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An obstruction to lifting to characteristic 0

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Abstract

We introduce a new obstruction to lifting smooth proper varieties from characteristic $p > 0$ to characteristic 0. It is based on Grothendieck's specialization homomorphism and the resulting discrete finiteness properties of étale fundamental groups.

1. Introduction

1.1 The first example and recent developments

Let A be a complete local noetherian domain with algebraically closed residue field k and field of fractions $A \subset K$. In [Ser61], Serre considers for a smooth proper variety X over k the question of whether X lifts to a smooth proper scheme X_A over $\mathrm{Spec}\,(A)$ for some A as above. To construct the first examples of varieties in characteristic p that do not lift to characteristic 0, he assumes that X admits a finite Galois étale cover $Y \to X$ by a complete intersection $Y \hookrightarrow \mathbb{P}^n$ of dimension at least 3 such that the action of the Galois group G extends to a linear action on projective space. It is then proven in [Ser61, Lemme] that a lift X_A implies a lift of the linear G-action to a representation $\rho_A \colon G \to \mathrm{PGL}_{n+1}(A)$. This relies on Grothendieck's isomorphism

$$\pi_1(X) \xrightarrow{\cong} \pi_1(X_A) \tag{1.1}$$

between étale fundamental groups as defined in [SGA1] and here denoted by π_1. If k has characteristic $p > 0$ and G has a 'large' p-Sylow subgroup, then the deformation ρ_A cannot exist and the variety X does not lift.

Serre's pioneer examples and methods have been largely amplified since then. For example, Achinger and Zdanowicz construct in [AZ17] non-liftable varieties whose motive is of Tate type. Moreover, Van Dobben de Bruyn proved in [vDdB21, Theorem 2] that if X lifts to characteristic 0 and is endowed with a morphism $X \to C$, where C is a smooth projective curve of genus $\geqslant 2$, then the morphism itself lifts to characteristic 0 after an inseparable base change over C. This enabled him to find examples of smooth projective varieties X such that no alteration of X lifts to characteristic 0; see [vDdB21, Theorem 1].

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1.2 The new obstruction

In this note we construct a new obstruction to the existence of a lift to characteristic 0.

Let $\overline{K}$ be an algebraic closure of K, the field of fractions of A as above with residue field k of characteristic $p > 0$, and let $X_{\overline{K}}$ be the corresponding geometric generic fibre of the deformation X_A. Recall that Grothendieck's isomorphism (1.1) is the key point to defining Grothendieck's specialization homomorphism

$$\mathrm{sp}\colon \pi_1(X_{\overline{K}}) \longrightarrow \pi_1(X)\,,$$

which is surjective and an isomorphism on the pro-p'-completion; see [SGA1, Exposé XIII, § 2.10 and Corollaire 2.12]. Here the pro-p'-completion of a profinite group is the canonical continuous quotient obtained by the projective limit of all finite continuous quotients of order not divisible by p. On the other hand, if $\bar{\eta}\colon \mathrm{Spec}\,(\mathbb{C}) \to \mathrm{Spec}\,(K) \to \mathrm{Spec}\,(A)$ is a complex generic point and $X_{\bar{\eta}} = X_A \times_{\mathrm{Spec}\,(A),\bar{\eta}} \mathrm{Spec}\,(\mathbb{C})$, then by the Riemann existence theorem [SGA1, Exposé XII, Théorème 5.1], the étale fundamental group $\pi_1(X_{\bar{\eta}})$ is the profinite completion of the topological fundamental group $\pi_1^{\mathrm{top}}(X_{\bar{\eta}}(\mathbb{C}))$, and the base change homomorphism $\pi_1(X_{\bar{\eta}}) \to \pi_1(X_{\overline{K}})$ is an isomorphism. As $X_{\bar{\eta}}(\mathbb{C})$ is homotopy equivalent to a finite CW-complex (by, for example, Morse theory), the discrete group $\Gamma = \pi_1^{\mathrm{top}}(X_{\bar{\eta}}(\mathbb{C}))$ is finitely presented as a discrete group. Thus those data yield a finitely presented group Γ together with a group homomorphism

$$\Gamma \longrightarrow \pi_1(X)\,,$$

which is surjective on the profinite completion and an isomorphism on the pro-p'-completion. In addition, those properties propagate naturally for any finite étale cover $X_U \to X$ associated with a finite-index open subgroup $U \subseteq \pi_1(X)$.

This suggests the following definition.

Definition A (Definition 2.4). A profinite group π is said to be *p'-discretely finitely generated* (respectively, *p'-discretely finitely presented*) if there exist a finitely generated (respectively, finitely presented) discrete group Γ together with a group homomorphism $\gamma\colon \Gamma \to \pi$ such that

(i) the profinite completion $\hat{\gamma}\colon \hat{\Gamma} \to \pi$ is surjective;

(ii) for any open subgroup $U \subset \pi$ with $\Gamma_U := \gamma^{-1}(U)$, the restriction $\gamma_U\colon \Gamma_U \to U$ induces a continuous group isomorphism on pro-p'-completions

$$\gamma_U^{(p')}\colon \Gamma_U^{(p')} \longrightarrow U^{(p')}\,.$$

We remark that, albeit named *p'-discretely* finitely generated/presented, such a profinite group π is still only topologically (and not discretely) generated by the image of the map $\Gamma \to \pi$, which is part of the structure. The main point here is that the claimed presentation requires finite words in the generators only as opposed to properly speaking profinite words that are allowed in the notion of topologically finitely presented profinite groups (or pro-p' groups).

Thus Grothendieck's theory of specialization for fundamental groups implies the following.

Proposition B (Proposition 2.7). *Let X be a smooth proper scheme defined over an algebraically closed field k of characteristic p. If $\pi_1(X)$ is not p'-finitely presented, for example if $\pi_1(X)$ is not even p'-finitely generated, then X is not liftable to characteristic* 0.

This is the announced obstruction to lifting based on discrete finiteness properties of the étale fundamental group. Proposition B shows a fundamental difference between the *virtual prime-to-p homotopy type* of varieties in characteristic $p > 0$ (that is, the prime-to-p completion of the étale homotopy types of a finite étale cover) and that of varieties in characteristic 0. The full étale homotopy type was already known to behave rather differently in positive characteristic because all connected affine varieties are of type $\mathrm{K}(\pi, 1)$, as was shown in [Ach17].

As the properties of Grothendieck's specialization homomorphism also hold for smooth quasi-projective varieties over A with a good relative simple normal crossings compactification with values in the tame étale fundamental group, we can apply the notion in this case as well; see Example 2.8.

We prove that our definition indeed yields an obstruction to the lifting property.

Theorem C (Main result, see Theorem 5.1 and Corollary 5.2). *Let k be an algebraically closed field of characteristic $p > 0$. Then there are smooth projective varieties X over k such that $\pi_1(X)$ is not even p'-discretely finitely generated. In particular, X does not lift to characteristic 0.*

Let us remark at this point that Theorem 1.1 of [ESS22] asserts that $\pi_1(X)$ is a finitely presented profinite group if X is smooth projective. More precisely, loc. cit. asserts that the same holds, more generally, for $\pi_1^t(X)$ when X is smooth quasi-projective and admits a good relative simple normal crossings compactification. Thus Theorem C shows as well that, in general, there is no finitely presented discrete group, which can explain the main result of [ESS22].

1.3 Outline

We now describe our method to prove Theorem C. Over $k = \bar{\mathbb{F}}_p$, let C be a smooth projective curve of genus $g \geqslant 2$ with $G = \mathrm{Aut}(C)$ its finite group of automorphisms. Let P be a simply connected variety on which G acts freely. We define

$$X = (C \times_k P)/G\,,$$

where G acts diagonally. Then G is a finite quotient of $\pi_1(X)$, and the associated Galois cover $C \times_k P$ has fundamental group equal to $\pi_1(C)$. If $\pi_1(X)$ were p'-discretely finitely generated by some $\Gamma \to \pi_1(X)$, then for any prime number $\ell \neq p$, the action

$$\rho_\ell \colon G \longrightarrow \mathrm{GL}\left(\mathrm{H}^1(C, \mathbb{Q}_\ell)\right)$$

of G on ℓ-adic cohomology $\mathrm{H}^1(C, \mathbb{Q}_\ell)$ would be defined over $\mathbb{Q}$; see Proposition 3.7. *We construct a curve C for which this rationality property fails.*

The representation ρ_ℓ is faithful, see Proposition 4.1, and by Proposition 4.6, the character of ρ_ℓ is $\mathbb{Z}$-valued for $\ell \neq p$. It turns out that the rationality property fails if for all $\ell \neq p$, the representation ρ_ℓ is absolutely irreducible; see Section 4.3. Indeed, the absolute irreducibility implies that C is supersingular, see Proposition 4.4, but then the Schur index of ρ_ℓ turns out to be 2. This prevents ρ_ℓ from being defined over $\mathbb{Q}$. It then remains to construct such a curve. We show that the Roquette curve

$$y^2 = x^p - x$$

discussed in Section 4.2 has the required property. For this we have to make explicit the structure of its group of automorphisms; see the appendix.

2. Profinite groups with p'-approximation

2.1 Finiteness properties

Let p be a prime number. For any group H, the pro-p'-completion of H is defined as

$$H^{(p')} := \varprojlim_{H \twoheadrightarrow Q} Q\,,$$

where $H \twoheadrightarrow Q$ ranges through all finite quotients with order $|Q|$ coprime to p. In case H is already a profinite group, we only consider continuous quotients $H \twoheadrightarrow Q$, that is, with open kernel. If $\alpha\colon H_1 \to H_2$ is a group homomorphism (continuous if the H_i are profinite), we denote the induced continuous homomorphism between the pro-p'-completions by

$$\alpha^{(p')}\colon H_1^{(p')} \longrightarrow H_2^{(p')}\,.$$

Remark 2.1. Let Γ be a discrete group. Recall that any presentation of $\Gamma = \langle S \mid R \rangle$ with set of generators S and set of relations R gives rise to a *presentation complex* $X_{S,R}$ with a single 0-cell $*$, a 1-cell for each $s \in S$ and a 2-cell for each relation in R; see, for example, [Hat02, Corollary 1.28] for a description of the attaching maps. It follows from loc. cit. that, naturally,

$$\pi_1^{\mathrm{top}}(X_{S,R}, *) = \Gamma\,.$$

The proof shows in particular that the fundamental group of a CW-complex with finitely many 1-cells (respectively, finite 2-skeleton) is finitely generated (respectively, of finite presentation).

Recall the following well-known proposition; see, for example, [LS01, Proposition II.4.2] and [MKS04, Corollaries 2.7.1 and 2.8] for the forward direction, or [Sch27, Introduction, p. 162] for the claim on finite generation of subgroups (implicit by the formula for the number of generators of a subgroup of the free group) and [Sch27, § 3], what became known as the Reidemeister–Schreier rewriting process, for the claim on finite presentation of subgroups.

Proposition 2.2 (Reidemeister–Schreier). *Let Γ be a discrete group, and let $\Gamma_\circ \subseteq \Gamma$ be a subgroup of finite index.*

(i) *The group Γ is finitely generated if and only if $\Gamma_\circ$ is finitely generated.*

(ii) *The group Γ is finitely presented if and only if $\Gamma_\circ$ is finitely presented.*

Proof. If Γ is finitely generated (respectively, finitely presented), then there is a presentation complex X for Γ with finitely many 1-cells (respectively, finite 2-skeleton). The finite-index subgroup $\Gamma_\circ$ agrees with the fundamental group of a finite covering space $Y \to X$. The complex Y then also has finitely many 1-cells (respectively, a finite 2-skeleton) as the number of cells multiplies by the degree of the cover. Thus $\Gamma_\circ$ is also finitely generated (respectively, finitely presented).

For the converse direction, we assume that $\Gamma_\circ$ is finitely generated by $u_1, \ldots, u_n \in \Gamma_\circ$. Then Γ is finitely generated by the generators of $\Gamma_\circ$ and representatives x_t for each coset $t \in \Gamma/\Gamma_\circ$. Now let $\Gamma_\circ = \langle u_1, \ldots, u_n \mid r_1, \ldots, r_m \rangle$ be, moreover, finitely presented. We may assume that $\Gamma_\circ$ is normal by first passing to $\bigcap_{t \in \Gamma/\Gamma_\circ} x_t \Gamma_\circ x_t^{-1}$, which is also of finite index and thus finitely presented by the first part of the proof. There are $a_{s,t} \in \Gamma_\circ$ for all $s, t \in \Gamma/\Gamma_\circ$ such that

$$x_s x_t = a_{s,t} x_{st}\,, \tag{2.1}$$

and for all $t \in \Gamma/\Gamma_\circ$ and all $1 \leqslant i \leqslant n$, there are $b_{i,t} \in \Gamma_\circ$ such that

$$x_t u_i x_t^{-1} = b_{i,t}\,. \tag{2.2}$$

We write $a_{s,t}$ and $b_{i,t}$ as words in the u_i. In this sense, Γ is then finitely presented by

$$\Gamma = \langle u_1, \ldots, u_n, x_t;\ t \in \Gamma/\Gamma_\circ \mid r_1, \ldots, r_m,\ (2.1),\ (2.2)\rangle\,.$$

Indeed, if we denote the right-hand side by $\widetilde{\Gamma}$, then there is a surjective group homomorphism $\widetilde{\Gamma} \twoheadrightarrow \Gamma$ because all relations of the presentation of $\widetilde{\Gamma}$ hold in Γ. Let $\widetilde{\Gamma}_\circ$ be the subgroup of $\widetilde{\Gamma}$ generated by the u_i. Then the natural map

$$\Gamma_\circ \twoheadrightarrow \widetilde{\Gamma_\circ} \hookrightarrow \widetilde{\Gamma} \twoheadrightarrow \Gamma$$

is the identity onto $\Gamma_\circ \subseteq \Gamma$. We may thus identify $\Gamma_\circ$ with $\widetilde{\Gamma}_\circ$. Moreover, by (2.1) and (2.2), any element of $\widetilde{\Gamma}$ can be put in a form ux_t with $u \in \Gamma_\circ$ and $t \in \Gamma/\Gamma_\circ$. So the index of $\Gamma_\circ = \widetilde{\Gamma_\circ}$ in $\widetilde{\Gamma}$ is less than or equal to the index $(\Gamma : \Gamma_\circ)$. Therefore, $\widetilde{\Gamma} \to \Gamma$ is an isomorphism. □

The profinite version of Proposition 2.2 holds as well.

Proposition 2.3. *Let π be a profinite group, and let $U \subseteq \pi$ be an open subgroup. Then the following hold:*

(i) *The group π is topologically finitely generated if and only if U is topologically finitely generated.*

(ii) *The group π is topologically finitely presented if and only if U is topologically finitely presented.*

Proof. If U is topologically finitely generated (respectively, finitely presented), then the same holds for π with a proof analogous to that of Proposition 2.2. For the converse direction in part (i), we refer to [Wil98, Proposition 4.3.1]. The converse direction in part (ii) follows from the criterion in [Lub01, Theorem 0.3] thanks to Shapiro's lemma. □

Recall the central definition of this note from the introduction.

Definition 2.4. A profinite group π is said to be *p'-discretely finitely generated* (respectively, *p'-discretely finitely presented*) if there is a finitely generated (respectively, presented) discrete group Γ together with a group homomorphism $\gamma\colon \Gamma \to \pi$ such that

(i) the profinite completion $\hat{\gamma}\colon \hat{\Gamma} \to \pi$ is surjective;

(ii) for any open subgroup $U \subset \pi$ with $\Gamma_U := \gamma^{-1}(U)$, the restriction $\gamma_U\colon \Gamma_U \to U$ induces a continuous group isomorphism on pro-p'-completions

$$\gamma_U^{(p')}\colon \Gamma_U^{(p')} \longrightarrow U^{(p')}\,.$$

Remark 2.5. We refer to [Lub01, § 1] for basic definitions of profinite presentations. A p'-discretely finitely generated (respectively, finitely presented) profinite group π has, in particular, by definition the property that π is topologically finitely generated (respectively, $\pi^{(p')}$ is finitely presented as a pro-p' group; due to [Lub01, Corollary 1.4], the group $\pi^{(p')}$ is also finitely presented as a profinite group).

Remark 2.6. Condition (i) in Definition 2.4 implies that for any U as in condition (ii), the map $\hat{\gamma}_U\colon \hat{\Gamma}_U \to U$ is surjective as well. Indeed, we must show that for all open normal subgroups $V \subseteq U$, the composition $\Gamma_U \to U \to U/V$ is surjective. Cofinally among these V are open subgroups that are even normal in π. Now $\Gamma \twoheadrightarrow \pi/V$ is surjective by assumption, and the preimage of U/V is Γ_U.

2.2 Finiteness properties of fundamental groups

Of primary interest for us are the (tame) fundamental groups of smooth projective varieties (respectively, smooth varieties with a good compactification).

Proposition 2.7. *Let X be a connected smooth proper scheme defined over an algebraically closed field k of characteristic p. If $\pi_1(X)$ is not p'-discretely finitely presented, for example if $\pi_1(X)$ is not even p'-discretely finitely generated, then X is not liftable to characteristic 0.*

Proof. We argue by contradiction. If X lifts to characteristic 0, then there is a smooth proper X_V over a complete discrete valuation ring V of mixed characteristic $(0, p)$ with residue field k, such that $X = X_k$ is the special fibre.

Let $\mathrm{Spec}(K) \to V$ be a geometric generic point and $K_0 \subset K$ be the algebraic closure of a finitely generated algebraically closed field over which the geometric generic fibre X_K is defined as $X_{K_0} \otimes_{K_0} K = X_K$. Let $K_0 \hookrightarrow \mathbb{C}$ be a complex embedding. Let $\Gamma := \pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C}))$ be the topological fundamental group, which is finitely presented. We first compose the profinite completion map for the topological fundamental group

$$\Gamma = \pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C})) \longrightarrow \widehat{\pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C}))}$$

with the comparison isomorphism [SGA1, Exposé XII, Théorème 5.1] of the Riemann existence theorem comparing the completion with the étale fundamental groups $\pi_1(X_{K_0})$ and, using [SGA1, Exposé X, Corollaire 1.8], also $\pi_1(X_K)$

$$\widehat{\pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C}))} \xrightarrow{\sim} \pi_1(X_{K_0}) \xleftarrow{\sim} \pi_1(X_K)\,.$$

We then compose with Grothendieck's specialization homomorphism [SGA1, Exposé X, Corollaire 2.4]

$$\mathrm{sp}\colon \pi_1(X_K) \longrightarrow \pi_1(X_{\bar{k}})$$

to obtain a homomorphism

$$\gamma\colon \Gamma \longrightarrow \pi_1(X_{\bar{k}})\,.$$

The specialization map sp is surjective, and its pro-p' completion $\mathrm{sp}^{(p')}$ is an isomorphism by [SGA1, Exposé X, Théorème 3.8] or rather [SGA1, Exposé X, Corollaire 3.9].[1] It follows that $\hat{\gamma}$ is surjective and $\gamma^{(p')}$ is an isomorphism.

We now show that the pro-p'-isomorphism property also holds for finite-index open subgroups $U \subset \pi_1(X)$. Associated with any such subgroup is a connected finite étale cover $f\colon X_U \to X$ with $\pi_1(X_U) = U$. The surjectivity of the specialization map sp is essentially proven based on [EGAIV_4, Théorème 18.1.2] by providing a formal lift of the cover that algebraizes to a connected étale cover $f_V\colon X_{U,V} \to X_V$. The base-changed cover $f_V \otimes_V K$ is still defined over K_0 and gives rise to a complex connected finite étale cover

$$f_{K_0} \otimes_{K_0} \mathbb{C}\colon X_{U,K_0} \otimes_{K_0} \mathbb{C} \longrightarrow X_{K_0} \otimes_{K_0} \mathbb{C}\,.$$

The restriction of $\gamma\colon \Gamma \to \pi_1(X_{\bar{k}})$ to $\gamma^{-1}(U) = \Gamma_U$ as a map $\gamma_U\colon \Gamma_U \to U$ identifies with the top

[1] Beware that [SGA1, Exposé X, Corollaire 3.9] writes $\pi^{(p)}$ for the pro-p' completion.

arrow in the commutative diagram

$$\begin{array}{ccc} \pi_1^{\mathrm{top}}(X_{U,K_0}(\mathbb{C})) & \longrightarrow & \pi_1(X_{U,\bar{k}}) \\ {\scriptstyle \mathrm{inj}}\downarrow & & \downarrow{\scriptstyle \mathrm{inj}} \\ \pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C})) & \longrightarrow & \pi_1(X_{\bar{k}})\,. \end{array}$$

Therefore, γ_U is the analogue of the map γ but constructed for X_U, so it is an isomorphism for pro-p' completions, again by [SGA1, Exposé X, Théorème 3.8]. This finishes the proof. □

Example 2.8. The criterion of Proposition 2.7 holds more generally for the tame fundamental group of a smooth connected variety with a normal crossing compactification. Let V be a complete discrete valuation ring of mixed characteristic $(0, p)$ with residue field k. Let X_V be a smooth scheme over V with geometrically connected fibres such that there is a compactification $X_V \hookrightarrow \bar{X}_V$ over V, where $\bar{X}_V \setminus X_V$ is a relative normal crossing divisor. We prove in this example that the tame fundamental group $\pi_1^t(X_{\bar{k}})$ is p'-discretely finitely presented.

We use the notation and construction of the proof of Proposition 2.7. Mutatis mutandis, we find a finitely presented group $\Gamma := \pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C}))$ and a homomorphism

$$\gamma\colon \Gamma = \pi_1^{\mathrm{top}}(X_{K_0}(\mathbb{C})) \longrightarrow \pi_1^{\mathrm{top}}\widehat{(X_{K_0}(\mathbb{C}))} \xrightarrow{\sim} \pi_1(X_{K_0}) \xleftarrow{\sim} \pi_1(X_K) = \pi_1^t(X_K) \xrightarrow{\mathrm{sp}^t} \pi_1^t(X_{\bar{k}})\,,$$

where we replace sp with Grothendieck's specialization homomorphism [SGA1, Exposé XIII, § 2.10]

$$\mathrm{sp}^t\colon \pi_1^t(X_K) \longrightarrow \pi_1^t(X_{\bar{k}})$$

of tame fundamental groups. The argument for curves given in [SGA1, Exposé XIII, Corollaire 2.12] extends mutatis mutandis[2] to X_V and shows that sp^t is surjective and the pro-p' completion $\mathrm{sp}^{t,(p')}$ is an isomorphism. It follows that $\hat{\gamma}$ is surjective and $\gamma^{(p')}$ is an isomorphism.

We now show the pro-p'-isomorphism property for the restriction $\gamma_U\colon \Gamma_U = \gamma^{-1}(U) \to U$ for any open subgroup $U \subseteq \pi_1^t(X_{\bar{k}})$. As for all Galois categories, there is an associated connected finite étale cover $X_U \to X_{\bar{k}}$, which extends to a tamely ramified cover $\bar{X}_U \to \bar{X}_{\bar{k}}$, where $\bar{X}_U$ is the normalization of $\bar{X}_{\bar{k}}$ in $K(X_U)$. The surjectivity of sp^t is proven as for sp by the algebraization of a formal deformation to yield a finite étale cover $X_{U,V} \to X_V$ which extends to a tamely ramified cover $\bar{X}_{U,V} \to \bar{X}_V$, where $\bar{X}_{U,V}$ is the normalization of $\bar{X}_V$ in $K(X_{U,V})$. If $\bar{X}_{U,V}$ is a relative normal crossing compactification of $X_{U,V}$, then we can argue as for X_V to finish the proof. If not, we sketch two ways to overcome this issue. The first pedestrian approach works for projective X_V, while the second approach uses logarithmic geometry.

Sketch 1: We assume in addition that X_V is projective. We may then reduce to $\dim(X_{\bar{k}}) = 2$, by the usual generic hyperplane section argument in $\bar{X}_V$ relative to V and transversal to the boundary. See [EK16] for the tame Lefschetz argument saying that the tame fundamental group of the special fibre does not change.

By [KS10, Theorem 4.4], tame covers of $X_{\bar{k}}$ relative $\bar{X}_{\bar{k}}$ in the sense of [SGA1, Exposé XIII] agree with finite covers of $X_{\bar{k}}$ that are *curve tame* [KS10, Definition, § 4, p. 653]. Curve tameness also applies to finite étale covers X_U, as associated above with an open subgroup $U \subseteq \pi_1^t(X_{\bar{k}})$. As remarked in [KS10, § 7], curve-tame covers form a Galois category. The Galois category of

[2]The key input is the more general [SGA1, Exposé XIII, Corollaire 2.8].

curve-tame covers of U defines and determines the tame fundamental group $\pi_1^t(X_U)$, which then equals U as a subgroup of $\pi_1^t(X_{\bar{k}})$.

By Abhyankar's lemma [SGA1, Exposé XIII, § 5], the compactification $X_{U,V} \hookrightarrow \bar{X}_{U,V}$ has a tame cyclic quotient singularity locally in each of the double points of the boundary. That is, the singularity is étale-locally isomorphic to Spec $\big(V[\zeta_n, x, y]^G\big)$ with $G \simeq \mu_n$ acting by scaling the coordinates by powers of nth roots of unity. The analogues for complex surfaces are Hirzebruch–Jung singularities with an explicit resolution by canonically subdividing dual cones; see for example [Ful93, § 2.6], [BHP$^+$04, § III.5], or [Alt98, § 2]. This toric resolution works equally well relative to Spec (V) and globally.

Thus $X_{U,V} \hookrightarrow \bar{X}_{U,V}$ is still toroidal and admits a resolution as a relative normal crossing compactification. The modification does not alter the (curve-)tame fundamental group, which for the resolution with relative normal crossing is again defined in the sense of [SGA1, Exposé XIII]. We conclude by applying the original argument to this modification.

Sketch 2: Alternatively, we may consider $\bar{X}_V$ as a log-regular fs-log-scheme with respect to the log-structure induced by the normal crossing divisor $\bar{X}_V \setminus X_V$. The resulting log-scheme is log-smooth over V endowed with the trivial log-structure. Then purity for the log-fundamental group due to K. Fujiwara and K. Kato, as originally stated in unpublished work (1995) and reiterated without proof in [Kat21, Remark 10.3], shows

$$\pi_1^t(X_{\bar{k}}) = \pi_1^{\log}(\bar{X}_{\bar{k}}) \quad \text{and} \quad \pi_1^t(X_V) = \pi_1^{\log}(\bar{X}_V)\,.$$

(See, however, Hoshi [Hos09, Proposition B.7 and Remark B.2] for a proof of the statement that we need based on an independent proof of the purity theorem due to Mochizuki [Moc99, Theorem 3.3].)

This shows, in particular, that $\bar{X}_{U,V} \to \bar{X}_V$ can be enriched to a finite Kummer étale cover of fs-log-schemes. Hence $\bar{X}_{U,V}$ is also log-smooth over V with $U = \pi_1^{\log}(\bar{X}_U)$. Now the claim follows from the theory of the log-specialization map, a particular case of which (over the standard log-structure on V) was worked out by I. Vidal in her thesis (Université de Paris-Sud, 2001). The essential ingredient is the topological invariance of $\pi_1^{\log}$ of [Vid01, Théorème 0.1] that implies the log-analogue of [EGAIV$_4$, Théorème 18.1.2]. We therefore have that

$$\mathrm{sp}^{\log} \colon \pi_1^{\log}(X_K) \longrightarrow \pi_1^{\log}(X_{\bar{k}})$$

is surjective, and an isomorphism after pro-p'-completion. Moreover, the same applies to the covering described by open subgroups $U \subseteq \pi_1^{\log}(X_{\bar{k}})$. This completes the discussion.

Recall from [ESS22] that, as a profinite group, $\pi_1^t(X)$ is finitely presented. It is natural to ask whether without the liftability assumption, $\pi_1^t(X_{\bar{k}})$ is always p'-discretely finitely presented. We shall prove in Section 5 that it is even not necessarily p'-discretely finitely generated, producing thereby *a new liftability obstruction, notably for smooth proper varieties.*

Remark 2.9. For a given profinite group π that is p'-discretely finitely presented, the discrete group Γ that realizes the discrete finite presentation by $\Gamma \to \pi$ is not uniquely determined by the group π. Serre constructs in [Ser64] an algebraic variety X over a number field k that upon different complex embeddings $\sigma, \tau \colon k \to \mathbb{C}$ yields non-homeomorphic complex manifolds $X^\sigma(\mathbb{C})$, $X^\tau(\mathbb{C})$. Their algebraic origin shows that the étale fundamental groups $\pi_1(X^\sigma_{\mathbb{C}}) \simeq \pi_1(X^\tau_{\mathbb{C}})$ are isomorphic, but their topological fundamental groups are not.

3. Independence of ℓ and rationality

3.1 Rationality of representations

Let G be a finite group. We recall how to decide if a complex linear representation of G is defined over $\mathbb{Q}$; see, for example, [Ser77, Chapter 12]. The ring of complex-valued characters R_G has subrings

$$R_G(\mathbb{Q}) \subseteq \bar{R}_G(\mathbb{Q}) \subseteq R_G\,,$$

where $R_G(\mathbb{Q})$ is the ring of characters defined over $\mathbb{Q}$ and $\bar{R}_G(\mathbb{Q})$ is the ring of $\mathbb{Q}$-valued characters. Wedderburn's theorem decomposes the group ring $\mathbb{Q}[G]$ of G according to the distinct irreducible representations V_i of G in $\mathbb{Q}$-vector spaces as

$$\mathbb{Q}[G] = \prod_{i=1}^{r} \operatorname{End}_{D_i}(V_i) \tag{3.1}$$

with simple factors isomorphic to matrix rings $\mathrm{M}_{d_i}(D_i)$ over skew fields $D_i = \operatorname{End}_G(V_i)$ with centre K_i. Let $\chi_i\colon G \to \mathbb{Q}$ be the character of V_i as a G-representation over $\mathbb{Q}$. These χ_i form a basis of $R_G(\mathbb{Q})$.

Next, using the reduced trace $\operatorname{End}_{D_i}(V_i) \to K_i$ composed with an embedding $\sigma\colon K_i \hookrightarrow \mathbb{C}$ instead, we obtain a complex character $\psi_{i,\sigma}\colon G \to \mathbb{C}$. The $\psi_{i,\sigma}$ for all i and all σ form a basis of R_G, and the $\psi_i = \sum_\sigma \psi_{i,\sigma}$ form a basis of $\bar{R}_G(\mathbb{Q})$ according to [Ser77, Proposition 35]. Now $\dim_{K_i}(D_i) = m_i^2$ is the square of the index of D_i as a skew field over K_i. The Schur index of the representation V_i is this m_i. By [Ser77, Chapter 12], we have $\chi_i = m_i\psi_i$ and so

$$\bar{R}_G(\mathbb{Q})/R_G(\mathbb{Q}) = \bigoplus_{i=1}^{r} \mathbb{Z}/m_i\mathbb{Z}\,.$$

This means that a general complex-valued character $\chi = \sum_{i,\sigma} d_{i,\sigma}\psi_{i,\sigma}$ arises from a representation defined over $\mathbb{Q}$ if and only if the following two conditions are satisfied:

(i) The character must be Galois invariant: the values lie in $\mathbb{Q}$. That is, the coefficients $d_{i,\sigma}$ are independent of σ; say $\chi = \sum_i d_i\psi_i$.

(ii) The coefficients d_i must be divisible by the Schur index m_i.

Remark 3.1. Since G is a finite group, any representation in a $\mathbb{Q}$-vector space stabilizes a $\mathbb{Z}$-lattice (for example, the lattice $\Lambda = \sum_{s\in G} s\Lambda_0$ generated by the G-translates of any lattice Λ_0) and hence is even definable over $\mathbb{Z}$. So integrality is no further constraint for a representation of a finite group G.

3.2 Independence of ℓ

Let π be a profinite group, and let $\varphi\colon \pi \twoheadrightarrow G$ be a finite quotient with kernel $U_\varphi = \ker(\varphi)$. We denote its abelianization by U_φ^{ab}. Then conjugation induces a commutative diagram

$$\begin{array}{ccccc} \pi & \longrightarrow & \operatorname{Aut}(U_\varphi) & \longrightarrow & \operatorname{Aut}\left(U_\varphi^{\mathrm{ab}}\right) \\ \downarrow{\scriptstyle\varphi} & & \downarrow & \nearrow & \\ G & \longrightarrow & \operatorname{Out}(U_\varphi)\,. & & \end{array} \tag{3.2}$$

If π is finitely generated, then U_φ is finitely generated by Proposition 2.3. We deduce that U_φ^{ab} is a finitely generated $\hat{\mathbb{Z}}$-module. The resulting G-representations with values in finite-dimensional $\mathbb{Q}_\ell$-vector spaces are denoted by

$$\rho_{\varphi,\ell}\colon G \longrightarrow \mathrm{GL}\left(U_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}_\ell\right), \tag{3.3}$$

with character

$$\chi_{\varphi,\ell} = \mathrm{tr}(\rho_{\varphi,\ell})\colon G \longrightarrow \mathbb{Q}_\ell\,.$$

Definition 3.2. A profinite group π is said to satisfy *independence of ℓ with the exception of the prime number p* if

(i_p) as a profinite group, π is finitely generated, and

(ii_p) for all continuous finite quotients $\varphi\colon \pi \twoheadrightarrow G$, the following holds: for all prime numbers $\ell \neq p$, the characters $\chi_{\varphi,\ell}$ have values in $\mathbb{Z}$ and are independent of ℓ.

A profinite group π is said to satisfy *independence of ℓ* if

(i) as a profinite group π is finitely generated, and

(ii) for all continuous finite quotients $\varphi\colon \pi \twoheadrightarrow G$, the following holds: for all prime numbers ℓ, the characters $\chi_{\varphi,\ell}$ have values in $\mathbb{Z}$ and are independent of ℓ.

Remark 3.3. For a profinite group as in Definition 3.2, we define a variant (ii'_p) of condition (ii_p).

(ii'_p) For each $\ell \neq p$, fix an embedding $\mathbb{Q}_\ell \subset \mathbb{C}$. Then the $\rho_{\varphi,\ell}$, viewed by scalar extension as representations of G in $\mathbb{C}$ vector spaces, are all isomorphic for all $\ell \neq p$.

Then condition (ii_p) is equivalent to condition (ii'_p). As G is finite and $\mathbb{C}$ is of characteristic 0, the representations are semisimple and thus determined by their characters. Consequently, condition (ii_p) implies condition (ii'_p).

Conversely, if condition (ii'_p) is satisfied, then all characters $\chi_{\varphi,\ell}\colon G \to \mathbb{Q}_\ell$ agree after composition with the chosen embedding $\mathbb{Q}_\ell \subset \mathbb{C}$ with a complex-valued character $\chi\colon G \to \mathbb{C}$. Let $F \subseteq \mathbb{C}$ be the subfield generated by the values of χ. This is an abelian number field since all eigenvalues are roots of unity. Moreover, the field F is contained in $\mathbb{Q}_\ell \subset \mathbb{C}$ for all $\ell \neq p$; that is, F has a split place above ℓ. It follows that $F/\mathbb{Q}$ is completely split over all $\ell \neq p$, and thus $F = \mathbb{Q}$ by Cebotarev's theorem. Therefore, all $\chi_{\varphi,\ell}$ take values in rational algebraic integers, that is, in $\mathbb{Z}$, and these values are independent of $\ell \neq p$.

We formulated condition (ii_p) rather than condition (ii'_p) because it suggests a motivic flavour.

Proposition 3.4. *Let p be a prime number. Let π be a profinite group which is p'-discretely finitely generated via $\Gamma \to \pi$. Then π satisfies independence of ℓ with the exception of p.*

Proof. Let $\varphi\colon \pi \twoheadrightarrow G$ be a finite continuous quotient. The composite map $f\colon \Gamma \to G$ defines similarly with $\Gamma_\varphi = \ker(f)$ and $\Gamma_\varphi^{\mathrm{ab}}$ a representation

$$\rho_\varphi\colon G \longrightarrow \mathrm{GL}\left(\Gamma_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}\right) \tag{3.4}$$

in a finite-dimensional $\mathbb{Q}$-vector space $\Gamma_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}$. The assumption on $\Gamma \to \pi$ yields that $\Gamma_\varphi \to U_\varphi$ is an isomorphism on pro-p' completion. Hence, in particular, for all $\ell \neq p$, the homomorphism $\Gamma_\phi \to U_\phi$ induces a G-equivariant isomorphism

$$\Gamma_\varphi^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}_\ell = U_\varphi^{\mathrm{ab}} \otimes_{\hat{\mathbb{Z}}} \mathbb{Q}_\ell\,.$$

Thus the $\rho_{\varphi,\ell}$, for the various $\ell \neq p$, are compatible and *even definable over* $\mathbb{Q}$. □

Proposition 3.5. *Let k be an algebraically closed field of characteristic* 0 *(respectively,* $p>0$*), and let X/k be a smooth proper variety over k. Then $\pi_1(X)$ satisfies independence of ℓ (respectively, independence of ℓ with the exception of the prime number p).*

Proof. If X lifts to characteristic 0, then the combination of Propositions 2.7 and 3.4 shows the claim.

Now let k be of positive characteristic p, and let X be arbitrary. Let $\varphi\colon \pi_1(X) \twoheadrightarrow G$ be a finite continuous quotient, and let $Y \to X$ be the corresponding G-Galois étale cover. Then the G-representation $U_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}_\ell$ is dual to the natural G-representation on $\mathrm{H}^1(Y, \mathbb{Q}_\ell)$. There is a scheme S of finite type over $\mathbb{F}_p$ such that Y and the graphs $\mathrm{graph}(g) \subset Y \times_k Y$, for all $g \in G$, have smooth proper models Y_S and $\mathrm{graph}(g)_S$ with $\mathrm{graph}(g)_S \subset Y_S \times_S Y_S$. By proper base change for étale cohomology [SGA4-3, Exposé XII, Théorème 5.1], we reduce to the case where $k = \bar{\mathbb{F}}_p$.

Let $\alpha\colon Y \to A$ be the Albanese morphism of Y. Then it follows from the comment after [SGA1, Exposé XI, Corollaire 6.6] – for details, see, for example, [Sti13, Proposition 69] – that the induced map

$$\pi_1^{\mathrm{ab}}(Y) \longrightarrow\!\!\!\!\!\rightarrow \pi_1(A)$$

is surjective with finite kernel. Thus $\alpha^*\colon \mathrm{H}^1(A, \mathbb{Q}_\ell) \to \mathrm{H}^1(Y, \mathbb{Q}_\ell)$ is a G-equivariant isomorphism (note that G does act on A by automorphisms that do not necessarily fix the origin). We may thus replace Y with A and therefore, in particular, assume that Y is a smooth projective variety.[3] As any $g \in G$ acts via correspondences, the characteristic polynomial of each g acting on $\mathrm{H}^1(Y, \mathbb{Q}_\ell)$ lies in $\mathbb{Z}[T]$ and is independent of ℓ; see [KM74, Theorem 2(2)]. □

3.3 The obstruction imposed by the Schur index

Let π be a profinite group that satisfies independence of ℓ with the exception of the prime number p. This means that for a finite quotient $\varphi\colon \pi \twoheadrightarrow G$, the character

$$\chi_\varphi = \chi_{\varphi,\ell} = \mathrm{tr}(\rho_{\varphi,\ell})\colon G \longrightarrow \mathbb{Q}_\ell\,.$$

has values in $\mathbb{Z}$ and is independent of $\ell \neq p$. This character χ_φ belongs to $\bar{R}_G(\mathbb{Q})$, and the *Schur index obstruction* in the proper sense is its class

$$[\chi_\varphi] \in \bar{R}_G(\mathbb{Q})/R_G(\mathbb{Q})\,.$$

This is the obstruction for the representation associated with χ_φ to actually be defined as a linear representation of G in a $\mathbb{Q}$-vector space.

Definition 3.6. We say that a profinite group π satisfying independence of ℓ with the exception of the prime number p is *(Schur) rational* if for all finite continuous quotients $\varphi\colon \pi \twoheadrightarrow G$, the Schur index obstruction class $[\chi_\varphi]$ is trivial; that is, there is an actual G-representation in a $\mathbb{Q}$-vector space V_φ that gives rise, for all $\ell \neq p$, to the ℓ-adic representations

$$U_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}_\ell \simeq V_\varphi \otimes_{\mathbb{Q}} \mathbb{Q}_\ell\,.$$

The following proposition was actually proved within the given proof of Proposition 3.4.

Proposition 3.7. *Let π be a profinite group which is p'-discretely finitely generated. Then π satisfies independence of ℓ with the exception of p and moreover is rational.*

[3] We reduce to the projective case in order to be able to cite Katz–Messing [KM74] directly. The argument of [KM74] also applies to proper smooth varieties and étale cohomology in view of purity of weights.

Not being Schur rational is inherited for fundamental groups in the following geometric context. We may focus on positive characteristic because fundamental groups of smooth proper varieties in characteristic 0 satisfy independence of ℓ and are rational due to Proposition 3.7.

Proposition 3.8. *Let k be an algebraically closed field, and let X and Y be smooth proper varieties over k. If $\pi_1(X)$ is not Schur rational, then $\pi_1(X \times_k Y)$ is not Schur rational either, and in particular $X \times_k Y$ does not lift to characteristic 0.*

Proof. Let $\varphi\colon \pi_1(X) \twoheadrightarrow G$ be a finite quotient such that the corresponding character χ_φ has non-trivial class in $\bar{R}_G(\mathbb{Q})/R_G(\mathbb{Q})$. As X (in fact even X and Y) is proper, we have the Künneth formula, see [SGA1, Exposé X, Corollaire 1.7],

$$\pi_1(X \times_k Y) = \pi_1(X) \times \pi_1(Y)\,.$$

Composition with the first projection $\varphi \circ \mathrm{pr}_1 \colon \pi_1(X \times_k Y) \twoheadrightarrow G$ leads to the character

$$\chi_{\varphi\circ\mathrm{pr}_1} = \chi_\varphi + \dim_{\mathbb{Q}_\ell} \mathrm{H}^1(Y, \mathbb{Q}_\ell) \cdot \mathbf{1}_G\,,$$

where $\mathbf{1}_G$ is the trivial character of G. Because $\mathbf{1}_G$ is defined over $\mathbb{Q}$, it follows that $\chi_{\varphi\circ\mathrm{pr}_1}$ has the same class in $\bar{R}_G(\mathbb{Q})/R_G(\mathbb{Q})$ as χ_φ. This proves the claim. □

4. Curves with many automorphisms

4.1 Action on H^1

In this section, we consider a specific curve C defined over a finite field with a very large group G of automorphisms, and we single out a property of the representation of G on its first ℓ-adic cohomology $H^1(C, \mathbb{Q}_\ell)$ which prevents a variety X constructed in the style of Serre to lift to characteristic 0.

We start with the well-known fact that this action is faithful.

Proposition 4.1. *Let C be a smooth projective curve of genus $g \geqslant 2$ over an algebraically closed field k. Then, for all ℓ different from the characteristic of k, the representation*

$$\rho_\ell \colon \mathrm{Aut}(C) \hookrightarrow \mathrm{GL}\left(\mathrm{H}^1(C, \mathbb{Q}_\ell)\right)$$

is faithful.

Proof. Let $s \in G$ be non-trivial and in the kernel. Then the graph of s in $C \times_k C$ and the diagonal intersect in a scheme of dimension 0, the degree of which we can compute cohomologically by the Grothendieck–Lefschetz formula as

$$|\text{degree of the fixed point scheme of } s \text{ on } C| = \mathrm{tr}\left(s^* | \mathrm{H}^*(C, \mathbb{Q}_\ell)\right) = 2 - 2g < 0\,.$$

This is absurd. □

4.2 The Roquette curve

In [Roq70, § 4], Roquette defines the smooth projective curve $C_{\mathbb{F}_p}$ over $\mathbb{F}_p$ which is the smooth projective compactification of the affine curve defined by

$$C_{\mathbb{F}_p} \colon y^2 = x^p - x\,,$$

which we call the *Roquette curve* in this note. The map $(x, y) \mapsto x$ defines $C_{\mathbb{F}_p}$ as a double cover $C_{\mathbb{F}_p} \to \mathbb{P}^1$. It follows that, for $p = 2$, the curve $C_{\mathbb{F}_p}$ is rational, and thus we shall consider only the case $p > 2$ from now on. For $p \neq 2$, the hyperelliptic cover $C_{\mathbb{F}_p} \to \mathbb{P}^1$ considered above is tame, and the Riemann–Hurwitz formula immediately yields the genus $g = g(C_{\mathbb{F}_p})$ as $2g = p-1$. In particular, the Roquette curve has genus $g \geqslant 2$ if and only of $p \geqslant 5$.

We set $C = C_{\bar{\mathbb{F}}_p} := C_{\mathbb{F}_p} \otimes \bar{\mathbb{F}}_p$. By [Roq70, § 4], the group of automorphisms $\mathrm{Aut}(C)$ over $\bar{\mathbb{F}}_p$ is of cardinality equal to

$$|\,\mathrm{Aut}(C)| = 2 \cdot |\,\mathrm{PGL}_2(\mathbb{F}_p)| = 2p\big(p^2-1\big)\,.$$

For $p \geqslant 5$, the size of $\mathrm{Aut}(C)$ exceeds the Hurwitz bound $84(g-1)$, which bounds from above the order of automorphism groups of curves of genus $g \geqslant 2$ in characteristic 0. Actually, Roquette proved in [Roq70] that among curves of genus g with $p > g+1$, the Roquette curve is the only curve that fails the Hurwitz bound.

We shall use the precise group structure of $\mathrm{Aut}(C)$ as sketched in [Hor12, § 1.2] and also that all automorphisms are defined over $\mathbb{F}_{p^2}$ on $C_{\mathbb{F}_{p^2}} := C_{\mathbb{F}_p} \otimes \mathbb{F}_{p^2}$; see Proposition A.3. As we could not find in the existing literature proofs for the precise structure of this group and, more importantly, the necessary representation theory, we refer to the appendix for this.

Proposition 4.2. *For all $\ell \neq p$, the representation*

$$\rho_\ell \colon \mathrm{Aut}(C) \longrightarrow \mathrm{GL}\left(\mathrm{H}^1(C, \mathbb{Q}_\ell)\right)$$

is absolutely irreducible.

Proof. We denote by N a p-Sylow subgroup of $\mathrm{Aut}(C)$. The dimension of $\mathrm{H}^1(C, \mathbb{Q}_\ell)$ is $2g = (p-1)$, so that by Proposition A.7, it is enough to check that $\rho_\ell|_N$ contains a non-trivial character or, equivalently, that $\rho_\ell|_N$ is not trivial. This follows immediately from Proposition 4.1. □

It turns out that all we need from the Roquette curve is the absolute irreducibility proven in Proposition 4.2.

4.3 Curves with Schur obstruction

Let C be a smooth projective curve over $\bar{\mathbb{F}}_p$ of genus $g \geqslant 2$ such that the following holds:

($\star$) for all $\ell \neq p$, the representation of $G = \mathrm{Aut}(C)$ on $\mathrm{H}^1(C, \mathbb{Q}_\ell)$ is absolutely irreducible.

Let $J = J(C)$ be the Jacobian of C, and let $\mathrm{V}_\ell(J) = \mathrm{T}_\ell(J) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ be the rational Tate module of J.

Lemma 4.3. *Let C be a smooth projective curve with ($\star$). For all $\ell \neq p$, the natural map $\mathbb{Q}_\ell[G] \to \mathrm{End}(V_\ell(J))$ is surjective.*

Proof. Since $\mathrm{H}^1(C, \mathbb{Q}_\ell) = \mathrm{Hom}(V_\ell(J), \mathbb{Q}_\ell)$, the representation $G \to \mathrm{GL}(V_\ell(J))$ is dual to the representation on $\mathrm{H}^1(C, \mathbb{Q}_\ell)$, which we assume to be absolutely irreducible. The claim follows from standard representation theory of finite groups. □

The following result is well known for the Roquette curve ([Eke87, § 2, p. 172] using slopes in crystalline cohomology) and in fact is a property shared by many curves with exceptionally large automorphism group.

Proposition 4.4. *Let C be a smooth projective curve with ($\star$). Then C is supersingular.*

Proof. Let $C_0/\mathbb{F}_q$ be a model of C such that all automorphisms of C are defined as automorphisms of C_0 over $\mathbb{F}_q$. Let J_0 be the Jacobian of C_0, so that $J = J_0 \otimes_{\mathbb{F}_q} \bar{F}_p$. The geometric q-Frobenius of C_0 acts on $V_\ell(J_0) = V_\ell(J)$ commuting with G. The centralizer of the image of G in $\mathrm{End}(V_\ell(J))$ consists only of scalars due to Lemma 4.3.

It follows that the q-Weil numbers associated with J as the Jacobian of the curve C defined over $\mathbb{F}_q$ are contained in a number field that admits an embedding to $\mathbb{Q}_\ell$ for all $\ell \neq p$. This must be $\mathbb{Q}$. The only q-Weil numbers that are rational are $\pm\sqrt{q}$, and q must be a square. Since the Frobenius map acts as a scalar, only one of the possible Weil numbers occurs as eigenvalue of the Frobenius map. By Honda–Tate theory, and because q is a square, there is a supersingular elliptic curve E_0 over $\mathbb{F}_q$ with the same Weil number. We set $E = E_0 \otimes_{\mathbb{F}_q} \bar{\mathbb{F}}_p$. It follows that

$$V_\ell(E^g) \simeq V_\ell(J)$$

as Galois representations. By the Tate conjecture [Tat66, Theorem 1], we find that J and E^g are isogenous, and that proves the claim. □

Remark 4.5. Our main example will be the Roquette curve, for which Proposition 4.4 has the following elementary shortcut. The hyperelliptic double cover $C_{\mathbb{F}_p} \to \mathbb{P}^1$ allows one to count

$$\#C_{\mathbb{F}_p}(\mathbb{F}_p) = p + 1\,.$$

Concerning rational points over $\mathbb{F}_{p^2}$, we note that (1) they all lie over points in $\mathbb{P}^1(\mathbb{F}_{p^2}) \setminus \mathbb{P}^1(\mathbb{F}_p)$ and (2) the action of $G = \mathrm{Aut}(C)$ on $\mathbb{P}^1$ by the group $\mathrm{PGL}_2(\mathbb{F}_p)$ of Möbius transformations (see Lemma A.2) permutes all these possible images transitively. Since the hyperelliptic involution acts transitively on all fibres, we find that $C_{\mathbb{F}_p}(\mathbb{F}_{p^2}) \setminus C_{\mathbb{F}_p}(\mathbb{F}_p)$ either is empty or consists of $2\big(p^2 - p\big)$ points. A precise calculation (which we omit because the precise description of when which case occurs is irrelevant to us) shows that

$$\#C_{\mathbb{F}_p}(\mathbb{F}_{p^2}) = \begin{cases} p+1 & \text{if } p \equiv 1 \pmod 4\,, \\ 2p^2 - p + 1 & \text{if } p \equiv 3 \pmod 4\,. \end{cases}$$

In any case, the Hasse–Weil bound for $C_{\mathbb{F}_p}$ and $\mathbb{F}_{p^2}$-rational points is sharp:

$$\big|\#C_{\mathbb{F}_p}(\mathbb{F}_{p^2}) - \big(1 + p^2\big)\big| = (p-1)\cdot p = 2g\sqrt{p^2}\,.$$

In other words, the Roquette curve is minimal/maximal over $\mathbb{F}_{p^2}$, and this is only possible if the Frobenius eigenvalues are all p or all $-p$. From here, we argue as in the proof of Proposition 4.4.

Proposition 4.6. *Let C be a smooth projective curve with ($\star$). For $\ell \neq p$, the representation*

$$\rho_{J,\ell}\colon G \longrightarrow \mathrm{GL}(\mathrm{V}_\ell(J))$$

has character with values in $\mathbb{Z}$ that is independent of ℓ with the exception of the prime number p but is not defined over $\mathbb{Q}$. The Schur index over $\mathbb{Q}$ is equal to 2.

Proof. Since $V_\ell(J)$ is dual to $\mathrm{H}^1(C, \mathbb{Q}_\ell)$, the character has values in $\mathbb{Z}$ and is independent of ℓ, for $\ell \neq p$, by the same argument as in the proof of Proposition 3.5.

Let E be a supersingular elliptic curve as in the proof of Proposition 4.4 such that J is isogenous to E^g. We denote by $D = \mathrm{End}^0(E)$ the endomorphisms of E over $\bar{\mathbb{F}}_p$ up to isogeny. This is the unique quaternion algebra over $\mathbb{Q}$ ramified in p and ∞ only;[4] see [Deu41, § 8].

Due to the Tate conjecture [Tat66, Theorem 1], under extension of scalars to $\mathbb{Q}_\ell$, the natural representation

$$\mathbb{Q}[G] \longrightarrow \mathrm{End}^0(J) \simeq \mathrm{M}_g\big(\mathrm{End}^0(E)\big) = \mathrm{M}_g(D)$$

[4]Indeed, the action on the 2-dimensional $\mathrm{H}^1(E_{\bar{\mathbb{F}}_p}, \mathbb{Q}_\ell)$ shows that $D \otimes \mathbb{Q}_\ell \simeq \mathrm{M}_2(\mathbb{Q}_\ell)$ for all $\ell \neq p$, and since D is a skew field (E is simple) and not commutative (that is in fact one possible definition of a supersingular elliptic curve, see [Deu41, § 7]), there is no other central simple algebra over $\mathbb{Q}$ of dimension 4 due to the local global principle for central simple algebras; see Brauer–Hasse–Noether [BNH32, Hauptsatz, Reduction 1].

becomes

$$\mathbb{Q}_\ell[G] \longrightarrow \mathrm{End}^0(J) \otimes \mathbb{Q}_\ell = \mathrm{End}_{\mathrm{Gal}}\big(V_\ell(J)\big) \subseteq \mathrm{End}\big(V_\ell(J)\big)\,.$$

Here Gal indicates Galois-invariant endomorphisms. We know from Lemma 4.3 that the composition is surjective. So the inclusion on the right is in fact an equality (which also follows because the Frobenius map was identified with a scalar in the proof of Proposition 4.4). It follows that $\mathbb{Q}[G] \twoheadrightarrow \mathrm{M}_g(D)$ is surjective and identified with the component of the Wedderburn decomposition (3.1) of the group ring corresponding to the irreducible representation underlying the $\rho_{J,\ell}$. Its Schur index is the Schur index of D, which indeed is 2. □

5. A non-p'-discretely finitely generated fundamental group

The example presented in this section rests on Serre's construction [Ser58, § 15] (which he attributes to Weil [Wei38, Chapitre III]). Let C be a smooth projective curve of genus $g \geqslant 2$ over $\bar{\mathbb{F}}_p$ that satisfies condition ($\star$) of Section 4.3, and let $G = \mathrm{Aut}(C)$ be its group of automorphisms. As a concrete example, we can use the Roquette curve as discussed in Section 4.2. Let P be a smooth projective, connected and simply connected variety over $\bar{\mathbb{F}}_p$ such that G acts freely on P; see [Ser58, Proposition 15]. We define

$$X = (C \times_k P)/G\,,$$

where the action of G on $C \times_k P$ is the diagonal action.

Theorem 5.1. *The fundamental group $\pi_1(X)$ is not p'-discretely finitely presented.*

Applying Proposition 2.7, we obtain the following.

Corollary 5.2. *The variety X does not lift to characteristic* 0.

In particular, the condition for $\pi_1(X)$ to be p'-discretely finitely presented is a (new) obstruction for a characteristic p smooth proper geometrically irreducible variety defined over an algebraically closed characteristic $p > 0$ field to be liftable to characteristic 0.

Proof of Theorem 5.1. As G acts freely on P, the finite morphism $C \times_k P \to X$ is Galois étale with Galois group G. Since $\pi_1(C \times_k P) = \pi_1(C)$, due to the Künneth formula, Galois theory induces an exact sequence

$$1 \longrightarrow \pi_1(C) \longrightarrow \pi_1(X) \xrightarrow{\varphi} G \longrightarrow 1\,.$$

Conjugation defines the outer action $\rho\colon G \to \mathrm{Out}\big(\pi_1(C)\big)$ on $U_\varphi := \pi_1(C)$ considered in (3.2). This outer action agrees with the natural action of G acting on C by applying the functor π_1 as follows. For $s \in G$, we can consider the covering transformation $f_s = (s,s)\colon C \times_k P \to C \times_k P$ and the automorphism $s\colon C \to C$. With a lift $\gamma_s \in \varphi^{-1}(s)$, we obtain a diagram of isomorphisms that commutes (and is only well defined) up to inner automorphisms:

$$\begin{array}{ccccc}
\pi_1(C \times_k P) & = & \pi_1(C \times_k P) & = & \pi_1(C) \\
\Big\downarrow{\scriptstyle \gamma_s(-)\gamma_s^{-1}|_{\dots}} & & \Big\downarrow{\scriptstyle \pi_1(f_s)} & & \Big\downarrow{\scriptstyle \pi_1(s)} \\
\pi_1(C \times_k P) & = & \pi_1(C \times_k P) & = & \pi_1(C)\,.
\end{array}$$

The associated ℓ-adic representations $\rho_\ell\colon G \to \mathrm{GL}(U_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}_\ell)$ as considered in (3.3) therefore agree with the natural representations on the rational Tate module of the Jacobian J of C

$$V_\ell(J) = \pi_1(J) \otimes \mathbb{Q}_\ell = \pi_1^{\mathrm{ab}}(C) \otimes \mathbb{Q}_\ell = U_\varphi^{\mathrm{ab}} \otimes \mathbb{Q}_\ell\,.$$

It follows from Proposition 4.6 that ρ_ℓ is independent of ℓ with the exception of the prime number p but is of Schur index 2. So $\pi_1(X)$ fails to be rational, and Proposition 3.7 shows that $\pi_1(X)$ is not p'-discretely finitely generated. □

Appendix. The automorphism group of the Roquette curves

Recall from Section 4 that the Roquette curve $C_{\mathbb{F}_p}$ over $\mathbb{F}_p$ is the smooth hyperelliptic curve obtained as the compactification of the affine curve defined by the equation

$$y^2 = x^p - x\,.$$

The Roquette curve $C_{\mathbb{F}_p}$ has genus $g = (p-1)/2$, so $g \geqslant 2$ if and only if $p \geqslant 5$. We are going to construct a finite group G, define an action of G on $C_{\mathbb{F}_{p^2}} = C_{\mathbb{F}_p} \otimes \mathbb{F}_{p^2}$ and show that G is the full group of automorphisms of $C_{\bar{\mathbb{F}}_p} = C_{\mathbb{F}_p} \otimes \bar{\mathbb{F}}_p$.

A.1 The automorphisms

From now on we assume $p \geqslant 5$. The group of square roots

$$(\mathbb{F}_p^\times)^{1/2} := \left\{\lambda \in \bar{\mathbb{F}}_p^\times;\ \lambda^2 \in \mathbb{F}_p^\times\right\}$$

is a cyclic subgroup of $\mathbb{F}_{p^2}^\times$ of order $2(p-1)$. We define the group $\widetilde{G}$ as the fibre product[5]

$$\widetilde{G} := \left\{(A,\lambda) \in \mathrm{GL}_2(\mathbb{F}_p) \times (\mathbb{F}_p^\times)^{1/2};\ \det(A) = \lambda^2\right\}.$$

The action of $\widetilde{G}$ on $C_{\mathbb{F}_{p^2}}$ arises as follows. Let $g = (A,\lambda) \in \widetilde{G} \subseteq \mathrm{GL}_2(\mathbb{F}_p) \times \mathbb{F}_{p^2}^\times$ with matrix part $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Then we denote by α_g the map $C_{\mathbb{F}_{p^2}} \to C_{\mathbb{F}_{p^2}}$ defined in coordinates by

$$\alpha_g^*(x) := A(x) := \frac{ax+b}{cx+d}\,, \quad \alpha_g^*(y) := \frac{\lambda \cdot y}{(cx+d)^{(p+1)/2}}\,.$$

Here $A(x)$ is the usual Möbius action.

Proposition A.1. *The map $g \mapsto \alpha_g$ defined above yields a group homomorphism*

$$\alpha\colon \widetilde{G} \longrightarrow \mathrm{Aut}_{\mathbb{F}_{p^2}}(C_{\mathbb{F}_{p^2}})\,.$$

Proof. For $g = (A,\lambda) \in \widetilde{G}$, with $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, indeed α_g defines a map $C_{\mathbb{F}_{p^2}} \to C_{\mathbb{F}_{p^2}}$:

$$\begin{aligned}\alpha_g^*(y)^2 &= \frac{\lambda^2 \cdot y^2}{(cx+d)^{p+1}} = \frac{\det(A)\cdot(x^p - x)}{(cx+d)^{p+1}} = \frac{(ax^p+b)(cx+d) - (ax+b)(cx^p+d)}{(cx^p+d)(cx+d)} \\ &= \frac{ax^p+b}{cx^p+d} - \frac{ax+b}{cx+d} = \alpha_g^*(x)^p - \alpha_g^*(x)\,.\end{aligned}$$

For another element $h = (B,\mu) \in \widetilde{G}$ with matrix part $B = \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right)$, we compute

$$\alpha_h^*(\alpha_g^*(x)) = \alpha_h^*(A(x)) = A(\alpha_h^*(x)) = A(B(x)) = (AB)(x) = \alpha_{gh}^*(x)$$

[5] It has been brought to our attention by the referee that the construction of $\widetilde{G}$ is contained in [Hor12, § 1.2].

because GL_2 acts by Möbius transformations on $\mathbb{P}^1$. Moreover,

$$\alpha_h^*\big(\alpha_g^*(y)\big) = \alpha_h^*\left(\frac{\lambda\cdot y}{(cx+d)^{(p+1)/2}}\right) = \frac{\lambda\cdot\frac{\mu\cdot y}{(c'x+d')^{(p+1)/2}}}{(c\alpha_h^*(x)+d)^{(p+1)/2}} = \frac{\lambda\mu\cdot y}{\big((c\alpha_h^*(x)+d)(c'x+d')\big)^{(p+1)/2}}$$
$$= \frac{\lambda\mu\cdot y}{\big((c(a'x+b')+d(c'x+d')\big)^{(p+1)/2}} = \frac{\lambda\mu\cdot y}{\big((ca'+dc')x+(cb'+dd')\big)^{(p+1)/2}} = \alpha_{gh}^*(y)\,.$$

Since $\alpha_{(\mathbb{I},1)}$ is the identity on $C_{\mathbb{F}_{p^2}}$, where $\mathbb{I}$ is the unit matrix, the above shows simultaneously that α_g is an automorphism and that α is a homomorphism. □

Let $\iota\colon C_{\mathbb{F}_p}\to C_{\mathbb{F}_p}$ be the hyperelliptic involution $(x,y)\mapsto(x,-y)$. Since ι acts as -1 on the Jacobian of $C_{\mathbb{F}_p}$, it centralizes all automorphisms of $C_k = C_{\mathbb{F}_p}\otimes k$ for any field k. In particular, any automorphism $f\colon C_k\to C_k$ descends to a map $\bar f\colon \mathbb{P}^1_k\to\mathbb{P}^1_k$. Since the ramification locus of the hyperelliptic covering $x\colon C_k\to\mathbb{P}^1_k$ consists of all $\mathbb{F}_p$-rational points, the induced map $\bar f$ must permute these. Therefore, the Möbius transformation describing $\bar f$ has matrix entries in $\mathbb{F}_p$ due to the following lemma.

Lemma A.2. *Let k be a field of characteristic p. The group of automorphisms of $\mathbb{P}^1_k$ that permutes the subset $\mathbb{P}^1(\mathbb{F}_p)$ consists of the Möbius transformations from $\mathrm{PGL}_2(\mathbb{F}_p)$.*

Proof. The group $\mathrm{PGL}_2(k)$ acts sharply 3-transitively on $\mathbb{P}^1(k)$ for all fields k. □

Let k be a field containing $\mathbb{F}_{p^2}$. Then we deduce from Proposition A.1 and Lemma A.2 a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \langle(\mathbb{I},-1)\rangle & \longrightarrow & \widetilde{G} & \xrightarrow{\mathrm{pr}_1} & \mathrm{GL}_2(\mathbb{F}_p) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
1 & \longrightarrow & \langle\iota\rangle & \longrightarrow & \mathrm{Aut}_k(C_k) & \xrightarrow{f\mapsto\bar f} & \mathrm{PGL}_2(\mathbb{F}_p) & \longrightarrow & 1\,.
\end{array} \tag{A.1}$$

Here pr_1 is the projection $(A,\lambda)\mapsto A$.

Proposition A.3. *Both rows of (A.1) are exact, and the vertical maps are surjective. In particular, all the automorphisms of a Roquette curve are defined over $\mathbb{F}_{p^2}$.*

Proof. The top row is exact because squaring is surjective as a map $(\mathbb{F}_p^\times)^{1/2}\twoheadrightarrow\mathbb{F}_p^\times$. The bottom row is left exact by Galois theory of the hyperelliptic cover $C_k\to\mathbb{P}^1_k$, and we are going to show that the map $f\mapsto\bar f$ is also surjective. The left vertical map is an isomorphism because of $\alpha((\mathbb{I},-1))=\iota$. The right vertical map is the natural projection and thus also surjective. It follows that the bottom row is also exact and that α is surjective. □

Let $\left(\frac{\lambda}{p}\right)=\lambda^{(p-1)/2}\in\{\pm1\}$ denote the Legendre quadratic residue symbol modulo p. Then

$$\lambda^{(p+1)/2} = \left(\frac{\lambda}{p}\right)\lambda\,,$$

and we have an injective group homomorphism

$$\mathbb{F}_p^\times \longrightarrow \widetilde{G}\,,\quad \lambda\longmapsto\left(\lambda\mathbb{I},\left(\frac{\lambda}{p}\right)\lambda\right)$$

because $\det(\lambda\mathbb{I}) = \lambda^2 = \big((\frac{\lambda}{p})\lambda\big)^2$. All $\big(\lambda\mathbb{I}, (\frac{\lambda}{p})\lambda\big)$ are contained in the kernel of α. So a diagram chase with (A.1) shows the following.

Proposition A.4. *Let k be a field containing $\mathbb{F}_{p^2}$. The homomorphism α induces an isomorphism*

$$G := \widetilde{G} \Big/ \left\{ \left(\lambda\mathbb{I}, \left(\frac{\lambda}{p}\right)\lambda\right) ; \lambda \in \mathbb{F}_p^\times \right\} \xrightarrow{\sim} \mathrm{Aut}_k(C_k)\,.$$

It follows that the Roquette curve C has $2p\big(p^2-1\big)$ automorphisms; see [Roq70, § 4]. The main result of loc. cit. shows that among all curves with $p > g+1$, the Roquette curve is the only curve violating the Hurwitz bound $84(g-1)$ for the order of the automorphism group.

A.2 Basic representation theory of G

We denote by N the image in G of the group of upper-triangular unipotent matrices

$$N = \mathrm{im}\left(\left\{ \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} ; u \in \mathbb{F}_p \right\} \hookrightarrow \mathrm{SL}_2(\mathbb{F}_p) \xrightarrow{A \mapsto (A,1)} \widetilde{G} \twoheadrightarrow G \right).$$

The group N is cyclic of order p and thus a p-Sylow of G.

Lemma A.5. *All elements of order p in G are conjugate to one another.*

Proof. The computation in $\mathrm{GL}_2(\mathbb{F}_p)$

$$\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} m^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$$

shows that in $\mathrm{GL}_2(\mathbb{F}_p)$, all elements of order p are conjugate to one another. Indeed, any element of order p is conjugate to an element of the upper-triangular unipotent matrices by Sylow's theorems, and the computation explains the rest.

The same holds in G although $\mathrm{GL}_2(\mathbb{F}_p)$ is not a subgroup of G. Again by Sylow's theorems, we only have to prove the lemma for non-trivial $(s,1),(t,1) \in N$. Then, from the GL_2-result, we know that there is a matrix $A \in \mathrm{GL}_2(\mathbb{F}_p)$ with $AsA^{-1} = t$. Now we choose a square root λ of $\det(A)$. The element $g \in G$ which is the image of $(A,\lambda) \in \widetilde{G}$ does the job:

$$g(s,1)g^{-1} = (A,\lambda)(s,1)\big(A^{-1},\lambda^{-1}\big) = \big(AsA^{-1},1\big) = (t,1)\,. \qquad \square$$

Let K be a field of characteristic 0. We consider a representation $\rho\colon G \to \mathrm{GL}(V)$ with a finite-dimensional K vector space V. For simplicity, we assume that K contains the pth roots of unity. The restriction $V|_N$ to N decomposes into a direct sum of 1-dimensional representations, according to the K-valued characters $\psi\colon N \to K^\times$ on N.

Proposition A.6. *In the situation above, the multiplicity of ψ occurring in (V,ρ) is the same for all non-trivial 1-dimensional representations ψ.*

Proof. Let χ be the character of ρ as a representation of G. By Lemma A.5, the value of χ on $N \setminus \{1\}$ is constant, say $\chi(s) = n_\chi$. The multiplicity of ψ in $V|_N$ is computed as

$$\begin{aligned} \langle \mathsf{res}_N(\chi), \psi\rangle_N &= \frac{1}{|N|} \cdot \sum_{s \in N} \chi(s)\psi\big(s^{-1}\big) = \frac{1}{|N|}(\chi(1) - n_\chi) + n_\chi \frac{1}{|N|} \cdot \sum_{s\in N} \psi\big(s^{-1}\big) \\ &= \frac{1}{|N|}(\chi(1)-n_\chi) + n_\chi \langle \mathbf{1}, \psi\rangle_N = \frac{1}{|N|}(\chi(1)-n_\chi)\,. \end{aligned}$$

Here $\mathbf{1}$ is the trivial representation, and the vanishing of $\langle \mathbf{1}, \psi\rangle_N$ follows from the orthogonality relations since ψ is non-trivial, or even from more elementary facts on characters. $\square$

Proposition A.7. *Let (V, ρ) be a representation of G such that the restriction $V|_N$ is not the trivial representation. Then $\dim_K(V) \geqslant (p-1)$, and if equality occurs, then ρ is an absolutely irreducible representation.*

Proof. The assumption $V|_N$ non-trivial means that there is a non-trivial character ψ of N that occurs on $V|_N$. There are $(p-1)$ non-trivial characters of N, and each occurs in $V|_N$ with the same multiplicity according to Proposition A.6. The dimension estimate follows at once.

We can apply the same reasoning to an irreducible subrepresentation $W \subseteq V$, and we may choose one which contains a non-trivial character ψ of N. The dimension estimate in case $\dim_K(V) = p-1$ shows $V = W$; hence V itself is irreducible. The same argument applies after scalar extension to an algebraic closed field; hence the representation is even absolutely irreducible. □

Remark A.8. Proposition A.7 applies in particular to a faithful G-representation.

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