the answer to this new higher-dimensional version of the Nandakumar & Ramana Rao problem is positive whenever m is such that there exists no $\mathfrak{S}_m$-equivariant map from the configuration space $\mathrm{Conf}(m, \mathbb{R}^n)$ of m points in $\mathbb{R}^n$ to $(\mathbb{R}^{n-1})^m$ which misses the diagonal $\Delta(\mathbb{R}^{n-1})$. The group action in question is induced by an action of the symmetric group $\mathfrak{S}_m$ on $\mathbb{R}^m$ by permuting the components.

Using this criterion, one can attack what was originally a discrete geometry problem with methods from algebraic topology, such as a homological analogue of obstruction theory, used by Karasev, Hubard & Aronov in [KHA14] to give a positive answer for all primes m, or cohomological index theory, which, as demonstrated later by Blagojević, Lück & Ziegler, in [BLZ15] leads to the same result, or equivariant obstruction theory, used by Blagojević & Ziegler in [BZ14] to expand the positive answer to all partitions into a prime power number of pieces. In this thesis, we will apply the same criterion by Karasev, Hubard & Aronov, suitably adjusted for our purposes, combined with the cohomological index approach of Blagojević, Lück & Ziegler, to a parameterised version of Problem 1.2.

Among more recent developments in this field, Blagojević & Sadovek [BS23] use little cube operads to prove an iterated partitions version of the Nandakumar & Ramana Rao problem, once again for partitions into a prime power number of pieces.

Notice, however, that so far we talked only about the version of the problem where in addition to volume one would like to equalise no more than $n-1$ functions, n being the dimension of the initial polytope. What if we wish to equalise more than $n-1$ functions? This is not always possible. For example, if one takes f_i, $1 \leqslant i \leqslant n$ to be the function that sends each convex body in $\mathbb{R}^n$ to the value of the i-th coordinate of its barycentre, there is no such equipartition for this choice of functions.

What about families of full-dimensional compact convex bodies?

Problem 1.3 (Transversal Nandakumar & Ramana Rao problem)**.** *Let K be a full-dimensional compact convex body in $\mathbb{R}^N$ containing the origin in its interior. Consider a family of n-dimensional convex bodies, arising as sections of K by n-dimensional linear subspaces in $\mathbb{R}^N$. Let us fix the desired number m of pieces in a partition and consider any $j > n-1$ pairwise distinct continuous functions defined on the set of full-dimensional compact convex bodies in $\mathbb{R}^n$. For this family and this set of functions, is it true that there exists at least one convex body in the family that admits a convex equipartition onto m pieces in the sense of the classical Nandakumar & Ramana Rao problem, but for $j > n-1$ functions?*

The family of convex bodies we just described can be viewed as parameterised by points of the real Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ of n-dimensional linear subspaces in $\mathbb{R}^N$, while each convex body in this family belongs to its own fibre of the tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$.

Following this logic, the next natural step is to replace the Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ with any suitably well-behaved topological space B, and the tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ with some real rank n vector bundle E over B. The precise statement of this question can be found in Problem 2.5.

The historical background – other problems of the same flavour. The Nandakumar & Ramana Rao problem is just one of many examples of questions in discrete geometry where the solution can be found using methods from the equivariant algebraic topology. One of the most famous and probably the earliest of such examples is the Ham Sandwich theorem [Mau81, Problem 123], stated around 1938 and attributed by different sources to either Steinhaus or Ulam [BZ04]. In its simplest form, the Ham Sandwich theorem states that given a piece of bread, a slice of ham, and a slice of cheese, it is always possible to find one cut that simultaneously divides each of the ingredients into parts of equal volume. Its higher-dimensional generalisation asserts the existence of such a bisection for n-masses in n-dimensional Euclidean space $\mathbb{R}^n$. The three-dimensional version was proved by Banach using the Borsuk–Ulam theorem, which states that there is no $\mathbb{Z}_2$-equivariant map from the sphere S^n to the sphere S^{n-1}. Traditionally, one considers the antipodal action of $\mathbb{Z}_2$ on both spheres, although the statement of the theorem stays true for any two free $\mathbb{Z}_2$-actions. The Borsuk–Ulam theorem and its generalisations have proved to be sufficiently useful tools, that there is a whole book [MBZ03] dedicated to them, which we highly recommend. The Ham Sandwich theorem can be seen as a special case of the Nandakumar & Ramana Rao problem corresponding to the case when $m = 2$ and all $d-1$ functions are induced by some measures on $\mathbb{R}^d$.

Banach's proof establishes a strategy that can be applied to many other problems: Given a geometric or combinatorial problem, one creates a topological space that in some way encodes the set of all potential solutions to this problem (*"configuration space"*). For example, in the case of the Ham Sandwich theorem, this is a three-dimensional sphere S^3, and in the case of the approach of Karasev, Hubard & Aronov to the Nandakumar & Ramana Rao problem, it is the configuration space $\mathrm{Conf}(m, \mathbb{R}^n)$ of m distinct points in $\mathbb{R}^n$. In the next step, one shows that if the problem does not have a solution, or if the converse of the statement holds true, then there exists a *"test"* map from that space of potential solutions to some other topological *"target"* space, such as the space $\mathbb{R}^3 \setminus \{0\}$ for the Ham Sandwich theorem, or the complement of the diagonal $\Delta(\mathbb{R}^{n-1})$ in $(\mathbb{R}^{n-1})^m$ in the case of the Nandakumar & Ramana Rao problem. Usually, the initial geometric problem has some symmetry, reflected by group actions on the test- and target space, and the chosen test map is equivariant reflecting this symmetry. For example, in both the Ham Sandwich theorem and the Nandakumar & Ramana Rao problem there is an action of the symmetric group, $\mathfrak{S}_2$ or $\mathfrak{S}_m$ respectively, permuting the components of the partitions. Finally, to reach a contradiction, one needs to show that no such map exists using some topological properties of these spaces, as did Banach via the Borsuk-Ulam theorem.

Since Banach's proof of the Ham Sandwich theorem, many more applications of algebraic topology to discrete geometry and combinatorics were found, prompting the creation of a new field of topological combinatorics. The Ham Sandwich theorem and the Nandakumar & Ramana Rao problem represent a big family of fair-partition problems, see [RPS22] for a more detailed overview. Other examples include applications to graph colouring problems, of which Kneser's conjecture [Kne56] is probably the most famous one, we direct the reader to [dL04] for a survey of its history and the role it has played in topological combinatorics; evasiveness of graphs, such as, for example, the proof of Karp's conjecture by Kahn, Saks & Sturtevant [KSS84]; embeddings and mapping problems, such as Tverberg-type problems, see [BS18] for their history. Another very old problem is the square peg problem together with its relatives, such as the rectangular peg problem. Proposed in 1911 by Toeplitz [Toe11], its most general form still remains unsolved today, see [Mat14] for the survey of its history. For more details about applications of topological methods in discrete geometry, we refer the reader to [Bjö95, dL12].

Many of the problems that we have listed above follow a common pattern. In these problems, we are given two groups of numerical parameters: The condition parameters (such as the prescribed dimension of a polytope P and a number of functions for the Nandakumar & Ramana Rao problem, or the number of measures and a dimension of the ambient space for the Ham Sandwich theorem) and the target parameters (number of pieces in an equipartition for the Nandakumar & Ramana Rao problem or the Ham Sandwich theorem). The problem then asks whether it is possible, starting from some geometric object characterised by these initial conditions, to achieve a result satisfying the target set of parameters.

The natural question one might ask is whether it is possible to achieve "better" results (such as equiparting more functions or more masses) by asking to do so for only one object of some big family. This question gives rise to the *parameterised* generalisations of classical problems. Schnider [Sch20] shows that given n continuous mass distributions on the k-dimensional linear subspaces of $\mathbb{R}^n$ (informally, one can think of n masses in each of the subspaces such that the masses change continuously when passing from one subspace to another), there is always a subspace in which it is possible to bisect all n masses by just one hyperplane. In contrast, in the "classical" Ham Sandwich theorem case, one can find $k+1$ masses in $\mathbb{R}^k$ such that there is no hyperplane which equiparts them all at the same time, such as for example $k+1$ balls of a small diameter located at the vertices of a k-simplex. We refer to this generalisation as *parameterised*, since one can consider this new problem as a collection of the classical Ham Sandwich theorems, one for each point in the Grassmannian $\mathrm{Gr}(\mathbb{R}^k, \mathbb{R}^n)$, but with additional ingredients added to the sandwich. Notice, that the tautological bundle $\gamma(\mathbb{R}^k, \mathbb{R}^n)$ over $\mathrm{Gr}(\mathbb{R}^k, \mathbb{R}^n)$ can be seen in this case as an ambient space for all mass assignments at the same time. A different parameterised generalisation of the Ham Sandwich theorem was established by Axelrod-Freed & Soberon in [AFS22].

Blagojević, Calles, Crabb & Dimitrijević Blagojević [BCLCDB], formulate a parameterised version of the Grünbaum–Hadwiger–Ramos problem, another classical fair-partition-type problem. They consider the parameterisations by Grassmannians, and, in a follow-up paper, Blagojević & Crabb [BC23] study the case of a general real bundle E over B.

Inspired by the aforementioned results, in this thesis, we study the parameterised version of the Nandakumar & Ramana Rao problem, which we already outlined briefly at the beginning of the section. We state it properly in Problem 1.3 (for the tautological bundle over a Grassmannian) and in Problem 2.5 (for a general bundle).

The historical background – a topological point of view. So far, we have discussed only one side of the interaction between equivariant topology and discrete geometry, namely how the topological methods help to solve geometric problems. In turn, discrete geometry provides topology with completely new settings for its classical objects and theorems, and sets up new computational challenges.

An important example of this is the Borsuk–Ulam theorem. Since its origin in the 1930s ([Bor33], see [MBZ03, p. 25] for the history), it has been a starting point for multiple generalisations. One of the directions of such generalisations starts with the equivalent statements of the original theorem, namely that given any pair of integers $n > m$, for any $\mathbb{Z}_2$-equivariant map f from S^n to $\mathbb{R}^m$ (where $\mathbb{Z}_2$ acts freely on both spaces) there exists a point x_0 in S^n such that $f(-x_0) = -f(x_0)$. The fact that f is $\mathbb{Z}_2$-equivariant implies that $f(x_0) = f(-x_0) = 0$. Let A_f be a subspace of the sphere S^n consisting of all points $x \in S^n$ such that $f(x) = f(-x)$. The Borsuk–Ulam theorem states that A_f is non-empty for all $\mathbb{Z}_2$-equivariant maps f. Twenty years later, Bourgin [Bou55] and Yang [Yan54, Yan55] showed that the dimension of the space A_f is at least $n - m - 1$. These results were generalised by Jaworowski [Jaw81, Jaw04] and Dold [Dol88]. In their work, they consider $\mathbb{Z}_2$-equivariant maps from the sphere bundle $S(E)$ associated with some rank $(n+1)$ real vector bundle E over the base B to a rank n real vector bundle E' over the same base. Once again, the action of $\mathbb{Z}_2$ is assumed to be free on both spaces. Jaworowski and Dold proved a series of lower bounds on the cohomological dimension of the space A_f by comparing the equivariant cohomology of the space $A_f/\mathbb{Z}_2$ with that of B. Fadell [Fad86] and then Fadell & Husseini [FH87, FH88] developed an *ideal valued index theory*, a powerful method, that can be applied to prove more *parameterised Borsuk–Ulam–Bourgin–Yang-type* theorems, that is, theorems about the size of the set A_f associated with the equivariant, fibre-preserving map f between two fibre bundles equipped with a fibrewise action of some group G. Notice, that, in a sense, the word "parameterised" in this case carries the same meaning as in "parameterised" versions of classical geometric problems we discussed in the previous paragraph. Today, many theorems of the fibrewise Borsuk–Ulam–Bourgin–Yang-type exist. The most relevant in the context of this thesis are the results by Volovikov [Vol80] and Jaworowski [Jaw04] where the group G is the cyclic group $\mathbb{Z}_p$ of prime order p. In fact, in this thesis, we use the result of Jaworowski to compute indices of the fibrewise configuration spaces $\mathrm{Fconf}(m, E)$ of m distinct points in fibres of the vector bundle E in the case when m equals an odd prime p and prove a parameterised Borsuk–Ulam-type theorem that provides a range of cases when A_f is non-empty for $\mathbb{Z}_p$-equivariant fibre-preserving maps from the fibre bundle $\mathrm{Fconf}(p, E)$ to the certain trivial vector bundles.

The fibrewise configuration spaces $\mathrm{Fconf}(m, E)$ in the bundle E, are fibrewise versions of a very important class of spaces, namely ordered configuration spaces $\mathrm{Conf}(m, M)$ of m distinct points in a manifold M. Although the origins of the non-fibrewise configuration spaces can be traced back at least as early as Artin's works [Art25, Art47a, Art47b] in the first half of the twentieth century, the systematic study of configuration spaces starts in 1962 with works of Fadell & Neuwirth [FN62a] as well as Fox & Neuwirth [FN62b]. The ordered configuration space $\mathrm{Conf}(m, \mathbb{C})$ of m points on the complex line can be interpreted in many ways. One of them is to view it as a $K(\pi, 1)$ space for a group of pure braids on m strings. As a part of his research related to the algebraic form of Hilbert's 13th problem, Arnold was interested in cohomology rings of braid groups, and in 1969 he published a paper [Arn69] in which the integral cohomology of the ordered configuration space $\mathrm{Conf}(m, \mathbb{C})$ is described for the first time. In this paper, he attributes the idea of the main computation to Fuchs. As a result of their collaboration (refer to [KT12] for the historical details due to Fuchs), a year later, Fuchs [Fuc70] computed the cohomology ring of the unordered configuration space $\mathrm{Conf}(m, \mathbb{C})/\mathfrak{S}_m$ using methods which paved the path for the later configuration spaces–related computations. Among many later results related to the configuration spaces the most relevant in the context of this thesis are computations by Cohen [CLM76, Thm. 5.2 and Thm. 5.3] of cohomology rings of $\mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p$ for a prime p with the integer and twisted coefficients. In particular, the value of the index of $\mathrm{Conf}(p, \mathbb{R}^n)$ can be seen as an immediate consequence of these computations. We will refer to these results on multiple occasions throughout the thesis.

To study such problems as the parameterised Ham Sandwich or the parameterised Nandakumar & Ramana Rao problem, the notion of fibrewise configuration spaces is useful. Informally, one can think about a fibrewise configuration space $\mathrm{Fconf}(m, E)$ of m points in a fibre bundle $F \to E \to B$ as a fibre bundle obtained from E by substituting each of its fibres with $\mathrm{Conf}(m, F)$. We will introduce this notion properly in Definition 2.8. The idea to consider fibrewise configuration space seems to appear first in the work of Duvall & Husch [DH79], motivated by their interest in embeddings of finite covering spaces into bundles. One can see $\mathrm{Fconf}(m, E)$ as a space of all embeddings over B of a trivial covering space $[m] \times B$ into E. In contrast, the main interest of Duvall & Husch lies in embedding non-trivial coverings into trivial bundles. The first systematic treatment of fibrewise configuration spaces as objects of their own was done by Crabb & James [CJ92]. They introduce fibrewise configuration spaces and reprove for the fibrewise case some of the classical results of Fadell and Neuwirth concerning sequences of fibrations. In their later book [CJ98], the same authors extend their results and consider fibrewise configuration spaces in the context of fibrewise homotopy theory. In this thesis, motivated by the parameterised version of the Nandakumar & Ramana Rao problem, we look at the Fadell-Husseini index of $\mathrm{Fconf}(p, E)$ for some real vector bundle E and an odd prime number p. We consider mainly the index with respect to the symmetric group $\mathfrak{S}_p$, and, in the special case when E admits two linearly independent nowhere zero sections, with respect to $\mathbb{Z}_p$. For the precise statements of these results see Section 1.2.

Case when m an even vs. an odd prime. The Ham Sandwich theorem can be considered to be a special case of the Nandakumar & Ramana Rao problem, corresponding to the case when $m = 2$ and all the functions we consider come from some measures on $\mathbb{R}^n$. Although the Nandakumar & Ramana Rao problem for $m = 2$ holds for a bigger class of possible functions, the proof of these two theorems are identical: Both follow directly from the Borsuk–Ulam theorem. Indeed, according to the strategy by Karasev, Hubard & Aronov, in order to prove the generalised Nandakumar & Ramana Rao problem in this case, we need to show that there exists no $\mathbb{Z}_2$-equivariant map from $\mathrm{Conf}(2, \mathbb{R}^n)$ to $S(\mathbb{R}^{n-1})$. Observe, that the configuration space $\mathrm{Conf}(2, \mathbb{R}^n)$ is homotopy-equivalent to the sphere S^{n-1} and the claim follows.

The same holds for the parameterised case: The parameterised Nandakumar-Ramana Rao problem follows from the parameterised Borsuk–Ulam theorem for sphere bundles with $\mathbb{Z}_2$-actions. This is exactly the case considered by Jaworowski [Jaw81, Jaw04] and Dold [Dol88]. The parameterised Ham Sandwich result proved by Schnider [Sch20] is a special case of this.

Thus, for the rest of this thesis, we consider only the cases when m is an odd prime.

1.2 Statements of the main results, examples, and overview

Notation 1.4. We identify the ring $H^*(\mathrm{B}\mathbb{Z}_p)$ with $\Lambda[e] \otimes \mathbb{F}_p[t]$, where e has the degree 1 and t has the degree 2. We identify the ring $H^*(\mathrm{B}\mathfrak{S}_p)$ with $\Lambda[a] \otimes \mathbb{F}_p[b]$, where a has the degree $2p - 3$ and b has the degree $2p - 2$, see [Knu18, Thm. 8.1.3, due to Nakaoka].

Let $\mathbb{Z}_p \xrightarrow{reg} \mathfrak{S}_p$ be the regular embedding. It induces a map between classifying spaces $\mathrm{B}\mathbb{Z}_p \to \mathrm{B}\mathfrak{S}_p$, and, in turn, a map reg^* in cohomology, from $H^*(\mathrm{B}\mathfrak{S}_p)$ to $H^*(\mathrm{B}\mathbb{Z}_p)$. We choose generators e, t, a, b such that $reg^*(a) = et^{p-2}$ and $reg^*(b) = t^{p-1}$.

Traditionally one uses $\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$ as the name for reg^*, reflecting the fact that this homomorphism corresponds to the restriction of the symmetric $\mathfrak{S}_p$ onto the group $\mathbb{Z}_p$. However, by abuse of notation, we denote by $\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$ the map

$$H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p) \xrightarrow{\mathrm{id} \otimes reg^*} H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$$

for any space B, since it will be always clear from the context which particular B we are using.

For more details about the regular embedding and related map in cohomology see [AM13, Ex. 2.7 on p. 100] or a brief overview in Section 3.1.

Since the results unavoidably use definitions appearing in the later parts of the thesis, we repeat these definitions and provide a reference to the place in the subsequent chapters where they appear naturally.

Definition 4.1. Let $\mathbb{F}^n \to E \to B$ be a vector bundle, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Recall that any vector bundle can be equipped with a Euclidean structure ([Hat17, Prop. 1.2]), allowing us to take orthogonal complements. Define W_pE to be a vector bundle over B, with total space W_pE that is the orthogonal complement of the diagonal ΔE in E^p. Equip W_pE with a $\mathfrak{S}_p$ group action, inherited from the action on E^p that permutes its components.

Definition 4.4. Given a complex vector bundle $\mathbb{C}^d \to E \to B$ and a prime number p define $\check{\mathrm{c}}\mathrm{h}(E) \in H^{2d(p-1)}(B \times \mathrm{B}\mathbb{Z}_p)$ by

$$\check{\mathrm{c}}\mathrm{h}(E) \coloneqq \prod_{1 \leqslant r \leqslant p-1} \big((rt)^d + c_1(E)(rt)^{d-1} + \cdots + c_d(E)\big),$$

where $c_i(E)$ is the i-th Chern class of E.

Example 1.5. Let E be a tautological bundle $\gamma\left(\mathbb{C}^1, \mathbb{C}^{n+1}\right)$ over the complex projective space $\mathbb{C}P^n$ viewed as the Grassmannian $\mathrm{Gr}(\mathbb{C}^1, \mathbb{C}^{n+1})$.

$$\check{\mathrm{c}}\mathrm{h}\big(\gamma\left(\mathbb{C}^1, \mathbb{C}^{n+1}\right)\big) = \prod_{1 \leqslant r \leqslant p-1} (rt + c_1) = -t^{p-1} + c_1^{p-1},$$

where c_1 is the first Chern class of the bundle $\gamma\left(\mathbb{C}^1, \mathbb{C}^{n+1}\right)$ and a generator of the cohomology ring of $\mathbb{C}P^n$, $H^*(\mathbb{C}P^n) = \mathbb{F}_p[c_1]/c_1^{n+1}$.

Theorems 4.5 & 4.7 combined with 4.29 (Indices of some sphere bundles)**.** *Let $\mathbb{R}^n \to E \to B$ be a real vector bundle and $\mathbb{C}^n \to E \otimes_{\mathbb{R}} \mathbb{C} \to B$ its complexification. Then the ideal $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big)$ in the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$ is equal*

$$\begin{aligned}\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big) &= \langle \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \rangle \\ &= \langle \big(\prod_{1 \leqslant r \leqslant p-1} \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i p_i(E)(rt)^{n-2i}\big)^{\frac{1}{2}} \rangle.\end{aligned}$$

In contrast, the index of $S\big(W_pE\big)$ with respect to $\mathfrak{S}_p$ depends on the parity of n:

1. *When n is even, $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}$ belongs to the image of $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$. Let us name ζ_E its preimage in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$. Then ζ_E generates the whole index ideal $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$ in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$.*

2. *When n is odd, the ideal* $\operatorname{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$ *is generated by two elements,* ε_E *and* θ_E *in* $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$, *such that*

$$\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}\varepsilon_E = et^{\frac{p-3}{2}}\check{\mathrm{c}}\mathrm{h}(E\otimes_{\mathbb{R}}\mathbb{C})^{\frac{1}{2}}\,,$$
$$\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}\theta_E = t^{\frac{p-1}{2}}\check{\mathrm{c}}\mathrm{h}(E\otimes_{\mathbb{R}}\mathbb{C})^{\frac{1}{2}}\,.$$

Notation 1.6. In the formula above and throughout the rest of the thesis we use the following conventions: Since we are working with $\mathbb{F}_p$ coefficients, via the Künneth formula $H^*(B\times\mathrm{B}\mathbb{Z}_p)\cong H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$. We identify $H^*(B)$ with

$$H^*(B)\otimes 1\subsetneq H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p),$$

and $H^*(\mathrm{B}\mathbb{Z}_p)$ with

$$1\otimes H^*(\mathrm{B}\mathbb{Z}_p)\subseteq H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p),$$

to simplify formulae visually and to emphasise that in a sense we are working with polynomials in t over $H^*(B)$, with an occasional shift by e.

Remark 1.7. Although the square root of $\check{\mathrm{c}}\mathrm{h}(E\otimes_{\mathbb{R}}\mathbb{C})$ is defined only up to a sign, since we are interested in the ideal, generated by it, the ambiguity of the sign does not matter.

Example 1.8.

- Consider the Grassmannian $\mathrm{Gr}(\mathbb{R}^2,\mathbb{R}^{N+2})$. Its cohomology ring with coefficients in $\mathbb{F}_p$ is isomorphic to

$$\mathbb{F}_p[p_1,\bar{p}_1,\dots,\bar{p}_{\lfloor N/2\rfloor}]/\langle(1+p_1)(1+\bar{p}_1+\cdots+\bar{p}_{\lfloor N/2\rfloor})-1\rangle\cong\mathbb{F}_p[p_1]/p_1^{\lfloor N/2\rfloor+1},$$

where p_i and $\bar{p}_i$ are ith Pontryagin classes of the bundles $\gamma(\mathbb{R}^2,\mathbb{R}^{N+2})$ and $\bar{\gamma}\left(\mathbb{R}^2,\mathbb{R}^{N+2}\right)$ respectively, see Fact 3.1. In particular, $\deg(p_i)=\deg(\bar{p}_i)=4i$.

Then the class $\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^2,\mathbb{R}^{2+N})\otimes_{\mathbb{R}}\mathbb{C}\big)$ by definition equals

$$\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^2,\mathbb{R}^{2+N})\otimes_{\mathbb{R}}\mathbb{C}\big)=\prod_{1\leqslant r\leqslant p-1}\big((rt)^2-p_1\big).$$

Using the fact that the polynomial $t^{p-1}-1$ over the field $\mathbb{F}_p$ can be expressed as

$$t^{p-1}-1=\prod_{1\leqslant r\leqslant p-1}(t+r),$$

we compute

$$\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^2,\mathbb{R}^{2+N})\otimes_{\mathbb{R}}\mathbb{C}\big)=t^{2(p-1)}-2t^{p-1}p_1^{(p-1)/2}+p_1^{p-1}.$$

Therefore the generator of the index ideal $\operatorname{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^2,\mathbb{R}^{N+2})} S\big(W_p\gamma(\mathbb{R}^2,\mathbb{R}^{N+2})\big)$ in the ring $H^*(\mathrm{Gr}(\mathbb{R}^2,\mathbb{R}^{N+2})\times \mathrm{B}\mathbb{Z}_p)$ equals, up to a sign,

$$\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^2,\mathbb{R}^{2+N})\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}} = t^{p-1} - p_1^{(p-1)/2}.$$

In particular, we see that it belongs to the image of $\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$, and

$$\zeta_{\gamma(\mathbb{R}^2,\mathbb{R}^{N+2})} = b - p_1^{(p-1)/2}$$

in this case.

- Consider the Grassmannian $\mathrm{Gr}(\mathbb{R}^3,\mathbb{R}^{N+3})$. The polynomial part of its cohomology ring is again isomorphic
$$\mathbb{F}_p[p_1]/p_1^{\lfloor N/2\rfloor+1},$$
where this time p_1 is the first Pontryagin class $p_1\left(\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})\right)$ of the tautological bundle $\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})$. We compute that $\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}}$ equals in this case, up to a sign,
$$\check{\mathrm{c}}\mathrm{h}\big(\gamma(\mathbb{R}^3,\mathbb{R}^{3+N})\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}} = \prod_{1\leqslant r\leqslant p-1}\big((rt)^3 - p_1 t\big)^{\frac{1}{2}} = t^{\frac{1}{2}}\big(t^{p-1} - p_1^{(p-1)/2}\big)$$
Therefore we compute the ideal
$$\operatorname{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^3,\mathbb{R}^{N+3})} S\big(W_p\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})\big) = \langle t^{\frac{3(p-1)}{2}} - t^{\frac{p-1}{2}} p_1^{(p-1)/2}\rangle$$
in $H^*(\mathrm{Gr}(\mathbb{R}^3,\mathbb{R}^{N+2})\times \mathrm{B}\mathbb{Z}_p)$. This element clearly does not belong to the image of $\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$. By definition in this case,
$$\varepsilon_{\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})} = ab - ap_1^{(p-1)/2},$$
$$\theta_{\gamma(\mathbb{R}^3,\mathbb{R}^{N+3})} = b^2 - bp_1^{(p-1)/2}.$$

To state the next group of theorems, we need one additional definition:

Definition 4.17. Let x be some element from the ring $H^*(B)$. Denote by $\operatorname{Ann}\big(x\big)$ the ideal in $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$ consisting of all elements Q satisfying the following two conditions:

- Q belongs to the subring $H^*(B)\otimes \mathbb{F}_p[b]$ of $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$.
- Using the previous point, consider Q as a polynomial in b. Its free term should belong to the annihilator of x in $H^*(B)$.

Example 1.9. In the case when the base $B = \mathrm{Gr}(\mathbb{R}^2, \mathbb{R}^{N+2})$ and we take bundle $E = \gamma(\mathbb{R}^2, \mathbb{R}^{N+2})$, the ideal $\mathrm{Ann}\big(p_1^{(p-1)/2}\big)$ consists of all polynomials in b with coefficients in the ring $H^*(B)$ such that their free term, up to a multiplication by a scalar $k \in \mathbb{F}_p$, equals p_1^i with $i \geq \lfloor N/2 \rfloor + 1 - \frac{p-1}{2}$.

Theorems 4.15, 4.18 and 4.22, combined with 4.29 (Indices of configuration spaces)**.** *Let $\mathbb{R}^n \to E \to B$ be a vector bundle over the base B,*

- *When n is odd,* $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E) = \mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p E\big)$.
- *When n is even, recall, that in Theorem 4.7 we established that there exists an element ζ_E in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ such that $\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} \zeta_E = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$.*

  The following bound on the ideal $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ holds

$$\langle \zeta_E \rangle \subsetneq \mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E) \subseteq \langle \zeta_E \rangle + ab^{-1}\Big(\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big).$$

  The sum above is understood as a sum of ideals.

- *There are two special families of vector bundles E of even rank for which we can we can show that the upper bound above can be achieved. In case E admits two linearly independent nowhere zero sections, that is $E = E' \oplus \underline{\mathbb{R}}^2$ or when the cohomological dimension of B (that is, the highest degree in wich it has non-zero cohomology) is smaller than $n(p-2)$, it holds that*

$$\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E) = \langle ab^{-1}\zeta_E, \zeta_E \rangle.$$

- *Additionally, in the case when E is isomorphic to the direct sum $E' \oplus \underline{\mathbb{R}}^2$ for some vector bundle E', non-dependent on the parity of the rank E we can compute the index with respect to $\mathbb{Z}_p$ action*

$$\mathrm{Index}_B^{\mathbb{Z}_p} \mathrm{Fconf}(p, E) = \langle et^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}, t^{\frac{p+1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}} \rangle.$$

Remark 1.10. Observe that when E is of an even rank and admits two linearly independent sections then its top Pontryagin class $p_{n/2}$ is zero, meaning it is annihilated by the whole ring $H^*(B)$. Also, observe, that the class $p_{n/2}(E)^{\frac{p-1}{2}}$ has degree $n(p-1)$, therefore if the cohomological dimension of B is smaller than $n(p-2)$ then this class is necessarily equals to zero. In both these cases the ideal $ab^{-1}\Big(\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big)$ is precisely the ideal generated by $ab^{-1}\zeta_E$.

Let us look again at some examples:

Example 1.11. Let E be a trivial bundle $\mathbb{R}^n$ over the base $B = \mathrm{pt}$. Then the fibrewise configuration space $\mathrm{Fconf}(p, E)$ coincides with the usual configuration space $\mathrm{Conf}(p, \mathbb{R}^n)$. Let us use our formulae for the indices of fibrewise configuration spaces and compare these results with computations by Cohen [CLM76, Thm. 5.2 and Thm. 5.3]. In this case the class $\check{\mathrm{c}}\mathrm{h}(\mathbb{R}^n \otimes_{\mathbb{R}} \mathbb{C})$ is simply $t^{n(p-1)}$. Trivially, $n(p-1)$ is bigger than the cohomological dimension of B for any positive n, therefore the index ideal of the configuration space $\mathrm{Fconf}(p, \mathbb{R}^n)$ equals

$$\mathrm{Index}_{\mathrm{pt}}^{\mathfrak{S}_p} \mathrm{Fconf}(p, \mathbb{R}^n) = \begin{cases} \langle ab^{\frac{n-1}{2}} b^{\frac{n+1}{2}} \rangle, & \text{when } n \text{ is odd,} \\ \langle ab^{\frac{n}{2}-1}, b^{\frac{n}{2}} \rangle, & \text{when } n \text{ is even.} \end{cases}$$

Also,

$$\mathrm{Index}_{pt}^{\mathbb{Z}_p} \mathrm{Fconf}(p, \mathbb{R}^n) = \langle et^{\frac{(n-1)(p-1)}{2}}, t^{\frac{(n-1)(p-1)}{2}+1} \rangle$$

This is exactly the result that follows from Cohen's work.

Example 1.12. Let us consider again the base $B = \mathrm{Gr}(\mathbb{R}^2, \mathbb{R}^{N+2})$ and the vector bundle $E = \gamma(\mathbb{R}^3, \mathbb{R}^{N+3})$. Then we see that when $p - 1 \geq 2\lfloor N/2 \rfloor + 2$, the element $p_1^{(p-1)/2}$ equals 0 in the ring $H^*(\mathrm{Gr}(\mathbb{R}^2, \mathbb{R}^{N+2}))$, therefore $\mathrm{Ann}\big(p_1^{(p-1)/2}\big)$ coincides with the whole ring.

The cohomological dimension of the Grassmannian $\mathrm{Gr}(\mathbb{R}^2, \mathbb{R}^{N+2})$ with $\mathbb{F}_p$-coefficients equals $4\lfloor N/2 \rfloor$. Therefore, precisely when $p - 1 \geq 2\lfloor N/2 \rfloor + 2$, we can compute the index of fibrewise configuration space $\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^2, \mathbb{R}^{N+2})\big)$. Under these conditions it equals

$$\mathrm{Index}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \mathrm{Fconf}\big(p, \gamma(\mathbb{R}^2, \mathbb{R}^{N+2})\big) = \langle a, b \rangle,$$

an ideal in the ring

$$H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p) \cong H^*(B) \otimes \Lambda[a] \otimes \mathbb{F}_p[b].$$

The computations of indices of fibrewise configuration spaces allow us to prove the following geometric result:

Theorem 5.4 (Transversal Nandakumar & Ramana Rao problem)**.** *The transversal Nandakumar & Ramana Rao Problem 1.3 admits solutions for all* $\big(\gamma(\mathbb{R}^n, \mathbb{R}^N), m = p, j\big)$ *such that* p *– an odd prime number,* $p \leqslant n$ *and* $j \leqslant N - 1$ *when* $N - n$ *is even, or* $j \leqslant N - 2$ *when* $N - n$ *is odd. Additionally, for* $p = n + 1$ *and odd* n *the bound on* j *is worse by* 1*.*

Example 1.13. The original Nandakumar & Ramana Rao problem corresponds to the case the base B is just a point, E is the vector space $\mathbb{R}^n$ considered as a vector bundle over a point and $n = N$. We consider a partition onto $m = p$ prime number of pieces. In this case, we get exactly the same result as was already known: $j \leqslant n - 1$, meaning that for any n-dimensional compact convex body K it is possible to find a partition onto p convex bodies such that any chosen in advance $n - 1$ appropriately defined functions are equalised in the sense of the Nandakumar & Ramana Rao problem. Although the statement of the theorem above has an additional condition that $p \leqslant n$, in this particular case the proof actually works for any p.

Example 1.14. Take any six-dimensional polytope $P \subsetneq \mathbb{R}^6$ centred at the origin. Theorem 5.4 tells us that one can choose a four-dimensional linear subspace $V \subset \mathbb{R}^6$, and an equipartition of $P \cap V$ into three convex pieces $P \cap V = P_1 \cup P_2 \cup P_3$ of equal volume such that all of the three pieces also have an equal surface area, diameter, and the first three coordinates (with respect to the standard coordinate system in $\mathbb{R}^6$) of their barycenters.

Remember that if one considers only one fixed four-dimensional polytope P, then the Nandakumar & Ramana Rao problem says that only three functions can be maximised in this way [BZ14, Thm. 1.3].

Notice also the restrictions of Theorem 5.4: First of all, the number of pieces in the partition should be small compared to the dimensions of the sections. If we are interested in partitions into five convex pieces, then since $p = 5 = n + 1$, Theorem 5.4 guarantees only four functions equalised in some four-dimensional section of P, and for any bigger prime number of pieces in partition, we can not say anything new compared to the original case.

Another important detail in this theorem is the parity of the difference in dimension between the polytope and its linear cuts. For example, substituting a six-dimensional polytope with a seven-dimensional one brings no gain on the number of functions that can be maximised, since $N - n = 7 - 4 = 3$ is odd in this case.

More generally, one can consider the family of full-dimensional compact convex bodies parameterised by some general bundle E over B of rank n and aim to find an equipartition in the sense of the Nandakumar & Ramana Rao problem for one of them. Let $f_1, \dots, f_j$ be continuous functions on the space $\mathcal{K}(E)$ of all compact n-dimensional convex bodies in fibres of E (see Definition 2.4). Let s be continuous from the base B to the space $\mathcal{K}(E)$ that matches a point $x \in B$ with a n-dimensional compact convex body $s(x)$ in the fibre of E over x. Fix an integer number m. We say that the parameterised generalisation of Fibrewise Nandakumar & Ramana Rao problem has a solution for this choice of functions $f_1, \dots, f_j$, the section s and the number m if there is $x \in B$, such that the convex body $s(x)$ admits a partition in p convex pieces $(K_1 \cup \dots \cup K_m)$ such that they all have equal volume and for any $1 \leqslant i \leqslant j$ the function f_i has an equal value on all these parts, that is $f_i(K_1) = \cdots = f_i(K_m)$.

It turns out that if we can prove that given a vector bundle E and natural numbers m and j there is a solution for some choice of s and $f_1, \ldots, f_j$, then there is a solution for any s and any j continuous functions from $\mathcal{K}(E)$ to $\mathbb{R}$. In this case, we say that the parameterised generalisation of the fibrewise Nandakumar & Ramana Rao problem has a solution for a tuple (E, p, j).

We prove the following theorem:

Theorem 5.1 (Parameterised Nandakumar & Ramana Rao problem)**.** *Fix an odd prime p. Consider a real vector bundle $\mathbb{R}^n \to E \to B$. Then the parameterised generalisation of fibrewise Nandakumar & Ramana Rao problem, stated in Problem 2.5 has a solution for all tuples (E, p, j) such that*

$$j \leqslant \begin{cases} \operatorname{rank} E + I - 2, & \textit{if } \operatorname{rank} E \textit{ is even,} \\ \operatorname{rank} E + I - 1, & \textit{if } \operatorname{rank} E \textit{ is odd.} \end{cases}$$

where I can be determined in one of the following ways:

- *Choose a vector bundle $\bar{E}$ such that the direct sum $E \oplus \bar{E}$ is isomorphic to a trivial bundle. Then*

$$I = \max_i \{ {}_p c_i(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}) \neq 0 \} = \max_i \{ \mathrm{P}^i \, u_{\bar{E}^{\mathbb{C}}} \neq 0 \}.$$

  Here P^i denotes i-th Steenrod mod p power, $u_{\bar{E}^{\mathbb{C}}}$ is the Thom class of the bundle $E \otimes_{\mathbb{R}} \mathbb{C}$ and ${}_p c_i(\bar{E} \otimes_{\mathbb{R}} \mathbb{C})$ are as defined in 3.10.

- *Alternatively, denote by $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ the inverse of $c(E \otimes_{\mathbb{R}} \mathbb{C})$ in the ring of infinite series $H^{\Pi}(B; \mathbb{F}_p)$ and by $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}(t) \in H^*(B; \mathbb{F}_p) \otimes \mathbb{F}_p[t]$ its homogenisation. Then $2I(p-1)$ is the biggest degree among non-zero coefficients of polynomial in t given by $\left(\prod_{1 \leqslant r \leqslant p-1} c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}(rt) \right)^{\frac{1}{2}}$.*

It turns out that I is always an even number. In the very special case when $\operatorname{rank} E$ is also even and E admits two linearly-independent nowhere zero sections, that is, $E \cong E' \oplus \underline{\mathbb{R}}^2$, this bound can be improved by 1. In this case, a maximal j attainable equals $\operatorname{rank} E + I - 1$ – an odd number, which we could not get in the more general case.

How this thesis is organised. The last section of this chapter contains a list of questions, arising from the results we just presented, that are left unanswered.

In Chapter 2 we formulate the two main questions that we investigate in the rest of this thesis. First, in Section 2.1 we make the statement of the parameterised Nandakumar & Ramana Rao problem precise. Then in Section 2.2 we look at how the original Nandakumar & Ramana Rao problem was translated to the language of equivariant algebraic topology and perform a similar reformulation with the parameterised Nandakumar & Ramana Rao problem.

In Chapter 3 we provide a brief introduction to some of the facts from algebraic topology that play a key part in our proofs.

Chapter 4 contains all index computations. We start by collecting a couple of simple observations in Section 4.1. The next Section 4.2 is concerned with indices of sphere bundles containing fibrewise configuration spaces: We extend the results of Jaworowski [Jaw04] and Crabb (private communication). Finally, in Section 4.3 we turn our attention to indices of fibrewise configuration spaces $\mathrm{Fconf}(p, E)$ for an odd prime p and a vector bundle E. We start with the assumption that E is orientable and consider three different cases. First, we compute the index with respect to the symmetric group action in the case when the rank E is odd. Our main tool is the Leray–Hirsch theorem. Then, we use these computations to find non-strict lower and upper bounds on this ideal for the case when the rank E is even. It turns out that the lower bound can be made strict by the cost of applying heavy machinery of an appropriate Leray–Serre spectral sequence. In the last part of this section, we compute the index with respect to $\mathbb{Z}_p$-action for all real vector bundles E that have two linearly-independent nowhere zero sections, using an appropriate spectral sequence in a similar way as we did for $\mathfrak{S}_p$-action and E of an even rank. In both parts involving spectral sequences, we make extensive use of the computations of the cohomology rings $H^*(\mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p)$ and $H^*(\mathrm{Conf}(p, \mathbb{R}^n)/\mathbb{Z}_p)$ done by Cohen [CLM76], applying some of his spectral sequences arguments to our case. In the last section of this chapter, Section 4.4, we prove that our results stay true for the case of a non-orientable bundle E.

In Chapter 5 we return to the parameterised Nandakumar & Ramana Rao problem. In Section 5.1, using index computations from the previous chapter, we obtain an algebraic criterion such that for any tuple (E, p, j) satisfying this criterion, Problem 1.2 has a solution. Afterwards, we apply these results in Section 5.2 to the particular case when E is a tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over the Grassmanninan $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$. The criterion for the general case is not optimal in the case when E has an even rank. In the case of the tautological bundle over the Grassmannian, we can improve our results for an even n by a trick specific to this situation.

1.3 Open questions

In this section, we collect the list of questions that have arisen during the work on this thesis and remain unanswered. Following the structure of this thesis, they fell into two categories – questions related to parameterised Nandakumar & Ramana Rao problem, and questions arising from the index computations.

Geometric question:

Open question 1. In the proof of Theorem 5.4 we have found a trick that helped us to get a better result than Theorem 5.1 provides for a general bundle E. However, from Remark 5.3 we know that there is still room for improvement. Is it possible to find some way to prove that there is always a solution to the Problem 1.3 for the set of parameters $(\gamma(\mathbb{R}^{p-1}, \mathbb{R}^N), p, N-1)$ when N is even, and for the set of parameters $(\gamma(\mathbb{R}^{p-1}, \mathbb{R}^N), p, N-1)$ when N is odd?

Currently, we have the restriction $p \leqslant n$, which feels like an artificial one, a byproduct of our choice of approach, rather than dictated by the geometry of the problem.

Open question 2. More generally, is it true that for any bundle E of an even rank n, there is a solution to Problem 1.2 with parameters $(E, p, n + I - 1)$ where p is an odd prime and I is the same as in Theorem 5.1?

We have seen two indications that this might be true. We know that this is true for all bundles E that admit two linear independent nowhere zero sections, and we have seen it for some tautological bundles over Grassmannians.

If it is not true for all bundles, is it possible to find a criterion that differentiates between bundles E when it holds and those when it does not?

Open question 3 (Proposed by F.Frick). In the current version of Theorem 5.4 there is a difference in the answer, depending on whether $N - n$ is even or odd. In the odd case, the statement is weaker. Can this be rectified by some geometric argument, using the results for the even case?

An example of the smallest dimension would be a three-polytope and its two-dimensional sections. However, in this case, $n = 2$, wich implies very tight restrictions on possible number p. The next example would be $p = 3, n = 3, N = 4$. In this case, Theorem 5.4 guarantees an equipartition of $4 - 2 = 3 - 1$ functions. This does not give us any gain compared to the original problem. Is it possible to show that there always exists an equipartition of any set of four appropriate functions? Alternatively, is there an example of a four-polytope P and four functions on the space of all three-dimensional convex bodies, such that no three-dimensional section of P admits an equipartition with respect to these four functions?

Open question 4 (compare with open question 7). In his parameterised Ham Sandwich theorem, Schnider [Sch20], not only proves that given a N continuous mass distributions in n-dimensional linear subspaces in $\mathbb{R}^N$, one can find such a subspace in which it is possible to bisect all N measures by one hyperplane but also that there is some degree of control over which linear subspace it is: If one aims to equipart $N-n+2$ masses, then one can choose this linear subspace to contain any chosen $k-1$ linearly independent vectors.

Is it possible to prove the results of a similar flavour for the transversal Nandakumar & Ramana Rao problem? For the parameterised Nandakumar & Ramana Rao problem?

Open question 5 (Proposed by F.Frick). What other families of convex bodies can be considered?

Algebraic topology questions:

Open question 6. Theorem 4.16 describes the $H^*(B)$-module structure of the cohomology of unordered fibrewise configuration space $\mathrm{Fconf}(p, E)$ in a vector bundle E of an odd rank. Is it possible to get a similar statement for the case when rank E is even? For example, is it possible to describe the structure of $H^*(\mathrm{Fconf}(p, E)/\mathfrak{S}_p)$ as a $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$-module?

The computations we did to prove Theorem 4.18 provide us with a description of the spectral sequence associated with the fibre bundle (4.12). However, this description is not complete. In notation from Picture 4.3 we can formulate questions one needs to answer to understand the module structure in question.

- What distinguishes cases with $P = 1$?
- More generally, what are the possible values of P for a given vector bundle E (remember that there can be many of them)? Notice, that although we have drawn an element $P \otimes X$ as if it has a bigger degree than $1 \otimes Y$, as we have seen in the proof of Theorem 4.18 P can be of any degree, including zero.
- What is the precise value of $d_{(n-1)(p-1)+1}(1 \otimes X)$? It will be interesting to see whether it indeed can be something else besides ζ_E and what influences this.
- What elements of the $(n-1)$ row are in the image of the differential d_n? Do they have a geometric interpretation?

Notice, that if the class $p_{n/2}^{p-1/2} \neq 0$, it follows that $ab^{-1}\zeta_E$ is not allowed in the ring $H^*(B) \otimes \mathrm{B}\mathfrak{S}_p$, therefore the element $1 \otimes X$ can not survive till the last page, implying that $d_{(n-1)(p-2)+1}(1 \otimes X)$ is non-zero. For example, this is the case when E is isomorphic to a tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ such that $p - 1 < N - n$.

Open question 7 (compare with open question 4). We have proved a Borsuk–Ulam-type result. Can we prove the corresponding Burgin-Yang-type result as well? In other words, given an odd prime p, a vector space E over base B and a trivial vector bundle W over the same base, what can be said about the size of the space of all points x in $\mathrm{Fconf}(p, E)$ such that their whole orbit (with respect to $\mathbb{Z}_p$ or $\mathfrak{S}_p$ action) is mapped to the zero section of W?

Open question 8. We have computed the index with respect to $\mathbb{Z}_p$ of the sphere bundle $S\big(W_pE\big)$. As one can see from the proof, this method works for the sphere bundle associated with any complex bundle of the form

$$\bigoplus_{i \in I} E_i \otimes L_i,$$

where I is a finite index set, E_i, $i \in I$ is a vector bundle over B, on which the trivial action of $\mathbb{Z}_p$ is assumed and L_i is a trivial line bundle over B with some action of $\mathbb{Z}_p$. Is it true, that any vector bundle E can be represented in this form?

Chapter 2

The parameterised Nandakumar & Ramana Rao problem, statements

2.1 The precise statement

We begin by fulfilling the promise from the introduction and make the statement of the parameterised version of Problem 1.2 precise.

Recall, that on the intuitive level, it asks whether given

- a family of full-dimensional compact convex bodies parameterised by some general bundle E over B of rank n,
- j continuous functions $f_1, \ldots, f_j$ be continuous functions on the space $\mathcal{K}(E)$ (which we will define soon properly) of all compact n-dimensional convex bodies in fibres of E,
- a continuous map $s\colon B \to \mathcal{K}(E)$ that matches a point $x \in B$ with a n-dimensional compact convex body $s(x)$ in the fibre of E over x,
- and a natural number m,

it is possible to find point $x \in B$ such that there exists an equipartition of $s(x)$ in the sense of the Nandakumar & Ramano Rao problem onto m pieces with respect to functions $f_1, \ldots, f_j$.

In these terms, Problem 1.1 corresponds to the case $B = \mathrm{pt}$, $E = \mathbb{R}^2$ and s that sends pt to the chosen convex body K. Problem 1.3 corresponds to the particular case when B is the Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$, E is the tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ and s matches a point

$[x]$ of the Grassmannian with $K \cap V_x$, where K is some fixed full-dimensional compact convex body in $\mathbb{R}^N$ and V_x is an n-dimensional linear subspace in $\mathbb{R}^N$ corresponding to x.

With these examples in mind, let us start the formal part of this section. Fix $n \in \mathbb{N}$. Let $\mathcal{K}$ denote the set of all full-dimensional compact convex bodies in $\mathbb{R}^n$. There are two standard choices of a metric on this space: the Hausdorff metric d_H and the symmetric difference d_S.

Definition 2.1 ([SW65])**.** For any two full-dimensional compact convex bodies K, L in $\mathcal{K}$ define

$$\begin{aligned} d_H(K, L) &\coloneqq \max\{\max_{x \in K} \operatorname{dist}(x, L), \max_{y \in L} \operatorname{dist}(K, y)\}, \\ d_S(K, L) &\coloneqq \mathcal{L}^n(K \Delta L), \end{aligned}$$

where dist is the usual Euclidean distance from a point to a non-empty closed set, $K \Delta L$ stands for the symmetric difference of the two sets K and L, and $\mathcal{L}^n$ denotes the Lebesgue measure on $\mathbb{R}^n$.

Remark 2.2. For a full-dimensional compact convex body $K \subsetneq \mathbb{R}^n$ we use the notion of the volume and the Lebesgue measure of the underlining set K interchangeably.

Lemma 2.3 ([SW65])**.**

- *Both the Hausdorff distance function d_H and the symmetric difference d_S define a metric on $\mathcal{K}$.*
- *As topological spaces, $(\mathcal{K}, d_H)$ and $(\mathcal{K}, d_S)$ are homeomorphic.*

From now on, we consider $\mathcal{K}$ to be a topological space with the topology induced by these metrics.

Let $\mathbb{R}^n \to E \xrightarrow{\pi} B$ be a vector bundle of rank n. Denote by $\mathcal{K}(E)$ the fibre bundle over B obtained by substituting each of the fibres of E by the space $\mathcal{K}$ of all full-dimensional compact convex bodies in $\mathbb{R}^n$. To be precise:

Definition 2.4. Let $\bigcup_{i \in I} U_i = B$ be an open cover of B such that E is trivialisable over each of the U_i, that is, for any $i \in I$ there exists a homeomorphism $\varphi_i \colon \pi^{-1}(U_i) \to U_i \times \mathbb{R}^n$ which is linear when restricted to $\pi^{-1}(x)$ for all $x \in U_i$. Denote by φ_{ij} an automorphism of $(U_i \cap U_j) \times \mathbb{R}^n$ defined as the composition of the restriction of $\varphi_i^{-1} \restriction_{\varphi_i(U_i \cap U_j)}$ with φ_j.

$$\varphi_{ij} \coloneqq \varphi_j \circ \varphi_i^{-1} \restriction_{\varphi_i(U_i \cap U_j)} \colon (U_i \cap U_j) \times \mathbb{R}^n \to (U_i \cap U_j) \times \mathbb{R}^n.$$

Notice that, for any $x \in U_i \cap U_j$, it holds that $\varphi_{ij}(x, -)$ is an automorphism of the vector space $\{b\} \times \mathbb{R}^n$. Therefore it sends full-dimensional compact convex bodies in $\{x\} \times \mathbb{R}^n$ to full-dimensional compact convex bodies. Consequently $\varphi_{ij}(x, -)$ induces a one-to-one map $\mathcal{K}(\varphi_{ij})(x, -)\colon \{x\} \times \mathcal{K} \to \{x\} \times \mathcal{K}$. Using the metric d_S we see that $\mathcal{K}(\varphi_{ij})(x, -)$ is continuous. Indeed, for any pair of full-dimensional compact convex bodies $K, L \subsetneq \mathbb{R}^n$, observe that,

$$\mathcal{L}^n(\varphi_{j,i}(K)\Delta\varphi_{j,i}(L)) = \mathcal{L}^n(\varphi_{j,i}(K\Delta L)) = |\det \varphi_{j,i}|\mathcal{L}^n(K\Delta L),$$

which implies continuity of the $\mathcal{K}(\varphi_{ij})(x, -)$. The same holds for $\mathcal{K}(\varphi_{ji})(x, -)$. Define $\mathcal{K}(\varphi_{ij})$ to be an automorphism of $(U_i \cap U_j) \times \mathcal{K}$ composed of the $\mathcal{K}(\varphi_{ij})(x, -)$ for $x \in (U_i \cap U_j)$.

Now we are ready to define $\mathcal{K}(E)$ properly: Consider a collection $\{U_i \times \mathcal{K}\}_{i \in I}$. For each pair of $i \neq j$, glue $U_i \times \mathcal{K} \to U_j \times \mathcal{K}$ by $\mathcal{K}(\varphi_{ij})$ and define $\mathcal{K}(E)$ to be the resulting space. One can check that $\mathcal{K}(E)$ inherits the structure of a fibre bundle over B from that of E. Let us name the resulting projection map $\pi_{\mathcal{K}}\colon \mathcal{K}(E) \to B$.

Equipped with this definition, we now formalise the notion of a family of full-dimensional compact convex bodies. From now on we consider only such families that arise from continuous sections $s\colon B \to \mathcal{K}(E)$ for some rank n vector bundle $E \to B$.

Problem 2.5 (Parameterised Nandakumar & Ramana Rao problem)**.** *Let $\mathbb{R}^n \to E \to B$ be a vector bundle and let $\mathcal{K}(E)$ be the associated fibre bundle from Definition 2.4. Let $s\colon B \to \mathcal{K}(E)$ be a continuous section. Fix $j \in \mathbb{N}$. Let $(f_1, \dots, f_j)$ be a j-tuple of continuous functions on the space $\mathcal{K}(E)$. For a given natural number m, does there exist a point $x \in B$ such that there exists a partition of the convex body $K = s(x)$ into m parts $(K_1, \dots, K_m)$, such that they all have equal volume, and for any $1 \leqslant i \leqslant j$ the function f_i has an equal value on all these parts, that is $f_i(K_1) = \dots = f_i(K_m)$?*

Remark 2.6. One might notice that section s is not part of statements of Theorems 5.1 and 5.4 that were presented in Section 1.2. This is due to the fact that the tools we use to search for solutions to Problems 1.3 and 2.5 are purely topological and do not distinguish between different continuous functions s. In the next section, we reformulate these problems in such a way that the section s completely disappears from the statement.

2.2 Equivariant algebraic topology perspective

Our next step is to translate Problem 2.5 into the language of algebraic topology. To do so, we use the same strategy as was used in [KHA14, BZ14] for the original problem, adjusting it for the parameterised case.

Notation 2.7. Let $E_i \xrightarrow{\pi_i} B, 1 \leqslant i \leqslant k$ be a collection of fibre bundles over the same base B. Denote by $\Delta^*(E_1 \times \cdots \times E_k) \xrightarrow{\Delta^*(\pi_1 \times \cdots \times \pi_k)} B$ the pullback of the product bundle $E_1 \times \cdots \times E_k \to B^k$ along the diagonal map $\Delta\colon B \to B^k$. In the case when all k bundles are isomorphic to the same bundle E, we abuse the notation and write E^k instead of $\Delta^*(E^k)$.

Definition 2.8 (Fibrewise configuration space [CJ92]). Let $F \to E \to B$ be a fibre bundle. The associated *fibrewise configuration space* of m points in E as a subspace of E^m is defined as

$$\mathrm{Fconf}(m, E) \coloneqq \{(x_1, \ldots, x_m) \in E^m \colon x_i \neq x_j, \pi^m(x_i) = \pi^m(x_j) \text{ for } i \neq j\}.$$

One can show that $\mathrm{Fconf}(m, E)$ is a fibre bundle over B with fibre $\mathrm{Conf}(m, F)$, that is, the usual configuration space of m distinct points in F. We denote the projection $\mathrm{Fconf}(m, E) \to B$ by π_{Conf}. The action of the symmetric group $\mathfrak{S}_m$ on $\mathrm{Conf}(m, F)$ induced from the action that permutes components in F^m, can be extended to a free fibrewise action on the total space of $\mathrm{Fconf}(m, E)$.

Notation 2.9. Denote by W_m^j the orthogonal complement of the diagonal $\Delta(\mathbb{R}^j) \subseteq (\mathbb{R}^j)^m$, that is, the subspace of $(\mathbb{R}^j)^m$ consisting of all tuples $(x_1, \ldots, x_m)$, with $x_i \in \mathbb{R}^j$ such that $x_1 + \cdots + x_m = 0$. We consider W_m^j together with the action of $\mathfrak{S}_m$ induced from the action of $\mathfrak{S}_m$ that permutes the components of $(\mathbb{R}^j)^m$.

The goal of this section is to prove the following Lemma:

Lemma 2.10 (Compare with [KHA14, BZ14]). *Let $\mathbb{R}^n \to E \xrightarrow{\pi} B$ be a vector bundle. If there does not exist a $\mathfrak{S}_m$-equivariant map from the fibrewise configuration space $\mathrm{Fconf}(m, E)$ to the trivial sphere bundle $S\big(\underline{W_m^j}\big) \coloneqq S\big(W_m^j \times B\big)$ in the category of fibre bundles over B, then there exists a solution to Problem 2.5 with a set of parameters (E, m, j) for all choices of a section s and a tuple of continuous functions $(f_1, \ldots, f_j)$ on the space $\mathcal{K}(E)$.*

This lemma allows us to look for solutions to Problem 2.5 by studying topological obstructions to the existence of $\mathfrak{S}_m$-equivariant fibre-preserving maps $\mathrm{Fconf}(m, E) \to S\big(\underline{W_m^j}\big)$. This approach proved to be fruitful for the original Nandakumar & Ramana Rao problem: In [BZ14], in order to study equivariant maps $\mathrm{Conf}(m, \mathbb{R}^n) \to S\big(W_m^{n-1}\big)$, equivariant obstruction theory was used, and it was shown that a solution to the classical Nandakumar & Ramana Rao problem exists for any m of the form $= p^k$ where p is a prime and any $n - 1$ suitable functions. However, there is little hope of getting a description of CW-decomposition of $\mathrm{Fconf}(m, E)$ that will be good enough to allow us to apply obstruction theory, especially since the cellular structure on $\mathrm{Conf}(m, \mathbb{R}^n)$ used in [BZ14] is coordinate-dependent and hence can not be applied to the fibrewise case. In [BLZ15], it was suggested to use cohomological index theory instead, and, as we will see, it also works quite well in the fibrewise case. Unfortunately, this comes with a cost. Since even the index of $\mathrm{Conf}(p^k, \mathbb{R}^n)$ for $k \geqslant 2$ is not fully known, if we use index theory, m has to be prime.

The proof of Lemma 2.10 follows the same steps as can be found in [BZ14], with suitable adjustments. We begin with a short reminder of how this transition to the language of equivariant algebraic topology looks in the classical case when B is a point.

Given a full-dimensional compact convex body $K \subsetneq \mathbb{R}^n$, the desired number of parts in the partition m, and a collection of functions $(f_1, \dots, f_{n-1})$, such that each f_i is a continuous function on the space $\mathcal{K}$ of full-dimensional compact convex bodies in $\mathbb{R}^n$, the translation of Problem 1.2 to the language of equivariant topology is done via the following steps:

1. Let $EVP(K) \subseteq \mathcal{K}^m$ stand for the subspace of all ordered equal volume partitions of K, that is, all such tuples $(K_1, \dots, K_m) \in \mathcal{K}^m$ that $K = K_1 \cup \dots \cup K_m$, the interiors of K_i, $1 \leqslant i \leqslant m$ are pairwise disjoint, and all of the K_i have the same volume.

   Define a continuous map $ev\colon EVP(K) \to W_m^{n-1}$. First, define a map
   $$ev'\colon EVP(K) \to (\mathbb{R}^{n-1})^m,$$
   associated with the tuple of functions $(f_1, \dots, f_{n-1})$ by the formula
   $$ev'(K_1, \dots, K_m) = \{f_j(K_i)\}_{1 \leqslant j \leqslant n-1, 1 \leqslant i \leqslant m}.$$

2. Compose ev' with the projection $(\mathbb{R}^{n-1})^m \to W_m^{n-1}$ and name this new map ev. Effectively, the map ev measures for each equal volume partition $(K_1, \dots, K_m)$ how far it is from an equipartition in the sense of the Nandakumar & Ramana Rao problem. In particular, *there is a solution to Problem 1.2 if and only if the map ev hits the origin.*

   Consider the action of the symmetric group $\mathfrak{S}_m$ on $EVP(K)$ that renumbers the components of the convex partition $(K_1, \dots, K_m)$. With respect to this action, the map ev is $\mathfrak{S}_m$-equivariant.

3. If there are no solutions to Problem 1.2 for the given n, K, and m, then the map ev factors through $W_m^{n-1} \setminus \{0\}$ and therefore can be post-composed with the radial retraction $W_m^{n-1} \setminus \{0\} \to S\big(W_m^{n-1}\big)$, resulting in map $\mathfrak{S}_m$-equivariant map $ev : EVP(K) \to S\big(W_m^{n-1}\big)$.

4. It turns out, one can substitute the space $EVP(K)$ with a better-understood space, $\mathrm{Conf}(m,\mathbb{R}^n)$: On one hand, using generalised Voronoi diagrams, one can construct a continuous $\mathfrak{S}_m$-equivariant map v_K from the configuration space $\mathrm{Conf}(n,\mathbb{R}^n)$ to $EVP(K)$, see Definition 2.11 and Lemma 2.12 below. On the other hand, there exists a continuous $\mathfrak{S}_m$-equivariant map bar_K in the opposite direction, $EVP(K) \to \mathrm{Conf}(n,\mathbb{R}^n)$, that sends an equipartition $K = K_1 \cup \dots \cup K_m$ to the tuple of barycentres of $(K_1,\dots,K_m)$. Observe, that bar_K is $\mathfrak{S}_m$-equivariant by construction. For a proof of the continuity of bar_K, see Lemma 2.14.

   Using v_K and bar_K, we see that a $\mathfrak{S}_m$-equivariant map
   $$EVP(K) \to S\big(W_m^{n-1}\big)$$
   exists if and only if there exists a $\mathfrak{S}_m$-equivariant map
   $$\mathrm{Conf}(m,\mathbb{R}^n) \to S\big(W_m^{n-1}\big).$$
   Observe that, in the process, K disappears from the picture.

5. To conclude, if for a given m we can show that no $\mathfrak{S}_m$-equivariant map
   $$\mathrm{Conf}(m,\mathbb{R}^n) \to S\big(W_m^{n-1}\big)$$
   exists, then there exists a solution to Problem 1.2 for this number m, and any choice of K and functions $(f_1,\dots,f_{n-1})$.

We have promised to define the map v_K properly. First, we introduce a notion of the *generalised Voronoi diagram*.

Definition 2.11 ([AK00, 4.3.2])**.** For any point $x = (x_1,\dots x_m)$ in $\mathrm{Conf}(m,\mathbb{R}^n)$ and any vector $w = (w_1,\dots,w_m)$ in W_m define *generalised Voronoi diagram* or *power diagram* $C(x,w)$ with sites x and weights w as a collection of regions (*Voronoi cells*),
$$C(x,w) = \big(C_1(x,w),\dots,C_m(x,w)\big),$$
where $C_i(x,w)$ is defined as
$$C_i(x,w) = \{y \in \mathbb{R}^n \colon \|y - x_i\|^2 - w_i \leqslant \|y - x_j\|^2 - w_j \text{ for } 1 \leqslant j \leqslant m\}.$$

Let us quote the following results concerning power diagrams and the map v_K. We provide the references for these results after the lemma.

Lemma 2.12.

1. *For any $x \in \mathrm{Conf}(m, \mathbb{R}^n)$ and any $w \in W_m$, $C(x, w)$ provides a partition of $\mathbb{R}^n$ into m convex pieces with disjoint interiors. Some of these pieces might be empty and in general, x_i does not belong to $C_i(x, w)$.*

2. *Given any full-dimensional compact convex body $K \subsetneq \mathbb{R}^n$ and a point x in the configuration space $\mathrm{Conf}(m, \mathbb{R}^n)$ there exists a unique choice of weights $w(x, K) \in W_m$ such that*

$$vol\Big(K \cap C_1\big(x, w(x, K)\big)\Big) = \cdots = vol\Big(K \cap C_m\big(x, w(x, K)\big)\Big) = \frac{1}{m} vol(K).$$

3. *Denote by v_K a map $\mathrm{Conf}(m, \mathbb{R}^n) \to EVP(K)$ that maps a point x in $\mathrm{Conf}(m, \mathbb{R}^n)$ to*

$$\Big(K \cap C_1\big(x, w(x, K)\big), \ldots K \cap C_m\big(x, w(x, K)\big)\Big).$$

Denote by v a map $\mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K} \to \mathcal{K}^m$ defined as $v(x, K) \coloneqq v_K(x)$. Then v is continuous, and, consequently, v_K is continuous for every choice of $K \in \mathcal{K}$.

Remark 2.13. One can see v_K as a composition of two maps:

$$\mathrm{Conf}(m, \mathbb{R}^n) \to \mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K} \xrightarrow{v} \mathcal{K}^m,$$

where the first map sends a point x in configuration space $\mathrm{Conf}(m, \mathbb{R}^n)$ to the pair (x, K). We will use a similar idea later for the parameterised version.

Directions to the proof of Lemma 2.12. Parts 1- 2 can be proved by many different methods: Using optimal transport theory, discretisation theory or by minimising quadratic objection function. We refer the reader to the following sources: [GKPR13, Thm. 1 and Thm. 2] for the proof by minimising quadratic objection function; [BS23, Lem. 6.3] for a topological proof due to M.Fisching, and [BZ14, Sect. 2] for the overview of other methods of proofs.

A proof of part 3 can be found in [BS23, Thm. 6.6] □

Later in this section, we introduce a parameterised analogue of the map bar_K and prove its continuity using that of bar_K. For the sake of completeness, we provide here a proof of the continuity of bar_K itself, since it is not explicitly written in [KHA14, BZ14].

Lemma 2.14. *The map $bar_K \colon \mathcal{K}^m \to \mathrm{Conf}(m, \mathbb{R}^n)$, which sends a point $(K_1, \ldots, K_m)$ in $EVP(K)$ to the tuple of barycenters of K_i, $1 \leqslant i \leqslant k$ is continuous.*

Proof. Observe, that bar_K can be extended to the map $\mathcal{K}^m \xrightarrow{bar^m} (\mathbb{R}^n)^m$, where

$$bar \colon \mathcal{K} \to \mathbb{R}^n$$

is a map that sends a full-dimensional compact convex body $K \in \mathcal{K}$ to its barycentre. We prove that *bar* is continuous, which implies continuity of bar_K for each K since the topology on $EVP(K)$ was defined as a subspace topology induced by that of on $\mathcal{K}^m$.

Let $K_t, t \in \mathbb{N}$ be a sequence of full-dimensional compact convex bodies in $\mathbb{R}^n$ such that $\lim_{t\to\infty} K_t \coloneqq K$. Our goal is to prove that the sequence $bar(K_t)$ converges to $bar(K)$. For this proof we will use that $\mathcal{K}$ is a metric space with a metric given by d_s, in particular $\lim_{t\to\infty} \mathcal{L}^n(K\Delta K_t) = 0$.

Let $(c_1, \dots, c_n)$ be the coordinates in $\mathbb{R}^n$ of the barycentre of K, and $(c_1^t, \dots, c_n^t) \in \mathbb{R}^n$ be the coordinates of barycenters of each of K_t.

These coordinates satisfy the formula [PM64, p.512]

$$c_i^t = \frac{1}{vol(K_t)} \int y\mathcal{L}^{n-1}(K_t \cap H_{i,y})dy, \tag{2.1}$$

where $H_{i,y}$ is an affine hyperplane in $\mathbb{R}^n$ with an equation $x_i = y$ and x_i is i-th standard coordinate in $\mathbb{R}^n$.

Applying formula (2.1), we can compute,

$$\begin{aligned} c_i - c_i^t &= \frac{1}{vol(K)} \int y\mathcal{L}^{n-1}(K \cap H_{i,y})dy - \frac{1}{vol(K_t)} \int y\mathcal{L}^{n-1}(K_t \cap H_{i,y})dy \\ &= \frac{1}{vol(K)} \Big(\int y\mathcal{L}^{n-1}(K \cap H_{i,y})dy - \int y\mathcal{L}^{n-1}(K_t \cap H_{i,y})dy \Big) \\ &\quad + \frac{vol(K_t) - vol(K)}{vol(K)vol(K_t)} \Big(\int y\mathcal{L}^{n-1}(K_t \cap H_{i,y})dy \Big) \\ &\leqslant \frac{1}{vol(K)} \int |y| \left| \mathcal{L}^{n-1}(K \cap H_{i,y}) - \mathcal{L}^{n-1}(K_t \cap H_{i,y}) \right| dy + \frac{vol(K_t) - vol(K)}{vol(K)} c_i^t. \end{aligned}$$

Let us look closer at the summands in the last row. We prove that the value of each of them converges to 0 as t goes to infinity.

Since $\lim_{t\to\infty} K_t = K$ all K_t and K lie in some closed ball $B_R \subset \mathbb{R}^n$ of sufficiently big radius R, centred at the origin.

Analysing the first summand we see that,

$$\begin{aligned} 0 &\leqslant \int |y| \left| \mathcal{L}^{n-1}(K \cap H_{i,y}) - \mathcal{L}^{n-1}(K_t \cap H_{i,y}) \right| dy \\ &\leqslant R \int \left| \mathcal{L}^{n-1}(K \cap H_{i,y}) - \mathcal{L}^{n-1}(K_t \cap H_{i,y}) \right| dy, \end{aligned}$$

and

$$0 \leqslant \lim_{t\to\infty} \int \left| \mathcal{L}^{n-1}(K \cap H_{i,y}) - \mathcal{L}^{n-1}(K_t \cap H_{i,y}) \right| dy \leqslant \lim_{t\to\infty} \mathcal{L}^n(K\Delta K_t) = 0.$$

To prove that the value of the second summand also converges to 0, notice that $|c_i^t| \leqslant R$ for any t and any i, which implies,

$$\lim_{t\to\infty} \frac{vol(K_t) - vol(K)}{vol(K)} |c_i^t| \leqslant \lim_{t\to\infty} \frac{vol(K_t) - vol(K)}{vol(K)} R = 0,$$

since $vol(K_t)$ converges to $vol(K)$. □

Remark 2.15. Observe that although we used coordinates in $\mathbb{R}^n$ to prove the continuity of bar, we did not need coordinates in the target space to define it. The independence of bar from the choice of coordinates allows us to define its counterpart in the fibrewise case, bar_E.

For the rest of this section fix a real rank n vector bundle E over B and s - a continuous section $s\colon B \to \mathcal{K}(E)$.

Definition 2.16. Define $EVP(s) \subset \mathcal{K}(E)^m$ to be a subspace of the total space of $\mathcal{K}(E)^m$, such that for any base point $x \in B$,

$$(\pi_{\mathcal{K}}^m)^{-1}(x) \cap EVP(s) = EVP\big(s(x)\big)$$

and equip $EVP(s)$ with the subset topology. Observe, that $EVP(s)$ inherits from $\mathcal{K}(E)^m$ a projection to B, which we denote by $\pi_{EVP(s)}$.

Below we repeat the steps 1-5 of the translation of Problem 1.2 to the language of equivariant topology, this time for the parameterised case. This involves defining maps ev_E, bar_s and v_s analogous to ev, bar_K and v_K respectively, but this time there should be maps between fibre bundles.

Given j continuous functions $(f_1, \ldots, f_j)$ on $\mathcal{K}(E)$, we define an $\mathfrak{S}_m$-equivariant map

$$ev_E\colon EVP(s) \to W_m^j \times B,$$

which generalises the map ev. We do so in three steps. First, consider a map from $\mathcal{K}(E)^m$ to $(\mathbb{R}^j)^m$ that sends a point $(K_1, \ldots, K_m)$ in $\mathcal{K}(E)^m$ to $\{f_k(K_i)\}_{1\leqslant k\leqslant j, 1\leqslant i\leqslant m}$. Combine this map with a projection of $(\mathbb{R}^j)^m$ onto W_m^j. Let us name this composition ev'_E. Define the map ev_E as $(ev'_E, \pi_{\mathcal{K}})$. By construction, the map ev_E is continuous and $\mathfrak{S}_p$-equivariant.

Therefore, similar to the step 3, we see that for a given tuple (E, s, m, j) Problem 2.5 has a solution if and only if the map ev_E hits 0_B - a 0-section of a trivial vector bundle $\underline{W_m^j} := W_m^j \times B$. Consequently, when no solution exists, ev_E can be extended to a map $EVP(s) \to S\big(\underline{W_m^j}\big)$.

To define bar_s and v_s we use the strategy of gulling the global map from local pieces.

Let $\{U_i\}_I$ and $\varphi_i, i \in I$ be as in the Definition 2.4. Then $\mathrm{Fconf}(m, E)$, $\mathcal{K}(E)$, and $\mathcal{K}(E)^m$ are locally trivial over the same family $\{U_i\}_{i\in I}$, with the following trivialising homeomorphisms

$$\begin{aligned} \mathcal{K}(\varphi_i) &: \pi_{\mathcal{K}}^{-1}(U_i) \cong U_i \times \mathcal{K} \\ \mathcal{K}(\varphi_i)^m &: (\pi_{\mathcal{K}}^m)^{-1}(U_i) \cong U_i \times \mathcal{K}^m \\ \varphi_i^m &: \pi_{\mathrm{Conf}}^{-1}(U_i) \cong U_i \times \mathrm{Conf}(m, \mathbb{R}^n) \end{aligned}$$

Remember that $\mathcal{K}(\varphi_i)$ stands for an automorphism of $\mathcal{K}$ induced by the automorphism φ_i of the ambient space.

Denoting the trivialising homeomorphism for $\mathrm{Fconf}(m, E)$ by φ_i^m involves a slight abuse of notation. Strictly speaking, φ_i^m is a trivialising homeomorphism of E^m over U_i, and $\mathrm{Fconf}(m, E)$ is only a subspace of E^m. Observe, however, that any isomorphism $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ induces a homeomorphism between configuration spaces $\mathrm{Conf}(m, \mathbb{R}^n)$, therefore the restriction of φ_i^m on $\pi_{\mathrm{Conf}}^{-1}(U_i)$ indeed is a fibrewise homeomorphism onto $U_i \times \mathrm{Conf}(m, \mathbb{R}^n)$. We choose to omit restriction signs to have easier-to-read formulae.

To define bar_s it is easier to define a map $bar_E^m \colon \mathcal{K}(E)^m \to E^m$ first, and then take its restriction onto $EVP(s)$

Definition 2.17. For any U_i, $i \in I$, define

$$bar_E^m(U_i) \colon \big(\mathcal{K}(E)^m\big)^{-1}(U_i) \to U_i \times (\mathbb{R}^n)^m$$

as a composition $(id \times bar) \circ \mathcal{K}(\varphi_i)^m$.

Lemma 2.18. *The collection of maps $\{bar_E^m(U_i)\}$ can be glued to become a global continuous map*

$$bar_E^m \colon \mathcal{K}(E)^m \to E^m.$$

Proof. Proving that is equivalent to proving that the following diagram commutes for any pair $i, j \in I$.

$$\begin{tikzcd} & (U_i \cap U_j) \times \mathcal{K}^m \arrow[r, "id\times bar^m"] \arrow[d, "\mathcal{K}(\varphi_{ij})^m"] & (U_i \cap U_j) \times (\mathbb{R}^n)^m \arrow[d, "\varphi_{ij}^m"] \\ (\mathcal{K}(E)^m)^{-1}(U_i \cap U_j) \arrow[ur, "\mathcal{K}(\varphi_i)^m"] \arrow[r, "\mathcal{K}(\varphi_j)^m"] & (U_i \cap U_j) \times \mathcal{K}^m \arrow[r, "id\times bar"] & (U_i \cap U_j) \times (\mathbb{R}^n)^m \end{tikzcd}$$

Remember that by φ_{ij} we agreed to denote the composition $\varphi_j \circ \varphi_i^{-1}$. The commutativity of the diagram follows from the definition of $\mathcal{K}(E)$ and its trivialising maps.

Observe that each of $bar_E^m(U_i)$ is continuous. Therefore if bar_E^m is correctly defined, then it immediately follows that it is continuous as well. □

Definition 2.19. Define bar_s to be the restriction of bar_E^m onto $EVP(s)$

Now let us turn to v_s. In the spirit of Remark 2.13, we construct v_s as a composition of two maps and prove the continuity of each of them.

Let us try to get an intuition for what these maps should look like by guessing them from their analogy in Remark 2.13. Naturally, $\mathrm{Fconf}(m, E)$ substitutes $\mathrm{Conf}(m, \mathbb{R}^n)$ and $\mathcal{K}(E)^m$ substitutes $\mathcal{K}^m$. We want all our maps to be maps in the category of fibre bundles over the base B. Therefore a natural candidate on place $\mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K}$ is $\Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big)$ - a pullback of the product bundle $\mathrm{Fconf}(m, E) \times \mathcal{K}(E)$ over the base $B \times B$ along the diagonal embedding of B into $B \times B$. Its fibre is exactly $\mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K}$.

Let $\tilde{x}$ be a point in $\mathrm{Fconf}(m, E)$ in a fibre over the point $x \in B$, that is, $\pi_{\mathrm{Conf}}(\tilde{x}) = x$. The first map we construct, from $\mathrm{Fconf}(m, E)$ to $\Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big)$, sends $\tilde{x}$ to a pair $\big(\tilde{x}, s(x)\big)$, which belongs to the fibre of $\Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big)$ over the point x. We name this map φ. The second map we construct sends $\big(\tilde{x}, s(x)\big)$ to the equipartition of $s(x)$ provided by the generalised Voronoi diagram with sites $\tilde{x}$ in the ambient space $\pi^{-1}(x)$. Let us name this map v_E, it plays the role of v Remark 2.13.

Definition 2.20. Define v_s as the composition $v_E \circ \varphi$.

All maps that we have just described are built pointwise and do not take into account fibre bundles' structures. Below we define φ and v_E properly.

Definition 2.21. Consider a composition $s \circ \pi_{\mathrm{Conf}}$, which is a map from $\mathrm{Fconf}(m, E)$ to $\mathcal{K}(E)$. By construction, this is a continuous map between fibre bundles over B. Denote by $\varphi_\times$ the map $(id, s \circ \pi_{\mathrm{Conf}}^{-1})$ from the fibre bundle $\mathrm{Fconf}(m, E)$ to the product bundle $\mathrm{Fconf}(m, E) \times \mathcal{K}(E)$.

Let Δ be the diagonal embedding of B into $B \times B$. Notice, that $\varphi_\times$ factors through $\Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big)$:

$$\mathrm{Fconf}(m, E) \xrightarrow{\varphi} \Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big) \xrightarrow{\Delta^*} \mathrm{Fconf}(m, E) \times \mathcal{K}.$$

Define φ to be the map such that $\varphi^\times = \varphi \circ \Delta^*$. We see that φ is continuous.

Definition 2.22. Observe that, $\mathrm{Fconf}(m, E) \times \mathcal{K}(E)^m$ admits a trivialisation over $U_i \times U_i$ for each i, with a trivialising homomorphism $\varphi_i^m \times \mathcal{K}(\varphi_i)^m$. It follows that $\Delta^*\big(\mathrm{Fconf}(m, E) \times \mathcal{K}(E)\big)$ is trivialisible over U_i for each $i \in I$, and the trivialisation is provided by $\big(\varphi_i^m \times \mathcal{K}(\varphi_i)^m\big) \circ \Delta^*$.

For each i in I define a map $v_E(U_i)\colon \big(\Delta^*(\pi_{\mathrm{Conf}\times\pi_{\mathcal{K}}})\big)^{-1}(U_i) \to U_i \times \mathcal{K}^m$ as the composition $(id \times v) \circ (\varphi_i^m \times \mathcal{K}(\varphi_i)^m) \circ \Delta^*$.

Lemma 2.23. *The collection of maps $v_E(U_i)$, $i \in I$ can be glued together to get a continuous map*

$$v_E\colon \mathrm{Fconf}(m, E) \to EVP(s).$$

Proof. To check that v_E is correctly defined, we need to show that the external circuit of the following diagram commutes for any choice of the pair $i, j \in I$.

$$\begin{tikzcd}
 & (U_i \cap U_j) \times (\mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K}) \arrow[r, "\mathrm{id}\times v"] \arrow[d, "\varphi_{ij}^m \times \mathcal{K}(\varphi_{ij})"] & (U_i \cap U_j) \times \mathcal{K}^m \arrow[d, "\mathcal{K}(\varphi_{ij})^m"] \\
(\Delta^*(\pi_{\mathrm{Conf}} \times \mathcal{K}^m))^{-1}(U_i \cap U_j) \arrow[ur, "((\varphi_i)^m \times \mathcal{K}(\varphi_i))\circ\Delta^*"] \arrow[r, "((\varphi_j)^m \times \mathcal{K}(\varphi_j))\circ\Delta^*"] & (U_i \cap U_j) \times (\mathrm{Conf}(m, \mathbb{R}^n) \times \mathcal{K}) \arrow[r, "\mathrm{id}\times v"] & (U_i \cap U_j) \times \mathcal{K}^m.
\end{tikzcd}$$

The commutativity of the left triangle follows again from the construction of $\mathcal{K}(E)$.

Remember that all φ_i, $i \in I$ can be chosen to be orthogonal transformations of $\mathbb{R}^n$ when restricted to any point x in U_i ([Hat17, Prop. 1.2]). Let us consider only such sets of $\{\varphi_i\}$, $i \in I$. Using the construction of v vie generalised Voronoi diagrams and the fact that orthogonal transformations preserve distances, we see that the right square on the diagram above also commutes. This finishes the proof that v_E is correctly defined. We conclude that v_E is continuous since it is continuous when restricted to the fibre over any U_i, $i \in I$.

Observe, that by the way it is constructed, v_E factors through $EVP(s)$. □

Observation 2.24. *One can check that all maps constructed above, and most importantly, bar_s and v_s are $\mathfrak{S}_m$-equivariant.*

To summarise: We defined a pair of continuous $\mathfrak{S}_p$-equivariant maps. The map

$$bar_s\colon EVP(s) \to \mathrm{Fconf}(m, E)$$

and the map

$$v_s\colon \mathrm{Fconf}(m, E) \to EVP(s).$$

By analogy with step 5, we conclude that *if there is no $\mathfrak{S}_m$-equivariant fibrewise map* $\mathrm{Fconf}(m, E) \to S\big(\underline{W}_m^j\big)$*, then there exists a solution to the Problem 2.5.*

Remark 2.25. After the statement of Problem 1.2 we mentioned that it is possible to substitute in its statement a compact full-dimensional compact convex body with an absolutely continuous probability measure. What changes in that case? First of all, one would need a stronger version of Lemma 2.12. In fact, in the references we provided for this Lemma, it is stated already for any such measure. Then, one would need to adjust the formula for barycentric coordinates, which is also not a problem. All other steps of the transition from geometry to topology for the Problem 2.12 stay the same, and one concludes, that the solutions we found for it are also solutions for the version with measure.

The same holds for the parameterised version. The only additional ingredient one needs is a notion of a *mass assignment*, see [Sch20], [BC23, Sec. 1.2], then the translation of the problem can be done exactly in the same way.

Chapter 3

Preliminaries

This chapter contains a collection of facts from algebraic topology, that are important for understanding the main part of the thesis, and which we feel might be less known to a reader. The selection is of course purely subjective. We assume that the reader feels comfortable with Chern and Pontryagin classes (the traditional references are [MS74, Hat17]), and has seen Leray–Serre spectral sequences (see [McC01]) before.

3.1 Key cohomology rings

Cohomology of finite Grassmannians. To solve Problem 1.3 we need a knowledge of the cohomology ring of the finite real Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ with coefficients in $\mathbb{F}_p$, so we cite it here:

Fact 3.1 ([He17])**.** *Consider a real finite Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$. Let $i^\perp$ be an involution that sends a point $[V]$ in $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ represented by an n-dimensional linear subspace V in $\mathbb{R}^N$ to the point $[V^\perp]$ of the Grassmannian $\mathrm{Gr}(\mathbb{R}^{N-n}, \mathbb{R}^N)$ represented by $(N-n)$-dimensional orthogonal complement of V in $\mathbb{R}^N$. Let p_i denote the i-th Pontryagin class of a tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$, and $\bar{p}_i$ the i-th Pontryagin class of the bundle $\bar{\gamma}\left(\mathbb{R}^n, \mathbb{R}^N\right)$, defined as a pullback of $\gamma(\mathbb{R}^{N-n}, \mathbb{R}^n)$ along the involution $i^\perp$. Recall that for any i, p_i and $\bar{p}_i$ has a degree $4i$. The ring $H^*(\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N))$ depends on the parity of $N-n$:*

- *When $N-n$ is even, the cohomology ring is*

$$H^*(\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)) \cong \mathbb{F}_p[p_1, \ldots, p_{\lfloor n/2 \rfloor}, \bar{p}_1, \ldots, \bar{p}_{\lfloor (N-n)/2 \rfloor}]/I.$$

- *When $N-n$ is odd, the cohomology ring is*

$$H^*(\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)) \cong \Lambda[r] \otimes \mathbb{F}_p[p_1,\dots,p_{\lfloor n/2\rfloor},\bar{p}_1,\dots,\bar{p}_{\lfloor (N-n)/2\rfloor}]/I.$$

Here I is the ideal generated by homogeneous relations coming from the equality

$$(1+p_1+\cdots+p_{\lfloor n/2\rfloor})(1+\bar{p}_1+\cdots+\bar{p}_{\lfloor (N-n)/2\rfloor}) = 1$$

and r is an element of degree $n-1$.

Remark 3.2. To recover this information from [He17] use Theorem 1.6 (originally due to Ehresman [Ehr37]) and the universal coefficients theorem to show that integer cohomology of any real finite Grassmannian has only two-torsion. In this case the result for the cohomology with coefficients in $\mathbb{F}_p$ where p is an odd prime follows from the results for cohomology with coefficients in $\mathbb{Q}$. (For the case when $n(N-n)$ is even the latter follows from more general results of Borel [Bor53] and Leray [Ler49], the result in the case when $n(N-n)$ is odd is due to Takeuchi [Tak62]).

We also refer the reader to a very helpful answer by M.Wendt [hw] on math overflow.

In the proof of Theorem 5.4 an important role is played by the following lemma:

Lemma 3.3.

$$\bar{p}^i_{\lfloor (N-n)/2\rfloor} \neq 0 \in H^*(\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)) \iff i \leqslant \lfloor \frac{n}{2} \rfloor.$$

Proof. This might be not the optimal proof, but surely there should be proof that directly works with real Grassmannians and Pontryagin classes. However, since $\mathbb{F}_p$-cohomology of a finite real Grassmannian seems to be not very well presented in the literature, we opt here for an indirect proof via the cohomology of an appropriate complex Grassmannian. A very good overview of different approaches to the cohomology of complex Grassmannians can be found in [Hil82, Ch. III], in particular, we used it as a source for all complex Grassmannian-related statements below.

Recall that

$$H^*(\mathrm{Gr}(\mathbb{C}^d,\mathbb{C}^{d+k});\mathbb{Z}) \cong \mathbb{Z}[c_1,\dots,c_d,\bar{c}_1,\dots,\bar{c}_k]/I,$$

where I the ideal generated by homogeneous equations induced by the equality

$$(1+c_1+\cdots+c_d)(1+\bar{c}_1+\cdots+\bar{c}_k) = 1$$

There is a standard CW-structure on complex Grassmannian $\mathrm{Gr}(\mathbb{C}^d,\mathbb{C}^{d+k})$, consisting of so-called Schubert cells. Using it, one can show that $\bar{c}_k^d \neq 0$ in the integer cohomology.

This is a direct application of Pieri formula - $\bar{c}_k$ corresponds to $(0,\ldots,k)$ Schubert special class in $\mathrm{Gr}(\mathbb{C}^d,\mathbb{C}^{d+k})$, and for $1 \le n \le d$ it follows by induction that $\bar{c}_k^n$ corresponds to Schubert class $(0,\ldots,0,k,\ldots,k)$, where k appears n times, a good exposition of this theory can be found in [Hil82, Ch. III].

Also, since all cells in that CW-structure are of even dimensions, it follows that cohomology ring $H^*(\mathrm{Gr}(\mathbb{C}^d,\mathbb{C}^{d+k});\mathbb{Z})$ has no torsion part, therefore the same formula holds for the cohomology with $\mathbb{F}_p$ coefficients, namely

$$H^*(\mathrm{Gr}(\mathbb{C}^d,\mathbb{C}^{d+k});\mathbb{F}_p) \cong \mathbb{F}_p[c_1,\ldots,c_d,\bar{c}_1,\ldots,\bar{c}_k]/I.$$

In particular, it follows that the mod-p reduction of $\bar{c}_k^n$ is non-zero.

Observe that when $n(N-n)$ is even, there is an obvious isomorphism of rings (forgetting their grading) between $H^*(\mathrm{Gr}(\mathbb{C}^{\lfloor \frac{n}{2}\rfloor},\mathbb{C}^{\lfloor \frac{N-n}{2}\rfloor}))$ and $H^*(\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N))$, that sends c_i to p_i, for $1 \leqslant i \leqslant \lfloor \frac{n}{2}\rfloor$ and $\bar{c}_i$ to $\bar{p}_i$ for $1 \leqslant i \leqslant \lfloor \frac{N-n}{2}\rfloor$. And when $n(N-n)$ is odd the same map is an injective homomorphism that is a surjection onto the polynomial part of the latter ring. In either case, an element $\bar{c}_{\lfloor \frac{N-n}{2}\rfloor}^{\lfloor \frac{n}{2}\rfloor}$ is mapped to $\bar{p}_{\lfloor \frac{N-n}{2}\rfloor}^{\lfloor \frac{n}{2}\rfloor}$ and the statement of the lemma follows. □

Cohomology of $\mathbb{Z}_p$ and $\mathfrak{S}_p$ The next theorem describes cohomology rings of $\mathbb{Z}_p$ and $\mathfrak{S}_p$ with coefficients in $\mathbb{F}_p$. Recall, that one way to define the cohomology of a group G is to set them to be equal to the cohomology of its universal space BG (see [AM13, Ch. II §3]).

Theorem 3.4 (Cohomology of $\mathbb{Z}_p$ and $\mathfrak{S}_p$, [Knu18, Thm. 8.1.3, due to Nakaoka])**.**

1. *$H^*(\mathrm{B}\mathbb{Z}_p) \cong \Lambda[e] \otimes \mathbb{F}_p[t]$, where e has degree 1, and t has degree 2.*
2. *$H^*(\mathrm{B}\mathfrak{S}_p) \cong \Lambda[a] \otimes \mathbb{F}_p[b]$, where a has degree $2p-3$ and b has degree $2(p-1)$.*
3. *There exists a monomorphism from $H^*(\mathrm{B}\mathfrak{S}_p)$ to $H^*(\mathrm{B}\mathbb{Z}_p)$ that sends a to et^{p-2} and b to t^{p-1}.*

We would like to have some control over the homeomorphism from point 3 of Theorem 3.4. We start by fixing an embedding of $\mathbb{Z}_p$ into $\mathfrak{S}_p$.

Notation 3.5 ([AM13, Ex. II.27])**.** Let $\mathrm{E}\mathfrak{S}_p$ be a contractible space with free $\mathfrak{S}_p$ action. The regular embedding identifies $\mathbb{Z}_p$ with a subgroup of $\mathfrak{S}_p$, inducing an action of $\mathbb{Z}_p$ on $\mathrm{E}\mathfrak{S}_p$. Then one can take the respective quotient spaces, $\mathrm{E}\mathfrak{S}_p/\mathfrak{S}_p$ and $\mathrm{E}\mathfrak{S}_p/\mathbb{Z}_p$ as $\mathrm{B}\mathfrak{S}_p$ and $\mathrm{B}\mathbb{Z}_p$. In this case, $\mathrm{B}\mathbb{Z}_p$ is a $(p-1)!$-fold cover of $\mathrm{B}\mathfrak{S}_p$. Let us call the projection map provided by this covering *reg*. Then we choose generators in the cohomology rings of $\mathbb{Z}_p$ and $\mathfrak{S}_p$ such that the monomorphism 3 is provided by the induced map *reg**.

Remark 3.6.

- Observe that if one already knows the first two parts of the theorem above, then one can actually take the reasoning above as a proof of point 3 of Theorem 3.1. Indeed, since the map reg is a $(p-1)!$-covering, it follows that the induced map in cohomology in injective, see [Hat02, Prop. 3.G.1] (be aware that the other half of this proposition holds only if the covering is normal, which is not true in our case).

- In fact, the map reg^* can be used to compute cohomology of $\mathfrak{S}_p$ if cohomology of $\mathbb{Z}_p$ is already known (one possible reference is [AM13, Cor. II.4.3]). To do so, one needs a statement, analogous to that of [Hat02, Prop. 3.G.1], but for not-normal coverings, see [AM13, Lem. II.3.1], [BCC$^+$21, Lem. 7.23]. See [AM13, Ch. VI] for extensive treatment of this topic.

3.2 Local coefficients and Künneth theorem

The starting point for most of the proofs in this thesis is the Künneth theorem for local coefficients. Its statement and proof can be found in [Gre06]. For the sake of completeness, we repeat it here, in the minimal generality needed.

Usually, we use this theorem to compute the E_2-page of some spectral sequence as a module over its zero-row, meaning that every time we use not only Theorem 3.8 itself but also Lemma 3.9 that describes what this module structure looks like after the application of the Künneth formula in these cases.

Definition 3.7 (Local coefficients, see [Hat02, Sect. 3.H])**.** Let X be a CW-complex and $\tilde{X}$ be its universal cover. Denote by $C_*^{cell}(X)$ the chain complex of all cellular chains of X. Observe, that $C_*^{cell}(X)$ has a structure of a module over $\mathbb{F}_p[\pi_1(X)]$. Let $\mathcal{M}$ be another $\mathbb{F}_p[\pi_1(X)]$ module. We define cohomology of X with coefficients in the local system $\mathcal{M}$, $H^*(X;\mathcal{M})$, as homology of the hom complex $hom_{\pi_1(X)}(C_*^{cell}(X),\mathcal{M})$, of all $\mathbb{F}_p[\pi_1(X)]$-module homomorphisms from $C_*^{cell}(X)$ to $\mathcal{M}$.

Theorem 3.8 (Künneth theorem for local coefficients, Theorem 1.7 in [Gre06])**.** *Let $K_i, i \in \{1,2\}$ be two connected CW-complexes such that they have a finite number of cells in each dimension and $\mathcal{R}_i$ be left $\mathbb{F}_p[\pi_1(K_i)]$ modules over the respective fundamental groups $\pi_1(K_i)$. Then there is a natural isomorphism of rings*

$$H^*(K_1 \times K_2; R_1 \otimes R_2) \cong H^*(K_1;\mathcal{R}_1) \otimes H^*(K_2;R_2),$$

where $\mathbb{F}_p[\pi_1(K_1 \times K_2)] \cong \mathbb{F}_p[\pi_1(K_1)] \otimes \mathbb{F}_p[\pi_1(K_2)]$ acts on $\mathcal{R}_1 \otimes \mathcal{R}_2$ diagonally.

Proof. Our strategy is to reduce this theorem to the case of the standard Künneth formula.

Given a CW-complex K and some $\mathbb{F}_p[\pi_1(K)]$-module $\mathcal{R}$ note that by one of the definitions of cohomology with local coefficients (see, for example [Hat02, §3.H])

$$H^*(K;\mathcal{R}) = H(\hom_{\pi_1(K)}(C_*^{cell}(\tilde{K}),\mathcal{R})),$$

where $C_*^{cell}(\tilde{K})$ is the complex of the cellular chains of the universal cover of K with coefficients in $\mathbb{F}_p$.

We would like to apply the standard Künneth formula to the two chain complexes $\hom_{\pi_1(K_1)}(C_*^{cell}(\tilde{K}_1),\mathcal{R}_1)$ and $\hom_{\pi_1(K_2)}(C_*^{cell}(\tilde{K}_2),\mathcal{R}_2)$. These are chain complexes consisting of $\mathbb{F}_p$-modules. Any module is free over $\mathbb{F}_p$ and no torsion could possibly arise, therefore the Künneth formula takes its simplest form,

$$\begin{aligned} &H(\hom_{\pi_1(K_1)}(C_*^{cell}(\tilde{K}_1),\mathcal{R}_1)) \otimes H(\hom_{\pi_1(K_2)}(C_*^{cell}(\tilde{K}_2),\mathcal{R}_2)) \cong \\ &\qquad \cong H(\hom_{\pi_1(K_1)}(C_*^{cell}(\tilde{K}_1),\mathcal{R}_1) \otimes \hom_{\pi_1(K_2)}(C_*^{cell}(\tilde{K}_2),\mathcal{R}_2)). \end{aligned}$$

with the isomorphism given by sending $[\alpha]\otimes[\beta]$ to $[\alpha\otimes\beta]$ for any pair of cocycles

$$\begin{aligned} \alpha &\in \hom_{\pi_1(K_1)}(C_*^{cell}(\tilde{K}_1),\mathcal{R}_1), \\ \beta &\in \hom_{\pi_1(K_2)}(C_*^{cell}(\tilde{K}_2),\mathcal{R}_2). \end{aligned}$$

The last step in the proof of the theorem is to notice that since each of $C_*^{cell}(\tilde{K}_i)$ is a free $\mathbb{F}_p[\pi_{K_i}]$-module, finitely generated in each dimension, it holds

$$\begin{aligned} &\hom_{\pi_1(K_1)}(C_*^{cell}(\tilde{K}_1),\mathcal{R}_1) \otimes \hom_{\pi_1(K_2)}(C_*^{cell}(\tilde{K}_2),\mathcal{R}_2) \\ &\qquad \cong \hom_{\pi_1(K_1\times K_2)}(C_*^{cell}(\tilde{K}_1\times\tilde{K}_2),\mathcal{R}_1\otimes\mathcal{R}_2), \end{aligned}$$

were we denote by $\tilde{K}_1\times\tilde{K}_2$ the universal cover of $K_1\times K_2$ and use the fact that

$$C_*^{cell}(\tilde{K}_1)\otimes C_*^{cell}(\tilde{K}_2) = C_*^{cell}(\tilde{K}_1\times\tilde{K}_2).$$

In preparation for the Lemma 3.9 it is useful to note, that in the same notation as before the last isomorphism sends $\alpha\otimes\beta$ to the class, that is equal $\alpha(a)\otimes\beta(b)$ for any element $a\otimes b\in C_*(\tilde{K}_1)\otimes C_*(\tilde{K}_2)$ - a cellular chain of $\tilde{K}_1\times\tilde{K}_2$. □

Let $K_1, K_2, \mathcal{R}_1, \mathcal{R}_2$ be same as above. Let $\Delta_i\colon K_i\to K_i\times K_i$ be the diagonal embedding.

Observe that $H^*(K_i;\mathcal{R}_i)$ carries naturally the structure of a module over $H^*(K_i;\mathbb{F}_p)$, given by the composition of maps

$$H^*(K_i;\mathbb{F}_p)\otimes H^*(K_i;\mathcal{R}_i) \xrightarrow[\cong]{\varkappa} H^*(K_i\times K_i;\mathbb{F}_p\otimes\mathcal{R}_i) \xrightarrow{\Delta^*} H^*(K_i;\mathcal{R}_i).$$

Here the first isomorphism is the one from Theorem 3.8. Similarly $H^*(K_1\times K_2;\mathcal{R}_1\otimes\mathcal{R}_2)$ carries a natural structure of a module over $H^*(K_1\times K_2;\mathbb{F}_p)$.

On the other hand, once $H^*(K_1\times K_2;\mathcal{R}_1\otimes\mathcal{R}_2)$ and $H^*(K_1\times K_2;\mathbb{F}_p)$ are transformed by the Künneth theorem into $H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2;R_2)$ and $H^*(K_1;\mathbb{F}_p)\otimes H^*(K_2;\mathbb{F}_p)$ respectively, therefore there is another natural way to define a module structure:

$$H^*(K_1;\mathbb{F}_p)\otimes H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2;R_2)\otimes H^*(K_2;\mathbb{F}_p)\to H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2;R_2).$$

using module structures on each of $H^*(K_i;\mathcal{R}_i)$ separately.

The relationship between these two module structures is represented by the following diagram:

$$\begin{tikzcd}
H^*(K_1\times K_2;\mathcal{R}_1\otimes\mathcal{R}_2) & H^*(K_1\times K_2\times K_1\times K_2;\mathbb{F}_p\otimes\mathbb{F}_p\otimes\mathcal{R}_1\otimes\mathcal{R}_2) \arrow[l, dashed, "\Delta^*_{12}"] \\
H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2;\mathcal{R}_2) \arrow[u, "\varkappa_{12}"] & \\
H^*(K_1\times K_1;\mathbb{F}_p\otimes\mathcal{R}_1)\otimes H^*(K_2\times K_2;\mathbb{F}_p\otimes\mathcal{R}_2) \arrow[u, dashed, "\Delta_1^*\otimes\Delta_2^*"'] & H^*(K_1\times K_2)\otimes H^*(K_1\times K_2;\mathcal{R}_1\otimes\mathcal{R}_2) \arrow[uu, dashed, "\varkappa_{1212}"'] \\
H^*(K_1)\otimes H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2)\otimes H^*(K_2;\mathcal{R}_2) \arrow[u, dashed, "\varkappa_1\otimes\varkappa_2"'] \arrow[r, "\cong"] & H^*(K_1)\otimes H^*(K_2)\otimes H^*(K_1;\mathcal{R}_1)\otimes H^*(K_2;\mathcal{R}_2). \arrow[u, "\varkappa_{12}\otimes\varkappa 12"']
\end{tikzcd}$$

In this diagram, $\varkappa$ always stands for the isomorphism provided by the Künneth formula, where the subscript indicates to which spaces it is applied. The bottom horizontal line is the natural isomorphism that swaps components of the tensor product.

Observe, that compositions of dash-lined maps are the respective module structures. They are connected by Künneth isomorphism maps.

Theorem 3.9 (Module structure and Künneth formula)**.** *The two module structures on $H^*(K_1\times K_2;\mathcal{R}_1\otimes\mathcal{R}_2)$ described above agree, that is Diagram (3.2) is commutative.*

Proof. This can be proved by a straightforward check on the level of cochains - we have seen in the proof of Theorem 3.8 how Künneth isomorphism looks like on a cochain level, and both the bottom horizontal map and all Δ^*-maps have a nice description in terms of cochains as well. We leave details to the reader. $\square$

3.3 Steenrod operations and characteristic classes

Steenrod powers. Below is a short summary of the facts about Steenrod powers we use in this thesis. A proper introduction to the topic can be found in [Hat02, § 4.L]

For any space X and for any prime number p there are associated homomorphisms in cohomology with coefficients in $\mathbb{F}_p$ called Steenrod squares for $p = 2$ and Steenrod powers for an odd prime p.

$$\begin{aligned} \mathrm{Sq}^i \colon H^n(X;\mathbb{Z}_2) &\to H^{n+i}(X;\mathbb{Z}_2), \\ \mathrm{P}^i \colon H^n(X;\mathbb{F}_p) &\to H^{n+2i(p-1)}(X;\mathbb{F}_p). \end{aligned}$$

For our purposes, the following basic properties of Steenrod powers are especially important:

1. For any continuous $f \colon X \to Y$, and a class $\alpha \in H^*(Y)$, $\mathrm{P}^i(f^*(\alpha)) = f^*(\mathrm{P}^i(\alpha))$.
2. For any $\alpha, \beta \in H^n(X)$, $\mathrm{P}^i(\alpha + \beta) = \mathrm{P}^i(\alpha) + \mathrm{P}^i(\beta)$.
3. For any $\alpha \in H^*(X)$, $P^i(\alpha) = \alpha^p$ if $2i = |\alpha|$ and $P^i(\alpha) = 0$ if $2i > |\alpha|$.
4. Let $P \coloneqq P^0 + P^1 + \dots$. Due to the previous property, only finitely many of these summands correspond to a non-trivial homomorphism with a domain $H^n(X)$ for each n. For any pair $\alpha, \beta \in H^*(X)$, $\mathrm{P}(\alpha \smile \beta) = \mathrm{P}(\alpha) \smile \mathrm{P}(\beta)$.
5. $\mathrm{P}^0 = id$ - the identity homeomorphism.

The Steenrod squares have similar properties, the only difference is in the third property: $\mathrm{Sq}^i(\alpha) = \alpha^2$ if $i = \deg(\alpha)$ and $\mathrm{Sq}^i(\alpha) = 0$ for any $i > \deg(\alpha)$. This property explains the names “Steenrod powers” and “Steenrod squares”.

Classes ${}_pc_j(E)$. *We are grateful to M. Crabb for introducing us to this topic. We provide here the definition of these characteristic classes and describe their relation to other objects in this thesis, as told by M. Crabb in private communication.*

Let $\mathbb{C}^d \to E \xrightarrow{\pi} B$ be a complex vector bundle. Fix an odd prime p. Let $E_0 = E \setminus 0_B$ denote the total space E with its zero section removed. Then the Thom isomorphism theorem [MS74, Thm 9.1] says that there exists a class $u_E \in H^{2d}(E, E_0)$, called the Thom class, such that the following composition of maps is an isomorphism:

$$T \colon H^*(B) \xrightarrow[\cong]{\pi^*} H^*(E) \xrightarrow[\cong]{-\smile u_E} H^{*+2d}(E, E_0).$$

In the second isomorphism, we consider the relative cup product

$$H^*(E) \otimes H^*(E, E_0) \xrightarrow{\smile} H^*(E, E_0).$$

Definition 3.10 (M. Crabb, private communication)**.** For any $j \geqslant 0$ define

$${}_pc_j(E) \in H^{2j(p-1)}(B)$$

to be a unique class such that

$$\mathrm{P}^j\, u_E = (-1)^j \pi^*({}_pc_j(E)) \smile u_E.$$

It follows from the definition that ${}_pc_0(E) = 1$ and ${}_pc_j(E) = 0$ for $j > d$.

Observe that if one takes $p = 2$ instead and considers E as a real vector bundle of rank $2d$, then this formula gives one of the definitions of Stiefel–Whitney classes [MS74, Ch. 8]:

$$\mathrm{Sq}^j\, u_E = \pi^*(\omega_j(E)) \smile u_E.$$

However, when $p > 2$, the relation between classes ${}_pc_j(E)$ and Chern classes $c_j(E)$ is more complicated:

Lemma 3.11 (M. Crabb, private communication)**.**

$$T^{-1}(\mathrm{P}\, u_E) = (-1)^d \prod_{1 \leqslant r \leqslant p-1} \big(r^d + c_1(E) r^{d-1} + \cdots + c_d(E)\big).$$

The last formula has a geometric interpretation:

Theorem 4.5.

$$e(W_p E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \prod_{1 \leqslant r \leqslant p-1} \big((rt)^d + c_1(E)(rt)^{d-1} + \cdots + c_d(E)\big).$$

To get the version of Theorem 4.5 as it is stated in the Section 4.2, use the relationship between indices and Euler classes, see Section 3.4 for a brief overview.

Combining the last two results together, we see that ${}_pc_j(E)$ is exactly the coefficient in front of $t^{(d-j)(p-1)}$ in $(-1)^d e(W_p E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \in H^*(B)[t] \subset H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$

Proof of the Lemma 3.11 provided Theorem 4.5 holds. For a class

$$\alpha \in H^*(B) \otimes \mathbb{F}_p[t] \subsetneq H^*(B \times \mathrm{B}\mathbb{Z}_p),$$

denote by $\alpha \restriction_{t=1}$ the image of α under the homomorphism of rings

$$H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p) \to H^*(B),$$

which is defined on generators by $(1 \otimes t) \restriction_{t=1} = 1$ and $(\alpha \otimes 1) \restriction_{t=1} = \alpha$ for any cohomology class $\alpha \in H^*(B)$.

Using Theorem 4.5 we see that the equality we would like to prove is equivalent to

$$\mathrm{P}\, u_E = (-1)^d \pi^*(e(W_p E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \restriction_{t=1}) \smile u_E.$$

Let us start with the trivial observation that Lemma 3.11 obviously holds for $j = 0$.

Step one: Line bundles. Let $\mathbb{C} \to E \xrightarrow{\pi} B$ be a complex line bundle. Then

$$(-1)^d e(W_p E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = - \prod_{1 \leqslant r \leqslant p-1} \big(rt + c_1(E)\big) = t^{p-1} - c_1(E)^{p-1}.$$

We need to show that ${}_p c_1(E) = -c_1(E)^{p-1}$.

Denote by i the natural embedding $(E, \emptyset) \subset (E, E_0)$. Due to the relationship between the Thom class, Euler class and the first Chern class for line bundles, we see that

$$\pi^*(c_1(E)) = i^*(u_E) \in H^2(E),$$

therefore $\pi^*(c_1(E)^{p-1}) = i^*(u_E^{p-1})$. Using the property 3 of Steenrod operations, we compute $\mathrm{P}^1(u_E) = u_E^p$. Combined together we see that

$$\mathrm{P}^1(u_E) = u_E^p = j^*(u_E^{p-1}) \smile u_E = \pi^*(c_1(E)^{p-1}) \smile u_E.$$

Step two: Sums of line bundles. Let $\mathbb{C}^d \to E \xrightarrow{\pi} B$ be such that $E = E_1 \oplus \cdots \oplus E_d$ with each of E being a complex line bundle $\mathbb{C} \to E_i \xrightarrow{\pi_i} B$.

Since the Euler class is multiplicative,

$$\begin{aligned}\mathrm{P}\, u_E = \prod_{i=1}^{d} \mathrm{P}\, u_{E_i} &= \prod_{i=1}^{d} \Big(-\pi^*\big(e(W_p E_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \restriction_{t=1}\big) \smile u_{E_i}\Big) \\ &= (-1)^d \pi^*\big(e(W_p E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \restriction_{t=1}\big) \smile u_E.\end{aligned}$$

Step three: Splitting principle. Let $\mathbb{C}^d \to E \xrightarrow{\pi} B$ be any complex vector bundle. Consider the flag fibre bundle $\mathrm{Fl}\, E$ associated with E and let f be a natural projection from $\mathrm{Fl}\,E$ to B. It is known [BT+82, § 21] that f^* is injective and f^*E splits into the sum of d line bundles, as represented on the diagram below

$$\begin{tikzcd} E_1 \oplus E_2 \oplus \cdots \oplus E_d \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{Fl}\,E \arrow[r, "f"] & B. \end{tikzcd}$$

Consequently $f^*W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ splits into the sum of vector bundles $f^*W_pE_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ over $\mathrm{Fl}\,E \times \mathrm{B}\mathbb{Z}_p$.

Since f^*E is isomorphic to the direct sum of line bundles, the total Steenrod power $\mathrm{P}\,u_{f^*E}$ equals

$$\mathrm{P}\,u_{f^*E} = (-1)^d \pi^*(e(W_p f^*E \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)\restriction_{t=1}) \smile u_{f^*E}$$

Observe that $f^*(u_E) = u_{f^*E}$, $f^*e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = e(f^*W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ and $f^*(\mathrm{P}\,u_E) = \mathrm{P}\,u_{f^*E}$. Since the map f^* is injective by construction, the statement of Lemma 3.11 follows. □

Remark 3.12. The proofs of Lemma 3.11 and Theorem 4.5 follow the same pattern and in principle, it should be possible to run both arguments in parallel and get a proof of both simultaneously.

3.4 Equivariant cohomology and Fadell–Husseini index

In this section, we make a very brief introduction to the Fadell–Husseini index. For more details see the original paper [FH88].

Cohomological index as a kernel.

Definition 3.13 (Borel construction, [AM13, Ch, V, Def. 0.2])**.** Let X be a space equipped with a left action of some group G. Then its Borel construction is

$$X \times_G \mathrm{E}G \coloneqq (X \times \mathrm{E}G)/G$$

where $\mathrm{E}G$ is the universal covering of the classifying space $\mathrm{B}G$, that is, a contractible CW-complex with a free left action of G and we consider the diagonal action of G on $X \times \mathrm{E}G$.

Observe that when the action of G on X is free, then projection $X \times \mathrm{E}G \to X$ induces a homotopy equivalence $X \times_G \mathrm{E}G \simeq X/G$. On the other hand, if the action of G on X is trivial, then $X \times_G \mathrm{E}G = X \times \mathrm{B}G$, where BG is a classifying space for G. An important example of this form is $pt \times_G \mathrm{E}G = \mathrm{B}G$.

Definition 3.14 (Equivariant cohomology, [AM13, Ch. V, Def. 0.5])**.** For a pair of a topological space X and a group G that acts on it, define G-equivariant cohomology $H^*_G(X; R)$ of X with coefficients in R as $H^*(X \times_G \mathrm{E}G; R)$.

In particular $H^*_G(pt; R) = H^*(\mathrm{B}G; R)$ is the cohomology of the group G (see [AM13, Ch. II §3]

Definition 3.15 (In bigger generality [FH88, Def. 2.1])**.** Fix a group G. Let $F \to X \xrightarrow{\pi} B$ be a fibre bundle, such that G acts trivially on B and fibrewise on X, implying that π is an equivariant map. Then an application of Borel construction to both X and B gives rise to a new fibre bundle $F \to X \times_G \mathrm{E}G \xrightarrow{\pi_{Borel}} B \times \mathrm{B}G$. The cohomological index of X over B with respect to G-action is defined as

$$\mathrm{Index}^G_B\, X := \ker \pi^*_{Borel}.$$

The property of this index that is most important for us is that it is *monotone*: Take any pair of fibre bundles with a fibrewise action of a group G and a G-equivariant map f between them

$$\begin{tikzcd} F_1 \arrow[d] & F_2 \arrow[d] \\ X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ B_1 \arrow[r, "f"] & B_2. \end{tikzcd}$$

It follows from definition that $\mathrm{Index}^G_{B_1}\, X_1 \supseteq \mathrm{Index}^G_{B_2}\, X_2$. This allows us to use the index as an obstruction to the existence of equivariant maps.

The following lemma provides another important property of the index:

Lemma 3.16 ([FH88, Prop. 3.1])**.** *Let X be a trivial fibre bundle over B, that is X is isomorphic to a bundle $F \times B$ for some fibre F. Suppose a group G acts continuously on F and trivially on B. Consider the diagonal action of G on X. With respect to this action, the index of X can be computed via the formula*

$$\mathrm{Index}^G_B\, X = H^*(B) \otimes \mathrm{Index}^G_{pt}\, F \subseteq H^*(B) \otimes H^*(\mathrm{B}G).$$

For the computations, it is often very useful to think about $\mathrm{Index}^G_B\, X$ under various different angles.

Cohomological index via spectral sequence. In this paragraph we describe our main method for index computations. Notice that since $F \to X \times_G \mathrm{E}G \xrightarrow{\pi_{Borel}} B$ is a fibre bundle, one can try to compute $H^*(X \times_G \mathrm{E}G; R)$ using Leray–Serre spectral sequence. In this case its second page equals $E_2^{q,s} = H^q(B \times \mathrm{B}G; \mathcal{H}^s(F; R))$ and the composition of maps

$$E_2^{*,0} = H^*(B \times \mathrm{B}G; R) \hookrightarrow E_2^{*,*} \to E_\infty = H^*(X \times_G \mathrm{E}G; R)$$

is exactly π^*_{Borel}. This gives us a new description of $\mathrm{Index}_B^G X$ - to this ideal belong exactly those elements of $H^*(B \times \mathrm{B}G; R) = E_2^{*,0}$ that do not survive to the last page of the spectral sequence, that is, exactly those elements such that they belong to the image of some differential d_t in the spectral sequence, $2 \leqslant t$. In particular, when $X \to B$ is a sphere bundle and the system of local coefficients $\mathcal{H}^b(F; R)$ is simple, this allows us to see $\mathrm{Index}_B^G X$ in yet another way:

Cohomological index via an Euler class, [BZ11, Prop. 3.11]. From now on, we consider only $R = \mathbb{Z}$ or $R = \mathbb{F}_p$ for p an odd prime. Consider a sphere bundle $X = S(E)$ obtained from some vector bundle $\mathbb{R}^n \to \mathrm{E} \to B$. Assume there exists fibrewise and orthonormal on each fibre action of G on E. In this case

$$S(E) \times_G \mathrm{E}G = S(E \times_G \mathrm{E}G).$$

Suppose E is such that $\mathbb{R}^n \to E \times_G \mathrm{E}G \to B \times \mathrm{B}G$ is orientable. Then from the Gysin sequence for sphere bundles [Hat17, p. 88], it follows that

$$\mathrm{Index}_B^G S(E) = \langle e(E \times_G \mathrm{E}G) \rangle,$$

where $e(E \times_G \mathrm{E}G)$ is an Euler class of the vector bundle $E \times_G \mathrm{E}G$.

Viewing the index as the Euler class allows us to take advantage of all Euler class properties, making it in some cases the most convenient choice of definition for the index.

Chapter 4

Index computations

4.1 Two useful index inclusions

In this section, we make a couple of observations about the relationship between indices of configuration spaces and indices of sphere bundles. Indices of spheres were already known for some cases, (see Section 4.2 for the literature review), and, in general, are much easier to compute. Our global strategy is to use them to compute in turn indices of fibrewise configuration spaces.

All results in this section hold for both real and complex vector bundles and for any subgroup G of the symmetric group $\mathfrak{S}_p$.

Definition 4.1 (W_pE and $\perp$)**.** Let $\mathbb{F}^n \to E \to B$ be a vector bundle, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$. Recall, ([Hat17, Prop. 1.2]), that any vector bundle can be equipped with a Euclidean structure, allowing us to take orthogonal complements. Define W_pE to be a vector bundle over B, isomorphic to the orthogonal complement of the diagonal ΔE in E^p. The isomorphism class of this bundle does not depend on the choice of the Euclidean structure.

Observe that E^p carries a natural action of the symmetric group $\mathfrak{S}_p$ that permutes its components and ΔE is the subbundle fixed by this action, therefore W_pE inherits $\mathfrak{S}_p$ action, and, in turn, $\mathbb{Z}_p \subsetneq \mathfrak{S}_p$ action, where we always consider the regular embedding of $\mathbb{Z}_p$ (see [AM13, Ex. 2.7 on p. 100] or a brief overview in Section 3.1).

Write $\perp\colon E^p \to W_pE$ for the orthogonal projection along the diagonal. The map $\perp$ is equivariant by construction.

Observation 4.2. *Notice, that fibrewise configuration space* $\mathrm{Fconf}(p, E)$ *is contained in* $E^p \setminus \Delta(E)$*, and this inclusion is* $\mathfrak{S}_p$*-equivariant. The bundle* $E^p \setminus \Delta(E)$ *is in turn*

$\mathfrak{S}_p$-equivariantly homotopy equivalent to the sphere bundle $S\big(W_pE\big)$. Therefore by the monotonicity property of the index, the following inclusion of ideals holds in the ring $H^*(B)\otimes H^*(\mathrm{B}G)$:

$$\mathrm{Index}_B^G\, S(W_pE) \subseteq \mathrm{Index}_B^G\, \mathrm{Fconf}(p, E)\,.$$

Lemma 4.3 (Inspired by p.8 of [BLZ16]). *Let E_1, E_2 be two vector bundles over the same base B. Then the following inclusion of the ideals in $H^*(B)\otimes H^*(\mathrm{B}G)$ holds*

$$\mathrm{Index}_B^G\, \mathrm{Fconf}(p, E_2)\cdot \mathrm{Index}_B^G\, S\big(W_pE_1\big) \subseteq \mathrm{Index}_B^G\, \mathrm{Fconf}(p, E_1\oplus E_2)\,.$$

Proof. Consider the projection pr_1 from the direct sum bundle $E_1\oplus E_2$ to its first component E_1. This projection induces a map pr_1^p of fibre bundles

$$pr_1^p\colon \mathrm{Fconf}(p, E_1\oplus E_2) \to E_1^p.$$

Define φ as a composition of maps pr_1^p and $\perp$, so φ is a map

$$\varphi\colon \mathrm{Fconf}(p, E_1\oplus E_2) \to W_pE_1.$$

By construction, φ is a $\mathfrak{S}_p$-equivariant map between fibre bundles over the base B. Notice that the 0-section of the bundle W_pE_1 (let us denote it simply as 0_B) is fixed by the action of $\mathfrak{S}_p$ (consequently by $\mathbb{Z}_p$ action too). This allows us to apply the Borsuk–Ulam–Bourgin–Yang theorem [FH88, Thm. 4.1] to the preimage of 0_B with respect to φ. It states that the following inclusion holds

$$\mathrm{Index}_B^G\, \varphi^{-1}(0_B)\cdot \mathrm{Index}_B^G(W_pE\setminus 0_B) \subseteq \mathrm{Index}_B^G\, \mathrm{Fconf}(p, E_1\oplus E_2)\,.$$

Let us look closer at the components of this equation. The fibre bundle $W_pE\setminus 0_B$ is fibrewise and $\mathfrak{S}_p$-equivariantly homotopy equivalent to the sphere bundle $S\big(W_pE\big)$.

The only thing left in the proof of Lemma 4.3 is to show that $\varphi^{-1}(0_B)$ is fibrewise and equivariantly homotopy equivalent to the fibre bundle $\mathrm{Fconf}(p, E_2)$. Observe, that the subspace $\varphi^{-1}(0_B)$ can be described as a set of points

$$\{(v_1^x, w_1^x, \dots, v_p^x, w_p^x; x)\} \in E_1^p\oplus E_2^p$$

such that $(v_i^x, w_i^x)\neq (v_j^x, w_j^x),\ \text{for } i\neq j,\ \text{and } v_1^x=\dots=v_p^x := v^x\}$. This description can be simplified to become

$$\{(v^x, w_1^x, \dots, w_p^x; x)\in E_1\oplus E_2^p\colon (v^x, w_i^x)\neq (v, w_j^x),\ \text{for } i\neq j\}.$$

Therefore, we compute

$$\begin{aligned}\varphi^{-1}(0_B) &= \{(v^x, w_1^x, \dots, w_p^x; x) \in E_1 \oplus E_2^p \colon (v^x, w_i^x) \neq (v, w_j^x),\ \text{for } i \neq j\} \\ &= \Delta^*\big(E_1 \times \mathrm{Fconf}(p, E_2)\big) \\ &\simeq \mathrm{Fconf}(p, E_2)\,,\end{aligned} \tag{4.1}$$

where x stands for a choice of point in B, v_i^x, w_i^x denote vectors in the fibre over x in E_1 and E_2 respectively, $\Delta\colon B \to B \times B$ is the diagonal map and $\Delta^*\big(E_1 \times \mathrm{Fconf}(p, E_2)\big)$ is a pullback along Δ of the fibre bundle $E_1 \times \mathrm{Fconf}(p, E_2)$ over the base $B \times B$.

Notice, that all relations in the equation (4.1) hold equivariantly, taking into account that E_1 inherits a trivial action from $\Delta(E_1)$ in E_1^p. □

4.2 Indices of sphere bundles containing configuration spaces

In this section, we compute the value of $\mathrm{Index}_B^G S\big(W_pE\big)$ for a vector bundle E (real or complex) over the base B, with the group $G = \mathbb{Z}_p$ and $G = \mathfrak{S}_p$. In the case $G = \mathbb{Z}_p$, similar computations appear in Jaworowski [Jaw04]. He computes index $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(E \otimes L\big)$, where $\mathbb{Z}_p$ acts trivially on a vector bundle E, by multiplication with $e^{\frac{2\pi i}{p}}$ on linear bundle $L \cong \mathbb{C}$ and diagonally on their tensor product. We are grateful to M. Crabb for suggesting us an alternative proof of this result, that allows also to compute $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big)$. We present his proof below.

Section specific notation and assumptions. TIn the next two sections (4.2-4.3) we assume that the bundle E is orientable. In Section 4.4 we will see that the same results hold for non-orientable E as well.

Recall, that earlier we introduced the following definition to announce the results of this chapter.

Definition 4.4. Given a complex vector bundle $\mathbb{C}^d \to E \to B$ and a prime number p, define the element $\check{\mathrm{c}}\mathrm{h}(E)$ in $H^{2d(p-1)}(B \times \mathrm{B}\mathbb{Z}_p)$ by

$$\check{\mathrm{c}}\mathrm{h}(E) \coloneqq \prod_{1 \leqslant r \leqslant p-1} \big((rt)^d + c_1(E)(rt)^{d-1} + \dots + c_d(E)\big).$$

Note, that the multiplicativity property of the total Chern class with respect to a direct sum operation, implies $\check{\mathrm{c}}\mathrm{h}()$ is multiplicative as well, that is,

$$\check{\mathrm{c}}\mathrm{h}\big(E \oplus E'\big) = \check{\mathrm{c}}\mathrm{h}(E) \cdot \check{\mathrm{c}}\mathrm{h}\big(E'\big)\,.$$

This definition is motivated by the next two theorems.

Theorem 4.5 (Complex case, in $\mathbb{Z}_p$ case - due to M.Crabb, private communication)**.** *Let $\mathbb{C}^d \to E \to B$ be a complex vector bundle. Then*

$$\mathrm{Index}_B^{\mathbb{Z}_p} S(W_pE) = \langle \check{\mathrm{c}}\mathrm{h}(E)\rangle \subsetneq H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p).$$

Moreover, there exists the unique element $\varsigma_E \in H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$, such that

$$\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \varsigma_E = \check{\mathrm{c}}\mathrm{h}(E)\,.$$

This element generates the index of $S(W_pE)$ with respect to the action of symmetric group $\mathfrak{S}_p$, that is,

$$\mathrm{Index}_B^{\mathfrak{S}_p} S(W_pE) = \langle \varsigma_E\rangle \subsetneq H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p).$$

Remark 4.6.

- Remember that we have seen in Section 3.3 that provided the Theorem 4.5 holds, the coefficients of $\check{\mathrm{c}}\mathrm{h}(E)$ viewed as a polynomial in t are ${}_pc_j(E)$ and can be described in terms of Steenrod operations on the Thom class u_E.
- To see that $\check{\mathrm{c}}\mathrm{h}(E)$ is indeed contained in the image of $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$, observe that $\check{\mathrm{c}}\mathrm{h}(E)$ is invariant with respect to the substitution $t \to kt$, for any $k \in \mathbb{F}_p^{\times}$, therefore it is a polynomial in $t^{(p-1)}$ and $t^{p-1} = \mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} b$.
- The preimage of $\check{\mathrm{c}}\mathrm{h}(E)$ with respect to $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ is uniquely defined since $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ is injective.

Theorem 4.7 (Real case)**.** *Let $\mathbb{R}^n \to E \to B$ be a real orientable vector bundle and $\mathbb{C}^n \to E \otimes_{\mathbb{R}} \mathbb{C} \to B$ its complexification. Then*

$$\begin{aligned}\mathrm{Index}_B^{\mathbb{Z}_p} S(W_pE) &= \langle \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\rangle \\ &= \langle \Big(\prod_{1\leqslant r\leqslant p-1} \sum_{i=0}^{\lfloor n/2\rfloor} (-1)^i p_i(E)(rt)^{n-2i}\Big)^{\frac{1}{2}}\rangle.\end{aligned}$$

In contrast, the index of $S(W_pE)$ with respect to $\mathfrak{S}_p$ depends on the parity of n:

1. *When n is even, $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}$ belongs to the image of $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$. Let us name ζ_E its preimage in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$. Then ζ_E is the unique generator of the whole index of* $\mathrm{Index}_B^{\mathfrak{S}_p} S(W_pE)$.

2. *When n is odd,* $\operatorname{Index}_B^{\mathfrak{S}_p} S\big(W_p E\big)$ *is generated by two elements,* ε_E *and* θ_E *in* $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$*, such that*

$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \varepsilon_E = e t^{\frac{p-3}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,,$$
$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \theta_E = t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,.$$

Remark 4.8. The square root of $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})$ is defined only up to a sign, however, we are interested only in the ideal, generated by this element, which is independent of the sign choice.

Remark 4.9. Let us check that for E of an odd rank $t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ indeed belongs to the image of $\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$. First of all, let us fix a choice of a square root for this particular E for the rest of this paragraph. Without loss of generality assume that the leading coefficient of $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ is $t^{\frac{n(p-1)}{2}}$. Notice, that

$$\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(kt) \cdot \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(kt) = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})(kt) = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})(t),$$

for any $k \in \mathbb{F}_p^{\times}$. Therefore,

$$\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(kt) = \alpha_k \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(t),$$

for some $\alpha_k \in \{-1, 1\} \in \mathbb{F}_p$. In particular, this holds for the leading term of this polynomial, that is,

$$(kt)^{\frac{n(p-1)}{2}} = \alpha_k \cdot t^{\frac{n(p-1)}{2}}.$$

Since n is odd, we see that $\alpha_k = k^{\frac{p-1}{2}}$. This implies that $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(t)$ is a polynomial containing only odd powers of $t^{\frac{p-1}{2}}$, therefore $t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(t)$ is a polynomial in $t^{(p-1)}$ and hence

$$t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{1/2}(t) \in \operatorname{im} \operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} H^*(B \times \mathrm{B}\mathfrak{S}_p).$$

The rest of this section is organised in the following way: We start Subsection 4.2.1 by presenting the proof of Theorem 4.5 for the case of $\mathbb{Z}_p$ due to M. Crabb. Then we notice that $S\big(W_p E\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ is a $(p-1)!$-fold covering of $S\big(W_p E\big) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p$ and prove the case of $\mathfrak{S}_p$-action by a short spectral sequences argument.

In Subsection 4.2.2 to prove Theorem 4.7 in $\mathbb{Z}_p$ action case we use complexification of E and reduce this case to that of Theorem 4.5 which we have just proved. The case for real E and $\mathfrak{S}_p$-action follows from the same observation about $(p-1)!$-fold covering but with a little more of spectral sequences work involved.

4.2.1 Complex case

Cyclic group action. To start with the proof, consider the Borel construction with respect to $\mathbb{Z}_p$ action associated with the sphere bundle $S(W_pE)$, assuming the trivial action of $\mathbb{Z}_p$ on B. We obtain the fibre bundle

$$S(W_p\mathbb{C}^d) \to S(W_pE) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \to B \times \mathrm{B}\mathbb{Z}_p.$$

Observe that when p is odd, any free $\mathbb{Z}_p$ action on any vector bundle does not change its orientation, since all elements of $\mathbb{Z}_p$ have an odd order. From this and the fact that any complex bundle E is orientable, we conclude that the vector bundle $W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ over $B \times \mathrm{B}\mathbb{Z}_p$ is orientable as well. As was mentioned in Section 3.4, page 46, in this case $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big)$ is generated by the Euler class of $W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$,

$$\mathrm{Index}_B^{\mathbb{Z}_p} S(W_pE) = \langle e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)\rangle \subsetneq H^*(B \times \mathrm{B}\mathbb{Z}_p).$$

Our first Lemma on the way to the proof of Theorem 4.5 is

Lemma 4.10 (M.Crabb, private communication)**.** *Let L be a one-dimensional complex representation of $\mathbb{Z}_p$ such that 1, considered as an additive generator of $\mathbb{Z}_p$, acts on L by sending z to $e^{2\pi i/p}z$ for any $z \in \mathbb{C} \cong L$. Denote by $\underline{L}$ the trivial bundle $L \times B \to B$. Then*

$$e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \prod_{1\leqslant j\leqslant p-1} e\big((E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big).$$

Proof of Lemma 4.10. Observe that W_pE is $\mathbb{Z}_p$-equivariantly isomorphic as a complex vector bundle to $E \otimes \underline{W_p}\mathbb{C}$, where we assume the trivial action on E and consider the diagonal action on the tensor product. Following our standard notation, $\underline{W_p}\mathbb{C}$ is a trivial bundle over B with a fibre $W_p\mathbb{C}$. The latter splits into the sum of trivial one-dimensional $\mathbb{Z}_p$ representations:

$$W_p\mathbb{C} \cong \bigoplus_{1\leqslant j\leqslant p-1} L^{\otimes j},$$

(if $\alpha = e^{2\pi i/p}$, then for any $1 \leqslant j \leqslant p-1, t \neq 0$, $(z, \alpha^{(p-1)j}z, \alpha^{(p-1)(j-1)}z, \dots, \alpha^{p-1}z)$ in $W_p\mathbb{C}$ spans a complex one-dimensional $\mathbb{Z}_p$-representation isomorphic to $L^{\otimes j}$). This entails the following $\mathbb{Z}_p$-equivariant isomorphisms of vector bundles over the base B:

$$W_pE \cong E \otimes \bigoplus_{1\leqslant j\leqslant p-1} \underline{L}^{\otimes j} \cong \bigoplus_{1\leqslant j\leqslant p-1} E \otimes \underline{L}^{\otimes j}.$$

Once again, the action of $\mathbb{Z}_p$ is assumed to be trivial on E and diagonal on all tensor products. Passing on to the Borel construction we get

$$W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \cong \Big(\bigoplus_{1 \leqslant j \leqslant p-1} E \otimes \underline{L}^{\otimes j}\Big) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \cong \bigoplus_{1 \leqslant j \leqslant p-1} (E \ \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p.$$

Using the multiplicativity property of the Euler class we see

$$e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \prod_{1 \leqslant j \leqslant p-1} e\big((E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big). \tag{4.2}$$

The Euler class of $E \otimes \underline{L}^{\otimes j}$ is correctly defined for the same reason as for $W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$, that is since any complex bundle E is orientable and $\mathbb{Z}_p$ action can change the orientation of a fibre of $E \otimes \underline{L}^{\otimes j}$ either. □

We turn our attention to the individual elements in the product in equation (4.2).

Theorem 4.11 (M.Crabb, private communication)**.** *For any $1 \leqslant j \leqslant p-1$, E and $\underline{L}$ as in the previous lemma, the Euler class of $(E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ can be computed via formula*

$$e((E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \sum_{k=0}^{d} c_k(E) \cdot c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)^{d-k}.$$

In this case $c_k(E)$, $0 \leqslant k \leqslant d$ belong to $H^*(B)$ which we identify with the subring $H^*(B) \otimes 1$ of $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$. Since $\underline{L}^{\otimes j}$ is a trivial bundle over B, $c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ belong to $1 \otimes H^*(\mathrm{B}\mathbb{Z}_p)$.

For the proof of Theorem 4.11 as well our main Theorem 4.5 the following technical lemma is useful:

Lemma 4.12. *Let M and M' be vector bundles over the base B equipped with free fibrewise $\mathbb{Z}_p$-actions. Consider a diagonal $\mathbb{Z}_p$ action on $M \otimes M'$. There exists an isomorphism of vector bundles over $B \times B\mathbb{Z}$*

$$(M \otimes M') \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \cong (M \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \otimes (M' \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p).$$

Proof. We build the following diagram and prove that it is commutative.

$$\begin{tikzcd}
\Delta_B^*(M \times M') \times \mathrm{E}\mathbb{Z}_p \arrow[r, "\exists\varphi", "\cong"'] \arrow[d, "\otimes"] & \Delta^*_{B\times \mathrm{E}\mathbb{Z}_p}\big((M \times \mathrm{E}\mathbb{Z}_p) \times (M \times EZ)\big) \arrow[d, "\otimes"] \\
(M \otimes M') \times \mathrm{E}\mathbb{Z}_p \arrow[r, "\varphi_\otimes"] \arrow[d, "/\mathbb{Z}_p"] & (M \times \mathrm{E}\mathbb{Z}_p) \otimes (M' \times \mathrm{E}\mathbb{Z}_p) \arrow[d, "/\mathbb{Z}_p"] \\
(M \otimes M') \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \arrow[r, dashed, "\varphi_\otimes/\mathbb{Z}_p"] & (M \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \otimes (M' \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p),
\end{tikzcd}$$

where Δ_B is a diagonal embedding of B into $B\times B$ and $\Delta_{B\times \mathrm{E}\mathbb{Z}_p}$ is the diagonal embedding of $B \times \mathrm{E}\mathbb{Z}_p$ into $(B \times \mathrm{E}\mathbb{Z}_p) \times (B \times \mathrm{E}\mathbb{Z}_p)$. We assume the diagonal action of $\mathbb{Z}_p$ on all direct products.

We prove that there exists $\mathbb{Z}_p$-equivariant isomorphism φ of vector bundles over $B \times \mathrm{E}\mathbb{Z}_p$ between the elements of the first row. Using this isomorphism, one can pass in each fibre from the direct sum to the tensor product, which induces a $\mathbb{Z}_p$-equivariant isomorphism $\varphi_\otimes$ between the bundles from row two. To construct $\varphi_\otimes$, recall that the tensor product of vector spaces is a continuous functor, allowing us to build a tensor product of two vector bundles fibre by fibre, see [MS74, Ch. 3f]. It follows from the universal property of the tensor product that $\varphi_\otimes$ is, in fact, an isomorphism. If φ is equivariant then so does $\varphi_\otimes$. Applying the quotient by the fibrewise action of $\mathbb{Z}_p$ on both sides, we get the desired isomorphism $\varphi_\otimes/\mathbb{Z}_p$.

Let us now show that such φ exists. Notice that the map $\Delta_{B\times \mathrm{E}\mathbb{Z}_p}$ can be viewed as a composition of the map $\Delta_B \times \mathrm{id}_{\mathrm{E}\mathbb{Z}_p}$ from the space $B \times \mathrm{E}\mathbb{Z}_p$ to $B \times B \times \mathrm{E}\mathbb{Z}_p$ witch the map $id_{B\times B} \times \Delta_{\mathrm{E}\mathbb{Z}_p}$ from $B \times B \times \mathrm{E}\mathbb{Z}_p$ to $B \times B \times \mathrm{E}\mathbb{Z}_p \times \mathrm{E}\mathbb{Z}_p$. In this notation the bundle $\Delta_B^*(M \times M') \times \mathrm{E}\mathbb{Z}_p$ can be seen as a pullback of the bundle $(M \times M) \times \mathrm{E}\mathbb{Z}_p$ along the map $\Delta_B \times \mathrm{id}_{\mathrm{E}\mathbb{Z}_p}$. On the other hand, the bundle $(M \times M) \times \mathrm{E}\mathbb{Z}_p$ is itself a pullback of the bundle $(M \times \mathrm{E}\mathbb{Z}_p) \times (M \times EZ)$ along the map $id_{B\times B} \times \Delta_{\mathrm{E}\mathbb{Z}_p}$, up to homomorphisms that permute components of the product. Combining these arguments together we get the following chain of isomorphisms of vector bundles:

$$\begin{aligned}
\Delta_B^*(M \times M') \times \mathrm{E}\mathbb{Z}_p &\cong (\Delta_B \times \mathrm{id}_{\mathrm{E}\mathbb{Z}_p})^*\big((M \times M') \times \mathrm{E}\mathbb{Z}_p\big) \\
&\cong (\Delta_B \times \mathrm{id}_{\mathrm{E}\mathbb{Z}_p})^* \circ (\mathrm{id}_{B\times B} \times \Delta_{\mathrm{E}\mathbb{Z}_p})^*\big((M \times \mathrm{E}\mathbb{Z}_p) \times (M' \times \mathrm{E}\mathbb{Z}_p)\big) \\
&\cong \Delta^*_{B\times \mathrm{E}\mathbb{Z}_p}\big((M \times \mathrm{E}\mathbb{Z}_p) \times (M' \times \mathrm{E}\mathbb{Z}_p)\big).
\end{aligned}$$

All these isomorphisms are equivariant by construction. □

Proof of the Theorem 4.11. Let $\mathrm{Fl}\, E$ be the flag fibre bundle associated with E and f a projection map from $\mathrm{Fl}\, E$ to B. Then by splitting principle [BT+82, § 21] f^*E splits into the sum of d complex line bundles. Let us name them $E_1, \ldots, E_d$.

$$\begin{tikzcd} E_1 \oplus E_2 \oplus \cdots \oplus E_d \arrow[r] \arrow[d] & E \arrow[d] \\ \mathrm{Fl}\,E \arrow[r, "f"] & B. \end{tikzcd}$$

Then the pullback of $(E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ along f^* splits into the sum of d complex bundles bundles $(E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$, $1 \leqslant i \leqslant d$,

$$\begin{tikzcd} \bigoplus_{i=1}^{d} (E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \arrow[r] \arrow[d] & (E \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \arrow[d] \\ \mathrm{Fl}\,E \times \mathrm{B}\mathbb{Z}_p \arrow[r, "f \times \mathrm{id}"] & B \times \mathrm{B}\mathbb{Z}_p. \end{tikzcd}$$

Let us look at individual summands of this splitting. Using Lemma 4.12 we rewrite $(E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ as

$$(E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \cong (E_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \otimes (\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p).$$

Observe that both $(E_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ and $(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ are line bundles, therefore their first Chern classes are related by the formula (see [BT+82, § 20])

$$\begin{aligned} e\big((E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) &= c_1\big((E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) \\ &= c_1(E_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) + c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) \\ &= c_1(E_i) \otimes 1 + c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p). \end{aligned}$$

The last equality holds since we assumed trivial action on E_i implying the equality

$$E_i \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p = E_i \times \mathrm{B}\mathbb{Z}_p.$$

as a bundle over $B \times \mathrm{B}\mathbb{Z}_p$.

Combining all computations together, we get

$$\begin{aligned} e\Big(\bigoplus_{i=1}^{d} (E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\Big) &= \prod_{i=1}^{d} e\big((E_i \otimes \underline{L}^{\otimes j}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) \\ &= \prod_{i=1}^{d} \big(c_1(E_i) + c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)\big). \end{aligned}$$

The result of the theorem follows using the relation between the first Chern classes of E_i and that of E and the fact that f^* is injective. $\square$

Finally, we are ready to prove Theorem 4.5:

Proof of the Theorem 4.5, the cyclic group case, following the suggestion by M. Crabb. From Lemma 4.12 it follows that

$$\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \cong (\underline{L} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)^{\otimes j},$$

therefore

$$c_1(\underline{L}^{\otimes j} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = j c_1(\underline{L} \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p).$$

Now if we would know that $c_1(L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = t \in H^2(\mathrm{B}\mathbb{Z}_p)$, by the combination of Lemmas 4.11 and 4.10 the result would follow.

Since $L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ is a complex line bundle over $\mathrm{B}\mathbb{Z}_p$, there exists the unique up to a homotopy map φ such that the following diagram is a pullback diagram:

$$\begin{tikzcd} L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \arrow[r] \arrow[d] & \gamma\left(\mathbb{C}^1, \mathbb{C}^\infty\right) \arrow[d] \\ \mathrm{B}\mathbb{Z}_p \arrow[r, "\varphi"] & \mathrm{Gr}(\mathbb{C}^1, \mathbb{C}^\infty). \end{tikzcd}$$

and by the definition $c_1(L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \varphi^* c_1(\gamma\left(\mathbb{C}^1, \mathbb{C}^\infty\right))$.

Observe that $\mathrm{Gr}(\mathbb{C}^1, \mathbb{C}^\infty) = \mathbb{C}P^\infty$ as a model for BS^1 and an infinite lens space L_p^∞ can be chosen as a model for $\mathrm{B}\mathbb{Z}_p$. From this point of view $\gamma\left(\mathbb{C}^1, \mathbb{C}^\infty\right) \cong L' \times_{S^1} \mathrm{E}S^1$, where L' is one dimensional complex representation of S^1, acting by rotations. Now if we identify S^1 with $\{z \in \mathbb{C} : |z| = 1\}$ we can *choose* the embedding $\mathbb{Z}_p \hookrightarrow S^1$ to be the one that sends 1 – the generator of $\mathbb{Z}_p$ – to $e^{2\pi i/p}$. Once we fix an embedding $\mathbb{Z}_p \hookrightarrow S^1$, ES^1 can be used as a model for $\mathrm{E}\mathbb{Z}_p$. After these choices are taken into account, the diagram above becomes

$$\begin{tikzcd} L \times_{\mathbb{Z}_p} ES^1 \arrow[r, "/S^1"] \arrow[d] & L' \times_{S^1} \mathrm{E}S^1 \arrow[d] \\ L_p^\infty = \mathrm{B}\mathbb{Z}_p \arrow[r, "\varphi=/S^1"] & \mathbb{C}P^\infty = BS^1, \end{tikzcd}$$

where the horizontal maps are coverings induced by quotiening out the action of the group $S^1/\mathbb{Z}_p \cong S^1$.

Suppose the isomorphism $H^*(BS^1) \cong \mathbb{F}_p[t]$ is fixed. Then we can *choose* an isomorphism $H^*(\mathrm{B}\mathbb{Z}_p) \cong \mathbb{F}_p[t] \otimes \Lambda[e]$ such that φ^* sends t to t, as would be expected from the notation. Then it follows that

$$c_1(L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p) = \varphi^* c_1\Big(\gamma\left(\mathbb{C}^1, \mathbb{C}^\infty\right)\Big) = t.$$

This finishes the computation of $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big)$. $\square$

Remark 4.13. Observe that all choices mentioned above do not influence the resulting value of $\check{\mathrm{c}}\mathrm{h}(E)$. Any equivalent choice re-scales the value of $c_1(L \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ by an element $\mathbb{F}_p^\times$ and $\check{\mathrm{c}}\mathrm{h}(E)$ is invariant with respect to such rescaling.

Remark 4.14. Observe that as a by-product of this proof, we get a new interpretation of the class $\check{\mathrm{c}}\mathrm{h}(E)$, namely as the Euler class of the bundle $W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$.

Symmetric group action.

Proof of Theorem 4.5, the symmetric group case. To compute $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$ we consider an appropriate fibre bundle:

$$S\big(W_pE\big) \to S\big(W_pE\big) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to B \times \mathrm{B}\mathfrak{S}_p.$$

The regular embedding $\mathbb{Z}_p \hookrightarrow \mathfrak{S}_p$ allows us to see $S\big(W_pE\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathfrak{S}_p$ as a $(p-1)!$-fold cover over $S\big(W_pE\big) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p$. In this case, we use $\mathrm{E}\mathfrak{S}_p$ as a model for $\mathrm{E}\mathbb{Z}_p$ in the Borel construction $S\big(W_pE\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$. Let us name the covering map π_T ("T" stands for "total space"). In a similar way, $B \times \mathrm{B}\mathbb{Z}_p$ is a $(p-1)!$-fold cover over $B \times \mathrm{B}\mathfrak{S}_p$ with a covering map π_B. Observe that π_T is a map of fibre bundles, and the induced map between their bases is exactly π_B.

For clarity, we have depicted these maps on commutative diagram (4.3).

$$\begin{tikzcd}
S\big(W_pE\big) \arrow[d] & S\big(W_pE\big) \arrow[d] \\
S\big(W_pE\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathfrak{S}_p \arrow[r, "\pi_T"] \arrow[d] & S\big(W_pE\big) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \arrow[d] \\
B \times \mathrm{B}\mathbb{Z}_p \arrow[r, "\pi_B"] & B \times \mathrm{B}\mathfrak{S}_p.
\end{tikzcd} \tag{4.3}$$

Let us consider the map between Leray–Serre spectral sequences associated with this map of fibre bundles. See Figure 4.1 for a sketch.

Applying Künneth formula with local coefficients ([Gre06] or Theorem 3.8) we see that the second pages of both of these spectral sequences can be described as

$$E_2^{*,q} \cong H^*(B) \otimes H^*\Big(\mathrm{B}G; \mathcal{H}^q\big(S\big(W_p\mathbb{C}^d\big); \mathbb{F}_p\big)\Big),$$

where G is either $\mathbb{Z}_p$ or $\mathfrak{S}_p$, and $\mathcal{H}^q\big(S\big(W_p\mathbb{C}^d\big); \mathbb{F}_p\big)$ local coefficients system consisting of $H^q\big(S\big(W_p\mathbb{C}^d\big); \mathbb{F}_p\big)$ viewed as a $\pi_1(B \times \mathrm{B}G)$-module. It follows immediately that both

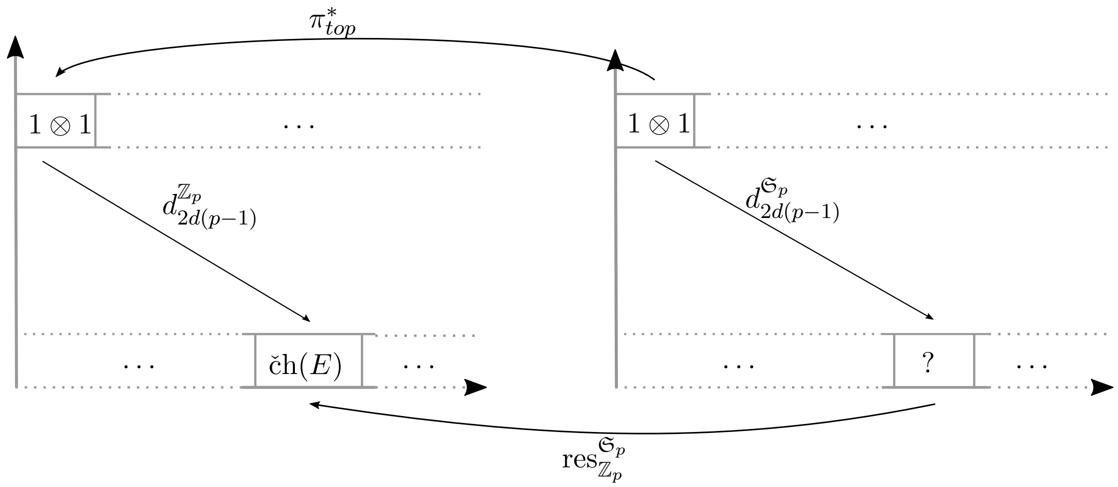

FIGURE 4.1: The map between $E_{2d(p-1)}$-pages induced by (4.3), complex E

spectral sequences have just two non-zero rows: when $q = 0$ and when $b = 2q(p-1) - 1$. It follows that nothing changes till the page $d(p-1)$,

$$E_2^{*,*} = \cdots = E_{2d(p-1)}^{*,*}.$$

Our next step is to see that for both spectral sequences, their respective local coefficient systems are simple.

When $G = \mathbb{Z}_p$ this follows from the orientability of $W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$ which we have already shown.

To see this for $G = \mathfrak{S}_p$, remember, that E is a complex vector bundle and thus is orientable. Also, since real rank E is even, an automorphism of E^p induced by swapping any two copies of E in E^p does not change an orientation of E^p. Therefore $\pi_1(B \times \mathrm{B}\mathfrak{S}_p)$ acts trivially on the cohomology of the fibre of $S\big(W_pE\big) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p$.

Consequently, both spectral sequences have an element $(1 \otimes 1) \in E_2^{*,2d(p-1)-1}$ that generates their respective top rows as $H^*(B \times BG)$-module on all pages till the last page $E_{2d(p-1)}$. Notice also that $\pi^*_{top}(1 \otimes 1) = 1 \otimes 1$, where π^*_{top} is the restriction of the map between spectral sequences to the row $E_{2d(p-1)}^{*,2d(p-1)-1}$.

Let us denote by $d^{\mathbb{Z}_p}_{2d(p-1)}$ and $d^{\mathfrak{S}_p}_{2(p-1)+1}$ differentials on the last pages of spectral sequences associated with $\mathbb{Z}_p$ and $\mathfrak{S}_p$ action respectively.

The fact that $\mathrm{Index}^{\mathbb{Z}_p}_B\, S\big(W_pE\big) = \langle \check{\mathrm{ch}}(E) \rangle$ implies that $d^{\mathbb{Z}_p}_{2d(p-1)}(1 \otimes 1) = \check{\mathrm{ch}}(E)$.

Then

$$\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{2d(p-1)}(1 \otimes 1) = d^{\mathbb{Z}_p}_{2d(p-1)} \pi^*_{top}(1 \otimes 1) = d^{\mathbb{Z}_p}_{2d(p-1)}(1 \otimes 1) = \check{\mathrm{ch}}(E)\,.$$

Remember that $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ is an injection (since it is a map induced in cohomology with coefficients in $\mathbb{F}_p$ from a $(p-1)!$-covering map) and therefore the second part of the statement of Theorem 4.5 follows. □

Let us prove Theorem 4.7 next.

4.2.2 Real case

Let $\mathbb{R}^n \to E \to B$ be a real orientable vector bundle. Similar to the case when E was a complex bundle, to compute $\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big)$ and $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$ we consider fibre bundles

$$S(W_p\mathbb{R}^n) \to S(W_pE) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \to B \times \mathrm{B}\mathbb{Z}_p,$$
$$S(W_p\mathbb{R}^n) \to S(W_pE) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to B \times \mathrm{B}\mathfrak{S}_p.$$

In the same way, as we did in the case when E was a complex bundle, we will first prove the theorem for the $\mathbb{Z}_p$ group action and then use a comparison of spectral sequences to get results for the index with respect to the action of $\mathfrak{S}_p$.

Cyclic group action. The trick is to reduce the real case to the complex one, so we can use Theorem 4.5. As the appearance of the Pontryagin classes suggests, we will use a standard for such theorems trick, and consider a complexification of a suitable bundle.

Proof of the Theorem 4.7, cyclic group case. We start with a complexification. Observe that there exists an isomorphism of real vector bundles

$$W_pE \otimes_{\mathbb{R}} \mathbb{C} \cong_{\mathbb{R}} W_pE \otimes \mathbb{R}^2 \cong_{\mathbb{R}} W_pE \oplus W_pE.$$

This isomorphism is $\mathbb{Z}_p$-equivariant, where we assume that $\mathbb{Z}_p$ acts trivially on $\mathbb{C} \cong_{\mathbb{R}} \mathbb{R}^2$ and diagonally on tensor products.

Since E is orientable and p is an odd prime, cyclic permutations do not change the orientation of E^p, and we are once again in a situation when $\pi_1(B \times \mathrm{B}\mathbb{Z}_p)$ acts trivially on the cohomology of fibres. In particular $e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)$ is correctly defined.

From the multiplicativity of an Euler class and the isomorphism we established above it follows that

$$e\big((W_pE \otimes_{\mathbb{R}} \mathbb{C}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) = e(W_pE \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p)^2.$$

To compute the Euler class on the LHS of the equation above, observe that

$$\begin{aligned} W_pE \otimes_{\mathbb{R}} \mathbb{C} &\cong (E \otimes_{\mathbb{R}} W_p\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \\ &\cong (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} (W_p\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}) \\ &\cong (E \otimes_{\mathbb{R}} \mathbb{C}) \otimes_{\mathbb{C}} W_p\mathbb{C} \\ &\cong W_p(E \otimes_{\mathbb{R}} \mathbb{C}). \end{aligned}$$

We can apply Theorem 4.5 to $W_p(E \otimes_{\mathbb{R}} \mathbb{C})$ and compute the Euler class of the vector bundle $W_p(E \otimes_{\mathbb{R}} \mathbb{C}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$:

$$e\big((W_pE \otimes_{\mathbb{R}} \mathbb{C}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) = e\big(W_p(E \otimes_{\mathbb{R}} \mathbb{C}) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p\big) \coloneqq \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})\,.$$

This finishes the $\mathbb{Z}_p$ part of the theorem. □

Symmetric group action.

Proof of the Theorem 4.7, symmetric group case. To get results for the $\mathfrak{S}_p$, consider again the diagram (4.3), assuming now that E is a real bundle. Applying Künneth formula with local coefficients ([Gre06] or Theorem 3.8) we see that the second pages of both of these spectral sequences can be described as

$$E_2^{*,q} = H^*(B) \otimes H^*\Big(\mathrm{B}G; \mathcal{H}^q\big(S\big(W_p\mathbb{R}^d\big); \mathbb{F}_p\big)\Big).$$

Such a spectral sequence again has just two non-zero rows, when $q = 0$ and when $q = n(p-1)-1$. In particular, this implies that $E_2 = \cdots = E_{n(p-1)}$.

Notice, that to prove in the complex case that both spectral sequences have simple local coefficients systems we used two facts: that E is orientable and that E is even dimensional. We never used the complex structure itself. Therefore when n is even, exactly the same arguments as in the case of complex vector bundle lead to the equality

$$\mathrm{Index}_B^{\mathbb{Z}_p} S\big(W_pE\big) = \langle \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \rangle.$$

However, for an odd n the coefficient system $\mathcal{H}^b\big(S\big(W_p\mathbb{R}^n\big); \mathbb{F}_p\big)$ is not simple. Indeed, any transposition that swaps two components of E^p changes the orientation of W_pE. Consequently in the respective spectral sequence, via Künneth formula for local coefficients, we see that

$$E_2^{*,n(p-1)-1} \cong H^*(B; \mathbb{F}_p) \otimes H^*(\mathrm{B}\mathfrak{S}_p; \mathcal{F}_p),$$

where $\mathcal{F}_p$ is a one-dimensional sign representation of $\mathfrak{S}_p$. Here we used once again that E is orientable and therefore no $\pi_1(B)$ action on the cohomology of the fibre arises.

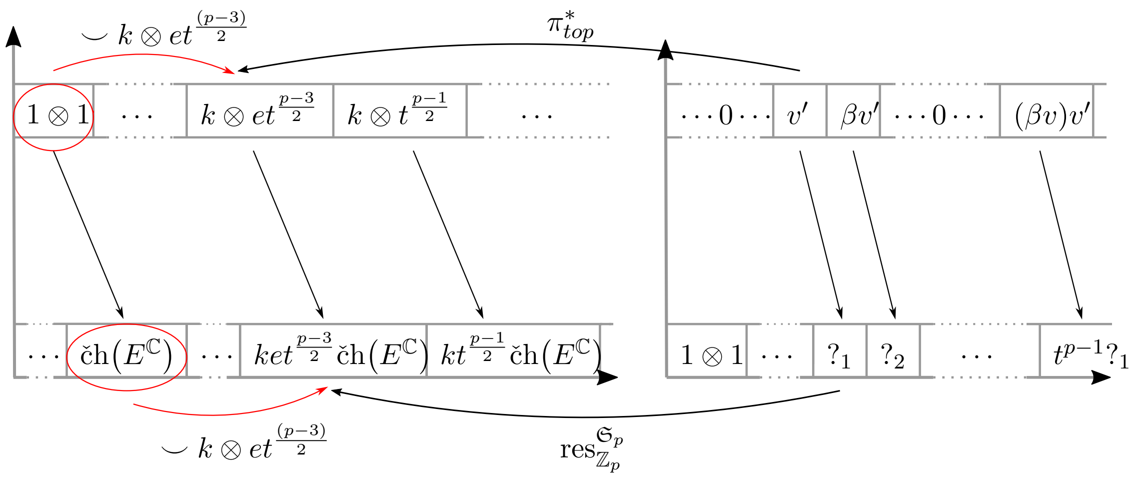

FIGURE 4.2: The map between $E_{d(p-1)}$-pages induced by (4.3), real E

From now on we assume that n is odd.

Remember that $\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$ is an injection. Similarly, the π^*_T induced from the map between total spaces of Borel constructions is an injection too. In the language of the maps between spectral sequences, these are injections on $E_2^{*,0} = \cdots = E_{n(p-1)}^{*,0}$ and E_∞ respectively.

Let us call again π^*_{top} the map between top rows on the respective $E_{n(p-1)}$-pages. It turns out it is injective too. This can be seen either directly by adapting our standard injectivity argument for local coefficients (see [Knu18, Lem. 8.1.1]) or one can show it indirectly. Suppose the opposite is true. Then there exists an element X in the top row of the $E_{n(p-1)}$-page of the spectral sequence for $\mathfrak{S}_p$-action, such that $\pi^*_{top}(X) = 0$. Then

$$d^{\mathbb{Z}_p}_{n(p-1)}\pi^*_{top}(X) = 0 = \mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{n(p-1)}(X).$$

Since $\mathrm{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$ is injective, it follows that $d^{\mathfrak{S}_p}_{n(p-1)}(X) = 0$ and X survives to E_∞. Since π^*_T is injective, this implies that there is an element from the top row in the spectral sequence associated with $\mathbb{Z}_p$ action that survives to E_∞ as well. Therefore we came to a contradiction – we already know that this is not the case.

Equipped with these observations, we look closer at the map of the respective spectral sequences. Some of the details of the argument that is to follow are represented in Figure 4.2.

Consider the top row in the spectral sequence for $S(W_pE) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p$. As we have already shown,

$$E_{d(p-1)}^{*,d(p-1)-1} \cong H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p; \mathcal{F}_p).$$

It is known (see [CLM76, Prop. 5.1]) that $H^*(\mathrm{B}\mathfrak{S}_p; \mathcal{F}_p)$ has an additive basis given by the set of elements $\{(\beta v)^s \beta^\varepsilon v'\}$ where β is the Bockstein homomorphism, $\varepsilon = 0, 1$, $s \geqslant 0$,

v is of degree $2(p-1)-1$, and v' is of degree $(p-2)$. In particular, in all degrees, this cohomology is either zero or one-dimensional $\mathbb{F}_p$-vector space. The smallest non-zero degrees are $p-2$ and $p-1$, generated by $\beta^\varepsilon v'$.

Observe that a map between these spectral sequences is the map of $H^*(B)$-modules, in particular, π^*_{top} sends an element of the form $1 \otimes x \in H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p; \mathcal{F}_p)$ to an element of the form $1 \otimes x' \in H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$.

Combining this observation with the fact that π^*_{top} is injective, we see that it should send an element of the form $1 \otimes (\beta v)^s \beta^\varepsilon v'$ of degree $(p-2)+\varepsilon+2s(p-1)$ to the element of the same degree in $1 \otimes H^*(\mathrm{B}\mathbb{Z}_p)$. Since in each degree $H^*(\mathrm{B}\mathbb{Z}_p)$ is one-dimensional as $\mathbb{F}_p$-vector space, there is a unique option up to a non-zero scalar k where such an element can be sent, namely $1 \otimes k e^{1-\varepsilon} t^{(2s+1)(p-1)/2-(1-\varepsilon)}$.

This allows us to compute

$$\begin{aligned}
\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{n(p-1)}\big(1 \otimes (\beta v)^s \beta^\varepsilon v'\big) &= d^{\mathbb{Z}_p}_{n(p-1)} \pi^*_{top}\big(1 \otimes (\beta v)^s \beta^\varepsilon v'\big) \\
&= d^{\mathbb{Z}_p}_{n(p-1)}\big(1 \otimes k e^{1-\varepsilon} t^{(2s+1)(p-1)/2-(1-\varepsilon)}\big) \\
&= k e^{1-\varepsilon} t^{s(p-1)} t^{(p-1)/2-(1-\varepsilon)} d^{\mathbb{Z}_p}_{n(p-1)}(1 \otimes 1) \\
&= k e^{1-\varepsilon} t^{s(p-1)} t^{(p-1)/2-(1-\varepsilon)} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,.
\end{aligned}$$

Observe that for admissible s and ε, $\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{(p-1)d}\big(1 \otimes (\beta v)^s \beta^\varepsilon v'\big)$ belongs to the ideal generated by the images under $d^{\mathfrak{S}_p}_{(p-1)d}$ of $1 \otimes v'$ and $1 \otimes \beta v'$, these are

$$\begin{aligned}
\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{d(p-1)}(1 \otimes v') &= e t^{\frac{p-3}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,, \\
\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} d^{\mathfrak{S}_p}_{d(p-1)}(1 \otimes \beta v') &= t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,.
\end{aligned}$$

Since $1 \otimes v'$ and $1 \otimes \beta v'$ have the smallest degrees among all non-zero elements of the top row, and $\{1 \otimes (\beta v)^s \beta^\varepsilon v'\}$ is an additive basis for this row, the statement of the theorem follows.

□

Sanity check. Let

$$\mathbb{C}^d \to E_{\mathbb{C}} \to B$$

be a complex bundle. Forgetting its complex structure we obtain a real vector bundle

$$\mathbb{R}^{2d} \to E_{\mathbb{R}} \to B$$

Obviously $E_{\mathbb{C}} = E_{\mathbb{R}}$ as topological spaces, we use subscripts purely to keep track of which additional structure we equip them with in each situation. Let us check that our recipes for the index of $S(W_pE)$ give identical results for $E_{\mathbb{R}}$ and $E_{\mathbb{C}}$. Since E is of even rank, if we check this for the $\mathbb{Z}_p$ group, the same result will follow for $\mathfrak{S}_p$.

For the complex case according to Theorem 4.5 we have $\operatorname{Index}_B^{\mathbb{Z}_p} S(W_pE_{\mathbb{C}}) = \langle \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}})\rangle$.

On the other hand $E_{\mathbb{R}}$ has an even rank, therefore point 1 of Theorem 4.7 is applicable, implying

$$\operatorname{Index}_B^{\mathbb{Z}_p} S(W_pE_{\mathbb{R}}) = \langle \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\rangle.$$

Observe, that there is an isomorphism of complex bundles $E_{\mathbb{R}} \otimes \mathbb{C} \cong E_{\mathbb{C}} \oplus E_{\mathbb{C}}^*$, where $E_{\mathbb{C}}^*$ is a complex conjugate of the bundle $E_{\mathbb{C}}$ (see [MS74, § 14]). Let J be a complex structure on $E_{\mathbb{C}}$. It defines an operator J on $E_{\mathbb{R}} \otimes \mathbb{C}$ by $J(x \otimes z) \coloneqq J(x) \otimes z$. Observe that $J^2 = -\mathrm{id}$. In this case, $E_{\mathbb{C}} \oplus E_{\mathbb{C}}^*$ can be seen as a decomposition of $E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ into a sum of eigenspaces of the operator J, corresponding to the eigenvalues $\pm i$. On the level of fibres of the vector bundle, this is a standard fact from linear algebra. It can be extended to the whole $E \otimes_{\mathbb{R}} \mathbb{C}$ since both J and the complex conjugation operator are defined globally. For any j, it holds that $c_j(E_{\mathbb{C}}) = (-1)^j c_j(E_{\mathbb{C}}^*)$ (see [MS74, Lem. 14.9]. However, in cohomology with $\mathbb{F}_p$ coefficients $c_j(E_{\mathbb{R}} \otimes \mathbb{C}) = 0$ for all odd j (see [Hat17, Thm. 3.16]). Therefore there is an equality of total Chern classes $c(E_{\mathbb{C}}) = c(E_{\mathbb{C}}^*)$, implying the equality

$$\check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}}) = \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}}^*)\,.$$

We compute

$$\begin{aligned}\operatorname{Index}_B^{\mathbb{Z}_p} S(W_pE_{\mathbb{R}}) &= \langle \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\rangle \\ &= \langle \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}})^{\frac{1}{2}}\, \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}}^*)^{\frac{1}{2}}\rangle \\ &= \langle \check{\mathrm{c}}\mathrm{h}(E_{\mathbb{C}})\rangle.\end{aligned}$$

This makes all our computations for complex vector bundles a particular case of computations for the real ones. Therefore in the next sections, we always state theorems for real bundles only.

4.3 Indices of fibrewise configuration spaces

4.3.1 Odd rank and symmetric group action

Consider a vector bundle $\mathbb{R}^n \to E \xrightarrow{\pi} B$ of an odd rank n, where we denote as usual the projection from E to B by π. Assume again that E is orientable. The goal of this section is to prove the following theorem:

Theorem 4.15. *With the assumptions above the ideal* $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ *equals*

$$\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E) = \mathrm{Index}_B^{\mathfrak{S}_p} S(W_pE) = \langle \varepsilon_E, \theta_E\rangle \subsetneq H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p).$$

The following lemma describes the shape of the index ideal - how many generators it has and of what degrees.

Lemma 4.16.

1. *There is an isomorphism* φ *of* $H^*(B)$*-modules*

$$H^*(B)\otimes H^*(\mathrm{Conf}(p,\mathbb{R}^n)/\mathfrak{S}_p) \underset{\cong}{\xrightarrow{\varphi}} H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p).$$

2. *Recall that* $H^*(\mathrm{B}\mathfrak{S}_p)\cong\Lambda[a]\otimes\mathbb{F}_p[b]$ *with* $\deg(a)=2(p-1)-1$ *and* $\deg(b)=2(p-1)$*. Then*

$$\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)=\langle ab^{\frac{n-1}{2}}-l_1, b^{\frac{n+1}{2}}-l_2\rangle,$$

where l_i *are linear (with coefficients from* $H^*(B)$*) combinations of elements*

$$\left\langle a,b,\ldots,ab^{\frac{n-3}{2}},b^{\frac{n-1}{2}}\right\rangle,$$

homogeneous with respect to the total degree.

Equipped with this knowledge, we prove Theorem 4.15 just in a few lines.

Proof of the Theorem 4.15. From Observation 4.2 we know that $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ contains the ideal $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$. In the previous section (Theorem 4.7) we have computed that $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_pE\big)$ is generated by two elements, ε_E and θ_E of degrees $(n+1)(p-1)-1$ and $(n+1)(p-1)$ respectively, which are exactly degrees of elements $ab^{\frac{n-3}{2}}-l_1$ and $b^{\frac{n+1}{2}}-l_2$ from Lemma 4.16 and the statement follows. □

Now it is time to prove Lemma 4.16.

Proof of Lemma 4.16. Consider the fibre bundle

$$\mathrm{Conf}(p,\mathbb{R}^n)\xrightarrow{i}\mathrm{Fconf}(p,E)\xrightarrow{\pi_{\mathrm{Conf}}} B.$$

Here i is some fixed till the end of this section inclusion of a fibre $\mathrm{Conf}(p, \mathbb{R}^n)$ over a point $x \in B$ into $\mathrm{Fconf}(p, E)$.

Since $\mathfrak{S}_p$ acts freely on $\mathrm{Conf}(p, \mathbb{R}^n)$ and, in turn, on $\mathrm{Fconf}(p, E)$, we have a choice between two fibre bundles when computing $H^*(\mathrm{Fconf}(p, E)/\mathfrak{S}_p)$.0 First, there is a bundle featuring the unordered fibrewise configuration space

$$\mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p \xrightarrow{i/\mathfrak{S}_p} \mathrm{Fconf}(p, E)/\mathfrak{S}_p \xrightarrow{\pi/\mathfrak{S}_p} B, \tag{4.4}$$

where the map $i/\mathfrak{S}_p$ is induced from i and $\pi/\mathfrak{S}_p$ is induced from π_{Conf} quotient by the symmetric group action. The second fibre bundle comes from the Borel construction associated with $\mathfrak{S}_p$ action, namely

$$\mathrm{Conf}(p, \mathbb{R}^n) \xrightarrow{i} \mathrm{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \xrightarrow{\pi_{Borel}} B \times \mathrm{B}\mathfrak{S}_p.$$

We denote by π_{Borel} the new projection map and keep the notation i for the inclusion of the fibre over a point (x, x') where x' is any chosen and fixed from that moment on point in $\mathrm{B}\mathfrak{S}_p$.

We use both of these fibre bundles in the proof of Lemma 4.16. We will see that $(i/\mathfrak{S}_p)^*$ is surjective and therefore the first of these fibre bundles satisfies conditions of the Leray–Hirsch theorem ([Hat02, Thm. 4D.1]), allowing us to compute $H^*(\mathrm{Fconf}(p, E)/\mathfrak{S}_p)$ as $H^*(B)$-module, thus proving the first point of the lemma. By the definition, the kernel of the map π^*_{Borel} equals precisely $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$, which explains the need to introduce the second of these fibre bundles as well.

Observe that that the projection of product bundle $\mathrm{Fconf}(p, E) \times \mathrm{E}\mathfrak{S}_p$ to its first component $\mathrm{Fconf}(p.E)$ induces a homotopy equivalence between the Borel construction $\mathrm{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p$ and the unordered fibrewise configuration space $\mathrm{Fconf}(p, E)/\mathfrak{S}_p$ which we name $pr_{1,E}$. Moreover, $pr_{1,E}$ is a map of fibre bundles. The relation of these fibre bundles to each other is shown in the diagram below

$$\begin{tikzcd}
\mathrm{Conf}(p, \mathbb{R}^n) \arrow[r] \arrow[d, "i"] & \mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p \arrow[d, "i/\mathfrak{S}_p"] \\
\mathrm{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \arrow[r, "pr_{1,E}", "\simeq"'] \arrow[d, "\pi_{Borel}"] & \mathrm{Fconf}(p, E)/\mathfrak{S}_p \arrow[d, "\pi/\mathfrak{S}_p"] \\
B \times \mathrm{B}\mathfrak{S}_p \arrow[r, "pr_B"] & B,
\end{tikzcd} \tag{4.5}$$

where the map pr_B is a projection of $B \times \mathrm{B}\mathfrak{S}_p$ to its first component.

However, to prove Lemma 4.16 we need to consider additionally a bigger diagram

$$\begin{tikzcd}
\mathrm{Conf}(p,\mathbb{R}^n)/\mathfrak{S}_p \arrow[r, "i/\mathfrak{S}_p"] & \mathrm{Fconf}(p,E)/\mathfrak{S}_p \arrow[r, "j/\mathfrak{S}_p"] & \mathrm{Conf}(p,\mathbb{R}^\infty)/\mathfrak{S}_p \\
\mathrm{Conf}(p,\mathbb{R}^n)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p \arrow[u, "\simeq"', "pr_{1,\mathbb{R}^n}"] \arrow[r, "i_{Borel}"] \arrow[d, "\pi_{fib}"] & \mathrm{Fconf}(p,E)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p \arrow[u, "\simeq"', "pr_{1,E}"] \arrow[r, "j_{Borel}"] \arrow[d, "\pi_{Borel}"] & \mathrm{Conf}(p,\mathbb{R}^\infty)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p \arrow[u, "\simeq"', "pr_{1,\infty}"] \arrow[d, "\simeq"', "pr_2"] \\
\{x\}\times\mathrm{B}\mathfrak{S}_p \arrow[r, "i_{\mathrm{B}\mathfrak{S}_p}"] & B\times\mathrm{B}\mathfrak{S}_p \arrow[r, "pr_{\mathrm{B}\mathfrak{S}_p}"] & \{x\}\times\mathrm{B}\mathfrak{S}_p.
\end{tikzcd} \tag{4.6}$$

Let us build this diagram map by map.

The first row is inspired by the construction of Chern classes of a complex bundle via the Leray–Hirsch theorem applied to its projectivisation, described, for example in [Hat17, Proof of Thm. 3.1 and 3.2]). We apply the same idea to $\mathrm{Fconf}(p,E)/\mathfrak{S}_p$, in particular, we would like to see $(i/\mathfrak{S}_p)^*$ as the last part in some composition of maps with the classifying space $\mathrm{B}\mathfrak{S}_p$ as a domain.

Similar to the construction of Chern classes, we start by choosing an embedding j of E to $\mathbb{R}^\infty$ such that j is injective on fibres of E (see [Hat17, Proof of Thm. 1.16] for the proof that such a map always exists). Due to its injectivity property j induces a map $\mathrm{Conf}(j)$ from $\mathrm{Fconf}(p,E)$ to $\mathrm{Conf}(p,\mathbb{R}^\infty)$. The symmetric group $\mathfrak{S}_p$ acts freely on both of them and $\mathrm{Conf}(j)$ is $\mathfrak{S}_p$-equivariant, therefore it induces a map $j/\mathfrak{S}_p$ between unordered configuration spaces, $\mathrm{Fconf}(p,E)/\mathfrak{S}_p \to \mathrm{Conf}(p,\mathbb{R}^\infty)/\mathfrak{S}_p$. Since $\mathrm{Conf}(p,\mathbb{R}^\infty)$ is contractible it can be chosen as a model for $\mathrm{E}\mathfrak{S}_p$, and the space $\mathrm{Conf}(p,\mathbb{R}^\infty)/\mathfrak{S}_p$ as a model for $\mathrm{B}\mathfrak{S}_p$. The reason we keep an abstract $\mathrm{B}\mathfrak{S}_p$ in parts of Diagram 4.6 is to have a starting point that does not depend on any choices that we have made, such as a choice of the map j.

Our next goal is to incorporate π_{Borel} into this diagram. We start by defining homotopy equivalences $pr_{1,\mathbb{R}^n}$ from $\mathrm{Conf}(p,\mathbb{R}^n)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p$ to $\mathrm{Conf}(p,\mathbb{R}^n)/\mathfrak{S}_p$ and $pr_{1,\infty}$ from $\mathrm{Conf}(p,\mathbb{R}^\infty)$ to $\mathrm{Conf}(p,\mathbb{R}^\infty)/\mathfrak{S}_p$ in the same way as we defined $pr_{1,E}$.

Observe, that maps i and $\mathrm{Conf}(j)$ induce maps between the respective Borel constructions:

$$i_{Borel}\colon \mathrm{Conf}(p,\mathbb{R}^n)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p \to \mathrm{Fconf}(p,E)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p$$

and

$$j_{Borel}\colon \mathrm{Fconf}(p,E)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p \to \mathrm{Conf}(p,\mathbb{R}^\infty)\times_{\mathfrak{S}_p}\mathrm{E}\mathfrak{S}_p.$$

This finishes our description of the second row of the diagram.

Notice that i_{Borel} and j_{Borel} can be seen as maps between fibre bundles. The third row, therefore, consists of their respective base spaces and the maps induced on the base space level by i_{Borel} and j_{Borel} - a map $i_{\mathrm{B}\mathfrak{S}_p}$ that includes $\{x\} \times \mathrm{B}\mathfrak{S}_p$ into $B \times \mathrm{B}\mathfrak{S}_p$ and a map $pr_{\mathrm{B}\mathfrak{S}_p}$ that collapses $B \times \mathrm{B}\mathfrak{S}_p$ onto $\{x\} \times \mathrm{B}\mathfrak{S}_p$.

It follows from the construction that this diagram is commutative.

Remember, that our first goal is to show that $(i/\mathfrak{S}_p)^*$ is surjective. We show this by proving instead that the composition

$$\{x\} \times \mathrm{B}\mathfrak{S}_p \xrightarrow{(i/\mathfrak{S}_p)^*(j/\mathfrak{S}_p)^*(pr^*_{1,\infty})^{-1}pr^*_2} \mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p.$$

is surjective. Indeed, from the commutativity of the diagram (4.6) it follows that

$$(i/\mathfrak{S}_p)^*(j/\mathfrak{S}_p)^*(pr^*_{1,\infty})^{-1}pr^*_2 = (pr^*_{1,\mathbb{R}^n})^{-1}\pi^*_{fib}\, i^*_{\mathrm{B}\mathfrak{S}_p} pr^*_{\mathrm{B}\mathfrak{S}_p} = (pr^*_{1,\mathbb{R}^n})^{-1}\pi^*_{fib}$$

In the last step an observation $pr_{\mathrm{B}\mathfrak{S}_p} \circ i_{\mathrm{B}\mathfrak{S}_p} = id_{\{x\}\times \mathrm{B}\mathfrak{S}_p}$ is used.

Since $pr_{1,\mathbb{R}^n}$ homotopy equivalence, $(pr^*_{1,\mathbb{R}^n})^{-1}$ is an isomorphism. Therefore $(i/\mathfrak{S}_p)^*$ is surjective if and only π^*_{fib} is surjective.

For an odd n it is known [CLM76, Thm. 5.3] that

$$H^*(\mathrm{Conf}(p, \mathbb{R}^n) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p) \cong H^*(\mathrm{B}\mathfrak{S}_p)/H^{>(n-1)(p-1)}(\mathrm{B}\mathfrak{S}_p),$$

with π^*_{fib} exactly the natural projection map

$$H^*(\mathrm{B}\mathfrak{S}_p) \to H^*(\mathrm{B}\mathfrak{S}_p)/H^{>(n-1)(p-1)}(\mathrm{B}\mathfrak{S}_p).$$

In particular π^*_{fib} is surjective and the claim follows.

Now if we fix an isomorphism $H^*(\{x\} \times \mathrm{B}\mathfrak{S}_p) \cong \Lambda[a] \otimes \mathbb{F}_p[b]$ in the right bottom corner of the diagram (4.6), we can use $(pr^*_{1,\mathbb{R}^n})^{-1}\pi^*_{fib}$ to identify the additive generators of $H^*(\mathrm{Conf}(p, \mathbb{R}^n)/\mathfrak{S}_p)$ with the set

$$\{a, b, ab, b^2 \ldots, ab^{\frac{n-3}{2}}, b^{\frac{n-1}{2}}\}.$$

Since $(i/\mathfrak{S}_p)^*$ is surjective, there exist elements α, β in $H^*(\mathrm{Fconf}(p, E)\,/\mathfrak{S}_p)$ such that $(i/\mathfrak{S}_p)^*(\alpha) = a$ and $(i/\mathfrak{S}_p)^*(\beta) = b$

Applying the Leray–Hirsch theorem [Hat02, Thm. D4.1] to the fibre bundle (4.4) we get that $H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p)$ is free $H^*(B)$-module with a basis

$$\langle \alpha, \beta, \alpha\beta, \beta^2 \dots, \alpha\beta^{\frac{n-3}{2}}, \beta^{\frac{n-1}{2}} \rangle$$

In particular, there exist linear combinations l_1 and l_2 of these basis vectors (with coefficients in $H^*(B)$ and homogeneous with respect to the total degree) such that

$$\begin{aligned} \alpha\beta^{\frac{n-1}{2}} &= l_1(\alpha, \beta, \dots, \alpha\beta^{\frac{n-3}{2}}) \in H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p), \\ \beta^{\frac{n+1}{2}} &= l_2(\alpha, \beta, \dots, \alpha\beta^{\frac{n-3}{2}}) \in H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p). \end{aligned}$$

The equations above can be interpreted in the following way: Consider the map

$$H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p) \xrightarrow{(\pi/\mathfrak{S}_p)^* \otimes (j/\mathfrak{S}_p)^* pr_{1,\infty}^* pr_2^*} H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p).$$

Then the kernel of this map is exactly an ideal

$$\langle ab^{\frac{n-1}{2}} - l_1(a, b, \dots, ab^{\frac{n-3}{2}}), b^{\frac{n+1}{2}} - l_2(a, b, \dots, ab^{\frac{n-3}{2}}) \rangle.$$

Using the commutativity of the diagrams (4.5) and (4.6), we notice that for the maps $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p) \to H^*(\mathrm{Fconf}(p,E)\,/\mathfrak{S}_p)$ the following sequence of equalities holds

$$\begin{aligned} (\pi/\mathfrak{S}_p)^* \otimes (j/\mathfrak{S}_p)^* pr_{1,\infty}^*\, pr_2^* &= (\pi/\mathfrak{S}_p)^* \otimes (pr_{1,E}^*)^{-1} j_{Borel}^*\, pr_2^* \\ &= (pr_{1,E}^*)^{-1} (pr_{1,E}^* (\pi/\mathfrak{S}_p)^* \otimes j_{Borel}^*\, pr_2^*) \\ &= (pr_{1,E}^*)^{-1} (\pi_{Borel}^*\, pr_B^* \otimes \pi_{Borel}^*\, pr_{\mathrm{B}\mathfrak{S}_p}^*) \\ &= (pr_{1,E}^*)^{-1} \pi_{Borel}^*. \end{aligned}$$

Since $pr_{1,E}$ is a homotopy equivalence, $pr_{1,E}^*$ is an isomorphism, and we conclude that

$$\begin{aligned} \mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p,E) &= \ker \pi_{Borel}^* \\ &= \ker\big((\pi/\mathfrak{S}_p)^* \otimes (j/\mathfrak{S}_p)^* pr_{1,E}^*\, pr_2^*\big) \\ &= \langle ab^{\frac{n-1}{2}} - l_1, b^{\frac{n+1}{2}} - l_2 \rangle. \end{aligned}$$

□

4.3.2 Even rank and symmetric group action

Now we turn our attention to the case when n is an even number. To formulate our results we need a new definition.

Definition 4.17. Let x be an element from the ring $H^*(B)$. Denote by $\operatorname{Ann}(x)$ an ideal in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ consisting of all elements Q satisfying the following two conditions:

- Q belongs to the subring $H^*(B) \otimes \mathbb{F}_p[b]$ of $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$.
- If, using the previous point, we write Q as a polynomial in b, then its free term should belong to the annihilator of x in $H^*(B)$.

Theorem 4.18. *Let $\mathbb{R}^n \to E \to B$ be a vector bundle of even rank n. Recall, that in Theorem 4.7 we established that there exists an element ζ_E in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ such that $\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} \zeta_E = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$.*

- *The following bound for the ideal $\operatorname{Index}^{\mathfrak{S}_p}_B \operatorname{Fconf}(p, E)$ in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ holds*

$$\langle \zeta_E \rangle \subsetneq \operatorname{Index}^{\mathfrak{S}_p}_B \operatorname{Fconf}(p, E) \subseteq \langle \zeta_E \rangle + ab^{-1}\Big(\operatorname{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big).$$

  The sum above is understood as a sum of ideals.

- *There are two special cases when we are able to compute the index ideal precisely. If vector bundle E admits two linearly independent nowhere zero sections, that is $E = E' \oplus \underline{\mathbb{R}}^2$ or if the cohomological dimension (that is, the highest non-zero degree in cohomology) of B satisfies the inequality $\operatorname{coh-dim} B < n(p-2)$ then the upper bound from the last point is achieved and the index of fibrewise configuration space $\operatorname{Fconf}(p, E)$ with respect to $\mathfrak{S}_p$ equals precisely*

$$\operatorname{Index}^{\mathfrak{S}_p}_B \operatorname{Fconf}(p, E) = \langle ab^{-1}\zeta_E, \zeta_E \rangle. \tag{4.7}$$

In the rest of this section, we present two proofs. The first of them combines the result of Theorem 4.15 with Lemma 4.3. In particular, it has the advantage that it does not use spectral sequences, at least explicitly, they are hidden inside the Leray–Hirsch theorem used in the proof of Theorem 4.15. Unfortunately, it gives a weaker version of Theorem 4.18 - in this way, we can not show the strict inclusion of the ideal $\langle \zeta_E \rangle$ into the ideal $\operatorname{Index}^{\mathfrak{S}_p}_B \operatorname{Fconf}(p, E)$, that is, guarantee that there exists at least one generator of $\operatorname{Index}^{\mathfrak{S}_p}_B \operatorname{Fconf}(p, E)$ other than ζ_E. With this approach, we also can prove the second part of the theorem only for the case when E admits two sections, but not for the case when B has a small cohomological dimension.

The second proof is based on the analysis of the spectral sequence for the fibre bundle $\operatorname{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to B \times \mathrm{B}\mathfrak{S}_p$. We will see that although our understanding of this spectral sequence is good enough to get the statement of the theorem, we can not describe the differentials of that spectral sequence fully. In particular, we do not have enough information for a precise computation of $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ or the $H^*(B)$-module structure on the cohomology of unordered fibrewise configuration space $\mathrm{Fconf}(p,E)/\mathfrak{S}_p$ - the structure we have a full description of in an odd rank case (see Open question 6).

Proof of Theorem 4.18 using the result for the odd rank case, resulting in non-strict bound. The non-strict inclusion of the ideal $\langle\zeta_E\rangle$ into $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ follows immediately from Observation 4.2 and Theorem 4.7 that computes the index of the real sphere bundle.

To prove the upper bound on $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$, let us add a trivial one dimensional bundle $\underline{\mathbb{R}}$ to E, yielding a new vector bundle

$$\mathbb{R}^{n+1}\to E\oplus\underline{\mathbb{R}}\to B.$$

The bundle $E\oplus\underline{\mathbb{R}}$ has the advantage of being odd-dimensional, meaning Theorem 4.15 applies to it. It is also a direct sum of two other vector bundles, which allows us to use Lemma 4.3. Consequently there exists the following inclusion of ideals in $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$:

$$\mathrm{Index}_B^{\mathfrak{S}_p} S\big(\underline{W_p\mathbb{R}}\big)\cdot\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)\subseteq\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E\oplus\underline{\mathbb{R}})\,. \tag{4.8}$$

We already know the precise value of two of the three ideals in the formula above, $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p\underline{\mathbb{R}}\big)$ (see Theorem 4.7) and $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E\oplus\underline{\mathbb{R}})$ (see Theorem 4.15).

$$\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p\underline{\mathbb{R}}\big)=\langle a,b\rangle,$$
$$\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E\oplus\underline{\mathbb{R}})=\langle\varepsilon_{E\oplus\underline{\mathbb{R}}},\theta_{E\oplus\underline{\mathbb{R}}}\rangle.$$

Substituting these results to the equation (4.8) we establish the following conditions on elements of $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$.

1. For any X in $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ there exists P,Q from $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$ such that
$$bX=P\cdot\varepsilon_{E\oplus\underline{\mathbb{R}}}+Q\cdot\theta_{E\oplus\underline{\mathbb{R}}}.$$
2. For any X in $\mathrm{Index}_B^{\mathfrak{S}_p}\mathrm{Fconf}(p,E)$ there exists P,Q from $H^*(B)\otimes H^*(\mathrm{B}\mathfrak{S}_p)$ such that
$$aX=P\cdot\varepsilon_{E\oplus\underline{\mathbb{R}}}+Q\cdot\theta_{E\oplus\underline{\mathbb{R}}}.$$

Observe that we can represent X as a sum $X_1^{\mathfrak{S}_p} + X_2^{\mathfrak{S}_p}$ such that $X_1^{\mathfrak{S}_p}$ belongs to the polynomial subring $H^*(B)[b]$ and $X_2^{\mathfrak{S}_p}$ belongs to the ideal $\langle a \rangle$. In the same way, write $P = P_1^{\mathfrak{S}_p} + P_2^{\mathfrak{S}_p}$ and $Q = Q_1^{\mathfrak{S}_p} + Q_2^{\mathfrak{S}_p}$.

Let us apply $\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p}$ to both of these equalities. In this way, we can work with them directly in the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$, where explicit computations are easier.

As a preparation, we compute the value of $\check{\mathrm{ch}}\big((E \oplus \underline{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}\big)$.

$$\begin{aligned}\check{\mathrm{ch}}\big((E \oplus \underline{\mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}\big) &= \check{\mathrm{ch}}\big((E \otimes_{\mathbb{R}} \mathbb{C}) \oplus (\underline{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})\big) \\ &= \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})\, \check{\mathrm{ch}}(\underline{\mathbb{C}}) \\ &= t^{(p-1)} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})\,.\end{aligned}$$

According to the definition of ε_E and θ_E in this case

$$\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} \varepsilon_{E \oplus \underline{\mathbb{R}}} = e t^{\frac{p-3}{2}} \check{\mathrm{ch}}(E \oplus \underline{\mathbb{R}})^{\frac{1}{2}} = e t^{\frac{p-3}{2}} t^{\frac{p-1}{2}} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} = e t^{p-2} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,,$$

$$\operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} \theta_{E \oplus \underline{\mathbb{R}}} = t^{\frac{p-1}{2}} \check{\mathrm{ch}}(E \oplus \underline{\mathbb{R}})^{\frac{1}{2}} = t^{\frac{p-1}{2}} t^{\frac{p-1}{2}} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} = t^{p-1} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,.$$

Denote by by

$$\begin{aligned} X_1 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} X_1^{\mathfrak{S}_p}, \\ P_1 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} P_1^{\mathfrak{S}_p}, \\ Q_1 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} Q_1^{\mathfrak{S}_p}. \end{aligned}$$

Denote by X_2, P_2 and Q_2 the unique elements of the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$ such that

$$\begin{aligned} eX_2 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} X_2^{\mathfrak{S}_p}, \\ eP_2 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} P_2^{\mathfrak{S}_p}, \\ eQ_2 &\coloneqq \operatorname{res}^{\mathfrak{S}_p}_{\mathbb{Z}_p} Q_2^{\mathfrak{S}_p}. \end{aligned}$$

In this notation, condition (1) transforms into the condition that

$$t^{p-1}(X_1 + eX_2) = e t^{p-2} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot (P_1 + e\, P_2) + t^{p-1} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot (Q_1 + eQ_2).$$

After opening brackets and dividing both sides by t^{p-2} we get the equality

$$tX_1 + et\, X_2 = t \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\; Q_1 + e\big(\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot P_1 + t \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_2\big).$$

By separating parts with and without e we see that

$$tX_1 = t \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_1, \tag{4.9}$$

$$etX_2 = e \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot P_1 + et \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_2. \tag{4.10}$$

Equation (4.9) can be further simplified to become $X_1 = \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_1$.

which implies $X_1^{\mathfrak{S}_p} = \zeta_E \cdot Q_1^{\mathfrak{S}_p}$, and we conclude that $X_1^{\mathfrak{S}_p}$ belongs to the ideal $\langle \zeta_E \rangle$.

Now we consider equation (4.10). Observe, that its LHS has a zero free term as a polynomial in t.

When $p_{n/2}^{(p-1)/2}(E) = 0$, it follows $\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ has a zero free term and is divisible by t. In this case, one can see the equation (4.10) as

$$eX_2 = \left(et^{-1}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\right) \cdot P_1 + e \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_2,$$

which in the ring $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ corresponds to the equation

$$X_2^{\mathfrak{S}_p} = ab^{-1}\zeta_E \cdot P_1^{\mathfrak{S}_p} + \zeta_E \cdot Q_2^{\mathfrak{S}_p},$$

and implies

$$X_2^{\mathfrak{S}_p} \in \langle ab^{-1}\zeta_E, \zeta_E \rangle.$$

When $p_{n/2}^{(p-1)/2}(E) \neq 0$, it follows that $\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ has a non-zero free term, therefore for the equation (4.10) to hold true, $P_1 \cdot \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ has to have a zero free term as a polynomial in t. In the language of $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ this condition implies that $P_1^{\mathfrak{S}_p}$ belongs to $\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)$ and $X_2^{\mathfrak{S}_p} = ab^{-1}\big(\zeta_E P_1^{\mathfrak{S}_p}\big) + \zeta_E \cdot Q_1^{\mathfrak{S}_p}$, which implies

$$X_2^{\mathfrak{S}_p} \in \langle \zeta_E \rangle + ab^{-1}\Big(\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big) \cdot \zeta_E\Big).$$

Notice that we already got inclusions from the first two statements of the Theorem 4.18. However, we need to check that condition (2) does not add some new information.

Once more we apply $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ and keeping the notation from the previous part of the proof we see that

$$et^{p-2}(X_1 + eX_2) = et^{p-2}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot (P_1 + eP_2) + t^{p-1}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot (Q_1 + eQ_2),$$

which is equivalent the equality

$$et^{p-2}X_1 = et^{p-2}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot P_1 + et^{p-1}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_2 + t^{p-1}\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot Q_1.$$

The LHS in the last equation is divisible by e, therefore the RHS should be as well, which implies $Q_1 = 0$. Then the equation can be simplified and we see that

$$X_1 = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \cdot (P_1 + tQ_2),$$

which gives us the condition we have seen already, namely $X_1^{\mathfrak{S}_p} \in \langle \zeta_E \rangle$. This proves a weaker version of the first part of the theorem.

Let us show that (4.7) holds in the case when E admits two linearly-independent nowhere zero sections by applying Lemma 4.3 again: In this situation, we get that

$$\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}\big(p, \underline{\mathbb{R}}^2\big) \cdot \mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p E'\big) \subseteq \mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E) \tag{4.11}$$

Two of these three ideals are already known: $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}\big(p, \underline{\mathbb{R}}^2\big)$ was computed in [CLM76, Thm. 5.3] and $\mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p E'\big)$ is given by Theorem 4.7. More precisely,

$$\begin{aligned} \mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}\big(p, \underline{\mathbb{R}}^2\big) &= \mathrm{Index}_{pt}^{\mathfrak{S}_p} \mathrm{Conf}(p, \mathbb{R}^2) \otimes H^*(B) = \langle a, b \rangle, \\ \mathrm{Index}_B^{\mathfrak{S}_p} S\big(W_p E'\big) &= \langle \zeta_{E'} \rangle. \end{aligned}$$

Observe that $\zeta_E = b\,\zeta_{E'}$. This is easier to see in the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$. By definition

$$\begin{aligned} \mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \zeta_E &= \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}\,, \\ \mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} (b\,\zeta_{E'}) &= t^{p-1} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}\,. \end{aligned}$$

One can check that

$$\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} = \check{\mathrm{c}}\mathrm{h}\big(\mathbb{R}^2 \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}} \cdot \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}} = t^{p-1} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}\,.$$

Another observation we need is that $a\,\zeta_{E'} = ab^{-1}\big(b\,\zeta_{E'}\big) = ab^{-1}\zeta_E$.

Using these two observations, we deduce from (4.11) that

$$\langle ab^{-1}\zeta_E, \zeta_E \rangle \subseteq \mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)\,.$$

We have already proved inclusion the other way around, therefore equality (4.7) follows. This completes the proof of Theorem 4.18 □

Proof using a spectral sequence. Our second approach to the proof of Theorem 4.18 is to analyse the spectral sequence related to the fibre bundle

$$\mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to B \times \mathrm{B}\mathfrak{S}_p. \tag{4.12}$$

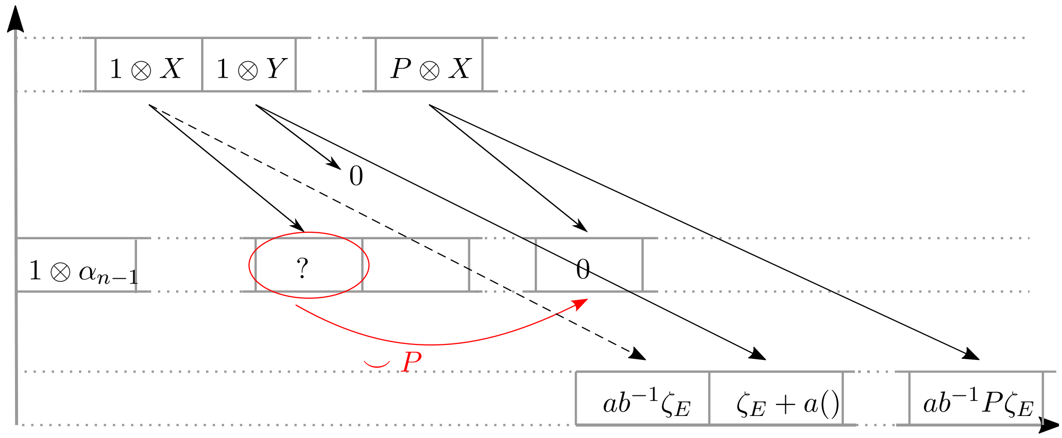

FIGURE 4.3: Spectral sequence for the fibre bundle (4.12)

Let us start with a spoiler - Picture 4.3 is a rough sketch that summarises all main properties of this spectral sequence.

We begin by writing down the E_2 page of this spectral sequence. Its rows are described by the formula

$$E_2^{*,b} \cong H^*\Big(B \times \mathrm{B}\mathfrak{S}_p; \mathcal{H}^b\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big).$$

where $\mathcal{H}^b\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)$ is a local coefficients system, arising from the action of the fundamental group $\pi_1(B \times \mathrm{B}\mathfrak{S}_p)$ on the cohomology of the fibre of this bundle. Since E is by assumption orientable, the non-trivial action comes from the subgroup $1 \times \pi_1(\mathrm{B}\mathfrak{S}_p) \cong \mathfrak{S}_p$. Using Künneth theorem with local coefficients (Theorem 3.8) we rewrite $E_2^{*,b}$ as

$$E_2^{*,b} \cong H^*(B) \otimes H^*\Big(\mathrm{B}\mathfrak{S}_p; \mathcal{H}^b\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big).$$

The second term of this tensor product can itself be seen as coming from the E_2-page of an appropriate fibre bundle, namely the bundle

$$\mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Conf}(p, \mathbb{R}^n) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to \mathrm{B}\mathfrak{S}_p. \tag{4.13}$$

It was extensively studied by Cohen [CLM76]. Below we quote some facts about the spectral sequence associated with the fibre bundle (4.13) that we use in the proof of Theorem 4.18.

Lemma 4.19 ([CLM76, Sec. 5, 8-11], [Knu18, §8]). *Consider the spectral sequence associated with the fibre bundle*

$$\mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Conf}(p, \mathbb{R}^n) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to \mathrm{B}\mathfrak{S}_p,$$

where n is even. Then the following holds for prime $p > 3$:

1. *Its E_2 page has only three non-zero rows: $E_2^{*,0}, E_2^{*,n-1}$ and $E_2^{*,(n-1)(p-1)}$.*
2. *$E_2^{*,n-1} \cong \Lambda[\alpha_{n-1}]$, where α_{n-1} is the basis vector of the one-dimensional $\mathbb{F}_p$-vector space $E_2^{0,n-1}$.*
3. *d_n is trivial, in particular, the only differential that hits the 0-row is $d_{(n-1)(p-1)+1}$. Also, it follows that nothing changes till the last pages, that is,*
$$E_2 = \cdots = E_{(n-1)(p-1)}.$$
4. *$E_2^{i,(n-1)(p-1)} = 0$ for $0 < i < p-3$. Since it is the top row, the same holds for the top rows of all consecutive pages, most importantly for us, it is true for the page $E_{(n-1)(p-1)+1}$.*
5. *The first two non-zero entries of the top row on the second page are $E_2^{p-3,(n-1)(p-1)}$ and $E_2^{p-2,(n-1)(p-1)}$. They are one-dimensional $\mathbb{F}_p$-vector spaces. Let us choose a basis vector in each and call them X and Y respectively. Additionally it holds that $d_{(n-1)(p-1)+1}(bX) = d_{(n-1)(p-1)+1}(aY)$.*
6. *X and Y generate the whole $E_2^{*,(n-1)(p-1)}$ as an $H^*(\mathrm{B}\mathfrak{S}_p)$-module.*
7. *The module action by b on the top row of the second page that maps $E_2^{i,(n-1)(p-1)}$ to $E_2^{i+2(p-1),(n-1)(p-1)}$ is an isomorphism for any choice of non-negative integer i.*

Remark 4.20. When $p = 3$ the $(p-1)(n-1) = 2n-2$ row of the E_2 page also has one-dimensional vector space in the position $(0, 2n-2)$ corresponding to the space of invariants $\mathrm{Conf}(3, \mathbb{R}^n)^{\mathfrak{S}_3}$ [Knu18, Ex.8.2.1]. In this case, one can think of this E_2 page as consisting of two parts - the part that satisfies all properties of the main part of the lemma plus an additional entry in $(0, 2n-2)$. The differential d_{2n-1} is trivial in this entry.

As a next step, we prove an analogous lemma about the spectral sequence associated with the fibre bundle (4.12).

Lemma 4.21. *Consider a spectral sequence associated with the fibre bundle*
$$\mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Fconf}(p, E) \times_{\mathfrak{S}_p} \mathrm{E}\mathfrak{S}_p \to B \times \mathrm{B}\mathfrak{S}_p$$
where n is even. Then the following holds for $p > 3$:

1. *Its E_2 page has only three non-zero rows: $E_2^{*,0}, E_2^{*,n-1}$ and $E_2^{*,(n-1)(p-1)}$.*

2. *$E_2^{*,n-1} \cong H^*(B) \otimes \Lambda[\alpha_{n-1}]$, where $1 \otimes \alpha_{n-1}$ is the basis vector of the one-dimensional $\mathbb{F}_p$-vector space $E_2^{0,n-1}$.*

3. *d_n is trivial and therefore the only nontrivial differentials are $d_{(n-1)(p-2)+1}$ and $d_{(n-1)(p-1)+1}$. Both could be non-trivial only on the top row as a domain, and only $d_{(n-1)(p-1)+1}$ hits the 0-row.*

4. *$E_2^{i,(n-1)(p-1)} = 0$ for $0 < i < p-3$. Since it is the top row, the same holds for all consecutive pages, most importantly for the last one, $E_{(n-1)(p-1)+1}$.*

5. *The first two non-zero entries of the top row on the second page are $E_2^{p-3,(n-1)(p-1)}$ and $E_2^{p-2,(n-1)(p-1)}$. The first of them, $E_2^{p-3,(n-1)(p-1)}$, is a one-dimensional $\mathbb{F}_p$-vector space, with a basis given by $1 \otimes X$. The second one, $E_2^{p-2,(n-1)(p-1)}$, has a basis consisting of $1 \otimes Y$ and all elements of the form $x \otimes X$ for some $x \in H^1(B)$. Additionally, $d_{(n-1)(p-1)+1}(1 \otimes bX) = d_{(n-1)(p-1)+1}(1 \otimes aY)$.*

6. *$1 \otimes X$ and $1 \otimes Y$ generate the whole $E_2^{*,(n-1)(p-1)}$ as an $H^*(B \times \mathrm{B}\mathfrak{S}_p)$-module.*

7. *The module action by $1 \otimes b$ provides an isomorphism between $E_2^{i,(n-1)(p-1)}$ and $E_2^{i+2(p-1),(n-1)(p-1)}$ for any choice of non-negative integer i.*

Proof. Almost all parts of this lemma follow immediately from Lemma 4.19 in combination with the Künneth theorem for local coefficients 3.8. The only part that needs an additional argument is part 3. It can be proved by a standard trick, used, for example, in [CLM76] in computations related to $\mathbb{Z}_p$ group. The key idea is to notice that the module action by $1 \otimes b$ sends any $Z \in E_n^{*,n-1}$ to zero. Indeed, from part 2 of the same lemma, we know that any such Z is of the form $z \otimes \alpha_{n-1}$ for some element z from $H^*(B)$. Remember that the module structure commutes with the Künneth formula. Therefore the following equation holds

$$(1 \otimes b)Z = z \otimes b\,\alpha_{n-1} = 0$$

by part 2 of Lemma 4.19.

Using the fact that differentials in spectral sequences respect this module structure, we see that

$$0 = d_n\big((1 \otimes b)Z\big) = (1 \otimes b)\, d_n(Z).$$

Observe, that multiplication by $1 \otimes b$ is injective on the zero row of E_n, since it is invective on $E_2^{*,0} = H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ and all differentials d_i for $i < n$ are trivial, so $E_2^{*,0} = E_n^{*,0}$. It follows that $d_n(Z)$ is necessary zero. □

Now we are ready to prove Theorem 4.18 using the spectral sequence for the fibration (4.12). We assume at first that $p > 3$, and then see what changes in the case $p = 3$.

Proof of Theorem 4.18. Assume $p > 3$. From Lemma 4.21 we know that the index $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ is generated by the images of the differential $d_{(n-1)(p-1)+1}$ in the spectral sequence in question. Let us analyse its $E_{(n-1)(p-1)+1}$ page.

To begin with, observe that for any element Z from the top row of the E_2 page that survives till the last page, it follows $d_{(n-1)(p-1)+1}(Z) \neq 0$. Indeed, if Z survive till the last page, that is, the only up to this moment non-trivial differential $d_{(n-1)(p-1)+1}$ sends Z to 0, then any elements of the form $b^s Z$, $s > 0$, survives as well. Suppose $d_{(n-1)(p-1)+1}(Z) = 0$. Then $d_{(n-1)(p-1)+1}(b^s Z)$ also equals zero for any $s > 0$. Consequently, the resulting cohomology of unordered fibrewise configuration space is infinite-dimensional, which is not possible since $\mathfrak{S}_p$ acts freely on $\mathrm{Fconf}(p, E)$ and $\mathrm{Fconf}(p, E)/\mathfrak{S}_p$ has a finite dimension.

From part 6 of Lemma 4.21 we know that the top row of E_2 page of this spectral sequence is generated as an $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$-module by elements $1 \otimes X$ and $1 \otimes Y$. Let us show that $1 \otimes Y$ survives till the last page. Suppose the opposite holds. Notice, that from Observation 4.2 in combination with Theorem 4.7 it follows that ζ_E belongs to the $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$, consequently it belongs to the image of $d_{(n-1)(p-1)+1}$. Therefore there exists an element Z such that $d_{(n-1)(p-1)+1}(Z) = \zeta_E$. Remember that $\deg(\zeta_E) = n(p-1)$. Therefore Z should belong to $E_{(n-1)(p-1)+1}^{p-2,(n-1)(p-1)}$. By our assumption, $1 \otimes Y$ did not survive till this page, meaning the only elements in $E_{(n-1)(p-1)+1}^{p-2,(n-1)(p-1)}$ are those of the form $H^1(B) \otimes X$. Therefore there exists some element Q in $H^1(B)$ such that $d_{(n-1)(p-1)+1}(Q \otimes X) = \zeta_E$. On the other hand, for dimensional reasons, there exists at least one element $P \in H^*(B)$ such that $d_{(n-1)(p-2)+1}(P \otimes Y) = 0$, meaning $P \otimes Y$ survives till the last page.

Then, from point 5 of Lemma 4.21 it follows that

$$d_{(n-1)(p-1)+1}(PQ \otimes bX) = d_{(n-1)(p-1)+1}(QP \otimes aY),$$

which implies

$$bP\,\zeta_E = aQ \cdot d_{(n-1)(p-1)+1}(P \otimes Y).$$

The left-hand side of the last equation has a leading term $Pb^{\frac{\operatorname{rank} E+2}{2}}$ and therefore does not equal zero. Moreover, it is fully contained in $H^*(B) \otimes \mathbb{F}_p[b]$ subring of $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$. On the other hand, the right-hand side of the last equation either equals zero or is a multiple of a, so we arrive at a contradiction.

Let us turn our attention to the element $1 \otimes X$. Again, by dimensional reasons, there exists at least one element P in $H^*(B)$ such that $d_{(n-1)(p-2)+1}(P \otimes X) = 0$, meaning $P \otimes X$ survives till the last page.

Applying point 5 of Lemma 4.21 we see that

$$d_{(n-1)(p-1)+1}(P \otimes bX) = d_{(n-1)(p-1)+1}(P \otimes aY),$$

which implies

$$bd_{(n-1)(p-1)+1}(P \otimes X) = aP \cdot d_{(n-1)(p-1)+1}(1 \otimes Y). \tag{4.14}$$

From this equality, it follows that $d_{(n-1)(p-1)+1}(P \otimes X)$ belongs to the ideal generated by a in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$. On the other hand, ζ_E belongs to the polynomial part of the same ring. Since the whole $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ is generated by $d_{(n-1)(p-1)+1}(1 \otimes Y)$ and $d_{(n-1)(p-1)+1}(P \otimes X)$ for various P, the only way ζ_E can be contained in this ideal is if up to a rescaling

$$d_{(n-1)(p-1)+1}(1 \otimes Y) = \zeta_E + aQ,$$

where Q is some element of $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ of an appropriate degree. Let us plug in this into the equation (4.14). We get that up to a scale factor

$$b\, d_{(n-1)(p-1)+1}(P \otimes X) = aP\, \zeta_E.$$

Notice, that the left-hand side of this equation is divisible by b, implying that $P\,\zeta_E$ must be divisible by b too, hence P belongs to $\mathrm{Ann}\big(p_{n/2}^{(p-1)/2}(E)\big)$. Transforming the last equation we get that

$$d_{(n-1)(p-1)+1}(P \otimes X) = ab^{-1}\, (P\, \zeta_E)\,.$$

We repeat our reasoning: since the whole $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ is generated by

$$d_{(n-1)(p-1)+1}(1 \otimes Y) \text{ and } d_{(n-1)(p-1)+1}(P \otimes X)$$

for various $P \in H^*(B)$, and now we know that $d_{(n-1)(p-1)+1}(P \otimes X)$ always has a form $ab^{-1}P\,\zeta_E$, the only way ζ_E can be contained in this ideal is if Q belongs to the ideal

$$b^{-1}\Big(\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big).$$

It follows that any element of $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ can be presented as a sum of an element from the ideal $\langle \zeta_E \rangle$ and an element from the ideal $ab^{-1}\Big(\mathrm{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big)$. Moreover, for dimensional reasons, there always exists an element in $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$

that is not fully contained in $\langle \zeta_E \rangle$ - the fact that we were not able to show by the first method of proof.

Let us prove that in the case coh-dim $B < n(p-2)$ the upper bound on the index ideal is achieved. Notice that the differential $d_{(n-1)(p-2)+1}$ maps $1 \otimes X$ to some element in position $(n-1, n(p-2))$ on the $E_{(n-1)(p-2)+1}$-page. Recall that $(n-1)$-st row of the second page of this spectral sequence equals $\Lambda[\alpha_{n-1}] \otimes H^*(B)$. In case when coh-dim $B < n(p-2)$ this implies that the position $E_2^{n(p-2),n-1}$ equals to 0. Consequently, the element $1 \otimes X$ belongs to the kernel of the differential $d_{(n-1)(p-2)+1}$ and survives till the last page. In the notation from above, in this case $P = 1$ and $d_{(n-1)(p-1)+1}(1 \otimes X) = ab^{-1}\zeta_E$. This finishes the proof for the case $p > 3$.

When $p = 3$, the top row of the E_2 page has, in addition to what we have described above, the part $\mathrm{Conf}(3, \mathbb{R}^n)^{\mathfrak{S}_3} \otimes H^*(B)$. Observe, that we can prove that differential d_{2n-1} is trivial on this part in the same way we have proved that d_n is trivial in all cases. Hence, this part of the spectral sequence does not influence the value of the index. □

4.3.3 Indices with respect to $\mathbb{Z}_p$, special case

It turns out that the case of $\mathbb{Z}_p$-group action is much harder than that of $\mathfrak{S}_p$. The only theorem we are able to prove about indices of fibrewise configuration spaces with respect to $\mathbb{Z}_p$-action is the following one

Theorem 4.22. *Let $\mathbb{R}^n \to E \to B$ and $\mathbb{R}^{n-2} \to E' \to B$ be real vector bundles such that $E \cong E' \oplus \underline{\mathbb{R}}^2$. Then*

$$\mathrm{Index}_B^{\mathbb{Z}_p} \mathrm{Fconf}(p, E) = \langle e t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}, t^{\frac{p+1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}} \rangle.$$

The proof is very similar to the proof of the analogous statement with the symmetric group action but with one major difference: In the case of a symmetric group we used the Leray–Hirsch theorem, and now we will work directly with the spectral sequence, resulting in the lemma below:

Lemma 4.23 (compare with in [CLM76, Thm. 8.2] or [BLZ15, Sec. 6.2])**.** *Let*

$$\mathbb{R}^n \to E \to B$$

be a real vector bundle of rank n. The Leray–Serre spectral sequence for the fibre bundle

$$\mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Fconf}(p, E) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \to B \times \mathrm{B}\mathbb{Z}_p$$

has the following properties:

1. *No differential hits the 0-row before the differential $d_{(n-1)(p-1)+1}$. In particular, $\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)$ does not contain elements of degrees $(n-1)(p-1)$ and smaller.*

2. *The top row on the second page of the spectral sequence, $E_2^{*,(n-1)(p-1)}$, is generated as a $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$ module by two elements: an element $1\otimes\tilde{e}\in E_2^{0,(n-1)(p-1)}$ and an element $1\otimes\tilde{t}\in E_2^{1,(n-1)(p-1)}$.*

3. *If $\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)$ happens to contain any generators of degree $(n-1)(p-1)+1$ then it has exactly one generator of that degree and no more than one generator in degree $(n-1)(p-1)+2$.*

4. *If $\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)$ has a generator in degree $(n-1)(p-1)+1$ and a generator in degree $(n-1)(p-1)+2$ (the unique one according to the previous point) then these two in fact generate the whole $\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)$.*

Let us do the easier part first and prove Theorem 4.22. After that, we will prove Lemma 4.23.

Proof of the Theorem 4.22. Lemma 4.3 provides us with the following inclusion of the ideals in $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$:

$$\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}\big(p,\underline{\mathbb{R}}^2\big)\cdot\mathrm{Index}_B^{\mathbb{Z}_p}\,S\big(W_PE'\big)\subseteq\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)\,.$$

Using results from [BZ14, Thm. 6.1] (alternatively [CLM76, Thm. 8.2]) and Theorem 4.7) we can compute two of these ideals, namely

$$\begin{aligned}\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}\big(p,\underline{\mathbb{R}}^2\big)&=\mathrm{Index}_{pt}^{\mathbb{Z}_p}\,\mathrm{Conf}(p,\mathbb{R}^2)\otimes H^*(B)=\langle et^{\frac{p-1}{2}},t^{\frac{p+1}{2}}\rangle,\\ \mathrm{Index}_B^{\mathbb{Z}_p}\,S\big(W_PE'\big)&=\langle\check{\mathrm{c}}\mathrm{h}\big(E'\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}}\rangle.\end{aligned}$$

Therefore the following inclusion holds

$$\langle et^{\frac{p-1}{2}}\check{\mathrm{c}}\mathrm{h}\big(E'\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}},t^{\frac{p+1}{2}}\check{\mathrm{c}}\mathrm{h}\big(E'\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}}\rangle\subseteq\mathrm{Index}_B^{\mathbb{Z}_p}\,\mathrm{Fconf}(p,E)\,.$$

Notice, that the degree of $et^{\frac{p-1}{2}}\check{\mathrm{c}}\mathrm{h}(E'\otimes_{\mathbb{R}}\mathbb{C})$ is $(n-1)(p-1)+1$ and the degree of $t^{\frac{p+1}{2}}\check{\mathrm{c}}\mathrm{h}(E'\otimes_{\mathbb{R}}\mathbb{C})^{\frac{1}{2}}$ is $(n-1)(p-1)+2$.

An application of point 4 of Lemma 4.23 completes the proof. □

Proof of Lemma 4.23 Consider the fibre bundle

$$\xi_B:\mathrm{Conf}(p,\mathbb{R}^n)\to\mathrm{Fconf}(p,E)\times_{\mathbb{Z}_p}\mathrm{E}\mathbb{Z}_p\to B\times\mathrm{B}\mathbb{Z}_p.$$

The E_2 page of the corresponding Leray–Serre spectral sequence consists of the following rows

$$E_2^{*,q}(\xi_B) \cong H^*\Big(B \times \mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big),$$

where $\mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)$ is the local coefficients system, consisting, for each value of q, of $H^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)$ considered as an $H^*(B \times \mathrm{B}\mathbb{Z}_p)$ module.

Using Küunneth formula with local coefficients (Theorem 3.9) we can rewrite the expression above as following:

$$H^*\Big(B \times \mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big) \cong H^*(B; \mathbb{F}_p) \otimes H^*\Big(\mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big).$$

The second part of this tensor product, in turn, appears as $E_2^{*,q}(\xi_{pt})$ for the fibre bundle

$$\xi_{pt} \colon \mathrm{Conf}(p, \mathbb{R}^n) \to \mathrm{Conf}(p, \mathbb{R}^n) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p \to B\mathbb{Z}.$$

Remember, that for any Leray–Serre spectral sequence, for any fixed q, $E_2^{*,q}$ carries a structure of a left module over a $E_2^{*,0}$. Applied to the spectral sequence for ξ_{pt} this means that $E_2^{*,q}(\xi_{pt}) = H^*\Big(\mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big)$ carries a structure of $H^*(\mathrm{B}\mathbb{Z}_p)$-module, and for ξ_B – that $E_2^{*,q}(\xi_B) = H^*\Big(B \times \mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big)$ carries the structure of $H^*(B \times \mathrm{B}\mathbb{Z}_p)$ module.

With respect to the isomorphism provided by Künneth formula, the latter module structure becomes exactly as one would expect from it: as a tensor product of $H^*(B)$ left action on itself, and $H^*(\mathrm{B}\mathbb{Z}_p)$ action on $H^*\Big(\mathrm{B}\mathbb{Z}_p; \mathcal{H}^b\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)\Big)$ inherited from the spectral sequence associated with ξ_{pt}.

The spectral sequence associated with ξ_{pt} is very well studied, see [CLM76, Sect. 8]. In particular, the following is known:

Theorem 4.24 ([CLM76])**.** *There are the following isomorphisms of $H^*(\mathrm{B}\mathbb{Z}_p)$-modules:*

$$\begin{aligned} E_2^{*,0}(\xi_{pt}) &\cong H^*(\mathrm{B}\mathbb{Z}_p; \mathbb{F}_p), \\ E_2^{*,(n-1)(p-1)}(\xi_{pt}) &\cong \mathcal{H}^{(n-1)(p-1)}\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)^{\mathbb{Z}_p} \oplus H^{*+1}(\mathrm{B}\mathbb{Z}_p; \mathbb{F}_p), \\ E_2^{0,q}(\xi_{pt}) &\cong \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)^{\mathbb{Z}_p} \textit{ for } 0 \leqslant q < (n-1)(p-1), \end{aligned}$$

where $\mathcal{H}^*\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)^{\mathbb{Z}_p}$ stands for the set of invariants in $\mathcal{H}^*\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)$ with respect to $\mathbb{Z}_p$ action and contributes only to $E_2^{0,*}(\xi_{pt})$.

For any other pair (s, q), $H^s(\mathrm{B}\mathbb{Z}_p; \mathcal{H}^q\big(\mathrm{Conf}(p, \mathbb{R}^n); \mathbb{F}_p\big)) = 0$.

Theorem 4.24 appears in [BLZ15] as an intermediate step in describing the shape of the spectral sequence for ξ_{pt} and computing $\mathrm{Index}_{pt}^{\mathbb{Z}_p}\,\mathrm{Conf}(p,\mathbb{R}^n)$. Below we follow the same line of arguments, adjusting them to the fibrewise case when needed.

But before we begin the proof of Lemma 4.23 in earnest, let us collect another useful result about this spectral sequence.

Lemma 4.25 (compare [CLM76], [BLZ15])**.** *Let X be an element in $E_i^{s,q}(\xi_B)$, $i \geqslant 2$, $0 < q < (n-1)(p-1)$, $s \geqslant 0$. Remember that $E_i^{s,q}(\xi_B)$ has a structure of the left module over $H^*(B \times \mathrm{B}\mathbb{Z}_p)$. With respect to this module structure it holds that*

$$(1 \otimes t)X = 0$$

Proof. First, we prove it for elements of the form

$$x \otimes y \in E_2^{s,q}(\xi_B) = H^*(B;\mathbb{F}_p) \otimes \mathcal{H}^*\big(\mathrm{B}\mathbb{Z}_p;\mathcal{H}^b\big(\mathrm{Conf}(p,\mathbb{R}^n);\mathbb{F}_p\big)\big),$$

$0 < b < (n-1)(p-1)$. We know that $(1 \otimes t)(x \otimes y) = x \otimes ty$. Observe that $\deg(ty) > 0$, therefore according to Theorem 4.24 $ty = 0$.

This implies the result for all elements in $E_2^{a,b}(\xi_B)$ for the same range $0 < b < (n-1)(p-1)$. Since the module structure on the later pages $E_i^{*,q}$ is induced from that of $E_2(\xi_B)$, the statement follows. $\square$

Now we are ready to prove Lemma 4.23

Proof of Lemma 4.23. We will prove point 1 by induction on $0 < q < (n-1)(p-1)$. Unless specified otherwise, all pages of a spectral sequence mentioned correspond to $E(\xi_B)$ so we denote them simply by E.

Assume that no differential before d_{q+1} hits the zero row. In particular, this assumption holds for $q = n-1$, the smallest $q > 0$ such that $E_2^{*,q} \neq 0$. Suppose there is a pair (s,q), where s is a non-negative integer such that the differential $E_{q+1}^{s,q} \xrightarrow{d_{q+1}} E_{q+1}^{s+q+1,0}$ is non-trivial. It follows there exists $X \in E_{q+1}^{s,q}$ and $Y \in E_{b+1}^{s+q+1,0}$, with $Y \neq 0$, such that $d_{q+1}(X) = Y$.

Observe, that multiplication by $1 \otimes t$ is injective in $E_2^{*,0} = H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$, therefore using induction assumption it follows that it is also injective in $E_{q+1}^{*,0} = E_2^{*,0}$, implying

$$(1 \otimes t)d_{q+1}(X) = (1 \otimes t)Y \neq 0.$$

From Lemma 4.25 we know $(1 \otimes t)X = 0$. Then

$$d_{q+1}\big((1 \otimes t)X\big) = 0 = (1 \otimes t)d_{q+1}(X) = (1 \otimes t)Y,$$

and we came to a contradiction, meaning there exists no such Y and the induction step follows. Below is the same argument again in a more visual form:

$$\begin{tikzcd}
X \arrow[r, "\smile(1\otimes t)"] \arrow[dr, dashed, "d"] & 0 \arrow[dr, "d"] & \\
& Y \neq 0 \arrow[r, "\smile(1\otimes t)"] & ty \neq 0
\end{tikzcd}$$

Point 2 is a direct consequence of the respective statement that $\tilde{e}$ and $\tilde{t}$ generate the top row of $E_2(\xi_{pt})$.

To prove point 3 observe, that now we know that all elements of index come from the differential

$$d_{(n-1)(p-1)+1} \colon E^{*,(n-1)(p-1)}_{(n-1)(p-1)+1} \to E^{*+(n-1)(p-1)+1,0}_{(n-1)(p-1)+1}.$$

In particular, elements of the index of degrees $(n-1)(p-1)+1$ and $(n-1)(p-1)+2$ can only come from elements in the positions $(0,(n-1)(p-1))$ and $\big(1,(n-1)(p-1)\big)$ respectively.

Observe that $E_2^{*,(n-1)(p-1)}$ has two types of elements in it: The ones of the form

$$H^*(B) \otimes \mathcal{H}^{(n-1)(p-1)}\big(\mathrm{Conf}(p,\mathbb{R}^n);\mathbb{F}_p\big)^{\mathbb{Z}_p},$$

and those of the form

$$H^*(B) \otimes H^{*+1}(\mathrm{B}\mathbb{Z}_p;\mathbb{F}_p).$$

The differential $d_{(n-1)(p-1)+1}$ is trivial on the former, the proof is identical to that of 1. $H^*(B) \otimes H^{*+1}(\mathrm{B}\mathbb{Z}_p;\mathbb{F}_p)$ has a unique generator $1 \otimes \tilde{e}$ in position $\big(0,(n-1)(p-1)\big)$, therefore there could be no more than one element of the degree $(n-1)(p-1)+1$. Elements of index of degree $(n-1)(p-1)+2$ can come either as images of $x \otimes \tilde{e}$ for some $x \in H^1(B)$ or as an image of $1 \otimes \tilde{t}$. Using $H^*(B) \otimes H^*(B\mathbb{Z})$-structure, we compute

$$d_{(n-1)(p-1)+1}(x \otimes \tilde{e}) = x d_{(n-1)(p-1)+1}(1 \otimes \tilde{e}) \in \langle d_{(n-1)(p-1)+1}(1 \otimes \tilde{e})\rangle$$

Therefore if any new independent generator of index appears in degree $(n-1)(p-1)+2$ it can be only $d_{(n-1)(p-1)+1}(1 \otimes \tilde{e})$, which completes the proof of point 3.

Moreover, if $\mathrm{Index}_B^{\mathbb{Z}_p} \mathrm{Fconf}(p,E)$ happened to have generators in both degree $(n-1)(p-1)+1$ and $(n-1)(p-1)+1$, then both $1 \otimes \tilde{e}$ and $1 \otimes \tilde{t}$ had to survive till the

$\big((n-1)(p-1)+1\big)$-page. Since $1\otimes\tilde{e}$ and $1\otimes\tilde{t}$ generate the whole top row of the second page as $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$-module, if they survived, they generate the whole top row of $(n-1)(p-1)+1$ page as well. Therefore any other elements of $\operatorname{Index}_B^{\mathbb{Z}_p}\operatorname{Fconf}(p,\xi)$ belong to the ideal generated by $d_{(n-1)(p-1)+1}(1\otimes\tilde{e})$ and $d_{(n-1)(p-1)+1}(1\otimes\tilde{t})$.

□

4.4 What if E is non-orientable?

So far all results of this chapter worked on the condition that the bundle E is orientable. It turns out we can get exactly the same results for non-orientable bundles. In particular, we will see that Theorems 4.15, 4.18, and 4.22 also hold for tautological bundles over real Grassmannians, and therefore can be used for the solution of Problem 1.3.

Our plan is as follows: Given a non-orientable vector bundle $E\xrightarrow{\pi_B}B$ we construct its orientable double cover, that is, an orientable vector bundle $\tilde{E}$ over $\tilde{B}$ such that $\tilde{B}$ is a double cover of B and if κ is the covering map, then $\tilde{E}$ is a pullback of E along κ. Observe that in this case any fibrewise action by some group G on E induces a fibrewise action of G on $\tilde{E}$.

We would like to substitute $\tilde{E}$ for E, apply the index computation theorem we are interested in to the now orientable bundle and then relate the result we obtained to the original index we were interested in. With this plan in mind, we prove first Lemma 4.27 that describes the relationship between indices of fibre bundles over $\tilde{B}$ and B and then Lemma 4.28 that helps us to relate to each other the generators of these indices. We do not need the second part of Lemma 4.28 until the next section.

Remark 4.26. The construction of an oriented double cover of E is standard and is analogous to that of the oriented double cover of a non-orientable manifold [Dol88, Ch. VIII § 2]. As a set, $\tilde{B}$ can be described as

$$\tilde{B}\coloneqq\big\{(x,o_x)\colon x\in B, o_x\in\{\text{orientations of }\pi^{-1}(x)\}\big\}.$$

The basis of topology on $\tilde{B}$ is given by all sets $U^{\pm}$, defined in the following way: For any open set U in B such that the bundle E is trivialisable over U, we define two open sets in $\tilde{B}$, U^+ and U^-. Let $\varphi\colon\pi_B^{-1}(U)\to U\times\mathbb{R}^n$ be some trivialisation of $E\restriction_U$. Fix a standard orientation on $\mathbb{R}^n$ and transfer it to the orientation of each $\{x\}\times\mathbb{R}^n$ for $x\in U$. Define an open U^+ in $\tilde{B}$ as consisting of all pairs (x,o_x), such that $x\in U$ and o_x is an orientation of $\pi_B^{-1}(x)$ inherited via φ^{-1} from that of on $\{x\}\times\mathbb{R}^n$. Define U^- analogously but using the opposite orientation on $\mathbb{R}^n$ this time. The proof that this definition leads to a correct definition of topology on $\tilde{M}$ and that the map κ that sends a pair $(x, o_x) \in \tilde{M}$ to $x \in M$ is a covering map is analogous to how this is proved for an orientable cover of non-oriented manifold, see [Dol88, Ch. VIII § 2]. It follows from construction that $\kappa^* E$ is orientable.

Lemma 4.27. *Let M and $\tilde{M}$ be CW-complexes and a map κ from $\tilde{M}$ to M be a m-fold covering, where m is finite and $(m, p) = 1$. Let G be a group and $F \xrightarrow{\pi_M} M$ be a fibre bundle on which G acts fibrewise. Let $\kappa^* F \xrightarrow{\pi_{\tilde{M}}} \tilde{M}$ be a pullback of F along the covering map κ, equipped with an action of G inherited from F. As usual, we assume the trivial action of G on $\tilde{M}$ and M. By definition, $\operatorname{Index}_M^G F$is an ideal in the ring $H^*(M) \times H^*(\mathrm{B}G)$ and $\operatorname{Index}_{\tilde{M}}^G \kappa^* F$ is an ideal in the ring $H^*(\tilde{M}) \times H^*(\mathrm{B}G)$. Under the assumptions above they are connected by the formula*

$$(\kappa \times id)^* \operatorname{Index}_M^G F = \operatorname{Index}_{\tilde{M}}^G \kappa^* F \cap \operatorname{im}(\kappa \times id)^*.$$

Proof. Let us apply Borel construction to both fibre bundles. We get a new pair of bundles, $F \times_G \mathrm{E}G$ over $M \times \mathrm{B}G$ and $\kappa^* F \times_G \mathrm{E}G$ over $\tilde{B} \times \mathrm{B}G$. Notice, that the latter is a pullback of the former along the map $\kappa \times id$. Let us denote by pr_M and $pr_{\tilde{M}}$ the projections $F \times_G \mathrm{E}G \to M \times \mathrm{B}G$ and $\kappa^* F \times_G \mathrm{E}G \to \tilde{M} \times \mathrm{B}G$ respectively. Let κ_{Borel} be the map $\kappa^* F \times_G \mathrm{E}G \to F \times_G \mathrm{E}G$ induced from the pullback map $\kappa^* F \to F$. Below is the diagram that summarises these notation agreements.

$$\begin{tikzcd}
\kappa^* F \times_G \mathrm{E}G \arrow[r, "\kappa_{Borel}"] \arrow[d, "pr_{\tilde{M}}"] & F \times_G \mathrm{E}G \arrow[d, "pr_M"] \\
\tilde{M} \times \mathrm{B}G \arrow[r, "\kappa \times id"] & M \times \mathrm{B}G.
\end{tikzcd}$$

Using the diagram above, we see that the inclusion

$$(\kappa \times id)^* \operatorname{Index}_M^G F \subseteq \operatorname{Index}_{\tilde{M}}^G \kappa^* F$$

follows immediately from the monotonicity property of the index.

On the other hand, take any element $\tilde{X}$ in $H^*(\tilde{M} \times \mathrm{B}G)$ such that it belongs both to the ideal $\operatorname{Index}_{\tilde{M}}^G \kappa^* F$ and to the image of $(\kappa \times id)^*$, that is, $pr_{\tilde{M}}^*(\tilde{X}) = 0$ and there exists $X \in H^*(M \times \mathrm{B}G)$ such that $\tilde{X} = (\kappa \times id)^*(X)$. We would like to show that then X belongs to $\operatorname{Index}_M^G F$, that is, $\mathrm{pr}_M^*(X) = 0$.

Using the diagram above observe that

$$0 = pr_{\tilde{M}}^*(\tilde{X}) = pr_{\tilde{M}}^* \circ (\kappa \times id)^*(X) = \kappa_{Borel}^* \circ pr_M^*(X).$$

Observe that κ_{Borel} is also an m-fold covering, therefore κ^*_{Borel} is injective (for the proof see [Hat02, Prop. 3G.1]). Therefore we conclude that $pr^*_M(X) = 0$ implying that by definition X belongs to the ideal $\mathrm{Index}^G_M F$. □

Lemma 4.28. *Let $\tilde{M}$ be an m-fold normal cover over M with a covering map κ such that m is finite and $(m, p) = 1$. Then the following two statements hold*

1. *For any set $\{S_1, \ldots, S_l\}$ consisting of elements S_i, for $1 \leqslant i \leqslant l$ from $H^*(M)$ it holds that*

$$\kappa^*\langle S_1, \ldots, S_l\rangle = \langle \kappa^*(S_1), \ldots, \kappa^*(S_l)\rangle \cap \kappa^*\big(H^*(M)\big).$$

2. *For any two sets, $\{S_1, \ldots, S_l\}$ and $\{P_1, \ldots, P_{l'}\}$ consisting of elements S_i for $1 \leqslant i \leqslant l$ and P_j for $1 \leqslant j \leqslant l'$, all from $H^*(M)$, the inclusion*

$$\langle S_1, \ldots, S_l\rangle \supseteq \langle P_1, \ldots, P_{l'}\rangle$$

holds if and only if the inclusion

$$\langle \kappa^*(S_1), \ldots, \kappa^*(S_l)\rangle \supseteq \langle \kappa^*(P_1), \ldots, \kappa^*(P_{l'})\rangle$$

holds. In the first equation, we consider ideals in the ring $H^*(M)$ and in the second in the ring $H^*(\tilde{M})$.

Proof. In part one of this lemma, the inclusion $\kappa^*(\langle S_1, \ldots, S_l\rangle) \subseteq \langle \kappa^*(S_1), \ldots, \kappa^*(S_l)\rangle$ follows from the definition of the ideal in a ring. Our goal is to show inclusion the other way around.

Without loss of generality, we assume that $S_i \neq 0$ for any $1 \leqslant i \leqslant l$. Since $(m, p) = 1$, it follows that κ^* is injective, in particular $\kappa^*(S_i) \neq 0$ for all admissible i.

Let $\tilde{S_0} \in H^*(\tilde{M})$ and $S_0 \in H^*(M)$ be such that $\tilde{S_0} = \kappa^*(S_0) \neq 0$ and $\tilde{S_0}$ belongs to the ideal $\langle \kappa^*(S_1), \ldots, \kappa^*(S_l)\rangle$, that is, there exist elements $Q_i \in H^*(\tilde{M}), 1 \leqslant i \leqslant l$, such that $\tilde{S_0} = \kappa^*(S_1)Q_1 + \cdots + \kappa^*(S_l)Q_l$. We would like to show that S_0 belongs to the ideal $\langle S_1, \ldots, S_l\rangle$. Denote by $G := G(\tilde{M})$ a group of deck transformations of $\tilde{M}$. Let us act by each of the elements of this group on $\tilde{S_0}$ and sum up the results. Then we get that

$$\sum_{g\in G} \tilde{S_0}^g = \sum_{g\in G}\sum_{i=1}^{l} \kappa^*(S_i)^g Q_i^g. \tag{4.15}$$

It is a known fact [Hat02, Prop. 3G.1] that for a normal covering κ with a finite number of sheets coprime with the field characteristics, κ^* provides an isomorphism between the ring $H^*(M)$ and $H^*(\tilde{M})^G$, the subring of $H^*(\tilde{M})$ invariant under all automorphisms induced by the deck transformations. In particular, $\kappa^*(S_i)^g = \kappa^*(S_i)$ for any $0 \leqslant i \leqslant l$ and any element g in G. Due to the assumption that $\tilde{M} \to M$ is normal, $|G| = m$. Hence, combined with the re-scaling by factor $1/m = 1/|G|$ (remember that m is invertible since $(m,p) = 1$), the equation (4.15) transforms into the following equation

$$\kappa^*(S_0) = \sum_{i=1}^{l} \kappa^*(S_i)(\frac{1}{|G|}\sum_{g\in G} Q_i^g).$$

Since $\sum_{g\in G} Q_i^g$ for any $1 \leqslant i \leqslant l$ is invariant under all deck transformations, it follows that $\sum_{g\in G} Q_i^g$ belongs to the image of κ^* which finishes the first part of the proof.

The second part of the Lemma is an immediate consequence of the first one. In the following sequence of if and only if statements the first one follows from the injectivity of κ^* and the second one holds due to the first part of this lemma. The last if and only if holds because all of the ideal generators belong to the image of κ^*.

$$\begin{aligned}\langle S_1,\dots,S_l\rangle \supseteq \langle P_1,\dots,P_{l'}\rangle &\iff \kappa^*\langle S_1,\dots,S_l\rangle \supseteq \kappa^*\langle P_1,\dots,P_{l'}\rangle \\ &\iff \langle \kappa^*S_1,\dots,\kappa^*S_l\rangle \cap \operatorname{im}\kappa^* \supseteq \langle \kappa^*P_1,\dots,\kappa^*P_{l'}\rangle \cap \operatorname{im}\kappa^* \\ &\iff \langle \kappa^*S_1,\dots,\kappa^*S_l\rangle \supseteq \langle \kappa^*P_1,\dots,\kappa^*P_{l'}\rangle.\end{aligned}$$

□

Equipped with these two lemmas we prove the following theorem.

Theorem 4.29. *Let $\mathbb{R}^n \to E \to B$ be a non-orientable vector bundle. Then Theorems 4.5, 4.7, 4.15, 4.18 and 4.22 hold for E with the dropped requirement for E to be orientable.*

Proof of the theorem 4.29. Let $\mathbb{R}^n \to \tilde{E} \to \tilde{B}$ be the orientable double cover from the Remark 4.26.

Following the idea of the proof outlined at the beginning of the section, the first thing we need to check is that if E satisfies all but the orientability criteria of one index computation theorems, then $\tilde{E}$ satisfies all criteria for the same theorem.

For theorems listed in the statement we cared about the following data: That the bundle is orientable – $\tilde{E}$ satisfies this criterion; whether the bundle in question is real or complex, of odd or even rank – obviously these parameters are the same for $\tilde{E}$ as they are for E. If $E \cong E' \oplus \underline{\mathbb{R}}^2$ then $\tilde{E} \cong \tilde{E}' \oplus \underline{\mathbb{R}}^2$. Since $p_{n/2}(\tilde{E}) = \kappa^* p_{n/2}(E)$ and κ^* is injective, $p_{n/2}(\tilde{E}) = 0$ if and only if $p_{n/2}(E) = 0$.

In that group of theorems we worked with two types of constructions based on the bundle E: sphere bundle $S\big(W_pE\big)$ and fibrewise configuration space $\mathrm{Fconf}(p,E)$. For now, we concentrate on the case of $S\big(W_pE\big)$. The proof for $\mathrm{Fconf}(p,E)$ follows almost in the same way but with a couple of adjustments.

The map from κ^*E to E provided by the pullback induces $\mathfrak{S}_p$-equivariant map κ_{Borel} between the respective Borel constructions:

$$\begin{tikzcd} S\big(W_p\tilde{E}\big) \times_G \mathrm{E}G \arrow[r, "\kappa_{Borel}"] \arrow[d, "pr_{\tilde{B}}"] & S\big(W_pE\big) \times_G \mathrm{E}G \arrow[d, "pr_B"] \\ \tilde{B} \times BG \arrow[r, "\kappa\times id"] & B \times BG. \end{tikzcd}$$

We consider either $\mathbb{Z}_p$ or $\mathfrak{S}_p$ in the place of G. Observe that maps $\kappa \times id$ and κ_{Borel} are both 2-fold coverings. Since p is an odd prime it is coprime with 2. Applying Lemma 4.27 we see that

$$(\kappa \times id)^* \operatorname{Index}^G_B S\big(W_pE\big) = \operatorname{Index}^G_{\tilde{B}} S\big(W_p\tilde{E}\big) \cap \operatorname{im}(\kappa \times id)^*. \tag{4.16}$$

Notice, that any non-trivial double cover is normal, therefore we can apply Lemma 4.28 to $\kappa \times id$.

Observe, that in all cases $\operatorname{Index}^G_{\tilde{B}} S\big(W_p\tilde{E}\big)$ is generated by some elements which are polynomials in t and e when $G = \mathbb{Z}_p$ or a and b for when $G = \mathfrak{S}_p$ with coefficients that can be expressed in terms of the Pontryagin classes of $\tilde{E}$. By the naturality of the Pontryagin classes it follows, that any element of this form belongs to the image of $(\kappa\times id)^*$. Combining Lemma 4.28, equation (4.16) and the fact that $(\kappa\times id)^*$ is injective we see that $\operatorname{Index}^G_B S\big(W_pE\big)$ is generated by exactly the same polynomials in e and t, but with Pontryagin classes of $\tilde{E}$ substituted by Pontyagin classes of E. This finishes the proof for sphere bundles-related computations.

This proof also works for the set up of Theorems 4.15 and 4.22.

The only case when some additional work needs to be done: When E is a real vector bundle, $G = \mathfrak{S}_p$, and n is even, Theorem 4.18 give us not the precise value of $\operatorname{Index}^{\mathfrak{S}_p}_B \mathrm{Fconf}(p,E)$ but rather upper and lower bounds on it.

We would like to prove that in this case

$$\langle\zeta_E\rangle \subseteq \operatorname{Index}^{\mathfrak{S}_p}_B \mathrm{Fconf}(p,E) \subseteq ab^{-1}\Big(\operatorname{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\Big) + \langle\zeta_E\rangle,$$

using the fact that the same holds for $\tilde{E}$.

Indeed, we see that

$$
\begin{aligned}
(\kappa \times id)^* \langle \zeta_E \rangle &= \langle \zeta_{\tilde{E}} \rangle \cap \operatorname{im}(\kappa \times id)^* \\
&\subseteq \operatorname{Index}_{\tilde{B}}^{\mathfrak{S}_p} \operatorname{Fconf}\left(p, \tilde{E}\right) \cap \operatorname{im}(\kappa \times id)^* = (\kappa \times id)^* \operatorname{Index}_{B}^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)\,.
\end{aligned}
$$

The first equality follows from Lemma 4.28, the inclusion from Theorem 4.18 applied to the vector bundle $\tilde{E}$ and the last equality from Lemma 4.27. Similarly, we claim that

$$
\begin{aligned}
(\kappa \times id)^* \operatorname{Index}_{B}^{\mathfrak{S}_p} \operatorname{Fconf}(p, E) &= \operatorname{Index}_{\tilde{B}}^{\mathfrak{S}_p} \operatorname{Fconf}\left(p, \tilde{E}\right) \cap \operatorname{im}(\kappa \times id)^* \\
&\subseteq \left(ab^{-1}\left(\operatorname{Ann}\big(p_{n/2}(\tilde{E})^{\frac{p-1}{2}}\big)\zeta_{\tilde{E}}\right) + \langle \zeta_{\tilde{E}} \rangle\right) \cap \operatorname{im}(\kappa \times id)^* \\
&= (\kappa \times id)^* \left(ab^{-1}\left(\operatorname{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\right) + \langle \zeta_E \rangle\right).
\end{aligned}
$$

Indeed, the first equality follows again from Lemma 4.27, the inclusion follows from Theorem 4.18 applied to the vector bundle $\tilde{E}$. As for the last equality, it can be obtained from Lemma 4.28 but with a modification.

Let $\tilde{S}_0$ be an element from the intersection of ideal $ab^{-1}\left(\operatorname{Ann}\big(p_{n/2}(\tilde{E})^{\frac{p-1}{2}}\big)\zeta_{\tilde{E}}\right) + \langle \zeta_{\tilde{E}} \rangle$ and the image of $(\kappa \times id)^*$. Let S_0 be its preimage with respect to $(\kappa \times id)^*$. Our goal is to show that S_0 belongs to the ideal $ab^{-1}\left(\operatorname{Ann}\big(p_{n/2}(E)^{\frac{p-1}{2}}\big)\zeta_E\right) + \langle \zeta_E \rangle$. Although we do not have an exhaustive list of all generators of this ideal, one can see that any its element is of the form $ab^{-1}Q\zeta_{\tilde{E}} + P\zeta_{\tilde{E}}$, for some elements $\tilde{P}, \tilde{Q} \in H^*(\tilde{B} \otimes \mathrm{B}\mathfrak{S}_p)$ where $\tilde{Q}$ belongs to the ideal $\operatorname{Ann}\big(p_{n/2}(\tilde{E})\big)$. Let $\tilde{q}$ denote the free term of Q viewed as a polynomial in b. By the definition of $\operatorname{Ann}\big(p_{n/2}(\tilde{E})\big)$,

$$
\tilde{q} \cdot p_{n/2}(\tilde{E})^{\frac{p-1}{2}} = 0.
$$

Since $p_{n/2}(\tilde{E})$ belongs to the image of $(\kappa \times id)^*$, it is invariant under the deck transformations group action. Therefore for any element g of the deck transformations group $G(\tilde{B}) = \mathbb{Z}_2$ it holds that

$$
0 = \big(\tilde{q} \cdot p_{n/2}(\tilde{E})^{\frac{p-1}{2}}\big)^g = q^g \cdot p_{n/2}(\tilde{E})^{\frac{p-1}{2}}.
$$

Consequently, for any $g \in \mathbb{Z}_2$, $\tilde{Q}^g$ also belongs to $\operatorname{Ann}\big(p_{n/2}(\tilde{E})^{\frac{p-1}{2}}\big)$.

Now let us repeat the argument of Lemma 4.28 for $\tilde{S}_0$. Fix g to be the generator of $\mathbb{Z}_2$. We see that

$$
\tilde{S}_0 = ab^{-1}\frac{Q + Q^g}{2}\zeta_{\tilde{E}} + \frac{P + P^g}{2}\zeta_{\tilde{E}}.
$$

Observe, that $(\kappa \times id)^*$ maps ζ_E to $\zeta_{\tilde{E}}$. We know that $ab^{-1}\frac{Q+Q^g}{2}$ and $\frac{P+P^g}{2}$ belong to the image of $(\kappa \times id)^*$ as well and we have shown that $\frac{Q+Q^g}{2}$ belongs to $\operatorname{Ann}\big(p_{n/2}(\tilde{E})\big)$.

Since $(\kappa \times id)^*$ is injective the claim follows. $\square$

Chapter 5

Returning to the parameterised Nandakumar & Ramana Rao problem

5.1 Partial solution, general case

In this section, we finally prove Theorem 5.1. Then we apply it to the case when B is the Grassmannian $\mathrm{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ and E is the tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over it.

Theorem 5.1. *Fix an odd prime p. Let $\mathbb{R}^n \to E \to B$ be a real vector bundle. Then Problem 2.5 with parameters (E, p, j) has a solution whenever*

$$j \leqslant \begin{cases} \operatorname{rank} E + I - 2, & \text{if } \operatorname{rank} E \text{ is even,} \\ \operatorname{rank} E + I - 1, & \text{if } \operatorname{rank} E \text{ is odd.} \end{cases} \tag{5.1}$$

where I can be determined in one of the following ways:

- *Choose a real vector bundle $\bar{E}$ such that the direct sum $E \oplus \bar{E}$ is isomorphic to a trivial bundle. Then*

$$I := \max_i \{{}_p c_i(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}) \neq 0\} = \max_i \{\mathrm{P}^i\, u_{\bar{E}^{\mathbb{C}}} \neq 0\},$$

  were P^i denotes i-th Steenrod mod p operation, $u_{\bar{E}^{\mathbb{C}}}$ is the Thom class of the bundle $\bar{E} \otimes_{\mathbb{R}} \mathbb{C}$ and classes ${}_p c_j(\bar{E} \otimes_{\mathbb{R}} \mathbb{C})$ are as described in Definition 3.10.

- *Alternatively, denote by $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ the inverse of the total Chern class $c(E \otimes_{\mathbb{R}} \mathbb{C})$ in the ring of infinite series $H^{\Pi}(B)$ and by $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}(t) \in H^*(B) \otimes \mathbb{F}_p[t]$ its* *homogenisation using a variable t of degree 2. Then I is such that $2I(p-1)$ is the biggest degree among all non-zero coefficients of the polynomial in t given by*

$$\Big(\prod_{1\leqslant r\leqslant p-1} c(E\otimes_{\mathbb{R}}\mathbb{C})^{-1}(rt)\Big)^{\frac{1}{2}}.$$

It turns out that I is always an even number. In the special case when $\operatorname{rank} E$ is also even and E has two linearly independent nowhere zero sections, that is, $E \cong E' \oplus \underline{\mathbb{R}}^2$ for some vector bundle E', the bound given by (5.1) can be improved by 1: In this case, a maximal j attainable equals $\operatorname{rank} E + I - 1$. Notice that this is an odd number, while any upper bound on j that comes from the equation (5.1) is even.

We begin with a simple lemma that will help us to compare ideals in the cohomology ring $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$.

Lemma 5.2. *Let X be some element in $H^*(B)\otimes\mathbb{F}_p[t]$. Then for any positive integer M*

$$\big[\langle et^M, t^{M+1}\rangle \nsubseteq \langle eX, tX\rangle\big] \iff \big[\langle t^{M+1}\rangle \nsubseteq \langle tX\rangle\big],$$

where all ideals are considered inside the ring $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$.

Proof. The proof consists of two straightforward steps. First, notice that

$$\big[\langle t^{M+1}\rangle \subseteq \langle eX, tX\rangle\big] \iff \big[\langle t^{M+1}\rangle \subseteq \langle tX\rangle\big].$$

On the other hand,

$$\begin{aligned}
\big[\langle et^M\rangle \subseteq \langle eX, tX\rangle\big] &\iff \big[\exists Q, P_1, P_2\colon eXQ + tXP_1 + etXP_2 = et^M\big]\\
&\iff \big[\exists Q, P_2\colon eXQ + etXP_2 = et^M\big]\\
&\iff \big[\exists Q, P_2\colon X(Q + tP_2) = t^M\big]\\
&\iff \big[\langle t^M\rangle \subseteq \langle X\rangle\big]\\
&\iff \big[\langle t^{M+1}\rangle \subseteq \langle tX\rangle\big],
\end{aligned}$$

where we assume Q, P_1, P_2 belong to the polynomial subring $H^*(B)\otimes\mathbb{F}_p[t]$ o the ring $H^*(B)\otimes H^*(\mathrm{B}\mathbb{Z}_p)$.

□

Proof of the theorem 5.1. Our strategy is to find such combinations of E, p and j such that

$$\operatorname{Index}_B^{\mathfrak{S}_p} S\big(\underline{W_p^j}\big) \nsubseteq \operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E).$$

In this case, it follows from the monotonicity property of the index that no $\mathfrak{S}_p$-equivariant map exists from fibrewise configuration space $\mathrm{Fconf}(p, E)$ to the trivial sphere bundle $S(\underline{W_p^j})$. As we established in Lemma 2.10 this implies that Problem 2.5 has a solution for the set of parameters (E, p, j).

However, as we have seen already a couple of times, it is more convenient to work in the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$ than in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$.

The second part of Lemma 4.28 applied to $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ in a place of κ^* tells us that instead of comparing ideals directly in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ we can take generators of those ideals, push them by $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ to the ring $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$ and compare the ideals generated by their images in this ring.

Let us collect relevant results from previous sections. We have proved in Section 4.4 that all our index computations stay valid when E is non-orientable. From now on, every time we quote one of the index computation theorems, it is automatically assumed that it is combined with Theorem 4.29 in case E is non-orientable.

We begin with the sphere bundle. The relevant theorem, in this case, is Theorem 4.7. When j is even, $\mathrm{Index}_B^{\mathfrak{S}_p} S(\underline{W_p^j})$ is generated by the unique element $\zeta_{\underline{W_p^j}}$ such that $\mathrm{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ maps it to $t^{\frac{j}{2}(p-1)}$. When j is odd, there are two generators of $\mathrm{Index}_B^{\mathfrak{S}_p} S(\underline{W_p^j})$, $\varepsilon_{\underline{W_p^j}}$ and $\theta_{\underline{W_p^j}}$, which are in this case preimages of $et^{\frac{j+1}{2}(p-1)-1}$ and $t^{\frac{j+1}{2}(p-1)}$, respectively. We can simplify our task by considering only the latter of these generators. If for some tuple (E, p, j) with an odd j, $\theta_{\underline{W_p^j}}$ doesn't belong to $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$, then the same holds for the whole $\mathrm{Index}_B^{\mathfrak{S}_p} S(\underline{W_p^j})$ and we have found a solution to the problem. However, it might happen that the ideal $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ contains $\theta_{\underline{W_p^j}}$ but not $\varepsilon_{\underline{W_p^j}}$. In this case, we will lose a solution because of our simplification. Nevertheless, since we don't understand fully the ideal $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$ for bundles E of even rank, or, more precisely, since we don't have a full description of its part that belongs to the ideal generated by a in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$, we have to use some simplification. After this proof, in Remark 5.3 it is shown that in this way we miss potentially at most one value of j.

Notice that when j is odd, $\theta_{\underline{W_p^j}} = \zeta_{\underline{W_p^{j+1}}}$. Therefore if by using our method we have found out that there is a solution for some tuple (E, p, j) with an odd j, it automatically follows that there exists a solution for the tuple of parameters $(E, p, j+1)$ as well. Consequently, for a fixed pair (E, p) the maximal value of j we can find out with our strategy is always even.

Let us now summarise our knowledge of $\mathrm{Index}_B^{\mathfrak{S}_p} \mathrm{Fconf}(p, E)$. The relevant theorems, in this case, are Theorems 4.15 and 4.18. When the rank of E is odd, we know that

$\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ is generated by the elements ε_E and θ_E in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ such that

$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \varepsilon_E = e t^{\frac{p-3}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$$
$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \theta_E = t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} .$$

According to Lemma 5.2 $\operatorname{Index}_B^{\mathfrak{S}_p} S(\underline{W}_p^j)$ belongs to $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ if and only if $\zeta_{\underline{W}_p^j}$ or $\theta_{\underline{W}_p^j}$ for an even or odd values of j respectively belong to the ideal generated by θ_E. Therefore in this case we are not losing any information by disregarding non-polynomial ideals' generators.

When rank E is even, unfortunately, we don't know $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ fully. However, what we can tell in this case, is that among all its potential generators there is only one belonging to the subring $H^*(B) \otimes \mathbb{F}_p[b]$ of the ring $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$, namely ζ_E. Observe, that $\theta_{\underline{W}_p^j}$ belongs to $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ if and only if it belongs to the ideal generated by ζ_E. This is the reason we have chosen to disregard $\varepsilon_{\underline{W}_p^j}$ - we can't say much about whether it belongs to $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ or not if we don't know $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ better. However, we have enough information to check this for $\theta_{\underline{W}_p^j}$. In preparation for the next step, recall that

$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \zeta_E = \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} .$$

To summarise shortly our steps so far:

- We look for cases when $\operatorname{Index}_B^{\mathfrak{S}_p} S(\underline{W}_p^j)$ does not belong to $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$. Relevant results from previous sections are Lemma 2.10 that reformulates the problem, and Theorems 4.7, 4.15, and 4.18 that provide index computations. Theorem 4.29 allows us to work with non-orientable bundles.

- Instead of comparing ideals directly in $H^*(B) \otimes H^*(\mathrm{B}\mathfrak{S}_p)$ we map their generators to $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$ using inclusion $\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p}$ and compare ideals, generated by them in $H^*(B) \otimes H^*(\mathrm{B}\mathbb{Z}_p)$. The relevant result from previous sections is Lemma 4.28.

- We consider only generators belonging to $H^*(B) \otimes \mathbb{F}_p[b]$, equivalently, we consider only images of generators that belong to $H^*(B) \otimes \mathbb{F}_p[t]$. We are not losing any solutions this way for a vector bundle E of an odd rank, but we might lose one value of j if rank E is even.

- All previous points combined together transform the initial problem into the question of whether $t^{\lceil j/2 \rceil (p-1)}$ belongs to $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ when rank E is even, or t $t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C}))^{\frac{1}{2}}$ when rank E is odd.

Therefore, we can formulate our answer to Problem 2.5 in the following way.

Given a pair (E, p) let M be the minimal possible number such that

$$t^M \in \begin{cases} \langle t^{\frac{p-1}{2}} \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \rangle, & \text{if } \operatorname{rank} E \text{ is odd,} \\ \langle \check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \rangle, & \text{if } \operatorname{rank} E \text{ is even.} \end{cases} \tag{5.2}$$

Then there is a solution to the Problem 2.5 with parameters (E, p, j) for all j such that $j \leqslant 2j' \leqslant M - 1$ for some integer j'.

Our next goal is to determine how M depends on the bundle E and the value of p.

Let $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ denote the inverse of the total Chern class $c(E \otimes_{\mathbb{R}} \mathbb{C})$ in the ring of infinite series $H^{\Pi}(B)$. Observe, that it has a finite number of summands, all of which have even degrees. This follows from the fact that for any bundle E of a finite rank over a compact Hausdorff base, there exists a bundle $\bar{E}$ of a finite rank, such that the direct sum of E and $\bar{E}$ is isomorphic to a trivial bundle [Hat17, Prop. 1.4]. Consequently, by Whitney sum axiom for Chern classes, $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ equals the total Chern class of $\bar{E} \otimes_{\mathbb{R}} \mathbb{C}$, that is,

$$c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1} = c(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}) = 1 + c_1(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}) + \cdots + c_{d'}(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}).$$

In the expression above we assume that $c_{d'}$ is the last non-zero mod-p Chern class of $\bar{E} \otimes_{\mathbb{R}} \mathbb{C}$, or, in other words, that the biggest degree among summands of $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ is $2d'$. Observe that $c_i(\bar{E} \otimes_{\mathbb{R}} \mathbb{C})$ are non-zero only for even values of i. This follows from the fact that all odd Chern classes of any bundle arising as a complexification of some real bundle E are in $\mathbb{Z}_2$-torsion in $H^*(B; \mathbb{Z})$ and therefore their mod-p reductions are zero [Hat17, Thm. 3.16]

We are interested in $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ since it allows us to find at least some power of t in the ideal generated by $\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})$, and, as a consequence, in the ideals from the equation (5.2). Observe, that

$$c(E \otimes_{\mathbb{R}} \mathbb{C})(t) \cdot c(\bar{E} \otimes_{\mathbb{R}} \mathbb{C})(t) = (-t)^{\operatorname{rank} E + d'}. \tag{5.3}$$

In the equation above $c(E \otimes_{\mathbb{R}} \mathbb{C})(t) \in H^*(B \times \mathrm{B}\mathbb{Z}_p)$ stands for the formal homogenisation of $c(E \otimes_{\mathbb{R}} \mathbb{C})$ using a variable t of degree 2. Therefore $t^{(\operatorname{rank} E + d')(p-1)}$ belongs to the ideal generated by $\check{\mathrm{ch}}(E \otimes_{\mathbb{R}} \mathbb{C})$ in $H^*(B) \otimes \mathrm{B}\mathbb{Z}_p$. Are there any smaller powers of t contained in this ideal? Using Lemma 3.11 we can rewrite the second term of the product above as

$$\check{\mathrm{ch}}\big(\bar{E} \otimes_{\mathbb{R}} \mathbb{C}\big) = t^{d'(p-1)} + {}_pc_2(\bar{E})t^{(d'-2)(p-1)} + \cdots + {}_pc_{d'}(\bar{E})$$

Let I be the maximal number, such that the class ${}_pc_I(E)$ is non-zero. Then we can divide both sides of equation (5.3) by $t^{(d'-I)(p-1)}$. This shows that $t^{(\operatorname{rank} E+I)(p-1)}$ is also present in the ideal generated by $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})$. However, no smaller power of t can be contained in this ideal, due to our choice of I. Notice that I is always an even number since $c(\bar{E} \otimes_{\mathbb{C}} \mathbb{R})$ is the sum of elements of degrees divisible by 4, consequently ${}_pc_i(\bar{E}) = 0$ for any odd i.

We conclude that the minimal power of t present in the ideal generated by $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ is $t^{(\operatorname{rank} E+I)\frac{p-1}{2}}$, consequently the minimal power of t present in the ideal generated by $t^{\frac{p-1}{2}}\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$ is $t^{(\operatorname{rank} E+I+1)\frac{p-1}{2}}$. If its degree is bigger than $j(p-1)/2$ for an even j or $(j+1)(p-1)/2$ for an odd j, then we can deduce that there is a solution to the Problem 2.5 with parameters (E, p, j).

Therefore for a real vector bundle E the maximal value of j that we can get out of our proof such that Problem 2.5 with parameters (E, p, j) has a solution is

$$j \leqslant \begin{cases} \operatorname{rank} E + I - 2, & \text{if } \operatorname{rank} E \text{ is even,} \\ \operatorname{rank} E + I - 1 & \text{if } \operatorname{rank} E \text{ is odd.} \end{cases}$$

where I can be computed in two different ways. The first one is to see I as the biggest value of i such that ${}_pc_i(\bar{E})$ is non-zero, or, equivalently, such that the i-th Steenrod power of $u_{\bar{E}^{\mathbb{C}}}$ is non-zero. One can take as $\bar{E}$ any bundle such that $E \oplus \bar{E}$ is isomorphic to a trivial bundle. The value of I does not depend on this choice, since it can be also seen as the value such that the biggest degree among coefficients in the polynomial

$$\prod_{1 \leqslant r \leqslant p-1} c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}(at)$$

is equal $2I(p-1)$. Same as before, in the equation above $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}(t)$ stands for the formal homogenisation of $c(E \otimes_{\mathbb{R}} \mathbb{C})^{-1}$ using a variable t of degree 2.

This finishes the main part of the proof. Observe, however, that so far we have only used our knowledge of the Fadell-Husseini index with respect to $\mathfrak{S}_p$. In the special case when E admits two linearly independent nowhere zero sections, that is, it can be represented as a direct sum $E' \oplus \underline{\mathbb{R}}^2$ of some vector bundle E' with a trivial bundle of rank two, we can compute the index with respect to $\mathbb{Z}_p$-action as well. Using Theorem 4.7 and Theorem 4.22 we compute

$$\begin{aligned} \operatorname{Index}_B^{\mathbb{Z}_p} S(\underline{W}_p^j) &= \langle t^{\frac{j(p-1)}{2}} \rangle, \\ \operatorname{Index}_B^{\mathbb{Z}_p} \operatorname{Fconf}(p, E) &= \langle e t^{\frac{p-1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}, t^{\frac{p+1}{2}} \check{\mathrm{c}}\mathrm{h}\big(E' \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}} \rangle. \end{aligned}$$

In this case, we can use Lemma 2.10 and monotonicity property of Fadell-Husseini index immediately, without additional steps. Using the computations we did for the $\mathfrak{S}_p$ action case, we see that there is a solution for a tuple (E, p, j) whenever j is such that

$$\frac{j(p-1)}{2} < \frac{p+1}{2} + \frac{(\operatorname{rank} E' + I)(p-1)}{2}.$$

Substituting $\operatorname{rank} E'$ with $\operatorname{rank} E - 2$ we get the following bound on j:

$$j \leqslant \operatorname{rank} E + I - 1.$$

We abuse notation and do not specify whether I is computed for the bundle E or for the bundle E' since their total Chern classes are equal and therefore the value of I is the same in both cases.

Contrary to the computations with respect to $\mathfrak{S}_p$ there is no parity condition imposed on j. In particular, this result is better by 1 for the case when $\operatorname{rank} E$ is even than the one we got using $\mathfrak{S}_p$. □

Remark 5.3. As was promised, let us contemplate how many values of j we have potentially missed by ignoring non-polynomial generators of indices. If we knew the exact value of the ideal $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ and had a method to compare it with $\operatorname{Index}_B^{\mathfrak{S}_p} S(\underline{W_p^j})$, for any parities combinations of $\operatorname{rank} E$ and j, how much better could our results have been?

When E has an odd rank, we already know fully the ideal $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$. As was mentioned in the proof, Lemma 5.2 shows that we have lost no information in this case.

Consider now a vector bundle E of even rank. Let j_0 denote the maximum value of j we have obtained by applying our simplified proof strategy, meaning we have shown that $\zeta_{\underline{W_p^{j_0}}}$ does not belong to the ideal $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ but $\zeta_{\underline{W_p}^{j_0+2}}$ does. Remember that j_0 is necessarily even. Observe, that

$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \theta_{\underline{W_p^{j_0+3}}} = t^{\frac{(j_0+4)(p-1)}{2}} = t^{p-1} \operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \zeta_{\underline{W_p^{j_0+2}}}$$

and

$$\operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \varepsilon_{\underline{W_p^{j_0+3}}} = et^{\frac{(j_0+4)(p-1)}{2}-1} = et^{p-2} \operatorname{res}_{\mathbb{Z}_p}^{\mathfrak{S}_p} \zeta_{\underline{W_p^{j_0+2}}},$$

implying $\operatorname{Index}_B^{\mathfrak{S}_p} S(\underline{W_p^{j_0+3}})$ is contained in $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$. This shows that the only value of j which we potentially have lost since we don't know $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ fully is $j_0 + 1$.

Another question one might ask is whether we would be able to obtain better results if we could compute $\operatorname{Index}_B^{\mathbb{Z}_p} \operatorname{Fconf}(p, E)$ for any vector bundle E. From Observation 4.2

and Theorem 4.7 it follows that $\operatorname{Index}_B^{\mathbb{Z}_p} \operatorname{Fconf}(p, E)$ always contains the ideal generated by $\check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}}$.

As we computed above, $t^{(\operatorname{rank} E+I)\frac{p-1}{2}}$ belongs to the ideal $\langle \check{\mathrm{c}}\mathrm{h}(E \otimes_{\mathbb{R}} \mathbb{C})^{\frac{1}{2}} \rangle$. Since the ideal $\operatorname{Index}_B^{\mathbb{Z}_p} S(\underline{W_p^j})$ is generated by the unique element $t^{\frac{j(p-1)}{2}}$ non-dependent of the parity of j, the best possible value of j we could have hoped for is $j \leqslant \operatorname{rank} E + I - 1$. This is exactly the bound we have already got for all vector bundles E of an odd rank and is worse by 1 than what we have got for the case of even rank E.

To summarise: The fact that we don't know the precise value of $\operatorname{Index}_B^{\mathfrak{S}_p} \operatorname{Fconf}(p, E)$ and almost in no case know the value of $\operatorname{Index}_B^{\mathbb{Z}_p} \operatorname{Fconf}(p, E)$ doesn't change the result for vector bundles E of an odd rank, and potentially makes the result worse in cases when the rank of E is even, but just by one.

5.2 Partial solution, transversal case

In this subsection, we apply the above computations to the particular case when the base B is a real Grassmannian $\operatorname{Gr}(\mathbb{R}^n, \mathbb{R}^N)$ and E is a tautological bundle $\gamma(\mathbb{R}^n, \mathbb{R}^N)$ over it.

Theorem 5.4. *Fix an odd prime number p. For a given pair of integers, (n, N) such that $p \leqslant n \leqslant N$, the parameterised version of the Nandakumar & Ramana Rao problem 1.3 with parameters $(\gamma(\mathbb{R}^n, \mathbb{R}^N), p, j)$ has a solution whenever $j \leqslant N - 1$ and $N - n$ is even, or $j \leqslant N - 2$ and $N - n$ is odd. Additionally, for $p = n + 1$ and $n \leqslant N$ the upper bound on j is worse by 1.*

Proof. Let us apply Theorem 5.1 to this problem.

To compute I, observe that in this case a bundle $\bar{E}$ can be chosen to be a pullback of the tautological bundle $\gamma(\mathbb{R}^{N-n}, \mathbb{R}^N)$ over the Grassmannian $\operatorname{Gr}(\mathbb{R}^{N-n}, \mathbb{R}^N)$ along an involution $i\colon \operatorname{Gr}(\mathbb{R}^n, \mathbb{R}^N) \cong \operatorname{Gr}(\mathbb{R}^{N-n}, \mathbb{R}^N)$ that sends a class of a linear subspace in $\mathbb{R}^N$ to the class of its orthogonal complement. Denote the resulting bundle by $\bar{\gamma}(\mathbb{R}^n, \mathbb{R}^N)$. In this case $\operatorname{rank} E = n$ and $\operatorname{rank} \bar{E} = N - n$. By definition

$$\check{\mathrm{c}}\mathrm{h}\big(\bar{\gamma}(\mathbb{R}^n, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{C}\big) = \prod_{1 \leqslant a \leqslant p-1} \sum_{i=0}^{\lfloor (N-n)/2 \rfloor} (-1)^i p_i\big(\bar{\gamma}(\mathbb{R}^n, \mathbb{R}^N)\big)(at)^{(N-n)-2i}.$$

When $N - n$ is even, the element

$$\check{\mathrm{c}}\mathrm{h}\big(\bar{\gamma}(\mathbb{R}^n, \mathbb{R}^N) \otimes_{\mathbb{R}} \mathbb{C}\big)^{\frac{1}{2}}$$

has (up to a sign) the free term that equals

$$p^{(p-1)/2}_{(N-n)/2}\big(\gamma(\mathbb{R}^{N-n},\mathbb{R}^N)\big).$$

When $N-n$ is odd, the smallest non-zero power of t present in the polynomial

$$\check{\mathrm{ch}}\big(\bar{\gamma}(\mathbb{R}^n,\mathbb{R}^N)\otimes_{\mathbb{R}}\mathbb{C}\big)^{\frac{1}{2}}$$

is $t^{\frac{p-1}{2}}$, with the coefficient

$$p^{(p-1)/2}_{(N-n-1)/2}\big(\gamma(\mathbb{R}^{N-n},\mathbb{R}^N)\big).$$

For $p-1 \leqslant n$, the respective Pontryagin classes are non-zero due to Lemma 3.3. This means that I is equal to $N-n$ when $N-n$ is even, and I is $N-n-1$ when $N-n$ is odd.

Therefore, applying Theorem 5.1 we get the results for the following values of j:

	N is even	N is odd
n is even	$N-n$ is even, $I=N-n$ $\Longrightarrow$ $j\leqslant n+(N-n)-2=N-2$	$N-n$ is odd, $I=N-n-1$ $\Longrightarrow$ $j\leqslant n+(N-n-1)-2=N-3$
n is odd	$N-n$ is odd, $I=N-n-1$ $\Longrightarrow$ $j\leqslant n+(N-n-1)-1=N-2$	$N-n$ is even, $I=N-n$ $\Longrightarrow$ $j\leqslant n+(N-n)-1=N-1$

We could have stopped with our proof at this point, however, we know that if $\operatorname{rank}E=n$ is even, the results of Theorem 5.1 are not optimal and have a potential to be improved by 1. It turns out that this is also true for this particular case. We can improve the result from the table above for the case when n is even by reducing the problem to the one involving a bundle with two linearly independent sections, hence satisfying the additional condition in the Theorem 5.1.

Consider an embedding i_{+2} of the Grassmannian $\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2})$ into the Grassmannian $\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)$ defined in the following way: Let us choose two linearly independent vectors in $\mathbb{R}^N$ and name them e_{N-1} and e_N. Then $\mathbb{R}^N$ is isomorphic to a direct sum of $\mathbb{R}^{N-2}$ with a two-dimensional vector space $\langle e_{N-1},e_N\rangle$ spanned by the vectors e_{N-1} and e_N. For any point $[V]$ in the Grassmannian $\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2})$ corresponding to a $(n-2)$-dimensional linear subspace V in $\mathbb{R}^{N-2}$, define $i_{+2}(V)$ to be a point in the Grassmannian $\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)$, corresponding to the subspace $V\oplus\langle e_{N-1},e_N\rangle$ in $\mathbb{R}^N$. Notice that the pullback of a tautological bundle $\gamma(\mathbb{R}^n,\mathbb{R}^N)$ along i_{+2} has two linearly independent nowhere zero sections, more precisely

$$i_{+2}^*(\gamma(\mathbb{R}^n,\mathbb{R}^N)) = \gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \oplus \underline{\mathbb{R}}^2.$$

Let us apply the special case of the Theorem 5.1 to the fibre bundle

$$\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \oplus \underline{\mathbb{R}}^2\big).$$

The value of I for $\gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2})$ was already computed above: Provided $p-1 \leqslant n-2$, I is equal $N-n$ when both n and N are even, and $N-n-1$ when n is even and N is odd. An application of Theorem 5.1 (or, rather, its proof) gets us the following result: For j satisfying the criteria

$$j \leqslant \begin{cases} n+(N-n)-1 = N-1 & \text{when } n \text{ is even, } N \text{ is even.} \\ n+(N-n-1)-1 = N-2 & \text{when } n \text{ is even, } N \text{ is odd,} \end{cases} \tag{5.4}$$

an element $t^{j(p-1)/2}$ does not belong to the ideal

$$\mathrm{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2})}\,\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \oplus \underline{\mathbb{R}}^2\big).$$

The map i_{+2} induces a map $i_{+2} \times \mathrm{id}$

$$\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \times \mathrm{B}\mathbb{Z}_p \to \mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N) \times \mathrm{B}\mathbb{Z}_p.$$

Notice that the fibre bundle

$$\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \oplus \underline{\mathbb{R}}^2\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$$

is a pullback of the bundle

$$\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^n,\mathbb{R}^N)\big) \times_{\mathbb{Z}_p} \mathrm{E}\mathbb{Z}_p$$

along the embedding $i_{+2} \times \mathrm{id}$.

Since $(i_{+2} \times \mathrm{id})^*$ is obviously the identity on $H^*(\mathrm{B}\mathbb{Z}_p)$, it sends $t^{j(p-1)/2}$ to $t^{j(p-1)/2}$. Using once again the monotonicity property of the index, observe that the ideal

$$(i_{+2} \times \mathrm{id})^* \left(\mathrm{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)}\,\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^n,\mathbb{R}^N)\big)\right) \subsetneq H^*\big(\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \times \mathrm{B}\mathbb{Z}_p\big)$$

belongs to the ideal

$$\mathrm{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^{n-2},\mathbb{R}^{N-2})}\,\mathrm{Fconf}\big(p, \gamma(\mathbb{R}^{n-2},\mathbb{R}^{N-2}) \oplus \underline{\mathbb{R}}^2\big) \subsetneq H^*\big(\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N) \times \mathrm{B}\mathbb{Z}_p\big).$$

In particular, if the latter does not contain $t^{j(p-1)/2}$ for some specific value of j then neither does $\operatorname{Index}^{\mathbb{Z}_p}_{\mathrm{Gr}(\mathbb{R}^n,\mathbb{R}^N)} \operatorname{Fconf}\big(p, \gamma(\mathbb{R}^n, \mathbb{R}^N)\big)$.

Therefore we can conclude that Problem 1.3 has solutions for all combinations of values (n, N, p, j) such that n is even, $p \leqslant n-1$ and j is as specified in equation (5.4).

This result is better by 1 for any even n than the result we had from a straightforward application of Theorem 5.1, although it comes with a stricter restriction on p, namely $p \leqslant n-1$. $\square$

Declaration of Authorship

Surname: Levinson

Name: Tatiana

I declare to the Freie Universität Berlin that I have completed the submitted dissertation independently and without the use of sources and aids other than those indicated. The present thesis is free of plagiarism. I have marked as such all statements that are taken literally or in content from other writings. This dissertation has not been submitted in the same or similar form in any previous doctoral procedure.

I agree to have my thesis examined by a plagiarism examination software.

Date: ____________________ Signature: ____________________

Bibliography

[AM13] Alejandro Adem and R. James Milgram, *Cohomology of finite groups*, vol. 309, Springer Science & Business Media, 2013.

[Arn69] Vladimir I. Arnol'd, *The cohomology ring of the colored braid group*, Mathematical notes of the Academy of Sciences of the USSR **5** (1969), 138–140.

[Art25] Emil Artin, *Theorie der zopfe*, Hamburg Abh **4** (1925), 47–72.

[Art47a] ———, *Braids and permutations*, Annals of Mathematics (1947), 643–649.

[Art47b] ———, *Theory of braids*, Annals of Mathematics (1947), 101–126.

[AK00] Franz Aurenhammer and Rolf Klein, *Voronoi diagrams*, Handbook of computational geometry **5** (2000), no. 10, 201–290.

[AFS22] Ilani Axelrod-Freed and Pablo Soberón, *Bisections of mass assignments using flags of affine spaces*, Discrete & Computational Geometry (2022), 1–19.

[BBS10] Imre Bárány, V. M. Pavle Blagojević, and András Szűcs, *Equipartitioning by a convex 3-fan*, Adv. Math. **223** (2010), no. 2, 579–593.

[BS18] Imre Bárány and Pablo Soberón, *Tverberg's theorem is 50 years old: a survey*, Bulletin of the American Mathematical Society **55** (2018), no. 4, 459–492.

[BZ04] William A. Beyer and Andrew Zardecki, *The early history of the Ham–Sandwich theorem*, The American Mathematical Monthly **111** (2004), no. 1, 58–61.

[Bjö95] Anders Björner, *Topological methods*, vol. 2, Amsterdam, 1995.

[BCLCDB] Pavle V. M. Blagojević, Jaime Calles Loperena, Michael C. Crabb, and Aleksandra S. Dimitrijević Blagojević, *Topology of the Grünbaum–Hadwiger–Ramos problem for mass assignments*, Topol. Methods Nonlinear Anal., to appear, 22 pages, 2022, arXiv:2208.00666.

[BCC+21] Pavle V. M. Blagojević, Frederick R. Cohen, Michael C. Crabb, Wolfgang Lück, and Günter M. Ziegler, *Equivariant cohomology of configuration spaces* mod 2*—the state of the art*, Lecture Notes in Mathematics, vol. 2282, Springer, Cham, 2021.

[BC23] Pavle V. M. Blagojevic and Michael C. Crabb, *Many partitions of mass assignments*, 2023, Accessed on 18.06.23.

[BLZ15] Pavle V. M. Blagojević, Wolfgang Lück, and Günter M. Ziegler, *Equivariant topology of configuration spaces*, Journal of Topology **8** (2015), no. 2, 414–456.

[BLZ16] ———, *On highly regular embeddings*, Transactions of the American Mathematical Society **368** (2016), no. 4, 2891–2912.

[BS23] Pavle V. M. Blagojevic and Nikola Sadovek, *Convex equipartitions inspired by the little cubes operad*, 2023, Accessed on 18.06.23.

[BZ11] Pavle V. M. Blagojević and Günter M. Ziegler, *The ideal-valued index for a dihedral group action, and mass partition by two hyperplanes*, Topology Appl. **158** (2011), no. 12, 1326–1351.

[BZ14] Pavle V. M. Blagojević and Günter M. Ziegler, *Convex equipartitions via equivariant obstruction theory*, Israel journal of mathematics **200** (2014), no. 1, 49–77.

[Bor53] Armand Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes de groupes de lie compacts*, Annals of Mathematics (1953), 115–207.

[Bor33] Karol Borsuk, *Drei sätze über die n-dimensionale euklidische sphäre*, Fundamenta Mathematicae **20** (1933), no. 1, 177–190.

[BT+82] Raoul Bott, Loring W. Tu, et al., *Differential forms in algebraic topology*, vol. 82, Springer, 1982.

[Bou55] D. G. Bourgin, *On some separation and mapping theorems*, Comment. Math. Helv. **29** (1955), 199–214.

[CLM76] Frederick R. Cohen, Thomas J. Lada, and J. Peter May, *The homology of* c_{n+1} *spaces,* $n \geqslant 0$, The homology of iterated loop spaces, Springer, 1976, pp. 207–351.

[CJ92] Michael C. Crabb and Ioan M. James, *Fibrewise configuration spaces*, Bol. de la Soc. Mat. Mex **37** (1992), 83–97.

[CJ98] ———, *Fibrewise homotopy theory*, Springer Science & Business Media, 1998.

[dL04] Mark de Longueville, *25 years proof of the Kneser conjecture. The advent of topological combinatorics*, EMS Newsletter **53** (2004), 16–19.

[dL12] ———, *A course in topological combinatorics*, Springer Science & Business Media, 2012.

[Dol88] Albrecht Dold, *Parametrized Borsuk-Ulam theorems*, Comment. Math. Helv. **63** (1988), no. 2, 275–285. MR 948782

[DH79] P. F. Duvall and L. S. Husch, *Embedding finite covering spaces into bundles*, Topology Proceedings, vol. 4, 1979, pp. 361–370.

[Ehr37] Charles Ehresmann, *Sur la topologie de certaines variétés algébriques réelles*, J. math. pures appl **16** (1937), no. 9, 69–100.

[Fad86] Edward Fadell, *Cohomological methods in non-free g-spaces with applications to general Borsuk-Ulam theorems and critical point theorems for invariant functionals*, Nonlinear functional Analysis and its applications (1986), 1–45.

[FH87] Edward Fadell and Sufian Husseini, *Index theory for g-bundle pairs with applications to Borsuk-Ulam type theorems for g-sphere bundles*, Nonlinear analysis, World Scientific, 1987, pp. 307–336.

[FH88] ———, *An ideal-valued cohomological index theory with applications to Borsuk–Ulam and Bourgin-Yang theorems*, Ergodic Theory Dynam. Systems **8** (1988), no. Charles Conley Memorial Issue, 73–85.

[FN62a] Edward Fadell and Lee Neuwirth, *Configuration spaces*, Mathematica Scandinavica **10** (1962), 111–118.

[FN62b] Ralph Fox and Lee Neuwirth, *The braid groups*, Mathematica Scandinavica **10** (1962), 119–126.

[Fuc70] Dmitry B. Fuchs, *Cohomology of the braid group* mod 2, Funkcional. Anal. i Priložen. **4** (1970), no. 2, 62–73.

[GKPR13] Darius Geiß, Rolf Klein, Rainer Penninger, and Günter Rote, *Optimally solving a transportation problem using Voronoi diagrams*, Computational Geometry **46** (2013), no. 8, 1009–1016.

[Gre06] Robert Greenblatt, *Homology with local coefficients and characteristic classes*, Homology, Homotopy and Applications **8** (2006), no. 2, 91–103.

[Hat02] Allen Hatcher, *Algebraic topology*, Cambridge University Press, 2002.

[Hat17] ______, *Vector bundles and k-theory*, Unpublished book https://pi.math.cornell.edu/ hatcher/VBKT/VBpage.html, 2017, Accessed on 18.06.2023.

[He17] Chen He, *Torsions of integral homology and cohomology of real grassmannians*, 2017, Accessed on 18.06.23.

[Hil82] Howard Hiller, *Geometry of Coxeter groups*, vol. 54, Pitman Publishing, 1982.

[hw] Matthias Wendt (https://mathoverflow.net/users/50846/matthias wendt), *Integral or rational cohomology of real grassmannians*, MathOverflow, URL:https://mathoverflow.net/q/285068 (version: 2017-11-02), accessed on 18.06.23.

[Jaw81] Jan Jaworowski, *A continuous version of the Borsuk-Ulam theorem*, Proceedings of the American Mathematical Society **82** (1981), no. 1, 112–114.

[Jaw04] ______, *Bundles with periodic maps and mod p Chern polynomial*, Proceedings of the American Mathematical Society **132** (2004), no. 4, 1223–1228.

[KSS84] Jeff Kahn, Michael Saks, and Dean Sturtevant, *A topological approach to evasiveness*, Combinatorica **4** (1984), 297–306.

[KHA14] Roman Karasev, Alfredo Hubard, and Boris Aronov, *Convex equipartitions: the spicy chicken theorem*, Geom. Dedicata **170** (2014), 263–279.

[KT12] Boris Khesin and Serge Tabachnikov, *Memories of Vladimir Arnold*, Notices of the AMS **59** (2012), no. 4.

[Kne56] Martin Kneser, *Aufgabe 360*, Jahresbericht der DMV, 1956.

[Knu18] Ben Knudsen, *Configuration spaces in algebraic topology*, 2018, Accessed on 18.06.23.

[Ler49] Jean Leray, *Determination, dans les cas non exceptionnels, de lanneau de cohomologie de lespace homogene quotient dun groupe de lie compact par un sous-groupe de meme rang*, Comptes rendus hebdomadaires des seances de l academie des sciences **228** (1949), no. 25, 1902–1904.

[MBZ03] Jiří Matoušek, Anders Björner, and Günter M. Ziegler, *Using the Borsuk–Ulam theorem: lectures on topological methods in combinatorics and geometry*, vol. 2003, Springer, 2003.

[Mat14] Benjamin Matschke, *A survey on the square peg problem*, Notices of the American Mathematical Society **61** (2014), no. 4, 346–352.

[Mau81] Daniel R. Mauldin (ed.), *The Scottish Book*, Birkhäuser, Boston, Mass., 1981, Mathematics from the Scottish Café, Including selected papers presented at the Scottish Book Conference held at North Texas State University, Denton, Tex., May 1979.

[McC01] John McCleary, *A user's guide to spectral sequences*, no. 58, Cambridge University Press, 2001.

[MS74] John W. Milnor and James D. Stasheff, *Characteristic classes*, no. 76, Princeton university press, 1974.

[Nan06] R Nandakumar, *"Fair" partitions. Blog entry*, http://nandacumar.blogspot.com/2006/09/cutting-shapes.html, 2006, accessed on 18.06.23.

[NRR12] R. Nandakumar and N. Ramana Rao, *Fair partitions of polygons: An elementary introduction*, Proceedings-Mathematical Sciences **122** (2012), no. 3, 459–467.

[PM64] Murray H. Protter and Charles B. Morrey, *College calculus with analytic geometry*, Addison-Wesley, 1964.

[RPS22] Edgardo Roldán-Pensado and Pablo Soberón, *A survey of mass partitions*, Bulletin of the American Mathematical Society **59** (2022), no. 2, 227–267.

[Sch20] Patrick Schnider, *Ham-Sandwich cuts and center transversals in subspaces*, Discrete Comput. Geom. **64** (2020), no. 4, 1192–1209.

[SW65] Geoffrey C. Shephard and Roger J. Webster, *Metrics for sets of convex bodies*, Mathematika **12** (1965), no. 1, 73–88.

[Tak62] Masaru Takeuchi, *On Pontrjagin classes of compact symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I **9** (1962), no. 1962, 313–328.

[Toe11] Otto Toeplitz, *Über einige aufgaben der analysis situs*, Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn **4** (1911), no. 197, 29–30.

[Vol80] Aleksei Yu. Volovikov, *A generalization of the Borsuk-Ulam theorem*, Mathematics of the USSR-Sbornik **36** (1980), no. 2, 195.

[Yan54] Chung-Tao Yang, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. I*, Ann. of Math. (2) **60** (1954), 262–282.

[Yan55] ———, *On theorems of Borsuk-Ulam, Kakutani-Yamabe-Yujobô and Dyson. II*, Ann. of Math. (2) **62** (1955), 271–283.