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The Soundproof Model of an Acoustic–internal Waves System with Low Stratification

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Abstract. This work is devoted to investigating a compressible fluid system with low stratification, which is driven by fast acoustic waves and internal waves. The approximation using a soundproof model is justified. More precisely, the soundproof model captures the dynamics of both the non-oscillating mean flows and the oscillating internal waves, while filters out the fast acoustic waves, of the compressible system with or without initial acoustic waves. Moreover, the fast-slow oscillation structure is investigated.

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Keywords. Internal waves, Acoustic waves, Low stratification, Soundproof approximation.

1. Introduction

1.1. Motivations

The rigorous justification of the anelastic and pseudo-incompressible models for atmospheric flows [3, 11, 13] in the inviscid case remains a challenge for at least three technical reasons: First, under realistic conditions for the troposphere, the compressible flow model involves three asymptotically separated time scales, associated with advection (slow), internal gravity waves (intermediate), and acoustics (fast), respectively. The two sound-proof models still involve the slow and intermediate scales, see [10], and thus still depend on the scale separation parameter. In other words, the anelastic and pseudo-incompressible models are not "limit models" in the classical sense, e.g., of low Mach number analysis. The technical question to be rigorously answered therefore is: What is the relation between the compressible three-scale and the sound-proof pseudo-incompressible (or anelastic) two-scale models.

Secondly, realistic atmospheric background states feature temperatures and local Brunt–Väisälä or buoyancy frequencies that depend on the spatial position. This leaves the fast linear system describing acoustic and internal wave modes with non-constant, space-dependent coefficients. The control of derivatives for non-constant coefficient systems using techniques of energy estimates is substantially more difficult than it is in the constant coefficient case.

Thirdly, problems on the torus or in $\mathbb{T}^d$ ($d \in 2, 3$) are often technically easier to handle than bounded domain problems, except when the bounded domain problem has a natural extension through certain symmetries to the infinite or toroidal domain case. Owing to the presence of gravity, realistic atmospheric flows always include a bottom boundary of the critical type that does not lend itself to domain extensions that would preserve smoothness of solutions across the eliminated domain boundary.

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In this paper we make progress in addressing the first issue, i.e., model reduction from three to two asymptotically separated scales, while we avoid the non-constant coefficient problem and irregular behavior of solutions near the (bottom) boundary of the domain by introducing judicious simplifications in the original model, designed to render the physics of the scale interactions largely intact: Let us denote R the gas constant of the fluid, T the background temperature and g the typical gravitational acceleration, respectively. Then by (i) considering a fluid layer much thinner than the pressure scale height $h_{sc} = RT/g$, we guarantee that the leading-order temperature, and with it the leading order speed of sound, are constant. By (ii) assuming a particular vertical stratification of entropy (or potential temperature), we guarantee that the buoyancy (or Brunt–Väisälä) frequency is constant as well (see (H3) in page 6). This renders the linear fast system describing acoustic and internal waves with constant coefficients. Finally, by (iii) letting the gravitational acceleration decay to zero smoothly towards the top and bottom domain boundaries, while maintaining a constant buoyancy frequency by choice of the entropy stratification, we obtain a problem that has a regular extension to a vertically periodic domain problem [see (H1) and (H2) in page 6].

Under these conditions, our main results can be stated in an informal fashion as follows:

Theorem. *Consider the full compressible model [(17) in page 7] with both acoustic and internal waves, and the pseudo-incompressible model [(20) in page 7].*

- *Without initial acoustic waves, solutions of the compressible and pseudo-incompressible models remain asymptotically close over the slowest (advective) time scale as the small parameter representative of the Mach and Froude numbers vanishes. See Theorem 1 in page 8 for the detailed statement;*
- *Moreover, in the case with initial acoustic waves, the solutions of the pseudo-incompressible model capture the dynamics of the mean flows and the internal waves in the compressible model. See Theorem 2 in page 10 for more details.*

More details are given in the following section describing the relation between the compressible model (4) and the pseudo-incompressible system (20) through system (17).

An explanatory remark regarding our use of the notion of the "pseudo-incompressible" model is in order: By the assumption of a shallow domain, the formal leading order divergence constraint emerging from the pressure evolution equation is the *incompressibility constraint* $\nabla \cdot v = 0$ rather than the pseudo-incrompressibility constraint $\nabla \cdot (Pv) = 0$, where $P(z)$ is a function of the vertical coordinate only. We nevertheless speak of the pseudo-incompressible model in the last paragraph because we show in Sect. 4 that first order pseudo-incompressibility effects are important when closeness of the compressible and soundproof approximations are to be maintained over the slow advective time scale. In fact, in that section we study the intermediate model (51) in page 19, which we anticipate here in a notation similar to that of [10] which is likely more familiar to readers of the meteorological literature,

$$\frac{Dv}{Dt} + \nabla_h \pi = 0, \tag{1a}$$

$$\frac{Dw}{Dt} + \partial_z \pi = \frac{\theta}{\varepsilon^\nu}, \tag{1b}$$

$$\frac{D\theta}{Dt} = \frac{S^\varepsilon}{\varepsilon^\nu} w, \tag{1c}$$

$$\mathrm{div}_h\,(P^\varepsilon v) + \partial_z (P^\varepsilon w) = 0. \tag{1d}$$

where

$$\frac{D}{Dt} = \partial_t + v \cdot \nabla_h + w\,\partial_z\,, \tag{2}$$

and

$$P^\varepsilon(z) = 1 - \varepsilon \widetilde{P}^\varepsilon(z)\,, \qquad S^\varepsilon(z,\theta) = S_0^\varepsilon + \varepsilon^\nu S_\nu^\varepsilon(z)\theta\,. \tag{3}$$

The system in (1) captures part of the difference between incompressible and pseudo-incompressible dynamics. Thus, the divergence control in (1d) represents weak deviations from the constraint $\mathrm{div}_h\, v =$

0 of an incompressible flow that are due to the small but finite height of the flow domain. It is not equivalent to the pseudo-incompressible system, however, as it does not include its baroclinic nonlinearity which would be represented by pressure gradient terms $(\theta_0^\varepsilon(z) + \varepsilon^{\mu+\nu}\theta)(\nabla_h \pi, \partial_z \pi)$ in (1a) and (1b). Our main point in Sect. 4 will be to show that the weak deviation from incompressibility, even though small, significantly improves the system's agreement with the full compressible model relative to the incompressible model.

1.2. Description of the Problem

To model a compressible flow under the influence of an external force (e.g., earth gravity), the compressible Euler equations is considered. With low stratifications, the dimensionless system can be written as (see, e.g., [5]),

$$\begin{cases} \partial_t \rho + \operatorname{div}_h(\rho v) + \partial_z(\rho w) = 0, \\ \partial_t(\rho v) + \operatorname{div}_h(\rho v \otimes v) + \partial_z(\rho w v) + \dfrac{1}{\varepsilon^2}\nabla_h p = 0, \\ \partial_t(\rho w) + \operatorname{div}_h(\rho v w) + \partial_z(\rho w w) + \dfrac{1}{\varepsilon^2}\partial_z p + \dfrac{1}{\varepsilon}\rho G(z) = 0, \\ \partial_t p + v\cdot\nabla_h p + w\partial_z p + \gamma p(\operatorname{div}_h v + \partial_z w) = 0, \end{cases} \tag{4}$$

where $\varepsilon \in (0,1)$ denotes the small Mach number, and ρ, p, v, and w are the scalar density, the pressure potential, the horizontal velocity field, and the vertical velocity, respectively. Here $G(z)$ is the external force, causing stratification. As $\varepsilon \to 0^+$, system (4) describes flows in the low Mach number region with low stratification, i.e., the Boussinesq scale. The external force $\rho G(z)$ causes the flow to form stratification as $\varepsilon \to 0^+$. One particular stratification profile considered in this paper is characterised by

$$\partial_z \theta = \mathcal{O}(\varepsilon^\mu), \qquad \mu \in (0,1), \tag{5}$$

where θ is the potential temperature defined by

$$\theta := p^{1/\gamma}\rho^{-1}. \tag{6}$$

In addition, the Exner pressure, defined by

$$\varpi := \frac{\gamma}{\gamma-1} p^{\frac{\gamma-1}{\gamma}}, \tag{7}$$

is commonly used in meteorological study ([9,10] etc.). Then (4) is equivalent to, described by the new unknowns (ϖ, θ, v, w),

$$\begin{cases} \partial_t \varpi + v\cdot\nabla_h \varpi + w\partial_z \varpi + (\gamma-1)\varpi(\operatorname{div}_h v + \partial_z w) = 0, \\ \partial_t \theta^{-1} + v\cdot\nabla_h\theta^{-1} + w\partial_z\theta^{-1} = 0, \\ \theta^{-1}(\partial_t v + v\cdot\nabla_h v + w\partial_z v) + \dfrac{1}{\varepsilon^2}\nabla_h\varpi = 0, \\ \theta^{-1}(\partial_t w + v\cdot\nabla_h w + w\partial_z w) + \dfrac{1}{\varepsilon^2}\partial_z\varpi + \dfrac{1}{\varepsilon}\theta^{-1}G(z) = 0. \end{cases} \tag{8}$$

In order the investigate the stratification with (5), the following ansatz is introduced:

$$\varpi := \varpi_0 + \varepsilon\widetilde{\varpi}, \quad \theta^{-1} := \theta_0^{-1} + \varepsilon^\mu G^{-1}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}G^{-1}\widetilde{\mathcal{H}}, \tag{9}$$

where ϖ_0, θ_0 are constant, and $\overline{\mathcal{H}}_0 = \overline{\mathcal{H}}_0(z)$. Then from (8), one can derive, with

$$\mu + 2\nu = 1, \tag{10}$$

$$\begin{cases} \dfrac{1}{(\gamma-1)\varpi}(\partial_t\widetilde{\varpi} + v\cdot\nabla_h\widetilde{\varpi} + w\partial_z\widetilde{\varpi}) + \dfrac{1}{\varepsilon}(\operatorname{div}_h v + \partial_z w) = 0, \\ \dfrac{1}{G\partial_z(G^{-1}\overline{\mathcal{H}}_0)}(\partial_t\widetilde{\mathcal{H}} + v\cdot\nabla_h\widetilde{\mathcal{H}} + Gw\partial_z(G^{-1}\widetilde{\mathcal{H}})) + \dfrac{1}{\varepsilon^\nu}w = 0, \\ \theta^{-1}(\partial_t v + v\cdot\nabla_h v + w\partial_z v) + \dfrac{1}{\varepsilon}\nabla_h\widetilde{\varpi} = 0, \\ \theta^{-1}(\partial_t w + v\cdot\nabla_h w + w\partial_z w) + \dfrac{1}{\varepsilon}(\partial_z\widetilde{\varpi} + \theta_0^{-1}G(z) + \varepsilon^\mu\overline{\mathcal{H}}_0) + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}} = 0. \end{cases} \tag{11}$$

After denoting by

$$\widetilde{q} := \widetilde{\varpi} + \theta_0^{-1}\int_0^z G(z')\,dz' + \varepsilon^\mu\int_0^z \overline{\mathcal{H}}_0(z')\,dz' \tag{12}$$

and multiplying the first equation of (11) with ϖ/ϖ_0, we arrive at

$$\begin{cases} \dfrac{1}{(\gamma-1)\varpi_0}(\partial_t\widetilde{q} + v\cdot\nabla_h\widetilde{q} + w\partial_z\widetilde{q} - \theta_0^{-1}Gw - \varepsilon^\mu\overline{\mathcal{H}}_0 w) \\ \qquad + \dfrac{1}{\varepsilon}(\operatorname{div}_h v + \partial_z w) = -\varpi_0^{-1}\widetilde{q}(\operatorname{div}_h v + \partial_z w) \\ \qquad + \varpi_0^{-1}(\operatorname{div}_h v + \partial_z w)(\theta_0^{-1}\int_0^z G(z')\,dz' + \varepsilon^\mu\int_0^z\overline{\mathcal{H}}_0(z')\,dz'), \\ -\dfrac{1}{G\partial_z(G^{-1}\overline{\mathcal{H}}_0)}(\partial_t\widetilde{\mathcal{H}} + v\cdot\nabla_h\widetilde{\mathcal{H}} + w\partial_z\widetilde{\mathcal{H}} - \dfrac{\partial_z G}{G}\widetilde{\mathcal{H}}w) - \dfrac{1}{\varepsilon^\nu}w = 0, \\ \theta^{-1}(\partial_t v + v\cdot\nabla_h v + w\partial_z v) + \dfrac{1}{\varepsilon}\nabla_h\widetilde{q} = 0, \\ \theta^{-1}(\partial_t w + v\cdot\nabla_h w + w\partial_z w) + \dfrac{1}{\varepsilon}\partial_z\widetilde{q} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}} = 0. \end{cases} \tag{13}$$

On the other hand, denote by

$$U := \begin{pmatrix} \widetilde{q} \\ \widetilde{\mathcal{H}} \\ v \\ w \end{pmatrix}, \quad \mathcal{L}_a U := \begin{pmatrix} \operatorname{div}_h v + \partial_z w \\ 0 \\ \nabla_h\widetilde{q} \\ \partial_z\widetilde{q} \end{pmatrix}, \quad \text{and } \mathcal{L}_g U := \begin{pmatrix} 0 \\ -w \\ 0 \\ \widetilde{\mathcal{H}} \end{pmatrix}. \tag{14}$$

Notice that operators $\mathcal{L}_a$ and $\mathcal{L}_g$ are anti-symmetric with respect to the L^2-inner product and induce oscillations, corresponding to acoustic waves and internal waves of solutions to system (13), respectively.

Unfortunately, in general, the anti-symmetry property does not hold for general boundary conditions and systems with non-constant coefficients in more regular Sobolev space, for instance H^s, $s > 0$. This is a major difficulty in the study of asymptotic limit of fast oscillation systems (see, e.g., [12]). To resolve this difficulty is beyond the scope of this paper. Instead, we will introduce a system closely related to system (13), which still captures the acoustic waves and the internal waves driven by $\mathcal{L}_a$ and $\mathcal{L}_g$, respectively. As explained in Sect. 1.1, we will assume the following hypothesis in this first work.

(H1) If one considers (13) in $\{(x,y,z) \in \mathbb{T}^2 \times 0.5\mathbb{T}\}$, and assumes that $G, \overline{\mathcal{H}}_0$ are odd in the z-variable, then the following symmetry invariance holds:

$$\begin{gathered} \widetilde{q}, \widetilde{\mathcal{H}}, v, \text{ and } w \text{ are even, odd, even, and odd, respectively,} \\ \text{with respect to the } z\text{-variable.} \end{gathered} \tag{SYM}$$

Therefore, by, in addition, assuming $G, \overline{\mathcal{H}}_0$ to be smooth enough in $\mathbb{T}^3$, one can consider (13) in $\mathbb{T}^3$.

(H2) Noticing that in $(13)_2$, the term $\widetilde{\mathcal{H}}w\partial_z G/G$ becomes singular when G approaches 0. The function $\frac{\partial_z G}{G}$ is replaced by another function $\widetilde{G}$, which is odd with respect to the z-variable and smooth in $\mathbb{T}^3$. For the same reason, we replace G^{-1} in (9) by $\widetilde{G}$.

(H3) The Brunt–Väisälä frequency $\mathfrak{N}$, defined by,

$$\mathfrak{N}^2 := -G\partial_z(G^{-1}\overline{\mathcal{H}}_0), \tag{15}$$

is constant.

Then, after denoting the positive constants

$$\mathcal{A} := \frac{1}{(\gamma-1)\varpi_0}, \quad \mathcal{B} := -\frac{1}{G\partial_z(G^{-1}\overline{\mathcal{H}}_0)}, \quad \mathcal{C} := \theta_0^{-1}, \tag{16}$$

we introduce the following system: in $\mathbb{T}^3$, with $\mu + 2\nu = 1$,

$$\begin{cases} \mathcal{A}\partial_t\widetilde{q} + \mathcal{A}v\cdot\nabla_h\widetilde{q} + \mathcal{A}w\partial_z\widetilde{q} + \dfrac{1}{\varepsilon}(\operatorname{div}_h v + \partial_z w) \\ \qquad = \mathcal{A}\mathcal{C}Gw + \varepsilon^\mu\mathcal{A}\overline{\mathcal{H}}_0 w - \varpi_0^{-1}\widetilde{q}(\operatorname{div}_h v + \partial_z w) \\ \qquad\quad + \varpi_0^{-1}(\operatorname{div}_h v + \partial_z w)(\mathcal{C}\int_0^z G(z')\,dz' + \varepsilon^\mu\int_0^z \overline{\mathcal{H}}_0(z')\,dz'), \\ \mathcal{B}\partial_t\widetilde{\mathcal{H}} + \mathcal{B}v\cdot\nabla_h\widetilde{\mathcal{H}} + \mathcal{B}w\partial_z\widetilde{\mathcal{H}} - \dfrac{1}{\varepsilon^\nu}w = \mathcal{B}\widetilde{G}\cdot\widetilde{\mathcal{H}}w, \\ \vartheta\partial_t v + \vartheta v\cdot\nabla_h v + \vartheta w\partial_z v + \dfrac{1}{\varepsilon}\nabla_h\widetilde{q} = 0, \\ \vartheta\partial_t w + \vartheta v\cdot\nabla_h w + \vartheta w\partial_z w + \dfrac{1}{\varepsilon}\partial_z\widetilde{q} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}} = 0, \end{cases} \tag{17}$$

where $\widetilde{q}$, $\widetilde{\mathcal{H}}$, v, w admit the symmetry (SYM), G, $\overline{\mathcal{H}}_0$, $\widetilde{G}$ are odd in the z-variable and smooth enough in $\mathbb{T}^3$, and $\vartheta = \mathcal{C} + \mathcal{O}(\varepsilon^\mu)$ are given by

$$\vartheta := \mathcal{C} + \varepsilon^\mu\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}}. \tag{18}$$

System (17) is complemented with initial data

$$(\widetilde{q}, \widetilde{\mathcal{H}}, v, w)|_{t=0} = (\widetilde{q}_{\text{in}}, \widetilde{\mathcal{H}}_{\text{in}}, v_{\text{in}}, w_{\text{in}}). \tag{19}$$

Accordingly, $([\partial_t^\alpha\widetilde{q}]_{\text{in}}, [\partial_t^\alpha\widetilde{\mathcal{H}}]_{\text{in}}, [\partial_t^\alpha v]_{\text{in}}, [\partial_t^\alpha w]_{\text{in}})$, $\alpha \in \mathbb{N}^+$, are defined inductively after shifting spatial derivatives to temporal derivatives using equations of (17).

Before stating our results, we would like to make a few perspective remarks. As one can see, in system (17), the linear oscillator is given by

$$\frac{1}{\varepsilon}\mathcal{L}_a + \frac{1}{\varepsilon^\nu}\mathcal{L}_g,$$

i.e., a combination of the acoustic oscillator and the internal wave oscillator. Moreover, since $\nu \in (0,1)$, as $\varepsilon \to 0^+$, the oscillation induced by $\frac{1}{\varepsilon}\mathcal{L}_a$ is much faster than that of $\frac{1}{\varepsilon^\nu}\mathcal{L}_g$. This means that the acoustic waves will be averaged out (or filtered out) before the internal waves. Owing to such a phenomena, we propose a pseudo-incompressible/soundproof model, similar to [10]:

$$\begin{cases} \operatorname{div}_h v_{\text{sp}} + \partial_z w_{\text{sp}} = 0, \\ \mathcal{B}\partial_t\widetilde{\mathcal{H}}_{\text{sp}} + \mathcal{B}v_{\text{sp}}\cdot\nabla_h\widetilde{\mathcal{H}}_{\text{sp}} + \mathcal{B}w_{\text{sp}}\partial_z\widetilde{\mathcal{H}}_{\text{sp}} - \dfrac{1}{\varepsilon^\nu}w_{\text{sp}} = \mathcal{B}\widetilde{G}\cdot\widetilde{\mathcal{H}}_{\text{sp}}w_{\text{sp}}, \\ \mathcal{C}\partial_t v_{\text{sp}} + \mathcal{C}v_{\text{sp}}\cdot\nabla_h v_{\text{sp}} + \mathcal{C}w_{\text{sp}}\partial_z v_{\text{sp}} + \nabla_h p_{\text{sp}} = 0, \\ \mathcal{C}\partial_t w_{\text{sp}} + \mathcal{C}v_{\text{sp}}\cdot\nabla_h w_{\text{sp}} + \mathcal{C}w_{\text{sp}}\partial_z w_{\text{sp}} + \partial_z p_{\text{sp}} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}}_{\text{sp}} = 0, \end{cases} \tag{20}$$

whose solutions will be an approximation to the solutions to system (17) minus the acoustic waves, with or without initial acoustic waves.

Aside from the soundproof approximation, we would like to investigate how the mixture of acoustic waves and internal waves with different frequencies affects the total oscillation of the system. To do so, we will first consider a linear system associated with (17) and the corresponding eigenvalue problem. By comparing the distribution of eigenvalues with that of eigenvalues associated with $\mathcal{L}_a$, we have a more precise description of how the internal waves intertwine with the acoustic waves at the level of eigenvalues. Based on the understanding of the linear theory, we will discuss the fast-slow wave interaction of system (17) in the end.

We would like to mention, our current study is strongly motivated by previous study on flows with strong stratification (see, e.g., [9,10]), to which we refer readers for more metrological perspectives. A

Table 1. *Waves in the initial data*

Waves in the initial data	Theorem 1	Theorem 2
Mean flows	✓	✓
Internal waves	✓	✓
Acoustic waves	✗	✓

recent paper [2] focuses on the soundproof model with stratification to better understand the internal waves.

The justification of singular limits is rooted back to:

- Fast oscillation limit with only one parameter can be found in [15–17]. For geophysical purposes, see for instance [6,7,12].
- Fast oscillation limit with several parameters linked together can be found in [18]. For geophysical purposes, see, for instance, [4,5] for weak solutions, and [1] for strong solutions.

In this work, we do **not** perform fast oscillation limit. Instead, we want to prove that the non-oscillating mean flows and the oscillating internal waves of solutions of two singular systems (the compressible and pseudo-incompressible/soundproof models) remain asymptotically close over the slowest time scale. The main theorems in this paper consider the initial data of the following types in the full compressible system (17), as in Table 1:
and compare the solutions to those of the soundproof model (20). In both cases, we justify the rigidity of capturing the dynamics of the mean flows and the internal waves of the full compressible system using the soundproof approximation.

More precisely, our first result provides the comparison of solutions to the two singular systems in the well-prepared data (without acoustic waves) case:

Theorem 1 (Mean flows + Internal waves). *Let $0 < 2\nu < 1$. Denote the initial data to the intermediate model (51), below in page 19, as*

$$(\widetilde{\mathcal{H}}_{\mathrm{ms,in}}, v_{\mathrm{ms,in}}, w_{\mathrm{ms,in}}) \quad \in H^3(\mathbb{T}^3),$$

and the initial data to the soundproof model (20) as

$$(\widetilde{\mathcal{H}}_{\mathrm{sp,in}}, v_{\mathrm{sp,in}}, w_{\mathrm{ps,in}}) \quad \in H^3(\mathbb{T}^3),$$

satisfying the pseudo-incompressible and incompressible conditions $(51)_1$ and $(20)_1$, respectively. Then there exist local-in-time solutions to the intermediate model (51), below, and the soundproof model (20), denoted as

$$(p_{\mathrm{ms}}(s), \widetilde{\mathcal{H}}_{\mathrm{ms}}(s), v_{\mathrm{ms}}(s), w_{\mathrm{ms}}(s)) \quad \text{and} \quad (p_{\mathrm{sp}}(s), \widetilde{\mathcal{H}}_{\mathrm{sp}}(s), v_{\mathrm{sp}}(s), w_{\mathrm{sp}}(s)),$$

respectively, in $L^\infty((0,T_{\mathrm{ms+sp}}), H^3(\mathbb{T}^3)) \cap C([0,T_{\mathrm{ms+sp}}), H^2(\mathbb{T}^3))$ for some $T_{\mathrm{ms+sp}} \in (0,\infty)$.

Meanwhile, denote by

$$(\widetilde{q}(s), \widetilde{\mathcal{H}}(s), v(s), w(s))$$

the solution to compressible system (17) in Proposition 3, below, with initial data satisfying (23) for any fixed $\sigma \in (0,\mu]$. Then there exist $T_{\mathrm{app}} \in (0,\infty)$ and $\mathcal{C}_{\mathrm{app}} \in (0,\infty)$, depending only on the initial data above, such that, for $\varepsilon \in (0,1)$,

$$\begin{aligned} &\sup_{0\le s\le T_{\mathrm{app}}} \left\| \widetilde{q}(s) - \varepsilon p_{\mathrm{sp}}(s), \widetilde{\mathcal{H}}(s) - \widetilde{\mathcal{H}}_{\mathrm{sp}}(s), v(s) - v_{\mathrm{sp}}(s), w(s) - w_{\mathrm{sp}}(s) \right\|_{L^2(\mathbb{T}^3)} \\ &\quad \le \mathcal{C}_{\mathrm{app}} \Big(\varepsilon^{\max\{\mu-\nu,\mu-\sigma\}} \\ &\qquad + \left\| \widetilde{\mathcal{H}}_{\mathrm{ms,in}} - \widetilde{\mathcal{H}}_{\mathrm{sp,in}}, v_{\mathrm{ms,in}} - v_{\mathrm{sp,in}}, w_{\mathrm{ms,in}} - w_{\mathrm{sp,in}} \right\|_{L^2(\mathbb{T}^3)} \\ &\qquad + \left\| \widetilde{q}_{\mathrm{in}} - \varepsilon p_{\mathrm{ms,in}}, \widetilde{\mathcal{H}}_{\mathrm{in}} - \widetilde{\mathcal{H}}_{\mathrm{ms,in}}, v_{\mathrm{in}} - v_{\mathrm{ms,in}}, w_{\mathrm{in}} - w_{\mathrm{ms,in}} \right\|_{L^2(\mathbb{T}^3)} \Big). \end{aligned} \tag{21}$$

Here p_{ms}, $p_{\text{ms,in}}$, and p_{sp} are given by solutions to elliptic problems (52) and (73), and can be estimated as in (55) and (75), below in pages 19, 25, 20, and 25, respectively. Here, recall that $\mu + 2\nu = 1$.

The uniform-in-ε estimate of solutions to (20) and (51) can be found in (74), (75), (60), and (55), respectively. In particular, with $\max\{\mu-\nu, \mu-\sigma\} = \max\{1-3\nu, 1-2\nu-\sigma\} > 0$ and proper initial data (so that the initial data on the right hand side of (21) is small), (21) provides the error estimates and convergence rate of the soundproof approximation with "well-prepared" initial data.

The term $\varepsilon^{\max\{\mu-\nu,\mu-\sigma\}}$ in the error estimate (21) results from the comparison between the terms $(\mathcal{C} + \mathcal{O}(\varepsilon^{\mu}))(\partial_t v, \partial_t w)$ and $\mathcal{C}(\partial_t v_{\text{ms}}, \partial_t w_{\text{ms}})$ in Sect. 4.2, which can be either written as

$$(\mathcal{C} + \mathcal{O}(\varepsilon^{\mu}))(\partial_t(v - v_{\text{ms}}), \partial_t(w - w_{\text{ms}})) + \mathcal{O}(\varepsilon^{\mu})(\partial_t v_{\text{ms}}, \partial_t w_{\text{ms}})$$

or

$$\mathcal{C}(\partial_t(v - v_{\text{ms}}), \partial_t(w - w_{\text{ms}})) + \mathcal{O}(\varepsilon^{\mu})(\partial_t v, \partial_t w).$$

See (66) and (70), respectively, for details. Since $(\partial_t v_{\text{ms}}, \partial_t w_{\text{ms}}) \simeq (\partial_t v_{\text{sp}}, \partial_t w_{\text{sp}}) \simeq \mathcal{O}(\varepsilon^{-\nu})$ [as can be seen through (20)] and $(\partial_t v, \partial_t w) \simeq \mathcal{O}(\varepsilon^{-\sigma})$ thanks to Proposition 3, this results in our freedom of choice in the error estimate (21). For more details, we refer readers to the estimate of $\mathfrak{I}_3$ in (68) in page 24 and (72) in page 25 of the proof of the theorem.

Heuristically speaking, for larger $\nu \in [1/3, 1/2)$, the oscillating rates of the internal gravity waves and the acoustic waves (if non-trivial, of $\mathcal{O}(\varepsilon^{-\nu})$ and $\mathcal{O}(\varepsilon^{-1})$, respectively) are closer to each other. In order to control the error $\varepsilon^{\max\{\mu-\nu,\mu-\sigma\}} = \varepsilon^{\max\{1-3\nu,1-2\nu-\sigma\}}$, we need $\sigma < \mu = 1 - 2\nu$, i.e., smaller value of σ (hence weaker acoustic waves in the full compressible system), to avoid strong interaction between acoustic waves and internal waves in the full compressible system.

We would also like to point out that the constraint $0 < 2\nu < 1$ is physical [see the formal deviation between (8) and (13)].

The second result will provide the convergence in the ill-prepared data (with acoustic waves) case:

Theorem 2. Mean flows + Internal waves + Acoustic waves *Under the same assumptions as in Theorem 1, denote by $U = (\widetilde{q}, \widetilde{\mathcal{H}}, v, w)$, the solution to (17), and write $U = U_{\varepsilon}^{\text{mf}} + U_{\varepsilon}^{\text{gw}} + U_{\varepsilon}^{\text{aw}}$ as the summation of the mean flows, the internal waves, and the acoustic waves. Let $(p_{\text{sp}}, U_{\text{sp}}) = (p_{\text{sp}}, \widetilde{\mathcal{H}}_{\text{sp}}, v_{\text{sp}}, w_{\text{sp}})$ be the solution to the soundproof approximation (20) with initial data capturing the initial mean flows and internal waves of the full compressible system (17) [see (193) for the exact meaning of this statement]. Let $\mathcal{P}_{\text{rd}} : (\widetilde{q}, \widetilde{\mathcal{H}}, v, w) \mapsto (\widetilde{\mathcal{H}}, v, w)$ and $\{T_k\}_{k\in\mathbb{N}}$ be the vector-dimension reduction and finite dimension truncation defined in (145) and (163) of pages 45 and 48, respectively. Then for any positive integer K, one has*

$$\sup_{0<t<T_{\sigma,\text{mg}}} \left\| T_K \mathcal{P}_{\text{rd}}(U_{\varepsilon}^{\text{mf}} + U_{\varepsilon}^{\text{gw}})(t) - T_K U_{\text{sp}}(t) \right\|_{L^2}^2 \leq C_K(\mathcal{O}(\varepsilon^{2\mu-2\sigma}) + \mathcal{O}(\varepsilon)) + Err, \tag{22}$$

where $T_{\sigma,\text{mg}} \in (0,\infty)$ is the time of existence of solutions independent of ε and K, and Err is the truncation error which vanishes uniformly-in-ε as $K \to \infty$.

The physical rationale for the need to project out the pressure variable in the course of this estimate is as follows: By the non-dimensionalization underlying the full compressible system in (17), the small parameter ε is proportional to the Mach number. Then, under the assumption of initial velocities of order unity, acoustic pressure amplitudes will be of order $\mathcal{O}(\varepsilon)$ for otherwise general initial data, see (9) and, e.g., [7,8,14]. Similarly, internal waves inducing velocities of $\mathcal{O}(1)$ come with pressure perturbation amplitudes of order $\mathcal{O}(\varepsilon^{2-\nu})$, see [10], while slow, purely advective dynamics implies pressure amplitudes of $\mathcal{O}(\varepsilon^2)$ according to the classical scaling for incompressible flows. Therefore, when the contributions of the superimposed acoustic, gravity wave, and mean flow modes to the velocity field are comparable (e.g., of order unity), then their contributions to the pressure field have decidedly different amplitude scaling with $\delta p_{\text{aw}} \gg \delta p_{\text{gw}} \gg \delta p_{\text{mf}}$. That is, there are scaling regimes within which the influence of acoustics on the flow velocity and advected scalars is neglible compared to that of gravity waves and mean flow, although the pressure perturbations are still dominated by the acoustic modes. In these regimes, the projected variables $(\mathcal{H}, v, w)$ in the full compressible and pseudo-incompressible solutions are asymptotically close, whereas the pressure fields are not. Our theorem then states that the net effect of the larger acoustic pressure fluctuations rigorously average out at leading order and over the pertinent advective time scale. This generalizes related statements regarding acoustic averaging in the absence of gravity by Klainerman and Majda [7].

To get existence of solutions to (17), we need uniform-in-ε *a priori* estimate, namely:

Proposition 3. *Let $0 < 2\nu < 1$ and $0 < \varepsilon < 1$. Suppose that $(\widetilde{q}_{\rm in}, \widetilde{\mathcal{H}}_{\rm in}, v_{\rm in}, w_{\rm in})$ in (19) satisfies*

$$\sum_{\substack{\alpha,\beta\in\mathbb{N},\, \alpha+\beta\le 3,\\ \partial\in\{\partial_x,\partial_y,\partial_z\}}} \Big(\big\| [\partial^\beta(\varepsilon^\sigma\partial_t)^\alpha \widetilde{q}]_{\rm in}, [\partial^\beta(\varepsilon^\sigma\partial_t)^\alpha \widetilde{\mathcal{H}}]_{\rm in} \big\|^2_{L^2(\mathbb{T}^3)} + \big\| [\partial^\beta(\varepsilon^\sigma\partial_t)^\alpha v]_{\rm in}, [\partial^\beta(\varepsilon^\sigma\partial_t)^\alpha w]_{\rm in} \big\|^2_{L^2(\mathbb{T}^3)} \Big) \le \mathcal{C}_{\rm in}, \tag{23}$$

for some $\mathcal{C}_{\rm in} \in (0,\infty)$ and $\sigma \in (0,\mu]$, where $([\partial_t^\alpha \widetilde{q}]_{\rm in}, [\partial_t^\alpha \widetilde{\mathcal{H}}]_{\rm in}, [\partial_t^\alpha v]_{\rm in}, [\partial_t^\alpha w]_{\rm in})$, $\alpha \in \mathbb{N}^+$, are defined inductively after shifting spatial derivatives to temporal derivatives using equations of (17). Let $(\widetilde{q}(s), \widetilde{\mathcal{H}}(s), v(s), w(s))$ be the smooth solution to (17) with initial data $(\widetilde{q}_{\rm in}, \widetilde{\mathcal{H}}_{\rm in}, v_{\rm in}, w_{\rm in})$. Then there exist $T_\sigma \in (0,\infty)$, depending only on $\mathcal{C}_{\rm in}$, such that

$$\sup_{0\le s\le T_\sigma} \sum_{\substack{\alpha,\beta\in\mathbb{N},\, \alpha+\beta\le 3,\\ \partial\in\{\partial_x,\partial_y,\partial_z\}}} \Big(\big\| (\varepsilon^\sigma\partial_t)^\alpha \partial^\beta \widetilde{q}(s), (\varepsilon^\sigma\partial_t)^\alpha \partial^\beta \widetilde{\mathcal{H}}(s) \big\|^2_{L^2(\mathbb{T}^3)} + \big\| (\varepsilon^\sigma\partial_t)^\alpha \partial^\beta v(s), (\varepsilon^\sigma\partial_t)^\alpha \partial^\beta w(s) \big\|^2_{L^2(\mathbb{T}^3)} \Big) \le \mathcal{C}\mathcal{C}_{\rm in},$$

with some constant $\mathcal{C} \in (0,\infty)$, independent of ε.

We would like to mention that, with the *a priori* estimate, one can construct solutions locally in time to (17), and also show the well-posedness, i.e., uniqueness and continuous dependency on initial data. The construction and proof are standard, and we leave the details to readers.

After rescaling time at the same order, a uniform-in-ε estimate for a soundproof system similar to (20) was obtained by the authors of [2, Theorem 2] under the assumption that the Brunt-Väisälä frequency $\mathfrak{N}$ is constant, i.e., (H3). In the case when $\mathfrak{N}$ is not constant but depends on the vertical coordinate, z, the existence time is of $\mathcal{O}(\varepsilon^\nu)$ as shown in [2, Theorem 1] for the soundproof system. Moreover, a vertical mode decomposition based on modes obtained from the eigenfunctions of a Sturm-Liouville equation associated with the background stratification is introduced, and a formal derivation of (partial differential) evolution equations for these modes is provided. It is shown that the modes interact strongly with dispersive mixing when $\mathfrak{N}$ is not constant (see Proposition 4 in [2]). In contrast, when $\mathfrak{N}$ *is* constant, the vertical modes decouple (see their Proposition 5).

Notably, the vertical mode decomposition in [2] is not an eigenmode decomposition of the fast linear system describing its internal wave dynamics as developed in Sect. 5.2 of the present paper (see also [10]). In fact the eigenmodes of the fast system are sinusoidal in the horizontal direction and satisfy a Sturm–Liouville equation that is parameterized by the horizontal wave number. For non-constant $\mathfrak{N}$, the resulting vertical modes are not sinusoidal and their structure depends non-trivially on the horizontal wave number. As a consequence, the projections of the solution onto just the eigenmodes of the hydrostatic background in [2] will themselves be linear combinations of the eigenmodes of the full system and must reveal dispersive behavior. Moreover, in this case the modes of the background system will also generally be coupled, because their projection onto the eigenmodes of the full system will depend on the time evolving horizontal structure of the solution.

The present analysis for the pseudo-incompressible model reduces to that of the incompressible system studied in [2] for $P^\varepsilon = 1$ in (1). It would be interesting to compare the detailed analytical steps and accessible results when the solution decomposition in terms of a single family of vertical modes as invoked by Desjardins et al. [2] is replaced with a decomposition in terms of the full set of eigenmodes of the fast system as worked out here. As demonstrated in [10], that approach could also be transferred to the full compressible system (17) in which case the additional family of (even faster) acoustic eigenmodes and their potential interactions with the internal wave and advective modes will have to be accounted for.

To prove Theorem 2, we need to understand the distribution of eigenvalues and need to have comparison of eigenvectors, that is:

Proposition 4. *The eigenvalues of operator $\mathcal{L}_a + \varepsilon^{1-\nu}\mathcal{L}_g$ lie within the neighborhood of radius $\varepsilon^{1-\nu}$ of the eigenvalues of operator $\mathcal{L}_a$. More precisely, let $i\omega$ be an eigenvalue of $\mathcal{L}_a + \varepsilon^{1-\nu}\mathcal{L}_g$, then there exists $m \in \{0, 1, 2, \cdots\}$, such that*

$$|\omega^{\pm}_{\mathrm{ac},m}|^2 \leq |\omega|^2 \leq |\omega^{\pm}_{\mathrm{ac},m}|^2 + \varepsilon^{2-2\nu},$$

where $\{i\omega^{\pm}_{\mathrm{ac},m}\}_{m\in\{0,1,2,\cdots\}}$ are the eigenvalues of $\mathcal{L}_a$. Therefore, the eigenvalues of the linear oscillating operator

$$\frac{1}{\varepsilon}\mathcal{L}_a + \frac{1}{\varepsilon^{\nu}}\mathcal{L}_g,$$

to system (17), with $\mathcal{A} = \mathcal{B} = \mathcal{C} = 1$, can be classified into three families: mean flow frequency $|\iota_{\mathrm{mf}}| = 0$; internal wave frequency $|\iota_{\mathrm{gw}}| = \mathcal{O}(\varepsilon^{-\nu})$; perturbed acoustic wave frequency $|\iota_{\mathrm{aw}}| = \mathcal{O}(\varepsilon^{-1})$.

In addition, with Fourier representations, one can obtain more detailed and sharper comparison on the eigenvalues and eigenvectors, which are presented in Corollary 7 in page 43.

We refer readers to the representation of eigenvalue-eigenvector pairs to Proposition 5, below.

The rest of this paper is organized as follows. Section 2 will introduce some notations that have been and will be used in this paper, as well as some classic nonlinear and commutator estimates. Section 3 is devoted to uniform-in-ε energy estimates of solutions to (17), and thus proves Proposition 3. In Sect. 4, the rigidity of soundproof approximation is established, which proves Theorem 1. Notice that due to the stratification, we will introduce an intermediate model, i.e., (51), to establish the soundproof approximation. The aforementioned linear oscillating system is introduced in Sect. 5, where the eigenvalue problem is investigated. Using the Fourier representation, the eigenvalue-eigenvector pairs are identified. Thus Proposition 4 is proved. In Sect. 5.3, we further investigate the internal waves in the soundproof model (20) and compare them with those in the compressible system (17). In Sect. 6, we discuss the fast-slow wave interactions of nonlinear system (17), and establish Theorem 2.

2. Preliminaries

We assume that we are in $\mathbb{T}^3$ all the time. We use the notation $\partial \in \{\partial_x, \partial_y, \partial_z\}$ throughout the rest of the paper. The horizontal gradient, the horizontal divergence, and the horizontal laplacian operators are defined by

$$\nabla_h := \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix}, \quad \mathrm{div}_h := \nabla_h \cdot, \quad \text{and} \quad \Delta_h := \mathrm{div}_h \nabla_h,$$

respectively. By adding a subscript $_{\mathrm{in}}$ to any function u, we mean the initial data of u, i.e., $u\big|_{t=0} = u_{\mathrm{in}}$. By $A \lesssim B$, it means there exists a generic constant $C \in (0, \infty)$, different from lines to lines, such that $A \leq CB$. Whenever we would like to emphasize the dependency of the generic constant C on certain quantities, the depending quantities will be added as subscript, i.e., C_g means a constant depending on g. For any norm $\big\|\big\|_X$, we shorten the notation for norms of multiple functions as

$$\big\|A, B\big\|_X = \big\|A\big\|_X + \big\|B\big\|_X.$$

First, we introduce some nonlinear estimates, which are classic in the literature.

Lemma 1. *For $s \in \mathbb{N}^+$,*

$$\|uv\|_{H^s(\mathbb{T}^3)} \leq K\|u\|_{H^\eta(\mathbb{T}^3)}\|v\|_{H^s(\mathbb{T}^3)} + \|u\|_{H^s(\mathbb{T}^3)}\|v\|_{H^\eta(\mathbb{T}^3)}, \tag{24}$$

where

$$\eta := \max\{[s/2], 2\}, \tag{25}$$

and $K \in (0, \infty)$ depends on s.

Proof. The proof is straightforward, after applying Leibniz's formula, Hölder's inequality, and the Sobolev embedding inequality. Details are omitted here. □

Lemma 2. *For $s > 3/2$, $\sigma_1, \sigma_2 \in [0, s], \sigma_1 + \sigma_2 \leq s$, one has*

$$\|uv\|_{H^{s-\sigma_1-\sigma_2}(\mathbb{T}^3)} \leq K\|u\|_{H^{s-\sigma_1}(\mathbb{T}^3)}\|v\|_{H^{s-\sigma_2}(\mathbb{T}^3)}, \tag{26}$$

where $K \in (0, \infty)$ depends on s, σ_1, σ_2.

Proof. We sketch the estimate of $\|\partial^\alpha u \partial^\beta v\|_{L^2(\mathbb{T}^3)}$, with $\alpha + \beta \leq s - \sigma_1 - \sigma_2$. After applying Hölder's inequality and the Sobolev embedding inequality, one has

$$\begin{aligned} \|\partial^\alpha u \partial^\beta v\|_{L^2(\mathbb{T}^3)} &\lesssim \|\partial^\alpha u\|_{L^p(\mathbb{T}^3)}\|\partial^\beta v\|_{L^q(\mathbb{T}^3)} \\ &\lesssim \|u\|_{H^m(\mathbb{T}^3)}\|v\|_{H^n(\mathbb{T}^3)}, \end{aligned} \tag{27}$$

with certain

$$\begin{aligned} \frac{1}{2} = \frac{1}{p} + \frac{1}{q}, &\quad p, q \in (2, \infty] \\ \frac{1}{p} - \frac{\alpha}{3} \geq \frac{1}{2} - \frac{m}{3}, &\quad \frac{1}{q} - \frac{\beta}{3} \geq \frac{1}{2} - \frac{n}{3}. \end{aligned} \tag{28}$$

In order to have a non-empty set of (p, q) in (28), we require further that

$$\alpha \leq m, \quad \beta \leq n, \quad m + n \geq \alpha + \beta + \frac{3}{2}. \tag{29}$$

One can check, with $m = s - \sigma_1$ and $n = s - \sigma_2$, (29) are satisfied with $s > 3/2$. Therefore, (26) follows after taking the sum over α, β of (27). □

Next, we will introduce the some functional setups, and commutator estimates.
Let

$$\partial_\sigma^{\alpha,\beta} := \sum_{\partial \in \{\partial_x, \partial_y, \partial_z\}} (\varepsilon^\sigma \partial_t)^\alpha \partial^\beta. \tag{30}$$

Denote by

$$\|\cdot\|_{H^\beta_{\alpha,\sigma}} := \sum_{\iota \leq \beta} \|\partial_\sigma^{\alpha,\iota}(\cdot)\|_{L^2(\mathbb{T}^3)}. \tag{31}$$

The hyperbolic energy is defined as

$$\mathcal{E}_{\sigma,s}(\cdot) := \sum_{\alpha+\beta \leq s} \|\cdot\|_{H^\beta_{\alpha,\sigma}}. \tag{32}$$

Now, we are ready to establish some commutator estimates. The following lemma presents the estimate of $[\partial_\sigma^{\alpha,\beta}, f_1 \partial_t]$.

Lemma 3. *For $\alpha + \beta \geq 3$,*

$$\|[\partial_\sigma^{\alpha,\beta}, f_1 \partial_t] g_1\|_{L^2(\mathbb{T}^3)} \leq K \varepsilon^{-\sigma} \mathcal{E}_{\sigma,\alpha+\beta}(f_1) \mathcal{E}_{\sigma,\alpha+\beta}(g_1), \tag{33}$$

where $K \in (0, \infty)$ depends only on α, β.

Proof. It suffices to consider the estimate of

$$\begin{aligned}&\left\|\partial_\sigma^{\alpha_1,\beta_1} f_1 \partial_\sigma^{\alpha_2,\beta_2}\partial_t g_1\right\|_{L^2(\mathbb{T}^3)} = \varepsilon^{-\sigma}\left\|\partial_\sigma^{\alpha_1,\beta_1} f_1 \partial_\sigma^{\alpha_2+1,\beta_2} g_1\right\|_{L^2(\mathbb{T}^3)},\\ &\text{with} \quad \alpha_1+\alpha_2=\alpha, \quad \beta_1+\beta_2=\beta, \quad \alpha_1+\beta_1\geq 1.\end{aligned} \tag{34}$$

Since $\alpha+\beta-1\geq 2>3/2$, applying (26) with

$$\begin{aligned}&u=\partial_\sigma^{\alpha_1,\beta_1} f_1, \quad v=\partial_\sigma^{\alpha_2+1,\beta_2} g_1,\\ &s=\alpha+\beta-1, \quad \sigma_1=\alpha_1+\beta_1-1, \quad \sigma_2=\alpha_2+\beta_2,\end{aligned}$$

leads to

$$\begin{aligned}&\left\|\partial_\sigma^{\alpha_1,\beta_1} f_1 \partial_\sigma^{\alpha_2+1,\beta_2} g_1\right\|_{L^2(\mathbb{T}^3)} \lesssim \left\|\partial_\sigma^{\alpha_1,\beta_1} f_1\right\|_{H^{\alpha_2+\beta_2}(\mathbb{T}^3)}\\ &\quad\times\left\|\partial_\sigma^{\alpha_2+1,\beta_2} g_1\right\|_{H^{\alpha_1+\beta_1-1}(\mathbb{T}^3)} \leq \mathcal{E}_{\sigma,\alpha+\beta}(f_1)\mathcal{E}_{\sigma,\alpha+\beta}(g_1).\end{aligned} \tag{35}$$

Therefore (33) follows after summing over $(\alpha_1,\alpha_2,\beta_1,\beta_2)$ in (34) and (35). □

Next, we are going to establish the estimate of $[\partial_\sigma^{\alpha,\beta}, f_2\partial]$, with $\partial\in\{\partial_x,\partial_y,\partial_z\}$.

Lemma 4. *With $\partial\in\{\partial_x,\partial_y,\partial_z\}$ and $\alpha+\beta\geq 3$,*

$$\left\|[\partial_\sigma^{\alpha,\beta}, f_2\partial]g_2\right\|_{L^2(\mathbb{T}^3)} \leq K\mathcal{E}_{\sigma,\alpha+\beta}(f_2)\mathcal{E}_{\sigma,\alpha+\beta}(g_2), \tag{36}$$

where $K\in(0,\infty)$ depends only on α,β.

Proof. The proof is very much similar to that of Lemma 3. Therefore we leave it to readers. □

In addition, we would like to provide some nonlinear estimates.

Lemma 5. *For $\iota\geq 2$,*

$$\mathcal{E}_{\sigma,\iota}(fg)\leq K\mathcal{E}_{\sigma,\iota}(f)\mathcal{E}_{\sigma,\iota}(g), \tag{37}$$

where $K\in(0,\infty)$ depends only on ι.

Proof. Consider

$$\begin{aligned}&\left\|(\varepsilon^\sigma\partial_t)^{\alpha_1}\partial^{\beta_1} f\cdot(\varepsilon^\sigma\partial_t)^{\alpha_2}\partial^{\beta_2} g\right\|_{L^2(\mathbb{T}^3)},\\ &\text{with} \quad 2\leq\alpha_1+\alpha_2+\beta_1+\beta_2\leq\iota.\end{aligned} \tag{38}$$

Applying (26) with

$$\begin{aligned}&u=(\varepsilon^\sigma\partial_t)^{\alpha_1}\partial^{\beta_1} f, \quad v=(\varepsilon^\sigma\partial_t)^{\alpha_2}\partial^{\beta_2} g,\\ &s=\alpha_1+\alpha_2+\beta_1+\beta_2, \quad \sigma_1=\alpha_1+\beta_1, \quad \sigma_2=\alpha_2+\beta_2,\end{aligned}$$

leads to

$$\begin{aligned}&\left\|(\varepsilon^\sigma\partial_t)^{\alpha_1}\partial^{\beta_1} f\cdot(\varepsilon^\sigma\partial_t)^{\alpha_2}\partial^{\beta_2} g\right\|_{L^2(\mathbb{T}^3)} \lesssim \left\|(\varepsilon^\sigma\partial_t)^{\alpha_1}\partial^{\beta_1} f\right\|_{H^{\alpha_2+\beta_2}(\mathbb{T}^3)}\\ &\quad\times\left\|(\varepsilon^\sigma\partial_t)^{\alpha_2}\partial^{\beta_2} g\right\|_{H^{\alpha_1+\beta_1}(\mathbb{T}^3)} \leq \mathcal{E}_{\sigma,\iota}(f)\mathcal{E}_{\sigma,\iota}(g).\end{aligned} \tag{39}$$

The estimates of the case when $\alpha_1+\alpha_2+\beta_1+\beta_2=0,1$ is straightforwards and thus is omitted here. Therefore (37) follows after summing over $(\alpha_1,\alpha_2,\beta_1,\beta_2)$. □

3. Uniform *a Priori* Estimates

We are in the place to perform *a priori* energy estimates. That is, we will establish the Proof of Proposition 3 in this section. In particular, we will focus on the estimates of

$$\big\|\partial_\sigma^{\alpha,\beta}(\widetilde{q},\widetilde{\mathcal{H}},v,w)\big\|_{L^2(\mathbb{T}^3)}, \quad \text{with} \quad \alpha+\beta=3.$$

The case when $\alpha+\beta=0,1,2$ can be calculated in a similar, if not simpler, manner.

Applying $\partial_\sigma^{\alpha,\beta}$, $\alpha+\beta=3$, to (17) leads to

$$\begin{cases} \mathcal{A}\partial_t\partial_\sigma^{\alpha,\beta}\widetilde{q}+\mathcal{A}v\cdot\nabla_h\partial_\sigma^{\alpha,\beta}\widetilde{q}+\mathcal{A}w\partial_z\partial_\sigma^{\alpha,\beta}\widetilde{q} \\ \qquad +\dfrac{1}{\varepsilon}(\operatorname{div}_h\partial_\sigma^{\alpha,\beta}v+\partial_z\partial_\sigma^{\alpha,\beta}w)=\mathcal{I}_1+\mathcal{J}_1, \\ \mathcal{B}\partial_t\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}+\mathcal{B}v\cdot\nabla_h\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}+\mathcal{B}w\partial_z\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}-\dfrac{1}{\varepsilon^\nu}\partial_\sigma^{\alpha,\beta}w=\mathcal{I}_2+\mathcal{J}_2, \\ \vartheta\partial_t\partial_\sigma^{\alpha,\beta}v+\vartheta v\cdot\nabla_h\partial_\sigma^{\alpha,\beta}v+\vartheta w\partial_z\partial_\sigma^{\alpha,\beta}v+\dfrac{1}{\varepsilon}\nabla_h\partial_\sigma^{\alpha,\beta}\widetilde{q}=\mathcal{I}_3, \\ \vartheta\partial_t\partial_\sigma^{\alpha,\beta}w+\vartheta v\cdot\nabla_h\partial_\sigma^{\alpha,\beta}w+\vartheta w\partial_z\partial_\sigma^{\alpha,\beta}w \\ \qquad +\dfrac{1}{\varepsilon}\partial_z\partial_\sigma^{\alpha,\beta}\widetilde{q}+\dfrac{1}{\varepsilon^\nu}\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}=\mathcal{I}_4, \end{cases} \tag{40}$$

where

$$\begin{aligned} \mathcal{I}_1 &:= -\mathcal{A}[\partial_\sigma^{\alpha,\beta},v\cdot\nabla_h]\widetilde{q}-\mathcal{A}[\partial_\sigma^{\alpha,\beta},w\partial_z]\widetilde{q}, \\ \mathcal{J}_1 &:= \mathcal{A}\mathcal{C}\partial_\sigma^{\alpha,\beta}(Gw)+\varepsilon^\mu\mathcal{A}\partial_\sigma^{\alpha,\beta}(\overline{\mathcal{H}}_0w)-\varpi_0^{-1}\partial_\sigma^{\alpha,\beta}[\widetilde{q}(\operatorname{div}_h v+\partial_z w)] \\ &\quad +\varpi_0^{-1}\partial_\sigma^{\alpha,\beta}[(\operatorname{div}_h v+\partial_z w)(\mathcal{C}\int_0^z G(z')\,dz'+\varepsilon^\mu\int_0^z\overline{\mathcal{H}}_0(z')\,dz')], \\ \mathcal{I}_2 &:= -\mathcal{B}[\partial_\sigma^{\alpha,\beta},v\cdot\nabla_h]\widetilde{\mathcal{H}}-\mathcal{B}[\partial_\sigma^{\alpha,\beta},w\partial_z]\widetilde{\mathcal{H}}, \\ \mathcal{J}_2 &:= \mathcal{B}\partial_\sigma^{\alpha,\beta}(\widetilde{G}\cdot\widetilde{\mathcal{H}}w), \\ \mathcal{I}_3 &:= -[\partial_\sigma^{\alpha,\beta},\vartheta\partial_t]v-[\partial_\sigma^{\alpha,\beta},\vartheta v\cdot\nabla_h]v-[\partial_\sigma^{\alpha,\beta},\vartheta w\partial_z]v, \\ \mathcal{I}_4 &:= -[\partial_\sigma^{\alpha,\beta},\vartheta\partial_t]w-[\partial_\sigma^{\alpha,\beta},\vartheta v\cdot\nabla_h]w-[\partial_\sigma^{\alpha,\beta},\vartheta w\partial_z]w. \end{aligned}$$

After taking the L^2-inner product of (40) with $2\partial_\sigma^{\alpha,\beta}\widetilde{q}, 2\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}, 2\partial_\sigma^{\alpha,\beta}v, 2\partial_\sigma^{\alpha,\beta}w$, respectively, applying integration by parts to the resultant equations, summing the resultant equations, and applying Hölder's inequality and the Sobolev embedding inequalities, one can write down

$$\begin{aligned} &\frac{d}{dt}\Big(\mathcal{A}\big\|\partial_\sigma^{\alpha,\beta}\widetilde{q}\big\|_{L^2(\mathbb{T}^3)}^2+\mathcal{B}\big\|\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}\big\|_{L^2(\mathbb{T}^3)}^2+\big\|\vartheta^{1/2}\partial_\sigma^{\alpha,\beta}v\big\|_{L^2(\mathbb{T}^3)}^2 \\ &\quad +\big\|\vartheta^{1/2}\partial_\sigma^{\alpha,\beta}w\big\|_{L^2(\mathbb{T}^3)}^2\Big) \\ &\lesssim \big(\|\operatorname{div}_h v+\partial_z w\|_{H^2(\mathbb{T}^3)}\big)\big(\|\partial_\sigma^{\alpha,\beta}\widetilde{q}\|_{L^2(\mathbb{T}^3)}^2+\|\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}\|_{L^2(\mathbb{T}^3)}^2\big) \\ &\quad +\big(\|\partial_t\vartheta\|_{H^2(\mathbb{T}^3)}+\|\operatorname{div}_h(\vartheta v)\|_{H^2(\mathbb{T}^3)}+\|\partial_z(\vartheta w)\|_{H^2(\mathbb{T}^3)}\big) \\ &\quad \times\big(\|\partial_\sigma^{\alpha,\beta}v\|_{L^2(\mathbb{T}^3)}^2+\|\partial_\sigma^{\alpha,\beta}w\|_{L^2(\mathbb{T}^3)}^2\big) \\ &\quad +\big(\|\mathcal{I}_1\|_{L^2(\mathbb{T}^3)}+\|\mathcal{J}_1\|_{L^2(\mathbb{T}^3)}\big)\|\partial_\sigma^{\alpha,\beta}\widetilde{q}\|_{L^2(\mathbb{T}^3)} \\ &\quad +\big(\|\mathcal{I}_2\|_{L^2(\mathbb{T}^3)}+\|\mathcal{J}_2\|_{L^2(\mathbb{T}^3)}\big)\|\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}\|_{L^2(\mathbb{T}^3)} \\ &\quad +\|\mathcal{I}_3\|_{L^2(\mathbb{T}^3)}\|\partial_\sigma^{\alpha,\beta}v\|_{L^2(\mathbb{T}^3)}+\|\mathcal{I}_4\|_{L^2(\mathbb{T}^3)}\|\partial_\sigma^{\alpha,\beta}w\|_{L^2(\mathbb{T}^3)}. \end{aligned} \tag{41}$$

Now we present the estimates of $\mathcal{I}_j$'s and $\mathcal{J}_j$'s. Applying (36) yields that,

$$\begin{aligned}\|\mathcal{I}_1\|_{L^2(\mathbb{T}^3)} &\lesssim \|[\partial_\sigma^{\alpha,\beta}, v\cdot\nabla_h]\widetilde{q}\|_{L^2(\mathbb{T}^3)} + \|[\partial_\sigma^{\alpha,\beta}, w\partial_z]\widetilde{q}\|_{L^2(\mathbb{T}^3)}\\ &\lesssim \mathcal{E}_{\sigma,3}(v,w)\mathcal{E}_{\sigma,3}(\widetilde{q}).\end{aligned} \tag{42}$$

The estimates of $\mathcal{I}_j$, $j\in\{2,3,4\}$, are similar. In fact, since $\vartheta = \mathcal{C} + \mathcal{O}(\varepsilon^\mu)$ as in (18), applying (33), (36), and (37), one will arrive at

$$\|\mathcal{I}_2,\mathcal{I}_3,\mathcal{I}_4\|_{L^2(\mathbb{T}^3)} \lesssim (\varepsilon^{\mu-\sigma}+1)\big((\mathcal{E}_{\sigma,3}(\widetilde{\mathcal{H}},v,w))^3 + \mathcal{E}_{\sigma,3}(\widetilde{\mathcal{H}},v,w)\big). \tag{43}$$

On the other hand, the estimates of $\mathcal{J}_j$, $j\in\{1,2\}$, are straightforward, thanks to (37), which are

$$\|\mathcal{J}_1,\mathcal{J}_2\|_{L^2(\mathbb{T}^3)} \lesssim (\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w))^2 + \mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w). \tag{44}$$

The rest terms on the right hand side of (41) can be handled in a similar manner. We record the estimates below:

$$\begin{aligned}&\|\operatorname{div}_h v + \partial_z w\|_{H^2(\mathbb{T}^3)} + \|\partial_t\vartheta\|_{H^2(\mathbb{T}^3)}\\ &\quad + \|\operatorname{div}_h(\vartheta v)\|_{H^2(\mathbb{T}^3)} + \|\partial_z(\vartheta w)\|_{H^2(\mathbb{T}^3)}\\ &\lesssim (\varepsilon^{\mu+\nu-\sigma}+1)\big((\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w))^2 + \mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)\big).\end{aligned} \tag{45}$$

Consequently, integrating (41) in the temporal variable yields,

$$\begin{aligned}&\mathcal{A}\|\partial_\sigma^{\alpha,\beta}\widetilde{q}\|^2_{L^2(\mathbb{T}^3)}(t) + \mathcal{B}\|\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}\|^2_{L^2(\mathbb{T}^3)}(t) + \|\vartheta^{1/2}\partial_\sigma^{\alpha,\beta}v\|^2_{L^2(\mathbb{T}^3)}(t)\\ &\quad + \|\vartheta^{1/2}\partial_\sigma^{\alpha,\beta}w\|^2_{L^2(\mathbb{T}^3)}(t) \le \mathcal{A}\|\partial_\sigma^{\alpha,\beta}\widetilde{q}_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^3)}\\ &\quad + \mathcal{B}\|\partial_\sigma^{\alpha,\beta}\widetilde{\mathcal{H}}_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^3)} + \|\vartheta_{\mathrm{in}}^{1/2}\partial_\sigma^{\alpha,\beta}v_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^3)} + \|\vartheta_{\mathrm{in}}^{1/2}\partial_\sigma^{\alpha,\beta}w_{\mathrm{in}}\|^2_{L^2(\mathbb{T}^3)}\\ &\quad + (\varepsilon^{\mu-\sigma}+1)\int_0^t\Big(\big[\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s)\big]^4 + \big[\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s)\big]^2\Big)\,ds.\end{aligned} \tag{46}$$

While we only show (46) with $\alpha+\beta=3$, it holds with $\alpha+\beta=0,1,2$, which can be shown in a similar, if not simpler, way. Therefore, one can conclude from (46) that,

$$\begin{aligned}&\sup_{0\le s\le t}[\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s)]^2 \le C[\mathcal{E}_{\sigma,3}(\widetilde{q}_{\mathrm{in}},\widetilde{\mathcal{H}}_{\mathrm{in}},v_{\mathrm{in}},w_{\mathrm{in}})]^2\\ &\quad + C(\varepsilon^{\mu-\sigma}+1)\int_0^t\Big(\big[\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s)\big]^4\\ &\quad + \big[\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s)\big]^2\Big)\,ds,\end{aligned} \tag{47}$$

for some constant C, independent of ε. Recall that $\mu\in(0,1)$. Consequently, for $\sigma\in(0,\mu]$, after applying Grönwall's inequality to (47), there exists $T_\sigma\in(0,\infty)$, depending only on $\mathcal{E}_{\sigma,3}(\widetilde{q}_{\mathrm{in}},\widetilde{\mathcal{H}}_{\mathrm{in}},v_{\mathrm{in}},w_{\mathrm{in}})$, such that

$$\sup_{0\le s\le T_\sigma}\mathcal{E}_{\sigma,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s) \le C\mathcal{E}_{\sigma,3}(\widetilde{q}_{\mathrm{in}},\widetilde{\mathcal{H}}_{\mathrm{in}},v_{\mathrm{in}},w_{\mathrm{in}}), \tag{48}$$

with some constant $C\in(0,\infty)$, independent of ε.

In particular, let $\sigma=\mu$, and denote by $T := T_\mu$. We have shown that

$$\sup_{0\le s\le T}\mathcal{E}_{\mu,3}(\widetilde{q},\widetilde{\mathcal{H}},v,w)(s) \le C\mathcal{E}_{\mu,3}(\widetilde{q}_{\mathrm{in}},\widetilde{\mathcal{H}}_{\mathrm{in}},v_{\mathrm{in}},w_{\mathrm{in}}), \tag{49}$$

with some constant $C\in(0,\infty)$, independent of ε. Proposition 3 follows from (48).

4. The Soundproof Approximation

In this section, we focus on the proof of our first main theorem, i.e., Theorem 1. As mentioned in the introduction, the motivation of the soundproof approximation is due to the fact that the acoustic oscillator $\mathcal{L}_a$ induces a faster oscillation than the internal wave oscillator $\mathcal{L}_g$ in system (17), which leads to faster averaging of acoustic waves. Our soundproof model (20) preserves the internal gravity waves while filtering out the acoustic waves. In particular, if initial data do not carry any acoustic waves, solutions driven by (17) and (20) with the same initial data should produce solutions close to each other. Proving this statement is the main objective of this section.

However, to achieve our goal, we will need to introduce an intermediate model in Sect. 4.1. This is to handle the terms on the right hand side of (17) due to stratification, in contrast to [6,7]. Therefore, the soundproof approximation is done in two steps: approximation by the intermediate model of (17) in sects. 4.1 and 4.2; approximation by the soundproof model of the intermediate model in sects. 4.3 and 4.4.

4.1. The Intermediate Model

Here we analyse the intermediate model already introduced in the introduction in (1). In terms of the current notation, we utilise the replacements

$$\begin{gathered}\pi = \frac{p_{\mathrm{ms}}}{\mathcal{C}}\,, \qquad \theta = -\frac{\widetilde{\mathcal{H}}_{\mathrm{ms}}}{\mathcal{C}}\,, \qquad S^\varepsilon = -\frac{1}{\mathcal{C}}\left(\frac{1}{\mathcal{B}} + \varepsilon^\nu \widetilde{G}\widetilde{\mathcal{H}}_{\mathrm{ms}}\right)\,, \\ P^\varepsilon = 1 - \varepsilon \widetilde{P}^\varepsilon(z)\,, \qquad \frac{1}{P^\varepsilon}\frac{d\widetilde{P}^\varepsilon}{dz} = \mathcal{A}(\mathcal{C}G + \varepsilon^\mu \overline{\mathcal{H}}_0)\,, \end{gathered} \tag{50}$$

together with the obvious replacements $(v, w) = (v_{\mathrm{ms}}, w_{\mathrm{ms}})$, and obtain

$$\begin{cases} \mathrm{div}_h\, v_{\mathrm{ms}} + \partial_z w_{\mathrm{ms}} = \varepsilon\mathcal{A}(\mathcal{C}G + \varepsilon^\mu \overline{\mathcal{H}}_0) w_{\mathrm{ms}}, \\ \mathcal{B}\partial_t \widetilde{\mathcal{H}}_{\mathrm{ms}} + \mathcal{B} v_{\mathrm{ms}} \cdot \nabla_h \widetilde{\mathcal{H}}_{\mathrm{ms}} + \mathcal{B} w_{\mathrm{ms}} \partial_z \widetilde{\mathcal{H}}_{\mathrm{ms}} - \dfrac{1}{\varepsilon^\nu} w_{\mathrm{ms}} = \mathcal{B}\widetilde{G} \cdot \widetilde{\mathcal{H}}_{\mathrm{ms}} w_{\mathrm{ms}}, \\ \mathcal{C}\partial_t v_{\mathrm{ms}} + \mathcal{C} v_{\mathrm{ms}} \cdot \nabla_h v_{\mathrm{ms}} + \mathcal{C} w_{\mathrm{ms}} \partial_z v_{\mathrm{ms}} + \nabla_h p_{\mathrm{ms}} = 0, \\ \mathcal{C}\partial_t w_{\mathrm{ms}} + \mathcal{C} v_{\mathrm{ms}} \cdot \nabla_h w_{\mathrm{ms}} + \mathcal{C} w_{\mathrm{ms}} \partial_z w_{\mathrm{ms}} + \partial_z p_{\mathrm{ms}} + \dfrac{1}{\varepsilon^\nu} \widetilde{\mathcal{H}}_{\mathrm{ms}} = 0, \end{cases} \tag{51}$$

where p_{ms} is determined by, after calculating $\mathrm{div}_h\,(\phi_\varepsilon (51)_3) + \partial_z(\phi_\varepsilon (51)_4)$,

$$\begin{aligned} &-\Big(\mathrm{div}_h\,(\phi_\varepsilon \nabla_h p_{\mathrm{ms}}) + \partial_z(\phi_\varepsilon \partial_z p_{\mathrm{ms}})\Big) = \mathcal{C}\Big(\phi_\varepsilon (\nabla_h v_{\mathrm{ms}})^\top : \nabla_h v_{\mathrm{ms}} \\ &\quad + 2\phi_\varepsilon \partial_z v_{\mathrm{ms}} \cdot \nabla_h w_{\mathrm{ms}} + \phi_\varepsilon (\partial_z w_{\mathrm{ms}})^2 \\ &\quad - w_{\mathrm{ms}} \partial_z \phi_\varepsilon \mathrm{div}_h\, v_{\mathrm{ms}} - (w_{\mathrm{ms}})^2 \partial_z^2 \phi_\varepsilon \\ &\quad - w_{\mathrm{ms}} \partial_z \phi_\varepsilon \partial_z w_{\mathrm{ms}}\Big) + \frac{1}{\varepsilon^\nu} \partial_z(\phi_\varepsilon \widetilde{\mathcal{H}}_{\mathrm{ms}}), \\ &\int p_{\mathrm{ms}}\, d\vec{x} = 0, \end{aligned} \tag{52}$$

with

$$\phi_\varepsilon := \phi_\varepsilon(z) = e^{-\varepsilon\mathcal{A}\int_0^z (\mathcal{C}G(z') + \varepsilon^\mu \overline{\mathcal{H}}_0(z'))\, dz'}. \tag{53}$$

Notice that, $(51)_1$ is equivalent to

$$\mathrm{div}_h\,(\phi_\varepsilon v_{\mathrm{ms}}) + \partial_z(\phi_\varepsilon w_{\mathrm{ms}}) = 0. \tag{54}$$

We list some estimates of p_{ms}, induced by the elliptic estimates on (52):

$$\big\|p_{\mathrm{ms}}\big\|_{H^4(\mathbb{T}^3)} \lesssim \big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|^2_{H^3(\mathbb{T}^3)} + \frac{1}{\varepsilon^\nu}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}, \tag{55}$$

$$\begin{aligned}\big\|\partial_t p_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)} &\lesssim \big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}\big\|\partial_t v_{\mathrm{ms}}, \partial_t w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}\\ &\quad + \frac{1}{\varepsilon^\nu}\big\|\partial_t\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}.\end{aligned} \tag{56}$$

In addition, from $(51)_2$, $(51)_3$, and $(51)_4$, one can establish that

$$\big\|\partial_t\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)} \lesssim \big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)} + \frac{1}{\varepsilon^\nu}\big\|w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}, \tag{57}$$

$$\big\|\partial_t v_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)} \lesssim \big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}\big\|v_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)} + \big\|p_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}, \tag{58}$$

$$\begin{aligned}\big\|\partial_t w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)} &\lesssim \big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}\big\|w_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)} + \big\|p_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}\\ &\quad + \frac{1}{\varepsilon^\nu}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|_{H^2(\mathbb{T}^3)}.\end{aligned} \tag{59}$$

We point out here, the terms of $\mathcal{O}(\frac{1}{\varepsilon^\nu})$, above, although singular, will be used later together with multiplier ε or ε^μ (for instance, see $\mathfrak{I}_3$ of (67), below), which corresponds to the error $\mathcal{O}(\varepsilon^{\mu-\nu})$ in Theorem 1.

We claim that for any initial data $(\widetilde{\mathcal{H}}_{\mathrm{ms,in}}, v_{\mathrm{ms,in}}, w_{\mathrm{ms,in}}) \in H^3$, satisfying the pseudo-incompressible condition $(51)_1$, there is $T_{\mathrm{ms}} \in (0,\infty)$, depending only on $\big\|\widetilde{\mathcal{H}}_{\mathrm{ms,in}}, v_{\mathrm{ms,in}}, w_{\mathrm{ms,in}}\big\|_{H^3(\mathbb{T}^3)}$, such that

$$\sup_{0\le s\le T_{\mathrm{ms}}}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}, v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}(s) \le C\big\|\widetilde{\mathcal{H}}_{\mathrm{ms,in}}, v_{\mathrm{ms,in}}, w_{\mathrm{ms,in}}\big\|_{H^3(\mathbb{T}^3)} \tag{60}$$

where $C \in (0,\infty)$ is independent of ε. The proof of (60) follows from standard energy estimates. In fact, applying ∂^j, $j = 0,1,2,3$, to (60), after taking the L^2-inner product of the resultant equations with

$$2\partial^j p_{\mathrm{ms}}, 2\partial^j\widetilde{\mathcal{H}}_{\mathrm{ms}}, 2\partial^j v_{\mathrm{ms}}, 2\partial^j w_{\mathrm{ms}},$$

respectively, one can conclude that the summation of the resultant estimates is, thanks to (55),

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{B}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}\big\|^2_{H^3(\mathbb{T}^3)} + \mathcal{C}\big\|v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|^2_{H^3(\mathbb{T}^3)}\Big) \le C\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}, v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|^3_{H^3(\mathbb{T}^3)}\\ &\qquad + 2\varepsilon\mathcal{A}\sum_{j=0}^{3}\int\partial^j\big((\mathcal{C}G + \varepsilon^\mu\overline{\mathcal{H}}_0)w_{\mathrm{ms}}\big)\partial^j p_{\mathrm{ms}}\\ &\quad \le C\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}, v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|^3_{H^3(\mathbb{T}^3)} + C\big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|^2_{H^3(\mathbb{T}^3)},\end{aligned} \tag{61}$$

where we have used the fact that $0 < \varepsilon < 1$ and $0 < \nu < 1$. We would like to point out that, while we have omitted the details in (61), the quadratic terms in (51) are handled in the same manner as in Sect. 3. Namely, we use the following commutator estimate: for $\beta \ge 3$

$$\big\|[\partial^\beta, f_3\partial]g_3\big\|_{L^2(\mathbb{T}^3)} \le K\big\|f_3\big\|_{H^\beta(\mathbb{T}^3)}\big\|g_3\big\|_{H^\beta(\mathbb{T}^3)}, \tag{62}$$

where $K \in (0,\infty)$. The proof of (62) is similar to that of (33), and thus is omitted here. In addition, after the first inequality of (61), the coefficient ε in the second term guarantees that even though $p_{\mathrm{ms}} = \mathcal{O}(\frac{1}{\varepsilon^\nu})$ according to (55), there is no singular coefficient in the estimates. Thus (60) follows after applying Grönwall's inequality.

4.2. Intermediate Approximation

We would like to compare the solutions to (17) and (51). Denote by

$$\begin{aligned} \widetilde{q}_{\mathrm{ms},\delta} &:= \widetilde{q} - \varepsilon p_{\mathrm{ms}}, \\ \widetilde{\mathcal{H}}_{\mathrm{ms},\delta} &:= \widetilde{\mathcal{H}} - \widetilde{\mathcal{H}}_{\mathrm{ms}}, \\ v_{\mathrm{ms},\delta} &:= v - v_{\mathrm{ms}}, \\ w_{\mathrm{ms},\delta} &:= w - w_{\mathrm{ms}}. \end{aligned} \tag{63}$$

Also, we use $\mathcal{K}_1$ to represent the total bound of solutions to (17) and (51), i.e., for any fixed $\sigma \in (0, \mu]$,

$$\sup_{0\leq s\leq T} \mathcal{E}_{\sigma,3}(\widetilde{q}, \widetilde{\mathcal{H}}, v, w)(s) + \sup_{0\leq s\leq T_{\mathrm{ms}}} \big\|\widetilde{\mathcal{H}}_{\mathrm{ms}}, v_{\mathrm{ms}}, w_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)}(s) \leq \mathcal{K}_1, \tag{64}$$

which are obtained in (48) and (60).

In this section, we prove

$$\begin{aligned} &\sup_{0\leq s\leq T_{\mathrm{ms},\delta}} \big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}(s) \\ &\quad \leq C_{\mathcal{K}_1}\big(\big\|\widetilde{q}_{\mathrm{ms},\delta,\mathrm{in}}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta,\mathrm{in}}, v_{\mathrm{ms},\delta,\mathrm{in}}, w_{\mathrm{ms},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)} + (\varepsilon + \varepsilon^{\max\{\mu-\nu,\mu-\sigma\}})\big). \end{aligned} \tag{65}$$

Regime 1: $\mu - \nu \geq \mu - \sigma$. We first, by multiplying the first equation in (17) with ϖ_0/ϖ, recalling ϖ as given in (9) and (12) [i.e., reversing the reformulation of the $\widetilde{q}$ equation from (11) to (13)], system (17) can be written as

$$\begin{cases} \mathcal{A}\varpi_0\varpi^{-1}\partial_t\widetilde{q} + \mathcal{A}\varpi_0\varpi^{-1}v\cdot\nabla_h\widetilde{q} + \mathcal{A}\varpi_0\varpi^{-1}w\partial_z\widetilde{q} \\ \qquad + \dfrac{1}{\varepsilon}(\mathrm{div}_h\, v + \partial_z w) = \mathcal{A}\varpi_0\mathcal{C}G\varpi^{-1}w + \varepsilon^\mu\mathcal{A}\varpi_0\overline{\mathcal{H}}_0\varpi^{-1}w, \\ \mathcal{B}\partial_t\widetilde{\mathcal{H}} + \mathcal{B}v\cdot\nabla_h\widetilde{\mathcal{H}} + \mathcal{B}w\partial_z\widetilde{\mathcal{H}} - \dfrac{1}{\varepsilon^\nu}w = \mathcal{B}\widetilde{G}\cdot\widetilde{\mathcal{H}}w, \\ \vartheta\partial_t v + \vartheta v\cdot\nabla_h v + \vartheta w\partial_z v + \dfrac{1}{\varepsilon}\nabla_h\widetilde{q} = 0, \\ \vartheta\partial_t w + \vartheta v\cdot\nabla_h w + \vartheta w\partial_z w + \dfrac{1}{\varepsilon}\partial_z\widetilde{q} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}} = 0, \end{cases} \tag{17'}$$

After comparing (17') and (51), one can derive that $(\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta})$ satisfies

$$\begin{cases} \mathcal{A}\varpi_0\varpi^{-1}\partial_t\widetilde{q}_{\mathrm{ms},\delta} + \mathcal{A}\varpi_0\varpi^{-1}v\cdot\nabla_h\widetilde{q}_{\mathrm{ms},\delta} + \mathcal{A}\varpi_0\varpi^{-1}w\partial_z\widetilde{q}_{\mathrm{ms},\delta} \\ \qquad + \dfrac{1}{\varepsilon}(\mathrm{div}_h\, v_{\mathrm{ms},\delta} + \partial_z w_{\mathrm{ms},\delta}) \\ \quad = -\varepsilon\mathcal{A}\varpi_0\big(\varpi^{-1}\partial_t p_{\mathrm{ms}} + \varpi^{-1}v\cdot\nabla_h p_{\mathrm{ms}} + \varpi^{-1}w\partial_z p_{\mathrm{ms}}\big) \\ \qquad + \mathcal{A}\varpi_0\mathcal{C}G(\varpi^{-1} - \varpi_0^{-1})w + \mathcal{A}\mathcal{C}Gw_{\mathrm{ms},\delta} + \varepsilon^\mu\mathcal{A}\varpi_0\overline{\mathcal{H}}_0(\varpi^{-1} - \varpi_0^{-1})w \\ \qquad + \varepsilon^\mu\mathcal{A}\overline{\mathcal{H}}_0 w_{\mathrm{ms},\delta}, \\ \mathcal{B}\partial_t\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} + \mathcal{B}v\cdot\nabla_h\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} + \mathcal{B}w\partial_z\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} - \dfrac{1}{\varepsilon^\nu}w_{\mathrm{ms},\delta} \\ \quad = -\mathcal{B}\big(v_{\mathrm{ms},\delta}\cdot\nabla_h\widetilde{\mathcal{H}}_{\mathrm{ms}} + w_{\mathrm{ms},\delta}\partial_z\widetilde{\mathcal{H}}_{\mathrm{ms}}\big) \\ \qquad + \mathcal{B}\widetilde{G}\cdot\big(\widetilde{\mathcal{H}}w_{\mathrm{ms},\delta} + \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}w_{\mathrm{ms}}\big), \\ \vartheta\partial_t v_{\mathrm{ms},\delta} + \vartheta v\cdot\nabla_h v_{\mathrm{ms},\delta} + \vartheta w\partial_z v_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon}\nabla_h\widetilde{q}_{\mathrm{ms},\delta} \\ \quad = \mathcal{C}^{-1}\big(\varepsilon^\mu\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}}\big)\nabla_h p_{\mathrm{ms}} \\ \qquad - \vartheta v_{\mathrm{ms},\delta}\cdot\nabla_h v_{\mathrm{ms}} - \vartheta w_{\mathrm{ms},\delta}\partial_z v_{\mathrm{ms}}, \\ \vartheta\partial_t w_{\mathrm{ms},\delta} + \vartheta v\cdot\nabla_h w_{\mathrm{ms},\delta} + \vartheta w\partial_z w_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon}\partial_z\widetilde{q}_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} \\ \quad = \mathcal{C}^{-1}\big(\varepsilon^\mu\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}}\big)\big(\partial_z p_{\mathrm{ms}} + \dfrac{1}{\varepsilon^\nu}\widetilde{\mathcal{H}}_{\mathrm{ms}}\big) \\ \qquad - \vartheta v_{\mathrm{ms},\delta}\cdot\nabla_h w_{\mathrm{ms}} - \vartheta w_{\mathrm{ms},\delta}\partial_z w_{\mathrm{ms}}. \end{cases} \tag{66}$$

Now, we consider the L^2-inner product of equations in (66) with

$$2\widetilde{q}_{\mathrm{ms},\delta}, 2\widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, 2v_{\mathrm{ms},\delta}, 2w_{\mathrm{ms},\delta},$$

respectively. After applying integration by parts and summing up the resultant equations, one can write down that

$$\begin{aligned} &\frac{d}{dt}\Big(\mathcal{A}\varpi_0\big\|\varpi^{-1/2}\widetilde{q}_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} + \mathcal{B}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} + \big\|\vartheta^{1/2}v_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} \\ &\quad + \big\|\vartheta^{1/2}w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)}\Big) = \sum_{j=1}^{4}\mathfrak{I}_j, \end{aligned} \tag{67}$$

where

$$\begin{aligned} \mathfrak{I}_1 &:= \int\Big(\mathcal{A}\varpi_0\partial_t(\varpi^{-1})|\widetilde{q}_{\mathrm{ms},\delta}|^2 + \partial_t\vartheta|v_{\mathrm{ms},\delta}|^2 + \partial_t\vartheta|w_{\mathrm{ms},\delta}|^2\Big)\,d\vec{x}, \\ \mathfrak{I}_2 &:= \int\Big(\mathcal{A}\varpi_0\big(\mathrm{div}_h\,(\varpi^{-1}v) + \partial_z(\varpi^{-1}w)\big)|\widetilde{q}_{\mathrm{ms},\delta}|^2 + \mathcal{B}\big(\mathrm{div}_h\,v + \partial_z w\big)|\widetilde{\mathcal{H}}_{\mathrm{ms},\delta}|^2 \\ &\quad + \big(\mathrm{div}_h\,(\vartheta v) + \partial_z(\vartheta w)\big)\big(|v_{\mathrm{ms},\delta}|^2 + |w_{\mathrm{ms},\delta}|^2\big)\Big)\,d\vec{x}, \\ \mathfrak{I}_3 &:= -2\varepsilon\mathcal{A}\varpi_0\int\big(\varpi^{-1}\partial_t p_{\mathrm{ms}} + \varpi^{-1}v\cdot\nabla_h p_{\mathrm{ms}} + \varpi^{-1}w\partial_z p_{\mathrm{ms}}\big)\widetilde{q}_{\mathrm{ms},\delta}\,d\vec{x} \\ &\quad + 2\varepsilon^{\mu}\mathcal{C}^{-1}\int\big(\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\nu}\widetilde{G}\widetilde{\mathcal{H}}\big)\big(\nabla_h p_{\mathrm{ms}}\cdot v_{\mathrm{ms},\delta} + \partial_z p_{\mathrm{ms}} w_{\mathrm{ms},\delta} \\ &\quad + \frac{1}{\varepsilon^{\nu}}\widetilde{\mathcal{H}}_{\mathrm{ms}} w_{\mathrm{ms},\delta}\big)\,d\vec{x}, \\ \mathfrak{I}_4 &:= 2\mathcal{A}\varpi_0\int\Big(\mathcal{C}G(\varpi^{-1} - \varpi_0^{-1})w\widetilde{q}_{\mathrm{ms},\delta} + \varepsilon^{\mu}\overline{\mathcal{H}}_0(\varpi^{-1} - \varpi_0^{-1})w\widetilde{q}_{\mathrm{ms},\delta} \\ &\quad + \mathcal{C}G\varpi_0^{-1}w_{\mathrm{ms},\delta}\widetilde{q}_{\mathrm{ms},\delta} + \varepsilon^{\mu}\overline{\mathcal{H}}_0\varpi_0^{-1}w_{\mathrm{ms},\delta}\widetilde{q}_{\mathrm{ms},\delta}\Big)\,d\vec{x} \\ &\quad + 2\mathcal{B}\int\Big(-(v_{\mathrm{ms},\delta}\cdot\nabla_h\widetilde{\mathcal{H}}_{\mathrm{ms}}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} + w_{\mathrm{ms},\delta}\partial_z\widetilde{\mathcal{H}}_{\mathrm{ms}}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta}) \\ &\quad + \widetilde{G}\cdot(\widetilde{\mathcal{H}}w_{\mathrm{ms},\delta}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} + \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}w_{\mathrm{ms}}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta})\,d\vec{x} \\ &\quad - 2\int\vartheta\Big(v_{\mathrm{ms},\delta}\cdot\nabla_h v_{\mathrm{ms}}\cdot v_{\mathrm{ms},\delta} + w_{\mathrm{ms},\delta}\partial_z v_{\mathrm{ms}}\cdot v_{\mathrm{ms},\delta} \\ &\quad + v_{\mathrm{ms},\delta}\cdot\nabla_h w_{\mathrm{ms}}w_{\mathrm{ms},\delta} + w_{\mathrm{ms},\delta}\partial_z w_{\mathrm{ms}}w_{\mathrm{ms},\delta}\Big)\,d\vec{x}. \end{aligned}$$

Owing to (9), (12), and (18),

$$\big\|\partial_t(\varpi^{-1}), \partial_t\vartheta\big\|_{L^\infty(\mathbb{T}^3)} \leq \varepsilon C_{\mathcal{K}_1}\big\|\partial_t\widetilde{q}\big\|_{H^2(\mathbb{T}^3)} + \varepsilon^{\mu+\nu}C_{\mathcal{K}_1}\big\|\partial_t\widetilde{\mathcal{H}}\big\|_{H^2(\mathbb{T}^3)}.$$

Therefore,

$$\mathfrak{I}_1 \leq C_{\mathcal{K}_1}(\varepsilon^{1-\sigma} + \varepsilon^{\mu+\nu-\sigma})\big\|\widetilde{q}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)}.$$

Similarly,

$$\mathfrak{I}_2 \leq C_{\mathcal{K}_1}\big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)},$$

and

$$\mathfrak{I}_4 \leq C_{\mathcal{K}_1}\big(\big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} + \varepsilon\big\|\widetilde{q}_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}\big).$$

To estimate $\mathfrak{I}_3$, owing to (55), (56), (57), (58), and (59), after a tedious but straightforward calculations, one can conclude that, since $\mu + 2\nu = 1$,

$$\begin{aligned}\mathfrak{I}_3 \leq C_{\mathcal{K}_1}\big(\varepsilon\big\|\partial_t p_{\mathrm{ms}}\big\|_{H^3(\mathbb{T}^3)} + (\varepsilon+\varepsilon^{\mu})\big\|p_{\mathrm{ms}}\big\|_{H^4(\mathbb{T}^3)}\big)\big\|\widetilde{q}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)} \\ + \varepsilon^{\mu-\nu}C_{\mathcal{K}_1}\big\|w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)} \leq C_{\mathcal{K}_1}(\varepsilon+\varepsilon^{\mu-\nu})\big\|\widetilde{q}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}.\end{aligned} \tag{68}$$

Therefore, one can conclude from (67) that, provided $\varepsilon \ll 1$ small enough, for any $t \in (0, \min\{T, T_{\mathrm{ms}}\}]$, since $\sigma \leq \mu < 1$,

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{A}\varpi_0\big\|\varpi^{-1/2}\widetilde{q}_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} + \mathcal{B}\big\|\widetilde{\mathcal{H}}_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} + \big\|\vartheta^{1/2}v_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} \\ &\quad + \big\|\vartheta^{1/2}w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)}\Big) \leq C_{\mathcal{K}_1}\big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|^2_{L^2(\mathbb{T}^3)} \\ &\quad + C_{\mathcal{K}_1}(\varepsilon+\varepsilon^{\mu-\nu})\big\|\widetilde{q}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}.\end{aligned}$$

Therefore, after applying Grönwall's inequality, there exists $T_{\mathrm{ms},\delta} \in (0, \min\{T, T_{\mathrm{ms}}\}]$, depending only on

$$\big\|\widetilde{q}_{\mathrm{ms},\delta,\mathrm{in}}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta,\mathrm{in}}, v_{\mathrm{ms},\delta,\mathrm{in}}, w_{\mathrm{ms},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)},$$

such that

$$\begin{aligned}&\sup_{0\leq s\leq T_{\mathrm{ms},\delta}}\big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}(s) \\ &\quad \leq C_{\mathcal{K}_1}\big(\big\|\widetilde{q}_{\mathrm{ms},\delta,\mathrm{in}}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta,\mathrm{in}}, v_{\mathrm{ms},\delta,\mathrm{in}}, w_{\mathrm{ms},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)} + (\varepsilon+\varepsilon^{\mu-\nu})\big).\end{aligned} \tag{69}$$

Here $C_{\mathcal{K}_1} \in (0, \infty)$ is a constant depending only on $\mathcal{K}_1$ given in (64).

Regime 2: $\mu - \nu < \mu - \sigma$. Recalling that $\vartheta = \mathcal{C} + \mathcal{O}(\varepsilon^{\mu})$ as in (18), instead of (66), one can write down the following system by rewriting the v and w components:

$$\begin{cases}\text{The first and second equations as in (66)} \\ \mathcal{C}\partial_t v_{\mathrm{ms},\delta} + \mathcal{C}v\cdot\nabla_h v_{\mathrm{ms},\delta} + \mathcal{C}w\partial_z v_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon}\nabla_h\widetilde{q}_{\mathrm{ms},\delta} \\ \qquad = \big(\varepsilon^{\mu}\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}}\big)(\partial_t v + v\cdot\nabla_h v + w\partial_z v) \\ \qquad\quad -\mathcal{C}v_{\mathrm{ms},\delta}\cdot\nabla_h v_{\mathrm{ms}} - \mathcal{C}w_{\mathrm{ms},\delta}\partial_z v_{\mathrm{ms}}, \\ \mathcal{C}\partial_t w_{\mathrm{ms},\delta} + \mathcal{C}v\cdot\nabla_h w_{\mathrm{ms},\delta} + \mathcal{C}w\partial_z w_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon}\partial_z\widetilde{q}_{\mathrm{ms},\delta} + \dfrac{1}{\varepsilon^{\nu}}\widetilde{\mathcal{H}}_{\mathrm{ms},\delta} \\ \qquad = \big(\varepsilon^{\mu}\widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}}\big)\big(\partial_t w + v\cdot\nabla_h w + w\partial_z w\big) \\ \qquad\quad -\mathcal{C}v_{\mathrm{ms},\delta}\cdot\nabla_h w_{\mathrm{ms}} - \mathcal{C}w_{\mathrm{ms},\delta}\partial_z w_{\mathrm{ms}}.\end{cases} \tag{70}$$

Then similar arguments as before will yield

$$\begin{aligned}&\sup_{0\leq s\leq T_{\mathrm{ms},\delta}}\big\|\widetilde{q}_{\mathrm{ms},\delta}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta}, v_{\mathrm{ms},\delta}, w_{\mathrm{ms},\delta}\big\|_{L^2(\mathbb{T}^3)}(s) \\ &\quad \leq C_{\mathcal{K}_1}\big(\big\|\widetilde{q}_{\mathrm{ms},\delta,\mathrm{in}}, \widetilde{\mathcal{H}}_{\mathrm{ms},\delta,\mathrm{in}}, v_{\mathrm{ms},\delta,\mathrm{in}}, w_{\mathrm{ms},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)} + (\varepsilon+\varepsilon^{\mu-\sigma})\big).\end{aligned} \tag{71}$$

Indeed, only the corresponding $\mathfrak{I}_3$ estimate is different, where the control of

$$\nabla_h p_{\mathrm{ms}}\cdot v_{\mathrm{ms},\delta} + \partial_z p_{\mathrm{ms}} w_{\mathrm{ms},\delta} + \frac{1}{\varepsilon^{\nu}}\widetilde{\mathcal{H}}_{\mathrm{ms}} w_{\mathrm{ms},\delta} = \mathcal{O}(\varepsilon^{-\nu})$$

is replaced by

$$(\partial_t v + v\cdot\nabla_h v + w\partial_z v)\cdot v_{\mathrm{ms},\delta} + (\partial_t w + v\cdot\nabla_h w + w\partial_z w)w_{\mathrm{ms},\delta} = \mathcal{O}(\varepsilon^{-\sigma}). \tag{72}$$

Estimate (65) follows from (69) and (71).

4.3. The Soundproof Model

For convenience of the reader, we recall that the soundproof model reads

$$\begin{cases} \operatorname{div}_h v_{\mathrm{sp}} + \partial_z w_{\mathrm{sp}} = 0, \\ \mathcal{B}\partial_t \widetilde{\mathcal{H}}_{\mathrm{sp}} + \mathcal{B} v_{\mathrm{sp}} \cdot \nabla_h \widetilde{\mathcal{H}}_{\mathrm{sp}} + \mathcal{B} w_{\mathrm{sp}} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp}} - \dfrac{1}{\varepsilon^\nu} w_{\mathrm{sp}} = \mathcal{B}\widetilde{G} \cdot \widetilde{\mathcal{H}}_{\mathrm{sp}} w_{\mathrm{sp}}, \\ \mathcal{C}\partial_t v_{\mathrm{sp}} + \mathcal{C} v_{\mathrm{sp}} \cdot \nabla_h v_{\mathrm{sp}} + \mathcal{C} w_{\mathrm{sp}} \partial_z v_{\mathrm{sp}} + \nabla_h p_{\mathrm{sp}} = 0, \\ \mathcal{C}\partial_t w_{\mathrm{sp}} + \mathcal{C} v_{\mathrm{sp}} \cdot \nabla_h w_{\mathrm{sp}} + \mathcal{C} w_{\mathrm{sp}} \partial_z w_{\mathrm{sp}} + \partial_z p_{\mathrm{sp}} + \dfrac{1}{\varepsilon^\nu} \widetilde{\mathcal{H}}_{\mathrm{sp}} = 0, \end{cases} \tag{20}$$

where p_{sp} is determined by

$$\begin{aligned} -\Delta p_{\mathrm{sp}} = {} & \mathcal{C}((\nabla_h v_{\mathrm{sp}})^\top : \nabla_h v_{\mathrm{sp}} + 2\partial_z v_{\mathrm{sp}} \cdot \nabla_h w_{\mathrm{sp}} + (\partial_z w_{\mathrm{sp}})^2) \\ & + \frac{1}{\varepsilon^\nu} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp}}, \qquad \int p_{\mathrm{sp}} \, d\vec{x} = 0. \end{aligned} \tag{73}$$

Then, following similar, if not simpler, arguments to those in Sect. 4.1 leads to the conclusion that: there exists $T_{\mathrm{sp}} \in (0, \infty)$, depending only on $\|\widetilde{\mathcal{H}}_{\mathrm{sp,in}}, v_{\mathrm{sp,in}}, w_{\mathrm{sp,in}}\|_{H^3(\mathbb{T}^3)}$, such that

$$\sup_{0 \le s \le T_{\mathrm{sp}}} \|\widetilde{\mathcal{H}}_{\mathrm{sp}}, v_{\mathrm{sp}}, w_{\mathrm{sp}}\|_{H^3(\mathbb{T}^3)}(s) \le C \|\widetilde{\mathcal{H}}_{\mathrm{sp,in}}, v_{\mathrm{sp,in}}, w_{\mathrm{sp,in}}\|_{H^3(\mathbb{T}^3)}, \tag{74}$$

with some constant $C \in (0, \infty)$, independent of ε.

Now we list the estimate of p_{sp}, induced by the elliptic estimates on (73):

$$\|p_{\mathrm{sp}}\|_{H^4(\mathbb{T}^3)} \le \|v_{\mathrm{sp}}, w_{\mathrm{sp}}\|^2_{H^3(\mathbb{T}^3)} + \frac{1}{\varepsilon^\nu} \|\widetilde{\mathcal{H}}_{\mathrm{sp}}\|_{H^3(\mathbb{T}^3)}. \tag{75}$$

4.4. Soundproof Approximation

Now we are ready to estimate the difference of solutions to (51) and (20). Denote by

$$\begin{aligned} p_{\mathrm{sp},\delta} &:= p_{\mathrm{ms}} - p_{\mathrm{sp}}, \\ \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} &:= \widetilde{\mathcal{H}}_{\mathrm{ms}} - \widetilde{\mathcal{H}}_{\mathrm{sp}}, \\ v_{\mathrm{sp},\delta} &:= v_{\mathrm{ms}} - v_{\mathrm{sp}}, \\ w_{\mathrm{sp},\delta} &:= w_{\mathrm{ms}} - w_{\mathrm{sp}}. \end{aligned} \tag{76}$$

Also, we use $\mathcal{K}_2$ to represent the total bound of solutions to (51) and (20), i.e.,

$$\sup_{0 \le s \le T_{\mathrm{ms}}} \|\widetilde{\mathcal{H}}_{\mathrm{ms}}, v_{\mathrm{ms}}, w_{\mathrm{ms}}\|_{H^3(\mathbb{T}^3)} + \sup_{0 \le s \le T_{\mathrm{sp}}} \|\widetilde{\mathcal{H}}_{\mathrm{sp}}, v_{\mathrm{sp}}, w_{\mathrm{sp}}\|_{H^3(\mathbb{T}^3)} \le \mathcal{K}_2, \tag{77}$$

which are obtained in (60) and (74).

After comparing (51) and (20), one can derive that $(p_{\mathrm{sp},\delta}, \widetilde{\mathcal{H}}_{\mathrm{sp},\delta}, v_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta})$ satisfies

$$\begin{cases} \operatorname{div}_h v_{\mathrm{sp},\delta} + \partial_z w_{\mathrm{sp},\delta} = \varepsilon \mathcal{A}(\mathcal{C}G + \varepsilon^\mu \overline{\mathcal{H}}_0) w_{\mathrm{ms}}, \\ \mathcal{B}\partial_t \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} + \mathcal{B} v_{\mathrm{ms}} \cdot \nabla_h \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} + \mathcal{B} w_{\mathrm{ms}} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} - \dfrac{1}{\varepsilon^\nu} w_{\mathrm{sp},\delta} \\ \qquad = -\mathcal{B}(v_{\mathrm{sp},\delta} \cdot \nabla_h \widetilde{\mathcal{H}}_{\mathrm{sp}} + w_{\mathrm{sp},\delta} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp}}) \\ \qquad\qquad + \mathcal{B}\widetilde{G} \cdot (\widetilde{\mathcal{H}}_{\mathrm{ms}} w_{\mathrm{sp},\delta} + \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} w_{\mathrm{sp}}), \\ \mathcal{C}\partial_t v_{\mathrm{sp},\delta} + \mathcal{C} v_{\mathrm{ms}} \cdot \nabla_h v_{\mathrm{sp},\delta} + \mathcal{C} w_{\mathrm{ms}} \partial_z v_{\mathrm{sp},\delta} + \nabla_h p_{\mathrm{sp},\delta} \\ \qquad = -\mathcal{C}(v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp}} + w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp}}), \\ \mathcal{C}\partial_t w_{\mathrm{sp},\delta} + \mathcal{C} v_{\mathrm{ms}} \cdot \nabla_h w_{\mathrm{sp},\delta} + \mathcal{C} w_{\mathrm{ms}} \partial_z w_{\mathrm{sp},\delta} + \partial_z p_{\mathrm{sp},\delta} + \dfrac{1}{\varepsilon^\nu} \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} \\ \qquad = -\mathcal{C}(v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp}} + w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp}}). \end{cases} \tag{78}$$

To write down the equation of $p_{\mathrm{sp},\delta}$, instead of using (52) and (73), we first rewrite

$$\begin{aligned} & v_{\mathrm{ms}} \cdot \nabla_h \begin{pmatrix} v_{\mathrm{sp},\delta} \\ w_{\mathrm{sp},\delta} \end{pmatrix} + w_{\mathrm{ms}} \partial_z \begin{pmatrix} v_{\mathrm{sp},\delta} \\ w_{\mathrm{sp},\delta} \end{pmatrix} \\ & = v_{\mathrm{sp},\delta} \cdot \nabla_h \begin{pmatrix} v_{\mathrm{sp},\delta} \\ w_{\mathrm{sp},\delta} \end{pmatrix} + w_{\mathrm{sp},\delta} \partial_z \begin{pmatrix} v_{\mathrm{sp},\delta} \\ w_{\mathrm{sp},\delta} \end{pmatrix} \\ & \quad + \begin{pmatrix} \mathrm{div}_h\,(v_{\mathrm{sp},\delta} \otimes v_{\mathrm{sp}}) + \partial_z (w_{\mathrm{sp}} v_{\mathrm{sp},\delta}) \\ \mathrm{div}_h\,(w_{\mathrm{sp},\delta} v_{\mathrm{sp}}) + \partial_z (w_{\mathrm{sp}} w_{\mathrm{sp},\delta}) \end{pmatrix}, \end{aligned}$$

and then after applying

$$\begin{pmatrix} \mathrm{div}_h \\ \partial_z \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} (78)_3 \\ (78)_4 \end{pmatrix},$$

one can derive that

$$\begin{aligned} & -\Delta p_{\mathrm{sp},\delta} = \varepsilon \mathcal{A}\mathcal{C}(\mathcal{C}G + \varepsilon^\mu \overline{\mathcal{H}}_0)\partial_t w_{\mathrm{ms}} + \mathcal{C}\Big(\mathrm{div}_h\, \mathrm{div}_h\,(v_{\mathrm{sp},\delta} \otimes v_{\mathrm{sp}}) \\ & \quad + \mathrm{div}_h\, \partial_z (w_{\mathrm{sp}} v_{\mathrm{sp},\delta}) + \partial_z \mathrm{div}_h\,(w_{\mathrm{sp},\delta} v_{\mathrm{sp}}) + \partial_z^2 (w_{\mathrm{sp}} w_{\mathrm{sp},\delta}) \\ & \quad + \mathrm{div}_h\,(v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp},\delta}) + \mathrm{div}_h\,(w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp},\delta}) + \partial_z (v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp},\delta}) \\ & \quad + \partial_z (w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp},\delta}) \Big) + \mathcal{C}\Big(\mathrm{div}_h\,(v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp}}) + \mathrm{div}_h\,(w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp}}) \\ & \quad + \partial_z (v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp}}) + \partial_z (w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp}}) \Big) + \frac{1}{\varepsilon^\nu} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp},\delta}, \\ & \int p_{\mathrm{sp},\delta}\, d\vec{x} = 0. \end{aligned} \tag{79}$$

Consequently, applying the standard elliptic estimate on (79) yields that

$$\begin{aligned} & \|p_{\mathrm{sp},\delta}\|_{L^2(\mathbb{T}^3)} \lesssim \varepsilon \|\partial_t w_{\mathrm{ms}}\|_{L^2(\mathbb{T}^3)} + \|v_{\mathrm{sp},\delta} \otimes v_{\mathrm{sp}}, w_{\mathrm{sp}} v_{\mathrm{sp},\delta}\|_{L^2(\mathbb{T}^3)} \\ & \quad + \|w_{\mathrm{sp},\delta} v_{\mathrm{sp}}, w_{\mathrm{sp}} w_{\mathrm{sp},\delta}, v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp},\delta}\|_{L^2(\mathbb{T}^3)} \\ & \quad + \|v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp},\delta}, v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp}}, w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp}}\|_{L^2(\mathbb{T}^3)} \\ & \quad + \|v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp}}, w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp}}\|_{L^2(\mathbb{T}^3)} + \frac{1}{\varepsilon^\nu} \|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}\|_{L^2(\mathbb{T}^3)} \\ & \lesssim C_{\mathcal{K}_2} \big(1 + \varepsilon^{-\nu}\big) \|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}, v_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta}\|_{L^2(\mathbb{T}^3)} + C_{\mathcal{K}_2} \big(\varepsilon + \varepsilon^{1-\nu}\big), \end{aligned} \tag{80}$$

where we have applied (55), (59), and the Sobolev embedding inequality in the last inequality.

Now we are ready to estimate the L^2 norm of $(p_{\mathrm{sp},\delta}, \widetilde{\mathcal{H}}_{\mathrm{sp},\delta}, v_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta})$. Indeed, after applying the L^2-inner product of equations in (78) with $2p_{\mathrm{sp},\delta}, 2\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}, 2v_{\mathrm{sp},\delta}, 2w_{\mathrm{sp},\delta}$, respectively, applying integration by parts, and summing up the resultant equations, one has

$$\frac{d}{dt}\Big(\mathcal{B} \|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}\|^2_{L^2(\mathbb{T}^3)} + \mathcal{C} \|v_{\mathrm{sp},\delta}, w_{\mathrm{sp},\delta}\|^2_{L^2(\mathbb{T}^3)} \Big) = \sum_{j=5}^{7} \mathfrak{I}_j$$

where

$$\begin{aligned} & \mathfrak{I}_5 := \int (\mathrm{div}_h\, v_{\mathrm{ms}} + \partial_z w_{\mathrm{ms}}) \big(\mathfrak{B} |\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}|^2 + \mathcal{C} |v_{\mathrm{sp},\delta}|^2 + \mathcal{C} |w_{\mathrm{sp},\delta}|^2 \big)\, d\vec{x}, \\ & \mathfrak{I}_6 := -2 \int \Big(\mathfrak{B} (v_{\mathrm{sp},\delta} \cdot \nabla_h \widetilde{\mathcal{H}}_{\mathrm{sp}} \widetilde{\mathcal{H}}_{\mathrm{sp},\delta} + w_{\mathrm{sp},\delta} \partial_z \widetilde{\mathcal{H}}_{\mathrm{sp}} \widetilde{\mathcal{H}}_{\mathrm{sp},\delta}) \\ & \quad + \mathcal{C} (v_{\mathrm{sp},\delta} \cdot \nabla_h v_{\mathrm{sp}} \cdot v_{\mathrm{sp},\delta} + w_{\mathrm{sp},\delta} \partial_z v_{\mathrm{sp}} \cdot v_{\mathrm{sp},\delta}) \\ & \quad + \mathcal{C} (v_{\mathrm{sp},\delta} \cdot \nabla_h w_{\mathrm{sp}} w_{\mathrm{sp},\delta} + w_{\mathrm{sp},\delta} \partial_z w_{\mathrm{sp}} w_{\mathrm{sp},\delta}) \Big)\, d\vec{x} \end{aligned}$$

$$+ 2\int \mathcal{B}\widetilde{G}\cdot(\widetilde{\mathcal{H}}_{\mathrm{ms}}w_{\mathrm{sp},\delta}+\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}w_{\mathrm{sp}})\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}\,d\vec{x},$$

$$\mathfrak{I}_7 := 2\varepsilon\mathcal{A}\int(\mathcal{C}G+\varepsilon^{\mu}\overline{\mathcal{H}}_0)w_{\mathrm{ms}}p_{\mathrm{sp},\delta}\,d\vec{x}.$$

Thanks to (80), one has

$$\mathfrak{I}_7 \le C_{\mathcal{K}_2}(\varepsilon+\varepsilon^{1-\nu})\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta},v_{\mathrm{sp},\delta},w_{\mathrm{sp},\delta}\big\|_{L^2(\mathbb{T}^3)}+C_{\mathcal{K}_2}(\varepsilon^2+\varepsilon^{2-\nu}),$$

while the estimates of $\mathfrak{I}_5$ and $\mathfrak{I}_6$ are straightforward. Hence, we have shown that

$$\begin{aligned}&\frac{d}{dt}\Big(\mathcal{B}\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta}\big\|^2_{L^2(\mathbb{T}^3)}+\mathcal{C}\big\|v_{\mathrm{sp},\delta},w_{\mathrm{sp},\delta}\big\|^2_{L^2(\mathbb{T}^3)}\Big)\\&\le C_{\mathcal{K}_2}\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta},v_{\mathrm{sp},\delta},w_{\mathrm{sp},\delta}\big\|^2_{L^2(\mathbb{T}^3)}\\&\quad+C_{\mathcal{K}_2}(\varepsilon+\varepsilon^{1-\nu})\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta},v_{\mathrm{sp},\delta},w_{\mathrm{sp},\delta}\big\|_{L^2(\mathbb{T}^3)}+C_{\mathcal{K}_2}(\varepsilon^2+\varepsilon^{2-\nu}).\end{aligned}$$

Consequently, after applying Grönwall's inequality, one can conclude that, there is $T_{\mathrm{sp},\delta}\in(0,\min\{T_{\mathrm{ms}},T_{\mathrm{sp}}\}]$, depending only on $\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta,\mathrm{in}},v_{\mathrm{sp},\delta,\mathrm{in}},w_{\mathrm{sp},\delta,\mathrm{in}}\|_{L^2(\mathbb{T}^3)}$, such that

$$\begin{aligned}&\sup_{0\le s\le T_{\mathrm{sp},\delta}}\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta},v_{\mathrm{sp},\delta},w_{\mathrm{sp},\delta}\big\|_{L^2(\mathbb{T}^3)}(s)\\&\quad\le C_{\mathcal{K}_2}\big(\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta,\mathrm{in}},v_{\mathrm{sp},\delta,\mathrm{in}},w_{\mathrm{sp},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)}+(\varepsilon+\varepsilon^{1-\nu})\big).\end{aligned}\tag{81}$$

Here $C_{\mathcal{K}_2}\in(0,\infty)$ is a constant depending only on $\mathcal{K}_2$ given in (77). In particular, (65), (80), and (81) imply that, since $\mu+2\nu=1$,

$$\begin{aligned}&\sup_{0\le s\le\min\{T_{\mathrm{ms},\delta}T_{\mathrm{sp},\delta}\}}\big\|\widetilde{q}-\varepsilon p_{\mathrm{sp}},\widetilde{\mathcal{H}}-\widetilde{\mathcal{H}}_{\mathrm{sp}},v-v_{\mathrm{sp}},w-w_{\mathrm{sp}}\big\|_{L^2(\mathbb{T}^3)}\\&\le C_{\mathcal{K}_1,\mathcal{K}_2}\big(\varepsilon^{\max\{\mu-\nu,\mu-\sigma\}}+\big\|\widetilde{q}_{\mathrm{ms},\delta,\mathrm{in}},\widetilde{\mathcal{H}}_{\mathrm{ms},\delta,\mathrm{in}},v_{\mathrm{ms},\delta,\mathrm{in}},w_{\mathrm{ms},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)}\\&\quad+\big\|\widetilde{\mathcal{H}}_{\mathrm{sp},\delta,\mathrm{in}},v_{\mathrm{sp},\delta,\mathrm{in}},w_{\mathrm{sp},\delta,\mathrm{in}}\big\|_{L^2(\mathbb{T}^3)}\big).\end{aligned}\tag{82}$$

Theorem 1 follows from (55), (60), (74), (75), and (82)

5. Fast-slow Decompositions: the Linear Theory

Our goal is to decompose the solution to (17) into waves with different frequencies. Ideally, due the appearance of two different scales of oscillation, we are expecting at least three waves.

(H4) To simplify our presentation, we will, from now on, assume that

$$\mathcal{A}=\mathcal{B}=\mathcal{C}=1.\tag{83}$$

A linear system associated with (17) is introduced in this section, using two oscillation operators, corresponding to the acoustic waves and the internal waves, respectively.

In addition, we will investigate an ε-dependent linear oscillation operator, associated with the linear system, which can be treated as a perturbation of the acoustic wave operator. The eigenvalue-eigenvector pairs associated with such oscillation operator will be investigated.

To be more precise, we introduce the following linear system:

$$\partial_t U+\frac{1}{\varepsilon}\mathcal{L}_a U+\frac{1}{\varepsilon^{\nu}}\mathcal{L}_g U=0,\tag{84}$$

where $U,\mathcal{L}_a$, $\mathcal{L}_g$ are defined as in (14). Roughly speaking, $\frac{1}{\varepsilon}\mathcal{L}_a U$ and $\frac{1}{\varepsilon^{\nu}}\mathcal{L}_g U$ are the driving forces of acoustic waves and internal waves, respectively. One can immediately see from (84), that, as $\varepsilon\to 0^+$, the oscillation induced by operator $\frac{1}{\varepsilon}\mathcal{L}_a$ is faster than the one induced by $\frac{1}{\varepsilon^{\nu}}\mathcal{L}_g$, meaning that the acoustic waves will oscillate faster and thus will be averaged out before the internal waves dissipate. This is exactly why we can use the soundproof system (20) as an approximation to (17).

In the following subsections, we will investigate the acoustic waves, internal waves, and mean flows, in the linear system (84).

5.1. Perturbed Acoustic Waves

In this subsection, we consider the following perturbed acoustic wave operator

$$\mathcal{L}_\varepsilon := \mathcal{L}_a + \varepsilon^{1-\nu}\mathcal{L}_g. \tag{85}$$

Then (84) is equivalent to

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U = 0. \tag{86}$$

Notice that $\mathcal{L}_\varepsilon$ can be viewed as a perturbation of $\mathcal{L}_a$. An ad hoc analysis will be that, the eigenvalues of $\mathcal{L}_\varepsilon$ lie within neighborhoods with width $\mathcal{O}(\varepsilon^{1-\nu})$ of the eigenvalues of $\mathcal{L}_a$. In particular, the eigenvalues corresponding to the acoustic free vector fields lies in an neighborhood with width $\mathcal{O}(\varepsilon^{1-\nu})$ of the origin. In view of (86), one can decompose the eigenvalues of $\frac{1}{\varepsilon}\mathcal{L}_\varepsilon$, corresponding to the wave decomposition of solutions to (86), into three kinds: the zero eigenvalue; the eigenvalues of $\mathcal{O}(\varepsilon^{-\nu})$ near the origin; the eigenvalues of $\mathcal{O}(\varepsilon^{-1})$ ($\mathcal{O}(\varepsilon^{-1}) \pm \mathcal{O}(\varepsilon^{-\nu})$ to be more precise). We will refer the waves corresponding to these three kinds of eigenvalues as the **mean flows**, the **perturbed internal waves**, and the **perturbed acoustic waves**, respectively. In the following, we shall make the above ad hoc discussion rigid.

Let

$$\begin{aligned}\mathfrak{V} := \big\{&U = (\widetilde{q}, \widetilde{\mathcal{H}}, v = (v_1, v_2)^\top, w)^\top \in C^\infty(\mathbb{T}^3; \mathbb{R}^5) |\\ &\text{Symmetry (SYM) is satisfied.}\big\}.\end{aligned} \tag{87}$$

We first investigate $\ker \mathcal{L}_\varepsilon$, i.e., the space associated with the zero eigenvalue. Let

$$\mathcal{P}_{\varepsilon,\mathrm{mf}} : \mathfrak{V} \mapsto \ker \mathcal{L}_\varepsilon. \tag{88}$$

Then

$$\begin{aligned}\ker \mathcal{L}_\varepsilon = \big\{&U_{\varepsilon,\mathrm{mf}} = (\widetilde{q}_{\varepsilon,\mathrm{mf}}, \mathcal{H}_{\varepsilon,\mathrm{mf}}, v_{\varepsilon,\mathrm{mf}}, w_{\varepsilon,\mathrm{mf}})^\top \in \mathfrak{V} |\\ &\widetilde{q}_{\varepsilon,\mathrm{mf}} = \widetilde{q}_{\varepsilon,\mathrm{mf}}(z) \in C^\infty(\mathbb{T}; \mathbb{R}), \widetilde{\mathcal{H}}_{\varepsilon,\mathrm{mf}} = -\varepsilon^{\nu-1}\partial_z \widetilde{q}_{\varepsilon,\mathrm{mf}},\\ &\mathrm{div}_h\, v_{\varepsilon,\mathrm{mf}} = 0, w_{\varepsilon,\mathrm{mf}} = 0\big\}.\end{aligned} \tag{89}$$

Denote by

$$U_{\varepsilon,\mathrm{mf}} = (\widetilde{q}_{\varepsilon,\mathrm{mf}}, \widetilde{\mathcal{H}}_{\varepsilon,\mathrm{mf}}, v_{\varepsilon,\mathrm{mf}}, w_{\varepsilon,\mathrm{mf}})^\top = \mathcal{P}_{\varepsilon,\mathrm{mf}}(U = (\widetilde{q}, \widetilde{\mathcal{H}}, v, w)^\top).$$

Then, to look for the representation of $\mathcal{P}_{\varepsilon,\mathrm{mf}}$, we calculate the following functional: for any $V = (a, b, \xi, \eta)^\top \in \ker \mathcal{L}_\varepsilon$

$$\begin{aligned}\big\|V - U\big\|^2_{L^2(\mathbb{T}^3)} = &\int \big(|\widetilde{q} - a|^2 + |\varepsilon^{\nu-1}\partial_z a + \widetilde{\mathcal{H}}|^2\big)\, d\vec{x}\\ &+ \int \big(|\xi - v|^2 + |w|^2\big)\, d\vec{x}.\end{aligned}$$

Then $U_{\varepsilon,\mathrm{mf}}$ should be the minimizer of the above functional subject to the condition $U_{\varepsilon,\mathrm{mf}} \in \ker \mathcal{L}_\varepsilon$. Then calculating the Euler-Lagrangian equations yields that $U_{\varepsilon,\mathrm{mf}} = \mathcal{P}_{\varepsilon,\mathrm{mf}}(U)$ is given by

$$\begin{aligned}&\widetilde{\mathcal{H}}_{\varepsilon,\mathrm{mf}} \equiv -\varepsilon^{\nu-1}\partial_z \widetilde{q}_{\varepsilon,\mathrm{mf}},\\ &v_{\varepsilon,\mathrm{mf}} \equiv v - \nabla_h \psi_v,\\ &w_{\varepsilon,\mathrm{mf}} \equiv 0,\end{aligned}$$

where $\widetilde{q}_{\varepsilon,\mathrm{mf}}$, ψ_v are solutions to

$$
\begin{aligned}
&-\varepsilon^{2(\nu-1)}\partial_{zz}\widetilde{q}_{\varepsilon,\mathrm{mf}} + \widetilde{q}_{\varepsilon,\mathrm{mf}} - \varepsilon^{\nu-1}\partial_z \int \widetilde{\mathcal{H}}\, dxdy(z) \\
&\quad - \int \widetilde{q}\, dxdy(z) = 0, \qquad \int \widetilde{q}_{\varepsilon,\mathrm{mf}}\, dz = \int \widetilde{q}\, d\vec{x}, \\
&\quad \text{and} \quad \Delta_h \psi_v = \mathrm{div}_h\, v, \qquad \int \psi_v\, dxdy = 0.
\end{aligned} \tag{90}
$$

We remind readers that $\ker \mathcal{L}_\varepsilon$ is nothing but the space of eigenfunctions corresponding to the zero eigenvalue of $\mathcal{L}_\varepsilon$. Next we focus on the non-zero eigenvalue problem of $\mathcal{L}_\varepsilon$, i.e., the structure of $(\ker \mathcal{L}_\varepsilon)^\perp$. Since $\mathcal{L}_\varepsilon$ is anti-symmetric, it suffices to investigate the pure imaginary eigenvalues with non-zero imaginary part, i.e.,

$$
i\omega U_\omega = \mathcal{L}_\varepsilon U_\omega, \quad \omega \neq 0, \quad U_\omega = (\widetilde{q}_\omega, \widetilde{\mathcal{H}}_\omega, v_\omega, w_\omega)^\top \in \mathfrak{V}. \tag{91}
$$

We will not discuss the representations of the solutions to the eigenvalue problem in this section. Instead, we would like to estimate the value of the eigenvalues, assuming we have found the eigenvalue-eigenvector pairs. The exact quantity calculation will be postponed in the next section using Fourier representation.

If $\widetilde{q}_\varepsilon \equiv 0$, then the eigenvalue problem (91) is reduced to

$$
v_\varepsilon = 0, \quad \partial_z w_\varepsilon = 0, \quad -\eta w_\varepsilon = i\omega \widetilde{\mathcal{H}}, \quad \eta \widetilde{\mathcal{H}} = i\omega w,
$$

where, hereafter, $\eta := \varepsilon^{1-\nu}$, which yields $U_\varepsilon = 0$ due to symmetry (SYM).

In the following, we assume, without loss of generality, $\widetilde{q}_\varepsilon \not\equiv 0$. Direct calculation of the eigenvalue problem (91) shows that

$$
-(\omega^2 - \eta^2)\Delta_h \widetilde{q}_\varepsilon - \omega^2 \partial_{zz}\widetilde{q}_\varepsilon = \omega^2(\omega^2 - \eta^2)\widetilde{q}_\varepsilon. \tag{92}
$$

Notice that when $\omega \simeq \eta$, (92) admits strong degeneracy.

In addition, we introduce the following eigenvalue problem:

$$
-\Delta_h \widetilde{q}_{\mathrm{ac}} - \partial_{zz}\widetilde{q}_{\mathrm{ac}} = \omega_{\mathrm{ac}}^2 \widetilde{q}_{\mathrm{ac}}. \tag{93}
$$

In fact, (93) can be seen as the counter-part of (92) from (91) for $\mathcal{L}_a$, i.e., the eigenvalue problem of the acoustic operator. Unsurprisingly, (93) is just (92) when $\eta = 0$, at least formally. We denote the eigenvalue-eigenfunction pairs of (93) as $(\omega_{\mathrm{ac},n}^\pm \widetilde{q}_{\mathrm{ac},n})|_{n=0,1,2,\cdots}$, where $\omega_{\mathrm{ac},0}^\pm = 0$, $|\omega_{\mathrm{ac},1}^\pm| < |\omega_{\mathrm{ac},2}^\pm| < \cdots$, $\int |\widetilde{q}_{\mathrm{ac},n}|^2\, d\vec{x} = 1$. Then it is easy to check

$$
\begin{aligned}
&\int \widetilde{q}_{\mathrm{ac},m}\widetilde{q}_{\mathrm{ac},n}\, d\vec{x} = \delta_{m,n}, \qquad \int \nabla_h \widetilde{q}_{\mathrm{ac},m} \cdot \nabla_h \widetilde{q}_{\mathrm{ac},n}\, d\vec{x} = \big\| \nabla_h \widetilde{q}_{\mathrm{ac},m} \big\|_{L^2(\mathbb{T}^3)}^2 \delta_{m,n}, \\
&\int \nabla_h \widetilde{q}_{\mathrm{ac},m} \cdot \nabla_h \widetilde{q}_{\mathrm{ac},n}\, d\vec{x} + \int \partial_z \widetilde{q}_{\mathrm{ac},m} \partial_z \widetilde{q}_{\mathrm{ac},n}\, d\vec{x} = |\omega_{\mathrm{ac},m}^\pm|^2 \delta_{m,n}, \\
&m, n \in \cup\{0, 1, 2, \ldots\}.
\end{aligned} \tag{94}
$$

Then, one can represent solution $\widetilde{q}_\varepsilon$ to (92) as

$$
\widetilde{q}_\varepsilon = \sum_{n=0,1,2,\cdots} Q_n \widetilde{q}_{\mathrm{ac},n}, \quad Q_n \in \mathbb{R}. \tag{95}
$$

After taking the L^2-inner product of (92) with $\widetilde{q}_{\mathrm{ac},m}$, it follows that, thanks to (94),

$$
\begin{aligned}
&Q_m \times [\omega^2(\omega^2 - \eta^2) - \omega^2 |\omega_{\mathrm{ac},m}^\pm|^2 + \eta^2 \big\| \nabla_h \widetilde{q}_{\mathrm{ac},m} \big\|_{L^2(\mathbb{T}^3)}^2] = 0, \\
&\quad \text{for any } m \in \{0, 1, 2, \cdots\}.
\end{aligned}
$$

Suppose that for some m, $Q_m \neq 0$. Then

$$
\omega^2(\omega^2 - \eta^2) - \omega^2 |\omega_{\mathrm{ac},m}^\pm|^2 + \eta^2 \big\| \nabla_h \widetilde{q}_{\mathrm{ac},m} \big\|_{L^2(\mathbb{T}^3)}^2 = 0. \tag{96}
$$

We **claim** that

$$|\omega_{\mathrm{ac},m}^{\pm}|^2 \le |\omega|^2 \le |\omega_{\mathrm{ac},m}^{\pm}|^2 + \eta^2. \tag{97}$$

The rest of this section is devoted to the proof of (97). Notice that, if $m = 0$, we have $\omega_{\mathrm{ac},0}^{\pm} = 0$, $\nabla_h \widetilde{q}_{\mathrm{ac},0} = 0$, which implies $\omega = 0$ or $|\omega| = \eta$. In particular $0 \le |\omega| \le \eta$, i.e., (97) holds.

Without loss of generality, we assume that $m \ge 1$ and $|\omega| > \eta$, below. Since, from (94), $\left\|\nabla_h \widetilde{q}_{\mathrm{ac},m}\right\|_{L^2(\mathbb{T}^3)}^2 \le |\omega_{\mathrm{ac},m}^{\pm}|^2$, one has, from (96), that

$$\omega^2(\omega^2 - \eta^2) \ge |\omega_{\mathrm{ac},m}^{\pm}|^2(\omega^2 - \eta^2),$$

which implies

$$|\omega| \ge |\omega_{\mathrm{ac},m}^{\pm}|. \tag{98}$$

On the other hand, (96) can be written as

$$\omega^2 - |\omega_{\mathrm{ac},m}^{\pm}|^2 = \eta^2 \left(1 - \frac{\left\|\nabla_h \widetilde{q}_{\mathrm{ac},m}\right\|_{L^2(\mathbb{T}^3)}^2}{\omega^2}\right) \le \eta^2.$$

Together with (98), this proves (97).

Therefore, we have proved the following lemma:

Lemma 6. *Let $i\omega$ be an eigenvalue of operator $\mathcal{L}_\varepsilon$. Then, there exists $m \in \{0, 1, 2, \cdots\}$, such that*

$$|\omega_{\mathrm{ac},m}^{\pm}|^2 \le \omega^2 \le |\omega_{\mathrm{ac},m}^{\pm}|^2 + \varepsilon^{2-2\nu},$$

where $\{i\omega_{\mathrm{ac},m}^{\pm}\}_{m\in\{0,1,2,\cdots\}}$ are the eigenvalues of $\mathcal{L}_a$.

In particular, Lemma 6 confirms the ad hoc analysis at the beginning of this section.

We remark that, (96) can be solved explicitly for ω^2. Indeed, there exist exactly two solutions $(\omega^2)_1$ and $(\omega^2)_2$ satisfying (97). We will make it more clear using Fourier representations in the next subsection.

5.2. Fourier Representations

Owing to the symmetry (SYM), we consider the follow Fourier expansion of U:

$$U = \sum_{k_h \in 2\pi\mathbb{Z}^2, k_z \in 2\pi\mathbb{N}^+ \cup\{0\}} \begin{pmatrix} Q_{(k_h,k_z)} e^{ik_h\cdot x} \cos(k_z z) \\ H_{(k_h,k_z)} e^{ik_h\cdot x} \sin(k_z z) \\ V_{(k_h,k_z)} e^{ik_h\cdot x} \cos(k_z z) \\ W_{(k_h,k_z)} e^{ik_h\cdot x} \sin(k_z z) \end{pmatrix}, \tag{99}$$

with

$$F_{(-k_h,k_z)} = F_{(k_h,k_z)}, \qquad F \in \{Q, H, V, W\}.$$

Then, with $\eta = \varepsilon^{1-\nu} \ll 1$, the eigenvalue problem (91) can be written as:

$$\begin{aligned} \omega_{(k_h,k_z)} Q_{(k_h,k_z)} &= k_h \cdot V_{(k_h,k_z)} - ik_z W_{(k_h,k_z)}, \\ \omega_{(k_h,k_z)} H_{(k_h,k_z)} &= i\eta W_{(k_h,k_z)}, \\ \omega_{(k_h,k_z)} V_{(k_h,k_z)} &= Q_{(k_h,k_z)} k_h, \\ \omega_{(k_h,k_z)} W_{(k_h,k_z)} &= ik_z Q_{(k_h,k_z)} - i\eta H_{(k_h,k_z)}. \end{aligned} \tag{100}$$

We investigate the solutions to (100) in the following three cases:

Case 1: $\omega_{(k_h,k_z)} = 0$. Then if $k_h \ne (0,0)$, one can easily check that

$$\begin{pmatrix} Q_{(k_h,k_z)} \\ H_{(k_h,k_z)} \\ V_{(k_h,k_z)} \\ W_{(k_h,k_z)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_{(k_h,k_z)} \\ 0 \end{pmatrix}, \quad k_h \cdot V_{(k_h,k_z)} = 0,$$

or, equivalently,

$$\begin{pmatrix} Q_{(k_h,k_z)} \\ H_{(k_h,k_z)} \\ V_{(k_h,k_z)} \\ W_{(k_h,k_z)} \end{pmatrix} = \pm |V_{(k_h,k_z)}| \begin{pmatrix} 0 \\ 0 \\ \dfrac{k_h^\perp}{|k_h|} \\ 0 \end{pmatrix}.$$

On the other hand, $k_h = (0,0)$ imply that

$$\begin{pmatrix} Q_{((0,0),k_z)} \\ H_{((0,0),k_z)} \\ V_{((0,0),k_z)} \\ W_{((0,0),k_z)} \end{pmatrix} = \begin{pmatrix} Q_{((0,0),k_z)} \\ \dfrac{k_z}{\eta} Q_{((0,0),k_z)} \\ V_{((0,0),k_z)} \\ 0 \end{pmatrix} = Q_{((0,0),k_z)} \begin{pmatrix} 1 \\ \dfrac{k_z}{\eta} \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ V_{((0,0),k_z)} \\ 0 \end{pmatrix}.$$

Case 2: $|\omega_{(k_h,k_z)}| = |\eta|$. If $k_z \neq 0$, it is easy to check that there is no non-trivial solution to (100). Thus $k_z = 0$, and one can find the following solution: $|\omega_{(k_h,0)}| = |\eta| \ll 1$, and

$$\begin{pmatrix} Q_{(k_h,0)} \\ H_{(k_h,0)} \\ V_{(k_h,0)} \\ W_{(k_h,0)} \end{pmatrix} = \begin{pmatrix} 0 \\ H_{(k_h,0)} \\ 0 \\ -iH_{(k_h,0)} \dfrac{\omega_{(k_h,0)}}{\eta} \end{pmatrix} = H_{(k_h,0)} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -i\dfrac{\omega_{(k_h,0)}}{\eta} \end{pmatrix}, \quad k_h \neq (0,0).$$

Case 3: $\omega_{(k_h,k_z)} \neq 0$ **nor** $|\omega_{(k_h,k_z)}| \neq |\eta|$. Then from (100), one can derive

$$\begin{aligned} &\big((\omega^2_{(k_h,k_z)} - \eta^2)|k_h|^2 + \omega^2_{(k_h,k_z)}|k_z|^2 - (\omega^2_{(k_h,k_z)} - \eta^2)\omega^2_{(k_h,k_z)}\big) \\ &\times Q_{(k_h,k_z)} = 0. \end{aligned} \tag{101}$$

Notice that (101) is just the Fourier representation of (92). If $Q_{(k_h,k_z)} = 0$, one can easily check from (100), only when $|\omega_{(k_h,k_z)}| = |\eta|$ or 0, there will be non-trivial solutions, which is already covered in the previous case. Therefore, we focus on (101) when $Q_{(k_h,k_z)} \neq 0$, which leads to the algebraic equation

$$\omega^4_{(k_h,k_z)} - (|k_h|^2 + |k_z|^2 + \eta^2)\omega^2_{(k_h,k_z)} + \eta^2|k_h|^2 = 0. \tag{102}$$

Notice that (102) is nothing but (96). Thus, the solutions to (102) are given by

$$\begin{aligned} \omega^2_{(k_h,k_z)} &= \frac{|k_h|^2 + |k_z|^2 + \eta^2 + \sqrt{A}}{2}, \quad \text{or} \\ \omega^2_{(k_h,k_z)} &= \frac{|k_h|^2 + |k_z|^2 + \eta^2 - \sqrt{A}}{2} \\ &= \frac{2\eta^2|k_h|^2}{|k_h|^2 + |k_z|^2 + \eta^2 + \sqrt{A}} \in [0, \eta^2], \end{aligned} \tag{103}$$

where $A := (|k_h|^2+|k_z|^2+\eta^2)^2 - 4\eta^2|k_h|^2 = (|k_h|^2-\eta^2)^2+|k_z|^4+2|k_h|^2|k_z|^2+2\eta^2|k_z|^2 \geq (|k_h|^2-\eta^2)^2 \geq 0$. Then the solution to (100) is given by

$$\begin{pmatrix} Q_{(k_h,k_z)} \\ H_{(k_h,k_z)} \\ V_{(k_h,k_z)} \\ W_{(k_h,k_z)} \end{pmatrix} = \begin{pmatrix} Q_{(k_h,k_z)} \\ H_{(k_h,k_z)} \\ \dfrac{1}{\omega_{(k_h,k_z)}} Q_{(k_h,k_z)} k_h \\ -i\dfrac{\omega_{(k_h,k_z)}}{\eta} H_{(k_h,k_z)} \end{pmatrix},$$

with $H_{(k_h,k_z)}$ satisfies

$$
\begin{aligned}
& k_z H_{(k_h,k_z)} = \eta \left(\frac{|k_h|^2}{\omega^2_{(k_h,k_z)}} - 1 \right) Q_{(k_h,k_z)} \qquad \text{and} \\
& \eta \left(1 - \frac{\omega^2_{(k_h,k_z)}}{\eta^2} \right) H_{(k_h,k_z)} = k_z Q_{(k_h,k_z)}.
\end{aligned} \tag{104}
$$

If $k_z = 0$, from (103) and (104),

$$
\omega^2_{(k_h,0)} = |k_h|^2 \quad \big(\text{or} \quad \omega^2_{(k_h,0)} = \eta^2 \quad (\text{discarded})\big)
$$

with $k_h \neq (0,0)$ (otherwise it is covered in previous case), and thus

$$
\begin{pmatrix} Q_{(k_h,0)} \\ H_{(k_h,0)} \\ V_{(k_h,0)} \\ W_{(k_h,0)} \end{pmatrix} = \begin{pmatrix} Q_{(k_h,0)} \\ 0 \\ Q_{(k_h,0)} \dfrac{k_h}{\omega_{(k_h,0)}} \\ 0 \end{pmatrix}.
$$

We remark that for η small enough, in oder to reach the endpoint values of $(103)_2$, i.e., $|\omega_{(k_h,k_z)}| = 0 \,\text{or}\, \eta$, the necessary condition will be $k_h = (0,0)$ or $k_z = 0$, respectively, while $k_h = (0,0)$ is also a sufficient condition for $\omega_{k_h,k_z} = 0$.

In summary, we have established the following eigenvalue-eigenvector pairs to (91):

Proposition 5. *There exist three classes of eigenvalue-eigenvector pairs to* (91)*: the mean flows, the perturbed internal waves, and the perturbed acoustic waves. They are given as below: with* $k_h \in 2\pi\mathbb{Z}^2$ *and* $k_z \in 2\pi\mathbb{N}$*,*

Mean flows: $\omega = 0$ *and the space of mean flows* $\mathfrak{E}_{0,\varepsilon}$ *is given by*

$$
\begin{aligned}
\mathfrak{E}_{0,\varepsilon} := \operatorname{Span} & \left\{ U^{\mathrm{mf}}_{1,(k_h,k_z)} := \begin{pmatrix} 0 \\ 0 \\ \dfrac{k_h^{\perp}}{|k_h|} e^{ik_h\cdot x} \cos(k_z z) \\ 0 \end{pmatrix}, k_h \neq (0,0) \right\} \\
\oplus \operatorname{Span} & \left\{ U^{\mathrm{mf}}_{2,((0,0),k_z)} := \begin{pmatrix} \cos(k_z z) \\ \dfrac{k_z}{\eta} \sin(k_z z) \\ 0 \\ 0 \end{pmatrix}, \right. \\
& \left. U^{\mathrm{mf}}_{\mathrm{j},((0,0),k_z)} := \begin{pmatrix} 0 \\ 0 \\ \cos(k_z z) \vec{e}_{\mathrm{j}-2} \\ 0 \end{pmatrix}, \ \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ \mathrm{j} = 3,4 \right\}.
\end{aligned} \tag{105}
$$

Perturbed internal waves: $\omega = \pm\omega^{\mathrm{gw}}_{(k_h,k_z)}$ *where*

$$
\omega^{\mathrm{gw}}_{(k_h,k_z)} = \left(\frac{2\eta^2 |k_h|^2}{|k_h|^2 + |k_z|^2 + \eta^2 + \sqrt{A}} \right)^{1/2} \tag{106}
$$

with $A = (|k_h|^2 + |k_z|^2 + \eta^2)^2 - 4\eta^2|k_h|^2$, $k_h \neq 0, k_z \neq 0$, *and the space of internal waves* $\mathfrak{E}_{\pm\omega^{\rm gw}_{(k_h,k_z)},\varepsilon}$ *is given by*

$$\mathfrak{E}_{\pm\omega^{\rm gw}_{(k_h,k_z)},\varepsilon} := \operatorname{Span}\left\{ U^{\rm gw}_{\pm,(k_h,k_z)} := \begin{pmatrix} e^{ik_h\cdot x}\cos(k_z z) \\ \frac{\eta}{k_z}\left(\frac{|k_h|^2}{(\omega^{\rm gw}_{(k_h,k_z)})^2} - 1\right) e^{ik_h\cdot x}\sin(k_z z) \\ \pm\frac{1}{\omega^{\rm gw}_{(k_h,k_z)}} k_h e^{ik_h\cdot x}\cos(k_z z) \\ \mp i\frac{\omega^{\rm gw}_{(k_h,k_z)}}{k_z}\left(\frac{|k_h|^2}{|\omega^{\rm gw}_{(k_h,k_z)}|^2} - 1\right) e^{ik_h\cdot x}\sin(k_z z) \end{pmatrix}, \quad k_h \neq (0,0), k_z \neq 0 \right\}. \tag{107}$$

Perturbed acoustic waves: $\omega = \pm\omega^{\rm aw}_{(k_h,k_z)}$ *where*

$$\omega^{\rm aw}_{(k_h,k_z)} = \left(\frac{|k_h|^2 + |k_z|^2 + \eta^2 + \sqrt{A}}{2}\right)^{1/2} \tag{108}$$

with A *as above,* $(k_h, k_z) \neq ((0,0),0)$, *and the space of perturbed acoustic waves* $\mathfrak{E}_{\pm\omega^{\rm aw}_{(k_h,k_z)},\varepsilon}$ *is given by*

$$\begin{aligned} \mathfrak{E}_{\pm\omega^{\rm aw}_{(k_h,k_z)},\varepsilon} &:= \operatorname{Span}\left\{ U^{\rm aw}_{\pm,(k_h,k_z)} := \begin{pmatrix} e^{ik_h\cdot x}\cos(k_z z) \\ \frac{\eta}{k_z}\left(\frac{|k_h|^2}{(\omega^{\rm aw}_{(k_h,k_z)})^2} - 1\right) e^{ik_h\cdot x}\sin(k_z z) \\ \pm\frac{1}{\omega^{\rm aw}_{(k_h,k_z)}} k_h e^{ik_h\cdot x}\cos(k_z z) \\ \mp i\frac{\omega^{\rm aw}_{(k_h,k_z)}}{k_z}\left(\frac{|k_h|^2}{|\omega^{\rm aw}_{(k_h,k_z)}|^2} - 1\right) e^{ik_h\cdot x}\sin(k_z z) \end{pmatrix}, k_z \neq 0 \right\} \\ &\oplus \operatorname{Span}\left\{ U^{\rm aw}_{\pm,(k_h,0)} := \begin{pmatrix} e^{ik_h\cdot x} \\ 0 \\ \pm\frac{1}{\omega^{\rm aw}_{(k_h,0)}} k_h e^{ik_h\cdot x} \\ 0 \end{pmatrix}, k_h \neq (0,0) \right\}. \end{aligned} \tag{109}$$

Here $\eta = \varepsilon^{1-\nu}$.

Moreover, since $\mathcal{L}_\varepsilon$ *is anti-symmetric, it is easy to check*

$$U^{\rm mf}_{n,(k_h,k_z)}, \quad U^{\rm gw}_{\pm,(k_h,k_z)}, \quad U^{\rm aw}_{\pm,(k_h,k_z)}, \quad n \in \{1,2,3,4\}, \\ (k_h, k_z) \in 2\pi\mathbb{Z}^2 \times 2\pi\mathbb{N},$$

form orthogonal basis with respect to the complex L^2*-inner product.*

5.3. Internal Waves in the Soundproof Model (20)

We have already known that in the full compressible system (17), the internal waves bear frequencies of order $\mathcal{O}(\varepsilon^{-\nu})$ from previous sections (see, e.g., Lemma 6 and Proposition 5). In this subsection, we would like to investigate the internal gravity waves in the soundproof model (20), and provide a comparison study with those in system (17).

Denote by

$$U_{\rm sp} := \begin{pmatrix} \widetilde{\mathcal{H}}_{\rm sp} \\ v_{\rm sp} \\ w_{\rm sp} \end{pmatrix}, \qquad \text{and} \qquad \mathcal{L}_{\rm sp} U_{\rm sp} := \begin{pmatrix} -w_{\rm sp} \\ 0 \\ \widetilde{\mathcal{H}}_{\rm sp} \end{pmatrix}. \tag{110}$$

Then we introduce the linear system associated with the soundproof model (20) as follows:

$$\partial_t U_{\rm sp} + \frac{1}{\varepsilon^\nu} \mathcal{L}_{\rm sp} U_{\rm sp} + \begin{pmatrix} 0 \\ \nabla_h p_{\rm sp} \\ \partial_z p_{\rm sp} \end{pmatrix} = 0, \qquad \operatorname{div}_h v_{\rm sp} + \partial_z v_{\rm sp} = 0, \tag{111}$$

with $p_{\rm sp}, \widetilde{\mathcal{H}}_{\rm sp}, v_{\rm sp}, w_{\rm sp}$ satisfying the same symmetries as $\widetilde{q}, \widetilde{\mathcal{H}}, v, w$, respectively, as in (SYM). We consider the following eigenvalue problem:

$$i\omega_{\rm sp} U_{\rm sp} = \eta \mathcal{L}_{\rm sp} U_{\rm sp} + \begin{pmatrix} 0 \\ \nabla_h(\varepsilon p_{\rm sp}) \\ \partial_z(\varepsilon p_{\rm sp}) \end{pmatrix}, \qquad \operatorname{div}_h v_{\rm sp} + \partial_z v_{\rm sp} = 0. \tag{112}$$

Recalling $\eta = \varepsilon^{1-\nu}$, our scale of $\omega_{\rm sp}$ in (112) is the same as ω in (91), for the sake of convenience for comparison. Direct calculation of (112) leads to the following differential equation:

$$\left(1 - \frac{\omega_{\rm sp}^2}{\eta^2}\right) \Delta_h(\varepsilon p_{\rm sp}) - \frac{\omega_{\rm sp}^2}{\eta^2} \partial_{zz}(\varepsilon p_{\rm sp}) = 0. \tag{113}$$

It is obvious that (113) changes types according to $\omega_{\rm sp}^2/\eta^2 \in \{0\}$, or $(0,1)$, or $\{1\}$, or $(1,\infty)$, respectively. In particular, when $\omega_{\rm sp}^2/\eta^2 \in (1,\infty)$, (113) is a non-degenerate elliptic equation and has only 0 as the trivial solution. However, when $\omega_{\rm sp}^2/\eta^2 \in [0,1]$, unlike (93), (113) is a (degenerate) hyperbolic-type equation.

In the rest of this subsection, we shall use the Fourier representations to persuade further investigation. As in (99), let

$$\begin{aligned} \varepsilon p_{\rm sp} &= \sum_{k_h \in 2\pi\mathbb{Z}^2, k_z \in 2\pi\mathbb{N}} P_{{\rm sp},(k_h,k_z)} e^{ik_h \cdot x} \cos(k_z z), \\ U_{\rm sp} &= \sum_{k_h \in 2\pi\mathbb{Z}^2, k_z \in 2\pi\mathbb{N}} \begin{pmatrix} H_{{\rm sp},(k_h,k_z)} e^{ik_h\cdot x} \sin(k_z z) \\ V_{{\rm sp},(k_h,k_z)} e^{ik_h\cdot x} \cos(k_z z) \\ W_{{\rm sp},(k_h,k_z)} e^{ik_h\cdot x} \sin(k_z z) \end{pmatrix}, \end{aligned} \tag{114}$$

with

$$F_{{\rm sp},(-k_h,k_z)} = F_{{\rm sp},(k_h,k_z)}, \qquad F \in \{P, H, V, W\}.$$

Without loss of generality, we also assume that $P_{{\rm sp},(0,0)} = 0$.

Then (112) is equivalent to

$$\begin{aligned} i\omega_{{\rm sp},(k_h,k_z)} H_{{\rm sp},(k_h,k_z)} &= -\eta W_{{\rm sp},(k_h,k_z)}, \\ i\omega_{{\rm sp},(k_h,k_z)} V_{{\rm sp},(k_h,k_z)} &= iP_{{\rm sp},(k_h,k_z)} k_h, \\ i\omega_{{\rm sp},(k_h,k_z)} W_{{\rm sp},(k_h,k_z)} &= \eta H_{{\rm sp},(k_h,k_z)} - k_z P_{{\rm sp},(k_h,k_z)}, \\ ik_h \cdot V_{{\rm sp},(k_h,k_z)} + k_z W_{{\rm sp},(k_h,k_z)} &= 0, \end{aligned} \tag{115}$$

and (113) is equivalent to

$$\left(\left(1 - \frac{\omega_{{\rm sp},(k_h,k_z)}^2}{\eta^2}\right) |k_h|^2 - \frac{\omega_{{\rm sp},(k_h,k_z)}^2}{\eta^2} |k_z|^2\right) P_{{\rm sp},(k_h,k_z)} = 0. \tag{116}$$

Case 1: $P_{\mathrm{sp},(k_h,k_z)} = 0$. Then it is easy to verify that, the nontrivial solutions to (115) are given by

$$|\omega_{\mathrm{sp},(k_h,0)}| = \eta, \quad \begin{pmatrix} P_{\mathrm{sp},(k_h,0)} \\ H_{\mathrm{sp},(k_h,0)} \\ V_{\mathrm{sp},(k_h,0)} \\ W_{\mathrm{sp},(k_h,0)} \end{pmatrix} = \begin{pmatrix} 0 \\ H_{\mathrm{sp},(k_h,0)} \\ 0 \\ -i\dfrac{\omega_{\mathrm{sp},(k_h,0)}}{\eta} H_{\mathrm{sp},(k_h,0)} \end{pmatrix},$$

$$\text{or} \qquad \omega_{\mathrm{sp},(k_h,k_z)} = 0, \quad \begin{pmatrix} P_{\mathrm{sp},(k_h,k_z)} \\ H_{\mathrm{sp},(k_h,k_z)} \\ V_{\mathrm{sp},(k_h,k_z)} \\ W_{\mathrm{sp},(k_h,k_z)} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ V_{\mathrm{sp},(k_h,k_z)} \\ 0 \end{pmatrix}$$

$$\text{with} \quad k_h \cdot V_{\mathrm{sp},(k_h,k_z)} = 0.$$

Next, we focus on the cases when $P_{\mathrm{sp},(k_h,k_z)} \neq 0$. Then it must hold, from (116),

$$\left(1 - \frac{\omega^2_{\mathrm{sp},(k_h,k_z)}}{\eta^2}\right)|k_h|^2 - \frac{\omega^2_{\mathrm{sp},(k_h,k_z)}}{\eta^2}|k_z|^2 = 0. \tag{117}$$

Case 2: $k_h = (0,0)$. Solving (117) leads to either $\omega_{\mathrm{sp},(0,k_z)} = 0$ or $k_z = 0$. Then the nontrivial solutions to (115) are given by

$$\omega_{\mathrm{sp},(0,k_z)} = 0, \quad \begin{pmatrix} P_{\mathrm{sp},(0,k_z)} \\ H_{\mathrm{sp},(0,k_z)} \\ V_{\mathrm{sp},(0,k_z)} \\ W_{\mathrm{sp},(0,k_z)} \end{pmatrix} = \begin{pmatrix} P_{\mathrm{sp},(0,k_z)} \\ \dfrac{k_z}{\eta} P_{\mathrm{sp},(0,k_z)} \\ V_{\mathrm{sp},(0,k_z)} \\ 0 \end{pmatrix},$$

$$\text{or} \qquad |\omega_{\mathrm{sp},(0,0)}| = \eta, \quad \begin{pmatrix} P_{\mathrm{sp},(0,0)} \\ H_{\mathrm{sp},(0,0)} \\ V_{\mathrm{sp},(0,0)} \\ W_{\mathrm{sp},(0,0)} \end{pmatrix} = \begin{pmatrix} P_{\mathrm{sp},(0,0)} \\ H_{\mathrm{sp},(0,0)} \\ 0 \\ -i\dfrac{\omega_{\mathrm{sp},(0,0)}}{\eta} H_{\mathrm{sp},(0,0)} \end{pmatrix}.$$

Case 3: $k_h \neq (0,0)$. Solving (117) leads to

$$\frac{\omega^2_{\mathrm{sp},(k_h,k_z)}}{\eta^2} = \frac{|k_h|^2}{|k_h|^2 + |k_z|^2}.$$

Then solving (115) yields

$$|\omega_{\mathrm{sp},(k_h,0)}| = \eta, \quad \begin{pmatrix} P_{\mathrm{sp},(k_h,0)} \\ H_{\mathrm{sp},(k_h,0)} \\ V_{\mathrm{sp},(k_h,0)} \\ W_{\mathrm{sp},(k_h,0)} \end{pmatrix} = \begin{pmatrix} 0 \\ H_{\mathrm{sp},(k_h,0)} \\ 0 \\ -i\dfrac{\omega_{\mathrm{sp},(k_h,0)}}{\eta} H_{\mathrm{sp},(k_h,0)} \end{pmatrix} \quad \text{(discarded)},$$

$$\text{or} \quad k_z \neq 0, \quad |\omega_{\mathrm{sp},(k_h,k_z)}| = \frac{\eta|k_h|}{\sqrt{|k_h|^2 + |k_z|^2}},$$

$$\begin{pmatrix} P_{\mathrm{sp},(k_h,k_z)} \\ H_{\mathrm{sp},(k_h,k_z)} \\ V_{\mathrm{sp},(k_h,k_z)} \\ W_{\mathrm{sp},(k_h,k_z)} \end{pmatrix} = \begin{pmatrix} P_{\mathrm{sp},(k_h,k_z)} \\ \dfrac{\eta|k_h|^2}{k_z\omega^2_{\mathrm{sp},(k_h,k_z)}} P_{\mathrm{sp},(k_h,k_z)} \\ \dfrac{P_{\mathrm{sp},(k_h,k_z)}}{\omega_{\mathrm{sp},(k_h,k_z)}} k_h \\ -i\dfrac{|k_h|^2}{k_z\omega_{\mathrm{sp},(k_h,k_z)}} P_{\mathrm{sp},(k_h,k_z)} \end{pmatrix}.$$

In summary, we have established the following eigenvalue-eigenvector pairs to (112):

Proposition 6. *The mean flows and the internal waves in the eigenvalue problem* (112) *for the soundproof model are given as below: with* $k_h \in 2\pi\mathbb{Z}^2$ *and* $k_z \in 2\pi\mathbb{N}$,
Mean flows: $\omega_{\rm sp} = 0$ *and the space of mean flows* $\mathfrak{E}_{{\rm sp},0,\varepsilon}$ *is given by*

$$\begin{aligned}
\mathfrak{E}_{{\rm sp},0,\varepsilon} &:= \mathrm{Span}\Bigg\{\varepsilon p^{\rm mf}_{{\rm sp},1,(k_h,k_z)} := 0, \\
&\quad U^{\rm mf}_{{\rm sp},1,(k_h,k_z)} := \begin{pmatrix} 0 \\ \dfrac{k_h^{\perp}}{|k_h|} e^{ik_h\cdot x}\cos(k_z z) \\ 0 \end{pmatrix}, k_h \neq (0,0)\Bigg\} \\
&\oplus \mathrm{Span}\left\{\varepsilon p^{\rm mf}_{{\rm sp},2,((0,0),k_z)} := \cos(k_z z), U^{\rm mf}_{{\rm sp},2,((0,0),k_z)} := \begin{pmatrix} \dfrac{k_z}{\eta}\sin(k_z z) \\ 0 \\ 0 \end{pmatrix}\right\} \\
&\oplus \mathrm{Span}\left\{\varepsilon p^{\rm mf}_{{\rm sp},{\rm j},((0,0),k_z)} := 0, U^{\rm mf}_{{\rm sp},{\rm j},((0,0),k_z)} := \begin{pmatrix} 0 \\ \cos(k_z z)\vec{e}_{{\rm j}-2} \\ 0 \end{pmatrix},\right. \\
&\quad \left. \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \ {\rm j} = 3,4 \right\}.
\end{aligned} \tag{118}$$

Internal waves*:* $\omega_{\rm sp} = \pm\omega^{\rm gw}_{{\rm sp},(k_h,k_z)}$ *where*

$$\omega^{\rm gw}_{{\rm sp},(k_h,k_z)} := \frac{\eta|k_h|}{(|k_h|^2+|k_z|^2)^{1/2}}, \tag{119}$$

with $k_h \neq (0,0)$ *and* $k_z \neq 0$*, and the space of internal waves* $\mathfrak{E}_{{\rm sp},\pm\omega^{\rm gw}_{{\rm sp},(k_h,k_z)},\varepsilon}$ *is given by*

$$\begin{aligned}
\mathfrak{E}_{{\rm sp},\pm\omega^{\rm gw}_{{\rm sp},(k_h,k_z)},\varepsilon} &:= \mathrm{Span}\Bigg\{\varepsilon p^{\rm gw}_{{\rm sp},(k_h,k_z)} := e^{ik_h\cdot x}\cos(k_z z), \\
U^{\rm gw}_{{\rm sp},\pm,(k_h,k_z)} &:= \begin{pmatrix} \dfrac{\eta|k_h|^2}{k_z|\omega^{\rm gw}_{{\rm sp},(k_h,k_z)}|^2} e^{ik_h\cdot x}\sin(k_z z) \\ \pm\dfrac{1}{\omega^{\rm gw}_{{\rm sp},(k_h,k_z)}} k_h e^{ik_h\cdot x}\cos(k_z z) \\ \mp i\dfrac{|k_h|^2}{k_z\omega^{\rm gw}_{{\rm sp},(k_h,k_z)}} e^{ik_h\cdot x}\sin(k_z z) \end{pmatrix}, \\
&\quad k_h \neq (0,0), k_z \neq 0\Bigg\}.
\end{aligned} \tag{120}$$

Here $\eta = \varepsilon^{1-\nu}$.

5.4. Comparison with Limit Cases

In this subsection, we will quantitatively compare the linear dynamics of several reduced systems studied in this work. (i) We compare the internal waves between the full compressible and the pseudo-incompressible models as derived in Propositions 5 and 6 above. (ii) We compare the acoustic waves generated by the pure acoustic operator in (121) alone and by the full fast mode operator of the compressible system.

First, we summarize the eigenvalue-eigenvector pair of the pure acoustic system,

$$i\omega_{\rm a}U_{\rm a} = \mathcal{L}_a U_{\rm a}, \qquad U_{\rm a} = (\widetilde{q}_{\rm a}, \widetilde{\mathcal{H}}_{\rm a}, v_{\rm a}, w_{\rm a})^\top \in \mathfrak{V}. \tag{121}$$

That is,

Lemma 7. *There exist two classes of eigenvalue-eigenvector pairs to (121); the incompressible flows and the acoustic waves. They are given as below: with $k_h \in 2\pi\mathbb{Z}^2$ and $k_z \in 2\pi\mathbb{N}$,*

Incompressible flows: $\omega_{\mathrm{a}} = 0$ *and the space of incompressible flows $\mathfrak{E}_{\mathrm{a},0}$ is given by*

$$
\begin{aligned}
\mathfrak{E}_{\mathrm{a},0} := \mathrm{Span} & \left\{ U^{\mathrm{icf}}_{\mathrm{a},1,((0,0),0)} := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},\ U^{\mathrm{icf}}_{\mathrm{a},2,(k_h,k_z)} := \begin{pmatrix} 0 \\ e^{ik_h\cdot x}\sin(k_z z) \\ 0 \\ 0 \end{pmatrix} \right\} \\
\oplus & \left\{ U^{\mathrm{icf}}_{\mathrm{a},3,(k_h,k_z)} := \begin{pmatrix} 0 \\ 0 \\ \dfrac{k_h^\perp}{|k_h|} e^{ik_h\cdot x}\cos(k_z z) \\ 0 \end{pmatrix}, \right. \\
& \left. U^{\mathrm{icf}}_{\mathrm{a},4,(k_h,k_z)} := \begin{pmatrix} 0 \\ 0 \\ k_z \dfrac{k_h}{|k_h|} e^{ik_h\cdot x}\cos(k_z z) \\ -i|k_h| e^{ik_h\cdot x}\sin(k_z z) \end{pmatrix},\ k_h \neq (0,0) \right\} \\
\oplus & \left\{ U^{\mathrm{icf}}_{\mathrm{a},5,((0,0),k_z)} := \begin{pmatrix} 0 \\ 0 \\ \cos(k_z z)\vec{e}_h \\ 0 \end{pmatrix},\ \vec{e}_h = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \ \mathit{or} \ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}
\tag{122}
$$

Acoustic waves: $\omega_{\mathrm{a}} = \pm\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}$ *where*

$$
\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} := (|k_h|^2 + |k_z|^2)^{1/2}, \qquad (k_h,k_z) \neq ((0,0),0), \tag{123}
$$

and the space of acoustic waves $\mathfrak{E}_{\mathrm{a},\pm\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}}$ is given by

$$
\mathfrak{E}_{\mathrm{a},\pm\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}} := \mathrm{Span}\left\{ U^{\mathrm{aw}}_{\mathrm{a},\pm,(k_h,k_z)} := \begin{pmatrix} e^{ik_h\cdot x}\cos(k_z z) \\ 0 \\ \pm\dfrac{k_h}{\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}} e^{ik_h\cdot x}\cos(k_z z) \\ \pm\dfrac{ik_z}{\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}} e^{ik_h\cdot x}\sin(k_z z) \end{pmatrix},\ (k_h,k_z) \neq ((0,0),0) \right\}. \tag{124}
$$

In the following, we will compare the eigenvalue-eigenvector pairs obtained in Proposition 6 and Lemma 7 with those in Proposition 5.

Perturbed acoustic waves versus acoustic waves, i.e., $(\omega^{\mathrm{aw}}_{(k_h,k_z),\varepsilon}, U^{\mathrm{aw}}_{\pm,(k_h,k_z)})$ v.s. $(\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z),\varepsilon}, U^{\mathrm{aw}}_{\mathrm{a},\pm,(k_h,k_z)})$: Direct calculation, from (108) and (123), shows that, for $(k_h,k_z) \neq ((0,0),0)$,

$$
\omega^{\mathrm{aw}}_{(k_h,k_z)} = \omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} + \eta^2 \cdot \frac{|k_z|^2}{2(|k_h|^2 + |k_z|^2)^{3/2}} + \mathcal{O}(\eta^4). \tag{125}
$$

Meanwhile, owing to (109) and (124), one has, for $k_z \neq 0$,

$$U^{\text{aw}}_{\pm,(k_h,k_z)} - U^{\text{aw}}_{\text{a},\pm,(k_h,k_z)} = \begin{pmatrix} 0 \\ \dfrac{\eta}{k_z} \dfrac{|k_h|^2 - (\omega^{\text{aw}}_{(k_h,k_z)})^2}{(\omega^{\text{aw}}_{(k_h,k_z)})^2} e^{ik_h\cdot x} \sin(k_z z) \\ \pm k_h \dfrac{\omega^{\text{aw}}_{\text{a},(k_h,k_z)} - \omega^{\text{aw}}_{(k_h,k_z)}}{\omega^{\text{aw}}_{(k_h,k_z)} \omega^{\text{aw}}_{\text{a},(k_h,k_z)}} e^{ik_h\cdot x} \cos(k_z z) \\ \pm i \dfrac{(\omega^{\text{aw}}_{(k_h,k_z)})^2 \omega^{\text{aw}}_{\text{a},(k_h,k_z)} - |k_h|^2 \omega^{\text{aw}}_{\text{a},(k_h,k_z)} - |k_z|^2 \omega^{\text{aw}}_{(k_h,k_z)}}{k_z \omega^{\text{aw}}_{(k_h,k_z)} \omega^{\text{aw}}_{\text{a},(k_h,k_z)}} e^{ik_h\cdot x} \sin(k_z z) \end{pmatrix} \tag{126}$$

$$= \begin{pmatrix} 0 \\ -\Big(\eta \dfrac{k_z}{|k_h|^2 + |k_z|^2} + \eta^3 \dfrac{k_z |k_h|^2}{(|k_h|^2 + |k_z|^2)^3}\Big) e^{ik_h\cdot x} \sin(k_z z) + \mathcal{O}(\eta^5) \\ \mp \eta^2 \cdot \dfrac{k_h |k_z|^2}{2(|k_h|^2 + |k_z|^2)^{5/2}} e^{ik_h\cdot x} \cos(k_z z) + \mathcal{O}(\eta^4) \\ \pm \eta^2 \cdot i \dfrac{k_z (2|k_h|^2 + |k_z|^2)}{2(|k_h|^2 + |k_z|^2)^{5/2}} e^{ik_h\cdot x} \sin(k_z z) + \mathcal{O}(\eta^4) \end{pmatrix};$$

for $k_z = 0$, $k_h \neq (0,0)$,

$$U^{\text{aw}}_{\pm,(k_h,0)} - U^{\text{aw}}_{\text{a},\pm,(k_h,0)} = \begin{pmatrix} 0 \\ 0 \\ \mathcal{O}(\eta^4)) \\ 0 \end{pmatrix}. \tag{127}$$

Perturbed internal waves versus internal waves, i.e., $\Big(\omega^{\text{gw}}_{(k_h,k_z)}, U^{\text{gw}}_{\pm,(k_h,k_z)}\Big)$ v.s. $\left(\omega^{\text{gw}}_{\text{sp},(k_h,k_z)}, \begin{pmatrix} \varepsilon p^{\text{gw}}_{\text{sp},(k_h,k_z)} \\ U^{\text{gw}}_{\text{sp},\pm,(k_h,k_z)} \end{pmatrix}\right)$:

Direct calculation, from (106) and (119), shows that, for $k_h \neq (0,0)$, $k_z \neq 0$,

$$\frac{\omega^{\text{gw}}_{(k_h,k_z)}}{\eta} = \frac{\omega^{\text{gw}}_{\text{sp},(k_h,k_z)}}{\eta} - \eta^2 \cdot \frac{|k_h||k_z|^2}{2(|k_h|^2 + |k_z|^2)^{5/2}} + \mathcal{O}(\eta^4). \tag{128}$$

Meanwhile, owing to (107) and (120), one has,

$$U^{\text{gw}}_{\pm,(k_h,k_z)} - \begin{pmatrix} \varepsilon p^{\text{gw}}_{\text{sp},(k_h,k_z)} \\ U^{\text{gw}}_{\text{sp},\pm,(k_h,k_z)} \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{\eta}{k_z}\Big(\dfrac{|k_h|^2}{|\omega^{\text{gw}}_{(k_h,k_z)}|^2} - \dfrac{|k_h|^2}{|\omega^{\text{gw}}_{\text{sp},(k_h,k_z)}|^2} - 1\Big) e^{ik_h\cdot x} \sin(k_z z) \\ \pm\Big(\dfrac{k_h}{\omega^{\text{gw}}_{(k_h,k_z)}} - \dfrac{k_h}{\omega^{\text{gw}}_{\text{sp},(k_h,k_z)}}\Big) e^{ik_h\cdot x} \cos(k_z z) \\ \mp i \Big[\dfrac{|k_h|^2}{k_z}\Big(\dfrac{1}{\omega^{\text{gw}}_{(k_h,k_z)}} - \dfrac{1}{\omega^{\text{gw}}_{\text{sp},(k_h,k_z)}}\Big) - \dfrac{\omega^{\text{gw}}_{(k_h,k_z)}}{k_z}\Big] e^{ik_h\cdot x} \sin(k_z z) \end{pmatrix} \tag{129}$$

$$= \begin{pmatrix} 0 \\ -\eta \cdot \dfrac{|k_h|^2}{k_z(|k_h|^2 + |k_z|^2)} e^{ik_h\cdot x} \sin(k_z z) + \mathcal{O}(\eta^3) \\ \pm \eta \cdot \dfrac{|k_z|^2 k_h}{2|k_h|(|k_h|^2 + |k_z|^2)^{3/2}} e^{ik_h\cdot x} \cos(k_z z) + \mathcal{O}(\eta^3) \\ \pm i\eta \cdot \dfrac{|k_h|(2|k_h|^2 + |k_z|^2)}{2k_z(|k_h|^2 + |k_z|^2)^{3/2}} e^{ik_h\cdot x} \sin(k_z z) + \mathcal{O}(\eta^3) \end{pmatrix}.$$

Mean flows: It is obvious that $U^{\mathrm{mf}}_{j,(k_h,k_z)} = \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp},j,(k_h,k_z)} \\ U^{\mathrm{mf}}_{\mathrm{sp},j,(k_h,k_z)} \end{pmatrix}$, $j = 1, 2, 3$.

In summary, we have proved the following:

Corollary 7. *For* (k_h, k_z) *satisfying the corresponding restrictions, one has*

$$\mathfrak{E}_{0,\varepsilon} \equiv \mathfrak{E}_{\mathrm{sp},0,\varepsilon} \quad \textit{or equivalently}$$

$$U^{\mathrm{mf}}_{\mathrm{j},(k_h,k_z)} \equiv \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} \\ U^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} \end{pmatrix}, \quad \mathrm{j} = 1, 2, 3, 4, \tag{130}$$

$$0 < \omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)} - \omega^{\mathrm{gw}}_{(k_h,k_z)} = \mathcal{O}(\eta^3), \tag{131}$$

$$\left| U^{\mathrm{gw}}_{\pm,(k_h,k_z)} - \begin{pmatrix} \varepsilon p^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)} \\ U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \end{pmatrix} \right| = \mathcal{O}(\eta), \tag{132}$$

$$0 < \omega^{\mathrm{aw}}_{(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} = \mathcal{O}(\eta^2), \tag{133}$$

$$|U^{\mathrm{aw}}_{\pm,(k_h,k_z)} - U^{\mathrm{aw}}_{\mathrm{a},\pm,(k_h,k_z)}| = \mathcal{O}(\eta), \tag{134}$$

uniformly in (k_h, k_z). *Here* $\eta = \varepsilon^{1-\nu}$.

6. Fast-slow Waves Interactions: Soundproof Approximation With Ill-prepared Initial data

6.1. Nonlinear Equations

The Full Compressible System. With the understanding of the linear theory, we will discuss the nonlinear theory of fast-slow waves decompositions in system (17). Notice that, under assumption (83), (17) can be written as, with $U = (\widetilde{q}, \widetilde{\mathcal{H}}, v, w)^\top$, $\mathcal{L}_\varepsilon$ as in (85),

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U + \mathcal{N}(U) = \mathcal{M}(U) + \mathcal{K}_\varepsilon(U), \tag{135}$$

where

$$\mathcal{N}(U) := v \cdot \nabla_h U + w\partial_z U + \begin{pmatrix} \varpi_0^{-1}\widetilde{q}(\mathrm{div}_h\, v + \partial_z w) \\ -\widetilde{G} \cdot \widetilde{\mathcal{H}} w \\ 0 \\ 0 \end{pmatrix}, \tag{136}$$

$$\mathcal{M}(U) := \begin{pmatrix} Gw + \varpi_0^{-1} \int_0^z G(z')\, dz'(\mathrm{div}_h\, v + \partial_z w) \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{137}$$

$$\mathcal{K}_\varepsilon(U) := \begin{pmatrix} \varepsilon^\mu \overline{\mathcal{H}}_0 w + \varepsilon^\mu \varpi_0^{-1} \int_0^z \overline{\mathcal{H}}_0(z')\, dz'(\mathrm{div}_h\, v + \partial_z w) \\ 0 \\ -(\varepsilon^\mu \widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}})(\partial_t v + v \cdot \nabla_h v + w\partial_z v) \\ -(\varepsilon^\mu \widetilde{G}\overline{\mathcal{H}}_0 + \varepsilon^{\mu+\nu}\widetilde{G}\widetilde{\mathcal{H}})(\partial_t w + v \cdot \nabla_h w + w\partial_z w) \end{pmatrix}. \tag{138}$$

With estimate (48) and proper initial data, one can assume that $\mathcal{K}_\varepsilon(U) = \mathcal{O}(\varepsilon^{\mu-\sigma})$ in suitable Sobolev space (H^2 for instance). In particular, we choose $\sigma = \mu/2$, and thus $\mathcal{K}_\varepsilon(U)$ will be considered as an error term. For this reason, we write

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U + \mathcal{N}(U) = \mathcal{M}(U) + \mathcal{O}(\varepsilon^{\mu-\sigma}). \tag{139}$$

(H5) Furthermore, to simplify the presentation, we assume

$$G = \widetilde{G} = \sin(2\pi z). \tag{140}$$

We emphasise that with some modification, the following arguments work without assumption (H5). We will adopt the notation (99) for our solutions U.

Let

$$\mathcal{P}_\varepsilon^{\mathrm{mf}}, \qquad \mathcal{P}_\varepsilon^{\mathrm{gw}}, \quad \text{and} \quad \mathcal{P}_\varepsilon^{\mathrm{aw}}, \tag{141}$$

be the L^2-orthogonal projections to the spaces

$$\mathfrak{E}_\varepsilon^{\mathrm{mf}} := \mathfrak{E}_{0,\varepsilon}, \quad \mathfrak{E}_\varepsilon^{\mathrm{gw}} := \oplus_{k_h\neq(0,0),k_z\neq 0}\mathfrak{E}_{\pm\omega^{\mathrm{gw}}_{k_h,k_z},\varepsilon},$$
$$\text{and} \quad \mathfrak{E}_\varepsilon^{\mathrm{aw}} := \oplus_{k_z\neq 0}\mathfrak{E}_{\pm\omega^{\mathrm{aw}}_{(k_h,k_z)},\varepsilon} \oplus_{k_h\neq(0,0)} \mathfrak{E}_{\pm\omega^{\mathrm{aw}}_{(k_h,0)},\varepsilon},$$

respectively, given in Proposition 5.

The soundproof system. Similarly, denote by U_{sp} and $\mathcal{L}_{\mathrm{sp}}$ as in (110). Under assumption (H4) and the simplifying but not critical assumption (H5), (20) can be written as,

$$\mathrm{div}_h\, v_{\mathrm{sp}} + \partial_z w_{\mathrm{sp}} = 0, \tag{142}$$

$$\partial_t U_{\mathrm{sp}} + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}} U_{\mathrm{sp}} + \begin{pmatrix} 0 \\ \nabla_h p_{\mathrm{sp}} \\ \partial_z p_{\mathrm{sp}} \end{pmatrix} + \mathcal{N}_{\mathrm{sp}}(U_{\mathrm{sp}}) = 0. \tag{143}$$

Here, thanks to assumption (H5),

$$\mathcal{N}_{\mathrm{sp}}(U_{\mathrm{sp}}) := v_{\mathrm{sp}}\cdot\nabla_h U_{\mathrm{sp}} + w_{\mathrm{sp}}\partial_z U_{\mathrm{sp}} + \begin{pmatrix} -\sin(2\pi z)\cdot\widetilde{\mathcal{H}}_{\mathrm{sp}} w_{\mathrm{sp}} \\ 0 \\ 0 \end{pmatrix}. \tag{144}$$

Notice that equations (139) and (143) have different dimensions. In particular, (143) does not have an evolutionary equation of p_{sp}, corresponding to the $\widetilde{q}$-component of (139). For this reason, in order to investigate the rigidity of the soundproof approximation, we denote the dimension reduction projection $\mathcal{P}_{\mathrm{rd}}$, defined as

$$\mathcal{P}_{\mathrm{rd}} : \begin{pmatrix} \widetilde{q} \\ \widetilde{\mathcal{H}} \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} \widetilde{\mathcal{H}} \\ v \\ w \end{pmatrix}. \tag{145}$$

Notice that $\mathcal{P}_{\mathrm{rd}}$ is a bounded operator in any Sobolev space. Moreover, from (136) and (144), one can check that

$$\mathcal{P}_{\mathrm{rd}}\mathcal{N}(U) = \mathcal{N}_{\mathrm{sp}}(\mathcal{P}_{\mathrm{rd}} U). \tag{146}$$

6.2. Soundproof Approximation with Ill-prepared Initial Data

Compactness theory of solutions to (139) and finite dimension truncation. Denote by $\mathcal{S}_\varepsilon(t)$ the solving operator of $\partial_t + \mathcal{L}_\varepsilon$, $\mathcal{L}_\varepsilon$ as in (85); that is

$$\partial_t \mathcal{S}_\varepsilon(t) U_0 + \mathcal{L}_\varepsilon \mathcal{S}_\varepsilon(t) U_0 = 0. \tag{147}$$

Then Proposition 5 implies that

$$\begin{aligned} &\mathcal{S}_\varepsilon(t) U_\iota = e^{-i\omega_\iota t} U_\iota, \\ &(\omega_\iota, U_\iota) \in \big\{ (0, U^{\mathrm{mf}}_{\mathrm{j},(k_h,k_z)}),\ \mathrm{j}=1,2,3,4,\ (\pm\omega^{\mathrm{gw}}_{(k_h,k_z)}, U^{\mathrm{gw}}_{\pm,(k_h,k_z)}), \\ &(\pm\omega^{\mathrm{aw}}_{(k_h,k_z)}, U^{\mathrm{aw}}_{\pm,(k_h,k_z)}) \big\}. \end{aligned}$$

Then $\mathcal{S}_\varepsilon(t)$ is an isometry from $H^s(\mathbb{T}^3)$ to $H^s(\mathbb{T}^3)$, $\forall s$. Let

$$V_\varepsilon(t) := \mathcal{S}_\varepsilon\left(-\frac{t}{\varepsilon}\right) U(t), \tag{148}$$

where $U(t)$ is the solution to (139). Then it follows from (139) and (147) that

$$\partial_t V_\varepsilon + \mathcal{S}_\varepsilon\left(-\frac{t}{\varepsilon}\right)\mathcal{N}(U) = \mathcal{S}_\varepsilon\left(-\frac{t}{\varepsilon}\right)\mathcal{M}(U) + \mathcal{O}(\varepsilon^{\mu-\sigma}). \tag{149}$$

Owing to Proposition 3, it is straightforward to verify that, with the same initial data for (139) as stated in the proposition,

$$\sup_{0\leq t\leq T_\sigma}\left(\left\|\partial_t V_\varepsilon(t)\right\|^2_{H^2(\mathbb{T}^3)} + \left\|V_\varepsilon(t)\right\|^2_{H^3(\mathbb{T}^3)}\right) \lesssim \mathcal{C}\mathcal{C}_{\mathrm{in}}, \qquad \sigma\in(0,\mu]. \tag{150}$$

With Proposition 5, we can write V_ε as,

$$V_\varepsilon = V_\varepsilon^{\mathrm{mf}} + V_\varepsilon^{\mathrm{gw}} + V_\varepsilon^{\mathrm{aw}}, \tag{151}$$

with

$$\begin{aligned} V_\varepsilon^{\mathrm{mf}} :=& \sum_{k_h\neq(0,0)} \alpha^{\mathrm{mf}}_{1,(k_h,k_z),\varepsilon}(t) U^{\mathrm{mf}}_{1,(k_h,k_z)} + \sum_{\mathrm{j}=3,4,\ k_z\in\mathbb{Z}} \alpha^{\mathrm{mf}}_{\mathrm{j},((0,0),k_z),\varepsilon}(t) U^{\mathrm{mf}}_{\mathrm{j},((0,0),k_z)} \\ &+ \alpha^{\mathrm{mf}}_{2,((0,0),0),\varepsilon}(t) U^{\mathrm{mf}}_{2,((0,0),0)} \\ &+ \sum_{k_z\neq 0} \alpha^{\mathrm{mf}}_{2,((0,0),k_z),\varepsilon}(t)\varepsilon^{1-\nu} U^{\mathrm{mf}}_{2,((0,0),k_z)}, \end{aligned} \tag{152}$$

$$V_\varepsilon^{\mathrm{gw}} := \sum_{k_h\neq(0,0),k_z\neq 0} \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon}(t)\varepsilon^{1-\nu} U^{\mathrm{gw}}_{\pm,(k_h,k_z)}, \tag{153}$$

$$V_\varepsilon^{\mathrm{aw}} := \sum_{(k_h,k_z)\neq((0,0),0)} \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon}(t) U^{\mathrm{aw}}_{\pm,(k_h,k_z)}, \tag{154}$$

where the factor $\varepsilon^{1-\nu}$ plays the role of renormalization, such that for fixed $k=(k_h,k_z)$, $\varepsilon^{1-\nu}U^{\mathrm{mf}}_{2,((0,0),k_z)}\big|_{k_z\neq 0}$ and $\varepsilon^{1-\nu}U^{\mathrm{gw}}_{\pm,(k_h,k_z)}\big|_{k_h\neq(0,0),k_z\neq 0}$ are $\mathcal{O}(1)$.

Notice that the coefficients $\alpha^{\cdot}_{\cdot,\cdot,\cdot}(t)$'s in (152)–(154) are equicontinuous thanks to (150). Then, recalling (148), one has

$$U(t) = \mathcal{S}_\varepsilon(\frac{t}{\varepsilon})V_\varepsilon(t) = U_\varepsilon^{\mathrm{mf}}(t) + U_\varepsilon^{\mathrm{gw}}(t) + U_\varepsilon^{\mathrm{aw}}(t), \tag{155}$$

with

$$U_\varepsilon^{\mathrm{mf}} := V_\varepsilon^{\mathrm{mf}}, \tag{156}$$

$$U_\varepsilon^{\mathrm{gw}} := \sum_{k_h\neq(0,0),k_z\neq 0} e^{\mp i\frac{\omega^{\mathrm{gw}}_{(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{t}{\varepsilon^\nu}} \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon}(t)\varepsilon^{1-\nu} U^{\mathrm{gw}}_{\pm,(k_h,k_z)}, \tag{157}$$

$$U_\varepsilon^{\mathrm{aw}} := \sum_{(k_h,k_z)\neq((0,0),0)} e^{\mp i\omega^{\mathrm{aw}}_{(k_h,k_z)}\frac{t}{\varepsilon}} \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon}(t) U^{\mathrm{aw}}_{\pm,(k_h,k_z)}. \tag{158}$$

Meanwhile, let

$$\mathfrak{U}^{\mathrm{gw}} := \sum_{k_h\neq(0,0),k_z\neq 0} e^{\mp i\frac{\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{t}{\varepsilon^\nu}} \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon}(t)\varepsilon^{1-\nu} \begin{pmatrix} \varepsilon p^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)} \\ U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \end{pmatrix}, \tag{159}$$

$$\mathfrak{U}^{\mathrm{aw}} := \sum_{(k_h,k_z)\neq((0,0),0)} e^{\mp i\omega^{\mathrm{aw}}_{(k_h,k_z)}\frac{t}{\varepsilon}} \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon}(t) U^{\mathrm{aw}}_{\mathrm{a},\pm,(k_h,k_z)}. \tag{160}$$

Notice that $\mathfrak{U}^{\mathrm{gw}}$ and $\mathfrak{U}^{\mathrm{aw}}$ are obtained by changing the basis corresponding to the perturbed internal and acoustic waves in $U_\varepsilon^{\mathrm{gw}}$ and $U_\varepsilon^{\mathrm{aw}}$ to those corresponding to the non-perturbed ones, respectively. We don't need similar representation for $\mathfrak{U}^{\mathrm{mf}}$ thanks to (130). However, to simplify the representation later on, we denote

$$\mathfrak{U}^{\mathrm{mf}} := U_\varepsilon^{\mathrm{mf}}. \tag{161}$$

On the other hand, notice that $\mathcal{N}(U) = \mathcal{B}(U,U)$, with bilinear form $\mathcal{B}(\cdot,\cdot)$ defined by

$$\mathcal{B}(U_1,U_2) := v_1 \cdot \nabla_h U_2 + w_1 \partial_z U_2 + \begin{pmatrix} \varpi_0^{-1} \widetilde{q}_1 (\operatorname{div}_h v_2 + \partial_z w_2) \\ -\widetilde{G} \cdot \widetilde{\mathcal{H}}_1 w_2 \\ 0 \\ 0 \end{pmatrix}, \tag{162}$$

where $U_j = (\widetilde{q}_j, \widetilde{\mathcal{H}}_j, v_j, w_j)^\top$, $j = 1,2$. Then one can write

$$\begin{aligned} \mathcal{N}(U) = {} & \mathcal{N}(U_\varepsilon^{\mathrm{mf}} + U_\varepsilon^{\mathrm{gw}}) \\ & + \mathcal{B}(U_\varepsilon^{\mathrm{mf}} + U_\varepsilon^{\mathrm{gw}}, U_\varepsilon^{\mathrm{aw}}) + \mathcal{B}(U_\varepsilon^{\mathrm{aw}}, U_\varepsilon^{\mathrm{mf}} + U_\varepsilon^{\mathrm{gw}}) \\ & + \mathcal{N}(U_\varepsilon^{\mathrm{aw}}). \end{aligned}$$

In addition, let T_k, $k \in \mathbb{N}^+$, be a finite dimensional truncation defined as

$$T_k U := \sum_{|k_h| \leq k, |k_z| \leq k} \begin{pmatrix} Q_{(k_h,k_z)} e^{i k_h \cdot x} \cos(k_z z) \\ H_{(k_h,k_z)} e^{i k_h \cdot x} \sin(k_z z) \\ V_{(k_h,k_z)} e^{i k_h \cdot x} \cos(k_z z) \\ W_{(k_h,k_z)} e^{i k_h \cdot x} \sin(k_z z) \end{pmatrix} \tag{163}$$

for U in (99). For the sake of clear representation, we assume that T_k applies to U_{sp} in a similar method.

Then thanks to the uniform estimates obtained in Proposition 3, $\left\| U(t) - T_k U(t) \right\|_{H^1(\mathbb{T}^3)} \to 0$, as $k \to \infty$, and the convergence is uniform-in-ε. Therefore, to analyze $\mathcal{N}(U)$, it suffices to analyze $\mathcal{N}(T_k U)$.

Let us begin with $\mathcal{N}(T_k U_\varepsilon^{\mathrm{aw}})$. In particular, thanks to (124) and (134), by denoting $T_k \mathfrak{U}^{\mathrm{aw}} = (Q_k, 0, \nabla_h P_k, \partial_z P_k)^\top$, one has

$$\begin{aligned} \mathcal{N}(T_k U_\varepsilon^{\mathrm{aw}}) &= \mathcal{N}(T_k \mathfrak{U}^{\mathrm{aw}}) + \mathcal{O}(\varepsilon^{1-\nu}) \\ &= \mathcal{N}\left(\begin{pmatrix} Q_k \\ 0 \\ \nabla_h P_k \\ \partial_z P_k \end{pmatrix} \right) + \mathcal{O}(\varepsilon^{1-\nu}) = \begin{pmatrix} (\nabla P_k \cdot \nabla) Q_k + \varpi_0^{-1} Q_k \Delta P_k \\ 0 \\ \frac{1}{2} \nabla_h |\nabla P_k|^2 \\ \frac{1}{2} \partial_z |\nabla P_k|^2 \end{pmatrix} + \mathcal{O}(\varepsilon^{1-\nu}). \end{aligned} \tag{164}$$

Moreover,

$$\begin{aligned} \mathcal{N}(T_k U_\varepsilon^{\mathrm{aw}}) = {} & \sum_{|k_h|,|k_z|,|k_h'|,|k_z'| \leq k} e^{\mp i (\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} + \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')}) \frac{t}{\varepsilon}} \\ & \times e^{\mp i (\omega^{\mathrm{aw}}_{(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} + \omega^{\mathrm{aw}}_{(k_h',k_z')} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')}) \frac{t}{\varepsilon}} \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon} \alpha^{\mathrm{aw}}_{\pm,(k_h',k_z'),\varepsilon} \\ & \times \mathcal{B}(U^{\mathrm{aw}}_{\pm,(k_h,k_z)}, U^{\mathrm{aw}}_{\pm,(k_h',k_z')}) \\ & + \sum_{|k_h|,|k_z|,|k_h'|,|k_z'| \leq k} e^{\mp i (\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')}) \frac{t}{\varepsilon}} \\ & \times e^{\mp i ((\omega^{\mathrm{aw}}_{(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}) - (\omega^{\mathrm{aw}}_{(k_h',k_z')} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')})) \frac{t}{\varepsilon}} \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon} \alpha^{\mathrm{aw}}_{\mp,(k_h',k_z'),\varepsilon} \\ & \times \mathcal{B}(U^{\mathrm{aw}}_{\pm,(k_h,k_z)}, U^{\mathrm{aw}}_{\mp,(k_h',k_z')}). \end{aligned} \tag{165}$$

Therefore, the possible resonances are determined by $(k_h,k_z), (k_h',k_z')$ such that $\omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')} = 0$, i.e., $|k_h|^2 + |k_z|^2 = |k_h'|^2 + |k_z'|^2$, and

$$(\omega^{\mathrm{aw}}_{(k_h,k_z)} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h,k_z)}) - (\omega^{\mathrm{aw}}_{(k_h',k_z')} - \omega^{\mathrm{aw}}_{\mathrm{a},(k_h',k_z')}) = \begin{cases} \mathcal{O}(\varepsilon^{4-4\nu}) & \text{if } k_z = k_z', \\ \mathcal{O}(\varepsilon^{2-2\nu}) & \text{if } k_z \neq k_z', \end{cases} \tag{166}$$

thanks to (125). We remark that, since $\nu < 1/2$, (166) implies that there will be resonances in the second term of (165). However, according to (164), these resonances will form a gradient in the momentum equations, and therefore will converge to the Lagrangian multiplier ∇p_{sp} in the soundproof model. In fact, as we will see later, these resonances will not affect the dynamic of the soundproof waves. However, the same cannot be said about the $\widetilde{q}$ component, which does not exist in the soundproof model. We further remark this in the end of this paper.

On the other hand, thanks to (130), (132), and (134), one has

$$\begin{aligned} &\mathcal{N}(T_k U_\varepsilon^{\mathrm{mf}} + T_k U_\varepsilon^{\mathrm{gw}}) = \mathcal{N}(T_k \mathfrak{U}^{\mathrm{mf}} + T_k \mathfrak{U}^{\mathrm{gw}}) + \mathcal{O}(\varepsilon^{2-3\nu}) + \mathcal{O}(\varepsilon^{2-2\nu}),\\ &\mathcal{B}(T_k U_\varepsilon^{\mathrm{mf}} + T_k U_\varepsilon^{\mathrm{gw}}, T_k U_\varepsilon^{\mathrm{aw}}) + \mathcal{B}(T_k U_\varepsilon^{\mathrm{aw}}, T_k U_\varepsilon^{\mathrm{mf}} + T_k U_\varepsilon^{\mathrm{gw}})\\ &= \mathcal{B}(T_k \mathfrak{U}^{\mathrm{mf}} + T_k \mathfrak{U}^{\mathrm{gw}}, T_k \mathfrak{U}^{\mathrm{aw}}) + \mathcal{B}(T_k \mathfrak{U}^{\mathrm{aw}}, T_k \mathfrak{U}^{\mathrm{mf}} + T_k \mathfrak{U}^{\mathrm{gw}}) + \mathcal{O}(\varepsilon^{1-\nu}). \end{aligned} \tag{167}$$

Moreover,

$$\begin{aligned} \mathcal{B}(T_k U_\varepsilon^{\mathrm{gw}}, T_k U_\varepsilon^{\mathrm{aw}}) = &\sum_{\substack{k_h \neq (0,0), k_z \neq 0, (k_h', k_z') \neq ((0,0),0),\\ |k_h|,|k_z|,|k_h'|,|k_z'| \leq k}} e^{\mp i\left(\frac{\omega^{\mathrm{gw}}_{(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{1}{\varepsilon^\nu} + \omega^{\mathrm{aw}}_{(k_h',k_z')}\frac{1}{\varepsilon}\right)t}\\ &\times \mathcal{B}(U^{\mathrm{gw}}_{\pm,(k_h,k_z)}, U^{\mathrm{aw}}_{\pm,(k_h',k_z')})\\ &+ \sum_{\substack{k_h \neq (0,0), k_z \neq 0, (k_h', k_z') \neq ((0,0),0),\\ |k_h|,|k_z|,|k_h'|,|k_z'| \leq k}} e^{\mp i\left(\frac{\omega^{\mathrm{gw}}_{(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{1}{\varepsilon^\nu} - \omega^{\mathrm{aw}}_{(k_h',k_z')}\frac{1}{\varepsilon}\right)t}\\ &\times \mathcal{B}(U^{\mathrm{gw}}_{\pm,(k_h,k_z)}, U^{\mathrm{aw}}_{\mp,(k_h',k_z')}). \end{aligned} \tag{168}$$

Notice that $\frac{\omega^{\mathrm{gw}}_{(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{1}{\varepsilon^\nu} - \omega^{\mathrm{aw}}_{(k_h',k_z')}\frac{1}{\varepsilon} = \mathcal{O}(\frac{1}{\varepsilon})$, which implies that $\mathcal{B}(T_k U_\varepsilon^{\mathrm{gw}}, T_k U_\varepsilon^{\mathrm{aw}})$ oscillates in time with a rate of $\mathcal{O}(\frac{1}{\varepsilon})$, and thus weakly converges to zero as $\varepsilon \to 0^+$. Similar properties apply to $\mathcal{B}(T_k U_\varepsilon^{\mathrm{mf}}, T_k U_\varepsilon^{\mathrm{aw}}) + \mathcal{B}(T_k U_\varepsilon^{\mathrm{aw}}, T_k U_\varepsilon^{\mathrm{mf}} + T_k U_\varepsilon^{\mathrm{gw}})$.

Compactness theory of solutions to (143) and finite dimension truncation. We refer to the property of U_{sp} such that $\mathrm{div}_h\, v_{\mathrm{sp}} + \partial_z w_{\mathrm{sp}} = 0$ as the soundproof property. Also, let $\mathcal{P}_\sigma$ be the orthogonal projection of vector fields into the space with the soundproof property.

Denote by $\mathcal{S}_{\mathrm{sp}}(t)$ the solving operator of

$$\partial_t + \mathcal{L}_{\mathrm{sp}} + \begin{pmatrix} 0 \\ \nabla_h p \\ \partial_z p \end{pmatrix}$$

in the space with the soundproof property, $\mathcal{L}_{\mathrm{sp}}$ as in (110); that is

$$\partial_t \mathcal{S}_{\mathrm{sp}}(t) U_{\mathrm{sp},0} + \mathcal{L}_{\mathrm{sp}} \mathcal{S}_{\mathrm{sp}}(t) U_{\mathrm{sp},0} + \begin{pmatrix} 0 \\ \nabla_h p \\ \partial_z p \end{pmatrix} = 0 \tag{169}$$

for some p (as the Lagrangian multiplier, which might be different from lines to lines, hereafter) and $\mathrm{div}_h\, (\mathcal{S}_{\mathrm{sp}}(t) U_{\mathrm{sp},0})_{v_{\mathrm{sp}}} + \partial_z (\mathcal{S}_{\mathrm{sp}}(t) U_{\mathrm{sp},0})_{w_{\mathrm{sp}}}$. Here $(\cdot)_{v_{\mathrm{sp}}}$ and $(\cdot)_{w_{\mathrm{sp}}}$ represent the v_{sp} and w_{sp} component, respectively. Then Proposition 6 implies that

$$\begin{aligned} &\mathcal{S}_{\mathrm{sp}}(t) U_{\mathrm{sp},\iota} = e^{-i\omega_{\mathrm{sp},\iota} t/\eta} U_{\mathrm{sp},\iota},\\ &(\omega_\iota, U_\iota) \in \left\{ (0, U^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)}),\ \mathrm{j} = 1,2,3,4,\ (\pm\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}, U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}) \right\}. \end{aligned}$$

We remind readers that our choice of scale in Proposition 6 implies that $\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}/\eta = \mathcal{O}(1)$.

Then, it is easy to verify that $\mathcal{S}_{\mathrm{sp}}(t)$ is an isometry from H^s_σ to H^s_σ, $\forall s$. Here H^s_σ represents the H^s space with the soundproof property. Let

$$V_{\mathrm{sp}}(t) := \mathcal{S}_{\mathrm{sp}}\left(-\frac{t}{\varepsilon^\nu}\right) U_{\mathrm{sp}}(t), \tag{170}$$

where $U_{\mathrm{sp}}(t)$ is the solution to (143). Then it follows from (143) and (169) that

$$\partial_t V_{\mathrm{sp}}(t) + \mathcal{S}_{\mathrm{sp}}\left(-\frac{t}{\varepsilon^{\nu}}\right)\mathcal{P}_\sigma \mathcal{N}_{\mathrm{sp}}(U_{\mathrm{sp}}) = 0. \tag{171}$$

Thanks to the estimate (74), it is straightforward to verify that, with the same initial data as in Theorem 1 for (143), one has

$$\sup_{0\leq t\leq T_{\mathrm{app}}}\left(\left\|\partial_t V_{\mathrm{sp}}(t)\right\|^2_{H^2(\mathbb{T}^3)} + \left\|V_{\mathrm{sp}}(t)\right\|^2_{H^3(\mathbb{T}^3)}\right) \leq \mathcal{C}_{\mathrm{sp,in}}, \tag{172}$$

for some $\mathcal{C}_{\mathrm{sp,in}} \in (0,\infty)$ depending on the initial data.

Thanks to Proposition 6, we can write V_{sp} as,

$$V_{\mathrm{sp}} = V_{\mathrm{sp}}^{\mathrm{mf}} + V_{\mathrm{sp}}^{\mathrm{gw}}, \tag{173}$$

with

$$\begin{aligned} V_{\mathrm{sp}}^{\mathrm{mf}} := &\sum_{k_h\neq(0,0)} \alpha^{\mathrm{mf}}_{1,(k_h,k_z),\mathrm{sp}}(t) U^{\mathrm{mf}}_{\mathrm{sp},1,(k_h,k_z)} + \sum_{\mathrm{j}=3,4,\ k_z\in\mathbb{Z}} \alpha^{\mathrm{mf}}_{\mathrm{j},((0,0),k_z),\mathrm{sp}}(t) U^{\mathrm{mf}}_{\mathrm{sp,j},((0,0),k_z)} \\ &+ \alpha^{\mathrm{mf}}_{2,((0,0),0),\mathrm{sp}}(t)\underbrace{U^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),0)}}_{=0} \\ &+ \sum_{k_z\neq 0} \alpha^{\mathrm{mf}}_{2,((0,0),k_z),\mathrm{sp}}(t)\varepsilon^{1-\nu} U^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),k_z)}, \end{aligned} \tag{174}$$

$$V_{\mathrm{sp}}^{\mathrm{gw}} := \sum_{k_h\neq(0,0),k_z\neq 0} \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\mathrm{sp}}(t)\varepsilon^{1-\nu} U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}, \tag{175}$$

Thanks to (172), the coefficients $\alpha^{\cdot}_{\cdot,\cdot,\cdot}(t)$'s in (174)–(175) are equicontinuous.

Then, one has

$$U_{\mathrm{sp}}(t) = \mathcal{S}_{\mathrm{sp}}(\frac{t}{\varepsilon^{\nu}})V_{\mathrm{sp}}(t) = U_{\mathrm{sp}}^{\mathrm{mf}}(t) + U_{\mathrm{sp}}^{\mathrm{gw}}(t), \tag{176}$$

with

$$U_{\mathrm{sp}}^{\mathrm{mf}} := V_{\mathrm{sp}}^{\mathrm{mf}}, \tag{177}$$

$$U_{\mathrm{sp}}^{\mathrm{gw}} := \sum_{k_h\neq(0,0),k_z\neq 0} e^{\mp i\frac{\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}}{\varepsilon^{1-\nu}}\frac{t}{\varepsilon^{\nu}}} \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\mathrm{sp}}(t)\varepsilon^{1-\nu} U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}. \tag{178}$$

On the other hand, similarly as before, $\mathcal{N}_{\mathrm{sp}}(U_{\mathrm{sp}}) = \mathcal{B}_{\mathrm{sp}}(U_{\mathrm{sp}}, U_{\mathrm{sp}})$, with the bilinear form $\mathcal{B}_{\mathrm{sp}}(\cdot,\cdot)$ defined by

$$\mathcal{B}_{\mathrm{sp}}(U_{\mathrm{sp},1}, U_{\mathrm{sp},2}) := v_{\mathrm{sp},1}\cdot\nabla_h U_{\mathrm{sp},2} + w_{\mathrm{sp},1}\partial_z U_{\mathrm{sp},2} + \begin{pmatrix} -\sin(2\pi z)\cdot\widetilde{\mathcal{H}}_{\mathrm{sp},1} w_{\mathrm{sp},2} \\ 0 \\ 0 \end{pmatrix}. \tag{179}$$

Similar to (146), one has, for U_j as in (162),

$$\mathcal{P}_{\mathrm{rd}}\mathcal{B}(U_1,U_2) = \mathcal{B}_{\mathrm{sp}}(\mathcal{P}_{\mathrm{rd}}U_1, \mathcal{P}_{\mathrm{rd}}U_2). \tag{180}$$

Estimate of $\mathcal{P}_{\mathrm{rd}}(U_\varepsilon^{\mathrm{mf}} + U_\varepsilon^{\mathrm{gw}}) - U_{\mathrm{sp}}$. Let $K\in\mathbb{N}^+$ be a fixed positive integer. Then thanks to the uniform estimates obtained in Proposition 3 and (74), as mentioned before, (139) and (143) can be written as

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U + \mathcal{N}(T_K U) = \mathcal{M}(T_K U) + \mathcal{O}(\varepsilon^{\mu-\sigma}) + Err \qquad \text{and} \tag{181}$$

$$\partial_t U_{\mathrm{sp}} + \frac{1}{\varepsilon^{\nu}}\mathcal{L}_{\mathrm{sp}}U_{\mathrm{sp}} + \begin{pmatrix} 0 \\ \nabla_h p_{\mathrm{sp}} \\ \partial_z p_{\mathrm{sp}} \end{pmatrix} + \mathcal{N}_{\mathrm{sp}}(T_K U_{\mathrm{sp}}) = Err, \tag{182}$$

respectively, where Err represents the truncation error, satisfying

$$\big\|Err\big\|_{H^1} \to 0 \qquad \text{uniformly-in} - \varepsilon \quad \text{as} \quad K \to \infty. \tag{183}$$

Recalling (152), (155), (156), and (157), one has

$$\begin{aligned}
\partial_t U_\varepsilon^{\mathrm{mf}} + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U_\varepsilon^{\mathrm{mf}} &= \sum_{k_h\neq(0,0)} \partial_t \alpha^{\mathrm{mf}}_{1,(k_h,k_z),\varepsilon}(t) \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp},1,(k_h,k_z)} \\ U^{\mathrm{mf}}_{\mathrm{sp},1,(k_h,k_z)} \end{pmatrix} \\
&\quad + \sum_{\mathrm{j}=3,4,\ k_z\in\mathbb{Z}} \partial_t \alpha^{\mathrm{mf}}_{\mathrm{j},((0,0),k_z),\varepsilon}(t) \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp,j},((0,0),k_z)} \\ U^{\mathrm{mf}}_{\mathrm{sp,j},((0,0),k_z)} \end{pmatrix} \\
&\quad + \partial_t \alpha^{\mathrm{mf}}_{2,((0,0),0),\varepsilon}(t) \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp},2((0,0),0)} \\ U^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),0)} \end{pmatrix} \\
&\quad + \sum_{k_z\neq 0} \partial_t \alpha^{\mathrm{mf}}_{2,((0,0),k_z),\varepsilon}(t)\varepsilon^{1-\nu} \begin{pmatrix} \varepsilon p^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),k_z)} \\ U^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),k_z)} \end{pmatrix},
\end{aligned} \tag{184}$$

$$\begin{aligned}
\partial_t U_\varepsilon^{\mathrm{gw}} + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U_\varepsilon^{\mathrm{gw}} &= \sum_{k_h\neq(0,0),k_z\neq 0} e^{\mp i \frac{\omega^{\mathrm{gw}}_{(k_h,k_z)}}{\varepsilon^{1-\nu}} \frac{t}{\varepsilon^\nu}} \partial_t \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon}(t)\varepsilon^{1-\nu} U^{\mathrm{gw}}_{\pm,(k_h,k_z)} \\
&= \sum_{k_h\neq(0,0),k_z\neq 0} \Big[e^{\mp i \frac{\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}}{\varepsilon^{1-\nu}} \frac{t}{\varepsilon^\nu}} + \mathcal{O}(\varepsilon^{2-3\nu}) \Big] \partial_t \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon}(t) \\
&\quad \times \varepsilon^{1-\nu} \left[\begin{pmatrix} \varepsilon p^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)} \\ U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \end{pmatrix} + \mathcal{O}(\varepsilon^{1-\nu}) \right],
\end{aligned} \tag{185}$$

$$\begin{aligned}
\partial_t U_\varepsilon^{\mathrm{aw}} + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U_\varepsilon^{\mathrm{aw}} &= \sum_{(k_h,k_z)\neq((0,0),0)} e^{\mp i \omega^{\mathrm{aw}}_{(k_h,k_z)} \frac{t}{\varepsilon}} \partial_t \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon}(t) U^{\mathrm{aw}}_{\pm,(k_h,k_z)} \\
&= \sum_{(k_h,k_z)\neq((0,0),0)} e^{\mp i \omega^{\mathrm{aw}}_{(k_h,k_z)} \frac{t}{\varepsilon}} \partial_t \alpha^{\mathrm{aw}}_{\pm,(k_h,k_z),\varepsilon}(t) \Big[U^{\mathrm{aw}}_{\mathrm{a},\pm,(k_h,k_z)} + \mathcal{O}(\varepsilon^{1-\nu}) \Big],
\end{aligned} \tag{186}$$

thanks to (130), (131), and (132).

On the other hand, one can check from Proposition 6, $\big\{(U^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)})_{\mathrm{j}=1,2,3,4}, U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}\big\}$ forms a orthogonal basis and satisfies the soundproof property. Denote by the projection operators to Span $\{U^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)}\}$ and $\mathrm{Span}\{U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}\}$, defined as

$$\begin{aligned}
\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp},1,(k_h,k_z)}(\cdot) &:= \mathrm{Proj}_{\mathrm{Span}\{U^{\mathrm{mf}}_{\mathrm{sp},1,(k_h,k_z)}\}}(\cdot), && k_h \neq (0,0), \\
\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),k_z)}(\cdot) &:= \mathrm{Proj}_{\mathrm{Span}\{U^{\mathrm{mf}}_{\mathrm{sp},2,((0,0),k_z)}\}}(\cdot), && k_z \neq 0, \\
\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},((0,0),k_z)}(\cdot) &:= \mathrm{Proj}_{\mathrm{Span}\{U^{\mathrm{mf}}_{\mathrm{sp,j},((0,0),k_z)}\}}(\cdot), && \mathrm{j} = 3,4, \\
\mathcal{P}^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}(\cdot) &:= \mathrm{Proj}_{\mathrm{Span}\{U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}\}}(\cdot), && k_h \neq (0,0), k_z \neq 0.
\end{aligned} \tag{187}$$

Now we are ready to filter out the acoustic waves in (181) by projections. In the following, we always assume $|k_h|, |k_z| \leq K$, and the restrictions on (k_h, k_z) as in (187) apply.

First, thanks to (124), (146), (164), (167), (184), (185), and (186), one can calculate that, recalling $0 \leq 2\nu \leq 1$,

$$\begin{aligned}
\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} \mathcal{P}_{\mathrm{rd}} \mathcal{N}(T_K U) &= \mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} \mathcal{N}_{\mathrm{sp}}(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}})) \\
&\quad + \mathcal{O}(\varepsilon^{1-\nu}) + \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right), \\
\mathcal{P}^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \mathcal{P}_{\mathrm{rd}} \mathcal{N}(T_K U) &= \mathcal{P}^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \mathcal{N}_{\mathrm{sp}}(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}}))
\end{aligned}$$

$$
\begin{aligned}
&\quad + \mathcal{O}(\varepsilon^{1-\nu}) + \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right), \\
&\mathcal{P}_{\mathrm{rd}}\mathcal{M}(T_K U) = 0, \\
&\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)}\mathcal{P}_{\mathrm{rd}}\left(\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U\right) = \partial_t \alpha^{\mathrm{mf}}_{\mathrm{j},(k_h,k_z),\varepsilon} U^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)}, \mathrm{j} \neq 2, \\
&\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp},2,(k_h,k_z)}\mathcal{P}_{\mathrm{rd}}\left(\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U\right) = \partial_t \alpha^{\mathrm{mf}}_{2,(k_h,k_z),\varepsilon} \varepsilon^{1-\nu} U^{\mathrm{mf}}_{\mathrm{sp},2,(k_h,k_z)}, \\
&\mathcal{P}^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)}\mathcal{P}_{\mathrm{rd}}\left(\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U\right) = e^{\mp i \frac{\omega^{\mathrm{gw}}_{\mathrm{sp},(k_h,k_z)}}{\varepsilon^{1-\nu}} \frac{t}{\varepsilon^\nu}} \partial_t \alpha^{\mathrm{gw}}_{\pm,(k_h,k_z),\varepsilon} \varepsilon^{1-\nu} U^{\mathrm{gw}}_{\mathrm{sp},\pm,(k_h,k_z)} \\
&\quad + \mathcal{O}(\varepsilon^{2-3\nu}).
\end{aligned}
$$

In particular, recalling $\mathfrak{U}^{\mathrm{mf}}$ and $\mathfrak{U}^{\mathrm{gw}}$ in (161) and (159), similar calculation as in (184)–(186) for $(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}})(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}}))$ yields that

$$
\begin{aligned}
&\sum_{\substack{\mathrm{j}=1,2,3,4,\mathrm{j}'=+,- \\ |k_h|,|k_z|\le K}} \left(\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} + \mathcal{P}^{\mathrm{gw}}_{\mathrm{sp,j}',(k_h,k_z)}\right)\mathcal{P}_{\mathrm{rd}}\left(\partial_t U + \frac{1}{\varepsilon}\mathcal{L}_\varepsilon U\right) \\
&\quad = \left(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}}\right)(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}})) + \mathcal{C}_K \mathcal{O}(\varepsilon^{2-3\nu}).
\end{aligned} \tag{188}
$$

Therefore, denote by

$$
\mathcal{P}^{\mathrm{mf+gw}}_{\mathrm{sp},K} := \sum_{\substack{\mathrm{j}=1,2,3,4,\mathrm{j}'=+,- \\ |k_h|,|k_z|\le K}} \left(\mathcal{P}^{\mathrm{mf}}_{\mathrm{sp,j},(k_h,k_z)} + \mathcal{P}^{\mathrm{gw}}_{\mathrm{sp,j}',(k_h,k_z)}\right). \tag{189}
$$

Applying

$$
\mathcal{P}^{\mathrm{mf+gw}}_{\mathrm{sp},K}\mathcal{P}_{\mathrm{rd}}
$$

to (181) yields, since $0 < 2\nu < 1$,

$$
\begin{aligned}
&\left(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}}\right)(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}})) + \mathcal{P}^{\mathrm{mf+gw}}_{\mathrm{sp},K}\mathcal{N}_{\mathrm{sp}}(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}})) \\
&\quad = \mathcal{C}_K \mathcal{O}(\varepsilon^{1-\nu}) + \mathcal{C}_K \mathcal{O}(\varepsilon^{\mu-\sigma}) + \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right) + Err.
\end{aligned} \tag{190}
$$

On the other hand, with similar calculation as in (184) and (185), one can conclude that

$$
T_K\left(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}}\right)U_{\mathrm{sp}} = \left(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}}\right)T_K U_{\mathrm{sp}}.
$$

Consequently, applying $\mathcal{P}^{\mathrm{mf+gw}}_{\mathrm{sp},K}$ to (182) yields

$$
\left(\partial_t + \frac{1}{\varepsilon^\nu}\mathcal{L}_{\mathrm{sp}}\right)T_K U_{\mathrm{sp}} + \mathcal{P}^{\mathrm{mf+gw}}_{\mathrm{sp},K}\mathcal{N}_{\mathrm{sp}}(T_K U_{\mathrm{sp}}) = Err. \tag{191}
$$

Here, although not exactly the same expression as before, Err satisfies (183).

Then, after subtracting (190) with (191), and taking the L^2-inner product of the resultant equations with $2(T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}}) - T_K U_{\mathrm{sp}})$, with similar calculation as in Sect. 4.4, we arrive at the estimate

$$
\begin{aligned}
&\frac{d}{dt}\left\|T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}}) - T_K U_{\mathrm{sp}}\right\|^2_{L^2} \le \mathcal{C}\left\|T_K \mathcal{P}_{\mathrm{rd}}(\mathfrak{U}^{\mathrm{mf}} + \mathfrak{U}^{\mathrm{gw}}) - T_K U_{\mathrm{sp}}\right\|^2_{L^2} \\
&\quad + \mathcal{C}_K \mathcal{O}(\varepsilon^{2-2\nu}) + \mathcal{C}_K \mathcal{O}(\varepsilon^{2\mu-2\sigma}) \\
&\quad + \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right) + Err,
\end{aligned} \tag{192}
$$

where we use the fact that

$$\int \left\{ \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right)\right\} \cdot \underbrace{(T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}}) - T_K U_{\text{sp}})}_{\text{oscillation at rate } \mathcal{O}\left(\frac{1}{\varepsilon^\nu}\right)} \, d\vec{x}$$

$$= \text{oscillation in time with rate } \mathcal{O}\left(\frac{1}{\varepsilon}\right).$$

We would like to emphasize that it is important that we get an estimate with coefficient $\mathcal{C}$ independent of K on the right hand side of (192). Otherwise when applying Grönwall's inequality, below, it would arrive at an estimate with uncontrollable Err. This is possible thanks to the soundproof property of $T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}}) - T_K U_{\text{sp}}$ and cancellation when applying integration by parts, as it is done in Sect. 4.4.

Then integrating (192) in time yields, since $2 - 2\nu > 1$, for $0 < t \le T_{\sigma,\text{mg}} < \min\{T_\sigma, T_{\text{sp}}\}$ with some $T_{\sigma,\text{mg}} \in (0, \infty)$,

$$\begin{aligned}
&\left\| T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}})(t) - T_K U_{\text{sp}}(t)\right\|_{L^2}^2 \\
&\quad \le \left\| T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}})(0) - T_K U_{\text{sp}}(0)\right\|_{L^2}^2 \\
&\qquad + \int_0^t \mathcal{C}\left\| T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}})(s) - T_K U_{\text{sp}}(s)\right\|_{L^2}^2 ds \\
&\qquad + C_K(\mathcal{O}(\varepsilon^{2\mu-2\sigma}) + \mathcal{O}(\varepsilon)) + Err.
\end{aligned} \tag{193}$$

We would like to remind readers that $\mathfrak{U}^{\text{gw}}$ and $\mathfrak{U}^{\text{mf}}$ as in (159) and (161), thanks to (130), (131), and (132), satisfy

$$T_K\mathcal{P}_{\text{rd}}\mathfrak{U}^{\text{mf}} = T_K\mathcal{P}_{\text{rd}}U_\varepsilon^{\text{mf}} \qquad \text{and} \qquad T_K\mathcal{P}_{\text{rd}}\mathfrak{U}^{\text{gw}} = T_K\mathcal{P}_{\text{rd}}U_\varepsilon^{\text{gw}} + \mathcal{C}_K\mathcal{O}(\varepsilon^{2-3\nu}), \tag{194}$$

and thus, since $4 - 6\nu = 1 + 3(1 - 2\nu) > 1$,

$$\left\| T_K\mathcal{P}_{\text{rd}}(\mathfrak{U}^{\text{mf}} + \mathfrak{U}^{\text{gw}}) - T_K\mathcal{P}_{\text{rd}}(U_\varepsilon^{\text{mf}} + U_\varepsilon^{\text{gw}})\right\|_{L^2}^2 = \mathcal{C}_{\text{K}}\mathcal{O}(\varepsilon^{4-6\nu}) \le \mathcal{C}_{\text{K}}\mathcal{O}(\varepsilon). \tag{195}$$

Consequently, after choosing appropriate initial data for U_{sp} which carries the initial mean flows and internal waves, one can derive from (193) that

$$\left\| T_K\mathcal{P}_{\text{rd}}(U_\varepsilon^{\text{mf}} + U_\varepsilon^{\text{gw}})(t) - T_K U_{\text{sp}}(t)\right\|_{L^2}^2 \le C_K(\mathcal{O}(\varepsilon^{2\mu-2\sigma}) + \mathcal{O}(\varepsilon)) + Err, \tag{196}$$

after applying Grönwall's inequality and (195). We remind readers that Err satisfies (183). Thus from (196), one can conclude Theorem 2.

6.3. Remarks

In Sect. 6.1, we introduce the dimension reduction operator $\mathcal{P}_{\text{rd}}$ in (145), which is used in Sect. 6.2 to prove Theorem 2; that is, the asymptotic behavior of the $\widetilde{\mathcal{H}}(\widetilde{\mathcal{H}}_{\text{sp}}), v(v_{\text{sp}}), w(w_{\text{sp}})$ components. However, the asymptotic behavior of the $\widetilde{q}$ component is not discussed.

In the case of well-prepared initial data, i.e., in Theorem 1, we choose initially $\widetilde{q}$ and $\varepsilon p_{\text{sp}}$ (equivalently $\widetilde{q}_{\text{in}}$ and $\varepsilon p_{\text{ms,in}}$) close. In particular, since $\int p_{\text{sp}}\, d\vec{x} = 0$ in the soundproof system, the well-prepared initial data should satisfy that $\int \widetilde{q}_{\text{in}}\, d\vec{x}$ is close to zero, which is **not** the case for the ill-prepared initial data. In particular, $\int \widetilde{q}\, d\vec{x} = 0$ is not a conservative property for the full system (135).

That is, the $\widetilde{q}$ component is **nontrivial** in both the slow waves and fast waves in the case of ill-prepared initial data (see, for instance, (164)). However, these nontrivial waves do not have influence on the $\widetilde{\mathcal{H}}(\widetilde{\mathcal{H}}_{\text{sp}}), v(v_{\text{sp}}), w(w_{\text{sp}})$ components of the mean flows and internal waves of the solutions to (135) [(143), respectively]. In particular, there is no $\widetilde{q}$ component in the solutions to (143). This is why our asymptotic analysis works and has to be done after applying the dimension reduction $\mathcal{P}_{\text{rd}}$ to system (135), in the case of ill-prepared initial data.

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Declarations

Conflict of interest The authors declare that there is no conflict of interest concerning this work.

7. Appendix

Finally, although it is straightforward, we would like to record the representation of the waves decomposition of the full compressible system. With the Fourier representations (99), we calculate the mean flow part first. When $k_h \neq (0,0)$, noticing that $|U^{\mathrm{mf}}_{1,(k_h,k_z)}|^2 = \varsigma := \int_0^1 \cos^2(k_z z)\, dz = \begin{cases} \frac{1}{2} & \text{if } k_z \neq 0 \\ 1 & \text{if } k_z = 0 \end{cases}$,

$$\frac{1}{|U^{\mathrm{mf}}_{1,(k_h,k_z)}|^2} \int U \cdot \overline{U^{\mathrm{mf}}_{1,(k_h,k_z)}}^c \, d\vec{x} = \frac{V_{(k_h,k_z)} \cdot k_h^{\perp}}{|k_h|}.$$

When $k_h = (0,0)$, noticing that $|U^{\mathrm{mf}}_{2,((0,0),k_z)}|^2 = \varsigma + \dfrac{k_z^2}{\eta^2}(1-\varsigma)$, $|U^{\mathrm{mf}}_{3,((0,0),k_z)}|^2 = |U^{\mathrm{mf}}_{4,((0,0),k_z)}|^2 = \varsigma$,

$$\begin{aligned}
\frac{1}{|U^{\mathrm{mf}}_{2,((0,0),k_z)}|^2} \int U \cdot \overline{U^{\mathrm{mf}}_{2,((0,0),k_z)}}^c \, d\vec{x} &= \frac{\eta^2 Q_{((0,0),k_z)}\varsigma + k_z \eta H_{((0,0),k_z)}(1-\varsigma)}{\eta^2\varsigma + k_z^2(1-\varsigma)}, \\
\frac{1}{|U^{\mathrm{mf}}_{3,((0,0),k_z)}|^2} \int U \cdot \overline{U^{\mathrm{mf}}_{3,((0,0),k_z)}}^c \, d\vec{x} &= (V_{((0,0),k_z)})_1, \\
\frac{1}{|U^{\mathrm{mf}}_{4,((0,0),k_z)}|^2} \int U \cdot \overline{U^{\mathrm{mf}}_{4,((0,0),k_z)}}^c \, d\vec{x} &= (V_{((0,0),k_z)})_2
\end{aligned}$$

Therefore, the mean flow projection of U is given by

$$
\begin{aligned}
U_\varepsilon^{\mathrm{mf}} := \mathcal{P}_\varepsilon^{\mathrm{mf}} U = & \sum_{k_h \in 2\pi\mathbb{Z}^2\setminus\{(0,0)\}, k_z \in 2\pi\mathbb{N}} \frac{V_{(k_h,k_z)} \cdot k_h^\perp}{|k_h|} U^{\mathrm{mf}}_{1,(k_h,k_z)} \\
& + \sum_{k_z \in 2\pi\mathbb{N}} \Bigg(\frac{\eta^2 Q_{((0,0),k_z)} \varsigma + k_z \eta H_{((0,0),k_z)}(1-\varsigma)}{\eta^2\varsigma + k_z^2(1-\varsigma)} U^{\mathrm{mf}}_{2,((0,0),k_z)} \\
& + (V_{((0,0),k_z)})_1 U^{\mathrm{mf}}_{3,((0,0),k_z)} + (V_{((0,0),k_z)})_2 U^{\mathrm{mf}}_{4,((0,0),k_z)} \Bigg).
\end{aligned}
\tag{197}
$$

Next, we calculate the internal wave part of U. Notice that for $k_h \neq (0,0), k_z \neq 0$,

$$
\begin{aligned}
|U^{\mathrm{gw}}_{\pm,(k_h,k_z)}|^2 &= \frac{1}{2}\left(1 + \frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2}\right) \\
&\quad + \frac{1}{2}\left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right)^2 \left(\frac{\eta^2 + (\omega^{\mathrm{gw}}_{(k_h,k_z)})^2}{(k_z)^2}\right) \\
&= 1 + \frac{\eta^2}{(k_z)^2}\left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right)^2,
\end{aligned}
$$

where $\omega^{\mathrm{gw}}_{(k_h,k_z)}$ is given by (106). Here we have used the fact that $\int U^{\mathrm{gw}}_{+,(k_h,k_z)} \cdot \overline{U^{\mathrm{gw}}_{-,(k_h,k_z)}}^c \, d\vec{x} = 0$, which yields

$$
\frac{1}{2}\left(1 - \frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2}\right) + \frac{1}{2}\left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right)^2 \left(\frac{\eta^2 - (\omega^{\mathrm{gw}}_{(k_h,k_z)})^2}{(k_z)^2}\right) = 0.
$$

In addition,

$$
\begin{aligned}
\frac{1}{|U^{\mathrm{gw}}_{\pm,(k_h,k_z)}|^2} \int U \cdot \overline{U^{\mathrm{gw}}_{\pm,(k_h,k_z)}}^c \, d\vec{x} = & \frac{1}{2 + \frac{2\eta^2}{(k_z)^2}\left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right)^2} \\
& \times \Bigg[Q_{(k_h,k_z)} \pm \frac{1}{\omega^{\mathrm{gw}}_{(k_h,k_z)}} V_{(k_h,k_z)} \cdot k_h \\
& + \frac{H_{(k_h,k_z)}\eta \mp i W_{(k_h,k_z)} \omega^{\mathrm{gw}}_{(k_h,k_z)}}{k_z} \left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right) \Bigg].
\end{aligned}
$$

Therefore, the internal wave projection of U is given by

$$
\begin{aligned}
U_\varepsilon^{\mathrm{gw}} := \mathcal{P}_\varepsilon^{\mathrm{gw}} U = & \sum_{k_h \in 2\pi\mathbb{Z}^2\setminus\{(0,0)\}, k_z \in 2\pi\mathbb{N}^+} \frac{1}{2 + \frac{2\eta^2}{(k_z)^2}\left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right)^2} \\
& \times \Bigg[Q_{(k_h,k_z)} \pm \frac{1}{\omega^{\mathrm{gw}}_{(k_h,k_z)}} V_{(k_h,k_z)} \cdot k_h \\
& + \frac{H_{(k_h,k_z)}\eta \mp i W_{(k_h,k_z)} \omega^{\mathrm{gw}}_{(k_h,k_z)}}{k_z} \left(\frac{|k_h|^2}{(\omega^{\mathrm{gw}}_{(k_h,k_z)})^2} - 1\right) \Bigg] U^{\mathrm{gw}}_{\pm,(k_h,k_z)}.
\end{aligned}
\tag{198}
$$

The calculation of the acoustic wave part of U is similar for $k_z \neq 0$, which is

$$
\frac{1}{|U^{\mathrm{aw}}_{\pm,(k_h,k_z)}|^2} \int U \cdot \overline{U^{\mathrm{aw}}_{\pm,(k_h,k_z)}}^c \, d\vec{x} = \frac{1}{2 + \frac{2\eta^2}{(k_z)^2}\left(\frac{(|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,k_z)})^2} - 1\right)^2}
$$

$$\times \Bigg[Q_{(k_h,k_z)} \pm \frac{1}{\omega^{\mathrm{aw}}_{(k_h,k_z)}} V_{(k_h,k_z)} \cdot k_h + \frac{H_{(k_h,k_z)}\eta \mp iW_{(k_h,k_z)}\omega^{\mathrm{aw}}_{(k_h,k_z)}}{k_z} \left(\frac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,k_z)})^2} - 1 \right) \Bigg].$$

On the other hand, when $k_z = 0$, $k_h \neq (0,0)$, we have $|U^{\mathrm{aw}}_{\pm,(k_h,0)}|^2 = 1 + \frac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,0)})^2}$, and

$$\frac{1}{|U^{\mathrm{aw}}_{\pm,(k_h,0)}|^2} \int U \cdot \overline{U^{\mathrm{aw}}_{\pm,(k_h,0)}}^c \, d\vec{x} = \frac{Q_{(k_h,0)} \pm \dfrac{k_h \cdot V_{(k_h,0)}}{\omega^{\mathrm{aw}}_{(k_h,0)}}}{1 + \dfrac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,0)})^2}}.$$

Consequently, the perturbed acoustic wave projection of U is given by

$$\begin{aligned} U^{\mathrm{aw}}_{\varepsilon} := \mathcal{P}^{\mathrm{aw}}_{\varepsilon} U = & \sum_{k_h \in 2\pi\mathbb{Z}^2, k_z \in 2\pi\mathbb{N}^+} \frac{1}{2 + \dfrac{2\eta^2}{(k_z)^2} \left(\dfrac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,k_z)})^2} - 1 \right)^2} \\ & \times \Bigg[Q_{(k_h,k_z)} \pm \frac{1}{\omega^{\mathrm{aw}}_{(k_h,k_z)}} V_{(k_h,k_z)} \cdot k_h \\ & + \frac{H_{(k_h,k_z)}\eta \mp iW_{(k_h,k_z)}\omega^{\mathrm{aw}}_{(k_h,k_z)}}{k_z} \left(\frac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,k_z)})^2} - 1 \right) \Bigg] U^{\mathrm{aw}}_{\pm,(k_h,k_z)} \\ & + \sum_{k_h \in 2\pi\mathbb{Z}^2 \setminus \{(0,0)\}} \frac{Q_{(k_h,0)} \pm \dfrac{k_h \cdot V_{(k_h,0)}}{\omega^{\mathrm{aw}}_{(k_h,0)}}}{1 + \dfrac{|k_h|^2}{(\omega^{\mathrm{aw}}_{(k_h,0)})^2}} U^{\mathrm{aw}}_{\pm,(k_h,0)}. \end{aligned} \tag{199}$$

Here $\omega^{\mathrm{aw}}_{(k_h,k_z)}$ is given as (108).

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