Chapter 2

Polyhedral surfaces

2.1 Metric structure

A *polyhedral surface*, or *Euclidean cone surface* M_h is the metric space obtained by gluing together flat Euclidean triangles isometrically along their edges. Henceforth, we only consider *finite triangulations* which are homeomorphic to compact, connected, and orientable 2-manifolds.

If $\gamma : [a,b] \to M_h$ is a continuous curve, then the *length* of γ is the supremum over all *admissible* partitions, $Z = \{t_0 = a \leq t_1 \leq ... \leq t_n = b\}$, of $[a,b]$:

$$l(\gamma) = \sup_Z \sum_{i=1}^{n} d_{\mathbb{E}^2}(\gamma(t_{i-1}), \gamma(t_i)).$$

A partition is *admissible* if $\gamma(t_i)$ and $\gamma(t_{i+1})$ lie in the same triangle, T_h, (possibly on ∂T_h). Here $d_{\mathbb{E}^2}$ denotes the Euclidean distance within each triangle. The curve is called *rectifiable* if $l(\gamma) < \infty$. Let x and y be two points in M_h. The distance between x and y is defined as

$$d(x,y) := \inf_\gamma l(\gamma), \tag{2.1}$$

the infimum taken over all continuous curves $\gamma : [a,b] \to M_h$ connecting x and y. Following Gromov [42], we call M_h a *length space*. On individual triangles the length metric coincides with the flat Euclidean metric. Across an edge of two adjacent triangles this metric is still flat since one can rotate those triangles about their common edge until they become coplanar. In other words, an intrinsic observer fails to note the existence of edges. The situation changes at vertices where the metric exhibits *cone points*, cf. [79].

Definition 2.1.1 (metric cone)**.** The set $C_\theta := \{(r,\varphi) |\, 0 \leq r;\ \varphi \in \mathbb{R}/\theta\mathbb{Z}\}/_\sim$ together with the (infinitesimal) metric

$$\mathrm{d}s = \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2} \tag{2.2}$$

is called a *metric cone* with cone angle θ. Here $(0,\varphi_1) \sim (0,\varphi_2)$ for any pair (φ_1,φ_2). The *cone point* is the coset consisting of all points $(0,\varphi) \in C_\theta$.

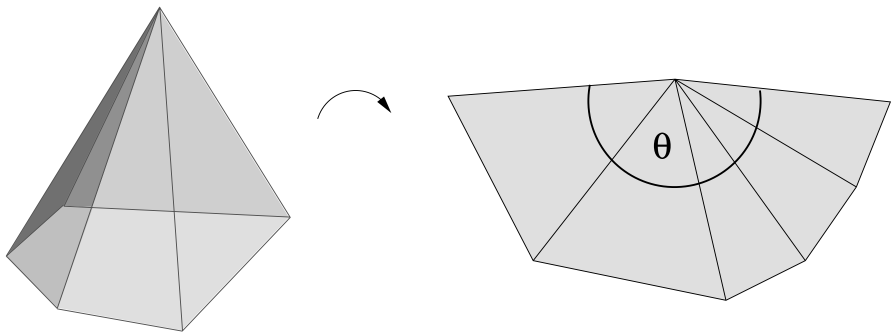


Figure 2.1: A neighborhood of a vertex with total vertex angle θ equipped with the length metric is isometric to a metric cone with cone angle θ.

A cone point is called *singular* if the cone angle does not equal 2π. A singular cone point is *spherical* if the cone angle is less than 2π; otherwise it is *hyperbolic*. We shall henceforth denote a polyhedral surface equipped with its *Euclidean cone metric* by

$$(M_h, g_{M_h}).$$

This metric coincides with the flat Euclidean metric outside cone singularities and is given by (2.2) in a neighborhood of cone singularities. We remark that the cone metric give rise to a (smooth) complex structure on M_h (cf. Troyanov [79]):

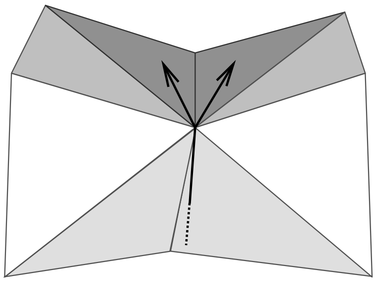
Proposition 2.1.1. *The complex plane* $\mathbb{C}$ *equipped with the metric* $\mathrm{d}s^2 = |z|^{2\beta}|\,\mathrm{d}z|^2$*, with* $\beta = (\theta/2\pi) - 1$*, is isometric to* C_θ*.*

Finally, a *geodesic* in M_h is a curve which locally minimizes the distance between any two points on its image. More precisely,

Definition 2.1.2 (minimizing geodesic)**.** A *minimizing geodesic* in a metric space (V, d) is a continuous curve $\gamma : [a, b] \to V$ such that $\mathrm{d}(\gamma(t), \gamma(t')) = |t' - t|$ for all t and t' in the interval $[a, b]$.

For a triangulated surface M_h a minimizing geodesics consists of straight line segments inside the triangles it crosses; and no minimizing geodesic passes through a spherical cone point (cf. Figure 2.2). The following theorem guarantees the existence of minimizing geodesics on the spaces we consider. In fact this theorem holds true in the wider class of locally compact complete *length spaces* (cf. Gromov [42]).

Theorem 2.1.1 (Hopf-Rinow)**.** *Let* M_h *be a metrically complete Euclidean cone surface.*



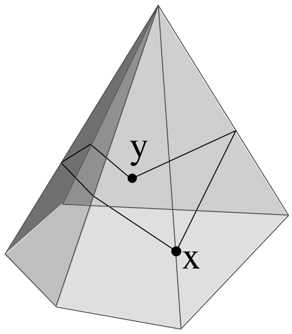


Figure 2.2: Left: A geodesics through a hyperbolic vertex ($\theta > 2\pi$); each incoming ray admits a family of outgoing rays spanning an angle of $\theta - 2\pi$. **Right**: For a neighborhood of a spherical vertex ($\theta < 2\pi$) and for each point x in that neighborhood there exists a corresponding point y such that there are two minimizing geodesics connecting x and y.

i *For all points $x, y \in M_h$ there exists a minimizing geodesic connecting them.*

ii *Each homotopy class of curves can be represented by a geodesic of minimal length.*

We will show in Section 3.3.6 that geodesics on Euclidean cone surfaces can be used to approximate smooth geodesics on smooth surfaces.

2.2 Sobolev theory on polyhedral surfaces

Here we outline the analytic preliminaries of Sobolev spaces over Euclidean cone surfaces. Although the theory is similar to well-established machinery on planar domains, there are subtle, yet crucial, differences due to the presence of cone singularities. Here we provide the necessary adjustments from the planar case.

In particular, we will carefully develop the theory of weak derivatives, show a Poincaré lemma, discuss the Dirichlet problem, and give an outlook on regularity theory. Many of these results could be deduced from the more general approach of considering Lipschitz manifolds (and in particular the fact that bi-Lipschitz maps leave invariant the Sobolev spaces $W^{1,p}$). For such a general treatment we refer to the books of Wloka [83] and Ziemer [86] and to an article by Cheeger [18]. We have chosen to develop the theory from scratch here because many results of this section will be used later in the text; moreover, we feel that a too general approach would obscure a clear understanding of the peculiarities of Euclidean cone surfaces.

2.2.1 L^p-spaces

Let $\mathrm{d}vol$ denote the volume form on the Euclidean cone surface (M_h, g_{M_h}). This volume form is defined (and smooth) outside cone singularities and induces a Borel regular measure on M_h. We let $\mathfrak{X}(M_h)$ denote the space of $\mathrm{d}vol$-measurable vector fields on M_h.

Definition 2.2.1 (L^p-spaces)**.** For $1 \leq p < \infty$ let $L^p(M_h)$ consist of all measurable functions u whose p^{th} power is integrable with respect to $\mathrm{d}vol$, that is,

$$\|u\|_p = \left(\int_{M_h} |u|^p \,\mathrm{d}vol\right)^{1/p} < \infty.$$

Similarly, for $1 \leq p < \infty$ let $L^p_{\mathfrak{X}}(M_h)$ consist of those $X \in \mathfrak{X}(M_h)$ for which

$$\|X\|_p = \left(\int_{M_h} g_{M_h}(X,X)^{p/2} \,\mathrm{d}vol\right)^{1/p} < \infty.$$

As usual, an element of L^p refers to a class of functions (resp. vector fields) which agree outside a set of measure zero.

For all $1 \leq p < \infty$, the spaces L^p are complete and the Hölder and Minkowski inequalities hold (a consequence of the Lemma of Fatou). Additionally, since M_h is compact, it follows that

$$L^p \subset L^1 \quad \text{for all} \quad p \geq 1,$$

a consequence of Hölder's inequality.

2.2.2 Calculus of variations

The main objective of this section is to define an appropriate class of *test functions* and *test vector fields* in the presence of cone singularities. In contrast to the smooth case, one needs to impose extra regularity conditions on the L^∞-norm of the divergence of test vector fields – in order to be able to treat integration by parts.

Definition 2.2.2 (Test functions)**.** The space of *test functions*, $C^\infty(M_h)$, consists of all *continuous* functions which are smooth outside the cone singularities and away from the boundary of M_h. $C^\infty_0(M_h)$ is the subspace of those functions which are compactly supported in $(M_h \setminus \partial M_h)$.

$\mathfrak{X}^\infty(M_h)$ is the space of *test vector fields*, X, which are smooth outside the cone singularities and away from the boundary of M_h, and whose pointwise norm, $\|X\|_{g_{M_h}}$, and divergence, $\operatorname{div} X$, (both classically defined a.e.) are in $L^\infty(M_h)$. $\mathfrak{X}_0^\infty(M_h)$ is the subspace of those vector fields which are compactly supported in $(M_h \setminus \partial M_h)$.

A theorem of the following kind is commonly referred to as the *fundamental theorem of calculus of variations*.

Theorem 2.2.1. *Let $u, v \in L^1(M_h)$ and assume*

$$\int_{M_h} u\varphi \,\mathrm{d}vol = \int_{M_h} v\varphi \,\mathrm{d}vol \quad \forall \quad \varphi \in C_0^\infty(M_h).$$

Then $u = v$ almost everywhere on M_h.

Proof. The standard proof (see e.g. Hörmander [49] Vol. I, Thm. 1.2.5) carries over to Euclidean cone surfaces: one shows that $w = u - v$ vanishes almost everywhere. Indeed, outside cone singularities, the set $(M_h \setminus \partial M_h)$ is locally indistinguishable from $\mathbb{R}^2$. By assumption, $w \in L^1(M_h)$ and hence we can apply Lebesgue's theorem which states that for almost every $x \in M_h$

$$\lim_{\delta \to 0} \frac{1}{\delta^2} \int\limits_{\mathrm{d}(x,y)<\delta} |w(x) - w(y)| \,\mathrm{d}y = 0.$$

Now consider a non-negative function $\rho \in C_0^\infty(\mathbb{R}^2)$ supported in the unit ball of $\mathbb{R}^2$ such that $\int \rho(x)\,\mathrm{d}x = 1$. For almost every $x \in M_h$ and small enough $\delta > 0$, every δ-ball around $x \in M_h$ is isometric to the δ-ball in $\mathbb{R}^2$. Hence for small enough δ and almost every x we have

$$\begin{aligned} w(x) &= \frac{1}{\delta^2} \int_{M_h} w(x)\rho(\,\mathrm{d}(x,y)/\delta)\,\mathrm{d}y \\ &= \frac{1}{\delta^2} \int_{M_h} (w(x) - w(y))\rho(\,\mathrm{d}(x,y)/\delta)\,\mathrm{d}y + \frac{1}{\delta^2} \int_{M_h} w(y)\rho(\,\mathrm{d}(x,y)/\delta)\,\mathrm{d}y. \end{aligned}$$

Now let $\delta \to 0$. The last term is zero by assumption (setting $\varphi = \rho$) and the preceding term tends to zero for almost every x by Lebesgue's theorem.
QED

Corollary 2.2.1. *Let $X, Y \in L^1_{\mathfrak{X}}(M_h)$ and assume*

$$\int_{M_h} g(X, Z)\,\mathrm{d}vol = \int_{M_h} g(Y, Z)\,\mathrm{d}vol \quad \forall \quad Z \in \mathfrak{X}_0^\infty(M_h).$$

Then $X = Y$ almost everywhere.

Proof. For small δ-balls which do not contain cone singularities, X and Y can be expressed in Euclidean coordinates. Apply the proof of the preceding theorem to each coordinate separately. QED

Corollary 2.2.2. *For every $1 \leq p < \infty$ the space of test functions, $C_0^\infty(M_h)$, is dense in $L^p(M_h)$; similarly, $\mathfrak{X}_0^\infty(M_h)$ is dense in $L^p_{\mathfrak{X}}(M_h)$.*

Proof. We are going to show the first part only. The proof of the second part is similar. Assume $C_0^\infty(M_h)$ was not dense in $L^p(M_h)$. Then by the Hahn-Banach theorem there exist $u_0 \in L^p(M_h)$ and a bounded linear functional $F : L^p(M_h) \to \mathbb{R}$ such that F vanishes on $C_0^\infty(M_h)$ but $F(u_0) \neq 0$. Then, by the Riesz representation theorem, there exists $f \in L^q(M_h)$ with $1/p+1/q = 1$ such that $F(u) = \int fu$ for all $u \in L^p(M_h)$. Since $F(\varphi) = 0$ for every $\varphi \in C_0^\infty(M_h)$, we can apply Theorem 2.2.1 which implies that $f = 0$, a contradiction to $0 \neq F(u_0) = \int fu_0$. QED

2.2.3 Weak derivatives

In this section we develop the concept of *weak derivatives* on polyhedra. Recall that the divergence, $\operatorname{div} X$, of a vector field, $X \in \mathfrak{X}_0^\infty(M_h)$, is classically well-defined outside the cone singularities of M_h.

Lemma 2.2.1. *Let $\varphi \in C^\infty(M_h)$. Then*

$$\int_{M_h} g_{M_h}(\nabla\varphi, X)\,\mathrm{d}vol = -\int_{M_h} \varphi \operatorname{div} X\,\mathrm{d}vol \quad \forall \quad X \in \mathfrak{X}_0^\infty(M_h).$$

Proof. Put small disks, D_i, around the cone singularities, $\{x_i\}$, of M_h. Since $M_h \backslash \cup D_i$ contains no cone singularities and carries the standard flat Euclidean metric, it follows that

$$\int_{M_h\backslash\cup D_i} g_{M_h}(\nabla\varphi, X)\,\mathrm{d}vol + \int_{M_h\backslash\cup D_i} \varphi \operatorname{div} X\,\mathrm{d}vol = \sum_i \oint_{\partial D_i} g_{M_h}(X, \eta_i)\varphi\,\mathrm{d}s.$$

Here η_i denotes the unit normal along ∂D_i. By definition, $\|X\|$ and $|\varphi|$ are bounded on M_h so that the right hand side approaches zero with D_i tending to the cone point, x_i. Moreover, by the definition of $\mathfrak{X}^\infty(M_h)$, we have $\operatorname{div} X \in L^\infty(M_h)$, so that the second integral stays bounded as the disks D_i become smaller. This proves the assertion. QED

The proof of the preceding lemma also implies the following *divergence theorem*.

Theorem 2.2.2 (Gauss' divergence theorem)**.** *For every open subdomain* $\Omega \subset M_h$ *with* $C^{0,1}$*-boundary,* $\partial\Omega$*, and every* $X \in \mathfrak{X}^\infty(M_h)$ *one has*

$$\int_\Omega \operatorname{div} X \, \mathrm{d}vol = \oint_{\partial\Omega} g_{M_h}(X,\eta)\, \mathrm{d}s,$$

where η *is the (almost everywhere defined) normal to* $\partial\Omega$*.*

This divergence theorem may be used as a tool to discriminate between appropriate and non-appropriate local charts for M_h. Indeed, the next example shows that *conformal charts* on polyhedra, as introduced in Proposition 2.1.1, are *not appropriate* for the theory of calculus of variations on polyhedral surfaces – in the sense that there exist vector fields for which the divergence theorem holds on M_h but fails in these charts. This can be viewed as an instance that cone singularities do indeed complicate the Sobolev theory.

Example (conformal charts are not appropriate)**.** Let C_θ be a cone with cone angle θ. By Proposition 2.1.1, C_θ is isometric to $\mathbb{C}$ equipped with the metric $\mathrm{d}s^2 = |z|^{2\beta}|\,\mathrm{d}z|^2$ with $\beta = (\theta/2\pi) - 1$. In the punctured complex plane, $\mathbb{C} \setminus \{0\}$, define a vector field X by

$$X(r,\varphi) = \frac{1}{r}\partial_r.$$

We claim that for any $\theta > 2\pi$, the field X satisfies the divergence theorem (for suitable Ω) with respect to the cone metric, $\mathrm{d}s^2$. Indeed, if $D_R \subset \mathbb{C}$ is a disk of Euclidean radius R, centered at the origin, then we have

$$\oint_{\partial D_R} g_{M_h}(X,\eta)\, \mathrm{d}s = 2\pi R^\beta.$$

For $\theta > 2\pi$ this integral converges to zero as $R \to 0$. Hence, the same argument as in the proof of Lemma 2.2.1 shows that the divergence theorem holds for X with respect to the cone metric. On the other hand X, having a distributional point source at the origin, does not satisfy the divergence theorem with respect to the standard Euclidean metric, $|\,\mathrm{d}z|^2$, on $\mathbb{C}$ since its singularity at the origin yields a non-vanishing contribution of 2π. Therefore, the conformal charts of Proposition 2.1.1 are not appropriate because there exist vector fields for which the divergence theorem holds on M_h but fails in these charts.

We now turn to the notion of weak derivatives.

Definition 2.2.3 (Sobolev spaces on polyhedra)**.** For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(M_h)$ consist of all $u \in L^p(M_h)$ for which there exists a *weak gradient*, $\nabla u \in L^p_{\mathfrak{X}}(M_h)$, such that

$$\int_{M_h} g_{M_h}(\nabla u, X)\,\mathrm{d}vol = -\int_{M_h} u \operatorname{div} X \,\mathrm{d}vol \quad \forall \quad X \in \mathfrak{X}_0^\infty(M_h).$$

Corollary 2.2.1 guarantees uniqueness of weak derivatives. Completeness of $W^{1,p}$ with respect to the metric

$$\|u\|_{1,p} = \|u\|_p + \|\nabla u\|_p$$

is a simple consequence of the completeness of L^p (cf. [50] Corollary 20.9).

The following result may be interpreted as an a posteriori justification of the choice of *test functions* made in this work.

Theorem 2.2.3. *$C^\infty(M_h) \cap W^{1,p}(M_h)$ is dense in $W^{1,p}(M_h)$ for $1 \leq p < \infty$.*

Proof. Consider a collection of metric disks, $\{U_i\}$, around the cone points and boundary vertices of M_h. Without loss of generality, assume that $\{U_i\}$ is a locally finite cover M_h (by introducing extra "flat cone points" if necessary). We require that no U_i contains more than one cone singularity and $U_i \cap U_j$ contains no cone singularity. Let (U_i, Φ_i) be charts of the following kind: for a metric cone of angle θ_i let

$$\Phi_i(r, \varphi) := (r, \frac{2\pi}{\theta_i}\varphi) \quad \text{with} \quad 0 \leq \varphi < \theta_i.$$

For a boundary vertex, $b_i \in \partial M_h$, with inner vertex angle θ_i set

$$\Phi_i(r, \varphi) := (r, \frac{\pi}{\theta_i}\varphi) \quad \text{with} \quad 0 \leq \varphi < \theta_i,$$

i.e., a neighborhood of b_i gets mapped to the half-disk in $\mathbb{R}^2$. The main property of these charts is that the differentials, $\mathrm{d}\Phi_i$, introduce no metric distortion in the radial direction and a constant distortion in the angular direction.

Let $\{\rho_i\}$ be a partition of unity subordinate to $\{U_i\}$. We can assume that $\rho_i \in C^\infty(M_h)$ and $\|\nabla \rho_i\| \in L^\infty(M_h)$. Then a straightforward calculation shows that $(\rho_i u) \in W^{1,p}(M_h)$ for every $u \in W^{1,p}(M_h)$. Next we claim that

$$\varphi := (\rho_i u) \circ \Phi_i^{-1} \in W^{1,p}(\Phi_i(U_i)),$$

where $\Phi_i(U_i)$ is equipped with the standard Euclidean metric in $\mathbb{R}^2$. To show this, we are going to establish that

$$\int_{\Phi_i(U_i)} \langle \nabla_{\mathbb{R}^2}\varphi, X\rangle_{\mathbb{R}^2}\, \mathrm{d}vol_{\mathbb{R}^2} = -\int_{\Phi_i(U_i)} \varphi \operatorname{div}_{\mathbb{R}^2} X \,\mathrm{d}vol_{\mathbb{R}^2} \tag{2.3}$$

for all smooth vector fields $X \in \mathfrak{X}(\mathbb{R}^2)$ which are compactly supported in $\Phi_i(U_i)$. Indeed, since Φ_i is smooth on U_i away from the cone point, and $\mathrm{d}\Phi_i^{-1}$ is bounded, it follows that $\mathrm{d}\Phi_i^{-1}(X) \in \mathfrak{X}_0^\infty(M_h)$. By Definition 2.2.3 we obtain

$$\int_{U_i} g_{M_h}(\nabla(\rho_i u),\, \mathrm{d}\Phi_i^{-1}(X))\, \mathrm{d}vol = -\int_{U_i} (\rho_i u) \operatorname{div}(\, \mathrm{d}\Phi_i^{-1}(X))\, \mathrm{d}vol. \tag{2.4}$$

Considering the metric distortion introduced by $\mathrm{d}\Phi_i$, it follows that the push-forward of the volume form, $\mathrm{d}vol$, to $\Phi_i(U_i)$ equals $C\,\mathrm{d}vol_{\mathbb{R}^2}$ for some positive constant, C. Moreover, we obtain

$$g_{M_h}(\nabla(\rho_i u),\, \mathrm{d}\Phi_i^{-1}(X)) = \langle \nabla_{\mathbb{R}^2}\varphi, X\rangle_{\mathbb{R}^2} \circ \Phi_i.$$

It follows that the left hand side of (2.4) equals

$$C \int_{\Phi_i(U_i)} \langle \nabla_{\mathbb{R}^2}\varphi, X\rangle_{\mathbb{R}^2}\, \mathrm{d}vol_{\mathbb{R}^2}.$$

Furthermore, since under a change of Riemannian metrics, the divergence operator transforms according to

$$\operatorname{div}_g X = \frac{1}{|\,\mathrm{d}vol_g|} \operatorname{div}(|\,\mathrm{d}vol_g| X),$$

and $|\,\mathrm{d}vol_g| = C$ in our case, it follows that the right hand side of (2.4) equals

$$-C \int_{\Phi_i(U_i)} \varphi \operatorname{div}_{\mathbb{R}^2} X\, \mathrm{d}vol_{\mathbb{R}^2}.$$

Consequently (2.3) holds and hence

$$(\rho_i u) \circ \Phi_i^{-1} \in W^{1,p}(\Phi_i(U_i)).$$

It is a classical result that $C^\infty(\Phi_i(U_i)) \cap W^{1,p}(\Phi_i(U_i))$ is dense in $W^{1,p}(\Phi_i(U_i))$, so that there exist smooth functions, $\psi_{i,j} \in C^\infty(\Phi_i(U_i))$, such that

$$\lim_{j\to\infty} \|\psi_{i,j} - (\rho_i u) \circ \Phi_i^{-1}\|_{W^{1,p}(\Phi_i(U_i))} = 0. \tag{2.5}$$

For interior points, ρ_i is compactly supported in U_i, so that we can assume $\psi_{i,j} \in C_0^\infty(\Phi_i(U_i))$ and hence $\psi_{i,j} \circ \Phi_i \in C_0^\infty(M_h)$. For boundary points we can assume that $\psi_{i,j} \circ \Phi_i$ is continuous up to the boundary (since we are dealing with piecewise linear boundary, cf. [83]). Consequently, we can assume $\psi_{i,j} \circ \Phi_i \in C^\infty(M_h)$. Moreover, since $\mathrm{d}\Phi_i$ is bounded, it follows that $\psi_{i,j} \circ \Phi_i \in W^{1,p}(M_h)$, and from (2.5) we obtain

$$\lim_{j\to\infty} \|\psi_{i,j} \circ \Phi_i - \rho_i u\|_{W^{1,p}(M_h)} = 0.$$

Using that $\{\rho_i\}$ is a partition of unity, we conclude that

$$\lim_{j\to\infty} \left\| u - \sum_i \psi_{i,j} \circ \Phi_i \right\|_{W^{1,p}(M_h)} \leq \lim_{j\to\infty} \sum_i \|\rho_i u - \psi_{i,j} \circ \Phi_i\|_{W^{1,p}(M_h)} = 0.$$

The assertion follows since $\sum_i \psi_{i,j} \circ \Phi_i \in C^\infty(M_h)$. QED

2.2.4 Rellich lemma and Poincaré inequality

In this section we prove *Poincaré's inequality* on Euclidean cone surfaces. This result will be useful later for a priori estimates. We provide the usual indirect proof of Poincaré's inequality built on Rellich's compactness lemma.

Lemma 2.2.2 (Rellich-Kondrachov). *Let M_h be a triangulated mesh. Then*

$$W^{1,p}(M_h) \subset\subset L^p(M_h)$$

is a compact embedding for all $1 \leq p < \infty$.

Proof. Let $\{u_n\}$ be a bounded sequence in $W^{1,p}(M_h)$. We have to show that there exists a subsequence, $\{u_{n_i}\}$, which converges in $L^p(M_h)$. Indeed, the Rellich lemma holds on individual triangles of M_h, and hence, for every triangle $T_h \subset M_h$ there exists a subsequence of $\{u_n\}$ which converges in $L^p(T_h)$. Since we consider only finite triangulations, there exists a subsequence $\{u_{n_i}\}$ which converges simultaneously on all triangles. Consequently, $\{u_{n_i}\}$ converges in $L^p(M_h)$. QED

Theorem 2.2.4 (Poincaré inequality). *Let M_h be a Euclidean cone surface. For every $1 \leq p < \infty$ there exists a constant C only depending on M_h and p such that*

$$\|u - \bar{u}\|_p \leq C\|\nabla u\|_p \tag{2.6}$$

for all $u \in W^{1,p}(M_h)$. As usual we let $\bar{u} := \frac{1}{|M_h|}\int_{M_h} u \,\mathrm{d}vol$.

Proof. We give the usual indirect proof of Poincaré's inequality. Assume the inequality is false. Then for each $n \in \mathbb{N}$ there exists $u_n \in W^{1,p}(M_h)$ such that

$$\|u_n - \bar{u}_n\|_p > n\|\nabla u_n\|_p.$$

Setting $v_n := u_n - \bar{u}_n$ and re-normalizing we can assume that

$$\int_{M_h} v_n \, \mathrm{d}vol = 0 \quad \text{and} \quad \|v_n\|_p = 1.$$

By our assumption it follows that $\|\nabla v_n\|_p < 1/n$ so that $\{v_n\}$ is bounded in $W^{1,p}(M_h)$. By the Rellich lemma, there exists a subsequence, $\{v_{n_i}\}$, which converges to $v \in L^p(M_h)$. Since $\|\nabla v_{n_i}\|_p \to 0$, it follows that $\{v_{n_i}\}$ forms a Cauchy sequence in $W^{1,p}(M_h)$, so that by completeness $v \in W^{1,p}(M_h)$ and $\nabla v = 0$. This implies that v is constant almost everywhere and since $\bar{v} = 0$ it follows that $v = 0$ (M_h is assumed to be connected); a contradiction to $\|v\|_p = 1$. QED

It is evident from the proof of the Poincaré inequality that one merely needs to *exclude constants* in order to bound the L^p-norm of a function by the L^p-norm of its weak derivative. Hence if M_h has non-empty boundary then the same argument as above shows that if u lies in the closure of $C_0^\infty(M_h) \subset W^{1,p}(M_h)$ then

$$\|u\|_p \le C\|\nabla u\|_p,$$

where the constant C is independent of u.

2.2.5 Laplace–Beltrami and Dirichlet problem

We define the Laplace–Beltrami operator and discuss the Dirichlet problem on Euclidean cone surfaces. First, let us recall the usual definition of the Sobolev space $H_0^1(M_h)$.

Definition 2.2.4. The space $H_0^1(M_h)$ is the closure of $C_0^\infty(M_h)$ in $W^{1,2}(M_h)$, where, in the case that M_h has no boundary, $u \in C_0^\infty(M_h)$ implies $\int_{M_h} u = 0$.

The inner product

$$(u, v)_{H_0^1(M_h)} := \int_{M_h} g_{M_h}(\nabla u, \nabla v) \, \mathrm{d}vol \tag{2.7}$$

induces a Hilbert space structure on $H_0^1(M_h)$ whose norm is equivalent to $\|\cdot\|_{1,2}$ by Poincaré's inequality.

Definition 2.2.5 (Laplace–Beltrami)**.** Let $H^{-1}(M_h)$ denote the *dual space* to $H_0^1(M_h)$, that is the space of bounded linear *functionals* on $H_0^1(M_h)$; and let $\langle\cdot|\cdot\rangle$ be the dual pairing between $H^{-1}(M_h)$ and $H_0^1(M_h)$. The *Laplace–Beltrami operator*

$$\Delta : H_0^1(M_h) \to H^{-1}(M_h)$$

is the negative of the *Riesz map* between $H_0^1(M_h)$ and $H^{-1}(M_h)$, that is

$$\langle \Delta u | v\rangle := -(u, v)_{H_0^1(M_h)}.$$

We can now treat existence and uniqueness of solutions to the Dirichlet problem. By Poincaré's inequality the embedding

$$E : H_0^1(M_h) \hookrightarrow L^2(M_h)$$

is continuous. Hence the dual operator, $E' : L^2(M_h) \to H^{-1}(M_h)$, defined by

$$\langle E'(f)|v\rangle = (f, E(v))_{L^2}$$

is also continuous.

Theorem 2.2.5 (Dirichlet problem)**.** *The equation*

$$-\Delta u = E'(f) \tag{2.8}$$

has a unique solution $u \in H_0^1(M_h)$ for every $f \in L^2(M_h)$. The solution u continuously depends on the data f.

Proof. Existence and uniqueness immediately follow from the fact that $-\Delta$ is the Riesz map. That u depends continuously on f follows from the continuity of E' and the continuity of the inverse operator, Δ^{-1}. QED

2.2.6 A GLIMPSE AT REGULARITY

We give an outlook on regularity of solutions to the Dirichlet problem on polyhedral surfaces. Without making any attempt to cover this issue in full generality, we observe here that the solutions are smooth (in the sense of Definition 2.2.2) provided that the right hand side is smooth.

Theorem 2.2.6. *Let M_h be a closed Euclidean cone surface. Let $u \in H_0^1(M_h)$ be the weak solution to the Dirichlet problem $-\Delta u = E'(f)$ for $f \in L^2(M_h)$. Then $f \in C^\infty(M_h)$ implies $u \in C^\infty(M_h)$.*

Proof. We have to show that u is continuous on M_h as well as smooth outside cone singularities. We first proof smoothness. Consider a simply connected domain $\Omega \subset M_h$ which does not touch the cone singularities of M_h (so that Ω can be developed into the plane). By assumption, $f \in C^\infty(\Omega)$, and whence local regularity implies $u \in C^\infty(\Omega)$ (cf. Gilbarg and Trudinger [35], Corollary 8.11). It follows that u is indeed smooth outside cone singularities.

What remains to be shown is that $u \in C^0(M_h)$. We show that u is continuous at every cone point. This will follow from the fact that in appropriate local charts the Laplacian on M_h can be written in divergence form, so that a classical result of De Giorgi and Nash implies that the solution to the Dirichlet problem is Hölder continuous.

As in the proof of Theorem 2.2.3, consider charts (U_i, Φ_i) of the following kind: for a metric cone of cone angle θ_i let

$$\Phi_i(r, \varphi) := (r, \frac{2\pi}{\theta_i}\varphi) \quad \text{with} \quad 0 \leq \varphi < \theta_i.$$

Let $(r, \bar{\varphi})$, with $0 \leq \bar{\varphi} < 2\pi$, denote polar coordinates on $\Phi_i(U_i) \subset \mathbb{R}^2$ (note that Φ_i introduces no distortion in the radial direction, so we deliberately leave the letter r here). In these coordinates, the Laplace–Beltrami operator on M_h locally takes the form

$$-\Delta v = \frac{1}{r}(r v_r)_r + \alpha^2 \frac{1}{r^2} v_{\bar{\varphi}\bar{\varphi}} \quad \text{for all} \quad v \in C^2(\mathbb{R}^2),$$

with $\alpha = 2\pi/\theta_i$. A straightforward calculation delivers that in the charts we consider, Δ can be written in divergence form with respect to the standard metric on $\mathbb{R}^2$,

$$-\Delta = \operatorname{div}_{\mathbb{R}^2}(A \nabla_{\mathbb{R}^2}),$$

where $\operatorname{div}_{\mathbb{R}^2}$ and $\nabla_{\mathbb{R}^2}$ denote the standard divergence and grad operators in $\mathbb{R}^2$. Indeed, one obtains that

$$A = Id + (\alpha^2 - 1) \begin{pmatrix} \sin^2 \bar{\varphi} & -\sin \bar{\varphi} \cos \bar{\varphi} \\ -\sin \bar{\varphi} \cos \bar{\varphi} & \cos^2 \bar{\varphi} \end{pmatrix},$$

with $\alpha = 2\pi/\theta_i$. In the chart (U_i, Φ_i) the Dirichlet problem hence takes the form

$$-\operatorname{div}_{\mathbb{R}^2}(A \nabla_{\mathbb{R}^2})(u \circ \Phi_i^{-1}) = (f \circ \Phi_i^{-1}).$$

The symmetric matrix A is positive definite and has measurable and bounded coefficients. It hence follows from classical results by De Giorgi and Nash that $(u \circ \Phi_i^{-1})$ is Hölder continuous (for a proof see [35], Theorem 8.22). Consequently, u is continuous since the maps Φ_i are continuous. QED

2.3 Discrete function spaces

Discretization of surfaces, in the point of view taken in this work, is equivalent to the study of finite-dimensional function spaces and bounded operators acting between them. A natural framework for this construction is the *finite element method* (FEM). The construction of the corresponding spaces on Euclidean cone surfaces turns out to be analogous to the planar case. For the planar case we refer to standard textbooks such as Braess [14], Ciarlet [20], and Strang and Fix [76].

In the current work, we use the concept of *intrinsic discretization* of surfaces, i.e., we do not make use of any parameterization. This intrinsic approach goes back to Dziuk ([27, 28]), who proved asymptotic convergence for elliptic equations and to Pinkall and Polthier [61] who introduced a FE approach to mean curvature discretization on Euclidean cone surfaces. For a different approach, where surfaces are sampled over triangulated planar domains by piecewise linear functions, we refer to e.g. Dziuk and Hutchinson [29, 30].

2.3.1 Conforming and nonconforming elements

As for the discretization of elliptic problems in the plane, one differentiates between *conforming elements*, where the finite-dimensional function spaces are subspaces of the Sobolev space H^1, and *nonconforming elements* (called variational crimes in older literature, cf. [76]), where the discrete spaces are no longer subspaces of H^1.

Definition 2.3.1 (conforming finite elements)**.** The space of *linear Lagrange elements* is defined as

$$S_h = \{u \in C^0(M_h) \,|\, u \text{ is linear on all triangles}\}.$$

A *basis* consists of all those functions, ϕ_p, for which $\phi_p(q) = \delta_{pq}$, where δ_{pq} denotes the Kronecker delta, cf. Figure 2.3. We define the subspace $S_{h,0} \subset S_h$ by

$$S_{h,0} = \begin{cases} \{u \in S_h \,|\, u = 0 \text{ along the boundary}\} & \text{if } \partial M_h \neq \emptyset \\ \{u \in S_h \,|\, \int_{M_h} u = 0\} & \text{if } \partial M_h = \emptyset. \end{cases}$$

We have the inclusions $S_h \subset H^1(M_h)$ and $S_{h,0} \subset H^1_0(M_h)$.

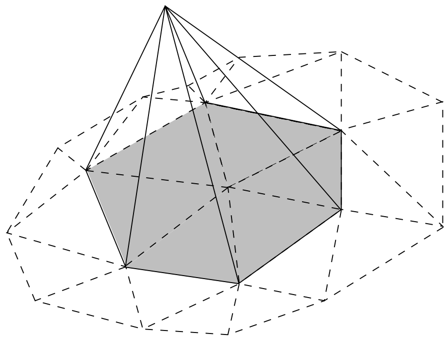


Figure 2.3: Conforming Lagrange basis function equaling 1 at a single vertex and 0 on all other vertices.

Definition 2.3.2 (nonconforming finite elements)**.** The space of *Crouzeix-Raviart elements* is defined as

$$S_h^* = \{u \in L^2(M_h) \,|\, u \text{ is linear on all triangles and continuous at edge midpoints}\}.$$

A *basis* consists of all those functions, ϕ_e, which take the value 1 at the midpoint of edge e and zero at all other edge midpoints of the mesh (cf. Figure 2.4). We define the subspace $S_{h,0}^* \subset S_h^*$ by

$$S_{h,0}^* = \begin{cases} \{u \in S_h^* \,|\, u = 0 \text{ at all edge midpoints along } \partial M_h\} & \text{if } \partial M_h \neq \emptyset \\ \{u \in S_h^* \,|\, \int_{M_h} u = 0\} & \text{if } \partial M_h = \emptyset. \end{cases}$$

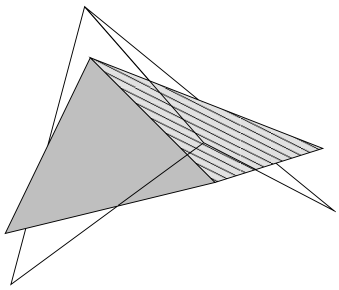


Figure 2.4: Nonconforming basis function equaling 1 at a single edge midpoint and 0 on all other edge midpoints.

It is evident from the definitions of S_h and S_h^* that

$$S_h \subset S_h^* \quad \text{and} \quad S_{h,0} \subset S_{h,0}^*;$$

however, S_h^* is no longer a subspace of $H^1(M_h)$. In particular, there is no unique extension of $F \in H^{-1}(M_h)$ to a functional on $H_0^1(M_h) + S_{h,0}^*$. Hence we will focus on a subclass of functionals which are extendable. For the FE part of this work, we restrict our attention to those functionals which arise from integration by parts. More precisely, we consider real-valued bounded linear operators F on the space $H_0^1(M_h) + S_h^*$ of the following kind:

$$\langle F|u\rangle = -\int_{M_h} g_{M_h}(X_F, \nabla u)\, \mathrm{d}vol + \int_{\partial M_h} u \cdot g_{M_h}(X_F, \eta)\, \mathrm{d}s, \tag{2.9}$$

for some *piecewise constant* vector field $X_F \in L^2_{\mathfrak{X}}(M_h)$. Here η denotes the (piecewise constant) normal along ∂M_h. The functional F is indeed well defined on $H_0^1(M_h)+S_h^*$ since the boundary term vanishes for any $u \in H_0^1(M_h)$ and the gradient of any $u \in S_h^*$ is well defined (and constant) on all triangles of M_h. The following criterion is hence a valid assumption in our case:

Criterion 1. For the FE part of this work, we restrict our attention to those $F \in H^{-1}(M_h)$ which take the form of equation (2.9).

Since piecewise constant vector fields will be used extensively, we fix the following notation:

Definition 2.3.3 (piecewise constant vector fields)**.** Let $\mathfrak{X}_h$ denote the space of vector fields which are constant on individual triangles.

The following lemma assures that the boundary term in (2.9) does not only vanish for every $u \in H_0^1(M_h)$, but also for every $u \in S_{h,0}^*$.

Lemma 2.3.1. *Assume $u \in S_{h,0}^*$, and let $Y \in \mathfrak{X}_h$ be piecewise constant. Then*

$$\int_{\partial M_h} u \cdot g_{M_h}(Y, \eta)\, \mathrm{d}s = 0.$$

Proof. Let e be an edge of the boundary, ∂M_h. Then $g_{M_h}(Y,\eta)$ is constant along e. Since $u \in S_{h,0}^*$, it follows that u vanishes at the midpoint of e. By linearity of u along e, we obtain

$$\int_e u \cdot g_{M_h}(Y, \eta)\, \mathrm{d}s = 0.$$

This completes the proof. QED

The significance of the last lemma lies in the fact that it provides justification for the *nonconforming versions* of the Dirichlet problem, divergence, curl, the Laplace–Beltrami operator, and the mean curvature vector.

2.3.2 Discrete Dirichlet problem

Given $f \in L^2(M_h)$, the discrete Dirichlet problem amounts to solving the variational problem

$$\int_{M_h} g_{M_h}(\nabla u_h, \nabla \phi_h)\, \mathrm{d}vol = \int_{M_h} f \phi_h \, \mathrm{d}vol \quad \forall\, \phi_h \tag{2.10}$$

under the requirement that *both the test functions*, ϕ_h, *and the solution*, u_h, are from the same space $S_{h,0}$ (resp. $S^*_{h,0}$). Due to the ellipticity of the problem, (2.10) always has a unique solution u_h. Furthermore, it follows from the choice of the inner product on H^1_0 (eq. (2.7)), that the *conforming solution*, $u_h \in S_{h,0}$, is the projection to $S_{h,0}$ of the solution $u \in H^1_0$ to the full Dirichlet problem (2.8).

2.3.3 Delaunay discretization

So far we have avoided questions of ambiguity of discretizing function spaces. But in fact there are several possible, and indeed plausible, discretizations of these spaces for any given polyhedron.

If we take the view of an intrinsic observer in M_h then there are several possible *intrinsic* ways to connect the vertices of M_h by *intrinsic edges*, i.e., geodesics of M_h. Notice that in this context an *edge* refers to an *intrinsic* straight line, rather than a straight line in ambient space. For example, consider the situation of Figure 2.5, where the original edge of a *hinge* (made up of two adjacent flat triangles of M_h) is replaced by another interior edge via an intrinsic *edge flip*. Intrinsic edge flips correspond to a *re-meshing* of M_h without changing the metric structure M_h.

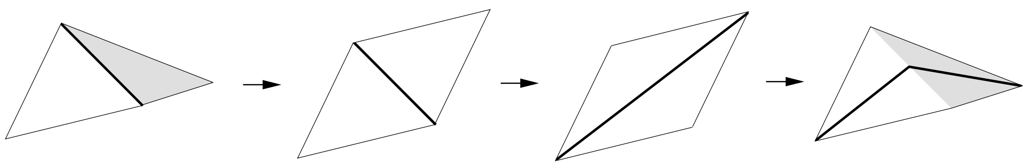


Figure 2.5: Intrinsic edge flip for a hinge of M_h. An intrinsic observer would experience this edge flip as depicted in the second and third picture.

Notice that the finite element spaces, S_h and S^*_h, will (in general) be different for different choices of intrinsic edges. From the numerical point of view, different edge choices effect the corresponding stiffness matrices: let $\{\phi_p\}$ be a nodal basis of S_h corresponding to a particular choice of edges. In order to simplify the discussion, we assume that M_h is closed. The *stiffness matrix* is given as

$$\mathfrak{L}_{pq} := \int_{M_h} g_{M_h}(\nabla\phi_p, \nabla\phi_q)\, \mathrm{d}vol.$$

By definition, $\mathfrak{L}_{pq} = -\langle\Delta\phi_p|\phi_q\rangle$, and $(\mathfrak{L}_{pq})$ is always *symmetric and positive semi-definite* (its kernel are the constants). Note that $\mathfrak{L}_{pp} > 0$ and $\sum_q \mathfrak{L}_{pq} = 0$. However, $(\mathfrak{L}_{pq})$ may contain positive off-diagonal entries . Indeed, let p and q share an edge. Then

$$\mathfrak{L}_{pq} = -\frac{1}{2}(\cot\alpha_{pq} + \cot\beta_{pq}) = -\frac{1}{2}\cdot\frac{\sin(\alpha_{pq}+\beta_{pq})}{\sin\alpha_{pq}\cdot\sin\beta_{pq}},$$

where α_{pq} and β_{pq} are the two angles opposite to the edge $\overline{pq}$. Hence $\mathfrak{L}_{pq} > 0$ if and only if $\alpha_{pq} + \beta_{pq} > \pi$. Numerically, positive off-diagonal entries effect the conditioning of the system. Geometrically, such entries are responsible for a violation of a discrete maximum principle (cf. [62]).

Bobenko and Springborn [10] recently observed that such positive entries can be avoided by considering *intrinsic Delaunay tessellations*. A choice of intrinsic edges is called *Delaunay* if the unfolding of any pair of adjacent (Euclidean) triangles satisfies the empty circumcircle property: none of the four vertices of the two unfolded triangles are contained in the interior of the two circumcircles of these triangles. Bobenko and Springborn show:

Theorem 2.3.1 (Delaunay discretization)**.** *Every compact Euclidean cone surface allows for an intrinsic Delaunay tessellation. Furthermore, the stiffness matrix corresponding to an intrinsic choice of edges has all non-positive off-diagonal entries if and only if the choice of edges is Delaunay.*

2.3.4 Mass matrices and discretized functionals

We conclude this section with a remark on scaling behavior. Because functionals scale differently from functions, one sometimes wishes to *interpret* a functional, $F \in H^{-1}(M_h)$, as a function in the space S_h or S_h^*. Notice that only because the dimension of the involved spaces is finite, a functional, $F \in H^{-1}(M_h)$, can be *discretized* to become a function $F_h \in S_h$ (resp. $F_h^* \in S_h^*$). There is no infinite-dimensional analogue of such a construction.

Definition 2.3.4 (mass matrix)**.** Let $\{\phi_p\}$ denote the nodal basis functions at vertices, and let $\{\phi_{e_i}\}$ denote the mid-edge basis functions at edges. Then

$$\mathcal{M}_{pq} = \int_{M_h} \phi_p \phi_q \, \mathrm{d}vol \quad \text{and} \quad \mathcal{M}^*_{ij} = \int_{M_h} \phi_{e_i} \phi_{e_j} \, \mathrm{d}vol$$

defines the *conforming* and the *nonconforming* mass matrix of the mesh.

Remark 2.3.1. Note that $\mathcal{M} = (\mathcal{M}_{pq})$ and $\mathcal{M}^* = (\mathcal{M}^*_{ij})$ are invertible because they represent the L^2 inner product.

Definition 2.3.5 (discretized functionals)**.** For any $F \in H^{-1}(M_h)$ which satisfies Criterion 1, define $F_h \in S_h$ and $F^*_h \in S^*_h$ by

$$\int_{M_h} F_h \phi_h \, \mathrm{d}vol = \langle F | \phi_h \rangle \quad \forall\, \phi_h \in S_h \quad \text{and}$$
$$\int_{M_h} F^*_h \phi^*_h \, \mathrm{d}vol = \langle F | \phi^*_h \rangle \quad \forall\, \phi^*_h \in S^*_h.$$

The functions F_h and F^*_h can be explicitly computed as follows:

$$F_h = \sum_{p,q} \langle F | \phi_p \rangle \mathcal{M}^{pq} \phi_q \qquad \text{and}$$
$$F^*_h = \sum_{i,j} \langle F | \phi_{e_i} \rangle (\mathcal{M}^*)^{ij} \phi_{e_j},$$

summing over all nodes p and q (resp. all edges e_i and e_j). Here $\mathcal{M}^{pq}$ and $(\mathcal{M}^*)^{ij}$ denote the inverse mass matrices.

Scaling: If the mesh M_h is scaled by a factor λ while the functional F is kept scale-free, then the functions, F_h and F^*_h, re-scale with $1/\lambda^2$.

2.4 Discrete differential operators

This section serves as summarizing a framework for weak versions of the following operators: divergence, curl, Laplace–Beltrami, and the mean curvature vector[1]. Most of the material here was initiated by Polthier (see for example [62]). Our development of a concise theory of Sobolev spaces on simplicial meshes (cf. Section 2.2) allows to give these operators a precise meaning. In particular, it makes possible the exact specification of the spaces these operators act on, a fact that is important for treating convergence later.

[1] Whereas divergence, curl, and the Laplacian are *intrinsic* notions, we assume that M_h is isometrically embedded into $\mathbb{R}^3$ whenever we talk about the mean curvature vector.

Observation (functions vs. functionals)**.** The operators considered here, such as divergence, curl, Laplace–Beltrami and the mean curvature vector, are *functionals*. By construction, they are elements of the Sobolev space $H^{-1}(M_h)$. Hence it makes in general no sense to speak about pointwise evaluation of these objects as if they were continuous functions. Instead, a functional $F \in H^{-1}(M_h)$ only gives a real number if paired with a function in H_0^1. In this sense the 'evaluation' of the functional F at an *interior* vertex p may be understood as the pairing $\langle F|\phi_p\rangle$ with the nodal basis function ϕ_p. Similarly, the 'evaluation' of the functional F at an *interior* edge e may be understood as the pairing $\langle F|\phi_e\rangle$ with the mid-edge basis function ϕ_e.

In what follows, we will provide two versions for each of the operators of interest – a conforming and a nonconforming one. The conforming version can be thought of as being *vertex-based*. Similarly, the nonconforming version is *edge-based*. The two versions are related by the following *averaging property*.

Lemma 2.4.1 (averaging property)**.** *Vertex-based quantities are obtained from edge-based quantities by summing over the edges incident to a particular vertex. By linearity, this follows from the identity*

$$\phi_p = \frac{1}{2}\sum_{e \ni p} \phi_e, \tag{2.11}$$

where ϕ_p is the Lagrange basis function of a vertex p, and ϕ_e denotes the mid-edge basis function for an edge e incident to p.

2.4.1 Complex structure

The cone metric on M_h induces a *complex structure*, $\mathbf{J}$, which on individual triangles acts by rotating tangential vectors counter-clockwise by $\pi/2$. Notice that except at cone singularities, $\mathbf{J}$ is well defined. Indeed, by making a pair of adjacent triangles coplanar, the action of $\mathbf{J}$ extends to the interior of edges.

The complex structure can be thought of as a version of a discrete Hodge star operator since for a pair (X, Y) of piecewise constant vector fields one has

$$\int_{M_h} X \wedge Y = \int_{M_h} g_{M_h}(\mathbf{J}X, Y)\, \mathrm{d}vol.$$

Later it will be shown, however, that one is often interested in *a discrete Hodge star operator which differs from complex multiplication.*

2.4.2 Divergence and Gauss' theorem

Gauss' theorem asserts that the divergence of a vector field, integrated over a volume, measures the flux through the boundary of this volume. This fact remains true in the weak setting. As usual, we define the divergence operator as the (negative) adjoint operator of

$$\nabla : H_0^1(M_h) \to L^2_{\mathfrak{X}}(M_h).$$

Let $X \in \mathfrak{X}_h \subset L^2_{\mathfrak{X}}(M_h)$ be a piecewise constant vector field. The *conforming version* of divergence is given by

$$\operatorname{div} X(p) := \langle \operatorname{div} X | \phi_p \rangle = -\int_{M_h} g_{M_h}(X, \nabla \phi_p)\, \mathrm{d}vol.$$

Here p is an *interior* vertex of M_h, and $\phi_p \in S_{h,0}$ denotes the linear Lagrange basis function at p. Similarly, for any *interior* edge (an edge is called *interior* if it is not a boundary edge), the *nonconforming* version reads

$$\operatorname{div}^* X(e) := \langle \operatorname{div} X | \phi_e \rangle = -\int_{M_h} g_{M_h}(X, \nabla \phi_e)\, \mathrm{d}vol.$$

Remark 2.4.1. The nonconforming version is justified by Lemma 2.3.1: the boundary contribution to integration by parts vanishes for $\phi_e \in S^*_{h,0}$.

Gauss' theorem holds in the sense that the conforming and nonconforming versions can be written, respectively, as boundary integrals over the boundary of the star of p (the star is the set of triangles containing p), and over the star of an interior edge e (the star is the set of triangles containing e). For a piecewise constant vector field, X, one obtains

$$\operatorname{div} X(p) = \frac{1}{2} \oint_{\partial\, star(p)} g_{M_h}(X, \eta)\, \mathrm{d}s = -\frac{1}{2} \sum_i g_{M_h}(X_i,\, \mathbf{J}\vec{e}_i).$$

Here η is the *outward* normal along the boundary, $\partial\, star(p)$, $\vec{e}_i$ denotes a boundary edge, oriented counter-clockwise, and X_i denotes the value of X inside the triangle formed by p and its opposite edge, e_i. The factor $1/2$ effectively means that the boundary integral is taken over the boundary of *half the star of* p, obtained by connecting up the midpoints of the edges emanating from p to a closed cycle.

The *nonconforming version* at an interior edge, e, can be rewritten as

$$\operatorname{div}^* X(e) = \oint_{\partial\, star(e)} g_{M_h}(X, \eta)\, \mathrm{d}s = g_{M_h}(X_{i+1} - X_i,\, \mathbf{J}\vec{e}_{pq}).$$

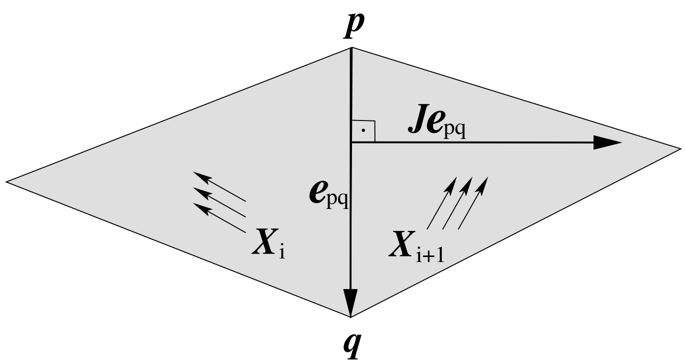


Figure 2.6: Notations for the nonconforming version of divergence and curl. The figure shows two adjacent triangles in the unfolded configuration.

Here η is the outer normal along the boundary, $\partial\, star(e)$, $\vec{e}_{pq}$ is oriented from p to q, and X_i and X_{i+1} denote the values of the piecewise constant field, X. The difference, $X_{i+1} - X_i$, is taken in the unfolded configuration, cf. Figure 2.6. We have the following interpretation.

Lemma 2.4.2 (normal jump)**.** $\operatorname{div}^* X(e)$ *measures the normal jump,* $[X]_{\text{nor}}(e)$*, of the piecewise constant field* X *at the edge* e*.*

The identity

$$\operatorname{div} X(p) = \frac{1}{2} \sum_{e \ni p} \operatorname{div}^* X(e)$$

constitutes the *averaging property* for any interior vertex p.

2.4.3 Curl and Stokes' theorem

The definition of curl on a $2D$ manifold can be based on the definition of div using complex multiplication,

$$\operatorname{curl} X = -\operatorname{div}(\mathbf{J}X).$$

Stokes' theorem holds in the sense that for any interior vertex curl can be expressed as a boundary integral. Indeed, let $X \in \mathfrak{X}_h$ be a piecewise constant vector field, and let p be an interior vertex. Then

$$\operatorname{curl} X(\mathrm{p}) = \frac{1}{2} \oint_{\partial\, star(p)} g_{M_h}(X, \tau)\, \mathrm{d}s = \frac{1}{2} \sum_i g(X_i, \vec{e}_i).$$

Here τ is the unit tangent vector along $\partial\, star(p)$, $\vec{e}_i$ denotes a boundary edge, oriented counter-clockwise, and X_i denotes the value of X inside the triangle formed by p and its opposite edge, e_i. Similarly,

$$\operatorname{curl}^* X(e) = \oint_{\partial\, star(e)} g_{M_h}(X,\tau)\,\mathrm{d}s = g(X_i - X_{i+1}, \vec{e}_{pq}),$$

where the edge, $\vec{e}_{pq}$, is oriented from p to q, and X_i, X_{i+1} denote the values of the piecewise constant field, X. The difference, $X_{i+1} - X_i$, is taken in the unfolded configuration, cf. Figure 2.6. This leads to the following interpretation.

Lemma 2.4.3 (tangential jump)**.** $\operatorname{curl}^* X(e)$ *measures the tangential jump,* $[X]_{\tan}(e)$*, of the piecewise constant field* X *at the edge* e*.*

The *averaging property* holds for the curl operators.

2.4.4 Laplace–Beltrami

The Laplace–Beltrami operator on polyhedral surfaces was introduced in Section 2.2. By construction we have

$$\Delta = \operatorname{div} \nabla.$$

From the preceding discussion we obtain two versions of discrete Laplacians: a conforming (vertex-based) and a nonconforming (edge-based) one.

Lemma 2.4.4 (vertex-based Laplace–Beltrami operator)**.** *Let* $u \in S_h$*, and let* p *be an interior vertex in* M_h*. Then*

$$\Delta u(\mathrm{p}) := \langle \Delta u | \phi_p \rangle = -\frac{1}{2} \sum_q (\cot \alpha_{pq} + \cot \beta_{pq})(u(p) - u(q))$$

is the conforming Laplace–Beltrami operator. The sum is taken over all vertices, q*, which share an edge with* p*, compare Figure 2.7. The vertex-based stiffness matrix evaluates to*

$$\mathfrak{L}_{pq} = -\Delta\phi_p(q) \quad \left(= -\frac{1}{2}(\cot \alpha_{pq} + \cot \beta_{pq}) \textit{ if } p \textit{ and } q \textit{ share an edge}\right),$$

where ϕ_p *and* ϕ_q *are the nodal basis functions at the vertices* p *and* q *and* α_{pq} *and* β_{pq} *are the two angles opposite to the edge* $\overline{pq}$*.*

The above formula is known as the *cotan formula*, as it appeared in the work of Pinkall and Polthier [61] in their discussion of discrete minimal surfaces, and earlier in Duffin's work [26] in the framework of electrical networks.

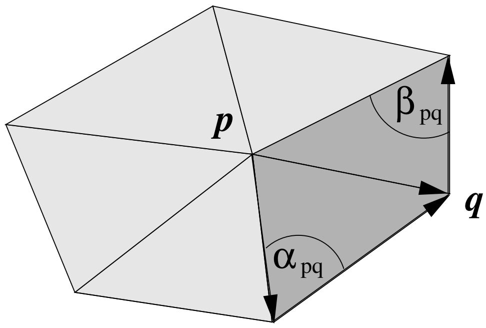


Figure 2.7: The *conforming* versions of the operators only see the vertex-star of p (the set of triangles containing p).

Lemma 2.4.5 (edge-based Laplace–Beltrami operator)**.** *Let $u \in S_h^*$, and let e be an interior edge in M_h. Then*

$$\Delta^* u(e) := \langle \Delta u | \phi_e \rangle = -2 \sum_j \cot \angle(e_i, e_j)(u_i - u_j)$$

is the nonconforming Laplace–Beltrami operator. The sum is taken over the four edges in $\partial\, star(e)$, and u_i, u_j denote the values of u at the edge-midpoints of e_i, e_j, respectively. The edge-based stiffness matrix is given by

$$\mathfrak{L}^*_{ij} := -\Delta^* \phi_{e_i}(e_j) \quad (= -2 \cot \angle(e_i, e_j) \text{ if } e_i \neq e_j \text{ belong to a single triangle}),$$

where ϕ_{e_i} is the mid-edge basis function corresponding to edge e_i.

The following theorem can be interpreted as a discrete equivalent of the fact that on a Riemannian manifold the first fundamental form is completely determined by the Laplace operator.

Theorem 2.4.1 (edge-Laplacian determines intrinsic metric)**.** *Let M_h be closed. Then, up to uniform re-scaling, the edge-based stiffness matrix, $(\mathfrak{L}^*_{ij})$, completely determines the first fundamental form of M_h.*

Proof. Since cot is bijective on $(0, \pi)$, one can recover all angles from $\mathfrak{L}^*_{ij} = -2 \cot \angle(e_i, e_j)$. QED

In particular, $(\mathfrak{L}^*_{ij})$ governs the entire theory of Sobolev spaces on closed triangular meshes.

2.4.5 Mean curvature

Analogous to the smooth setting, the mean curvature vector of a polyhedron is defined as the Laplace–Beltrami operator applied to the isometric embedding of M_h into $\mathbb{R}^3$, that is, as the $\mathbb{R}^3$-valued functional

$$\vec{H} = \Delta \vec{E} \in (H^{-1}(M_h))^3.$$

The mean curvature functional on Euclidean cone surfaces has given rise to a long and diverse list of applications over the past few years. For example, the vertex-based version was employed for isotropic mesh filtering by Desbrun et al. [23]; later the edge-based version was found to be useful for anisotropic filtering by Hildebrandt and Polthier [47]. Other applications cover modeling and animation of elastic materials (cf. Grinspun et al. [40]) as well as mesh editing and mesh compression (see the remarks following Theorem 2.4.2 below). Finally, in [61], Pinkall and Polthier for the first time started a systematic treatment of discrete minimal surfaces. Their approach has spawned a rich pool of explicitly computable examples (cf. [51] [44] [66]).

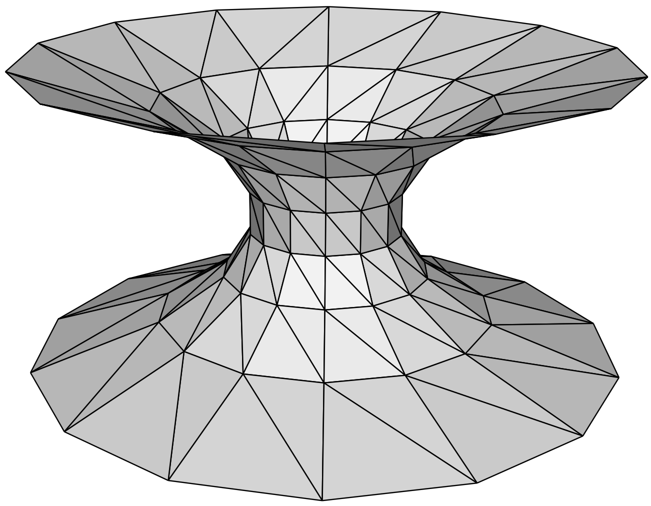


Figure 2.8: The *discrete catenoid* was the first example of an *explicitly computable* (conforming) discrete minimal surface (cf. Polthier and Rossman [66]).

Since the embedding of a piecewise linear surface is itself piecewise linear, we can define:

Definition 2.4.1 (discrete minimal surface)**.** A polyhedral surface is called (conforming) *minimal* if

$$\langle \vec{H} | \phi_p \rangle = 0 \quad \text{for all vertices} \quad p \in M_h \setminus \partial M_h.$$

Remark 2.4.2. Note carefully that discrete minimality means that $\langle \vec{H} | u_h \rangle = 0$ for all $u_h \in S_{h,0}$, so that it is a *weaker* condition than $\vec{H} = 0 \in H^{-1}(M_h)$. This is because (conforming) discrete minimality is a condition at vertices only; edges may still be bent, see Figure 2.8.

The *conforming version* of the mean curvature vector at an *interior* vertex p of M_h takes the form

$$\vec{H}(p) := \langle \vec{H} | \phi_p \rangle = -\frac{1}{2} \sum_q (\cot \alpha_{pq} + \cot \beta_{pq})(p - q),$$

the sum being taken over all vertices q which share an edge with p, compare Figure 2.7 for notation. The *conforming version* of the mean curvature vector is the negative area gradient of the Euclidean cone surface M_h for variations of its vertices (this is a simple consequence of the fact that all admissible variations are piecewise linear, cf. [61]). Hence the mean curvature vector at a vertex can be thought of as being *normal* to the polyhedron, while its length determines the velocity by which M_h needs to move to decrease its total area.

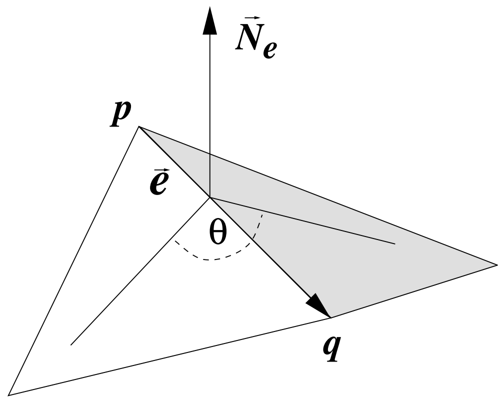


Figure 2.9: Ingredients for mean curvature vector at an edge.

The *nonconforming version* at an interior edge of M_h takes the form

$$\vec{H}^*(e) := \langle \vec{H} | \phi_e \rangle = -2 \cos \frac{\theta}{2} \, \|e\| \, \vec{N}_e,$$

where $\vec{N}_e$ denotes the outer angle-bisecting normal to M_h at the edge e and θ is the dihedral angle at e, compare Figure 2.9. The conforming and non-conforming versions are related by the *averaging property*.

The following result relates the vertex-based mean curvature vector to the embedding of M_h. It can be thought of as the uniqueness part of a discrete "fundamental theorem of surface theory", the smooth version of which asserts that there exists a unique embedding given the first and second fundamental forms. It is an interesting problem to find the discrete existence part - which in the smooth setting relates first and second fundamental form by certain integrability conditions – the Gauss and Codazzi-Mainardi equations.

Theorem 2.4.2. *Let M_h be closed. Then the vertex-based stiffness matrix $(\mathfrak{L}_{pq})$ together with the vertex-based mean curvature vectors $(\vec{H}(p))$ uniquely determine the embedding $\vec{E} : M_h \to \mathbb{R}^3$ up to a global translation in $\mathbb{R}^3$.*

Proof. The stiffness matrix $(\mathfrak{L}_{pq})$ is symmetric and positive semi-definite. Its kernel is the 1-dimensional space of constants. The embedding $\vec{E}$ of M_h can be written in terms of the nodal basis functions, $\vec{E} = \sum_q \phi_q q$, summing over all vertices of M_h. To recover the positions of the vertices of M_h, one has to solve the linear system

$$-\sum_{q \in M_h} \mathfrak{L}_{pq} q = \vec{H}(p). \tag{2.12}$$

Fixing the position of a single vertex makes the system full-rank. QED

Theorem 2.4.2 has interesting applications for shape editing and morphing: given the mean curvature vectors of an initial (undeformed) mesh, and altering a few vertex positions of that mesh, one solves for the remaining vertex positions using (a constrained version of) (2.12). In this view, mean curvature vectors take the role of 'mesh coordinates'. They were introduced under the name of *delta coordinates* by Alexa [2]. Applications of δ-coordinates range from single-resolution mesh editing (cf. Lipman et al. [52]) to mesh compression (cf. Sorkine et al. [74]). For an overview of recent developments using δ-coordinates, see [73].

Remark 2.4.3. We take an interpretation of δ-coordinates here which differs from their original definition. Originally these coordinates were not based on the *geometric* Laplace–Beltrami operator but rather on a *purely combinatorial* version corresponding to the incidence matrix of the mesh (where off-diagonal elements contain the entry 1 for each edge, and diagonal entries encode the degree of each vertex). The relation between geometric and combinatorial information of a mesh is an area of ongoing research, see e.g. Alliez and Gotsman [3].

2.5 Algebraic topology from FE

In this section we interpret the cohomological structure of a Euclidean cone surface in terms of chain complexes built from discrete differential operators

– in close analogy to the smooth de Rham complex. Later, in Chapter 3, we show that the operators and spaces considered here *converge* to their smooth counterparts. To simplify the discussion we are only going to deal with *closed surfaces.*

The discrete theory presented here builds on *mixing* conforming and nonconforming linear finite elements. This mixing yields *two distinct versions* of a discrete de Rham complex. The cohomology of each of these complexes is isomorphic to simplicial cohomology (Theorem 2.5.1). Moreover, mixing conforming and nonconforming elements gives rise to *two versions* of a discrete Hodge decomposition (Section 2.5.3). The observation that in the discrete case one obtains two distinct versions of a Hodge decomposition – a doubling which is absent in the smooth setting – is closely related to the cellular viewpoint taken by Mercat [54, 55] who builds his discretization upon simultaneously considering *two distinct grids* (a primal and a dual one) and obtains Riemann period matrices of *double the dimension* in his work on discrete conformal structures. Our view is also closely linked to that of Desbrun et al. [22] and Glickenstein [36, 37]. The *necessity* for such a doubling in the discrete case appears to be related to a certain impossibility of constructing a discrete Hodge star operator which at the same time takes the role of complex multiplication on 1-forms (i.e., acts by 90° rotation) and also isomorphically maps the $2\mathfrak{g}$-dimensional space of harmonic 1-forms to itself for a closed polyhedron of genus $\mathfrak{g}$. In the sequel we shall therefore carefully distinguish between complex multiplication and the Hodge star operator.

As before, we consider the space of *piecewise constant vector fields.* On this space, complex multiplication $\mathbf{J}$ acts by 90° rotation. In particular, $\mathbf{J}$ exchanges the spaces of conforming and nonconforming harmonic vector fields,

$$\mathbf{J} : \ker \mathrm{curl}_{\mathrm{h}}^{*} \cap \ker \mathrm{div}_{\mathrm{h}} \longrightarrow \ker \mathrm{curl}_{\mathrm{h}} \cap \ker \mathrm{div}_{\mathrm{h}}^{*} .$$

The (conforming) *Hodge star operator* is obtained by composing $\mathbf{J}$ with a projection (with respect to the L^2 inner product), mapping nonconforming harmonic fields back conforming ones:

$$\star = \Pi \circ \mathbf{J} : \ker \mathrm{curl}_{\mathrm{h}}^{*} \cap \ker \mathrm{div}_{\mathrm{h}} \circlearrowleft .$$

This Hodge star will be used to study *holomorphic* and *antiholomorphic* vector fields in Section 2.5.6.

Using piecewise constant vector fields is, in a sense, the simplest possible discretization of the space of smooth vector fields. Another common discretizations is based on (piecewise linear) *Whitney forms* which date back to the seminal work of Whitney [81]. For an excellent overview of the use of

Whitney forms in discretizing PDEs, see Arnold [5]. The reason to explore the theory and convergence of *piecewise constant fields* here is their widespread use in graphics applications, see Gu and Yau [45, 46], and Polthier and Preuss [65], as well as in the FE viewpoint of minimal surface theory, see [62]. The exact relations between piecewise constant fields and Whitney forms is established in Sections 3.4.1 and 3.4.2.

2.5.1 Discrete de Rham complex

The smooth setting. Recall the de Rham complex of a smooth Riemann surface (M, g):

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{\ \mathrm{d}\ } \Lambda^1(M) \xrightarrow{\ \mathrm{d}\ } \Lambda^2(M) \longrightarrow 0.$$

This is a chain complex ($\mathrm{d}^2 = 0$) for the Cartan outer differentials d acting on smooth q-forms, Λ^q. The *metric version* of the above complex is obtained by using the duality between 1-forms and vector fields (using the sharp operator), as well as the duality between 2-forms and functions (using the Hodge star). One obtains the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Lambda^0(M) & \xrightarrow{\ \mathrm{d}\ } & \Lambda^1(M) & \xrightarrow{\ \mathrm{d}\ } & \Lambda^2(M) & \longrightarrow & 0 \\ & & \big\downarrow Id & & \big\downarrow \sharp & & \big\downarrow \star & & \\ 0 & \longrightarrow & \Lambda^0(M) & \xrightarrow{\ \nabla\ } & \mathfrak{X}(M) & \xrightarrow{\ \mathrm{curl}\ } & \Lambda^0(M) & \longrightarrow & 0, \end{array}$$

where for $\sigma \in \Lambda^1(M)$, $\omega \in \Lambda^2(M)$, and $X \in \mathfrak{X}(M)$,

$$\sigma(X) = g(\sigma^\sharp, X) \quad \text{and} \quad \omega = (\star\omega)\, \mathrm{d}vol_g.$$

The discrete setting. We consider the following *metric version* of a discrete de Rham complex as well as its adjoint version with respect to the L^2 inner products on S_h, S_h^*, and $\mathfrak{X}_h$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_h & \xrightarrow{\ \nabla\ } & \mathfrak{X}_h & \xrightarrow{\ \mathrm{curl}_\mathrm{h}^*\ } & S_h^* & \longrightarrow & 0 \\ 0 & \longleftarrow & S_h & \xleftarrow{\ \mathrm{div}_\mathrm{h}\ } & \mathfrak{X}_h & \xleftarrow{\ \mathbf{J}\nabla\ } & S_h^* & \longleftarrow & 0. \end{array} \tag{2.13}$$

Recall that $\mathbf{J}$ denotes complex multiplication, and the subscript h denotes the discretization of functionals as in Definition 2.3.5, that is

$$(\mathrm{curl}_\mathrm{h}^* X, \phi)_{L^2} = \langle \mathrm{curl}^* X | \phi \rangle \quad \text{for all} \quad \phi \in S_h^* \quad \text{and}$$
$$(\mathrm{div}_\mathrm{h} X, \phi)_{L^2} = \langle \mathrm{div}\, X | \phi \rangle \quad \text{for all} \quad \phi \in S_h.$$

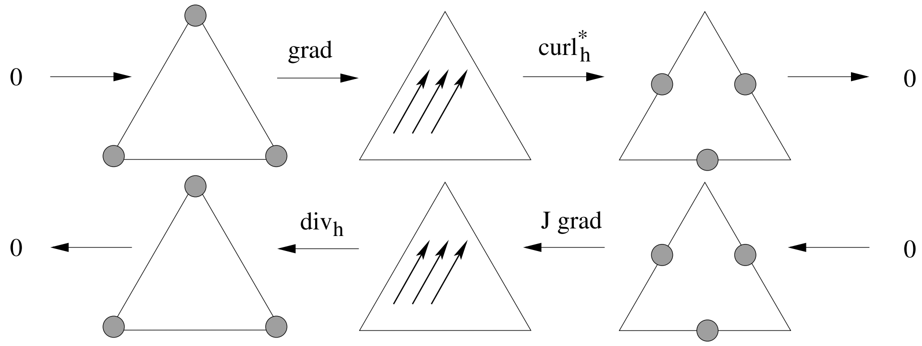


Figure 2.10: Top: Discrete de Rham complex from mixing conforming elements (where the degree of freedom is at the vertices) and nonconforming elements (where the degree of freedom is at edge midpoints). Bottom: The dual complex with respect to the L^2 inner products.

Lemma 2.5.1. *The complexes of Diagram (2.13) are chain complexes.*

Proof. By Lemma 2.4.3, $\operatorname{curl}^* X(e) = [X]_{\tan}(e)$ measures the tangential jump of the field X at the edge e. If $X = \nabla u$, with $u \in S_h$, then X is tangentially continuous at any edge and hence $\operatorname{curl}^* \nabla u(e) = 0$. This implies that the top row of (2.13) is a chain complex. The bottom row is the adjoint version of the top row – so it is a chain complex as well. QED

It is also useful the consider the **J**-transformed version of (2.14):

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_h & \xrightarrow{\mathbf{J}\nabla} & \mathfrak{X}_h & \xrightarrow{\mathrm{div}_h^*} & S_h^* & \longrightarrow & 0 \\ 0 & \longleftarrow & S_h & \xleftarrow{\mathrm{curl}_h} & \mathfrak{X}_h & \xleftarrow{\nabla} & S_h^* & \longleftarrow & 0. \end{array} \tag{2.14}$$

Considering both, (2.13) and (2.14), implies that any closed polyhedral surface gives rise to (at least) *two* pairs of de Rham chain complexes. Within each of these pairs, one complex is the L^2-adjoint of the other. Note that the above complexes *mix* conforming and nonconforming elements. The next section will illuminate why this is indeed necessary in the discrete setting.

2.5.2 de Rham cohomology

The most important feature of the chain complexes in Diagrams (2.13) and (2.14) is that they induce the same (co)homology as singular (co)homology. For the case of a *simply connected* M_h this result had been independently obtained by Arnold and Falk [6], and Polthier and Preuss [65]. Here we extend it to closed meshes of arbitrary genus.

Theorem 2.5.1 (de Rham cohomology from FE)**.** *Let M_h be a closed Euclidean cone surface. The i^{th} (co)homology ($i = 0, 1, 2$) of the chain complexes in Diagrams (2.13) and (2.14) are equal to the i^{th} singular (co)homology of M_h. In particular*

$$\mathcal{H}^1_{\text{sing}}(M_h; \mathbb{R}) \cong \frac{\ker \operatorname{curl}_h^*}{\operatorname{im} \nabla_{|S_h}} \cong \frac{\ker \operatorname{div}_h}{\operatorname{im} \mathbf{J}\nabla_{|S_h^*}} \cong \frac{\ker \operatorname{curl}_h}{\operatorname{im} \nabla_{|S_h^*}} \cong \frac{\ker \operatorname{div}_h^*}{\operatorname{im} \mathbf{J}\nabla_{|S_h}}.$$

Proof. We show that $\mathcal{H}^1_{\text{sing}}(M_h) = \ker \operatorname{curl}_h^* / \operatorname{im} \nabla_{|S_h}$. By Lemma 2.4.3 the elements of $\ker \operatorname{curl}_h^* \subset \mathfrak{X}_h$ are those piecewise constant vector fields which are tangentially continuous across every edge e of M_h. In particular, $X \in \ker \operatorname{curl}_h^*$ gives rise to a *simplicial* 1*-form* ω_X on M_h by

$$\omega_X(\vec{e}_{pq}) = g(X, \vec{e}_{pq}).$$

Since X is piecewise constant, ω_X is closed, that is $\delta^1 \omega_X = 0$, where

$$\delta^1 : C^1(M_h) \to C^2(M_h)$$

denotes the simplicial coboundary operator. Vice-versa, any closed simplicial 1-form ω gives rise to a piecewise constant vector field $X_\omega \in \ker \operatorname{curl}_h^*$ by dualization. We obtain

$$\ker \delta^1 \cong \ker \operatorname{curl}_h^*.$$

Now let $f \in C^0(M_h)$ be a simplicial 0-form, i.e. f gives answers to vertices of M_h. Consider the simplicial coboundary operator $\delta^0 : C^0(M_h) \to C^1(M_h)$. Extending f linearly across triangles we get

$$(\delta^0 f)(\vec{e}_{pq}) = f(q) - f(p) = g(\nabla f, \vec{e}_{pq}).$$

In other words, under the identification of 0-forms with piecewise linear functions we get

$$\operatorname{im} \delta^0 \cong \operatorname{im} \nabla_{|S_h}.$$

Together this shows

$$\mathcal{H}^1_{\text{sing}}(M_h; \mathbb{R}) = \frac{\ker \delta^1}{\operatorname{im} \delta^0} \cong \frac{\ker \operatorname{curl}_h^*}{\operatorname{im} \nabla_{|S_h}}.$$

It now follows from analogous considerations that the remaining quotients in the statement of the theorem are all equal to each other. QED

2.5.3 HODGE DECOMPOSITION

From the above de Rham complex one immediately obtains a discrete Hodge decomposition of the space of piecewise constant vector fields. Before we go into detail, we make a rather general note about Hodge decompositions for chain complexes involving *finite dimensional* metric spaces (cf. Eckmann [31]).

Lemma 2.5.2. *Let (U, g_U), (V, g_V) and (W, g_W) be finite-dimensional vector spaces equipped with positive symmetric inner products. Assume that*

$$U \xrightarrow{d_1} V \xrightarrow{d_2} W$$

is a chain complex. i.e. $d_2 \circ d_1 = 0$. Let $d_1^* : V \to U$ and $d_2^* : W \to V$ denote the adjoint operators to d_1 and d_2 (a finite-dimensional space and its dual are identified via the inner product). Then there exists a g_V-orthogonal decomposition

$$V = \operatorname{im} d_1 \oplus \operatorname{im} d_2^* \oplus \ker d_2 \cap \ker d_1^*.$$

Moreover, this decomposition only depends on the choice of the inner product g_V on V, as well the spaces $\operatorname{im} d_1 \subset V$ and $\ker d_2 \subset V$. The decomposition is otherwise independent of the choices of d_1, d_2, (U, g_U), and (W, g_W). Finally, the following spaces are isomorphic,

$$\frac{\ker d_2}{\operatorname{im} d_1} \cong \frac{\ker d_1^*}{\operatorname{im} d_2^*} \cong \ker d_2 \cap \ker d_1^*.$$

As usual, the space $\ker d_2 \cap \ker d_1^*$ is called harmonic.

Proof. One repeatedly uses the fact that the image of the adjoint of an operator equals the orthogonal complement of the kernel of the operator itself: $\mathrm{im}\ \mathrm{d}^* = (\ker\ \mathrm{d})^\perp$. QED

From Lemma 2.5.2, we deduce the following result for closed Euclidean cone surfaces.

Theorem 2.5.2 (Hodge decompositions from FE)**.** *Let M_h be closed. The space of piecewise constant vector fields can be decomposed according to the following (conforming and nonconforming) L^2-orthogonal splittings*

$$\begin{aligned}\mathfrak{X}_h &= \mathrm{im}\,\nabla_{|S_h} \oplus \mathrm{im}\,\mathbf{J}\nabla_{|S_h^*} \oplus \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h} \\ &= \mathrm{im}\,\mathbf{J}\nabla_{|S_h} \oplus \mathrm{im}\,\nabla_{|S_h^*} \oplus \ker \mathrm{div}_\mathrm{h}^* \cap \ker \mathrm{curl}_\mathrm{h}\,.\end{aligned}$$

Hence, in the discrete case there exist two versions of harmonic vector fields (given by $\ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h}$ and $\ker \mathrm{div}_\mathrm{h}^* \cap \ker \mathrm{curl}_\mathrm{h}$), and

$$\mathcal{H}^1_{\mathrm{sing}}(M_h;\mathbb{R}) \cong \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h} \cong \ker \mathrm{div}_\mathrm{h}^* \cap \ker \mathrm{curl}_\mathrm{h}\,.$$

In particular, the dimension of each of the spaces of harmonic vector fields equals twice the genus of M_h. We summarize the situation by a definition.

Definition 2.5.1 (conforming and nonconforming harmonic fields)**.** Let M_h be closed. The space of *conforming harmonic vector fields* is defined as

$$\mathcal{H}(M_h;\mathbb{R}) = \ker \mathrm{curl}_{\mathrm{h}}^{*} \cap \ker \mathrm{div}_{\mathrm{h}}\,.$$

The space of *nonconforming harmonic vector fields* is defined as

$$\mathcal{H}^{*}(M_h;\mathbb{R}) = \ker \mathrm{curl}_{\mathrm{h}} \cap \ker \mathrm{div}_{\mathrm{h}}^{*}\,.$$

The dimension of these spaces is $2\mathfrak{g}$ (twice the genus of M_h).

Interpretation in terms of 1-forms. The proof of Theorem 2.5.1 shows that every $\mathrm{curl}_{\mathrm{h}}^{*}$-free piecewise constant vector field, $X \in \ker \mathrm{curl}_{\mathrm{h}}^{*}$, is dual to a closed simplicial 1-form $\omega_X \in \ker \delta^1$:

$$X_\omega \in \ker \mathrm{curl}_{\mathrm{h}}^{*} \iff \omega_X \in \ker \delta^1,$$

where δ^1 is the usual simplicial co-boundary operator. If X is additionally *conforming harmonic* then $X \in \ker \mathrm{div}_{\mathrm{h}}$. This is equivalent to a condition at the vertices p of M_h, namely:

$$\sum_q (\cot \alpha_{pq} + \cot \beta_{pq}) g(X, \vec{e}_{pq}) = 0,$$

where the sum is taken over all edges $\vec{e}_{pq}$ emanating from p, and α_{pq} and $\cot \beta_{pq}$ are the two angles opposite to edge $\vec{e}_{pq}$. In terms of the simplicial 1-form ω_X and an appropriate dual operator $(\delta^1)^*$, this can be written as

$$0 = (\delta^1)^* \omega_X(p) = \sum_q (\cot \alpha_{pq} + \cot \beta_{pq}) \omega_X(\vec{e}_{pq}).$$

In other words, *discrete harmonicity* in the language of 1-forms is equivalent to

$$\delta^1 \omega_X = 0 \quad \text{and} \quad (\delta^1)^* \omega_X = 0,$$

i.e. a condition for faces ($\delta^1 \omega_X = 0$) and vertices ($(\delta^1)^* \omega_X = 0$), cf. Figure 2.11. The corresponding nonconforming version for harmonic vector fields has a similar interpretation in terms of 1-forms by integrating the normal component along edges. In the next subsection we will elaborate on discrete harmonic 1-forms.

Finally, we note that in the smooth setting there is only a single version of harmonic vector fields, and complex multiplication acts as an isometry on this space. In contrast, on polyhedral surfaces there are two versions of harmonic fields, and complex multiplication interchanges them:

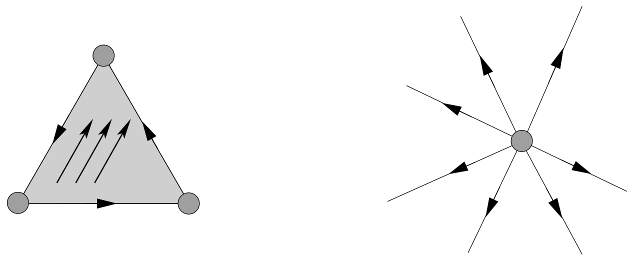


Figure 2.11: For a discrete 1-form to be *closed* is a condition on triangles: the answer to any simply connected closed loop must be zero. To be *co-closed* is a condition on vertices: the answer to a weighted sum over all edges emanating from a vertex must be zero.

Proposition 2.5.1. *Let M_h be closed. Complex multiplication induces an isomorphism between conforming and nonconforming harmonic vector fields:*

$$\mathbf{J} : \mathcal{H}(M_h;\mathbb{R}) \to \mathcal{H}^*(M_h;\mathbb{R}).$$

Moreover,

$$\mathcal{H}(M_h;\mathbb{R}) \cap \mathcal{H}^*(M_h;\mathbb{R}) = \{0\},$$

unless the cone angle at every vertex of M_h is an integer multiple of 2π.

Proof. Assume

$$0 \neq X \in \mathcal{H}(M_h;\mathbb{R}) \cap \mathcal{H}^*(M_h;\mathbb{R}) \subset \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h}^* .$$

Recall that $\ker \mathrm{curl}_\mathrm{h}^*$ consists of all edge-tangentially continuous piecewise constant vector fields and that $\ker \mathrm{div}_\mathrm{h}^*$ contains all edge-normally continuous ones. Hence a non-vanishing field $X \in \ker \mathrm{curl}_\mathrm{h}^* \cap \ker \mathrm{div}_\mathrm{h}^*$ would be everywhere parallel (wrt. intrinsic parallel transport). Such a field X exists if and only if the cone angle at any vertex of M_h is an integer multiple of 2π. QED

Remark 2.5.1. There is a third possible discretization of the de Rham complex on Euclidean cone surfaces which is entirely built from conforming elements:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & S_h & \xrightarrow{\nabla} & \mathfrak{X}_h & \xrightarrow{\mathrm{curl}_\mathrm{h}} & S_h & \longrightarrow & 0 \\ 0 & \longleftarrow & S_h & \xleftarrow{\mathrm{div}_\mathrm{h}} & \mathfrak{X}_h & \xleftarrow{\mathbf{J}\nabla} & S_h & \longleftarrow & 0. \end{array}$$

However, this complex *does not induce the correct (co)homology.* Indeed,

$$\dim(\ker \mathrm{curl}_\mathrm{h} \cap \ker \mathrm{div}_\mathrm{h}) = \#F + 4\mathfrak{g} - 2,$$

where $\mathfrak{g}$ is the genus of M_h, and $\#F$ denotes the number of faces (triangles) of M_h – in contrast to $\dim \mathcal{H}^1_{\mathrm{sing}}(M_h;\mathbb{R}) = 2\mathfrak{g}$.

2.5.4 LOCAL PARAMETERIZATION AND POINCARÉ INDEX THEOREM

In this subsection we review how harmonic vector fields can be used for mesh parameterization. In a later section we will refine these results and introduce *holomorphic vector fields* to get conformal parameterizations. As an application we mention that the Delaunay discretization of a Euclidean cone surface (cf. Section 2.3.3) provides a tool to obtain *locally injective parameterizations* in the FE setting.

As observed by Gortler, Gotsman and Thurston (cf. [39]), a certain class of harmonic 1-forms on meshes yields locally injective mesh parameterizations. In particular, they apply their results to obtain a simple proof of Tutte's celebrated *barycentric embedding theorem* for planar graphs [80]. The basis of their approach is to provide a discrete Poincaré-Hopf index theorem for 1-forms on *oriented meshes* $G = (V, E, F)$ with vertex set V, edge set E, and face set F.

A *one-form* on G is an assignment of a real number, $\omega(\vec{e}_{pq})$, to each oriented edge, $\vec{e}_{pq}$, such that $\omega(\vec{e}_{pq}) = -\omega(\vec{e}_{qp})$. Throughout it is assumed that this number is different from zero (if it is zero, remove the corresponding edge from E). A pair $(f, p) \in (F, V)$ with $p \in f$ is called a *corner* of ω if

$$\operatorname{sgn}\omega(\vec{e}_{pq}) \neq \operatorname{sgn}\omega(\vec{e}_{pr}),$$

for the unique pair of oriented edges $\vec{e}_{pq}$ and $\vec{e}_{pr}$ in f emanating from p. The *index* of a vertex p is the number

$$\operatorname{ind}(p) = 1 - \frac{\operatorname{corn}(p)}{2},$$

where $\operatorname{corn}(p)$ is the number of faces f such that (f, p) is a corner. The *index* of a face f is the number

$$\operatorname{ind}(f) = 1 - \frac{\text{non-corn}(f)}{2},$$

where non-corn(f) is the number of vertices p such that (f, p) is not a corner. A vertex (resp. face) is called *regular* if $\operatorname{ind}(v) = 0$ (resp. $\operatorname{ind}(f) = 0$), and a non-regular vertex (face) is called *singular*. A singular vertex of index 1 is either a *source* or a *sink*. A singular face of index 1 is called a *vortex*. All other singularities ($\operatorname{ind} < 0$) are called *saddles*. Gortler, Gotsman, Thurston show:

Theorem 2.5.3 (Poincaré-Hopf). *If $G = (V, E, F)$ has the topology of a closed oriented 2-manifold of genus $\mathfrak{g}$ then*

$$\sum_{p \in V} \operatorname{ind}(p) + \sum_{f \in F} \operatorname{ind}(f) = 2 - 2\mathfrak{g}.$$

As usual, a 1-form is called *closed* if

$$\delta\omega(f) = \sum_{\vec{e}\in\partial f} \omega(\vec{e}) = 0 \quad \forall\, f,$$

where ∂f is the boundary operator of the face f. Note that any closed 1-form can *locally* be integrated along edges to locally give a real-valued function on the vertices of G. A pair of (non-collinear) closed 1-forms can then be used to locally 'parameterize' G by using the corresponding functions on vertices to locally get a map to the plane. More formally,

Definition 2.5.2 (local parameterization)**.** Let ω_1 and ω_2 be two non-collinear *closed* 1-forms on G. Then locally integrating the complex-valued 1-form $(\omega_1 + i\omega_2)$ gives a *local parameterization* of G.

Let $k_{pq} = k_{qp}$ be a set of symmetric real-valued weights on edges. A 1-form is called *co-closed* with respect to $\{k_{pq}\}$ if

$$\delta^*\omega(p) = \sum_{\vec{e}_{pq}} k_{pq}\omega(\vec{e}_{pq}) = 0 \quad \forall\, p,$$

where $\vec{e}_{pq}$ runs over the (oriented) edges emanating from p. Finally, ω is called *harmonic* if it is closed and co-closed, that is

$$\delta\omega = 0 \quad \text{and} \quad \delta^*\omega = 0.$$

A parameterization is called *locally injective* if the one-ring of faces around each interior vertex maps homeomorphically to a disk in the plane (the term was coined by Floater, cf. [33]). The following result is shwon in [39]:

Theorem 2.5.4 (locally injective parameterizations)**.** *If the weights* $\{k_{pq}\}$ *in the definition of co-closed forms all have the same sign then any pair of (non-collinear) harmonic* 1*-forms yields a locally injective parameterization of* G*.*

Connection with harmonic vector fields. Recall that the space of conforming harmonic vector fields was defined as

$$\mathcal{H}(M_h;\mathbb{R}) = \ker \operatorname{curl}_{\mathrm{h}}^* \cap \ker \operatorname{div}_{\mathrm{h}}.$$

In the previous subsection we explained how to obtain a harmonic 1-form, ω_X, from a (conforming) harmonic vector field, X. In this context, the weights k_{pq} are given by the *cotan weights*

$$k_{pq} = \cot\alpha_{pq} + \cot\beta_{pq}.$$

By Section 2.3.3, these weights are all non-negative for the Delaunay discretization. Note that any edge $\vec{e}_{pq}$ for which $k_{pq} = 0$ can be removed without changing the conditions $\delta\omega = \delta^*\omega = 0$. As long as all faces stay simply connected after this removal, we get:

Corollary 2.5.1 (Delaunay discretization yields local injectivity)**.** *Let M_h be a closed Euclidean cone surface. If all edges with zero cotan weights are removed from the edge set of a Delaunay discretization of M_h and the remaining faces stay simply connected then any pair of (non-collinear) harmonic vector fields yields a locally injective parameterization.*

2.5.5 Hodge-star for harmonic vector fields

We introduce a Hodge-star operator which induces an isomorphism on the space of conforming harmonic vector fields,

$$\star : \mathcal{H}(M_h; \mathbb{R}) \circlearrowleft .$$

This is in contrast with complex multiplication on polyhedral meshes, which exchanges the space of conforming and nonconforming harmonic fields. Indeed, recall that complex multiplication induces an isomorphism

$$\mathbf{J} : \mathcal{H}(M_h; \mathbb{R}) \longrightarrow \mathcal{H}^*(M_h; \mathbb{R}).$$

Our construction of the Hodge star operator is similar to the construction of Wilson [82] who uses Whitney forms. In particular, we owe the discussion on Poincaré duality as well as Theorems 2.5.5 and 2.5.6 to this source. The similarity is due to the fact that harmonic Whitney 1-forms are in one-to-one correspondence with conforming harmonic vector fields (cf. Sections 3.4.1 and 3.4.2 for the precise relations). Furthermore, Gu and Yau [45] were the first to provide explicit formulas for computing a discrete Hodge star in the FE setting (without making explicit the spaces which $\star$ acts on). Their Hodge star coincides with the one discussed here.

Definition 2.5.3 (Hodge-star for conforming harmonic fields)**.** The Hodge-star operator on the space of conforming harmonic vector fields is a composition of complex multiplication with a projection,

$$\star := \Pi \circ \mathbf{J} : \mathcal{H}(M_h; \mathbb{R}) \circlearrowleft,$$

where $\Pi : \mathcal{H}^*(M_h; \mathbb{R}) \to \mathcal{H}(M_h; \mathbb{R})$ is the L^2-projection to the space of conforming harmonic vector fields.

The projection Π is responsible for the fact that $\star$ is no longer an isometry, in fact,

$$\star^2 \neq -Id.$$

Still, $\star$ remains an isomorphism as next theorem shows. To show that, we use *Poincaré duality on cohomology.* The argument is due to Wilson.

Poincaré duality on cohomology. For piecewise constant vector fields, Poincaré duality on cohomology can be stated in terms of the non-degeneracy *on cohomology* of the skew-symmetric product

$$X \cup Y = \int_{M_h} g(\,\mathbf{J}X, Y)\,\mathrm{d}vol.$$

To see that this is non-degenerate, let $X, Y \in \ker \mathrm{curl}_\mathrm{h}^*$ represent two cohomology classes. Recall that there is a $1:1$ correspondence between piecewise constant vector fields $X \in \ker \mathrm{curl}_\mathrm{h}^*$ and closed *simplicial* 1-forms ω_X. Hence it suffices to show that the above product is non-degenerate on the cohomology level for simplicial 1-forms. Indeed, this nondegeneracy follows from the fact that the above $\cup$-product yields the same product on cohomology as the Alexander-Whitney cup product (this follows by using the method of acyclic models, cf. Bredon [16]). Using the nondegeneracy of $\cup$, one obtains (cf. Wilson [82]):

Theorem 2.5.5. *The Hodge-star has the following properties:*

i *For $X, Y \in \mathcal{H}(M_h;\mathbb{R})$, one has $(\star X, Y)_{L^2} = -(X, \star Y)_{L^2}$.*

ii *$\star : \mathcal{H}(M_h;\mathbb{R}) \circlearrowleft$ is an isomorphism.*

Proof. The first statement follows from simple algebraic manipulations:

i For $X, Y \in \mathcal{H}(M_h;\mathbb{R})$ one has

$$\begin{aligned}(\star X, Y)_{L^2} &= (\Pi\,\mathbf{J}X, Y)_{L^2} = (\,\mathbf{J}X, Y)_{L^2}\\ &= -(X,\,\mathbf{J}Y)_{L^2} = -(X, \Pi\,\mathbf{J}Y)_{L^2}\\ &= -(X, \star Y)_{L^2}.\end{aligned}$$

ii The Hodge star is the composition of two isomorphisms, Poincaré duality and the inverse of a non-degenerate inner product. Indeed, let $X, Y \in \mathcal{H}(M_h;\mathbb{R})$ then

$$(\star X, Y)_{L^2} = (\,\mathbf{J}X, Y)_{L^2} = X \cup Y.$$

Since $\cup$ is non-degenerate on cohomology, and $(\cdot,\cdot)_{L^2}$ is non-degenerate, it follows that $\star$ has full rank.

This completes the proof. QED

2.5.6 Holomorphic and anti-holomorphic vector fields

The Hodge star can be used to obtain a splitting of the space of conforming (resp. nonconforming) harmonic vector fields into holomorphic and anti-holomorphic fields. We are only going to deal with the conforming version in this section; the nonconforming version is obtained from the conforming one by a transformation with $\mathbf{J}$.

In order to obtain such a splitting, one first complexifies the real vector space of conforming harmonic vector fields $\mathcal{H}(M_h;\mathbb{R})$, that is

$$\mathcal{H}(M_h;\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} \mathcal{H}(M_h;\mathbb{R}).$$

Next, one extends $\star$ linearly over $\mathbb{C}$ to this complexified space and extends $(\cdot,\cdot)_{L^2}$ to a Hermitian inner product

$$(X_1 + iY_1, X_2 + iY_2)^{\mathbb{C}}_{L^2} = (X_1, X_2)_{L^2} + (Y_1, Y_2)_{L^2} + i(Y_1, X_2)_{L^2} - i(X_1, Y_2)_{L^2},$$

where X_j, Y_j are real-valued conforming harmonic vector fields. Note that skew-symmetry

$$(\star X, Y)^{\mathbb{C}}_{L^2} = -(X, \star Y)^{\mathbb{C}}_{L^2}$$

implies that $\star$ has *purely imaginary eigenvalues*. Moreover, since $\star$ is the composition of an isometry (the operator $\mathbf{J}$) with a projection, the magnitude of these eigenvalues is less or equal to 1. The following definition of discrete (anti)holomorphic vector fields is due to Wilson:

Definition 2.5.4. Let M_h be closed. The space of *holomorphic vector fields* is generated by the eigenvectors of the Hodge star operator corresponding to negative imaginary eigenvalues:

$$\mathcal{H}^{1,0}(M_h) := \operatorname{span}\{X \in \mathcal{H}(M_h;\mathbb{C}) \mid \star X = -i\lambda X \text{ for some } 0 < \lambda \leq 1\}.$$

The space of *anti-holomorphic vector fields* is generated by the eigenvectors of $\star$ corresponding to positive imaginary eigenvalues:

$$\mathcal{H}^{0,1}(M_h) := \operatorname{span}\{X \in \mathcal{H}(M_h;\mathbb{C}) \mid \star X = i\lambda X \text{ for some } 0 < \lambda \leq 1\}.$$

Note that positive and negative imaginary eigenvalues come in pairs (of equal absolute value). The following theorem constitutes the splitting of the (complex) space of harmonic vector fields into holomorphic and anti-holomorphic fields. It is a simple consequence of Theorem 2.5.5.

Theorem 2.5.6. *There exists an orthogonal splitting with respect to* $(\cdot,\cdot)^{\mathbb{C}}_{L^2}$,

$$\mathcal{H}(M_h;\mathbb{C}) = \mathcal{H}^{1,0}(M_h) \oplus \mathcal{H}^{0,1}(M_h),$$

where each summand on the right hand side has complex dimension $\mathfrak{g}$ *and complex conjugation maps one space to the other.*

Holomorphic vector fields can be used to conformally parameterize M_h.

Definition 2.5.5 (local conformal parameterization)**.** Let the Euclidean cone surface M_h be closed and not spherical. A *local conformal parameterization* of M_h in the FE sense is obtained by integrating a holomorphic vector field.

Remark 2.5.2. Let the triple $(M, g_M, \star_M)$ represent a *smooth* Riemann surface with Riemannian metric g_M and star operator $\star_M$. Recall that

$$\star_M^2 = -Id \quad \text{and} \quad g_M(\star_M X, Y) = -g_M(X, \star_M Y),$$

so that star $\star_M$ induces a *complex structure* on M. The eigenvalues of $\star_M$ on the (complexified) space of harmonic vector fields (or 1-forms) are $-i$ and i. They occur with multiplicity $\mathfrak{g}$ (genus of M) for each of these two eigenvalues. The corresponding eigenspaces are called *holomorphic* and *anti-holomorphic*.

$$\mathcal{H}^{1,0}(M) = \{X \in \mathcal{H}(M;\mathbb{C}) \mid \star_M X = -iX\} \quad \text{and} \tag{2.15}$$
$$\mathcal{H}^{0,1}(M) = \{X \in \mathcal{H}(M;\mathbb{C}) \mid \star_M X = +iX\}. \tag{2.16}$$

We will later show that the discrete Hodge star $\star$ introduced in the last section is an approximation of the smooth operator $\star_M$ (under suitable assumptions). The *spectrum of* $\star$ is hence an approximation of the spectrum of $\star_M$. However, whereas the spectral decomposition of $\star_M$ consists of the two eigenspaces corresponding to $-i$ and $+i$, each of complex dimension $\mathfrak{g}$, the spectral decomposition of $\star$ does in general consist of $2\mathfrak{g}$ distinct eigenspaces, each of complex dimension 1 (compare Proposition 2.5.2).

2.5.7 CR equations and parameterization

In this subsection we are going to remark on a slightly different approach to surface parameterization. The approach taken by Gu and Yau [45, 46] falls into this category. We show that the space of vector fields considered by Gu and Yau for local parameterization is in general *larger* than the space of holomorphic vector fields (cf. Proposition 2.5.2). Our discussion will be based on discrete Cauchy-Riemann equations (CR equations) in the FE setting.

The smooth setting. Recall that for a smooth complex-valued function $f : \mathbb{C} \to \mathbb{C}$ with $f = u + iv$ (where u and v are real-valued), the CR equations are

$$\mathbf{J}\nabla u = \nabla v. \tag{2.17}$$

On a smooth *closed* Riemann surface $(M, g_M, \mathbf{J} = \star_M)$, the only solutions to equation (2.17) are constants (since u and v are harmonic). However, the complex-valued functions obtained by locally integrating (over any simply connected domain) a complex-valued field of the form

$$X + i \star_M X \quad \text{with} \quad X \in \mathcal{H}(M; \mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} (\ker \operatorname{curl}_M \cap \ker \operatorname{div}_M),$$

do (locally) satisfy the CR-equation. Moreover, since $\star_M^2 = -Id$ on a smooth Riemann surface, it follows that

$$\begin{aligned} \mathcal{H}^{1,0}(M) &= \{X + i \star_M X \mid X \in \mathcal{H}(M; \mathbb{C})\} \quad \text{and} \\ \mathcal{H}^{0,1}(M) &= \{X - i \star_M X \mid X \in \mathcal{H}(M; \mathbb{C})\}, \end{aligned}$$

where $\mathcal{H}^{1,0}(M)$ and $\mathcal{H}^{0,1}(M)$ are defined as in (2.15) and (2.16). Hence, the functions arising from locally integrating a complex-valued field $X \in \mathcal{H}(M; \mathbb{C})$ satisfy the CR equations if and only if $X \in \mathcal{H}^{1,0}(M)$, that is X is *holomorphic*.

The discrete setting. Drawing from this connection between CR equations and holomorphic vector fields for smooth Riemann surfaces, Gu and Yau [45, 46] define *discrete conformal parameterizations* on the Euclidean cone surface M_h by locally integrating complex fields of the form

$$X + i \star X \quad \text{with} \quad X \in \mathcal{H}(M_h; \mathbb{C}),$$

i.e. for X (conforming) harmonic. However, unlike in the smooth case, the complex vector space spanned by vectors of this kind *is in general larger* than the space of holomorphic vector fields $\mathcal{H}^{1,0}(M_h)$. This is due to the fact that in general $\star^2 \neq -Id$ in the discrete case. More precisely, the following result holds.

Proposition 2.5.2. *Let M_h be a closed Euclidean cone surface of genus $\mathfrak{g}$. Then*

$$\mathcal{H}^{1,0}(M_h) \subset \{X + i \star X \mid X \in \mathcal{H}(M_h; \mathbb{C})\},$$

but equality does in general not hold. Indeed, one has

$$\dim_{\mathbb{C}}\{X + i \star X \mid X \in \mathcal{H}(M_h; \mathbb{C})\} = 2\mathfrak{g} - \dim_{\mathbb{C}}\{X \mid \star X = iX\} \geq \mathfrak{g},$$

whereas $\dim_{\mathbb{C}} \mathcal{H}^{1,0}(M_h) = \mathfrak{g}$.

Proof. To show the inclusion, let $Y \in \mathcal{H}^{1,0}(M_h)$ such that $\star Y = -i\lambda Y$ for some $0 < \lambda \leq 1$. Then

$$Y = \frac{1}{1+\lambda}(Y + i \star Y) \in \{X + i \star X \mid X \in \mathcal{H}(M_h;\mathbb{C})\}.$$

To show the dimension count, let $\bar{Y} \in \mathcal{H}^{0,1}(M_h)$ be an anti-holomorphic vector field such that $\star\bar{Y} = i\lambda\bar{Y}$ for some $0 < \lambda \leq 1$. If $\lambda \neq 1$, we get

$$\bar{Y} = \frac{1}{1-\lambda}(\bar{Y} + i \star \bar{Y}) \in \{X + i \star X \mid X \in \mathcal{H}(M_h;\mathbb{C})\}.$$

Since by Theorem 2.5.6 we have

$$\mathcal{H}(M_h;\mathbb{C}) = \mathcal{H}^{1,0}(M_h) \oplus \mathcal{H}^{0,1}(M_h),$$

it follows that

$$\mathcal{H}(M_h;\mathbb{C}) = \{X + i \star X \mid X \in \mathcal{H}(M_h;\mathbb{C})\} \oplus \{X \mid \star X = iX\},$$

which completes the proof. QED

Remark 2.5.3. Polthier (cf. [62]) observed that a local version of the CR equations on Euclidean cone surfaces can be obtained by mixing conforming and nonconforming elements. Let Ω be a simply connected simplicial domain. Let $u \in S_h(\Omega)$ be harmonic, i.e. $\Delta u(p) = 0$ for all interior vertices $p \in \Omega$. Then there exists $v \in S_h^*(\Omega)$ such that

$$\mathbf{J}\nabla u = \nabla v$$

and $\Delta^* v(e) = 0$ for all interior edges e. The function v is uniquely determined up to an additive constant. Vice-versa, if $u \in S_h$ and $v \in S_h^*$ satisfy the discrete CR equations ($\mathbf{J}\nabla u = \nabla v$) then $\Delta u(p) = 0$ at all interior vertices and $\Delta^* v(e) = 0$ for all interior edges. This characterization leads to an interesting construction of discrete conjugate minimal surfaces in the FE sense (cf. [62]).