Model Order Reduction of Linear Control Systems: Comparison of Balance Truncation and Singular Perturbation Approximation with Application to Optimal Control

DISSERTATION
Zur Erlangung des Grades
eines Doktors der Naturwissenschaften (Dr. rer. nat.)
am Fachbereich Mathematik & Informatik
der Freien Universität Berlin

vorgelegt von
Adnan Farhan Sulaiman Daraghmeh

2016

Supervised By

Prof. Dr. Carsten Hartmann (Freie Universität Berlin, BTU Cottbus-Senftenberg)

Gutachter

1. Prof. Dr. Carsten Hartmann (Freie Universität Berlin, BTU Cottbus-Senftenberg)
2. Prof. Dr. Naji Qatanani (An-Najah National University)

Tag der Verteidigung: 18. 07. 2016

Selbständigkeitserklärung

I assure that all resources and aids that are used in this paper was authored independentily on this basis. My paper cannot have been submitted as part of an earlier doctoral procedure.

Berlin, den / /2016 Adnan Farhan Sulaiman Daraghmeh

Summary

In this thesis we have studied balanced model reduction techniques for linear control systems, specifically balanced truncation and singular perturbation approximation. A special feature of these methods, as compared to closely related rational approximation techniques for linear systems, is that they allow for an a priori L^2 and (frequency domain) H^∞ bounds of the approximation error. These methods have been successfully applied for system with homogeneous initial conditions but only little attention has been paid to systems with inhomogeneous initial conditions or feedback systems.

For open-loop control proplems, we have derived an L^2 error bound for balanced truncation and singular perturbation approximation for system with nonhomogeneous initial condition, extending research work by Antoulas etal. The theoretical results have been validated numerically with extensive comparison between different systems and balanced truncation and singular perturbation model reduction.

For closed-loop, one of the most important methods in control problems called linear quadratic regulator (LQR) has been introduced. This is used to find an optimal control that minimizes the quadratic cost function. In order to do that we have used formal asymptotics for the Pontryagin maximum principle (PMP) and the underlying algebraic Riccati equation. The outcome of this section are case description under which balanced truncation and the singular perturbation approximation give good closed-loop performance. The formal calculations are validated by numerical experiments, illustrating that the reduced-order can be used to approximate the optimal control of the original system.

Finally, we studied two different test cases to demonstrate the validity of the theoritical results.

Zusammenfassung

Diese Dissertation behandelt balancierte Modellreduktionsverfahren für lineare Differentialgleichungen, speziell das balancierte Abschneiden (”balanced truncation”) sowie die Approximation im Rahmen der Theorie singulär gestörter Systeme (”singular perturbation approximation”). Balancierte Modellreduktionsverfahren zeichnen sich gegenüber vergleichbaren rationalen Approximationsverfahren dadurch aus, dass sie a priori Fehlerschranken im L^2-Sinne sowie im H^∞ (Frequenzraum) für Systeme mit homogenen Anfangsbedingungen haben Allerdings gibt es bislang kaun Untersuchungen zu Systemen mit inhomogenen Anfangsbedingungen oder Feedback-Steuerung.

Im ersten Teil dieser Arbeit wurden ausgehend von Resultaten von Antoulas *et al.* L^2-Fehlerschranken für lineare gesteuerte Systeme (”open loop control”) mit inhomogenen Anfangswerten hergeleitet und für verschiedene Approximationen (”truncation”, ”singular perturbation approximation”) anhand numerischer Beispiele in Bezug auf den tatsächlichen Approximationsfehler miteinander verglichen.

Im zweiten Teil der Arbeit wurde untersucht, inwieweit balancierte Modellreduktionsverfahren im Zusammenhang mit linearen Regelungsproblemen (”closed loop control”) eingesetzt werden können. Dazu wurden reduzierte Modelle des linear quadratischen Reglers (LQ-Reglers) mit Hilfe formaler asymptotischer Methoden und dem Pontryagin’schen Maximumsprinzip hergeleitet. Als ein zentrales Resultat dieses Teils der Arbeit wurden verschiedene Parameterregime für das balancierte Ausgangsmodell identifiziert, in denen die formale Asymptotik für den LQ-Regler mit den Riccati-Gleichungen für die reduzierten Modelle aus dem ersten Teil der Arbeit übereinstimmt. Die formalen Argumente wurden mit numerischen Experimenten untermauert und zeigen, dass die reduzierten Modelle sehr gute Approximationen der optimalen Steuerung des vollen Systems liefern können.

Sämtliche theoretischen Resultate in der Arbeit wurden durch geeigntete numerische Testbeispiele validiert.

Acknowledgements

First of all, I would like to express my sincere thanks and gratitude to my main supervisor Prof. Dr. Carsten Hartmann for introducing me to the beautiful world of dynamical systems and for his continuous help and support, excellent guidance and encouragement. My sincere and deep gratitude goes also to my co-supervisor Prof. Dr. Naji Qatanani of An-Najah National University for his everlasting support and guidance, his invaluable advice and care. The distinguished personality of my supervisors both at personal and professional levels has contributed to finish my work successfully.

I am very grateful to the Erasmus Mundus Service for granting me the scholarship to pursue my doctoral studies.

I am deeply indebted to my home university, An - Najah National University for their encouragement and support, in particular to Dr. Mohammad Al-Amleh (the vice-president for academic affairs) for his friendly support.

Now, I would like to express my deep obligation to my father, mother, brothers, sisters and friends in Palestine. Their consistent moral supports have always been the strong source for me during my Ph.D.

I would never forget to thank my wife Anwar, son Mohammad and daughters Leen, Jana and Rand Without their love, patience, and understanding, this work would be unthinkable.

Dedicated to my Parents, my wife Anwar, son Mohammad and daughters Leen, Jana and Rand

Contents

Chapter 1

Introduction

Many physical, mechanical and artificial processes can be described by dynamical systems, which can be used for simulation or control. The modeling of many physical, chemical or biological phenomena resulting from discretized partial differential equations lead to the well-known representation of a linear time-invariant (LTI) system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ x(t_0) &= x_0 \end{aligned}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and $D^{p\times m}$ are constant matrices.
The order n of the system ranges from a few tens to several hundreds as in control problems for large flexible space structures. A common feature of the model used is that it is high-dimensional and displays a variety of time scales. If the time scales in the system are well separated, it is possible to eliminate the fast degrees of freedom and to derive low-ordered reduced models, using averaging and homogenization techniques. Homogenization of linear control systems has been widely studied by various authors [1, 4, 14, 26].
Linear systems have been under investigation for quite long time due to their wide range of applications in physics, mathematics and engineering. But the subject is such a fundamental and deep one that there is no doubt that linear systems will continue to be a main focus of study for long time to come.
Finite dimensional linear systems have been extensively studied since the early 1930s. The frequency-domain techniques that were commonly used often did not exploit the underlying finite dimensionality of the system involved. Moreover, almost all this work was for single-input, single-output systems and did not seem to extend satisfactorily to the multi-input, multi-output systems that become increasingly important in aerospace, process control and econometric applications in the late 1950s. This led to a special interest, sparked by the work of Bellman and Kalman in the state-space description of linear systems. This approach has

led to a more detailed examinations of the structure of finite-dimensional linear systems or linear dynamical systems and to questions of redundancy, minimality, controllability, observability, etc. For more details see [5, 29].
For linear systems, model order reduction [3] provides a rational basis for various approximation techniques that include easily computable error bounds [2, 16, 48]. The general idea of balanced model reduction is to restrict the system to the subspace of easily controllable and observable states which can be determined by the Hankel singular values associated with the system. Since many problems of dynamics in physics and engineering are modelled in terms of partial differential equations, the state space formula for such model requires infinite dimensionality. Design control for such state space is also of infinite dimension. For the purpose of computation, this is not practical. Hence it is important to find a low order controller for the infinite dimensional systems. Model reduction is one of the most important methods to obtain low order controller.
Related work:
A number of methods have been presented in the literature to reduce order of infinite dimensional linear time-invariant systems such as balanced truncation [13], Hankel norm approximation [42] and singular perturbation approximation [32].
All these methods give the stable reduced systems and guarantee the upper bound of the error reduction.
Although balanced truncation and singular perturbation approximation methods give the same of the upper bound of error reduction in the case when the dynamical system is homogeneous, but the characteristics of both methods are contrary to each other [27].
It has been shown that the reduced systems by balanced truncation have a smaller error at high frequencies, and tend to be larger at low frequencies. Furthermore, the reduced systems through the singular perturbation approximation method behave otherwise, i.e. the error goes to zero at low frequencies and tend to be large at high frequencies.
The balanced truncation and Hankel norm approximation techniques have been generalized to infinite dimensional systems [8, 38]. Curtain and Glover [8] generalized the balanced truncation techniques to infinite-dimensional systems and the upper bound of the error reduction can be found in [17].
In [9], it has been shown that the reduced systems through balanced truncation method in infinite dimensional systems preserve the behavior of the original system in infinite frequency. More often this condition is not desirable in applications. Therefore, it is necessary to improve the singular perturbation approximation method so that it can be applied to infinite dimensional systems.
Many of the properties of the singular perturbation approximation method can be connected through balanced reciprocal system as shown in [32].
For finite time-horizon optimal problems, among the most actively investigated singularly perturbed optimal control problems is the linear quadratic regulator

problems. Most of these approaches are based on the singularly perturbed differential Riccati equation. An alternative approach via boundary value problems is presented in [33]. Its relationship with the Riccati aproach is analyzed in [34].
Contribution of this thesis and outcome:
For open-loop control proplems, we have derived an L_2 error bound for balanced truncation and singular perturbation approximation for system with non-homogeneous initial condition, extending recent work by Antoulas etal. The theoretical results have been validated numerically with extensive comparison between different systems and balanced truncation and singular perturbation model reduction.
For closed-loop, one of the most important methods in control problems called linear quadratic regulator (LQR) has been introduced. This is used to find an optimal control that minimizes the quadratic cost function. In order to do that we have used formal asymptotics for the Pontryagin maximum principle (PMP) and the underlying algebraic Riccati equation. The outcome of this section are case description under which balanced truncation and the singular perturbation approximation give good closed-loop performance. The formal calculations are validated by numerical experiments, illustrating that the reduced-order can be used to approximate the optimal control of the original system.
Finally, we studied two different test cases to demonstrate the validity of the theoritical results.
This thesis is organized as follows:
Chapter (2) introduces the notions of state equations for the dynamical system, stability, controllability and observability matrices and gramians, Lyapunov equations, and Kalman canonical decomposition.
The question of reducing the homogeneous model of linear time-invarient continuous dynamical system on infinite-time horizon is addressed in Chapter (3). This involves the energy of controllability and observability, the balancing of linear systems using balanced truncation and the sinngular perturbation approximation.
Chapter (4) gives a detailed treatment of the non-homogeneous linear dynamical continuous system and the L_2 norm of the error bound between the outputs of the original and the reduced order model.
In Chapter (5) we present the LQR method for the closed-loop dynamical system. Feedback optimal control is used to minimize the quadratic cost function. In addition, an optimal control for the reduced model is obtained using the singular perturbation regulator and balanced truncation.
In Chapter (6) numerical experiments illustrate the performance of these techniques.

Chapter 2

Preliminaries

In this chapter we discuss some of the theoretical concepts of control systems. We present the state-space and the output equation for the dynamical system and their solutions [6]. We introduce the Laplace transform and its properties in this chapter. We discuss the characterization of a system in terms of its transfer function and the transition matrix. We introduce the basic concepts of controllability, observability, and stabilizability, and we clarify issues related to these concepts from the algebraic and engineering persepctive [48].

2.1 State equations for the dynamical system

To describe a linear dynamical system, we introduce the state space equations which is a set of first-order linear differential equations defined by:

$$\dot{x} = Ax + Bu \tag{2.1}$$

where

$$\dot{x} = \frac{dx}{dt}$$

denotes the derivative of x with respect to time t.
We call

$$x(t) = [x_1(t), x_2(t), x_3(t), ..., x_n(t)]^T \in \Re^n$$

the state vector of the dynamical system.
We denote by

$$u = u(t) \in \Re^m$$

the input function of the dynamical system.
The initial condition of this dynamical system is denoted by:

$$x(t_0) = x_0$$

A and B are constant matrices defining linear mappings and they determin the system structure [2].
In our study we consider the continuous linear – time invariant system. A time-invariant system is a system such that A, B, C and D are independent of time.
The representations of A, B, C and D are constant matrices with dimensions $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$ and $D \in \Re^{p\times m}$.

2.2 The output equation

A linear state equation gives a releationship between the input variable and the state variable. Now we are going to introduce the output equation of the system which is the system variable of interest. The output equation for the linear continuous dynamical system is given as:

$$y = Cx + Du \tag{2.2}$$

where y is a column vector of the output variables, and represents the response of the system. C and D are constant matrices such that $C : \Re^p \times \Re^n$ and $D : \Re^p \times \Re^n$ describe the dynamical system. C is called the output matrix and it describes the interaction between the system and the outside world.
Finally, D is a matrix of constant coefficient that describes the weight of the system input [2].
The following are linear differential equations with constant coefficients describing the finite dimensional linear time invariant *(FDLTI)* dynamical system. The equations are

$$\dot{x} = Ax + Bu \tag{2.3}$$

and

$$y = Cx + Du \tag{2.4}$$

where the system state is $x(t) \in \Re^n$, and the initial condition of the system is $x(t_0)$. The input of the system is $u(t) \in \Re^m$ and the output of the system is $y(t) \in \Re^p$ [48].
We can write the dynamical system described by (2.3) and (2.4) in general form by using the symbol $\mathbf{\Sigma}$.

Definition 2.2.1. [2] A linear system in state space description is a quadruple of linear maps represented as matrices.

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \tag{2.5}$$

The dimension of the system in (2.5) is the same as the dimension of the associated state spaces, that is :

$$Dim(\mathbf{\Sigma}) = n \tag{2.6}$$

In case where $D = 0$ or D is irrelevant we denote the system by:

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array} \right) \tag{2.7}$$

In this work we take the case $D = 0$.
If $D = 0$, then we write the linear system in the form

$$\dot{x} = Ax + Bu \tag{2.8}$$

$$y = Cx \tag{2.9}$$

where $x \in \Re^n$, $u \in \Re^m$, $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $C \in \Re^{p\times n}$ and the initial condition is taken as $x(t_0) = x_0$
One can write this system (2.8) and (2.9) in compact matrix form as:

$$\begin{pmatrix} \dot{x} \\ y \end{pmatrix} = \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \tag{2.10}$$

where

$$\begin{pmatrix} A & B \\ C & 0 \end{pmatrix}$$

is a block matrix.

Definition 2.2.2. Let

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right)$$

be a linear, continuous dynamical system, then $\mathbf{\Sigma}$ is called a *SISO* system if it has single input $(m = 1)$ and single output $(p = 1)$. Otherwise it is called *MIMO* system which has a multiple input and multiple output [48].

2.3 Stability of a continuous–time system

In this section we discuss the stability of a Continuous – Time system and introduce the following definitions and theorems related to the stability of the system.

Definition 2.3.1. [10] A matrix A is called a *stable matrix* if all the eigenvalues of A have strictly negative real parts.

Remark 2.3.2. A stable matrix is commonly known as a *Hurwitz matrix* in control literature [10, 48].

Definition 2.3.3. The system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right)$$

is called a symptotically stable if $\Re\{\lambda_i(A)\} < 0$ and is called stable if $\Re\{\lambda_i(A)\} \leq 0$ where $\Re\{\lambda\}$ denotes the real parts of λ, and λ are the eigenvalues of the matrix A.

2.4 The Laplace transform

Definition 2.4.1. [6, 11] Let $f(t)$ be a real-valued function defined for $t \geq 0$, then the Laplace transform of $f(t)$ denoted by $F(s)$ is given by:

$$\begin{aligned} L[f(t)] &= \int_0^\infty f(t)e^{-st}dt \\ &= F(s) \end{aligned} \tag{2.11}$$

where $s = \sigma + i\omega$, σ and ω are real variables [6, 11].

The inverse laplace transform of a function $F(s)$ is the unique function $f(t)$ that is continuous on $[0, \infty)$ and satisfies:

$$L^{-1}[F(s)] = f(t) \tag{2.12}$$

The following *theorems* and *properties* are used for computing the *Laplace Transform* [6][11].

Theorem 2.4.2. *(Linearity)*: If a is a constant or is independent of s and t then

$$\begin{aligned} L[af(t)] &= aL[f(t)] \\ &= aF(s) \end{aligned} \tag{2.13}$$

Theorem 2.4.3. *(Super-position)*: If $L[f_1(t)] = F_1(s)$, and $L[f_2(t)] = F_2(s)$, then:

$$\begin{aligned} L[f_1(t) + f_2(t)] &= L[f_1(t)] + L[f_2(t)] \\ &= F_1(s) + F_2(s) \end{aligned} \tag{2.14}$$

Theorem 2.4.4. *(Translation in time)*: If $L[f(t)] = F(s)$ and a is a positive real number, then the Laplace transform of the translated function $f(t-a)$ is given as:

$$L[f(t-a)] = e^{-as}F(s) \tag{2.15}$$

Theorem 2.4.5. *(Complex differentiation)*: If $L[f(t)] = F(s)$, then:

$$L[f(t)] = -\frac{d}{ds}F(s) \tag{2.16}$$

Theorem 2.4.6. *(Translation in the s domain)* If $L[f(t)] = F(s)$ and a is either real or complex, then

$$L\left[e^{-at}f(t)\right] = F(s-a) \tag{2.17}$$

Theorem 2.4.7. *(Real differentiation)*: If $L[f(t)] = F(s)$ and let $f'(t)$ be the first derivative of $f(t)$, then

$$L\left[f'(t)\right] = sF(s) - f(0) \tag{2.18}$$

Note that Theorem (2.4.7) can be generalized to the n^{th} derivative and we can write a general formula to find $L\left[f^{(n)}(t)\right]$

$$L\left[f^{(n)}(t)\right] = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots\cdots - sf^{(n-2)}(0) - f^{(n-1)}(0) \tag{2.19}$$

Theorem 2.4.8. *(Real integration)*: If $L[f(t)] = F(s)$, then its integral given by the following formula :

$$L[\int_0^t f(\tau)d\tau] = \frac{F(s)}{s} \tag{2.20}$$

Theorem 2.4.9. *(Final value)*: If $L[f(t)] = F(s)$ and $\lim_{t\to 0} f(t)$ exists then :

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) \tag{2.21}$$

Theorem 2.4.10. *(Initial value)* If $L[f(t)] = F(s)$ and $\lim_{s\to\infty} sF(s)$ exists then:

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) \tag{2.22}$$

Now, we come to our final theorem and tool in using the Laplace transform. It concerns the transform of a convolution of functions. The theorem provides a link between the notation of convolutions. It is a very important tool and is used throughout the theory

Theorem 2.4.11. [47] *(Convolution)*: We have

$$\begin{aligned} L[f(t) \star h(t)] &= L[f(t)]\, L[h(t)] \\ &= F(s)H(s) \end{aligned} \tag{2.23}$$

where the convolution operator is defined as:

$$
\begin{aligned}
(f \star h)(t) &= \int\limits_0^t f(\tau)h(t-\tau)d\tau \\
&= \int\limits_0^t f(t-\tau)h(\tau)d\tau
\end{aligned}
\tag{2.24}
$$

2.5 The derivative and integral of matrix and matrix exponential

In this section we introduce the derivative and integral of a matrix and discuss its properties and we define the exponential matrix and its representation and give the rules for its computation [6].

Definition 2.5.1. Let $A(t) = [a_{ij}(t)]$ be an $n \times n$ matrix where the entries of $A(t)$ are a function of time t, then :

1. The derivative of $A(t)$ denoted by $\frac{d}{dt}A(t)$ is:

$$
\begin{aligned}
\frac{d}{dt}A(t) &= \dot{A}(t) \\
&= \left(\frac{d}{dt}(a_{ij}(t))\right)
\end{aligned}
$$

2. The integral of $A(t)$ is :

$$
\begin{aligned}
\int A(t)dt &= \int \dot{A}(t)dt \\
&= \left(\int a_{ij}(t)dt\right)
\end{aligned}
$$

The differentiation or integration of any matrix can be computed by differentiating or integrating each element of the matrix. We have the following properties that are dependes on this definition and we can assume them as rules.
Let α and β be two constants and A and B two matrices, then :

- $\frac{d}{dt}(\alpha A) = \alpha \frac{d}{dt}A = \alpha \dot{A}$
- $\frac{d}{dt}(\alpha A + \beta B) = \alpha \frac{d}{dt}A + \beta \frac{d}{dt}B = \alpha \dot{A} + \beta \dot{B}$
- $\int\limits_a^b \alpha A dt = \alpha \int_a^b A dt$, where a and b are real numbers.

- $\int_a^b (\alpha A + \beta B)dt = \alpha \int_a^b Adt + \beta \int_a^b Bdt$
- $\frac{d}{dt}(AB) = A\frac{d}{dt}B + B\frac{d}{dt}A = A\dot{B} + \dot{A}B$
- $A^0 = I$
- $\frac{d}{dt}A^n \neq nA^{n-1}\frac{dA}{dt}$

Definition 2.5.2. Given a square matrix $A \in \Re^{n\times n}$ and $t \in \Re$. Then the matrix exponential of A is denoted by e^{At} and is a square matrix of the same order as A defined by :

$$e^{At} = I + At + \frac{A^2t^2}{2} + \frac{A^3t^3}{6} + \dots \tag{2.25}$$

For any square matrix, we can find the matrix exponential by using *Cayley-Hamilton Theorem.*
If A and B are two matrices and α and β are two constants, then the following rules hold for the matrix exponential.

- $e^{A0} = I$
- $e^{-A\alpha} = [e^{A\alpha}]^{-1}$
- $e^{A(\alpha+\beta)} = e^{A\alpha}e^{A\beta}$
- $e^{(A+B)\alpha} = e^{A\alpha}e^{B\beta}$, only if $AB = BA$
- $\frac{d}{dt}(e^{tA}) = Ae^{tA} = e^{At}A$
- $\int_0^\alpha e^{A\alpha}d\alpha = A^{-1}[e^{A\alpha} - I] = [e^{A\alpha} - I]A^{-1}$

2.6 The state transition matrix

Definition 2.6.1. The state transition matrix of a dynamical system is a matrix function denoted by $\phi(t,t_1)$ and a acts as a transformation from one state to another [11, 47, 48]. We define it by:

$$\phi(t,t_1) = e^{A(t-t_1)} \tag{2.26}$$

where A is a matrix

Properties of the state transition matrix Here we give a list of properties of the state transition matrix. The proofs of these facts are standard and can be found in many sources including [11].

- $\phi(t_2 - t_1)\phi(t_1 - t_0) = \phi(t_2 - t_0)$, for any t_0, t_1, t_2
- $\phi(t)\phi(t)\phi(t)\cdots\cdots\phi(t) = \phi^q(t) = \phi(qt)$, where q is a positive integer
- $\phi^{-1}(t) = \phi(-t)$
- $\phi(0) = I$, a unity matrix
- $\phi(t)$ is non-singlar for all finite values of t.

2.7 The transfer–function matrix of the dynamical system

The concept of a transfer function has an important use in the linear dynamical system, and it depends on the input condition [48][11][47].
Let

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

be a linear dynamical system, if we use the properties of the Laplace transform for the state and output equations (2.8) and (2.9), we have

$$\begin{aligned} L[\dot{x}] &= L[Ax] + L[Bu] \\ sX(s) - x(0) &= AX(s) + BU(s) \\ X(s) &= (sI - A)^{-1}BU(s) + (sI - A)^{-1}x(0) \end{aligned} \tag{2.27}$$

and

$$\begin{aligned} L[y] &= L[Cx] \\ Y(s) &= CX(s) \end{aligned} \tag{2.28}$$

The matrix $(sI - A)^{-1}$ is called the transition matrix or function matrix. Inserting equation (2.27) into equation (2.28) we obtain

$$Y(s) = C(sI - A)^{-1}BU(s) + C(sI - A)^{-1}x(0) \tag{2.29}$$

In case the initial condition is zero, i.e.,$x(0) = 0$, equation (2.29) becomes:

$$Y(s) = C(sI - A)^{-1}BU(s) \tag{2.30}$$

Definition 2.7.1. [2, 48] The transfer matrix or a function matrix $G(s)$ from u to y with zero initial condition is defined by:

$$Y(s) = G(s)U(s) \tag{2.31}$$

Therefore, we can define $G(s)$ as:

$$G(s) = \frac{Y(s)}{U(s)} \tag{2.32}$$

Moreover, if A is a stable matrix then the transfer function takes the form:

$$G(s) = C(sI - A)^{-1}B \tag{2.33}$$

Alternatively, the transfer matrix $G(s)$ in equation (2.32) can be written as:

$$\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right) = \frac{Y(s)}{U(s)} \tag{2.34}$$

2.8 Solution of the state and output space equations

To obtain a solution for the state space equation of the linear dynamical continuous system in equation (2.8), we consider the following steps:
multiply both sides of equation (2.8) by e^{At} giving:

$$\begin{aligned} e^{-At}\dot{x} &= e^{-At}Ax + e^{At}Bu \\ e^{-At}\dot{x} - e^{-At}Ax &= e^{-At}Bu \\ \frac{d}{dt}[e^{-At}x(t)] &= e^{-At}Bu \\ e^{-At}x(t) - e^{-At_0}x(t_0) &= \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau \\ e^{-At}x(t) - e^{-At_0}x_0 &= \int_{t_0}^{t} e^{-At}Bu(\tau)d\tau \\ x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau, \quad \forall t > t_0 \end{aligned} \tag{2.35}$$

This equation describes the change of state with respect to the input vector $u(t)$ and the initial condition $x(t_0)$.

From the solution of the state equation $x(t)$ and since $y = Cx(t)$, the solution of the output equation of the system is:

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.36}$$

In case where the initial time $t_0 = 0$, the solution of the dynamical system becomes:

$$x(t) = e^{At}x_0 + \int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.37}$$

$$y(t) = Ce^{At}x_0 + C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \tag{2.38}$$

Now, consider the linear dynamical system describe in equations (2.3)(2.4), it follows that the solution of the output equation with $D \neq 0$ is given as:

$$y(t) = Ce^{At}x_0 + C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{2.39}$$

We call equation (2.39) the convolution equation and the general form of the solution of the system can be represented by this equation. The input and the output of the dynamical system is jointly linear in booth the initial condition and the state [2].
The system time responses is determined by the state $x(t)$, the output $y(t)$, the control input $u(t)$ and the initial condition x_0 for $t \geq 0$.
For zero inout control and from equation (2.39), we obtain the response of the system as:

$$y(t) = Ce^{At}x_0 \tag{2.40}$$

For zero initial condition, the forced response of the dynamical system determin by the following equation:

$$y(t) = C\int_{0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau + Du(t) \tag{2.41}$$

Finally, we have the following case known as the impulse response and in this case we set $x_0 = 0$ and define the input control as:

$$u(t) = \delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases}$$

where $\delta(t)$ is the unit impulse or the Dirac delta function satisfy the diracc distribution:

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)dt = f(\tau)$$

and f is a continuous function at $t = \tau$.
Now, the impulse response is given as:

$$y(t) = \int_{0}^{t} \left(Ce^{A(t-\tau)}B + D\delta(t-\tau)\right) u(\tau)d\tau \tag{2.42}$$

The impulse response matrix of the dynamical system is defined as:

$$g(t) = Ce^{At}B + D\delta(t)$$

The relationship between the input and the output with zero initial condition can be described by the convolution equation [10, 48]

$$\begin{aligned} y(t) = (g \star u)(t) &= \int_{-\infty}^{\infty} g(t-\tau)u(\tau)d\tau \\ &= \int_{-\infty}^{t} g(t-\tau)u(\tau)d\tau \end{aligned} \tag{2.43}$$

2.9 Lyapunov equations

In this section we introduce a set of important equations in control theory called the Lyapunov equations. They defined as follows:

Definition 2.9.1. [2, 10] Let $M, M^T \in \Re^{n \times n}$, The matrix equation

$$AX + XA^T = -M \tag{2.44}$$

is called the *Lyapunov Equations*.

Theorem 2.9.2. [2, 10](Lyapunov Stability Theorem)[see:F2,P205] The system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is a sympotically stable if and only if for any symmetric positive definite matrix M, there exists a unique symmetric positive definite matrix X such that

$$AX + XA^T = -M \tag{2.45}$$

A full proof can be found in many sources and we refer the interested reader to [2, 10] for the detail.

We can write the solution of these equations in terms of an integral.
Consider the system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

where A is assumed to be stable and M is symmetric, positive definite, or semi definite, then :

1. The equation

   $$AX + XA^T = -M$$

   has a unique solution X such that:

   $$X = \int_0^\infty e^{At} M e^{A^T t} dt \tag{2.46}$$

   More detail can be found in [2] that explain the steps of finding the solution X.

2.10 Controllability and observability

In this section we introduce and discuss the concepts of *Controllability* and *Observability*, which are both fundamental in the study of continuous linear dynamical system.

Remark 2.10.1. The concepts of *Controllability* and *Reachability* are equivalent for continuous time systems [2][48].

We start by the following definition of the Controllability

Definition 2.10.2. [47, 48] The dynamical system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

or the pair (A, B) is said to be controllable if, for any initial state $x(0) = x_0$, $t_1 > 0$ and final state x_1 there exists a (piecewise continuous) input $u(\cdot)$ such that the solution of equation (2.1) satisfies $x(t_1) = x_1$.
Otherwise the system or the pair (A, B) is said to be uncontrollable .

Definition 2.10.3. [2, 48] Given the dynamical system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

then the controllability matrix of the system is defined by

$$C(A, B) = \begin{pmatrix} B & AB & A^2B \cdots\cdots A^{n-1}B \end{pmatrix} \tag{2.47}$$

where n is a positive integer.

Definition 2.10.4. [2, 48] The dynamical system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

or the pair (C, A) is said to be observable if , for any $t_1 > 0$, the initial state can be determined from the time history of the input $u(t)$ and the output $y(t)$ in the interval $[0, t_1]$.
Otherwise the system , or (C, A), is said to be unobservable.

Definition 2.10.5. [2, 48] The observability matrix of the dynamical system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is defined as:

$$O(C, A) = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ \vdots \\ CA^{n-1} \end{pmatrix} \tag{2.48}$$

where n is a positive integer.

We introduce two important matrices for the linear dynamical system the controllability and the observability Gramians. They are used in the Balance realization and model reduction method [2][2, 47]

Let

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

be a continuous –time linear system and assume that $\mathbf{\Sigma}$ is stable. We define the controllability and obsorvabelly gramians denoted by W_c and W_o respectively as follows

Definition 2.10.6. [2, 10] Let A be a stable matrix , then the matrix

$$W_c = \int_0^\infty e^{At} BB^T e^{A^T t} dt \tag{2.49}$$

is called the controllability gramian

Definition 2.10.7. [2, 10] Let A be a stable matrix , then the matrix

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt \tag{2.50}$$

is called the Observability Gramian

The two matrices W_c and W_o are the solution of the Lyapunov eguation, so we have:

$$AW_c + W_c A^T + BB^T = 0 \tag{2.51}$$

$$W_o A + A^T W_o + C^T C = 0 \tag{2.52}$$

Proposition 2.10.8. [10] Let

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

be a stable, continuous-time system and let W_c, W_o be the controllability and observability gramians of $\mathbf{\Sigma}$, then W_c and W_o satisfy the continuous time Lyapunov equations:

$$AW_c + W_c A^T + BB^T = 0 \tag{2.53}$$

$$W_o A + A^T W_o + C^T C = 0 \tag{2.54}$$

Proof. Since $\mathbf{\Sigma}$ is stable, then

$$
\begin{aligned}
AW_c + W_cA^T &= \int_0^\infty \left(Ae^{A\tau}BB^Te^{A^T\tau} + e^{A\tau}BB^Te^{A^T\tau}A^T \right) d\tau \\
&= \int_0^\infty \frac{d}{d\tau}\left(e^{A\tau}BB^Te^{A^T\tau} \right) d\tau \\
&= 0 - BB^T \\
&= -BB^T \\
AW_c + W_cA^T + BB^T &= 0
\end{aligned}
$$

The second equation can be proved in the same way as the first one. □

Definition 2.10.9. [2] A Hermition matrix $X = X^*$ is called positive semi-definite (or positive definite) if its eigenvalues are positive.

The controllability gramians have the following property that is holds for for continuous-time dynamical system.

$$W_c(t) = W_c^T(t) \geq 0, \;\; \forall t > 0 \tag{2.55}$$

Now, we introduce the following theorems to explain the relation between the controllability, obsorvability of the system and the solution of the Lyapunov equation .

Theorem 2.10.10. [2, 48] The system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array} \right)$$

or the pair $(A.B)$ is controllable if and only if W_c is positive definite for any $t > 0$ (W_c is non-singular).

Proof. ($\Leftarrow$) assume $W_c(t) > 0$ for some $t > 0$ and define the input

$$u(\tau) = -B^Te^{A^T(t_1-\tau)}W_c^{-1}(t_1)(e^{At_1}x - x_1) \tag{2.56}$$

the value $x(t_1)$ from the solution of $x(t)$ is

$$x(t_1) = e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau$$

If we substitute the value of $u(\tau)$ in the above equation, we have

$$\begin{aligned} x(t_1) &= e^{At_1}x_0 + \left(\int_0^{t_1} e^{A(t_1-\tau)}BB^T e^{A^T(t_1-\tau)}d\tau\right)W_c^{-1}(t_1)(e^{At_1}x_0 - x_1) \\ &= e^{At_1}x_0 - W_c(t_1)W_c^{-1}(t_1)(e^{At_1}x_0 - x_1) \\ &= e^{At_1}x_0 - e^{At_1}x_0 + x_1 \\ &= x_1 \end{aligned}$$

Since x_1 is arbitrary, (A,B) is controllable.
($\Rightarrow$) We use the proof by contradiction
assume that (A,B) is controllable. W_c is singular for some $t > 0$
since

$$e^{At}BB^T e^{A^T t} \geq 0, \ \ \forall t$$

and this mean there exists a vector

$$v \neq 0, v \in \Re^n$$

such that

$$v^T e^{At}B = 0$$

in the range $0 \leq t \leq t_1$
Now, assume that $x(t_1) = x_1 = 0$, this mean

$$\begin{aligned} x(t_1) &= e^{At_1}x_0 + \int_0^{t_1} e^{A(t_1-\tau)}Bu(\tau)d\tau \\ &= 0 \end{aligned}$$

and if we multiply the above equation by v^T, we obtain

$$v^T e^{At_1}x_0 = 0$$

Finally, multiplying by e^{At_1}, we get

$$x_0 = e^{At_1}v$$

If we choose the initial state to be $x_0 = e^{At_1}v$, then the value of $v = 0$ and this is a contradiction since $v \neq 0$ by assumption.
So the matrix W_c is nonsingular for any $t > 0$. □

Theorem 2.10.11. [2, 48] The system pair (A,B) is controllable if and only if the controllability matrix

$$C(A,B) = \begin{pmatrix} B & AB & A^2B\cdots\cdots A^{n-1}B \end{pmatrix}$$

has full row rank (i.e., $rank(C(A,B)) = n$).

Proof. If the system is controllable, then we have $W_c(t) > 0$, for all $t > 0$ and is a non-singular matrix.
Since $C(A,B)$ has no full row rank, this means there exists a nonzero vector, $k \in \Re^n$ such that

$$k^T C(A,B) = 0$$

which implise that

$$k^T A^s B = 0, \;\; s \leq 0$$

and for the case $0 \leq s \leq n-1$, this is obvious.
For the case $s \geq n$ By the Cayley– Hamilton theorem, it follow that

$$A^s = p(A)$$

where $p(A)$ is a polynomial in A of degree $n-1$.
From that, we get

$$k^T A^s B = k^T p(A) B = 0, \;\; \forall s \geq n$$

and this means that

$$k^T A^s B = 0, \;\; t \geq 0$$

Now, we have

$$\int_0^t k^T e^{A\tau} B B^T e^{A^T\tau} k d\tau = 0, \;\; t > 0$$

which is equivalent to

$$k^T W_c(t) k = 0$$

This is a contradiction, since $W_c(t) > 0$
Thus $C(A,B)$ has a full row rank.
For the converse, assume $C(A,B)$ has a full row rank and $W_c(t)$ is not positive for some $t_1 > 0$.
Then, there exists a vector $k \neq 0, k \in \Re^n$ such that

$$\int_0^{t_1} k^T e^{A\tau} B B^T e^{A^T\tau} k d\tau = \int_0^{t_1} \|B^T e^{A^T\tau} k\|_2^2 d\tau$$
$$= 0$$

We obtain

$$B^T e^{A^T t} k = 0, \;\; t > 0$$

If we transposing this equation, we get

$$k^T e^{At} B = 0$$

If we differentiating the last equation $(n-1)$ times with respect to t and evaluate it at $t = 0$, giving

$$k^T B = k^T AB = \ldots\ldots = k^T A^{n-1} B = 0$$

and this written as

$$k^T \begin{pmatrix} B & AB & A^2 B \cdots\cdots A^{n-1} B \end{pmatrix} = 0$$

and this mean $C(A,B)$ has no full row rank. □

Corollary 2.10.12. [2, 48] The system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array} \right)$$

is controllable if and only if the controllability matrix $C(A,B)$ has full row rank.

Theorem 2.10.13. (Controllability Conditions) The following statements are equivalent .

1. The pair (A,B), $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$ is controllable .
2. The rank of the controllability matrix is full i.e., $rank(C(A,B)) = n$
3. The controllability gramian is positive definite $W_c(t) > 0$, for some $t > 0$

For more details see [2, 48]

Theorem 2.10.14. The pair (C,A) is observable if and only if the matrix

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

is positive definite for any $t > 0$

For more details [48]

Theorem 2.10.15. [2] *Observabitility Conditions* The following statements are equivalent .

1. The pair (C,A), $A \in \Re^{n\times n}$, $C \in \Re^{p\times n}$ is observable .

2. The observability gramian is positive definite W_o for some $t > 0$

3. The rank of the observability matrix is full i.e., $rank(O(C,A)) = n$.

2.11 Kalman canonical decomposition

In this section, we look in to how we can change the coordinate system of a dynamical system to suit our needs. This is a powerful trick, in particular when the dynamical system is not completely controllable and or not completely observable [48]. According to the physical dynamical system we use different coordinate systems to make the analysis and synthesis of the system much easier. The key properties of a dynamical system are unchanged by this change of coordinates so we can still analyse it effectively [48].
In Kalman canonical decomposition, we choose a non-singular transformation to balance the system and determine the states which are not completely controllable and or not completely observable. These states have less effect on the dynamical system, so we can delete them.
In the next chapter, we introduce a model order reduction techniques to determine the states that have no effect on the dynamical system. This can be done by determining the so called Hankel singular values (HSVs) from the balancing Gramians of the system. The states corresponding to the smaller Hankel singular values have less effect to the dynamical system hence we can truncate them.
To convert the original system defined in equations (2.8) and (2.9) we define a non-singular transformation matrix $T \in \Re^{n\times n}$ and define z as:

$$z = Tx \tag{2.57}$$

If we multiply equation (2.57) by T^{-1}, we get:

$$x = T^{-1}z \tag{2.58}$$

and if we use the rule of differentiation for the matrix defined by equation (2.57) then we find the value of $\dot{z}$ as follwos

$$\begin{aligned}\dot{z} &= \frac{dz}{dt} \\ &= \frac{d}{dt}(Tx) \\ &= T\dot{x}\end{aligned}$$

Now, the value of $\dot{z}$ is defined by:

$$\dot{z} = T\dot{x} \tag{2.59}$$

and thus equation (2.59) can be written in terms of $\dot{z}$ in the form:

$$\dot{x} = T^{-1}\dot{z} \tag{2.60}$$

Now, for a given dynamical system describe by equations (2.8) and (2.9) if we substitute equation (2.58) and (2.60) in the system, we get

$$T^{-1}\dot{z} = AT^{-1}z + Bu \tag{2.61}$$

and

$$y = CT^{-1}z \tag{2.62}$$

Equations (2.61) and (2.62) can then be written in a new form as:

$$\dot{z} = TAT^{-1}z + TBu \tag{2.63}$$

and

$$y = CT^{-1}z \tag{2.64}$$

If we define

$$\bar{A} = TAT^{-1}, \;\; \bar{B} = TB, \;\; \bar{C} = CT^{-1}$$

and define the initial condition to be

$$z(t_0) = T^{-1}z_0$$

the system described by the equations (2.63) and (2.64), becomes:

$$\dot{z} = \bar{A}z + \bar{B}u \tag{2.65}$$

$$y = \bar{C}z \tag{2.66}$$

Remark 2.11.1. The two systems described in equations (2.8) (2.9) and (2.65) (2.66) are the same for any invartable matrix T [48].

To define the transfer fuction for the new system (2.65) and (2.66), we let

$$\bar{G}_z(s)$$

be the transfer function of the system, that is :

$$\bar{G}_z(s) = \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} \tag{2.67}$$

Since

$$G(s) = C\left(sI - A\right)^{-1}B$$

is the transfer function of the original system , we can show that the two transfer functions are equal by the following steps:

$$\begin{aligned}
\bar{G}_z(s) &= \bar{C}\left[sI - \bar{A}\right]^{-1}\bar{B} \\
&= CT^{-1}\left[sI - TAT^{-1}\right]^{-1}TB \\
&= CT^{-1}T\left[sI - A\right]^{-1}T^{-1}TB \\
&= C\left[sI - A\right]^{-1}B \\
&= G(s)
\end{aligned}$$

For the original systems

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right) = C\left[sI - A\right]^{-1}B$$

after the transformation, the system becomes:

$$\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right) = \left(\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & \end{array}\right) \tag{2.68}$$

The controllability and observability matrices for the system described in equation (2.68) is given by:

$$\bar{C}(\bar{A}, \bar{B}) = TC(A, B) \tag{2.69}$$

$$\bar{O}(\bar{C}, \bar{A}) = O(A, B)T^{-1} \tag{2.70}$$

where $C(A, B)$ and $O(C, A)$, is the controllability and observability matrices of the original system. We introduce now the following theorem which is related to the controllability and observability matrices after using the invertable transformation.

Theorem 2.11.2. [2, 48] The controllability and observability matrices are invariant under similarity transformation.

Using the fact in Theorm (2.11.2), we can introduce the following theorem.

Theorem 2.11.3. [2, 48] If the controllability matrix $C(A,B)$ has rank $m_1 < n$, then there exists a similarity transformation T and

$$\bar{x} = \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} = Tx$$

such that:

$$\begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{pmatrix} = \begin{pmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{B}_c \\ 0 \end{pmatrix} u \tag{2.71}$$

and

$$y = \begin{pmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{pmatrix} \tag{2.72}$$

where $\bar{A}_c \in C^{m_1 \times m_1}$ and $(\bar{A}_c, \bar{B}_c)$ are controllable.

For any one who interested in the proof see [48].
According to Theorem (2.11.3), the transfer function of the system described by equations (2.71) (2.72) is given by:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \bar{C}_c(sI - \bar{A}_c)^{-1}\bar{B}_c \end{aligned} \tag{2.73}$$

Theorem 2.11.4. [48] If the observability matrix $O(A,C)$ has rank $m_2 < n$, then there exists a similarity transformation T and

$$\bar{x} = \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} = Tx$$

such that:

$$\begin{pmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{pmatrix} = \begin{pmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{pmatrix} u \tag{2.74}$$

or equivalently

$$\left(\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & \end{array}\right) = \begin{pmatrix} \bar{A}_o & 0 & \bar{B}_o \\ \bar{A}_{21} & \bar{A}_{\bar{o}} & \bar{B}_{\bar{o}} \\ \bar{C}_o & 0 & \end{pmatrix} \tag{2.75}$$

where $\bar{A}_0 \in C^{m_2 \times m_2}$ and $(\bar{C}_o, \bar{A}_o)$ are observabile.

For the proof see [48].
The transfer function of the system defined by equation (2.74) or (2.75), can be written as:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \bar{C}_o(sI - \bar{A}_o)^{-1}\bar{B}_o \end{aligned} \tag{2.76}$$

The two theorems (2.11.3) and (2.11.5) can be combined in one theorem to give the *Kalman Canonical Decomposition*.

Theorem 2.11.5. [48] Let a linear time invariant dynamical system be described by equations (2.8) (2.9), then there exists a nonsingular coordinate transformation

$$\bar{x} = Tx$$

such that

$$\begin{pmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{pmatrix} = \begin{pmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix} u \tag{2.77}$$

$$y = \begin{pmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix} \tag{2.78}$$

or equivalently

$$\left(\begin{array}{c|c} TAT^{-1} & TB \\ \hline CT^{-1} & \end{array}\right) = \begin{pmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \\ \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{pmatrix} \begin{pmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix} \tag{2.79}$$

We mean by $\bar{x}_{co}$ that the state is controllable and observable, by $\bar{x}_{c\bar{o}}$ that the state is controllable but unobservable, by $\bar{x}_{\bar{c}o}$ that the stat is uncontrollable and observable, by $\bar{x}_{\bar{c}\bar{o}}$ that the stat is uncontrollable and unobservable [48].
The transfer function of the system described by equation (2.79) is given by:

$$\begin{aligned} G(s) &= C(sI - A)^{-1}B \\ &= \bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} \end{aligned} \tag{2.80}$$

We see from equation (2.80) that the transfer function of the dynamical system unchanged and is equal to the transfer function of the controllable and observable parts, that is:

$$\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right) = \left(\begin{array}{c|c} \bar{A}_{co} & \bar{B}_{co} \\ \hline \bar{C}_{co} & \end{array}\right) \tag{2.81}$$

For a simple system, consider the state space response given by equation (2.77). The solutions of the state space equation

$$\begin{pmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{pmatrix} = \begin{pmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{pmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix} + \begin{pmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{pmatrix} u$$

are written as:

$$\bar{x}_{co}(t) = e^{\bar{A}_{co}t}\bar{x}_{co}(0) + \int_0^t e^{\bar{A}_{co}(t-\tau)}\bar{B}_{co}u(\tau)d\tau \tag{2.82}$$

$$\bar{x}_{c\bar{o}}(t) = e^{\bar{A}_{c\bar{o}}t}\bar{x}_{c\bar{o}}(0) + \int_0^t e^{\bar{A}_{c\bar{o}}(t-\tau)}\bar{B}_{c\bar{o}}u(\tau)d\tau \tag{2.83}$$

$$\bar{x}_{\bar{c}o}(t) = e^{\bar{A}_{\bar{c}o}t}\bar{x}_{\bar{c}o}(0) \tag{2.84}$$

$$\bar{x}_{\bar{c}\bar{o}}(t) = e^{\bar{A}_{\bar{c}\bar{o}}t}\bar{x}_{\bar{c}\bar{o}}(0) \tag{2.85}$$

The solution of the output equation

$$y = \begin{pmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{pmatrix} \begin{pmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{pmatrix}$$

is given as:

$$y(t) = \bar{C}_{co}\bar{x}_{co} + \bar{C}_{\bar{c}o}\bar{x}_{\bar{c}o} \tag{2.86}$$

We see from the equation (2.82) to (2.86) that the input u has no effect in the states $\bar{x}_{\bar{c}o}$ and $\bar{x}_{\bar{c}\bar{o}}$. The states $\bar{x}_{\bar{c}\bar{o}}$ and $\bar{x}_{c\bar{o}}$ don't appear in the output equation (2.86). We explain in the following Remark the internal behaviors of the state response and the input and the output of the dynamical system according to the initial condition.

Remark 2.11.6. The internal behaviors of the two transfer functions are very different. The input and the output they are the same for zero initial condition, but they have very different behaviors with nonzero initial conditions [48].

To explain the idea of this remark, we compute the solution of the output equation (2.86) with non-zero initial condition.
If we set the initial condition equal to zero (.i.e.,$x(0) = 0$), then the output equation

(2.86) can be written in different form as:

$$y(t) = \int_0^t \bar{C}_{co} e^{\bar{A}_{co}(t-\tau)} \bar{B}_{co} u(\tau) d\tau \tag{2.87}$$

If we refer to equation (2.81), then we have the following output equation:

$$y(t) = \int_0^t C e^{A(t-\tau)} B u(\tau) d\tau \tag{2.88}$$

We see the output equation (2.86) is the same as the output equation (2.88) with zero initial condition, but they have different outputs with non-zero initial condition.

Chapter 3

Model Order Reduction of Linear Time-Invariant Continuous Homogeneous Dynamical System on Infinite-Time Horizon

3.1 State space realization for transfer function

In this section, we introduce the realization for a general dynamical system with a transfer function $G(s)$.
Let $G(s)$ be a proper (real rational) transfer function, then the state space model (A,B,C) given by

$$G(s) = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is a realization of $G(s)$.

Definition 3.1.1. A state space realization (A,B,C) of $G(s)$ is said to be a minimal realization of $G(s)$ if A has a smallest possible dimension .

Now, we have the following characterization of the minimal realization [48]

Theorem 3.1.2. A state space realization (A,B,C) of $G(s)$ is minimal if and only if (A,B) is controllable and (C,A) is observable.

Also, we have the following property of the minimal realization [48]

Theorem 3.1.3. Let $(A_1,B1,C_1)$ and $(A_2,B2,C_2)$ be two minimal realization of a real rational transfer function $G(s)$. Moreover, suppose that C_1,C_2,O_1 and O_2 are the corresponding controllability and observability matrices, respectively. Then there exists a unique non-singular matrix T such that

$$A_2 = TA_1T^{-1}, \quad B_2 = TB_1, \quad C_2 = C_1T^{-1}$$

Furthermore, T is given by

$$T = (O_2^T O_2)^{-1} O_2 O_1 \quad or T^{-1} = C_1 C_2^T (C_2 C_2^T)^{-1}$$

The balanced realization method is a numerically reliable method to eliminate the states that are uncontrollable and/or unobservable.

3.2 The amount of energy for controlling or an observing state

In this section we discuss one of the most important properties of a dynamical system which is used to classify the state of the system a according to the degree of controllability or the observability [2].
Consider the linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned}$$

and assume it is stable, controllable and observable.
We define the controllability and observability function at x_0 [7, 40] as follows:

Definition 3.2.1. The controllability function is defined as

$$L_c(x_0) = \min_{\substack{u \in L_2(-\infty,0) \\ x(-\infty)=0, x(0)=x_0}} \frac{1}{2} \int_{-\infty}^{0} \|u(t)\|^2 dt \tag{3.1}$$

Definition 3.2.2. The observability function is defined as

$$L_o(x_0) = \frac{1}{2} \int_0^{\infty} \|y(t)\|^2 dt, \quad x(0) = x_0, \quad u(t) = 0, \quad 0 \le t < \infty \tag{3.2}$$

The value of $L_c(x_0)$ is the minimum amount of control energy required to reach the state x_0.
The value of $L_o(x_0)$ is the amount of output energy generated by the state x_0.
To determine the degree of controllability and observability of a linear dynamical system, we introduce the following theorem :

Theorem 3.2.3. [40] Let

$$W_c = \int_0^{\infty} e^{At} B B^T e^{A^T t} dt$$

and

$$W_o = \int_0^\infty e^{A^T t} C^T C e^{At} dt$$

be the controllability and obesrvablity gramian respectively, where W_c, W_o are the unique positive-definite solutions of the Lyapunov equations

$$AW_c + W_c A^T = -BB^T$$

$$A^T W_o + W_o A = -C^T C$$

then $L_c(x_0)$ and $L_o(x_0)$ that described in equations (3.1), (3.2) can be written in terms of W_c, W_o to get:

$$L_c(x_0) = \frac{1}{2} x_0^T W_c^{-1} x_0$$
$$L_o(x_0) = \frac{1}{2} x_0^T W_o x_0$$

From theorem (3.2.3), the smallest amount of energy that is needed to steer the system from zero to the given state x_0 is given by $L_c(x_0)$ and the amount of energy with initial condition x_0 that is obtained from the output of the system is denoted by $L_o(x_0)$[2].

3.3 Balancing for linear system

In this section we introduce one of the most importants methods used to obtain a reduce order model from the original dynamical system. This is called the Balanced Truncation method [18, 36].
Consider the linear-time invariant continuous system written as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{3.3}$$

The concept of the *Balanced Truncation* method depends on the controllability and observablity gramians matrices W_c and W_o [39] which are the symmetric positive semi definite solution of the Lyapunov equations [see proposition (2.10.8)].

$$AW_c + W_c A^T + BB^T = 0$$

$$A^T W_o + W_o A + C^T C = 0$$

To obtain a reduced order model, we first balance the system then delete the states that are difficult to control(need large a mount of control energy) and difficult to observe (yield small amount of energy), these states are not important so they could not effect on the transfer function [18, 30, 39].
First, we introduce, in the following definition, the so called *The Hankel Singular Values (HSVs)* of the dynamical system.

Definition 3.3.1. [2, 18, 48] Let $\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ be controllable, observable and stable continuous-time system of dimension n, *The Hankel Singular Values (HSVs)*

$$\sigma_1 \geq \sigma_2 \geq \cdots\cdots \geq \sigma_n \geq 0$$

of $\mathbf{\Sigma}$ are the square roots of the eigenvalues of the product of $W_c W_o$ and given by:

$$\sigma_i(\mathbf{\Sigma}) = \sqrt{\lambda_i(W_c W_o)}$$

The diagonal matrix of *The Hankel Singular Values (HSVs)* is denoted by:

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix} \tag{3.4}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots\cdots \geq \sigma_n > 0$

Definition 3.3.2. [2, 35] The controllable, observable and stable system

$$\mathbf{\Sigma} = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is balanced if

$$W_c = W_o = \Sigma = diag(\sigma_1 \geq \sigma_2 \geq \cdots\cdots \geq \sigma_n)$$

The following theorem describes the method of balancing used to find a coordinate transformation S such that:

$$\bar{x} = S^{-1}x \tag{3.5}$$

in which the controllability, observability gramians become diagonal and equal [7, 18, 40].

Theorem 3.3.3. There exists a state space transformation $\bar{x} = S^{-1}x$ for the system

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

such that the transformed system

$$\begin{aligned}\dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x}\end{aligned} \tag{3.6}$$

is in balanced form and $\bar{A} = SAS^{-1}$, $\bar{B} = SB$ and $\bar{C} = CS^{-1}$

If we let $\bar{G}$ be the transfer function of the transformed system (3.6), then:

$$\bar{G} = \left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right) = \left(\begin{array}{c|c} SAS^{-1} & SB \\ \hline CS^{-1} & \end{array}\right) \tag{3.7}$$

Letting $\bar{W}_c$, $\bar{W}_o$ be the controllability and observability gramians of the balance system (3.6) we have that:

$$\bar{W}_c = S^{-1}W_cS^{-T} \tag{3.8}$$

and

$$\bar{W}_o = S^TW_oS \tag{3.9}$$

and since the two gramians are equal, then:

$$\bar{W}_c = \bar{W}_o = \Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix} \tag{3.10}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots\cdots \geq \sigma_n$.
The controllability and obsevability gramians in equation (3.10) satisfies the two Lyapunov equations:

$$\begin{aligned}\bar{A}\Sigma + \Sigma\bar{A}^T + \bar{B}\bar{B}^T &= 0 \\ \bar{A}^T\Sigma + \Sigma\bar{A} + \bar{C}^T\bar{C} &= 0\end{aligned}$$

For more detail, see equations (3.8) (3.9).
Since the two gramians W_c and W_o are posistive definite (or semi-definite), then we can decompose them according to:

$$\begin{aligned}W_c &= UU^T \\ W_o &= LL^T\end{aligned} \tag{3.11}$$

If we do a singular value decomposition of the matrix L^TU, we get:

$$L^TU = X\Sigma Y^T = \begin{pmatrix} X_1 & X_2 \end{pmatrix} \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix} \begin{pmatrix} Y_1^T \\ Y_2^T \end{pmatrix} \tag{3.12}$$

such that

$$\Sigma_1 = \sigma_1 \geq \sigma_2 \geq \cdots\cdots \geq \sigma_r$$

and

$$\Sigma_2 = \sigma_{r+1} \geq \sigma_2 \geq \cdots\cdots \geq \sigma_n$$

The other matrices satisfy

$$X_1^T X_1 = Y_1^T Y_1 = I_{r\times r}$$

and

$$X_2^T X_2 = Y_2^T Y_2 = I_{l\times l}$$

with $l = n - r$ [18].
We have the following Lemma that indicates the balancing transformation S and its inverse in terms of the singular value decomposition

Lemma 3.3.4. [2](Balancing transformation) Given the controllable, observable and stable system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ and the corresponding gramians W_c and W_o, a (principal axis) balancing transformation is given as:

$$\begin{aligned} S &= UY\Sigma^{\frac{-1}{2}} \\ S^{-1} &= \Sigma^{\frac{-1}{2}} X^T L^T \end{aligned} \tag{3.13}$$

Definition 3.3.5. The controllability and observability functions of the transformed system (3.6) are defined as :

$$\bar{L}_c(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma^{-1} \bar{x}_0 \tag{3.14}$$

and

$$\bar{L}_o(\bar{x}_0) = \frac{1}{2}\bar{x}_0^T \Sigma \bar{x}_0 \tag{3.15}$$

Now, if $\sigma_i \gg \sigma_{i+1}$ for $i = 1, 2, \cdots\cdots, n$, then the amount of control energy to reach the state $\bar{x}$ is large for small values of σ_i, and the output energy at $\bar{x}$ is small for large values of σ_i.
Hence, to reduce the number of states components of the system, we delete the state components x_{j+1} to x_n for which $\sigma_i \gg \sigma_{i+1}$ [39, 40, 48].
For the case when

$$G = \left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$$

is a minimal realization , we can use the following procedure to obtain balance realization [48].

1. Compute W_c and W_o for the system.

2. Find a matrix U such that $W_c = U^T R$.

3. Diagonalize $U^T W_o U$ to get

$$\begin{aligned} W_c &= R^T W_o R \\ &= L\Sigma^2 L^T \end{aligned}$$

4. Let

$$S^{-1} = U^T L \Sigma^{-\frac{1}{2}}$$

then

$$\begin{aligned} S^{-1} &= U^T L \Sigma^{-\frac{1}{2}} S W_c S^T \\ &= S^{-T} W_o S^{-1} \\ &= \Sigma \end{aligned}$$

and

$$\left(\begin{array}{c|c} SAS^{-1} & SB \\ \hline CS^{-1} & \end{array}\right)$$

is balanced.

3.4 Error bounds using balance truncation

Consider the linear time-invariant continuous system represented by the following state-space equation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{3.16}$$

where $x(t) \in \Re^n$ is the state vector, $u(t) \in \Re^m$ is the input control and $y(t) \in \Re^p$ is the output of the system.
Let

$$G(s) = C(sI - A)^{-1}B$$

be the transfer function of this system.
The system (3.16) is assumed to be asymptotically stable and $G(s)$ a minimal realization.

Assumptions 3.4.1. We assume that a system is asymptotically stable and the pair (A, B) is controllable and (A, C) is observable [41].

Since this system is controllable and observable, then the controllabilit and observability gramians W_c, W_o are positive semi-definite and satisfy the Lyapunov equatons

$$AW_c + W_c A^T + BB^T = 0$$
$$W_o A + A^T W_o + C^T C = 0$$

If we refer to theorem (3.3.3), we get the following balanced system

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{3.17}$$

where $\bar{A} = SAS^{-1}$, $\bar{B} = SB$ and $\bar{C} = CS^{-1}$.
Let us partition the balance system (A,B,C) as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad and \quad C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$$

The block matrices A_{11}, Σ_1 of order $r \times r$ respectively and A_{22} of order $n-r \times n-r$ and the other block matrices have order satisfing the original system.
Assuming that $\sigma_r > \sigma_{r+1}$, then the reduced order model obtained by the Balance Truncation method (BT) is represented by the following equation:

$$\begin{aligned} \dot{x}_r &= A_{11}x_r + B_1 u \\ y_r &= C_1 x_r \end{aligned} \tag{3.18}$$

and the transfer function of this reduced system is defined as:

$$G_r(s) = C_1(sI - A_{11})^{-1}B_1 \tag{3.19}$$

the subsystems (A_{11}, B_1, C_1) is a good approximation of the balanced system (A,B,C).
We have the following lemmas that characterizes the properties of these subsystems [48][2, 28].

Lemma 3.4.2. The subsystems (A_{ii}, B_i, C_i), $i = 1,2$ are internally balanced with gramian Σ_i, $i = 1,2$

Lemma 3.4.3. The matrices A_{ii}, $i = 1,2$ are asymptotically stable, i.e.

$$Re(\lambda_j\{A_{ii}\} < 0), \quad i = 1,2, \forall j$$

if Σ_1 and Σ_2 have no diagonal entries in common. Further, the subsystem (A_{ii}, B_i, C_i) is controllable and observable.

Finally, from the Lemmas (3.4.2) (3.4.3), we now introduce a very important result in control theory.
We explore the H_∞ norm of the transfer function of the model and compare the difference with the norm of the transfer function of our reduced order model obtained by balance truncation.
The H_∞ norm can be define as:

$$\|G(j\omega)\|_\infty = \sup_{\omega\in\Re} \sigma\{G(j\omega)\} \tag{3.20}$$

If we let $G(s)$ be the transfer function of the balanced system (A,B,C) and $G_r(s)$ be the transfer function of the reduced system (A_{11},B_1,C_1) then the upper bound for the approximation error is given in the following lemma [28, 41].
The balanced truncation has an important and useful property that has a priori a bounded error [2, 18].

Lemma 3.4.4. We have that

$$\|G-G_r\|_\infty \leq 2(\sigma_{r+1}+\sigma_{r+2}+\cdots\cdots+\sigma_n) \tag{3.21}$$

where σ_{r+1} is the first deleted (HSV) of $G(s)$.

3.5 The reciprocal system of a linear dynamical system

In this section we discuss some properties and some results related to the reciprocal system of the balanced realization for the infinite dimensional systems [37].
Let the linear continuous dynamical system represented by the equation

$$\dot{x} = Ax+Bu$$
$$y = Cx+Du$$

If the system (A,B,C,D) is balanced with gramian Σ, then we have

$$A\Sigma+\Sigma A^T+BB^T = 0$$
$$A^T\Sigma+\Sigma A+C^TC = 0$$

We let $G(s)$ to be the transfer fuction of the balanced system (A,B,C,D),then

$$G(s) = C(sI-A)^{-1}B+D$$

the reciprocal system of the balanced system (A, B, C, D) is denoted by $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and defined as [9, 37, 39]:

$$\begin{aligned} \hat{A} &= A^{-1} \\ \hat{B} &= A^{-1}B \\ \hat{C} &= -CA^{-1} \\ \hat{D} &= D - CA^{-1}B \end{aligned} \tag{3.22}$$

Remark 3.5.1. If we compute the value of $G(0)$, we have that:

$$G(0) = -CA^{-1}B + D = \hat{D}$$

Remark 3.5.2. Let a matrix A is given as:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

the inverse of A is:

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\ -A_{22}^{-1}A_{21}(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}$$

we also have

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{12}A_{11}^{-1} \\ -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}$$

Let $\hat{G}$ be the transfer fucnction of thr reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$, then:

$$\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B} + \hat{D} \tag{3.23}$$

the relation between the two transfer functions G and $\hat{G}$ is given as:

$$\begin{aligned} G(S) &= C(sI - A)^{-1}B + D \\ &= C(sI - A)^{-1}AA^{-1}B + D \\ &= C\frac{I}{s}(A^{-1} - \frac{I}{s})^{-1}A^{-1}B + D \\ &= -C(\frac{I}{s} - A^{-1} + A^{-1})(\frac{I}{s} - A^{-1})^{-1}A^{-1}B + D \\ &= -CA^{-1}B - CA^{-1}(\frac{I}{s} - A^{-1})^{-1}A^{-1}B + D \\ &= -CA^{-1}(\frac{I}{s} - A^{-1})^{-1}A^{-1}B + D - CA^{-1}B \\ &= \hat{C}(\frac{I}{s} - \hat{A})^{-1}\hat{B} + \hat{D} \\ &= \hat{G}(\frac{1}{s}) \end{aligned} \tag{3.24}$$

The following Lemma shows us the balanced realization of the reciprocal system [28, 37]

Lemma 3.5.3. Let the system (A, B, C, D) be the minimal and balanced realization with gramian Σ of a linear, time-ivariant and stable system, then the reciprocal system $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is also balanced with the same gramain Σ.

Proof. We know that Σ satisfies the Lypunove equations

$$\begin{aligned} A\Sigma + \Sigma A^T + BB^T = 0 \\ A^T\Sigma + \Sigma A + C^TC = 0 \end{aligned}$$

Thus multiplying the first equation from the right by A^{-1} and from the left by A^{-T} we get

$$\begin{aligned} A^{-1}(A\Sigma)A^{-T} + A^{-1}(\Sigma A^T)A^{-T} + A^{-1}(BB^T)A^{-T} = 0 \\ \Sigma A^{-T} + A^{-1}\Sigma + (A^{-1}B)(A^{-1}B)^T = 0 \end{aligned}$$

Substituting the values in equation (3.22), we have that

$$\hat{A}\Sigma + \Sigma\hat{A}^T + \hat{B}\hat{B}^T = 0$$

The second Lyapunov equation multiplyed by A^{-T} from the right and by A^{-1} from the left, gives us

$$\begin{aligned} A^{-T}(A^T\Sigma)A^{-1}+A^{-T}(\Sigma A)A^{-1}+A^{-T}(C^TC)A^{-1}&=0\\ \Sigma A^{-1}+A^{-T}\Sigma+(CA^{-1})^T(CA^{-1})&=0 \end{aligned}$$

In the same way from equation (3.22), we have

$$\hat{A}^T\Sigma+\Sigma\hat{A}+\hat{C}^T\hat{C}=0$$

This means that the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$ is balanced with the same gramian Σ. □

The reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$ and the gramian Σ are partitioned as

$$\hat{A}=\begin{pmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{pmatrix},\quad \hat{B}=\begin{pmatrix}\hat{B}_1\\ \hat{B}_2\end{pmatrix},\quad \hat{C}=\begin{pmatrix}\hat{C}_1 & \hat{C}_2\end{pmatrix},\quad \Sigma=\begin{pmatrix}\Sigma_1 & \mathbf{0}\\ \mathbf{0} & \Sigma_2\end{pmatrix} \tag{3.25}$$

then if we refer to lemmas (3.4.2) (3.4.3), we have the following [28]:

Lemma 3.5.4. Let the hypothesis of Lemma (3.5.3) hold and let the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$ be partitioned as in equation (3.25). Then the subsystems $(\hat{A}_{ii},\hat{B}_i,\hat{C}_i,\hat{D}),\ \ i=1,2$ are also internally balanced with gramian Σ_i, for $i=1,2$

Lemma 3.5.5. Let the hypothesis of Lemma (3.5.4) hold. Then the subsystem matrices $\hat{A}_{ii},\ \ i=1,2$ are asymptotically stable if Σ_1 and Σ_2 have no common diagonal element. Further, the subsystem $(\hat{A}_{ii},\hat{B}_i,\hat{C}_i,\hat{D}),\ \ i=1,2$ is controllable and observable.

In order to apply balance truncation to the reciprocal system $(\hat{A},\hat{B},\hat{C},\hat{D})$ we assume that the Hankel singular values σ_j for $j=1,2,\cdots\cdots,r$ are distinct and such that $\sigma_1>\sigma_2>\cdots\cdots>\sigma_n>0$ to have $\Sigma_1>0$, Then we have the following $r\times r$ reduced system $(\hat{A}_{11},\hat{B}_1,\hat{C}_1,\hat{D})$ with state space equation:

$$\begin{aligned} \dot{\hat{x}}&=\hat{A}_{11}\hat{x}+\hat{B}_1u\\ \hat{y}&=\hat{C}_1\hat{x}+\hat{D}u \end{aligned} \tag{3.26}$$

The values of $\hat{A}_{11}$, $\hat{B}_1$, $\hat{C}_1$ and $\hat{D}$ can be computed from equation (3.22) and remark (3.5.2), and they defined as:

$$\begin{aligned}\hat{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\\ \hat{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)\\ \hat{C}_1 &= (C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\\ \hat{D} &= D - CA^{-1}B\end{aligned} \tag{3.27}$$

The transfer function for the reduced system (3.26) is denoted by $\hat{G}_r$ and defined as:

$$\hat{G}_r(s) = \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1 + \hat{D} \tag{3.28}$$

We want, now, to find the H_∞ norm for the reduced reciprocal system.
The error bound according to Lemma (3.4.4) is represented in the following Lemma:

Lemma 3.5.6. We have

$$\|\hat{G} - \hat{G}_r\|_\infty \leq 2\sum_{i=r+1}^{n}\sigma_i \tag{3.29}$$

The proof of this Lemma can be found in [28].

3.6 Model reduction using singular perturbation approximation

In sections (3.3) and (3.4), we introduced a balanced truncation method to reduce the dimension of the original system and obtained an error bound.
In section (3.5) we introduced the properties of the reciprocal system and extend the error bound in section (3.4) to the reduced reciprocal system.
In this section we introduce another method to reduce the original system which is called the singular perturbation approximation method (SPAM).
The two methods give us the same error bounds. For the balanced truncation method the error is small at high frequencies and large at low frequencies, but for the singular perturbation approximation we have large error at high frequencies and small error at low frequencies.
Our goal is to find the error bound for the reduced order model using the singular perturbation approximation.
To obtain this error bound, we discuss the relationship between the reduced model of the reciprocal system and the reduced model when we use the singular perturbation method.

Consider the linear continuous sytem described by the equation

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{3.30}$$

We start from the balanced representation of the linear continuous system to derive a version of equation (3.30) with reduced dimension.
The controllability and observability gramians W_c and W_o respectively are positive semi-definite and can be decomposed as in equation (3.11).
The balanced gramain Σ is partitioned in the following form (see section (3.3))

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

The two partitions

$$\Sigma_1 = diag(\sigma_1, \sigma_2, \cdots\cdots, \sigma_r)$$

and

$$\Sigma_2 = diag(\sigma_{r+1}, \sigma_{r+2}, \cdots\cdots, \sigma_n)$$

show us the important singular values that we are interested in and the unimportant ones which we want to delete [35, 39].
Also we introduce, as in section (3.3), the balance transformation S that satisfies the equations

$$\begin{aligned} S &= UY\Sigma^{\frac{-1}{2}} \\ S^{-1} &= \Sigma^{\frac{-1}{2}} X^T L^T \end{aligned}$$

Now, if we suppose $\sigma_{r+1} << \sigma_r$ and we know that the Hankel singular values (HSVs) (see section (3.3)) are coordinate invariant, then a reduced dimension system with small parameters can be obtained since $\sigma_{r+1} > \sigma_{r+2} > \cdots\cdots > \sigma_n > 0$ [18].
To see where the small parameter Σ_2 enter the equation, we replace Σ_2 by $\epsilon\Sigma_2$ or in other words the small HSVs are scaled uniformly according to the equation

$$(\sigma_{r+1}, \sigma_{r+2}, \cdots\cdots, \sigma_n) \longmapsto \epsilon(\sigma_{r+1}, \sigma_{r+2}, \cdots\cdots, \sigma_n), \quad \epsilon > 0$$

We use the balance transformation $S(\epsilon)$ to change the coordinate such that

$$x \longmapsto S(\epsilon)x$$

If we let $S^{-1}(\epsilon) = T(\epsilon)$, then the balanced matrices are partitioned in the following form [18]:

$$S(\epsilon) = \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \tag{3.31}$$

and the inverse

$$T(\epsilon) = \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \tag{3.32}$$

If we use the balance transformation described in equations (3.31) (3.32), then a new balance coefficient is obtained and written as:

$$\begin{aligned} \tilde{A}(\epsilon) &= T(\epsilon)AS(\epsilon) \\ &= \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{A}_{11} & \frac{1}{\sqrt{\epsilon}}\tilde{A}_{12} \\ \frac{1}{\sqrt{\epsilon}}\tilde{A}_{21} & \frac{1}{\epsilon}\tilde{A}_{22} \end{pmatrix} \end{aligned} \tag{3.33}$$

$$\begin{aligned} \tilde{B}(\epsilon) &= T(\epsilon)B \\ &= \begin{pmatrix} T_{11} & T_{12} \\ \frac{1}{\sqrt{\epsilon}}T_{21} & \frac{1}{\sqrt{\epsilon}}T_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{B}_1 \\ \frac{1}{\sqrt{\epsilon}}\tilde{B}_2 \end{pmatrix} \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} \tilde{C}(\epsilon) &= CS(\epsilon) \\ &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} S_{11} & \frac{1}{\sqrt{\epsilon}}S_{12} \\ S_{21} & \frac{1}{\sqrt{\epsilon}}S_{22} \end{pmatrix} \\ &= \begin{pmatrix} \tilde{C}_1 & \frac{1}{\sqrt{\epsilon}}\tilde{C}_2 \end{pmatrix} \end{aligned} \tag{3.35}$$

If we set $\epsilon = 1$ in equation (3.33), then the value of $\tilde{A} = T(1)AS(1)$ is simply the balance matrix A.
We can rewrite the balancing transformations in the following form

$$S(\epsilon) = S(1)\chi(\epsilon)$$

and

$$T(\epsilon) = \chi(\epsilon)T(1)$$

where

$$\chi(\epsilon) = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{\epsilon}}\mathbf{I} \end{pmatrix}$$

In the next steps we omit the tilde from the balanced matrices, in order to have the following matrices:

$$A = \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}}A_{12} \\ \frac{1}{\sqrt{\epsilon}}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}}B_2 \end{pmatrix}, \quad C = \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}}C_2 \end{pmatrix}$$

Let us define the new variable $q = (q_1, q_2)$ which can be balanced using the balance transformation $T(\epsilon)$ and we write q in the balance form as:

$$q = T(\epsilon)x$$

Now, the linear dynamical system in equation (3.30) is converted to the singular perturbation system that is described in the following equation:

$$\begin{aligned} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & \frac{1}{\sqrt{\epsilon}}A_{12} \\ \frac{1}{\sqrt{\epsilon}}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\sqrt{\epsilon}}B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & \frac{1}{\sqrt{\epsilon}}C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{3.36}$$

Equation (3.36) can be written in another form:

$$\begin{aligned} \dot{q}_1 &= A_{11}q_1 + \frac{1}{\sqrt{\epsilon}}A_{12}q_2 + B_1u \\ \dot{q}_2 &= \frac{1}{\sqrt{\epsilon}}A_{21}q_1 + \frac{1}{\epsilon}A_{22}q_2 + \frac{1}{\sqrt{\epsilon}}B_2u \\ y &= C_1q_1 + \frac{1}{\sqrt{\epsilon}}C_2q_2 \end{aligned} \tag{3.37}$$

the variable q_2 is scaled as

$$q_2 \longmapsto \sqrt{\epsilon}q_2$$

then equation (3.37) becomes:

$$\begin{aligned} \dot{q}_1 &= A_{11}q_1 + A_{12}q_2 + B_1u \\ \epsilon\dot{q}_2 &= A_{21}q_1 + A_{22}q_2 + B_2u \\ y &= C_1q_1 + C_2q_2 \end{aligned} \tag{3.38}$$

This system can be written in matrix form as:

$$\begin{aligned} \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \end{aligned} \tag{3.39}$$

where the block matrices $A_{11}, A_{12}, \ldots$ are in balance form and ϵ is a small positive scalar that represent all small parameters to be neglected [18, 23].
To reduce the dimension of the original system and obtain a reduced order model, we set the singular perturbation $\epsilon = 0$.
The linear dynamical system has a multi-time behavior caused by the singular perturbation and this yeilds the slow and fast variable of the system. The quasi-steady-state for both slow and fast variables are found with more details in [23].
Now, to apply the singular perturbation approximation and obtain a reduced order model, we introduce the following two assumptions [23]:

Assumptions 3.6.1. The block matrix A_{22} is invertible and stable. i.e,

$$\Re\{\lambda(A_{22})\} < 0$$

Assumptions 3.6.2. The following equation has a distinct root when we set $\epsilon = 0$.

$$\epsilon \dot{q}_2 = A_{21}q_1 + A_{22}q_2 + B_2u \tag{3.40}$$

In our dynamical system described by equation (3.38), the slow variable (or dynamic) is q_1 and the fast variable (or dynamic) is q_2.
According to the two assumptions (3.6.1),(3.6.2) and from equation (3.38), if we set $\epsilon = 0$, then the root of equation (3.40) denoted by $\bar{q}_2$ is given as:

$$\bar{q_2} = -A_{22}^{-1}A_{21}\bar{x} - A_{22}^{-1}B_2u \tag{3.41}$$

If we substitute the value of $\bar{q}_2$ in the first part of equation (3.38), we obtain the reduced order model represented by the following state-space equation:

$$\begin{aligned} \dot{\bar{q}}_1 &= \bar{A}\bar{q}_1 + \bar{B}u \\ \bar{y} &= \bar{C}\bar{q}_1 + \bar{D}u \\ \bar{q}_1(0) &= q_1(0) \end{aligned} \tag{3.42}$$

where

$$\begin{aligned}\bar{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21}\\ \bar{B} &= B_1 - A_{12}A_{22}^{-1}B_2\\ \bar{C} &= C_1 - C_2A_{22}^{-1}A_{21}\\ \bar{D} &= -C_2A_{22}^{-1}B_2\end{aligned} \tag{3.43}$$

Let $\bar{G}$ be the transfer function of the reduced order model in equation (3.42), then:

$$\bar{G}(s) = \bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} \tag{3.44}$$

From the definition of the reduced reciprocal system (3.26) and the two equations (3.27) and (3.43), we obtain the following:

$$\begin{aligned}\hat{A}_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\\ &= (\bar{A})^{-1}\\ \hat{B}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}(B_1 - A_{12}A_{22}^{-1}B_2)\\ &= (\bar{A})^{-1}\bar{B}\\ \hat{C}_1 &= (C_1 - C_2A_{22}^{-1}A_{21})(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}\\ &= -\bar{C}(\bar{A})^{-1}\\ \hat{D} &= \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B}\end{aligned} \tag{3.45}$$

In virtue of equation (3.45), we have the following relationship between the two transfer functions $\bar{G}(s)$ and $\hat{G}_r(s)$ and written as:

$$\begin{aligned}
\bar{G}(s) &= \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} + \bar{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left(I - \frac{1}{s}\bar{A}\right)^{-1}\bar{B} + \bar{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1}\bar{A} - \frac{1}{s}\bar{A}\right)^{-1}\bar{B} + \bar{D} \\
&= \bar{C}\left(\frac{1}{s}\right)\left((\bar{A})^{-1} - \frac{I}{s}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \\
&= -\bar{C}\left(\frac{I}{s} - (\bar{A})^{-1} + \bar{A}\right)\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \qquad (3.46)\\
&= -\bar{C}(\bar{A})^{-1}\bar{B} - \bar{C}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} \\
&= -\bar{C}(\bar{A})^{-1}\left(\frac{I}{s} - (\bar{A})^{-1}\right)^{-1}(\bar{A})^{-1}\bar{B} + \bar{D} - \bar{C}(\bar{A})^{-1}\bar{B} \\
&= \hat{C}_1\left(\frac{I}{s} - \hat{A}_{11}\right)^{-1}\hat{B}_1 + \hat{D} \\
&= \hat{G}_r(\frac{1}{s})
\end{aligned}$$

Since the full system (A,B,C,D) is balanced and asymptotically stable and we have the balanced gramian $\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$, we introduce the following theorem for balancing of the reduced system $(\bar{A},\bar{B},\bar{C},\bar{D})$.

Theorem 3.6.3. [37] The reduced order model $(\bar{A},\bar{B},\bar{C},\bar{D})$ by singular perturbation approximation is balanced with Σ_1 and asymptotically stable.

Proof. We know from lemma (3.5.4) that the reduced system $(\hat{A}_{11},\hat{B}_1,\hat{C}_1,\hat{D})$ is balanced with Σ_1 which satisfy the Lypunove equations

$$\begin{aligned}
\hat{A}_{11}\Sigma_1 + \Sigma_1\hat{A}_{11}^T + \hat{B}_1\hat{B}_1^T = 0 \\
\hat{A}_{11}^T\Sigma_1 + \Sigma_1\hat{A}_{11} + \hat{C}_1^T\hat{C}_1 = 0
\end{aligned}$$

we multiply the first equation from the right by $\hat{A}_{11}^{-1}$ and from the left by $\hat{A}_{11}^{-T}$ to get

$$\begin{aligned}
\hat{A}_{11}^{-1}(\hat{A}_{11}\Sigma_1)\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}(\Sigma_1\hat{A}_{11}^T)\hat{A}_{11}^{-T} + A^{-1}(\hat{B}_1\hat{B}_1^T)A^{-T} = 0 \\
\Sigma_1\hat{A}_{11}^{-T} + \hat{A}_{11}^{-1}\Sigma_1 + (\hat{A}_{11}^{-1}\hat{B}_1)(\hat{A}_{11}^{-1}\hat{B}_1)^T = 0
\end{aligned}$$

substitute these values into equation (3.44) we obtain

$$\bar{A}\Sigma_1 + \Sigma_1 \bar{A}^T + \bar{B}\bar{B}^T = 0$$

If the second Lyapunov equation is multiplyed by $\hat{A}_{11}^{-T}$ from the right and by $\hat{A}_{11}^{-1}$ from the left, then we get

$$\begin{aligned} \hat{A}_{11}^{-T}(\hat{A}_{11}^T\Sigma_1)\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\Sigma_1\hat{A}_{11})\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}(\hat{C}_1^T\hat{C}_1)\hat{A}_{11}^{-1} = 0 \\ \Sigma_1\hat{A}_{11}^{-1} + \hat{A}_{11}^{-T}\Sigma_1 + (\hat{C}_1\hat{A}_{11}^{-1})^T(\hat{C}_1\hat{A}_{11}^{-1}) = 0 \end{aligned}$$

In the same way from equation (3.44), we have

$$\bar{A}^T\Sigma_1 + \Sigma_1\bar{A} + \bar{C}^T\bar{C} = 0$$

Finally, our reduced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is balanced with gramian Σ_1.
Since $\hat{A}_{11}$ is stable .i.e., $\Re\{\lambda(\hat{A}_{11})\} < 0$, where λ is an eigenvalue of $\hat{A}_{11}$, then the corresponding eigenvalue of $\bar{A}$ is $\frac{1}{\lambda}$ so we have $\Re\{\lambda_i(\hat{A}_{11})\} < 0$ which mean the reduced system $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is asymptotically stable □

If the hypothesis of Theorem (3.6.3) holds true, then there is an error bound avialable for the singular perturbation approximation $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ of the stable and balanced system (A, B, C, D).
In the form of the H_∞ norm, the error bound is given as [28]:

$$\|G - \bar{G}_r\|_\infty \leq 2 \sum_{r=i+1}^{n} \sigma_i \tag{3.47}$$

Proof. From Equations (3.24),(3.46) and Lemma (3.5.6), we have

$$\begin{aligned} \|G(s) - \bar{G}(s)\|_\infty &= \|G(s) - \hat{G}(\frac{1}{s}) + \hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s}) + \hat{G}_r(\frac{1}{s}) - \bar{G}(s)\|_\infty \\ &\leq \|G(s) - \hat{G}(\frac{1}{s})\|_\infty + \|\hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s})\|_\infty + \|\hat{G}_r(\frac{1}{s}) - \bar{G}(s)\|_\infty \\ &\leq \|\hat{G}(\frac{1}{s}) - \hat{G}_r(\frac{1}{s})\|_\infty \\ &\leq 2 \sum_{i=r+1}^{n} \sigma_i \end{aligned}$$

□

Chapter 4

Model Order Reduction of Linear Time-Invariant Continuous Non-Homogeneous Dynamical System on Infinite-Time Horizon

In this chapter we discuss a non-homogeneous linear dynamical continuous system and find the error bound between the input and the output of this system. We use the balance truncation model reduction to find the reduced order model of the full system and find the L_2 norm of the error bound.
We extend the approach and introduce the error bound for the reduced reciprocal system of the full system and use the result found for the reduced model by singular perturbation approximation method [20].

4.1 An error bound for non-homogeneous system using balance truncation model reduction (BTMR)

In this section we introduce the error bound between the output of the original and reduced system using the balanced truncation method.
Consider the following linear time-invariant continuous dynamical system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ with state-space equation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(t_0) &= x_0 \end{aligned} \tag{4.1}$$

where $A \in \Re^{n\times n}, \quad B \in \Re^{m\times n}, \quad C \in \Re^{p\times n}$.
The state x and the output y are defined on the domains

$$x : (t_0, \infty) \longrightarrow \Re^n$$

and

$$y : (t_0, \infty) \longrightarrow \Re^p$$

The input function u maps from $(t_0, \infty) \longrightarrow \Re^m$.
To reduce the full system described by equation (4.1), we choose $r < n$ and construct the two matrices $W, V \in \Re^{n\times r}$ such that $W^T V = I_r$ and the reduced system $\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right)$ obtained is written as :

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{4.2}$$

where $\bar{A} = W^T AV, \quad \bar{B} = W^T B, \quad \bar{C} = CV$.
The initial condition of the reduced system is

$$\bar{x}(t_0) = W^T x(t_0)$$

We denote by $\bar{y} \in L_2(t_0, \infty)$, the output of the reduced system [20].
If we apply the balanced truncation method with zero initial condition and for any $u \in L_2(t_0, \infty)$, then the error between the output of the original and reduced system is given as:

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq 2 \sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)} \tag{4.3}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values(HSVs) (for more details see lemma (3.4.4).
If we choose an initial condition that is different from zero then equation (4.3) does not apply.
As a resort, we introduce an approach called X_0-Balanced Truncation that can be used to derive an estimate for the norm of the output error.
The idea of this approach depends on a matrix $X_0 \in \Re^{n\times n_0}$ and assuming that the non-zero initial condition $x(t_0) = x_0$ satisfy the following property:

$$x_0 \in Im X_0$$

or in other words x_0 belongs to a subspace that is spanned by the columns of X_0.
Now, we want to extend the original system in equation (4.1) by replacing the old input B with new one given by:

$$B_e = [B \quad X_0] \in \Re^{n\times(m+n_0)} \tag{4.4}$$

The initial condition x_0 can be approximated by choosing a suitable L^2 input function $X_0u_0(t)$ written as:

$$x_0 = X_0u_0(t) \tag{4.5}$$

If we use the new input in equation (4.4) and the approximation of the non-zero initial condition in equation (4.5), then the extended system of the original system is written as:

$$\begin{aligned} \dot{x} &= Ax + \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u(t) \\ u_0(t) \end{pmatrix} \\ y &= Cx \end{aligned} \tag{4.6}$$

If we apply the balance truncation method to the system in equation (4.6), the reduced order model of size $r \times r$ is giving by:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix} \begin{pmatrix} u(t) \\ u_0(t) \end{pmatrix} \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{4.7}$$

where $\bar{A} = W^TAV, \quad \bar{B} = W^TB, \quad \bar{C} = CV$ and the initial condition of this reduced system is $\bar{X}_0 = W^TX_0$.
We let

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} \geq \cdots \sigma_n \geq 0$$

be the Hankel singular values of the system (4.6) and

$$\gamma = \sigma_{r+1} + \cdots + \sigma_n$$

Now, the error bound that can be obtained between the outputs of the two systems in equation (4.6) and (4.7) using the balanced truncation is given as:

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq 2\gamma \left(\|u\|_{L_2(t_0,\infty)} + \|u_0\|_{L_2(t_0,\infty)} \right) \tag{4.8}$$

The technique used to construct a reduced order model (4.2) from the orginal system (4.2) by applying the two projection $W, V \in \Re^{n\times r}$ resulted by applying the balanced truncation to system (4.6), is known as X_0-Balanced Truncation [20].
Our goal now is to use the X_0-Balanced Truncation and derive an estimate to the L_2 norm between the output of the original system (4.1) and the reduced order model (4.2). In the absence of auxiliary input function $u_0 \in L_2(t_0, \infty)$ and since the outputs of the original system and the extended system (4.1) and (4.6) respectively together with their reduced model (4.2) and (4.7) are equivalent, then the error bound obtained in equation (4.8) cannot be applied directly to get an error bound for the X_0-Balanced Truncation.

The following theorem contains the main result to find the error bound using the idea of the X_0-Balanced Truncation [20].

Theorem 4.1.1. Let $W,V \in \Re^{n\times r}$ be the projection matrices and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} \geq \cdots \geq \sigma_n \geq 0$ be the Hankel singular values, generated by applying balance truncation to (4.6). Let $\gamma = \sigma_{r+1} + \cdots + \sigma_n$. Moreover, let $\bar{\Sigma} = diag(\sigma_1, \cdots, \sigma_r)$ be the observability (or, equivalently, observability) Gramian of the reduced system (4.7) and let $Q = L^T L, L \in \Re^{n\times n}$, be a factorization of the observability Gramian Q of the extended system (4.6). If $x(t_0) = X_0 z_0$, then for all $u \in L_2(t_0,\infty)$ the error bound between the output $y \in L_2(t_0,\infty)$ of the full order model (4.1) and the output $\bar{y} \in L_2(t_0,\infty)$ of the reduced order model (4.2) satisfy

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq 2\gamma\|u\|_{L_2(t_0,\infty)} + 3.2^{\frac{-1}{2}}\left(\|LAX_0\|_2 + \|\bar{\Sigma}^{\frac{1}{2}}\bar{A}\bar{X}_0\|_2\right)^{\frac{1}{3}}\gamma^{\frac{2}{3}}\|z_0\|_2 \quad (4.9)$$

The proof of this theorem can be carried out by assuming that $t_0 = 0$, then by adding extra terms and regularization input function u_0 for the two cases $x_0 = 0$ and $u = 0$. To estimate each term we use the result found in (4.8) and finally the Taylor expansion.
For more details see [20].
If the extended system is balanceed, we introduce the following Corollary that includes the priori error bound [20]

Corollary 4.1.2. Let $X_0 \in \Re^{n\times n_0}$ be given and assume that the extended system (4.6) is balanced, Furthermore, let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > \sigma_{r+1} \geq \cdots \geq \sigma_n \geq 0$ be the Hankel singular values corresponding to (4.6), let $\Sigma = diag(\sigma_1, \cdots, \sigma_r)$ be its Gramian, and set $\gamma = \sigma_{r+1} + \cdots + \sigma_n$. If (4.2) is an rth order system obtained by X_0-Balanced Truncation, then for all $u \in L_2(t_0,\infty)$

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq 2\gamma\|u\|_{L_2(t_0,\infty)} + 3(\|\Sigma^{\frac{1}{2}}A\|_2)^{\frac{1}{3}}\|X_0\|_2)^{\frac{1}{3}}\gamma^{\frac{2}{3}}\|z_0\|_2 \quad (4.10)$$

Proof. The proof of this corollary depends on the extended system and the fact that $W = V = I_{n\times r}$. The matrix $I_{n\times r}$ consists of the first r columns of the identity matrix. The Gramian is given as $Q = \Sigma$ or we can write $L = \Sigma^{\frac{1}{2}}$. The following inequality holds

$$\|\bar{\Sigma}^{\frac{1}{2}}\bar{A}\bar{X}_0\|2 \leq \|\bar{\Sigma}^{\frac{1}{2}}\bar{A}\|2\|\bar{X}_0\|2 \leq \|\Sigma^{\frac{1}{2}}A\|2\|X_0\|2$$
$$\|\Sigma^{\frac{1}{2}}AX_0\|2 \leq \|\Sigma^{\frac{1}{2}}A\|2\|X_0\|2$$

By substituting these estimates into equation (4.9), we obtain the error bound

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq 2\gamma\|u\|_{L_2(t_0,\infty)} + 3(\|\Sigma^{\frac{1}{2}}A\|_2)^{\frac{1}{3}}\|X_0\|_2)^{\frac{1}{3}}\gamma^{\frac{2}{3}}\|z_0\|_2$$

□

4.2 New error bound for non-homogeneous system using the balance truncation model reduction (BTMR)

In section (4.1), we introduce an aproach called the X_0-Balanced Truncation and define a suitable L^2 input function. By this approach, we estimate the norm of the output error given in equation (4.9) and (4.10).
In this section, we want to use the idea of the X_0-Balanced Truncation and extend the original system using new input and non-zero initial condition. To obtain an error bound between the outputs of the original and its reduced system, we define the Driac delta function $\delta_0(t) \notin L^2$ in the extended system. The balance truncation method is applied to both original and extend reduced systems to obtain the error bound between their outputs.
Consider the initial value problem of linear continuous system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ of the form:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(t_0) &= x_0 \end{aligned} \tag{4.11}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ and $C \in \Re^{p\times n}$.
$u \in L_2(t_0, \infty) \longrightarrow \Re^m$ is the input control, $x \in L_2(t_0, \infty) \longrightarrow \Re^n$ is the state vector and $y \in L_2(t_0, \infty) \longrightarrow \Re^p$ is the output.

Assumptions 4.2.1. Throughout this section, the system described by equations (4.11) is Controllable, Observable and Asymptotically Stable

To reduce the system in equations (4.11) using the balance truncation model reduction, we choose a non-singular matrix $T \in \Re^{n\times n}$ such that the reduced order system $\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right)$ is asymptoticcaly stable and written as:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{4.12}$$

where

$$\bar{A} = TAT^{-1}, \quad \bar{B} = TB, \quad \bar{C} = CT^{-1}$$

and the initial condition of the reduced system is

$$\bar{x}(t_0) = T^{-1}x(t_0)$$

To obtain the error bound between the output of the original system (4.11) and its reduced system (4.12) with a non-zero initial condition, we extend the original system (4.11) to get the following extended system

$$\Sigma_e = \left(\begin{array}{c|cc} A & B & X_0 \\ \hline C & & \end{array}\right)$$

and the state-space equation of this system with zero initial condition can be written as:

$$\begin{aligned} \dot{x}_e &= Ax_e + \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} \\ y_e &= Cx_e \end{aligned} \tag{4.13}$$

Here A, B and C are defined the same as in (4.11) and the state vector is denoted by x_e , the output of the system is y_e.
We define X_0 as:

$$X_0 = x(t_0)$$

The Dirac delta function δ_0 is defined in chapter (2) section (2.9) and satisfies the dirac distribution:

$$\delta_0(t) = \lim_{\epsilon \to 0} \delta_\epsilon(t)$$

in the sense of the distribution .i.e.

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \varphi(t)\delta_\epsilon(t)dt = \varphi(0), \;\; \forall \varphi \in \Re$$

If we apply the balance truncation method and the balanced transformation T to the system in equations (4.13) we get the following reduced system

$$\left(\begin{array}{c|cc} \bar{A} & \bar{B} & \bar{X}_0 \\ \hline \bar{C} & & \end{array}\right)$$

with state-space equation:

$$\begin{aligned} \dot{\bar{x}}_e &= \bar{A}\bar{x}_e + \begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} \\ \bar{y}_e &= \bar{C}\bar{x}_e \end{aligned} \tag{4.14}$$

Here $\bar{A}$, $\bar{B}$ and $\bar{C}$ are the same as in equation (4.12) and the initial condition of this reduced system is

$$\bar{X}_0 = T^{-1}x(t_0)$$

The standard balanced truncation does not work here since the driac delta function $\delta_0(t) \notin L^2$.
To derive the error bound between the output of the original system and its reduced system, we start by computing the solution $x_e(t)$ of the extended system in equation (4.13) to obtain:

$$\begin{aligned}
x_e(t) &= \int_{t_0}^{t} e^{A(t-\tau)} \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau \\
&= \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + \int_{t_0}^{t} e^{A(t-\tau)} X_0 \delta_0(\tau) d\tau \\
&= \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau + e^{At} X_0 \\
&= e^{At} X_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \\
&= x(t)
\end{aligned} \tag{4.15}$$

We see that the solutions of the original system (4.11) and the extended system (4.13) are the same, hence the output for the two previous systems must be the same, that is:

$$y_e(t) = y(t) = C \left(e^{At} X_0 + \int_{t_0}^{t} e^{A(t-\tau)} Bu(\tau) d\tau \right) \tag{4.16}$$

Also we can compute the solution of the reduced system (4.14) which can be written as:

$$\begin{aligned}
\bar{x}_e(t) &= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix} \begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau \\
&= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau + \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{X}_0 \delta_0(\tau) d\tau \\
&= \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau + e^{\bar{A}t} \bar{X}_0 \\
&= e^{\bar{A}t} \bar{X}_0 + \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau \\
&= \bar{x}(t)
\end{aligned} \tag{4.17}$$

Since the solutions of the two reduced systems (4.12) and (4.14) are equal, hence the outputs of the two reduced systems must equal, so we get:

$$\bar{y}_e(t) = \bar{y}(t) = \bar{C} \left(e^{\bar{A}t} \bar{X}_0 + \int_{t_0}^{t} e^{\bar{A}(t-\tau)} \bar{B} u(\tau) d\tau \right) \tag{4.18}$$

We introduce now the controllability and the observability Gramians of the original system (4.11) and the extended system (4.13) and see how can they be related to each other.
Let W_c be the controllability Gramian of the full system in equation (4.11) and W_{ce} be the controllability Gramian of the extended system (4.13), then:

$$\begin{aligned}
W_{ce} &= \int_{t_0}^{\infty} e^{At} \begin{pmatrix} B & X_0 \end{pmatrix} \begin{pmatrix} B & X_0 \end{pmatrix}^T e^{A^T t} dt \\
&= \int_{t_0}^{\infty} e^{At} \begin{pmatrix} BB^T + X_0 X_0^T \end{pmatrix} e^{A^T t} dt \\
&= \int_{t_0}^{\infty} e^{At} BB^T e^{A^T t} dt + \int_{t_0}^{\infty} e^{At} X_0 X_0^T e^{A^T t} dt \\
&= W_c + \int_{t_0}^{\infty} e^{At} X_0 X_0^T e^{A^T t} dt
\end{aligned} \tag{4.19}$$

So we see that the two controllability matrices are not equal, but the observability

matrices are equal.
That is, if we let W_o be the observability Gramian of the full system and W_{oe} be the observability Gramian of the extended system, we have that:

$$W_{oe} = \int_{t_0}^{\infty} e^{A^T t} C^T C e^{At} dt = W_o \tag{4.20}$$

As before we let

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

where

$$\Sigma_1 = diag(\sigma_1, \cdots, \sigma_r)$$
$$\Sigma_2 = diag(\sigma_{r+1}, \cdots, \sigma_n), \quad r < n$$

and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values.
Since $W_{oe} = W_o$ then by using the balance transformation T, we have the following equation:

$$T^T W_o T = T^T W_{oe} T = \Sigma = diag(\sigma_1, \cdots, \sigma_n) \tag{4.21}$$

To derive the error bound between the output of the original and extended systems, we factorize the observability Gramian of the extended (-or original-) system as

$$W_{oe} = W_o = L^T L, \quad for\ L \in \Re^{n \times n}$$

and let

$$\Sigma_1 = diag(\sigma_1, \cdots, \sigma_r)$$

be the observability Gramian of the reduced system in equation (4.14).
The following theorem contains the new error bound obtained between the outputs of the original and its reduced system.

Theorem 4.2.2. Let $T \in \Re^{n \times n}$ be a non-singular transformation matrix and let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ be the Hankel singular values of the extended system (4.13). Then for all $u \in L_2(t_0, \infty)$ the error bound between $y \in L_2(t_0, \infty)$ and $\bar{y} \in L_2(t_0, \infty)$ is:

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \|LX_0\|_2^2 + \|\sqrt{\Sigma_1}\bar{X}_0\|_2^2 + 2 \sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)} \tag{4.22}$$

Proof. To obtain our error bound, we assume that the time $t_0 = 0$ and study the two cases $x_0 = 0$ and $u = 0$. We have that

$$\begin{aligned}
\|y-\bar{y}\|_{L_2(0,\infty)} &= \|y_e-\bar{y}_e\|_{L_2(0,\infty)} \\
&= \|\int_0^t Ce^{A(t-\tau)}\begin{pmatrix} B & X_0 \end{pmatrix}\begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau - \int_0^t \bar{C}e^{\bar{A}(t-\tau)}\begin{pmatrix} \bar{B} & \bar{X}_0 \end{pmatrix}\begin{pmatrix} u \\ \delta_0 \end{pmatrix} d\tau\| \\
&= \|Ce^{At}X_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - \bar{C}e^{\bar{A}(t}\bar{X}_0 - \bar{C}\int_0^t e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\| \\
&= \|Ce^{At}X_0 - \bar{C}e^{\bar{A}t}\bar{X}_0 + C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - \bar{C}\int_0^t e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\| \\
&\leq \|Ce^{At}X_0 - \bar{C}e^{\bar{A}t}\bar{X}_0\| + \|C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - \bar{C}\int_0^t e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\|
\end{aligned} \tag{4.23}$$

For the case $x_0 = 0$, we have the error bound

$$\|C\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau - \bar{C}\int_0^t e^{\bar{A}(t-\tau)}\bar{B}u(\tau)d\tau\| \leq 2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(0,\infty)}$$

For the case $u = 0$, we have

$$\|Ce^{At}X_0 - \bar{C}e^{\bar{A}t}\bar{X}_0\| \leq \|Ce^{At}X_0\| + \|\bar{C}e^{\bar{A}t}\bar{X}_0\|$$

and since W_o has the factorization as

$$L^T L = \int_0^\infty e^{A^T t}C^T Ce^{At}dt$$

then the estimate for the first term $\|Ce^{At}X_0\|_{L_2(0,\infty)}$ gives

$$\begin{aligned}
\|Ce^{At}X_0\|_{L_2(0,\infty)} &= X_0^T\left(\int_{t_0}^\infty e^{A^T t}C^T Ce^{At}dt\right)X_0 \\
&= \|X_0^T L^T L X_0\| \\
&= \|(LX_0)^T(LX_0)\| \\
&= \|(LX_0)^T\|\|LX_0\| \\
&= \|LX_0\|_2^2
\end{aligned}$$

Likewise, the second part $\|\bar{C}e^{\bar{A}(t}\bar{X}_0\|$ gives the bound

$$\|\bar{C}e^{\bar{A}(t}\bar{X}_0\|_{L_2(0,\infty)} = \|\sqrt{\Sigma_1}\bar{X}_0\|_2^2$$

If we substitute these values into equation (4.23), we get our error bound

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq \|LX_0\|_2^2 + \|\sqrt{\Sigma_1}\bar{X}_0\|_2^2 + 2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(t_0,\infty)}$$

□

In case where the extended system is balanced, we have the following Corollary.

Corollary 4.2.3. Let the extended system

$$\left(\begin{array}{c|cc} A & B & X_0 \\ \hline C & & \end{array}\right)$$

be balanced such that

$$W_{oe} = W_o = \Sigma = diag(\sigma_1, \cdots, \sigma_n)$$

is the observability Gramian and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values of the extended system. If

$$\left(\begin{array}{c|cc} \bar{A} & \bar{B} & \bar{X}_0 \\ \hline \bar{C} & & \end{array}\right)$$

is the r^{th} order reduced system obtained by balance truncation, then the error bound between the outputs of the original and its reduced system is:

$$\|y-\bar{y}\|_{L_2(t_0,\infty)} \leq \|\sqrt{\Sigma}\|_2^2\|X_0\|_2^2 + 2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(t_0,\infty)} \tag{4.24}$$

for all $u \in L_2(t_0,\infty)$

Proof. Since the system $\left(\begin{array}{c|cc} A & B & X_0 \\ \hline C & & \end{array}\right)$ is balanced for a given balance transformation $T \in \Re^{n\times n}$ and the observability Gramian

$$W_{oe} = \Sigma = diag(\sigma_1, \cdots, \sigma_n)$$

can be factorized as $W_{oe} = L^T L$, we obtain $L = \sqrt{\Sigma}$
We know that

$$\|\sqrt{\Sigma_1}\bar{X}_0\|_2^2 \leq \|\sqrt{\Sigma_1}\|_2^2 \|\bar{X}_0\|_2^2$$

and

$$\|\sqrt{\Sigma}X_0\|_2^2 \leq \|\sqrt{\Sigma}\|_2^2 \|X_0\|_2^2$$

since $\|\sqrt{\Sigma_1}\|_2^2 \leq \|\sqrt{\Sigma}\|_2^2\|$ and $\|\bar{X}_0\|_2^2 \leq \|X_0\|_2^2$
then we observe that

$$\|\sqrt{\Sigma_1}\bar{X}_0\|_2^2 \leq \|\sqrt{\Sigma}\|_2^2 \|X_0\|_2^2$$

If we substitute these values into equation (4.22) we obtain the error bound

$$\|y - \bar{y}\|_{L_2(t_0,\infty)} \leq \|\sqrt{\Sigma}\|_2^2 \|X_0\|_2^2 + 2 \sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)}$$

□

Finally, we see the error bound obtained in [20] can be approximated by adding an extra input of the initial condition. This new input can be regularized by choosing a suitable L^2 input function.
As the error bound which we have does not depend on the regularization parameters, we can interpolate the non-zero initial condition as an extra input and we choose the Driac delta fuction $\delta_0 \notin L^2$ to estimate the error bound by applying the triangle inequality and the two separated terms.

4.3 The reciprocal system of a linear continuous dynamical system

In this section we introduce the reciprocal system of the original (full) system and discuss some properties of this system. We want to find an error bound for the reduced reciprocal system by refering to the theorem and corollary that we deduced in Section (4.2).
Consider the linear continuous dynamical system with non-zero initial condition defined as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{4.25}$$

where $A \in \Re^{n\times n}$, $B \in \Re^{m\times n}$, $C \in \Re^{p\times n}$ are constant matrices.
The state x is defined as $x : (t_0, \infty) \longrightarrow \Re^n$ while the output y is defined on the form $y : (t_0, \infty) \longrightarrow \Re^p$.
The control input function u is given as $u : (t_0, \infty) \longrightarrow \Re^m$.

If we let W_c and W_o be the controllability and observability Gramians for the system in equations (4.25) that are defined in the same way as in definitions (2.10.6) (2.10.7), section (2.10) and satisfy the Lyapunov equations

$$AW_c + W_c A^T + BB^T = 0$$

$$A^T W_o + W_o A + C^T C = 0$$

We start by defining the reciprocal system denoted by $\left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right)$ of the linear continuous dynamical system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ described in equations (4.25). The matrices

$$\hat{A},\ \hat{B},\ \hat{C} \text{ and } \hat{D}$$

have the same definition as equations (3.22) in section (3.5).
The initial condition for this reciprocal system is defined as:

$$\hat{x}(t_0) = A^{-1} x(t_0) \tag{4.26}$$

The full system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ is balanced with Gramian

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

where

$$\Sigma_1 = diag(\sigma_1, \cdots, \sigma_r)$$

$$\Sigma_2 = diag(\sigma_{r+1}, \cdots, \sigma_n)$$

and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0,\ \ r < n$ are the Hankel singular values.
From lemma (3.5.3) in section (3.5), we deduced that the reciprocal system is balanced with the same Gramain Σ.
The reciprocal system can be partitioned in the same way as equation (3.25) in section (3.5).
The reduced reciprocal system $\left(\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D} \end{array}\right)$ of order $r \times r$ is balanced with Gramian Σ_1 and asymptotically stable (see Lemmas (3.5.4) (3.5.5) from section (3.5)).
The state space and output equations for the reciprocal system can be written as:

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}\hat{x} + \hat{B}u \\ \hat{y} &= \hat{C}\hat{x} + \hat{D}u \end{aligned} \tag{4.27}$$

where

$$\hat{A} = A^{-1}, \quad \hat{B} = A^{-1}B, \quad \hat{C} = CA^{-1}, \quad \hat{D} = D - CA^{-1}B$$

and the initial condition of this system is given as

$$\hat{x}(t_0) = A^{-1}x(t_0)$$

Also we can write the state and output equations for the reduced reciprocal system $\left(\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D} \end{array}\right)$ in the form:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{A}_{11}\hat{x}_1 + \hat{B}_1 u \\ \hat{y}_1 &= \hat{C}_1\hat{x}_1 + \hat{D}u \end{aligned} \tag{4.28}$$

where

$$\begin{aligned} \hat{A}_{11} &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \\ \hat{B}_1 &= \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}\left(B_1 - A_{12}A_{22}^{-1}B_2\right) \\ \hat{C}_1 &= \left(C_1 - C_2A_{22}^{-1}A_{21}\right)\left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1} \end{aligned}$$

and if the initial condition of the full system is given as

$$x(t_0) = \begin{pmatrix} x_1(t_0) \\ x_2(t_0) \end{pmatrix}$$

then the initial condition of the reduced reciprocal system is defined as

$$\hat{x}_1(t_0) = \left(A_{11} - A_{12}A_{22}^{-1}A_{21}\right)^{-1}\left(x_1(t_0) - A_{12}A_{22}^{-1}x_2(t_0)\right) \tag{4.29}$$

The observability Gramian W_o can be factorized as

$$W_o = L^T L, \;\; L \in \Re^{n\times n}$$

We now introduce the following Theorem which contains the error bound between the outputs of the reciprocal and its reduced systems.

Theorem 4.3.1. Given the full system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$, with non-zero initial condtion $x(t_0)$. Let the observability Gramian W_o be factorized as

$$W_o = L^T L, \;\; L \in \Re^{n\times n}$$

In addition, let

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

and

$$\begin{aligned} \Sigma_1 &= diag(\sigma_1, \cdots, \sigma_r) \\ \Sigma_2 &= diag(\sigma_{r+1}, \cdots, \sigma_n) \end{aligned}$$

where $r < n$ and $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the Hankel singular values.
If the non-zero initial conditions of the reciprocal and its reduced system is defined in equations (4.26) (4.29) respectively, then for all $u \in L_2(t_0, \infty)$, the error bound between the output $\hat{y}$ of the reciprocal system and the output $\hat{y}_1$ of its reduced system is given as:

$$\begin{aligned} \|\hat{y} - \hat{y}_1\|_{L_2(t_0,\infty)} \leq \|L\hat{A}x(t_0)\|_2^2 + \|\sqrt{\Sigma_1}\hat{A}_{11}\left(x_1(t_0) - A_{12}A_{22}^{-1}x_2(t_0)\right)\|_2^2 \\ + 2\sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)} \end{aligned} \tag{4.30}$$

Proof. We apply the result in theorem (4.2.2) to the reciprocal and reduced reciprocal systems with non-zero initial condition and use the factorization of W_o to get the error bound and the proof is concluded. □

Corollary 4.3.2. If the reciprocal system $\left(\begin{array}{c|c} \hat{A} & \hat{B} \\ \hline \hat{C} & \hat{D} \end{array}\right)$ is balanced, then the reduced reciprocal system

$$\left(\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & \hat{D} \end{array}\right)$$

is balanced with Σ_1, and the error bound between the outputs $\hat{y}$ and $\hat{y}_1$ is:

$$\|\hat{y} - \hat{y}_1\|_{L_2(t_0,\infty)} \leq \|\sqrt{\Sigma}\|_2^2 \|\hat{A}x(t_0)\|_2^2 + 2\sum_{i=r+1}^{n} \sigma_i \|u\|_{L_2(t_0,\infty)} \tag{4.31}$$

for all $u \in L_2(t_0, \infty)$

Proof. By refering to corollary (4.2.3) and using the initial condition $\hat{x}(t_0) = \hat{A}x(t_0)$ for the reciprocal system and the initial condition in equation (4.29) for the reduced reciprocal system and the fact that the observability Gramian can be factorized as $W_o = L^T L$, $L \in \Re^{n \times n}$ and $L = \sqrt{\Sigma}$, we obtain the error bound. □

4.4 Error bound of a non-homogeneous linear control system using the singular perturbation approximation method (SPA)

In this section we introduce an approch to find the error bound between the outputs of the original and the reduced systems with non-zero initial condition using the method of singular perturbation approximation (SPA).
To obtain such an error bound, we use the approach for the reciprocal system and extend it using the singular perturbatin approximaation.
Consider the Linear Dynamical System written in the form:

$$\begin{pmatrix} \dot{x} \\ \epsilon\dot{z} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \tag{4.32}$$

Here, once again $A \in \Re^{n\times n},\ B \in \Re^{n\times m},\ C \in \Re^{p\times n}$ and $x(t_0) = \begin{pmatrix} x_0 \\ z_0 \end{pmatrix}$ is the initial condition.
The scalar ϵ represents all the small parameters to be neglected.
The output equation of this system is:

$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{4.33}$$

If we use the singular perturbation technique to reduce the system (4.32), we choose $r < n$ such that the reduced system is given as:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}u \\ \bar{y} &= \bar{C}\bar{x} + \bar{D}u \end{aligned} \tag{4.34}$$

and

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}, \quad \bar{B} = B_1 - A_{22}^{-1}B_2, \quad \bar{C} = C_1 - C_2A_{22}^{-1}A_{21}, \quad \bar{D} = -C_2A_{22}^{-1}B_2$$

We assume that the block matrix A_{22} is bounded, invertable and stable matrix.
The relationship between the coefficient matrices of the reduced reciprocal system (4.28) and the reduced system (4.34) obtained by the singular perturbation approximation are given in equation (3.45).
From theorem (3.6.3), the reduced system (4.34) is balanced with Σ_1 and asymptotically stable.
We are now ready to introduce our main result to find the error bound between the output y of the original system and the output $\bar{y}$ of the reduced system using the singular perturbation approximation.
Let G be the transfer function of the original system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ and $\hat{G}$ be the

transfer function of the reciprocal system $\left(\begin{array}{c|c}\hat{A} & \hat{B} \\ \hline \hat{C} & \hat{C}\end{array}\right)$, then for zero-initial condition we have proved for the reduced system in section (3.5) equation (3.24) that

$$G(s) = \hat{G}(\frac{1}{s})$$

If we let $\bar{G}$ be the transfer function of the reduced system $\left(\begin{array}{c|c}\bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right)$ and $\hat{G}^r$ be the transfer function of the reduced reciprocal system $\left(\begin{array}{c|c}\hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & \end{array}\right)$, then from section (3.6) equation (3.44) we have:

$$\bar{G}(s) = \hat{G}^r(\frac{1}{s})$$

Now, for the non-zero initial condition $x(t_0)$, we have the following corollary for the transfer function $G_{x(t_0)}$ of the original system and the transfer function $\hat{G}_{x(t_0)}$ of the reciprocal systems.

Corollary 4.4.1. If the initial condition $x(t_0)$ is non-zero, then the relationship between the transfer function $G_{x(t_0)}$ of the original system and the transfer function $\hat{G}_{x(t_0)}$ of the reciprocal systems is given as:

$$G_{x(t_0)}(s) = \hat{G}_{x(t_0)}(\frac{1}{s}) \tag{4.35}$$

Proof. The transfer function of the original system with non-zero initial condition has the form

$$G_{x(t_0)}(s) = G_{x(t_0)}(s) + q_{x(t_0)}(s) \tag{4.36}$$

When $x(t_0) = 0$, we have:

$$\begin{aligned}G(s) &= C\,(sI - A)^{-1}\,B \\ &= CA^{-1}\left(sI - A^{-1}\right)^{-1}A^{-1}B \\ &= \hat{G}(\frac{1}{s})\end{aligned}$$

and for the value of $q_{x(t_0)}(s)$, we have:

$$\begin{aligned} q_{x(t_0)}(s) &= C\,(sI - A)^{-1}\,x(t_0) \\ &= CA^{-1}\left(sI - A^{-1}\right)^{-1} A^{-1}x(t_0) \\ &= \hat{q}_{\hat{x}(t_0)}(\frac{1}{s}) \end{aligned}$$

If we substitute these values into equation (4.36), we get:

$$\begin{aligned} G_{x(t_0)}(s) &= \hat{G}(\frac{1}{s})U(s) + \hat{q}_{\hat{x}(t_0)}(\frac{1}{s}) \\ &= \hat{G}_{\hat{x}(t_0)}(\frac{1}{s}) \end{aligned}$$

□

For the reduced system with non-zero initial condition, let $\bar{G}_{x(t_0)}(s)$ be the transfer function of the reduced system $\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right)$ and $\hat{G}^r_{x(t_0)}(\frac{1}{s})$ be the transfer function of the reduced reciprocal system $\left(\begin{array}{c|c} \hat{A}_{11} & \hat{B}_1 \\ \hline \hat{C}_1 & \end{array}\right)$, then we have the following corollary that includes the relationship between these transfer functions

Corollary 4.4.2. The two transfer functions $\bar{G}_{x(t_0)}(s)$ and $\hat{G}^r_{x(t_0)}(\frac{1}{s})$ for the reduced systems described in equations (4.34) (4.28) with non-zero initial condition satisfy the following result:

$$\bar{G}_{x(t_0)}(s) = \hat{G}^r_{\hat{x}(t_0)}(\frac{1}{s}) \tag{4.37}$$

Proof. The transfer function of the reduced system with non-zero initial condition is:

$$\bar{G}_{x(t_0)}(s) = \bar{G}(s) + \bar{q}_{\bar{x}(t_0)}(s) \tag{4.38}$$

In the case when the initial condition is zero, we have:

$$\begin{aligned} \bar{G}(s) &= \bar{C}\left(sI - \bar{A}\right)^{-1}\bar{B} \\ &= \hat{C}_1\hat{A}_1^{-1}\left(sI - \hat{A}_1^{-1}\right)^{-1}\hat{A}_1^{-1}\hat{B}_1 \\ &= \hat{G}^r(\frac{1}{s}) \end{aligned}$$

We can then write the value of $\bar{q}_{\bar{x}(t_0)}(s)$ as follows:

$$\begin{aligned}\bar{q}_{\bar{x}(t_0)}(s) &= \bar{C}\left(sI-\bar{A}\right)^{-1}\bar{x}(t_0)\\ &= \hat{C}_1\hat{A}_1^{-1}\left(sI-\hat{A}_1^{-1}\right)^{-1}\hat{A}_1^{-1}x_0\\ &= \hat{q}^r_{\hat{x}(t_0)}(\frac{1}{s})\end{aligned}$$

Substituting these values into equation (4.38), we get the result:

$$\begin{aligned}\bar{G}_{x(t_0)}(s) &= \hat{G}^r(\frac{1}{s})+\hat{q}^r_{\hat{x}(t_0)}(\frac{1}{s})\\ &= \hat{G}^r_{\hat{x}(t_0)}(\frac{1}{s})\end{aligned}$$

□

To find the error bound between the output of the original and the reduced order model by applying the singular perturbation approximation technique, we introduce the following Theorem:

Theorem 4.4.3. Let G be the transfer function of the original system and $\bar{G}$ be the transfer function of the reduced system using the singular perturbation approximation, then we have the following error bound between the output y of the full system and $\bar{y}$ of the reduced system:

$$\begin{aligned}|y-\bar{y}\|_{L_2(t_0,\infty)} \leq \|LA^{-1}x(t_0)\|_2^2+&\|\sqrt{\Sigma_1}(\bar{A})^{-1}\left(x_1(t_0)-A_{12}A_{22}^{-1}x_2(t_0)\right)\|_2^2\\ &+2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(t_0,\infty)}\end{aligned} \tag{4.39}$$

where $u \in L_2(t_0,\infty)$

Proof. Observe that

$$\begin{aligned}\|G-\bar{G}\|_{L_2(t_0,\infty)} &\leq \|G-\hat{G}\|_{L_2(t_0,\infty)}+\|\hat{G}-\hat{G}^r\|_{L_2(t_0,\infty)}+\|\hat{G}^r-\bar{G}\|_{L_2(t_0,\infty)}\\ &\leq \|\hat{G}-\hat{G}^r\|_{L_2(t_0,\infty)}\end{aligned}$$

but from section (4.3), we know that the error bound is given as:

$$\begin{aligned}\|\hat{G}-\hat{G}^r\|_{L_2(t_0,\infty)} &= \frac{\|\hat{y}-\hat{y}^r\|_{L_2(t_0,\infty)}}{\|u\|_{L_2(t_0,\infty)}} \\ &\leq \|L\hat{A}x(t_0)\|_2^2+\|\sqrt{\Sigma_1}\hat{A}_{11}\left(x_1(t_0)-A_{12}A_{22}^{-1}x_2(t_0)\right)\|_2^2+2\sum_{i=r+1}^{n}\sigma_i\end{aligned}$$

Then we have:

$$\begin{aligned}\|G-\bar{G}\|_{L_2(t_0,\infty)} &\leq \|LA^{-1}x(t_0)\|_2^2+\|\sqrt{\Sigma_1}(\bar{A})^{-1}\left(x_1(t_0)-A_{12}A_{22}^{-1}x_2(t_0)\right)\|_2^2 \\ &\quad +2\sum_{i=r+1}^{n}\sigma_i \\ \|y-\bar{y}\|_{L_2(t_0,\infty)} &\leq \|LA^{-1}x(t_0)\|_2^2+\|\sqrt{\Sigma_1}(\bar{A})^{-1}\left(x_1(t_0)-A_{12}A_{22}^{-1}x_2(t_0)\right)\|_2^2 \\ &\quad +2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(t_0,\infty)}\end{aligned}$$

□

In the case when the full system is balanced , we have the following Corollary to obtain the error bound between the outputs of the original and its reduced order system.

Corollary 4.4.4. If the system $\left(\begin{array}{c|c} A & B \\ \hline C & \end{array}\right)$ is balanced with

$$\Sigma = \begin{pmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \Sigma_2 \end{pmatrix}$$

where

$$\Sigma_1 = diag(\sigma_1,\cdots,\sigma_r)$$
$$\Sigma_2 = diag(\sigma_{r+1},\cdots,\sigma_n)$$

$\sigma_1 \geq \sigma_1 \geq \cdots \geq \sigma_n) > 0$ are the Hankel singular values, and reduced system $\left(\begin{array}{c|c} \bar{A} & \bar{B} \\ \hline \bar{C} & \end{array}\right)$ is balanced with Σ_1, then the error bound between the outputs y of the original system and $\bar{y}$ of thr reduced order system is:

$$\|y-\bar{y}\|_{L_2(t_0,\infty)}\leq \|\sqrt{\Sigma}\|_2^2\|A^{-1}\|_2^2\|x(t_0)\|_2^2+2\sum_{i=r+1}^{n}\sigma_i\|u\|_{L_2(t_0,\infty)} \tag{4.40}$$

for all $u \in L_2(t_0,\infty)$

Proof. By refering to section (4.3) and using the idea of proof (4.3.2), we can prove the corollary. □

Chapter 5

Optimal Control

In this chapter we introduce one of the most important methods in control problems that is *The Linear Quadratic Regulator*(LQR). We are interested in the case of the linear quadratic regulator with constrained states and inputs [19]. For closed-loop, we want to use the LQR to find an optimal control that minimizes the objective function which called "the quadratic cost function" with respect to the constraints on the states and the control input. In order to do that we have used formal asymptotes for the Pontryagin maximum principle (PMP) and we introduce an approach using the so called *The Hamiltonian Function* and the underlying algebraic Riccati equation. The outcome of this chapter are case description under which balanced truncation and the singular perturbation approximation give good closed-loop performance.

5.1 Linear quadratic regulator optimal control (LQR)

We start by considering the following continuous linear dynamical system defined as:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned} \tag{5.1}$$

where A,B and C are a constant matrices defined in chapter(2) section (2.1), and x, u are the state and the input of the system respictively and $x(0)$ represents the initial condition .

We assume that the linear system described by equation (5.1) is controllable and observable.

The quadratic cost function J is defined by the following equation :

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt \tag{5.2}$$

or, equivalently

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{5.3}$$

where $Q = C^T C \geq 0$ is a positive semi definite matrix representing the cost penalty of the states and $R > 0$ is a positive definite matrix that represents the cost penalty of the input.
We want to find an optimal control u that minimizes the quadratic cost function J subject to the constraint

$$\dot{x} = Ax + Bu$$

The optimal control can be denoted by u^* such that:

$$J(u^*) \leq J(u), \quad \forall u \in L^2$$

and the constraint equation $\dot{x} = Ax + Bu$ has a solution.
If we substitute the value of u^* in the coanstraint equation, we have that:

$$\dot{x} = Ax + Bu^*$$

and the optimal solution of this equation is denoted by x^*.
Now, we introduce an approach that depends on the Hamiltonian function defined in the following form:

$$H = \frac{1}{2}\left(x^T Qx + u^T Ru\right) + \lambda^T (Ax + Bu) \tag{5.4}$$

where $\lambda \in \Re^n$ is called the costate variable.
The following theorem describes the way in which we can find the optimal control that minimizes the quadratic cost function J in equation (5.2) and (5.3).

Theorem 5.1.1. [21, 31](Maximum Principle) If x^*,u^* is optimal (or a solution of the LQR), then there exists a solution $\lambda^* \in \Re^n$ such that:

$$\dot{x} = \frac{\partial H}{\partial \lambda} \tag{5.5}$$

$$\dot{\lambda} = -\frac{\partial H}{\partial x} \tag{5.6}$$

and the minimality condition of the Hamiltonian

$$H(x^*, u^*, \lambda^*) \leq H(x^*, u, \lambda^*)$$

holds for all $u \in \Re^m$

For more etails on the proof (see [21, 31]).
If H is a differentiable function, then to minimize H with respect to u we can find

our optimal control input.
The following condition must be true to find such u that is:

$$\frac{\partial H}{\partial u} = 0 \tag{5.7}$$

if we solve equation (5.7), we obtain the following control:

$$u = -R^{-1}B^T\lambda \tag{5.8}$$

From (5.1.1) and (5.8), we have the following canonical differential equations that form a linear system (or Hamiltonian system) written as:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial \lambda} \\ &= Ax - BR^{-1}B^T\lambda, \quad x(0) = x_0 \\ \dot{\lambda} &= -\frac{\partial H}{\partial x} \\ &= -Qx - A^T\lambda \end{aligned} \tag{5.9}$$

Since the terminal cost is not defined, then there is no constraint on the final value of λ.
This is a coupled system, linear in x and λ, of order $2n \times 2n$.
These control equations can be written in matrix form as:

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} \tag{5.10}$$

It is not easy to solve the system described in equation (5.10), so we guess the solution of this system or the relation between x and λ in the form:

$$\lambda = Px \tag{5.11}$$

where $P \in \Re^{n\times n}$.
We introduce now an important differential equation in the linear quadratic regulator problem that is called *Matrix Riccati Equation* (MRE) and to derive this

equation, we start from equation (5.11) and use (5.9) in the following way:

$$\begin{aligned}\lambda &= Px \\ \dot{\lambda} &= \dot{P}x + P\dot{X} \\ -Qx - A^T\lambda &= \dot{P}x + P(Ax - BR^{-1}B^T\lambda) \\ -Qx - A^TPx &= \dot{P}x + PAx - PBR^{-1}B^TPx \\ \dot{P}x + PAx + A^TPx - PBR^{-1}B^TPx + Qx &= 0\end{aligned}$$

From the final step, we obtain the MRE written as:

$$\dot{P} = -PA - A^TP + PBR^{-1}B^TP - Q \tag{5.12}$$

Since we have an infinite time horizon, there is no information about the terminal cost and hence λ has no constraint. In this case the steady state solution P of a so called Algebraic Riccati Equation (ARE) can be used instead of $P(t)$ [31].
In case when the time approaches infinty, we have:

$$\lim_{t \longrightarrow \infty} \dot{P} = 0$$

By using the limit above, we get another differential equation called *Algebraic Riccati Equation* (ARE), written as:

$$PA + A^TP - PBR^{-1}B^TP + Q = 0 \tag{5.13}$$

where P is the unique positive-definite solution.
We want now to find a state feedback control u that can be used to move any state x to the origin, so we let the system evolve in a closed-loop [15, 31].
If we find the solution P of the ARE (5.13), then the optimal control u that can be used to minimize the quadratic cost function J is written as:

$$u = -R^{-1}B^TPx \tag{5.14}$$

By substituting equation (5.14) into the original system described by equation (5.1), we get the following equation:

$$\dot{x} = (A - R^{-1}B^TP)x \tag{5.15}$$

Since the matrix $A - BK$ is stable, we have closed-loop poles formed by the eigenvalues of this matrix [15].
If we solve equation (5.15) and find the optimal solution x, then we can find our optimal control u that can be used to find a minimum value of the quadratic cost function J described in equation (5.2) (5.3).
We can summarized the LQR method as follows:

1. We start with the linear dynamical system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \\ x(0) &= x_0 \end{aligned}$$

2. We assume that this system is controllable.

3. We define the quadratic cost function as:

$$J = \frac{1}{2}\int_0^\infty (x^T Q x + u^T R u)dt$$

4. We choose $Q = Q^T \geq 0$ such that $Q = C^T C$ and $R = R^T > 0$

5. We find the constant solution P of the ARE:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0$$

6. We find the optimal control u such that:

$$u = -R^{-1}B^T Px$$

7. We write the original system in the form:

$$\dot{x} = (A - R^{-1}B^T P)x$$

5.2 Optimal control for reduced order model of different types

5.2.1 Singular perturbation requlator problem of type(1)

In this section we introduce the linear quadratic regulator problem for the reduced order model of a dynamical system [22].
Our goal is to find an optimal control for the reduced system using the singular perturbation a pproximation.
Consider the linear time-invariant dynamical system defined as:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \tag{5.16}$$

This system can be written in another form as :

$$\begin{array}{rcl} \dot{x} & = & A_{11}x + A_{12}z + B_1u \\ \epsilon\dot{z} & = & A_{21}x + A_{22}z + B_2u \end{array} \tag{5.17}$$

From section (5.1), we see that this system can be optimized according to the following quadratic cost function:

$$J = \frac{1}{2}\int_0^\infty (y^T y + u^T Ru)dt \tag{5.18}$$

or

$$J = \frac{1}{2}\int_0^\infty (x^T Qx + u^T Ru)dt \tag{5.19}$$

where $Q = C^T C \geq 0$ and $R > 0$.
The optimal control u is defined as:

$$u = -R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{5.20}$$

where P is the solution of the Algebraic Riccati Equation (ARE):

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{5.21}$$

The goal now is to solve the ARE and set $\epsilon = 0$ to obtain a reduced equation for the ARE.
If we substitute the matrices A,B,C and Q in equation (5.21), then we have the following new form of ARE:

$$\begin{array}{l} P\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix} P \\ -P\begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix} P + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{array} \tag{5.22}$$

A solution of equation (5.22) can be choosen as:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \tag{5.23}$$

so we can avoid the unboundness when we set $\epsilon \longrightarrow 0$ [22].
Substituting equation (5.23) into equation (5.22), we get :

$$\begin{aligned} &\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon}A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon}A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &- \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ \frac{1}{\epsilon}B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\epsilon}B_2^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &+ \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{5.24}$$

From equation (5.24), we get the following $(n+m)\times(n+m)$ equations:

$$\begin{aligned} 0 &= P_{11}A_{11} + P_{12}A_{21} + A_{11}^T P_{11} + A_{21}^T P_{12}^T - (P_{11}B_1 + P_{12}B_2)R^{-1}(B_1^T P_{11} + B_2^T P_{12}^T) \\ &\quad + C_1^T C_1 \end{aligned} \tag{5.25}$$

$$\begin{aligned} 0 &= P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^T P_{12} + A_{21}^T P_{22} - (P_{11}B_1 + P_{12}B_2)R^{-1}(\epsilon B_1^T P_{12} + B_2^T P_{22}) \\ &\quad + C_1^T C_2 \end{aligned} \tag{5.26}$$

$$\begin{aligned} 0 &= \epsilon P_{12}^T A_{11} + P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T - (\epsilon P_{12}^T B_1 + P_{22}B_2)R^{-1}B_1^T P_{11} + B_2^T P_{12}^T) \\ &\quad + C_2^T C_1 \end{aligned} \tag{5.27}$$

$$\begin{aligned} 0 &= \epsilon P_{12}^T A_{12} + P_{22}A_{22} + \epsilon A_{12}^T P_{12} + A_{22}^T P_{22} - (\epsilon P_{12}^T B_1 + P_{22}B_2)R^{-1}\epsilon B_1^T P_{12} + B_2^T P_{22}) \\ &\quad + C_2^T C_2 \end{aligned} \tag{5.28}$$

When we set $\epsilon = 0$ in equations (5.25)-(5.28) we obtain the following $m \times m$ reduced equation for $\bar{P}_{22}$ and written as:

$$\bar{P}_{22}A_{22} + A_{22}^T \bar{P}_{22} - \bar{P}_{22}W\bar{P}_{22} + C_2^T C_2 = 0 \tag{5.29}$$

where $W = B_2 R^{-1} B_2^T$.
Another $n \times n$ equation for $\bar{P}_{11}$ is obtained when we express $\bar{P}_{12}$ in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ and this equation takes the form:

$$\bar{P}_{11}\hat{A} + \hat{A}^T \bar{P}_{11}^T - \bar{P}_{11}\hat{B}R^{-1}\hat{B}^T \bar{P}_{11} + \hat{C}^T \hat{C} = 0 \tag{5.30}$$

where $\hat{A}$,$\hat{B}$ and $\hat{C}$ are defined in [25]. If $(\hat{A}, \hat{B})$ is controllable pair and $(\hat{A}, \hat{C})$ is observable pair, then applying the implicit function theorem to equation (5.22) with equation (5.23) [22, 23], we have:

$$P_{ij} = \bar{P}_{ij} + O(\epsilon), \quad i,j = 1,2 \tag{5.31}$$

If we use $\bar{P}_{ij}$ instead of P_{ij} in equation (5.31), then the feedback control in equation (5.20) becomes:

$$\begin{aligned} u &= -R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\epsilon} B_2^T \end{pmatrix} \begin{pmatrix} \bar{P}_{11} & \epsilon \bar{P}_{12} \\ \epsilon \bar{P}_{12}^T & \epsilon \bar{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})x - R^{-1}(\epsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})z \end{aligned} \tag{5.32}$$

From equation (5.32), the original system described by equation (5.16) becomes:

$$\begin{aligned} \dot{x} &= \Big(A_{11} - B_1 R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})\Big) x + \Big(A_{12} - B_1 R^{-1}(\epsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})\Big) z \\ \epsilon \dot{z} &= \Big(A_{21} - B_2 R^{-1}(B_1^T \bar{P}_{11} + B_2^T \bar{P}_{12})\Big) x + \Big(A_{22} - B_2 R^{-1}(\epsilon B_1^T \bar{P}_{12} + B_2^T \bar{P}_{22})\Big) z \end{aligned} \tag{5.33}$$

If this system is asymptotically stable then from equation (5.31), we have a solution $x(t)$ and $z(t)$ with $O(\epsilon)$ of the optimal solution [24]. If we assume that A_{22} is stable, then we can apply this assumption to the feedback system in equation (5.33).
If we reduce the full system in equation (5.17) using the singular approximation approximation, we obtain the following reduced order model:

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r u_r \\ y_r &= C_r x_r + D_r u_r \end{aligned} \tag{5.34}$$

where

$$\begin{aligned} A_r &= A_{11} - A_{12} A_{22}^{-1} A_{21} \\ B_r &= B_1 - A_{12} A_{22}^{-1} B_2 \\ C_r &= C_1 - C_2 A_{22}^{-1} A_{21} \\ D_r &= -C_2 A_{22}^{-1} B_2 \end{aligned}$$

We define the cost quadratic function of this reduced order system as:

$$J_r = \frac{1}{2} \int_0^\infty (y_r^T y_r + u_r^T R_r u_r) dt \tag{5.35}$$

or, equivalently

$$J_r = \frac{1}{2} \int_0^\infty (x_r^T Q_r x_r + 2 x_r^T C_r D_r u_r + u_r^T R_r u_r) dt \tag{5.36}$$

where $Q_r = C_r^T C_r$ and $R_r = R + D_r^T D_r$.
The optimal control for this reduced system defined as:

$$u_r = -R_r^{-1} B_r^T P_r x_r \tag{5.37}$$

where P_r is the constant solution of the following *Algebraic Riccati Equation* for the reduced system described by equations (5.34) given as:

$$P_r(A_r - B_r R_r^{-1} D_r^T C_r) + (A_r - B_r R_r^{-1} D_r^T C_r)^T P_r - P_r B_r R_r^{-1} B_r^T P_r + C_r^T (I + D_r R_r D_r^T)^{-1} C_r = 0 \quad (5.38)$$

We introduce now the following theorem that describes the relationship between the reduced Riccati Equation system (5.29)(5.30) for the full system (5.17) after putting $\epsilon = 0$ and the Riccati Equation (5.39) for the reduced system in equation (5.34) when we set $\epsilon = 0$

Theorem 5.2.1. If equation (5.31) holds and A_{22}^{-1} exists, then the solution P_r of equation (5.39) is identical to the solution $\bar{P}_{11}$ of equation (5.30)

For more details, see [23, 24].
According to theorem (5.2.1) and if we substitute the feedback optimal control u_r described by equation (5.37) into the reduced system equation (5.34), then we obtain the following system:

$$\dot{x}_r = (A_r - B_r R^{-1} B_r^T P_r) x_r \quad (5.39)$$

where $(A_r - B_r R^{-1} B_r^T P_r)$ is stable and the pair (A_r, B_r) controllable.
If we find the optimal solution x_r (5.39) and substitute the value into equation (5.37), then we find the optimal control for the reduced order model.

5.2.2 Singular perturbation requlator problem of type(2)

In this subsection, we introduce a linear dynamical continuous system with input matrix B that does not depends on ϵ. We want to find the optimal control for this dynamical system and then use the singular perturbation approximation to reduce this system and find the optimal control for the reduced order model.
Let us consider the following linear dynamical continuous system defined as:

$$\begin{aligned} \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} &= \begin{pmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\epsilon} & \frac{A_{22}}{\epsilon} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \end{aligned} \quad (5.40)$$

Another representation of the above system could be written as:

$$\begin{aligned} \dot{x} &= A_{11} x + A_{12} z + B_1 u \\ \epsilon \dot{z} &= A_{21} x + A_{22} z + \epsilon B_2 u \end{aligned} \quad (5.41)$$

If we assume that A_{22} is stable and A_{22}^{-1} exists, then we set $\epsilon = 0$ to obtain the following equation:

$$\bar{z} = A_{22}^{-1} A_{21} \bar{x} \tag{5.42}$$

When we substitute equation (5.42) into equation (5.41), we get the following reduced order model:

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{5.43}$$

where

$$\begin{aligned} \bar{A} &= A_{11} - A_{12} A_{22}^{-1} A_{21} \\ \bar{B} &= B_1 \\ \bar{C} &= C_1 - C_2 A_{22}^{-1} A_{21} \end{aligned}$$

Our goal now is to find the optimal control for the system in equation (5.40) that minimizes the quadratic cost function J defined by the following equations:

$$J = \frac{1}{2} \int_0^\infty (y^T y + u^T R u) dt \tag{5.44}$$

or equivalently

$$J = \frac{1}{2} \int_0^\infty (x^T Q x + u^T R u) dt \tag{5.45}$$

where $Q = C^T C \geq 0$ and $R > 0$.
The feedback optimal control u for the original system is defined as:

$$u = -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{5.46}$$

where P is the solution of the Algebraic Differential Equation defined below:

$$\begin{aligned} & P \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon} A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon} A_{22}^T \end{pmatrix} P \\ & - P \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P + \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{5.47}$$

We choose the solution of equation (5.47) as:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \tag{5.48}$$

to avoid the unboundness for $\epsilon = 0$.
Equation (5.48) together with equation (5.47) give the following equation:

$$\begin{aligned} &\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\epsilon} A_{21} & \frac{1}{\epsilon} A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & \frac{1}{\epsilon} A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon} A_{22}^T \end{pmatrix} \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &- \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \begin{pmatrix} B_1 \\ \frac{1}{\epsilon} B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & \frac{1}{\epsilon} B_2^T \end{pmatrix} \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &+ \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix} \begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{5.49}$$

Form equation (5.49), we obtain the following set of equations:

$$\begin{aligned} 0 \;=\;& P_{11}A_{11} + P_{12}A_{21} + A_{11}^T P_{11} + A_{21}^T P_{12}^T - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(B_1^T P_{11} + \epsilon B_2^T P_{12}^T) \\ &+ C_1^T C_1 \end{aligned} \tag{5.50}$$

$$\begin{aligned} 0 \;=\;& P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^T P_{12} + A_{21}^T P_{22} - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) \\ &+ C_1^T C_2 \end{aligned} \tag{5.51}$$

$$\begin{aligned} 0 \;=\;& \epsilon P_{12}^T A_{11} + P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}\epsilon B_1^T P_{11} + \epsilon B_2^T P_{12}^T) \\ &+ C_2^T C_1 \end{aligned} \tag{5.52}$$

$$\begin{aligned} 0 \;=\;& \epsilon P_{12}^T A_{12} + P_{22}A_{22} + \epsilon A_{12}^T P_{12} + A_{22}^T P_{22} - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) \\ &+ C_2^T C_2 \end{aligned} \tag{5.53}$$

When we set $\epsilon = 0$ in equations (5.50)-(5.53) we obtain the following reduced Riccati equations:

$$\bar{P}_{11}A_{11} + \bar{P}_{12}A_{21} + A_{11}^T \bar{P}_{11}^T + A_{21}^T \bar{P}_{12}^T - \bar{P}_{11}B_1 R^{-1} B_1^T \bar{P}_{11} + C_1^T C_1 = 0 \tag{5.54}$$

$$\bar{P}_{11}A_{12} + \bar{P}_{12}A_{22} + A_{21}^T \bar{P}_{22} + C_1^T C_2 = 0 \tag{5.55}$$

$$\bar{P}_{22}A_{21} + A_{12}^T \bar{P}_{11} + A_{22}^T \bar{P}_{12}^T + C_2^T C_1 = 0 \tag{5.56}$$

$$\bar{P}_{22}A_{22} + A_{22}^T \bar{P}_{22} + C_2^T C_2 = 0 \tag{5.57}$$

We write $\bar{P}_{12}$ and $\bar{P}_{12}^T$ in equations (5.55),(5.56) in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$ as follows:

$$\bar{P}_{12} = -(\bar{P}_{11}A_{12} + A_{21}^T \bar{P}_{22} + C_1^T C_2)A_{22}^{-1} \tag{5.58}$$

$$\bar{P}_{12}^T = -(A_{22}^T)^{-1}(\bar{P}_{22}A_{21} + A_{12}^T \bar{P}_{11} + C_2^T C_1) \tag{5.59}$$

Equation (5.57) can be expressed in different form as:

$$A_{21}^T(A_{22}^T)^{-1}\bar{P}_{22}A_{21} + A_{21}^T\bar{P}_{22}A_{22}^{-1}A_{21} = -A_{21}^T(A_{22}^T)^{-1}C_2^TC_2A_{22}^{-1}A_{21} \tag{5.60}$$

Substituting equations (5.58) and (5.59) into equation (5.54) and using equation (5.60) we obtain:

$$\bar{P}_{11}\hat{A} + \hat{A}^T\bar{P}_{11} - \bar{P}_{11}\hat{B}R^{-1}\hat{B}^T\bar{P}_{11} + \hat{C}^T\hat{C} = 0 \tag{5.61}$$

where

$$\begin{aligned} \hat{A} &= A_{11} - A_{12}A_{22}^{-1}A_{21} \\ \hat{B} &= B_1 \\ \hat{C} &= C_1 - C_2A_{22}^{-1}A_{21} \end{aligned} \tag{5.62}$$

If we assume the pair $(\hat{A}, \hat{B})$ is controllable, then the values of P_{ij} and $\bar{P}_{ij}$, $i,j = 1,2$ satisfy equation (5.31).
The feedback optimal control defined in equation (5.46) along with the result in equation (5.31) can be written as:

$$\begin{aligned} u &= -R^{-1}\begin{pmatrix} B_1^T & B_2^T \end{pmatrix}\begin{pmatrix} \bar{P}_{11} & \epsilon\bar{P}_{12} \\ \epsilon\bar{P}_{12}^T & \epsilon\bar{P}_{22} \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1}(B_1^T\bar{P}_{11} + \epsilon B_2^T\bar{P}_{12})x - R^{-1}(\epsilon B_1^T\bar{P}_{12} + \epsilon B_2^T\bar{P}_{22})z \end{aligned} \tag{5.63}$$

We can use the result found in equation (5.63) to write a new representation of the original system described by equation (5.41) as:

$$\begin{aligned} \dot{x} &= \Big(A_{11} - B_1R^{-1}(B_1^T\bar{P}_{11} + \epsilon B_2^T\bar{P}_{12})\Big)x + \Big(A_{12} - B_1R^{-1}(\epsilon B_1^T\bar{P}_{12} + \epsilon B_2^T\bar{P}_{22})\Big)z \\ \epsilon\dot{z} &= \Big(A_{21} - \epsilon B_2R^{-1}(B_1^T\bar{P}_{11} + \epsilon B_2^T\bar{P}_{12})\Big)x + \Big(A_{22} - \epsilon B_2R^{-1}(\epsilon B_1^T\bar{P}_{12} + \epsilon B_2^T\bar{P}_{22}\Big)z \end{aligned} \tag{5.64}$$

If the system in equation (5.64) is asymptotically stable and if equation (5.31)) holds, then we can compute the solution $x(t)$ and $z(t)$ within the $O(\epsilon)$ of the optimal control.
The next step now is to find a feedback optimal control for the reduced system defined in equation (5.43) that can be used to minimizes the quadratic cost function $\bar{J}$ defined as:

$$\bar{J} = \frac{1}{2}\int_0^\infty (\bar{y}^T\bar{y} + \bar{u}T\bar{R}\bar{u})dt \tag{5.65}$$

or equivalently

$$\bar{J} = \frac{1}{2}\int_0^\infty (\bar{x}^T\bar{Q}\bar{x} + \bar{u}^T\bar{R}\bar{u})dt \tag{5.66}$$

where $\bar{Q} = \bar{C}^T \bar{Q} \geq 0$ and $\bar{R} = R > 0$.
We define the optimal control for the reduced system (5.43) as:

$$\bar{u} = -\bar{R}^{-1} \bar{B}^T \bar{P} \bar{x} \tag{5.67}$$

where $\bar{P}$ is the solution of the following Algebraic Riccati Equation for the reduced system in equation (5.43), defined as:

$$\bar{P}\bar{A} + \bar{A}^T \bar{P} - \bar{P}\bar{B}\bar{R}^{-1}\bar{B}^T \bar{P} + \bar{C}^T \bar{C} = 0 \tag{5.68}$$

Since A_{22} is stable and A_{22}^{-1} is exists, then the solution of equation (5.68) is the same as the solution of equation (5.61), thus we have:

$$\bar{P} = \bar{P}_{11} \tag{5.69}$$

By usig the feedback optimal control in equation (5.67) and the solution $\bar{P}$ in equation (5.68), then we obtain the following reduced system derived from the reduced system in equation (5.43) that has the form:

$$\begin{aligned} \dot{\bar{x}} &= \left(\bar{A} - \bar{B}\bar{R}^{-1}\bar{B}^T \bar{P}\right) \bar{x} \\ \bar{y} &= \bar{C}\bar{x} \end{aligned} \tag{5.70}$$

where

$$\begin{aligned} \bar{A} &= A_{11} - A_{12} A_{22}^{-1} A_{21} \\ \bar{B} &= B_1 \\ \bar{C} &= C_1 - C_2 A_{22}^{-1} A_{21} \end{aligned}$$

We assume that the matrix $\bar{A} - \bar{B}\bar{R}^{-1}\bar{B}^T \bar{P}$ is stable and the pairs $(\bar{A}, \bar{B})$, $(\bar{A}, \bar{C})$ are controllable and observable respectively.
By solving the reduced system in equation (5.70), the solution $\bar{x}(t)$ is used to find the feedback control $\bar{u}$ which is important to find the minimum value of the quadratic cost function $\bar{J}$.

5.2.3 Singular perturbation requlator problem of type(3)

In section (5.2), we applied the singular perturbation linear quadratic regulator to find an optimal control for the reduced system.
In this section we introduce an approach to find the optimal control of the reduced system using the Balance Truncation optimal control.
Consider the full linear time-invariant dynamical system defined by the following

form:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u$$
$$y = \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \tag{5.71}$$

We can rewrite the original system in equation (5.71) in another form as :

$$\begin{aligned} \dot{x} &= A_{11}x + A_{12}z + B_1 u \\ \epsilon\dot{z} &= \epsilon A_{21}x + A_{22}z + \epsilon B_2 u \end{aligned} \tag{5.72}$$

If we apply the balanced truncation method to reduce the original system described by equation (5.72), we get the following reduced system form:

$$\begin{aligned} \dot{x}_r &= A_{11}x_r + B_1 u_r \\ y_r &= C_1 x_r \end{aligned} \tag{5.73}$$

Moreover, we can apply the singular perturbation approximation method to reduce the original system in equation (5.72) to obtain the reduced system:

$$\begin{aligned} \dot{\bar{x}} &= A_{11}\bar{x} + B_1\bar{u} \\ \bar{y} &= C_1\bar{x} \end{aligned} \tag{5.74}$$

From equations (5.73) and (5.74), we see the the two reduced systems have the same state space equation and this means that to find an optimal control for the reduced system in equation (5.73) using the balanced truncation method, we can use the singular perturbation method described in section (5.2).
We start by defining the quadratic cost function J for the original system (5.71) as:

$$J = \frac{1}{2}\int_0^{\infty} (y^T y + u^T R u)dt \tag{5.75}$$

or equivalently

$$J = \frac{1}{2}\int_0^{\infty} (x^T Q x + u^T R u)dt \tag{5.76}$$

where $Q = C^T C \geq 0$ and $R > 0$.
Our optimal control u for the original system is defined as :

$$u = -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} P \begin{pmatrix} x \\ z \end{pmatrix} \tag{5.77}$$

The matrix P is the solution of the follwoing Algebraic Riccati Equation:

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{5.78}$$

The next step now is to find a reduced Riccati equation for the full Riccati equation (5.78) when $\epsilon = 0$.
To avoid the unboundness when $\epsilon = 0$, we choose the solution P in the form:

$$P = \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \tag{5.79}$$

By substituting equation (5.79) into equation (5.78), we get:

$$\begin{aligned} &\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & \frac{1}{\epsilon}A_{22} \end{pmatrix} + \begin{pmatrix} A_{11}^T & A_{21}^T \\ A_{12}^T & \frac{1}{\epsilon}A_{22}^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &- \begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix}\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix}\begin{pmatrix} P_{11} & \epsilon P_{12} \\ \epsilon P_{12}^T & \epsilon P_{22} \end{pmatrix} \\ &+ \begin{pmatrix} C_1^T \\ C_2^T \end{pmatrix}\begin{pmatrix} C_1 & C_2 \end{pmatrix} = 0 \end{aligned} \tag{5.80}$$

After solving equation (5.80), we obtain the following equations:

$$\begin{aligned} 0 &= P_{11}A_{11} + \epsilon P_{12}A_{21} + A_{11}^T P_{11} + \epsilon A_{21}^T P_{12}^T - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(B_1^T P_{11} + \epsilon B_2^T P_{12}^T) \\ &\quad + C_1^T C_1 \end{aligned} \tag{5.81}$$

$$\begin{aligned} 0 &= P_{11}A_{12} + P_{12}A_{22} + \epsilon A_{11}^T P_{12} + \epsilon A_{21}^T P_{22} - (P_{11}B_1 + \epsilon P_{12}B_2)R^{-1}(\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) \\ &\quad + C_1^T C_2 \end{aligned} \tag{5.82}$$

$$\begin{aligned} 0 &= \epsilon P_{12}^T A_{11} + \epsilon P_{22}A_{21} + A_{12}^T P_{11} + A_{22}^T P_{12}^T - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}B_1^T P_{11} + \epsilon B_2^T P_{12}^T) \\ &\quad + C_2^T C_1 \end{aligned} \tag{5.83}$$

$$\begin{aligned} 0 &= \epsilon P_{12}^T A_{12} + P_{22}A_{22} + \epsilon A_{12}^T P_{12} + A_{22}^T P_{22} - (\epsilon P_{12}^T B_1 + \epsilon P_{22}B_2)R^{-1}\epsilon B_1^T P_{12} + \epsilon B_2^T P_{22}) \\ &\quad + C_2^T C_2 \end{aligned} \tag{5.84}$$

Now, if we set $\epsilon = 0$ in equations (5.81)-(5.84), we obtain the following reduced system Riccati Equations :

$$\bar{P}_{11}A_{11} + A_{11}^T \bar{P}_{11}^T - \bar{P}_{11}B_1R^{-1}B_1^T \bar{P}_{11} + C_1^T C_1 = 0 \tag{5.85}$$

$$\bar{P}_{11}A_{12} + \bar{P}_{12}A_{22} + C_1^T C_2 = 0 \tag{5.86}$$

$$A_{21}^T \bar{P}_{11} + A_{22}^T \bar{P}_{12} + C_2^T C_1 = 0 \tag{5.87}$$

$$\bar{P}_{22} A_{22} + A_{22}^T \bar{P}_{22} + C_2^T C_2 = 0 \tag{5.88}$$

Assumptions 5.2.2. The pair (A_{11}, B_1) is controllable and $\bar{P}_{11}$ is a unique positive semidefinite solution of equation (5.85) such that:

$$A_{11} - B_1 R^{-1} B_1^T \bar{P}_{11}$$

is stable.

According to equation (5.31) in section (5.2.1), we can use $\bar{P}_{ij}$ instead of P_{ij} to rewrite the feedback control in equation (5.77) as:

$$\begin{aligned} u &= -R^{-1} \begin{pmatrix} B_1^T & B_2^T \end{pmatrix} \begin{pmatrix} \bar{P}_{11} & \epsilon \bar{P}_{12} \\ \epsilon \bar{P}_{12}^T & \epsilon \bar{P}_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \\ &= -R^{-1}(B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12})x - R^{-1}(\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22})z \end{aligned} \tag{5.89}$$

Using equation (5.89), we obtain a new form of the original system described by equation (5.72) such that:

$$\begin{aligned} \dot{x} &= \Big(A_{11} - B_1 R^{-1}(B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12})\Big)x + \Big(A_{12} - B_1 R^{-1}(\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22})\Big)z \\ \epsilon \dot{z} &= \Big(\epsilon A_{21} - \epsilon B_2 R^{-1}(B_1^T \bar{P}_{11} + \epsilon B_2^T \bar{P}_{12})\Big)x + \Big(A_{22} - \epsilon B_2 R^{-1}(\epsilon B_1^T \bar{P}_{12} + \epsilon B_2^T \bar{P}_{22})\Big)z \end{aligned} \tag{5.90}$$

If the above system is asymptotically stable and equation (5.31) holds, then we have a solution $x(t)$ and $z(t)$ for this system with $O(\epsilon)$ of the optimal solution [24]. We are going now to define the quadratic cost function for the reduced order model system described in equation (5.74) or (5.74).
Let $\bar{J}$ be the quadratic cost function of the reduced system in equation (5.73) or (5.74) defined as:

$$\bar{J} = \frac{1}{2} \int_0^\infty (\bar{y}^T \bar{y} + \bar{u}^T \bar{R} \bar{u}) dt \tag{5.91}$$

or, equivalently

$$\bar{J} = \frac{1}{2} \int_0^\infty (\bar{x}^T \bar{Q} \bar{x} + \bar{u}^T \bar{R} \bar{u}) dt \tag{5.92}$$

where $\bar{Q} = \bar{C}^T \bar{C} \geq 0$ and $\bar{R} = R > 0$.
The optimal feedback control for the reduced order model is defined as:

$$\bar{u} = -\bar{R}^{-1} \bar{B}^T \bar{P} \bar{x} \tag{5.93}$$

where $\bar{P}$ is the solution of the Algebraic Riccati Equation for the reduced order model and given as:

$$\bar{P}A_1 + A_1^T\bar{P} - \bar{P}B_1\bar{R}^{-1}B_1^T\bar{P} + C_1^TC_1 = 0 \tag{5.94}$$

From theorem (5.2.1) in section (5.2.1), we see that the two solutions $\bar{P}_{11}$ and $\bar{P}$ are both identical.
Hence we conclude that $\bar{P}_{11}$ is the reduced Riccati Equation (5.85) and it is the same as $\bar{P}$ which is the solution of the reduced system.
By substituting the feedback control equation (5.93) into the reducd system (5.73), we get:

$$\dot{\bar{x}} = (A_{11} - B_1R^{-1}B_1^T\bar{P})x_r \tag{5.95}$$

where we have assumed that the matrix $(A_{11} - \bar{B}R^{-1}B_1^T\bar{P})$ is stable.
If we solve equation (5.95) of the reduced system, then we can use the solution $x(t)$ to find the optiaml control. This optimal control can be used to find the optimality of $\bar{J}$.

Chapter 6

Numerical Examples

In this chapter the construction of a low order model via balanced truncation and singular perturbation approximation for the mass spring damping and CD-player systems is demonstrated.

6.1 Mass-spring damping system

In this section we introduce a numerical example describing the behavior of the dynamical system.
As an application from the engeneering system, we take a mass-spring damping system.
For simplicity we start with three mass-spring damping and apply Newton's Second Law of motion to this masses.
Suppose that m_1,m_2 and m_3 are the masses described in figure (6.1)

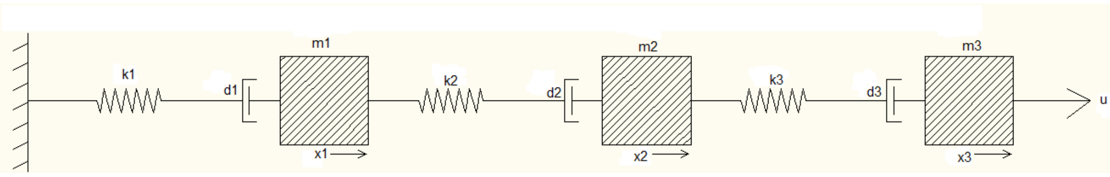


FIGURE 6.1: Three Mass-Spring Damping

where x_1,x_2 and x_3 are the positions of the masses m_1,m_2 and m_3 respectively and k_1,k_2,k_3 and d_1,d_2,d_3 are constants that represent the stiffness and the damping of the springs with u is the force acting on the mass m_3 .
For the mass m_1 in figure (6.2)

Applying Newton's Second Law we get the following differential equation:

$$\begin{aligned} m_1\ddot{x}_1 + d_1\dot{x}_1 + k_1x_1 + d_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) &= 0 \\ m_1\ddot{x}_1 + (d_1 + d_2)\dot{x}_1 + d_2\dot{x}_2 + (k_1 + k_2)x_1 - k_2x_2) &= 0 \end{aligned} \quad (6.1)$$

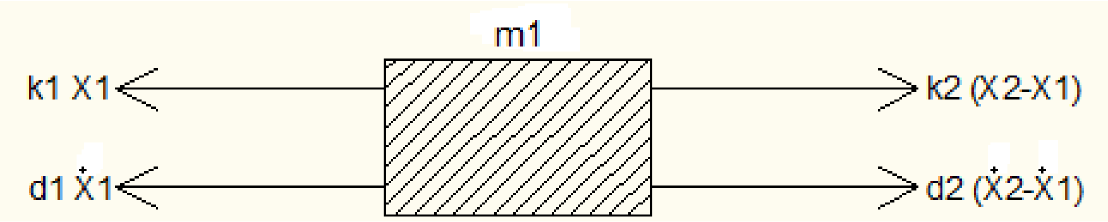


FIGURE 6.2: Mass1

Similarly for the mass m_2 in figure (6.3), we obtain:

$$\begin{aligned} m_2\ddot{x}_2 + d_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + d_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) &= 0 \\ m_2\ddot{x}_2 - d_2\dot{x}_1 + (d_2 + d_3)\dot{x}_2 - d_3\dot{x}_3 - k_2x_1 + (k_2 + k_3)x_2 - k_3x_3) &= 0 \end{aligned} \tag{6.2}$$

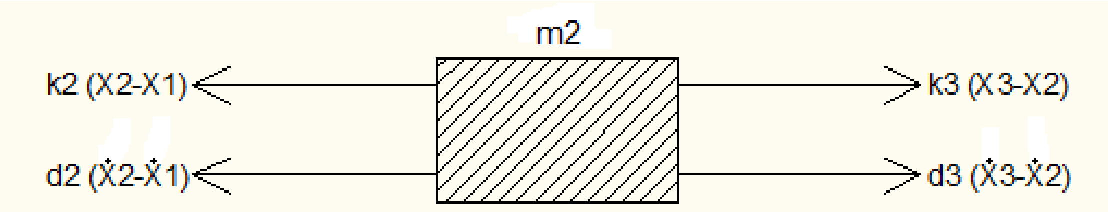


FIGURE 6.3: Mass2

Finally, the differential equation for the mass m_3 in figure (6.4) is:

$$\begin{aligned} m_3\ddot{x}_3 + d_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) &= u \\ m_3\ddot{x}_3 - d_3\dot{x}_2 + d_3\dot{x}_3 - k_3x_2 + k_3x_3 &= u \end{aligned} \tag{6.3}$$

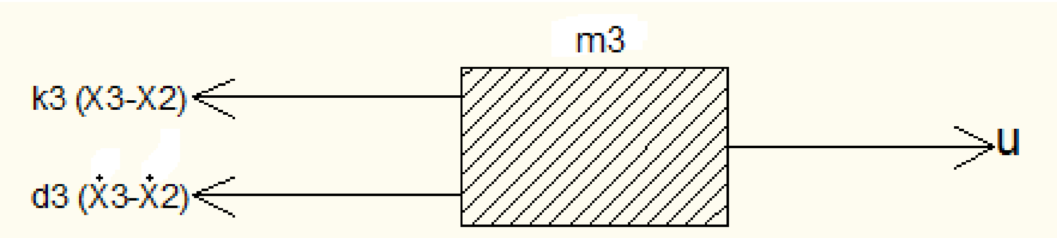


FIGURE 6.4: Mass3

The set of the differential equations in (6.1),(6.2) and (6.3) can be written in matrix form as:

$$\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} + \begin{pmatrix} d_1+d_2 & -d_2 & 0 \\ -d_2 & d_2+d_3 & -d_3 \\ 0 & -d_3 & d_3 \end{pmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} + \begin{pmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \tag{6.4}$$

and the differential equation that represents this system has the form:

$$M\ddot{x} + D\dot{x} + Kx = Lu \tag{6.5}$$

where M is the mass matrix, D is the damping matrix and K is the stiffnes matrix. and

$$L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

is an 3×1 colmun vector.
To find the state space equation for the previous linear continuous dynamical system we let:

$$\dot{x} = z$$

and

$$\ddot{x} = \dot{z}$$

if we assume that M^{-1} exists, then by substituting the above equations into equation (6.5), we get the following system:

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= -M^{-1}Kx - M^{-1}Dz + M^{-1}Lu \end{aligned} \tag{6.6}$$

and in matrix form we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -M^{-1}L \end{pmatrix} u \tag{6.7}$$

Let $A = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$ be of size (6×6) and $B = \begin{pmatrix} \mathbf{0} \\ -M^{-1}L \end{pmatrix}$ of size (6×1), then the state space equation for this system is :

$$\dot{X} = AX + Bu \tag{6.8}$$

where $X = \begin{pmatrix} x \\ z \end{pmatrix}$ is the state vector of the linear dynamical system of size (6×1).
Now, we are going to find the state space representation for any n mass-spring damping continuous system described in figure (6.5).
Using similar approach, we can derive the state space equation for n masses and apply Newton's Second Law on the mass m_i to obtain the following differential equation:

$$\begin{aligned} m_i\ddot{x}_i - d_i\dot{x}_{i-1} + (d_i + d_{i+1})\dot{x}_i - d_{i+1}\dot{x}_{i+1} - k_i x_{i-1} \\ + k_{i+1}(x_{i+1} - x_i) = bu \end{aligned} \tag{6.9}$$

where $i = 1, 2, 3, \ldots, n$
The value of b is zero when $i \neq n$ and one when $i = n$, but $x_0 = 0$ for $i = 1$ and $x_{n+1} = k_{n+1} = d_{n+1} = 0$ for $i = n$.
The matrix representation for the differential equation described by equation (6.9) can be expressed as:

$$M\ddot{x} + D\dot{x} + Kx = Lu \tag{6.10}$$

where

$$M = \begin{pmatrix} m_1 & 0 & 0 & \ldots & \ldots & 0 \\ 0 & m_2 & 0 & \ldots & \ldots & 0 \\ 0 & 0 & \ddots & 0 & \ldots & 0 \\ \vdots & \vdots & 0 & m_i & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & m_n \end{pmatrix}_{n \times n}$$

is called the mass matrix of the system.

$$D = \begin{pmatrix} d_1 + d_2 & -d_2 & 0 & \ldots & \ldots & 0 \\ -d_2 & d_2 + d_3 & -d_3 & 0 & \ldots & 0 \\ 0 & -d_3 & d_3 + d_4 & \ddots & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -d_n \\ 0 & 0 & 0 & \ldots & -d_n & d_n \end{pmatrix}_{n \times n}$$

is called the damping matrix and

$$K = \begin{pmatrix} k_1 + k_2 & -k_2 & 0 & \ldots & \ldots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 & \ldots & 0 \\ 0 & -k_3 & k_3 + k_4 & \ddots & \ldots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & -k_n \\ 0 & 0 & 0 & \ldots & -k_n & k_n \end{pmatrix}_{n \times n}$$

is called the stiffnes matrix. Finally, the vector

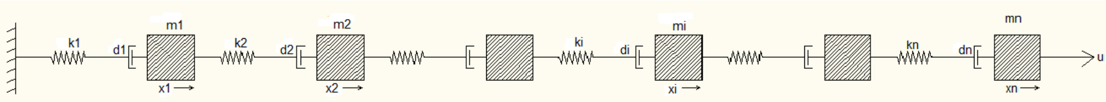


FIGURE 6.5: Multi Mass-Spring Damping

$$L = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}_{n\times 1}$$

representing the number of controllers that act on masses.
To find the state space equation to the linear continuous system, we let:

$$\dot{x} = z$$

and

$$\ddot{x} = \dot{z}$$

by substituting these equations into equation (6.10), we have the following system of differential equations:

$$\begin{aligned} \dot{x} &= z \\ \dot{z} &= -M^{-1}Kx - M^{-1}Dz + M^{-1}Lu \end{aligned} \tag{6.11}$$

and in matrix form, we have:

$$\begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & -M^{-1}D \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -M^{-1}L \end{pmatrix} u \tag{6.12}$$

if we let

$$A = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -M^{-1}K & -M^{-1}D \end{pmatrix}$$

of size $(2n \times 2n)$
and

$$B = \begin{pmatrix} \mathbf{0} \\ -M^{-1}L \end{pmatrix}$$

of size $(2n \times 1)$, then the state space equation for this system is :

$$\dot{X} = AX + Bu \tag{6.13}$$

where $X = \begin{pmatrix} x \\ z \end{pmatrix}$ is the state vector of size $(2n \times 1)$.
The following step is to find the H^∞ and L^2 bounds of the approximation error for the open-loop dynamical system. The numerical results are shown in section (6.3)

6.2 CD-player

In this section we introduce the CD-player as an application to a compact disc mechanism. The CD-player control task is to achieve track following, which basically amounts to pointing the laser spot to the track of pits on the CD that is rotating [43]. Figure (6.6) shows the mechanism treated here which consists of a swing arm on which a lens is mounted by means of two horizontal leaf spring. The rotation of the arm in the horizontal plane enables reading of the spiral-shaped disc track, and the suspended lens is used to focus the spot on the disc.

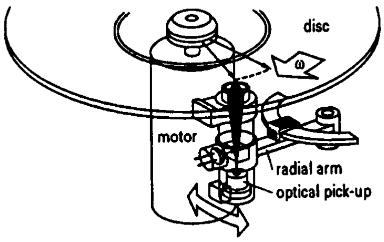


FIGURE 6.6: Schematic view of a rotating arm compact disc mechanism

We have two facts according to the disc, the first fact is that the disc is not perfectly flat and the second fact is the disc has irregularities in the spiral of pits on it. From these two facts, a feedback system is needed.
Our goal is to find a low-cost controller which makes the servo-system faster and less sensitive to external shocks [12, 44].
A detailed model is needed to describe the vibrational behaviour of the electromechanical system over a large frequency range in order to anticipate the interaction with a controller of possible high bandwidth [45].
The CD-player finite element model was built to describe the dynamics between the lens actuator and radial arm position of a portable compact disc player discussed in [46].
This model contains 60 vibration modes (n=120), and has two inputs (actuation of arm and of focus lens), and two outputs (tracking error and focus error) [45].
See section (6.3) for numerical results.

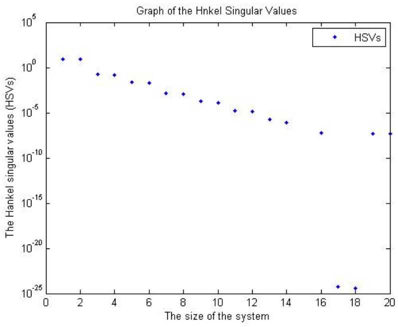
6.3 Numerical results

In this section we include all results obtained by the two approaches, namely; the balanced truncation (BT) and the singular perturbation approximation (SPA) techniques to determine the order of the reduced models.

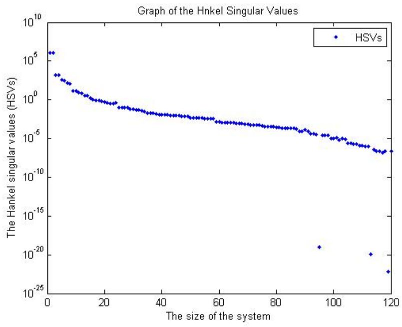
Open-Loop System:

We start by computing the Hankel singular values of the two dynamical systems illustrated in sections (6.1) and (6.2). Figures (6.7a) and (6.7b) represent the Hankel singular values (HSVs) for the mass-spring damping and the CD-player system of size $N_s = 10$ and $N_c = 120$ respectively.

For testing purposes, we apply the balanced truncation method for the two



(A) HSVs of the mass-spring damping



(B) HSVs of the CD-player

FIGURE 6.7: HSVs of the mass-spring damping and CD-player system

examples with zero-initial condition and compute the H^{∞} bound of the approximation error described in section (3.4) equation (3.21). The size of the mass-spring damping system is taken to be $N_s = 10$ and the size of the reduced model is $r_s = 2$. Figure (6.8) shows the maximum singular value decomposition (MSVD) σ_{max} of $(G - G_r)$, where G is the transfer function of the original system and G_r is the transfer function of the reduced order model, and the error bound is $2\sum_{i=3}^{10}\sigma_i$. Table (6.1) contains the values of $\|G - G_r\|_{\infty}$ and $2\sum_{i=r+1}^{10}\sigma_i$ computed for $N_s = 10$ and various values of r_s by applying the balanced truncation and singular perturbation approximation to the mass-spring damping system. To find an error bound for the CD-player, we take the size of the system to be $N_c = 120$ and for the reduced model is $r_c = 14$. By applying the balanced truncation, the maximum singular value decomposition of $(G - G_r)$ and the error bound $2\sum_{i=9}^{120}\sigma_i$ are shown in figure (6.9). For the singular perturbation approximation method, the two figures (6.10a) and (6.10b) describe the maximum singular value decomposition of $(G - G_r)$ and the

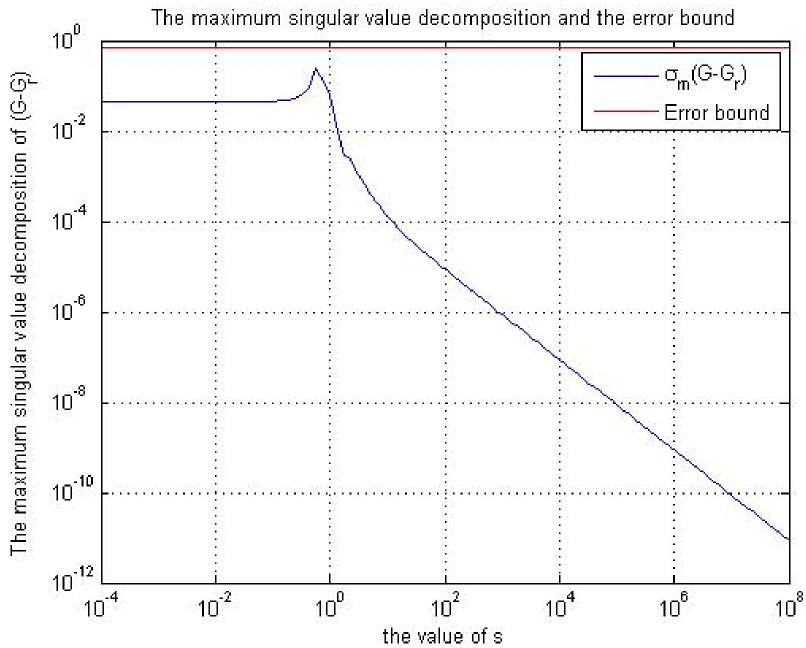


FIGURE 6.8: The MSVD and the error bound for the mass-spring damping for balanced truncation

TABLE 6.1: The H^{∞} norm of $(G - G_r)$ and the error bound.

r_s	$\|G-G_r\|_{\infty}$ by BT	$\|G-G_r\|_{\infty}$ by SPA	$2\sum\limits_{i=r+1}^{10}\sigma_i$
2	0.2368	0.2147	0.7025
4	0.0343	0.0374	0.0873
6	0.0026	0.0023	0.0061
8	3.1691×10^{-4}	2.6374×10^{-4}	6.7187×10^{-4}
10	3.0066×10^{-5}	2.5247×10^{-5}	6.4759×10^{-5}

error bounds $2\sum\limits_{i=3}^{10}\sigma_i$ and $2\sum\limits_{i=15}^{120}\sigma_i$ for the mass-spring damping and the CD-player systems respectively. Table (6.2) contains the values of $\|G-G_r\|_{\infty}$ and the error bound $2\sum\limits_{i=r+1}^{120}\sigma_i$ computed for $N_c = 120$ and various of r_c by using the balanced truncation and singular perturbation approximation techniques to the CD-player system.

We see clearly that both the balanced truncation and singular approximation methods give the same error bound in terms of the Hankel singular values. Furthermore, the balanced truncation method yields a reduced order model with smaller error at high frequencies and gives a larger error at low frequencies. Whereas, the singular perturbation approximation produces a reduced order model with an error

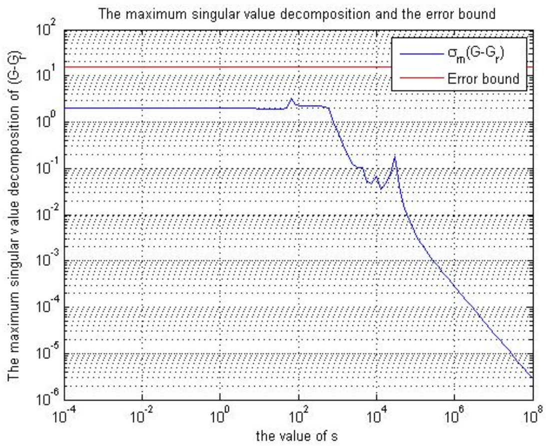
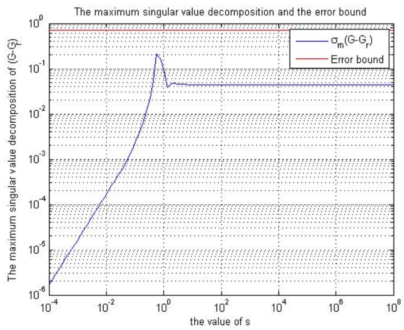
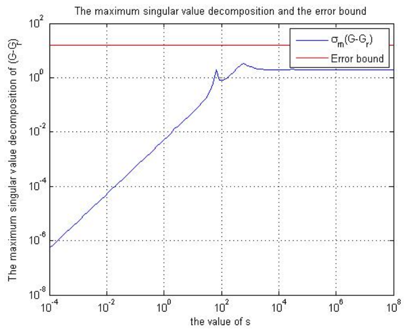


FIGURE 6.9: The MSVD and the error bound for the CD-player for balanced truncation



(A) The MSVD and the error bound for the mass-spring damping for SPA



(B) The MSVD and the error bound for the CD-player for SPA

FIGURE 6.10: The H^∞norm of $(G-G_r)$ and the error bound

tends to zero at low frequencies but the error becomes larger at high frequencies. Next, we want to compute the L^2 bound of the approximation error between the output y of the original system and the output y_r of the reduced system with non-zero initial condition. By applying the balanced truncation and singular perturbation approximation of the reduced order model, we have the formulas for the error bound (see section (4.1) equation (4.10)) and we denote it by $Error_{antb}$, the error bound (see section (4.2) equation (4.22)) denoted by $Error_{meb}$ and the error bound (see section (4.4) equation (4.39)) denoted by $Error_{spa}$.
Figures (6.11) and (6.12) contain the output y of the original system, the output y_r of the reduced model and the difference $y-y_r$. For the mass spring damping, let $N_s = 10$, $r_s = 2$ and for the CD-player $N_c = 120$ and $r_c = 14$.

The L^2 norm of $(y-y_r)$ can be computed for different r_s . Table (6.3) and (6.4) contain the values of $\|y-y_r\|_{L_2}$ and the error bounds for the mass-spring damping

TABLE 6.2: The H^∞ norm of $(G-G_r)$ and the error bound.

r_c	$\|G-G_r\|_\infty$ by BT	$\|G-G_r\|_\infty$ by SPA	$2\sum_{i=r+1}^{120}\sigma_i$
2	976.9395	$1.1726\times10^{+3}$	$8.8112\times10^{+3}$
4	472.2863	564.2177	$2.1307\times10^{+3}$
6	269.9229	266.4908	658.1466
8	22.2140	21.9202	117.6033
10	10.5536	10.9294	63.0871
12	3.1911	3.3050	30.4559
14	3.3078	3.3658	15.9412
16	1.4026	1.7800	10.3588

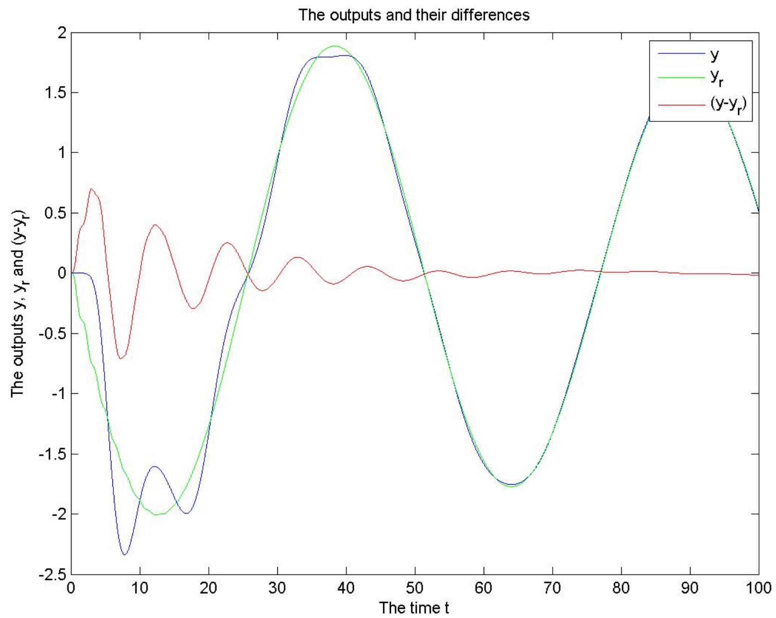


FIGURE 6.11: The outputs of the mass-spring damping for BT

and the CD-player systems.
We have the following table (6.4) that contains the $\|y-y_r\|_{L_2}$ norm and the error bounds for the CD-player system.
By applying the singular perturbation approximation technique to the two dynamical systems, we obtain the same plots in (6.11) and (6.12). From table (6.3) and (6.4), we see clearly that the balanced truncation and singular perturbation approximation methods give the same error bounds of the $\|y-y_r\|_{L_2}$ norm
Closed-Loop System:
To find an optimal control U_1 for the original system and U_r for the reduced order

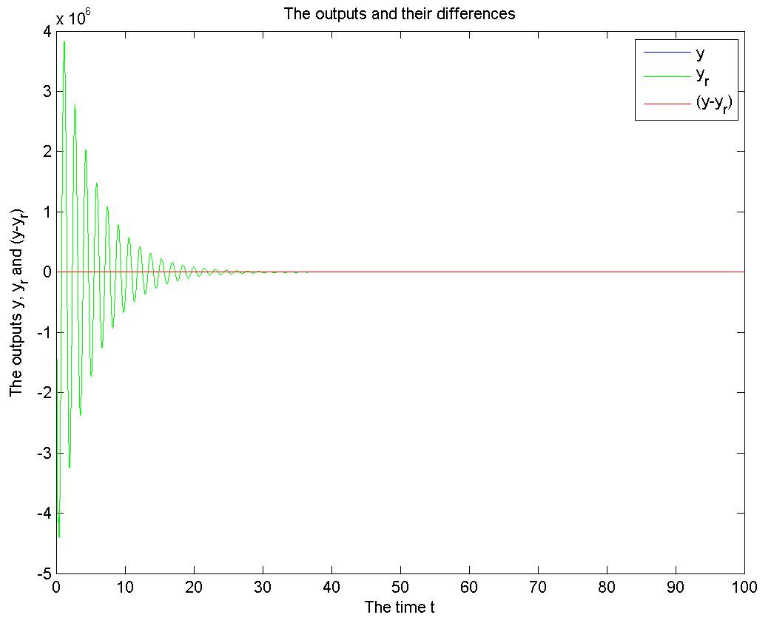


FIGURE 6.12: The outputs of the CD-player for BT

TABLE 6.3: The L^2 norm of $y - y_r$ and the error bounds of the mass-spring damping.

r_s	$\|y - y_r\|_{L_2}$ BT	$\|y - y_r\|_{L_2}$ SPA	$Error_{antb}$	$Error_{meb}$	$Error_{spa}$
2	5.1168×10^{-6}	1.4448×10^{-7}	0.7860	0.7844	0.7845
4	8.4209×10^{-9}	1.3092×10^{-10}	0.0979	0.0975	0.0975
6	8.6789×10^{-11}	7.8119×10^{-11}	0.0069	0.0068	0.0068
8	5.2488×10^{-11}	2.3580×10^{-13}	7.6560×10^{-4}	7.5046×10^{-4}	7.5161×10^{-4}
10	2.0546×10^{-13}	6.2379×10^{-15}	7.5386×10^{-5}	7.2544×10^{-5}	7.3692×10^{-5}

system, we apply the balanced truncation and the singular perturbation approximation methods to the mass damping system example. The size of the system is $N_s = 10$ and the size of the reduced model is $r_s = 2$. The optimal control is computed by using the results in sections (5.2) and (5.2.3). The solution of the Riccati equation P of the full system is computed and used to find the value of U_1. We apply the approaches in sections (5.2) and (5.2.3) to find the solution of the Riccati equation P_r of the reduced system. Since the first block P_{11} of P is equal to the value of P_r, so we can extended P using P_r as the first block and the rest blocks are zero to obtain a new solution of the Riccati equation denoted by $\tilde{P}_{11}$.

Another optimal control for the full system U_2 is found using the value of $\tilde{P}_{11}$, and hence we compute the $\|U_1 - U_2\|_{L_2}$ norm. Figure (6.13) represent the plots of the

TABLE 6.4: The L^2 norm of $y - y_r$ and the error bounds of the CD-player.

r_c	$\|y-y_r\|_{L_2}$ BT	$\|y-y_r\|_{L_2}$ SPA	$Error_{antb}$	$Error_{meb}$	$Error_{spa}$
2	1.9758×10^{-7}	530.9562	$1.4274\times10^{+4}$	$1.3932\times10^{+4}$	$1.3932\times10^{+4}$
4	4.7835×10^{-8}	13.6649	$3.5043\times10^{+3}$	$3.3690\times10^{+3}$	$3.3690\times10^{+4}$
6	3.7973×10^{-8}	13.6595	$1.1025\times10^{+3}$	$1.0407\times10^{+3}$	$1.0406\times10^{+3}$
8	1.1489×10^{-8}	1.0136×10^{-4}	205.5871	186.0009	185.9503
10	1.0735×10^{-8}	2.6581×10^{-4}	112.7179	99.8032	99.7527
22	6.7444×10^{-9}	1.4723×10^{-6}	6.7908	5.2021	5.1515
30	1.8227×10^{-9}	4.9888×10^{-8}	2.0099	1.3307	1.2801

two optimal controls U_1,U_2 and $(U_1 - U_2)$ using the balanced truncation and the singular approximation perturbation.

Finally, Table (6.5) contains the values of $\|U_1 - U_2\|_{L_2}$ and $\|P_{11} - \tilde{P}_{11}\|_{L_2}$ by

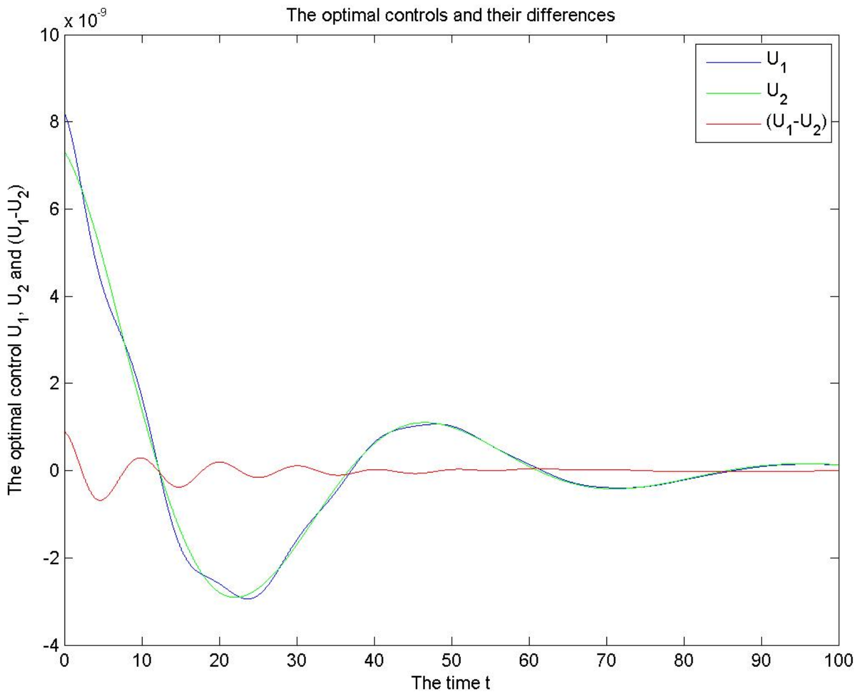


FIGURE 6.13: The optimal controls of the mass-spring damping

applying the balanced truncation and singular perturbation approximation to the mass-spring damping.

TABLE 6.5: The L^2 norm of $(U_1 - U_2)$ and $(P_{11} - \tilde{P}_{11})$ of the mass-spring.

r_s	$\|U_1 - U_2\|_{L_2}$ BT	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$	$\|U_1 - U_2\|_{L_2}$ SPA	$\|P_{11} - \tilde{P}_{11}\|_{L_2}$
2	4.0511×10^{-21}	0.0159	$4.2643\ \times10^{-21}$	0.0193
4	3.1877×10^{-23}	0.0049	3.4627×10^{-12}	0.0035
6	2.7572×10^{-25}	6.4136×10^{-4}	3.1813×10^{-25}	2.1445×10^{-4}
8	2.2720×10^{-28}	6.9710×10^{-4}	7.0385×10^{-27}	3.8847×10^{-5}
10	4.0902×10^{-28}	1.1015×10^{-4}	1.1291×10^{-29}	6.8224×10^{-6}

6.4 Conclusion

In this thesis we have studied balanced model reduction techniques for linear control systems, specifically balanced truncation and singular perturbation approximation. These methods have been successfully applied for system with homogeneous initial conditions but only little attention has been paid to systems with inhomogeneous initial conditions or feedback systems.
For open-loop control problems, we have derived an L^2 error bound for balanced truncation and singular perturbation approximation for system with non-homogeneous initial condition. The theoretical results have been validated numerically with extensive comparison between different systems and balanced truncation and singular perturbation model reduction.
For closed-loop, one of the most important methods in control problems called linear quadratic regulator (LQR) has been introduced. This is used to find an optimal control that minimizes the quadratic cost function. The formal calculations are validated by numerical experiments, illustrating that the reduced-order can be used to approximate the optimal control of the original system.
Finally, our suggestion for future work is to apply the balanced truncation and singular perturbation approximation methods to the linear dynamical systems with finite-time horizon and derive an error bound for stable systems. The two methods can be applied to obtain optimal control to the reduced order model for the infinite time-horizon. Moreover, another suggestion will be to derive an error bound and optimal control to the unstable linear dynamical system with finite or infinite time-horizon using the balanced truncation and singular perturbation approximation methods.

Bibliography

[1] O. Alvarez and M. Bardi, *Viscosity solutions methods for singular perturbations in deterministic and stochastic control*, SIAM journal on control and optimization, 40 (2002), pp. 1159–1188.

[2] A. C. Antoulas, *Approximation of large-scale dynamical systems*, Advances in design and control, Society for Industrial and Applied Mathematics, 2005.

[3] D. Bakry and M. Émery, *Hypercontractivité de semi-groupes de diffusion*, Comptes rendus des séances de l'Académie des sciences. Série 1, Mathématique, 299 (1984), pp. 775–778.

[4] A. Bensoussan and G. Blankenship, *Singular perturbations in stochastic control*, in Singular Perturbations and Asymptotic Analysis in Control Systems, Springer, 1987, pp. 171–260.

[5] N. Bose, *Special issue on multidimensional systems*, Proc. IEEE, 65 (1977), pp. 3–7.

[6] D. N. Burghes and A. Graham, *Control and Optimal Control Theories with Applications*, Woodhead Publishing, 2004.

[7] R. Condon, Marissa; Ivanov, *Nonlinear systems – algebraic gramians and model reduction*, COMPEL Int J for Computation and Maths in Electrical and Electronic Eng, 24 (2005), pp. 202–219.

[8] R. F. Curtain and K. Glover, *"balanced realisations for infinite dimensional systems",in operator theory and system*, Proc.Workshop Amesterdam, (1986), pp. 87–104.

[9] ——, *An Introduction to Infinite-Dimensional Systems*, Springer-Verlag. New York, 1995.

[10] B. Datta, *Numerical Methods for Linear Control Systems*, Academic Press, 2003.

[11] J. J. D'Azzo, C. H. Houpis, and S. N. Sheldon, *Linear Control System Analysis and Design with Matlab*, Control engineering series 14, M. Dekker, 5th ed., rev. and expanded ed., 2003.

[12] W. Draijer, M. Steinbuch, and O. H. Bosgra, *Adaptive control of the radial servo system of a compact disc player*, Automatica, 28 (1992), pp. 455–462.

[13] D. F. Enns, *Model reduction with balanced realizations: An error bound and a frequency weighted generalization*, in Decision and Control, 1984. The 23rd IEEE Conference on, IEEE, 1984, pp. 127–132.

[14] L. C. Evans, *The perturbed test function method for viscosity solutions of nonlinear pde*, Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 111 (1989), pp. 359–375.

[15] S. A. Ghoreishi, M. A. Nekoui, and S. O. Basiri, *Optimal design of lqr weighting matrices based on intelligent optimization methods*, International Journal of Intelligent Information Processing, 2 (2011).

[16] K. Glover, *All optimal hankel-norm approximations of linear multivariable systems and their l_∞-error bounds†*, International journal of control, 39 (1984), pp. 1115–1193.

[17] K. Glover, R. F. Curtain, and J. R. Partington, *Realisation and approximation of linear infinite-dimensional systems with error bounds*, SIAM Journal on Control and Optimization, 26 (1988), pp. 863–898.

[18] C. Hartmann, B. Schäfer-Bung, and A. Thöns-Zueva, *Balanced averaging of bilinear systems with applications to stochastic control*, SIAM Journal on Control and Optimization, 51 (2013), pp. 2356–2378.

[19] M. Hassan and E. Boukas, *Constrained linear quadratic regulator: continuous-time case*, Nonlinear Dynamics and Systems Theory, 8 (2008), pp. 35–42.

[20] M. Heinkenschloss, T. Reis, and A. Antoulas, *Balanced truncation model reduction for systems with inhomogeneous initial conditions*, Automatica, 47 (2011), pp. 559–564.

[21] G. E. Knowles, *An Introduction to Applied Optimal Control*, Mathematics in Science and Engineering 159, Academic Press, 1981.

[22] P. V. Kokotovic, *Singular perturbations in optimal control*, JOURNAL OF MATHEMATICS, 6 (1976).

[23] ——, *Applications of singular perturbation techniques to control problems*, SIAM review, 26 (1984), pp. 501–550.

[24] P. V. Kokotovic, R. E. O'malley, and P. Sannuti, *Singular perturbations and order reduction in control theory—an overview*, Automatica, 12 (1976), pp. 123–132.

[25] P. V. Kokotovic and R. A. Yackel, *Singular perturbation of linear regulators: basic theorems*, Automatic Control, IEEE Transactions on, 17 (1972), pp. 29–37.

[26] P.-L. Lions and P. E. Souganidis, *Correctors for the homogenization of hamilton-jacobi equations in the stationary ergodic setting*, Communications on pure and applied mathematics, 56 (2003), pp. 1501–1524.

[27] Y. Liu and B. D. Anderson, *Singular perturbation approximation of balanced systems*, International Journal of Control, 50 (1989), pp. 1379–1405.

[28] Y. LIU and B. D. O. ANDERSON, *Singular perturbation approximation of balanced systems*, International Journal of Control, 50 (1989), pp. 1379–1405.

[29] S. K. Mitra and M. P. Ekstrom, *Two-dimensional digital signal processing*, vol. 20, Hutchinson Ross Publishing Company, 1978.

[30] N. Monshizadeh, H. L. Trentelman, and M. Kanat Camlibel, *Stability and synchronization preserving model reduction of multi-agent systems*, Systems & Control Letters, 62 (2013), pp. 1–10.

[31] R. M. Murray, *Optimization-based control*, California Institute of Technology, CA, (2009).

[32] G. Muscato, G. Nunnari, and L. Fortuna, *Singular perturbation approximation of bounded real balanced and stochastically balanced transfer matrices*, International Journal of Control, 66 (1997), pp. 253–270.

[33] R. O'Malley, Jr, *The singularly perturbed linear state regulator problem*, SIAM Journal on Control, 10 (1972), pp. 399–413.

[34] R. E. O'Malley Jr, *On two methods of solution for a singularly perturbed linear state regulator problem*, SIAM Review, 17 (1975), pp. 16–37.

[35] C. W. ROWLEY, *Model reduction for fluids, using balanced proper orthogonal decomposition*, International Journal of Bifurcation and Chaos, 15 (2005), pp. 997–1013.

[36] H. Sandberg, *Model reduction of linear systems using extended balanced truncation*, in American Control Conference, 2008, IEEE, 2008, pp. 4654–4659.

[37] R. Saragih and Fatmawati, *Singular perturbation approximation of balanced infinite-dimentional system*, International Journal of Control and Automation, 6 (2013), pp. 409–420.

[38] A. Sasane, *Hankel norm approximation for infinite-dimensional systems*, vol. 277, Springer Science & Business Media, 2002.

[39] B. Schäfer-Bung, C. Hartmann, B. Schmidt, and C. Schütte, *Dimension reduction by balanced truncation: Application to light-induced control of open quantum systems*, The Journal of chemical physics, 135 (2011), p. 014112.

[40] J. Scherpen, *Balancing for nonlinear systems*, Systems & Control Letters, 21 (1993), pp. 143–153.

[41] D. Škatarić and N. Kovačević-Ratković, *The system order reduction via balancing in view of the method of singular perturbation*, FME Transactions, 38 (2010), pp. 181–187.

[42] S. Skogestad and I. Postlethwaite, *Multivariable feedback control: analysis and design*, vol. 2, Wiley New York, 2007.

[43] M. Steinbuch, P. J. Van Groos, G. Schootstra, P. M. Wortelboer, and O. H. Bosgra, *μ-synthesis for a compact disc player*, International Journal of Robust and Nonlinear Control, 8 (1998), pp. 169–189.

[44] P. M. Wortelboer, M. Steinbuch, and O. H. Bosgra, *Closed-loop balanced reduction with application to a compact disc mechanism*, Selected Topics in Identification, Modeling and Control, 9 (1996), pp. 47–58.

[45] ——, *Iterative model and controller reduction using closed-loop balancing, with application to a compact disc mechanism*, International Journal of Robust and Nonlinear Control, 9 (1999), pp. 123–142.

[46] P. M. R. Wortelboer, *Frequency-weighted balanced reduction of closed-loop mechanical servo-systems: theory and tools*, TU Delft, Delft University of Technology, 1994.

[47] D. Xue, Y. Chen, and D. P. Atherton, *Linear feedback control : analysis and design with MATLAB*, Advances in design and control, Society for Industrial and Applied Mathematics, 2007.

[48] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*, Prentice Hall, 1998.