Mean Curvature flow of Surfaces Asymptotic to Minimal Cones

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Erklärung

Ich bestätige hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

Unterschrift:

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Abstract

In this thesis we study the mean curvature flow of hypersurfaces asymptotic to the Simons' cone

$$\{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

From the work of Bomberi, De Giorgi and Giusti we know that $\mathbb{R}^8$ is foliated by family of minimal hypersurfaces which are asymptotic to the Simons cone.

We start our flow with a smooth hypersurface which lies underneath one of these foliating hypersurfaces (barrier) and which shares the symmetry property of the cone and this minimal foliation. Following the work of Ecker and Huisken, we show that given an initial bound on the gradient and sign on the mean curvature, we can obtain that the bound on the gradient is preserved by the flow and that the surface remains beneath the initial barrier minimal hypersurface. This in turns tells us that bounds on the second fundamental form and its derivatives are preserved. These combine to give us our main result:

Mean curvature flow, starting with a hypersurface as described above, will have a solution which exists for all time and which converges to a smooth minimal hypersurface asymptotic to the Simons' cone, belonging to the family of hypersurfaces found in by Bombieri, De Giorgi and Giusti.

Zusammenfassung

In dieser Arbeit untersuchen wir den mittleren Krümmungsfluss von Hyperflächen asymptotisch zum Simons Kegel

$$\{x \in \mathbb{R}^8 : x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2\}.$$

Aus den Arbeiten von Bomberi, De Giorgi und Giusti wissen wir, dass der $\mathbb{R}^8$ durch eine Familie von minimalen Hyperflächen asymptotisch zum Simons Kegel geblättert wird.

Wir beginnen unseren Fluss mit einer glatten Hyperfläche, die unterhalb einer blätternden Hyperfläche (Barriere) liegt und welche die Symmetrieeigenschaften des Kegels und der minimalen Blätterung hat. Den Arbeiten von Ecker und Huisken folgend zeigen wir: Ist eine Schranke an den Gradienten und ein Vorzeichen an die mittlere Krümmung gegeben, bleibt diese Schranke durch den Fluss erhalten und die Fläche bleibt unterhalb einer zu Anfung angenommen oberen minimalen Barrierefläche. Dies wiederum zeigt uns, dass Schranken an die zweite Fundamentalform und ihre Ableitungen erhalten bleiben. Zusammengenommen ergibt sich unser Hauptresultat:

Der mittlere Krümmungsfluss, beginnend mit einer Hyperfläche mit obigen Eigenschaften, hat eine Lösung, die für alle Zeiten existiert und die gegen eine glatte minimale Hyperfläche asymptotisch zum Simons Kegel konvergiert, welche zur von Bombieri, De Giorgi und Giusti gefundenen Familie von Hyperflächen gehört.

1 Introduction

Mean Curvature Flow (MCF) is the deformation in time of hypersurfaces in the direction of their mean curvature. That is, we take an immersion $\mathbf{F}_0 : M^n \mapsto \mathbb{R}^{n+1}$ and evolve it in time according to the equation

$$\frac{\partial}{\partial t} F_t(p) = \mathbf{H}(p,t), \quad \mathbf{F}(p,0) = \mathbf{F}_0(p), \quad p \in M \tag{1}$$

where $\mathbf{H}$ is the mean curvature vector $\mathbf{H} = -H\boldsymbol{\nu}$ and H denotes the mean curvature.

We write $M_0 = \mathbf{F}_0(M)$ and $M_t = \mathbf{F}_t(M)$.

In [6], Ecker & Huisken developed methods to study the MCF of graphs. They did this by first controlling key quantities such as the gradient and height of these graphs and using this to obtain a variety of results, in particular curvature bounds and conditions under which the flow would exist for all time.

A minimal hypersurface is one for which any sufficiently local, smooth deformation will result in an increase in area. These hypersurfaces have $H = 0$ and thus are stationary under MCF. Since MCF is the steepest descent flow of the area functional, it is the ideal flow to use when one is looking for minimal hypersurfaces.

Minimal hypersurfaces have also been studied within the more general class of objects called sets of finite perimeter [11] or integral currents [15]. Within this context, which uses the language of Geometric Measure Theory (GMT), one defines an area minimising hypersurface to be such an object whose area (mass) increases whenever any (not just small) compact region is altered.

In [3], the authors showed that the Simons cone

$$\left\{x \in \mathbb{R}^{2n} : x_1^2 + \ldots + x_n^2 = x_{n+1}^2 + \ldots + x_{2n}^2\right\}$$

which is smooth away from the origin, is area-minimising in the above sense, by proving the existence of a function, the level sets of which form a foliation of area-minimising hypersurfaces which includes the Simons cone as one of its leaves.

There is, in fact, a rich class of quadratic cones of the form

$$qy^2 - pz^2 = 0, \;\; y \in \mathbb{R}^{p+1}, z \in \mathbb{R}^{q+1} \text{ with } p+q \geq 7 \text{ or } p+q = 6 \;\&\; |p-q| \leq 4$$

which are minimal and for which the results of this thesis hold. In the interest of simplicity the body of the thesis contains the results only for the

Simons cone in $\mathbb{R}^8$. The main difference between these cones is their angle and therefore the gradient of the hypersurfaces asymptotic to them. Appendix B contains versions of key equations and quantities which become different when we consider these more general cones.

In [8], Hardt & Simon showed that these hypersurfaces are oriented, connected, embedded, real analytic and star-shaped. Furthermore and most importantly they are unique up to scaling, that is they can be represented by $\{\lambda x : x \in T^8\}$ where T^8 is the leaf of unit distance from the origin and $\langle \boldsymbol{\omega}, \bar{x} \rangle > 0$, where $\boldsymbol{\omega}$ is the normal to the hypersurface. Their methods rely heavily on tools from Geometric Measure Theory.

We would like to take a different approach by using MCF to evolve certain classes of initial hypersurface to members of the above foliations, that is in some sense construct them classically, without the use of GMT. We look at a class of hypersurfaces that share the symmetry properties of this foliation. The foliating function constructed in [3] can be written as a function of two variables $F(u, v)$ where $u = \sqrt{x_1^2 + \ldots + x_4^2}$ and $v = \sqrt{x_5^2 + \ldots + x_8^2}$. Thus these hypersurfaces are invariant under the groups $O(3) \times \{e(4)\}$ and $\{e(4)\} \times O(3)$. We regard these hypersurfaces being obtained by rotating as a contour curve in the u, v plane using these groups. We will call such hypersurfaces bi-rotationally symmetric.

Surfaces with a single rotational symmetry, in particular cylindrical graphs, have been studied by [1],[17]. They have a tendency of developing 'neck pinches' and becoming singular. These singularities are of interest and avoiding them requires terms in the evolution equations to be controlled somehow. Surprisingly, with the bi-rotational symmetry in our case, the geometry takes care of these bad terms and under certain conditions allows us to prove convergence results.

We draw inspiriation from the elegant and simple form of the mean curvature of these hypersurfaces. For a bi-rotationally symmetric hypersurface, the mean curvature takes the form

$$H = 3\frac{\nu_v}{v} + 3\frac{\nu_u}{u} + A_7^7$$

where $\nu_v = \langle \boldsymbol{\nu}, \hat{\mathbf{v}} \rangle$, $\hat{\mathbf{v}} = \frac{(0,\ldots,0,x_5,\ldots,x_8)}{v}$, ν_u defined analogously, and A_7^7 is the curvature of the contour curve.

We require initially that $H \leq$ and $\nu_v \geq C > 0$ We take ν_v^{-1} to be our gradient. We also assume that our initial hypersurface lies beneath a stationary hypersurface M^+ from the minimal foliation found in [3].

In chapter five we show using [7] that a solution will exist for short time and by a uniqueness argument based on a non-compact maximum principle from [7], it must remain bi-rotationally symmetric if the initial hypersurface has this symmetry. We show that the evolving hypersurface will stay above M_0 using the sign on the mean curvature. These results allow us to establish a global gradient bound. From here we obtain that the curvatures and their derivatives of all orders remain bounded which allows us to extend the short time solution to one which exists for all times. We then show under a certain condition that the flow remains below the barrier M^+, which allows us to find a limiting hypersurface M_∞. Finally we show that the mean curvature of these hypersurfaces must converge to zero everywhere, giving us that M_∞ is a minimal hypersurface. The uniqueness of the minimising hypersurfaces demonstrated in [16] tells us it must be a leaf of the foliation found in [3].

The following is our main theorem.

Theorem 1.1. *Let M_0 be a bi-rotationally symmetric hypersurface lying in the region $\{v > u\}$ with $\sup_{M_0} |\nabla^m A| \leq C_m < \infty$ for all $m \geq 0$, satisfying*

1) $H \leq 0$

2) $\nu_v \geq C > 0$.

Then there exists a smooth bi-rotationally symmetric solution of MCF which will remain in the region $\{v > u\}$ and exist for all time.

Additionally, if M_0

3) lies beneath a minimal hypersufaces $M^+ \subset \{v > u\}$ belonging to the foliation found in [3], and we are able to write M^+ as a graph over M_0, and if

4) $\inf_{M_0} \nu_v = \frac{1}{\sqrt{2}}$

the solution will remain beneath this minimal hypersurface, and converge to an area minimising hypersurface belonging to the family of hypersurfaces found in [3].

Under condition 4) we also obtain that the evolving hypersurfaces and their limit are star-shaped.

Condition 2) is a reasonable assumption since above the origin a smooth bi-rotationally symmetric hypersurface will have $\nu_v = 1$ and if it is asymptotic to the Simons cone it will have $lim_{|x| \to \infty} \nu_v = \frac{1}{\sqrt{2}}$ and thus must have a minimum as long as ν_v remains continuous.

Note that without the assumption on the sign of H, M_t may approach the origin in finite time and become singular there (see [18]). Without the upper barrier assumption one may have to use the full machinery of the minimal foliations construction combined with mean curvature flow to prevent M_t from moving off to infinity. Certainly, if we consider the flow in lower dimensions e.g. inside the cone $\{x \in \mathbb{R}^2 m, x_{m+1}^2 + \ldots + x_{2m}^2 > x_1 + \ldots + x_m^2\}$ for $m \leq 3$ the hypersurfaces M_t cannot obey a global height bound inside this set, since, by the Bernstein theorem, there are no non-trivial minimal hypersurfaces in these dimensions. Note that a global height bound from above combined with regularity estimates for M_t would imply the convergence to a minimal hypersurface (see chapter 9 of this thesis).

In view of the current state of research on MCF of hypersurfaces it also seems a very difficult task to do away with the graph assumption on M_0. We are not aware of any results in this directions. Note also that we do not assume M_0 to be close in any sense to a minimal hypersurface.

2 Definitions

We work in $\mathbb{R}^8$ and define $u = \sqrt{x_1^2 + \ldots + x_4^2}$ and $v = \sqrt{x_5^2 + \ldots + x_8^2}$

Since the Simons cone and the minimal surfaces that foliate $\mathbb{R}^8$, found in [3] are bi-rotationally symmetric, we study evolving surfaces which share the same symmetry. To this end we define the following:

At a given point we take $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_6, \hat{\mathbf{u}}, \hat{\mathbf{v}}\}$ as a basis of $\mathbb{R}^8$, where $\boldsymbol{\tau}_1$ to $\boldsymbol{\tau}_3$ and $\boldsymbol{\tau}_4$ to $\boldsymbol{\tau}_6$ are each the normal coordinate system for $\mathbb{S}^3$, and

$$\hat{\mathbf{u}} = \frac{(x_1, \ldots, x_4, 0, \ldots, 0)}{u}, \quad \hat{\mathbf{v}} = \frac{(0, \ldots, 0, x_5, \ldots, x_8)}{v}.$$

A bi-rotationally-symmetric surface is one which is invariant under the group $O(3) \times \{e_4\} \cup \{e_4\} \times O(3)$. Such a surface will include in its tangent basis $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_6\}$ and thus its normal (and τ_7) will *only have components in the* u,v *plane.*

We can decompose the normal to such a surface as $\boldsymbol{\nu} = \langle \boldsymbol{\nu}, \hat{\mathbf{u}} \rangle \, \hat{\mathbf{u}} + \langle \boldsymbol{\nu}, \hat{\mathbf{v}} \rangle \, \hat{\mathbf{v}}$. We denote $\nu_u = \langle \boldsymbol{\nu}, \hat{\mathbf{u}} \rangle$ and $\nu_v = \langle \boldsymbol{\nu}, \hat{\mathbf{v}} \rangle$ so that $\boldsymbol{\nu} = \nu_u \hat{\mathbf{u}} + \nu_v \hat{\mathbf{v}}$ and thus by a simple calculation $\boldsymbol{\tau}_7 = \nu_v \hat{\mathbf{u}} - \nu_u \hat{\mathbf{v}}$. We note that $\{\boldsymbol{\tau}_1, \ldots, \boldsymbol{\tau}_7\}$ forms an orthonormal basis for the tangent space. $\boldsymbol{\tau}_7$ will occasionally be referred to as the pseudo-radial direction.

The mean curvature H of a bi-rotationally symmetric surface can be calculated directly (see chapter 4 for a more detailed approach) by:

$$\begin{aligned} H &= \operatorname{div}(\boldsymbol{\nu}) \\ &= \operatorname{div}(\nu_u \hat{\mathbf{u}} + \nu_v \hat{\mathbf{v}}) \\ &= \nu_u \operatorname{div}\hat{\mathbf{u}} + \nu_v \operatorname{div}\hat{\mathbf{v}} + D(\nu_u) \cdot \hat{\mathbf{u}} + D(\nu_v) \cdot \hat{\mathbf{v}} \\ &= 3\frac{\nu_u}{u} + 3\frac{\nu_v}{v} + \frac{\partial \nu_u}{\partial u} + \frac{\partial \nu_v}{\partial v} \end{aligned}$$

where $\frac{\partial \nu_u}{\partial u} = D(\nu_u) \cdot \hat{\mathbf{u}}$, $\frac{\partial \nu_v}{\partial v} = D(\nu_v) \cdot \hat{\mathbf{v}}$.

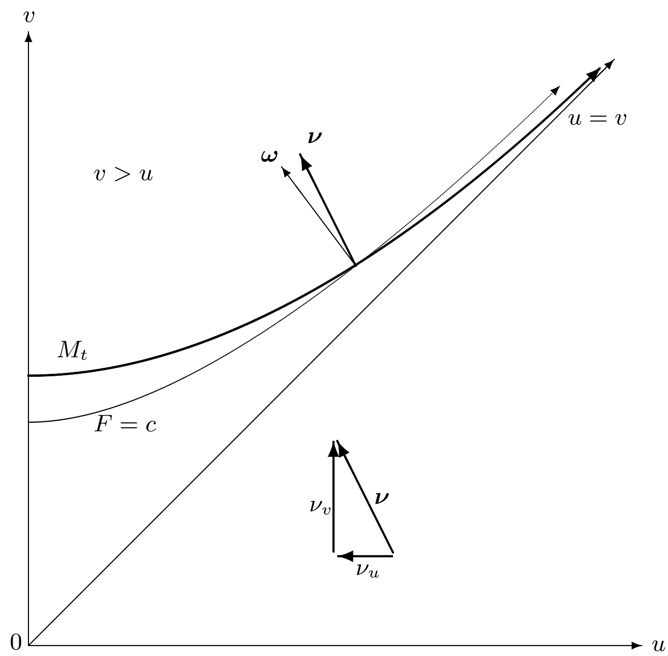
Now, since $\nu_u^2 + \nu_v^2 = 1$ we have

$$\nu_u D(\nu_u) = -\nu_v D(\nu_v)$$

and thus

$$\begin{aligned} H &= 3\frac{\nu_u}{u} + 3\frac{\nu_v}{v} + D(\nu_u)\cdot\hat{\mathbf{u}} - \frac{\nu_u}{\nu_v}D(\nu_u)\cdot\hat{\mathbf{v}} \\ &= 3\frac{\nu_u}{u} + 3\frac{\nu_v}{v} + \frac{1}{\nu_v}\left(D(\nu_u)\cdot(\nu_v\hat{\mathbf{u}} - \nu_u\hat{\mathbf{v}})\right) \\ &= 3\frac{\nu_u}{u} + 3\frac{\nu_v}{v} + \frac{1}{\nu_v}\left(D(\nu_u)\cdot\boldsymbol{\tau}_7\right) \end{aligned}$$

We note that the $(7,7)$-component of the second fundamental form is $A_7^7 = \frac{\partial\nu_u}{\partial u} + \frac{\partial\nu_v}{\partial v} = \frac{1}{\nu_v}\frac{\partial\nu_u}{\partial\tau_7} = -\frac{1}{\nu_u}\frac{\partial\nu_v}{\partial\tau_7}$ where $\frac{\partial\nu_u}{\partial\tau_7} = D(\nu_u)\cdot\boldsymbol{\tau}_7$ (see Chapter 4)



In the picture shown, $\nu_u = \langle\nu,\hat{\mathbf{u}}\rangle < 0$ and $\nu_v = \langle\nu,\hat{\mathbf{v}}\rangle > 0$

The contour curve shown in the above graph can be described by a function $v = h(u)$. It can be shown (chapter 4) that $\nu_v = \frac{1}{\sqrt{1+h'(u)^2}}$ and thus we consider the gradient to be ν_v^{-1}. Motivated by this expression, a hypersurface can be considered a graph if $\nu_v^{-1} < \infty$ everywhere.

3 The non-compact maximum principle

The following theorem is a slight modification of Theorem 4.3 from [7].

Theorem 3.1. *Suppose that a manifold M^n with Riemannian metric $g(t)$ satisfies a uniform volume growth restriction, namely:*

$$vol^t(B_r^t(p)) \leq C\exp(k(1+r^2))$$

holds for some point $p \in M^n$, for all $r > 0$ and a uniform constant $k > 0$ for all $t \in [0,T]$

Let f be a function on $M^n \times [0,T]$ which is smooth on $M^n \times (0,T]$ and continuous on $M^n \times [0,T]$. Assume that f and $g(t)$ satisfy:

(i)$\frac{\partial}{\partial t} f \leq \Delta^t f + \mathbf{a}\cdot\nabla f + bf$ where the functions a and b satisfy $\sup_{M^n\times[0,T]} |\mathbf{a}| \leq \alpha_1$ and $\sup_{M^n\times[0,T]} |b| \leq \alpha_2$ respectively for some $\alpha_1, \alpha_2 < \infty$,

(ii) $f(p,0) \leq 0$ for all $p \in M^n$,

(iii) $\int_0^T \left(\int_M \exp\left(-\alpha_2^2 r^t(p,y)^2\right) |\nabla f|^2 (y) d\mu_t\right) dt < \infty$, for some $\alpha_2 < \infty$,

(iv) $\sup_{M^n\times[0,T]} \left|\frac{\partial}{\partial t} g_{ij}\right| \leq \alpha_3$, for some $\alpha_3 < \infty$

Then we have $f \leq 0$ on $M^n \times [0,T]$

From [10] we know that $|\frac{\partial}{\partial t} g_{ij}| \leq |2Hh_{ij}| \leq C(n)\,|A|^2$ if our manifold evolves by mean curvature. As long as $|A|^2$ is bounded we therefore have that condition (iv) is satisfied.

The Gauss equations tells us that $Ric_{M_t} \geq -C(n)\,|A|^2$. The volume growth condition follows from this and the volume comparison theorem (1.2 in [13]). In fact this gives us that area$(\partial B_r(p)) \leq c_1 e^{c_2 r}$ where c_1 depends on n and c_2 depends on n and our bound for $|A|^2$. Since in the integral in condition (iii) we are integrating against an expression of the form $e^{-\alpha_2^2 r^2}$, this condition will be met by any function with gradient which grows at most exponentially in r.

Therefore for any function satisfying (ii) with a gradient growing at most exponentially in r and with $|A|^2 \leq C < \infty$ the noncompact maximum principle is applicable.

Corollary 3.2. *Suppose the conditions of Theorem 3.1 hold and the function $f = f(x,t)$ satisfies the inequality:*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \leq \mathbf{a}\cdot\nabla f - \delta^2 f^2 + C^2$$

for some vector field $\mathbf{a}$ with $\sup_{M\times[0,t_1)} |\mathbf{a}| < \infty$ for some time $t_1 > 0$, then f satisfies the estimates

$$\begin{array}{lll} (i) & f \leq C/\delta + 1/\left(\delta^2 t\right) & on M \times (0,t_1) \\ (ii) & f \leq C/\delta + \sup_{M_0} f & on M \times [0,t_1) \end{array}$$

Proof. Set $g = t\left(f - \frac{C}{\delta}\right)$ thus we get:

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_M\right) g &= f - \frac{C}{\delta} + t\left(\frac{\partial}{\partial t} - \Delta_M\right) f \leq t\mathbf{a}\cdot\nabla f + f - \frac{C}{\delta} - \delta^2 t f^2 + tC^2 \\ &= \mathbf{a}\cdot\nabla g - 2t\delta C f + 2tC^2 + t^{-1}g - \delta^2 t^{-1} g^2 \\ &= \mathbf{a}\cdot\nabla g - 2\delta C g + \delta^2 t^{-1} g\left(\frac{1}{\delta^2} - g\right).\end{aligned}$$

Let $g_\delta = \max\left(g - \frac{1}{\delta^2}, 0\right)$. We calculate in the set where $g_\delta > 0$. (The proof of Theorem 3.1 in [7] uses only that f is Lipschitz in the spatial variable, so we can consider the condition (i) of Theorem 3.1 above in the weak sense by multiplying by a test function. We are therefore allowed to consider g_δ^2 even though it is not C^2 on its zero set.)

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_M\right) g_\delta^2 &\leq 2g_\delta\mathbf{a}\cdot\nabla g - 4\delta C g_\delta g + 2\delta^2 t^{-1} g_\delta g\left(\frac{1}{\delta^2} - g\right) - 2\left|\nabla g_\delta\right|^2 \\ &\leq 2g_\delta\mathbf{a}\cdot\nabla g_\delta - 2\left|\nabla g_\delta\right|^2.\end{aligned}$$

since $g_\delta\left(g - \frac{1}{\delta^2}\right) = g_\delta^2$ and $g_\delta\nabla g = g_\delta\nabla g_\delta$ a.e. Then Young's inequality gives us

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) g_\delta^2 \leq \frac{1}{2}\sup_{M\times[0,t_0)} |a|^2 g_\delta^2.$$

Result (i) then follows from Theorem 3.1.

To prove (ii), we let $f_k = \max(f - k, 0)$, $k > 0$ to be chosen later. We can, on the set $\{f > k\}$ express f as $f = f_k + k$. Therefore we obtain from the equations for f

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f_k^2 \leq 2f_k\mathbf{a}\cdot\nabla f_k - 2|\nabla f_k|^2 - 2\delta^2 f_k f^2 + 2C^2 f_k$$

$$= 2f_k\mathbf{a}\cdot\nabla f_k - 2|\nabla f_k|^2 + 2f_k\left(-\delta^2 f_k^2 - 2\delta^2 f_k k - \delta^2 k^2 + C^2\right).$$

If $k \geq \frac{C}{\delta}$ we obtain

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f_k^2 \leq 2f_k\mathbf{a}\cdot\nabla f_k - 2|\nabla f_k|^2 \leq \frac{1}{2}\sup_{M^n\times[0,T]} |a|^2 f_k^2.$$

We now also need to assume that $k \geq \sup_{M_0} f$ to ensure that $f_k = 0$ at $t = 0$ and thus taking $k \geq \frac{C}{\delta} + \sup_{M_0} f$ we can apply the non-compact maximum principle. □

4 Bi-rotational graphs

Bi-rotational Graphs

We consider a bi-rotationally symmetric surface as a contour curve rotated around each axis using two $SO(3) \times \{e_4\}$ type actions. The metric, connection, curvatures and their derivatives can all be calculated in terms of this contour curve. To do this we define the following:

Let $(\phi_1, \phi_2, \phi_3) \mapsto \Phi(\phi_1, \phi_2, \phi_3)$, $(\psi_1, \psi_2, \psi_3) \mapsto \Psi(\psi_1, \psi_2, \psi_3)$ be 'standard' embeddings for spheres in polar co-ordinates, likewise let $\boldsymbol{\partial}_{\phi_1} = \frac{\partial \Phi}{\partial \phi_1}$ etc be their coordinate frame such that $\left\langle \boldsymbol{\partial}_{\phi_i}, \boldsymbol{\partial}_{\phi_j} \right\rangle = (g_{ij})^{\mathbb{S}^3}$. We will abbreviate $\Phi(\phi_1, \phi_2, \phi_3)$ as $\Phi(\phi)$ for convenience.

For any function $h : \mathbb{R}^+ \to \mathbb{R}^+$ we can define the embedding

$$F : \mathbb{R}^+ \times \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{R}^8 = (u\boldsymbol{\Phi}(\phi), h(u)\boldsymbol{\Psi}(\psi))$$

yielding the bi-rotationally symmetric surface formed by rotating the contour curve described by h around both axes.

Note that at any given point $\hat{\mathbf{u}} = (\boldsymbol{\Phi}(\phi), \mathbf{0})$ and $\hat{\mathbf{v}} = (\mathbf{0}, \boldsymbol{\Psi}(\psi))$.

The hypersurface has tangent vectors given by

$$(u\boldsymbol{\partial}_{\phi_i}, \mathbf{0}) \;,\; (\mathbf{0}, h(u)\boldsymbol{\partial}_{\psi_i}) \;,\; \hat{\mathbf{u}} + h'(u)\hat{\mathbf{v}}$$

and a metric of the form

$$(g_{ij}) = \operatorname{diag}\left(u^2 (g_{ij})^{\mathbb{S}^3}, h(u)^2 (g_{ij})^{\mathbb{S}^3}, 1 + \left(h'(u)\right)^2\right), \tag{2}$$

with inverse

$$(g^{ij}) = \operatorname{diag}\left(\frac{1}{u^2}(g^{ij})^{\mathbb{S}^3}, \frac{1}{h(u)^2}(g^{ij})^{\mathbb{S}^3}, \frac{1}{1 + (h'(u))^2}\right).$$

It is easy to see that the normal to such a surface will take the form

$$\boldsymbol{\nu} = \frac{-h'(u)\hat{\mathbf{u}} + \hat{\mathbf{v}}}{\sqrt{1 + h'(u)^2}} \tag{3}$$

with $\nu_v = \frac{1}{\sqrt{1+h'(u)^2}}$ and $\nu_u = \frac{-h'(u)}{\sqrt{1+h'(u)^2}}$, and

$$\boldsymbol{\tau}_7 = \nu_v \hat{\mathbf{u}} - \nu_u \hat{\mathbf{v}} = \frac{\hat{\mathbf{u}} + h'(u)\hat{\mathbf{v}}}{\sqrt{1 + h'(u)^2}}. \tag{4}$$

The second fundamental form is defined by

$$A_{ij} = \langle \partial_i \boldsymbol{\nu}, \boldsymbol{\partial}_j \rangle$$

which leads to a matrix of the form

$$\operatorname{diag}\left\{ \frac{-uh'(u)}{\sqrt{1+h'(u)^2}} \left(A_{ij}\right)^{\mathbb{S}^3}, \frac{h(u)}{\sqrt{1+h'(u)^2}} \left(A_{ij}\right)^{\mathbb{S}^3}, A_{77} \right\} \tag{5}$$

where the $\left(A_{ij}\right)^{\mathbb{S}^3}$ are 3×3 submatricies. In the (7,7)th position we calculate

$$\frac{\partial}{\partial u}\left(\frac{-h'(u)\hat{\mathbf{u}}+\hat{\mathbf{v}}}{\sqrt{1+h'(u)^2}}\right) = \frac{-h''(u)\left(1+h'(u)^2\right)\hat{\mathbf{u}} + \left(h'(u)\hat{\mathbf{u}}-\hat{\mathbf{v}}\right)h''(u)h'(u)}{\left(1+h'(u)^2\right)^{\frac{3}{2}}}$$

$$= \frac{-h''(u)(\hat{\mathbf{u}}+h'(u)\hat{\mathbf{v}})}{\left(1+h'(u)^2\right)^{\frac{3}{2}}} = \frac{-h''(u)}{\left(1+h'(u)^2\right)^{\frac{3}{2}}}\boldsymbol{\tau}_7$$

using (4), and $\frac{\partial \hat{\mathbf{u}}}{\partial u} = \frac{\partial \hat{\mathbf{v}}}{\partial u} = 0$. This gives us

$$A_{77} = \frac{-h''(u)}{\sqrt{1+h'(u)^2}} \tag{6}$$

and

$$A_7^7 = -\frac{h''(u)}{\left(1+h'(u)^2\right)^{\frac{3}{2}}}. \tag{7}$$

The mean curvature $H = \Sigma_{i=1}^{7} A_i^i$ is therefore given by

$$H = 3\frac{h'(u)}{u\sqrt{1+h'(u)^2}} + 3\frac{1}{h(u)\sqrt{1+h'(u)^2}} - \frac{h''(u)}{\left(1+h'(u)^2\right)^{\frac{3}{2}}} \tag{8}$$

which can be rewritten as

$$H = 3\frac{\nu_u}{u} + 3\frac{\nu_v}{v} + A_7^7. \tag{9}$$

Note we can also write $|A|^2$ as

$$|A|^2 = 3\frac{\nu_u^2}{u^2} + 3\frac{\nu_v^2}{v^2} + \left(A_7^7\right)^2 \tag{10}$$

We can also write

$$A_7^7 = -\frac{1}{\nu_u}\frac{\partial \nu_v}{\partial \boldsymbol{\tau}_7} = \frac{1}{\nu_v}\frac{\partial \nu_u}{\partial \boldsymbol{\tau}_7}$$

using the form of τ_7 in (4).

$$\frac{\partial \nu_v}{\partial \boldsymbol{\tau}_7} = D\nu_v \cdot \boldsymbol{\tau}_7 = \frac{\partial \nu_v}{\partial u}\boldsymbol{\tau}_7 \cdot \hat{\mathbf{u}} = -\frac{h'(u)h''(u)}{\left(1+h'(u)^2\right)^2}$$

and so

$$-\frac{1}{\nu_u}\frac{\partial \nu_v}{\partial \boldsymbol{\tau}_7} = -\frac{h''(u)}{\left(1+h'(u)^2\right)^{\frac{3}{2}}} = A_7^7.$$

For MCF our immersion is of the form

$$\mathbf{F}(p,t) = u(p,t)\hat{\mathbf{u}} + h(u(p,t),t)\hat{\mathbf{v}}.$$

Thus we have

$$\frac{\partial \mathbf{F}}{\partial t} = \frac{\partial u}{\partial t}(p,t)\hat{\mathbf{u}} + \left(h'(u(p,t),t)\frac{\partial u}{\partial t}(p,t) + \frac{\partial h}{\partial t}(p,t)\right)\hat{\mathbf{v}}$$

where h' refers to the spacial derivative.

Taking the inner product with $\boldsymbol{\nu}$ and using $\frac{\partial \mathbf{F}}{\partial t} = -H\boldsymbol{\nu}$ we get

$$-H = \frac{\partial u}{\partial t}\nu_u + h'\frac{\partial u}{\partial t}\nu_v + \frac{\partial h}{\partial t}\nu_v.$$

Since $\nu_u = -h'(u)\nu_v$ we get

$$\frac{\partial h}{\partial t}(u,t) = -\frac{H}{\nu_v}. \tag{11}$$

Derivative bounds on h versus derivative bounds on A

We wish to establish an equivalence between bounds on the derivatives of the second fundamental form and bounds on the derivatives of our height function $h(u)$. It is easy to see that bounds on $||h'||_{C^{m+1}}$ will give us bounds on $||A||_{C^m}$, for all $m \geq 0$. To see the other direction we first calculate the Christoffel Symbols.

Lemma 4.1. *For a bi-rotationally symmetric surface the Christoffel symbols are as follows:*

i) $\Gamma^k_{ij} = \left(\Gamma^k_{ij}\right)^{\mathbb{S}^3}$, $1 \leq i,j,k \leq 3$ *or* $4 \leq i,j,k \leq 6$

ii) $\Gamma^7_{ij} = \frac{u(g_{ij})^{\mathbb{S}^3}}{1+(h'(u))^2}$, $1 \leq j,k \leq 3$

iii) $\Gamma^7_{ij} = \frac{h(u)h'(u)(g_{ij})^{\mathbb{S}^3}}{1+(h'(u))^2}$, $4 \leq j,k \leq 6$

iv) $\Gamma^k_{7j} = \frac{1}{u}\delta^k_j$, $1 \leq j,k \leq 3$

v) $\Gamma^k_{7j} = \frac{h'(u)}{h(u)}\delta^k_j$, $4 \leq j,k \leq 6$

vi) $\Gamma^7_{77} = \frac{h''(u)h'(u)}{1+(h'(u))^2}$

vii) $\Gamma^k_{ij} = 0$ *otherwise*

Proof.

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}\left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij}\right)$$

i) Since the metric of the sphere appears as a sub-matrix in our metric this is clear.

ii & iii) Assuming $i, j \neq 7$ we get

$$\Gamma^7_{ij} = \frac{1}{2} g^{7l} \left(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij} \right) = \frac{1}{2} g^{77} \left(\partial_i g_{7j} + \partial_j g_{i7} - \partial_7 g_{ij} \right) = -\frac{1}{2} g^{77} \partial_7 g_{ij}$$

and the result follows.

iv) Again assuming $1 \leq j \leq 3$ we get

$$\Gamma^k_{7j} = \frac{1}{2} g^{kl} \left(\partial_7 g_{lj} + \partial_j g_{7l} - \partial_l g_{7j} \right) = \frac{1}{2} g^{kl} \partial_7 g_{lj} = \frac{1}{u^2} (g^{ik})^{\mathbb{S}^3} u (g_{ij})^{\mathbb{S}^3} = \frac{\delta^k_j}{u}$$

v) Is proved in the same way

vii)

$$\Gamma^k_{77} = \frac{1}{2} g^{kl} \left(\partial_7 g_{l7} + \partial_7 g_{7l} - \partial_l g_{77} \right)$$

Since $g_{l7} = g_{77,l} = g^{7k} = 0$ when $l, k \neq 7$ the only non zero term here is

$$\Gamma^7_{77} = \frac{1}{2} g^{77} \partial_7 g_{77} = \frac{h''(u) h'(u)}{1 + (h'(u))^2}$$

□

Note that $\Gamma^7_{77} \leq \frac{1}{2} |h''(u)|$ since $\frac{x}{1+x^2} \leq \frac{1}{2}$.

Lemma 4.2. *A surface, described by a contour curve rotated around each axis as defined previously, for which*

$$\nu_v^{-1} \leq C_1$$

and

$$||A||_{C^{m-1}} \leq C_2$$

hold, will satisfy

$$||h'||_{C^m} \leq \tilde{C}(C_1, C_2) < \infty.$$

Proof. The function $h(u)$ is bounded on compact sets of the positive u-axis. A bound on ν_v^{-1} gives a bound on the first derivative of h since $\nu_v^{-1} = \sqrt{1 + (h'(u))^2}$.

We have by (7)

$$h''(u) = \nu_v^{-3} A^7_7$$

and so

$$|h''(u)| \leq C_1^3 |A|.$$

For the higher derivatives we write out the covariant derivatives of A_{77}.

$$\nabla_k A_{ij} = \partial_k A_{ij} + \Gamma^m_{ki} A_{mj} + \Gamma^m_{kj} A_{im}$$

$$\nabla_7 A_{77} = \partial_7 A_{77} + \Gamma^k_{77} A_{7k} + \Gamma^k_{77} A_{k7} = \partial_7 A_{77} + 2\Gamma^k_{77} A_{7k} = \frac{\partial}{\partial u} A_{77} + 2\Gamma^7_{77} A_{77}$$

$$= -h'''(u)\nu_v + \left(h''(u)\right)^2 h'(u)\nu_v^3 + 2\Gamma^7_{77} A_{77}$$

where we have used the form of A_{ij} given in (5) and $A_{77} = \frac{-h''(u)}{\sqrt{1+h'(u)^2}}$ from (6).

Thus

$$h'''(u)\nu_v = -\nabla_7 A_{77} + \left(h''(u)\right)^2 h'(u)\nu_v^3 + 2\Gamma^7_{77} A_{77}.$$

Hence

$$|h'''(u)| \leq \nu_v^{-1} C\left(|A|^2, |\nabla A|^2, h''(u), h'(u)\right).$$

where C is an expression depending on the quantities indicated, having none of them in the denominator, where we used that $\Gamma^7_{77} \leq \frac{1}{2}|h''(u)|$.

Thus a bound on the first derivative of A coupled with a bound on the lower derivatives of h gives us a bound on the third derivative.

Likewise if we have $\nabla^m_7 A_{77} = \nu_v h^{(m+2)}(u) + F(\nabla^{m-1} A_{77}, \ldots, A_{77}, h^{m+1}, \ldots, h', m)$ where F is an expression in lower order quantities appearing as factors, and $h^{(m)}$ denotes the m-th derivative of h indicate, we will get

$$\nabla^{m+1}_7 A_{77} = \nabla_7 \nabla^m_7 A_{77} = \frac{\partial}{\partial u} \nabla^m_7 A_{77} + m\Gamma^k_{77} \nabla_k \nabla^{m-1}_7 A_{77} + 2\Gamma^k_{77} \nabla^m_7 A_{k7}$$

$$= \frac{\partial}{\partial u} \nabla^m_7 A_{77} + (m+2)\Gamma^7_{77} \nabla^m_7 A_{77}$$

$$= \nu_v h^{(m+3)} + G(\nabla^m A, \ldots, A, h^{(m+2)}, \ldots, h', m)$$

where G is some new expression of the quantities indicated.

Thus for all derivatives we can get a bound of the form.

$$|h^{(m)}(u)| \leq \nu_v^{-1} C\left(m, ||A||_{C^{m-2}}, ||h'||_{C^{m-2}}\right).$$

$\square$

5 Short term existence, uniqueness & height bounds

This chapter relies on major input from my PhD supervisor Klaus Ecker.

The following short time existence theorem is a special case of Theorem 4.2 in [7].

Theorem 5.1. *Let* $F_0 : M^n \to \mathbb{R}^{n+1}$ *satisfy* $\sup_{M^n} |\nabla^m A| \leq C_m < \infty$ *for all* $m \geq 0$. *Then the mean curvature flow problem*

$$\frac{\partial F}{\partial t}(p,t) = \mathbf{H}(p,t), \qquad p \in M^n, t > 0 \tag{12}$$

$$F(p,0) = F_0(p)$$

has a smooth solution on some time interval $(0, T_0)$ *which smoothly attains its initial data.*

In particular, $\sup_{t\in[0,T_0)} \sup_{M_t} |\nabla^m A|$ *is finite for all* $m \geq 0$. *This solution is a normal graph over* M_0.

In order to prove the uniqueness of this solution (which was not done in [7] but should also follow by adapting some relevant theorem in [12]), we first write out equation (12) in the case where the solution hypersurfaces M_t are written as normal graphs over M_0 with globally bounded second fundamental form A_{M_0}. Not that we can introduce a global Gaussian normal coordinate system around M_0 in this case.

We can alternatively think of M_0 as being the zero level set in a foliation of a uniform tubular neighbourhood of M_0 given by the normal distance to M_0. There exists a global unit vectorfield ω perpendicular to the equidistant level sets which equals the unit normal to M_0 on M_0. Then we are in an analogous situation to the one considered in [2] chapter 2 although there a Lorentzian ambient manifold is considered in contrast to the Euclidean space in our case. This however will have no effect on our argument. If some other complete hypersurface M can be written as a normal graph over M_0 we can consider the 'height function' $\gamma = d|_M$ on M where d is the distance function to M_0 (in [2] this corresponds to the time function to M called t there). Since we are in Euclidean space, the function α in [2] corresponds to the constant function 1 in our setting (since $\alpha = |\nabla^{\mathbb{R}^{n+1}} d|^{-1}$ for us).

A formula for the mean curvature of M can be found on page 160 in [2] (γ is called u there). The particular form of the inverse metric (in our case we use g^{ij} instead of $\bar{g}^{ij}$ used in [2]) is not important, but note that it is C^∞ in $\partial\gamma$, the n-tuple of coordinate derivatives of γ. We do have to interchange all the minus signs to plus and vice versa however in [2] (2.19) to account for the fact that we are in the Riemannian setting. The relevant equation in our situation is

$$H_M = \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \, g^{ij}(\partial\gamma)\partial_i\partial_j\gamma - \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \, a(\partial\gamma, A_{M_0}) + \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle \, H_{M_0} \tag{13}$$

where a is a smooth expression in $\partial\gamma$ and A_{M_0}, and g^{ij} is the inverse metric on M. In [2], the factor $-\langle N, T \rangle$ called $\boldsymbol{\nu}$ there has to be changed to $+\langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle$ again due to the Lorentzian situation in [2]. $\boldsymbol{\nu}$ is the unit normal to M in our setting. If (M_t) is a solution of MCF then γ will evolve according to the equation

$$\frac{\partial\gamma}{\partial t} = -\langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle^{-1} H$$

(derived in a similar manner to chapter 4 equation (11)), which leads to

$$\frac{\partial\gamma}{\partial t} = g^{ij}(\partial\gamma)\partial_i\partial_j\gamma + a(\partial\gamma, A_{M_0}) - H_{M_0}. \tag{14}$$

Remark: 1) One could have alternatively worked with the immersions

$$\mathbf{F}_M(p) = \mathbf{F}_{M_0}(p) + \gamma\boldsymbol{\omega}$$

where $\boldsymbol{\omega}$ is the unit normal to M_0 and calculated H_M directly from this. One obtains a similar equation to the one above.

2) In what follows we could even afford an addition dependence of a and g^{ij} on γ.

Theorem 5.2. *Under the conditions of Theorem 5.1 the solution found there is unique.*

Proof. We employ a standard comparison principle argument. However, since we are in a non-compact situation we have to be careful. The solution found in Theorem 5.1 is a normal graph over M_0. Let γ, $\tilde{\gamma}$ both be solutions to (14), corresponding to two solutions M_t and $\tilde{M}_t$ of (12). Set $f = \gamma - \tilde{\gamma}$. We want to show that f satisfies an equation of the form considered in Theorem 3.1. We can then write

$$\begin{aligned}\frac{\partial f}{\partial t} &= g^{ij}(\partial\gamma)\partial_i\partial_j\gamma + a(\partial\gamma, A_{M_0}) - H_{M_0} - g^{ij}(\partial\tilde{\gamma})\partial_i\partial_j\tilde{\gamma} - a(\partial\tilde{\gamma}, A_{M_0}) + H_{M_0} \\ &= g^{ij}(\partial\gamma)\partial_i\partial_j f + \left(g^{ij}(\partial\gamma) - g^{ij}(\partial\tilde{\gamma})\right)\partial_i\partial_j\tilde{\gamma} + a(\partial\gamma, A_{M_0}) - a(\partial\tilde{\gamma}, A_{M_0}).\end{aligned}$$

Note that the terms $|\partial\gamma|$, $|\partial\tilde{\gamma}|$, $|\partial^2\gamma|$, $|\partial^2\tilde{\gamma}|$ and $|A_{M_0}|$ are all bounded since our solutions retain bounded geometry on their common interval of existence. We now write (calling y_k the arguments of g^{ij})

$$g^{ij}(\partial\gamma) - g^{ij}(\partial\tilde{\gamma}) = \int_0^1 \frac{d}{ds} g^{ij}(s\partial\gamma + (1-s)\partial\tilde{\gamma})ds$$

$$= \left(\int_0^1 \frac{\partial}{\partial y_k} g^{ij}(s\partial\gamma + (1-s)\partial\tilde{\gamma}) ds \right) \partial_k f.$$

Hence

$$\left(g^{ij}(\partial\gamma) - g^{ij}(\partial\tilde{\gamma}) \right) \partial_i \partial_j \tilde{\gamma} = c^k(\partial\gamma, \partial\tilde{\gamma}, \partial^2\tilde{\gamma}) \partial_k f$$

for a suitable expression c depending smoothly on $\partial\gamma$, $\partial\tilde{\gamma}$ and $\partial^2\tilde{\gamma}$ since $g^{ij} = g^{ij}(y)$ is C^∞ in its dependence on y.

Similarly,

$$a(\partial\gamma, A_{M_0}) - a(\partial\tilde{\gamma}, A_{M_0}) = \left(\int_0^1 \frac{\partial}{\partial \gamma_k} a(s\partial\gamma + (1-s)\partial\tilde{\gamma}, A_{M_0}) ds \right) \partial_k f$$

and thus f satisfies an equation of the form

$$\frac{\partial f}{\partial t} = g^{ij}(\partial\gamma)\partial_i\partial_j f + \tilde{b}^k \partial_k f$$

where $|\tilde{b}|$ depends on $\partial\gamma$, $\partial\tilde{\gamma}$ and $\partial^2\tilde{\gamma}$ and is therefore bounded due to the estimates on the geometry of (M_t) and $(\tilde{M_t})$. Noting that the Γ^i_{jk}, the Christoffel symbols with respect to the metric g_{ij}, can be controlled in terms of $\partial\gamma$ and $\partial^2\gamma$, and using the expression $\Delta_M f = g^{ij}(\partial_i\partial_j - \Gamma^k_{ij}\partial_k f)$ we finally arrive at

$$\frac{\partial f}{\partial t} = \Delta_{M_t} f + \left\langle \mathbf{b}, \nabla^{M_t} f \right\rangle$$

where $\mathbf{b}$ is bounded on M_t. To obtain $\mathbf{b}$ we have multiplied $\tilde{b}$ by an expression involving $g^{ij}(\partial\gamma)$. $\langle \cdot, \cdot \rangle$ denotes the metric on M_t.

Since $f = 0$ at $t = 0$, applying Theorem 3.1 to both f and $-f$ yields $f = 0$ on the common time interval of existence. □

Corollary 5.3. *If M_0 is bi-rotationally symmetric then so is the unique solution to (12).*

Proof. This follows easily from the bi-rotational invariance of M_0 combined with the uniqueness result of Theorem 5.2. □

Theorem 5.4. *Suppose that $M_0 \subset \{v > u\}$ and suppose that $H_{M_0} \leq 0$. Then $v_0 = \inf_{M_0} v > 0$ and M_t will satisfy*

$$M_t \subset \{v > u\}$$

$$\inf_{M_t} v \geq v_0 > 0$$

for all time $t \in [0, T_0)$.

Proof. A surface M_0 which is smooth and lies within $\{v > u\}$ will satisfy $\inf_{M_0} v \geq v_0 > 0$: Consider the set $M_0 \cap \bar{B}_R(0)$. Within this compact set v must obtain a minimum, or the set is empty. In the latter case M_0 will lie within the set $\{v > u\} \cap \mathbb{R}^8 \setminus (B_R(0))$, and within this set $v \geq \frac{R}{\sqrt{2}}$ as one easily checks from the definitions of u and v.

By writing M_t as a normal graph of a function γ over M_0 we obtain

$$\frac{\partial \gamma}{\partial t} = - \langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle^{-1} H$$

where $\boldsymbol{\omega}$ is the unit normal field to M_0. The mean curvature of M_t remains non-positive (see Lemma 6.2 in the following chapter) and therefore γ is non-decreasing in t at every point in space. Hence γ is bounded below by its initial value at every point in space. Since $M_0 \subset \{v > u\}$ and M_t lies above M_0 we also have that $M_t \subset \{v > u\}$. $\square$

6 Evolution equations

In order to understand the evolution of our graphical hypersurfaces, we must study the evolution of certain key quantities, In [6] the two quantities studied were the height function and the gradient. What constitutes 'height' in our case could be thought of in different ways.

The distance from the 'axis' v=0 gives one notion for height, a second, as studied in chapter 5 gives the perpendicular distance from the initial hypersurfaces as the height. A third notion for height defined in terms of the value of the level set function which gives us the foliations either for the Bombieri, Di Giorgi, Giusti minimal foliation ($F_1(u,v)=\varepsilon$) or the $F_2(u,v)=v^2-u^2=\varepsilon$ foliation.

Likewise what constitutes gradient could be thought of in several ways. Like in [6] a gradient of the form $\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-1}$ is desirable, but the V can be chosen in a number of different ways. Firstly we can take $\mathbf{V}=\frac{DF}{|DF|}$ where F is the level set function for either foliation. This approach did not work, but gave us some partial results, which are included in Appendix C. These gradient functions correspond to the height functions above.

Instead, the approach we take looks at the evolving hypersurfaces as a graph simply over the $v=0$ plane with $F=v$ our "height" and $\mathbf{V}=\hat{\mathbf{v}}$ our reference vector field, as seen in Chapter 4. In this setting we have $\nu_v^{-1}=\langle \boldsymbol{\nu}, \hat{\mathbf{v}}\rangle^{-1}=\sqrt{1+(h')^2}$. This mimics the work of [1],[17] studying cylindrically symmetric hypersurfaces, but with additional suprising results.

Since bi-rotationally symmetric hypersurfaces can be described in a two dimensional picture by taking their contour curve and rotating around each axis using two $SO(3)\times\{e_4\}$ actions, the contour curves themselves becomes the focus of study for a great deal of this thesis.

Lemma 6.1.

$$\left(\frac{\partial}{\partial t}-\Delta_M\right)h^i_j=|A|^2\,h^i_j$$

Proof. From [10] we have

$$\frac{\partial}{\partial t}g_{ij}=-2Hh_{ij}$$

$$\left(\frac{\partial}{\partial t}-\Delta_M\right)h_{ij}=-2Hh_{ij}g^{lm}h_{mj}+|A|^2\,h_{ij}$$

Thus

$$\frac{\partial}{\partial t}g^{ij}=2Hg^{il}h_{lm}g^{mj}$$

Combining these two results we get

$$\left(\frac{\partial}{\partial t}-\Delta_M\right)g^{ik}h_{kj}=-2Hg^{ik}h_{kl}g^{lm}h_{mj}+|A|^2\,g^{ik}h_{kj}+2Hg^{il}h_{lm}g^{mj}h_{kj}$$

since $\nabla^M g_{ij} = \Delta g_{ij} = 0$.

□

Lemma 6.2. *If H is initially non-positive everywhere it remains so for all time.*

Proof. Summing over the indicies in the above equation for A^i_j we get

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) H = |A|^2 H$$

which satisfies condition (i) of Theorem 3.1, as long as the curvature remains bounded. Furthermore, since we have $|\nabla^m A| \leq C_m$ for all $m \geq 0$ we get that $|\nabla H|$ is bounded and satisfies condition (iii), and thus we can apply the non-compact maximum principle to this quantity. □

We next wish to calculate the evolution of the gradient, but first we need the following preliminary calculation. We refer here to chapter 3 of [2].

We consider $M \subset \mathbb{R}^{n+1}$ to be a complete hypersurface and V to be a smooth vectorfield on $\mathbb{R}^{n+1}$ everywhere transverse to M which admits a flow ϕ_τ in a uniform tubular neighbourhood Ω of M (note that M has such a neighbourhood if it has bounded second fundamental form). i.e.

$$\phi : \Omega \times (-\varepsilon, \varepsilon) \mapsto \Omega,\ \left.\frac{\partial \phi}{\partial \tau}\right|_{\tau=0} = V,\ \phi_0(x) = x.$$

Then we easily see that

$$[\mathbf{V}, \mathbf{e}_i] = 0$$

on all M_τ, $\tau \in (-\varepsilon, \varepsilon)$ where e_i form the frame for M_τ. Here $[\cdot,\cdot]$ denotes the lie bracket of vector fields. We now define as in chapter 2 of [2]

$$\mathbf{V}(H_\mathbf{V}) := \left.\frac{\partial}{\partial \tau}\right|_{\tau=0} H(M_\tau). \tag{15}$$

The following proposition is identity 2.9 from Proposition 2.1 in [2] which can be understood as a second variation formula for M with respect to the variation V.

Proposition 6.3.

$$\Delta_M \langle \boldsymbol{\nu}, \mathbf{V}\rangle = \langle \boldsymbol{\nabla} H, \mathbf{V}\rangle - \langle \boldsymbol{\nu}, \mathbf{V}\rangle |A|^2 - \mathbf{V}(H_\mathbf{V}) \tag{16}$$

Proof. We proceed as in [2] but note that we are in the Riemannian setting, hence the different signs compared to the one in equation (2.19) stated there. The quantity $\nu = -\langle T, N\rangle$ in [2] corresponds to our $\langle \boldsymbol{\nu}, \mathbf{V}\rangle$. We are working in an adapted frame where $\Delta = \mathbf{e}_i\mathbf{e}_i$, and we can write $\mathbf{V}(H_\mathbf{V}) = \mathbf{V}(g^{ij}h_{ij})$ where $\mathbf{V}(g^{ij}) = -\mathbf{V}(g_{ij})$ is for instance the variation of g^{ij} with respect to V

$$\begin{aligned}
\Delta\langle \boldsymbol{\nu}, \mathbf{V}\rangle &= \mathbf{e}_i\mathbf{e}_i\langle \boldsymbol{\nu}, \mathbf{V}\rangle \\
&= \mathbf{e}_i\left(\langle D_{\mathbf{e}_i}\boldsymbol{\nu}, \mathbf{V}\rangle + \langle \boldsymbol{\nu}, D_{\mathbf{e}_i}\mathbf{V}\rangle\right) \\
&= \langle D_{\mathbf{e}_i}D_{\mathbf{e}_i}\boldsymbol{\nu}, \mathbf{V}\rangle + 2\langle D_{\mathbf{e}_i}\boldsymbol{\nu}, D_{\mathbf{e}_i}\mathbf{V}\rangle + \langle \boldsymbol{\nu}, D_{\mathbf{e}_i}D_{\mathbf{e}_i}\mathbf{V}\rangle \\
&= \left\langle \boldsymbol{\nabla}H - |A|^2\boldsymbol{\nu}, \mathbf{V}\right\rangle + 2h_{ij}\langle \mathbf{e}_j, D_{\mathbf{e}_i}\mathbf{V}\rangle + \langle \boldsymbol{\nu}, D_{\mathbf{e}_i}D_{\mathbf{e}_i}\mathbf{V}\rangle \\
&= \langle \boldsymbol{\nabla}H, \mathbf{V}\rangle - \langle \boldsymbol{\nu}, \mathbf{V}\rangle|A|^2 + h_{ij}V(g_{ij}) + \langle \boldsymbol{\nu}, D_V D_{\mathbf{e}_i}\mathbf{e_i}\rangle \\
&= \langle \boldsymbol{\nabla}H, \mathbf{V}\rangle - \langle \boldsymbol{\nu}, \mathbf{V}\rangle|A|^2 - h_{ij}V(g^{ij}) + V(\langle \boldsymbol{\nu}, D_{\mathbf{e}_i}\mathbf{e_i}\rangle) - \langle D_\mathbf{V}\boldsymbol{\nu}, D_{\mathbf{e}_i}\mathbf{e_i}\rangle \\
&= \langle \boldsymbol{\nabla}H, \mathbf{V}\rangle - \langle \boldsymbol{\nu}, \mathbf{V}\rangle|A|^2 - h_{ij}V(g^{ij}) - g^{ij}V(h_{ij}) \\
&= \langle \boldsymbol{\nabla}H, \mathbf{V}\rangle - \langle \boldsymbol{\nu}, \mathbf{V}\rangle\,|A|^2 - \mathbf{V}(H_\mathbf{V}).
\end{aligned}$$

□

Corollary 6.4.

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \boldsymbol{\nu}, \mathbf{V}\rangle = \langle \boldsymbol{\nu}, \mathbf{V}\rangle\,|A|^2 + \mathbf{V}(H_\mathbf{V}) - H\left\langle \boldsymbol{\nu}, \frac{\partial \mathbf{V}}{\partial x_\alpha}\boldsymbol{\nu}_\alpha\right\rangle$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-1} = -2\langle \boldsymbol{\nu}, \mathbf{V}\rangle|\nabla(\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-1})|^2 - v|A|^2 - \langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-2}\left(\mathbf{V}(H_\mathbf{V}) - H\left\langle \boldsymbol{\nu}, \frac{\partial \mathbf{V}}{\partial x_\alpha}\boldsymbol{\nu}_\alpha\right\rangle\right)$$

Proof.

$$\begin{aligned}
\frac{\partial}{\partial t}\langle \boldsymbol{\nu}, \mathbf{V}\rangle &= \left\langle \frac{\partial \boldsymbol{\nu}}{\partial t}, \mathbf{V}\right\rangle + \left\langle \boldsymbol{\nu}, \frac{\partial \mathbf{V}}{\partial t}\right\rangle \\
&= \langle \nabla H, \mathbf{V}\rangle + \left\langle \boldsymbol{\nu}, \frac{\partial \mathbf{V}}{\partial x_\alpha}\frac{\partial x_\alpha}{\partial t}\right\rangle \\
&= \langle \nabla H, \mathbf{V}\rangle - H\left\langle \boldsymbol{\nu}, \frac{\partial \mathbf{V}}{\partial x_\alpha}\nu_\alpha\right\rangle
\end{aligned}$$

Combining this with Proposition 6.3 gives us our first result.

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-1} = -\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-2}\left(\frac{\partial}{\partial t} - \Delta_M\right)\langle \boldsymbol{\nu}, \mathbf{V}\rangle - 2\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-3}|\nabla\langle \boldsymbol{\nu}, \mathbf{V}\rangle|^2$$

Gives us our second result since $|\nabla\langle \boldsymbol{\nu}, \mathbf{V}\rangle|^2 = \langle \boldsymbol{\nu}, \mathbf{V}\rangle^4\,|\nabla(\langle \boldsymbol{\nu}, \mathbf{V}\rangle^{-1})|^2$ □

Since the gradient we wish to study is $\langle \boldsymbol{\nu}, \hat{\mathbf{v}} \rangle^{-1}$ we first study the evolution of $\nu_v = \langle \boldsymbol{\nu}, \hat{\mathbf{v}} \rangle$ to get some interesting results about the direction in which the normal points.

Lemma 6.5. *If M_t is bi-rotationally symmetric, we have*

$$\hat{\mathbf{v}}(H_{\hat{\mathbf{v}}}) = -3\frac{\nu_v}{v^2}.$$

Proof. We define $\delta_v^{\alpha\beta} = \begin{cases} 1 & \alpha = \beta,\ 5 \le \alpha, \beta \le 8 \\ 0 & \text{otherwise} \end{cases}$ and $\delta_u^{\alpha\beta}$ similarly, and write vectors as $\hat{\mathbf{v}} = \hat{v}^\alpha \mathbf{e}_\alpha$ where greek indices take the values 1 to 8.

$$\begin{aligned}
\hat{\mathbf{v}}(H_{\hat{\mathbf{v}}}) &= \hat{\mathbf{v}}\left(g^{ij}h_{ij}\right) = -\hat{\mathbf{v}}\left\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j \right\rangle h_{ij} - \hat{\mathbf{v}}\left\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i \right\rangle \\
&= -\hat{\mathbf{v}}\left\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j \right\rangle h_{ij} - \hat{\mathbf{v}}\left\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i \right\rangle \\
&= -2\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\hat{\mathbf{v}} \right\rangle h_{ij} - \left\langle \boldsymbol{\nu}, \mathrm{D}_{\hat{\mathbf{v}}}\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i \right\rangle - \left\langle \mathrm{D}_{\hat{\mathbf{v}}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i \right\rangle \\
&= -2\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\hat{\mathbf{v}} \right\rangle h_{ij} - \left\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\mathrm{D}_{\boldsymbol{\tau}_i}\hat{\mathbf{v}} \right\rangle - \left\langle \mathrm{D}_{\hat{\mathbf{v}}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i \right\rangle
\end{aligned}$$

Where we have used that $\hat{\mathbf{v}}(g^{ij}) = -\hat{\mathbf{v}}(g_{ij})$ and the commutative properties shown in Lemma 6.3.

$$(D_{\boldsymbol{\tau}_i}\hat{\mathbf{v}})^\alpha = \left(\frac{\partial \hat{\mathbf{v}}}{\partial x_\beta}\right)^\alpha (\tau_i)_\beta = \frac{\delta_v^{\alpha\beta} - \hat{v}^\alpha \hat{v}^\beta}{u}(\tau_i)_\beta = \begin{cases} 0 & 1 < i < 3 \\ \frac{\tau_i^\alpha}{u} & 4 < i < 6 \\ 0 & 7 \end{cases}$$

Thus $\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\hat{\mathbf{v}} \right\rangle = \frac{1}{v}$ for $4 < i = j < 6$ and 0 otherwise.

$D_{\boldsymbol{\tau}_i}D_{\boldsymbol{\tau}_i}\hat{\mathbf{v}} = D_{\boldsymbol{\tau}_i}\frac{\boldsymbol{\tau}_i}{v} = \frac{D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i}{v} - \frac{\boldsymbol{\tau}_i\langle\boldsymbol{\tau}_i, \hat{\mathbf{v}}\rangle}{v^2} = -\frac{\nu_v}{v^2}$ for $4 < i = j < 6$ and 0 otherwise.

Finally $D_{\hat{\mathbf{v}}}\boldsymbol{\nu} = 0$ and thus

$$\hat{\mathbf{v}}(H) = -3\frac{\nu_v}{v^2}.$$

□

Lemma 6.6. *If M_t is bi-rotationally symmetric, we have*

$$\left\langle \boldsymbol{\nu}, \frac{\partial \hat{\mathbf{v}}}{\partial x_\alpha}\nu_\alpha \right\rangle = 0.$$

Proof. $\frac{\partial \hat{\mathbf{v}}}{\partial x_\alpha}\nu_\alpha = \frac{\delta_v^{\alpha\beta} - \hat{v}^\alpha\hat{v}^\beta}{v}\nu_\alpha \mathbf{e}_\beta$

$\delta_v^{\alpha\beta}\nu_\alpha \mathbf{e}_\beta = \nu_v \hat{\mathbf{v}}$ since $\delta_v^{\alpha\beta}$ picks out the last four entries of $\boldsymbol{\nu}$.

$\hat{v}^\alpha\hat{v}^\beta\nu_\alpha\mathbf{e}_\beta = \hat{v}^\alpha\nu_\alpha\hat{\mathbf{v}} = \langle \hat{\mathbf{v}}, \boldsymbol{\nu}\rangle \hat{\mathbf{v}} = \nu_v\hat{\mathbf{v}}$ and thus we get the result. □

Combining these three lemmas gives us the following.

Lemma 6.7. *If M_t is bi-rotationally symmetric, we have*

$$\left(\frac{\partial}{\partial t} - \Delta_M\right)\nu_v = \nu_v\left(|A|^2 - 3\frac{1}{v^2}\right). \tag{17}$$

Lemma 6.8. *Suppose M_0 satisfies the conditions of Theorem 5.4 i.e. $H_{M_0} \leq 0$ and $M_0 \subset \{v > u\}$. Suppose furthermore that* $\inf_{M_0} \nu_v \geq C_1 > 0$ *and assume that M_0 is bi-rotationally symmetric. Then our unique bi-rotationally solution (M_t) satisfies $\nu_v \geq C_1$ on M_t for all $t \in (0, T_0)$.*

Proof. Theorem 5.1 implies that $|A|^2$ is uniformly bounded on $[0, T_0)$ while by Theorem 5.4 v is uniformly bounded below on M_t. Thus the coefficient function $|A|^2 - 3\frac{1}{v^2}$ is uniformly bounded. Therefore we can apply the non-compact maximum principle (Theorem 3.1) to $f = -\nu_v$, which is initially non-positive, to conclude that $\nu_v \geq 0$ for all time. We next write out the $|A|^2$ in full (see equation (10) in chapter 4) to obtain

$$\begin{aligned}\left(\frac{\partial}{\partial t} - \Delta_M\right)\nu_v &= \nu_v\left(3\frac{\nu_u^2}{u^2} + 3\frac{\nu_v^2}{v^2} - 3\frac{1}{v^2} + A_7^{7\,2}\right)\\ &= \nu_v\left(3\frac{\nu_u^2}{u^2} - 3\frac{\nu_u^2}{v^2} + A_7^{7\,2}\right)\\ &\geq 0.\end{aligned}$$

where we have used $\nu_u^2 = 1 - \nu_v^2$ and our result that $v > u$ (Theorem 5.4) on our evolving hypersurfaces, and we can again apply Theorem 3.1, this time to the function $f = C_1 - \nu_v$, to give us our result.

□

If in addition we have $\nu_v \geq \frac{1}{\sqrt{2}}$ on M_0 and therefore on M_t then the following corollary holds.

Corollary 6.9. *Suppose the conditions of Lemma 6.8 hold. If in addition* $\inf_{M_0} \nu_v = \frac{1}{\sqrt{2}}$ *(which gives that M_0 is starshaped - see below) then on M_t we have*

(i) $\nu_v \geq |\nu_u|$,

(ii) M_t remains a normal graph over M_0

(iii) $\langle \mathbf{x}, \boldsymbol{\nu} \rangle > 0$.

Proof. If $\nu_v \geq \frac{1}{\sqrt{2}}$ then we have $|\nu_u| = \sqrt{1 - \nu_v^2} \leq \frac{1}{\sqrt{2}}$, so $\nu_v \geq |\nu_u|$.

This inequality is true on both M_0 and M_t. We therefore have

$$\langle \boldsymbol{\nu}_{M_0}, \boldsymbol{\nu}_{M_t} \rangle = \nu_{v_{M_0}} \nu_{v_{M_t}} + \nu_{u_{M_0}} \nu_{u_{M_t}}$$

ν_v is positive for all time and the second term is smaller in magnitude than the first. Thus the inner product is always positive.

Futhermore since $v > u$ we have $v\nu_v > |u\nu_u|$ so we can write

$$\langle \mathbf{x}, \boldsymbol{\nu} \rangle = \langle u\hat{\mathbf{u}} + v\hat{\mathbf{v}}, \nu_u\hat{\mathbf{u}} + \nu_v\hat{\mathbf{v}} \rangle = u\nu_u + v\nu_v > 0.$$

□

So not only does our surface remain a graph in our sense (i.e. $\nu_v > 0$) it also remains a graph in both the sense of normal graphs over M_0 and in the sense of radial graphs, albeit the latter with an unbounded gradient.

7 Curvature estimates

Having established a lower bound for ν_v for all time we can now use this to derive a curvature bound for all time.

Theorem 7.1. *If M_0 satisfies the conditions of Lemma 6.8 (required for our gradient bound) and of the short time existence Theorem 5.1 then there exists $C_2 = C_2(\inf_{M_0} \nu_v, \inf_{M_0} v, \sup_{M_0} |A|^2)$ such that*

$$\sup_{M_t} |A|^2 \leq C_2$$

for all $t \in (0, T_0)$.

Proof. Theorems 5.4 and 6.8 give us a lower bound on v and an upper bound for ν_v^{-1} respectively for all $t \in (0, T_0)$. Theorem 5.1 tells us there is some bound on $|\nabla^m A|$ for all $m \geq 0$ for a short time at least, which allows us to apply the non-compact maximum principle to g considered below at the end of the proof.

We choose the same test function as for Theorem 3.1 in [7]. Let

$$\phi(r) = \frac{r}{1 - kr}$$

Note that if $1 \leq r \leq a$ and $k < \frac{1}{2a}$ then $1 \leq \phi(r) \leq 2a$

$$\phi'(r) = \frac{1}{1 - kr} + \frac{kr}{(1 - kr)^2} = \frac{1}{(1 - kr)^2}$$

$$\phi''(r) = \frac{2k}{(1 - kr)^3}$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) \nu_v^{-2} = 2\nu_v^{-2} \left(-|A|^2 + \frac{3}{v^2}\right) - 6 \left|\nabla \nu_v^- 2\right|^2$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |A|^2 \leq -2 \left|\nabla |A|\right|^2 + 2|A|^4$$

Let $g = \phi(\nu_v^{-2})\,|A|^2$. Choosing $k < \frac{\sup_{M_0}\nu_v^2}{2}$ we obtain

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_M\right) g &= \phi\left(\frac{\partial}{\partial t} - \Delta_M\right)|A|^2 + |A|^2\,\phi'\left(\frac{\partial}{\partial t} - \Delta_M\right)\nu_v^{-2} \\
&\quad -2\nabla\phi\cdot\nabla\,|A|^2 - \phi''\,|A|^2\left|\nabla\nu_v^{-2}\right|^2 \\
&\leq \phi\left(-2\,|\nabla|A||^2 + 2|A|^4\right) - 2\left(\frac{1}{1-k\nu_v^{-2}} + \frac{k\nu_v^{-2}}{\left(1-k\nu_v^{-2}\right)^2}\right)\nu_v^{-2}|A|^4 \\
&\quad \frac{|A|^2}{\left(1-k\nu_v^{-2}\right)^2}\left(2\nu_v^{-2}\left(\frac{3}{v^2}\right) - 6\left|\nabla{\nu_v}^{-1}\right|^2\right) - 2\nabla\phi\cdot\nabla\,|A|^2 - 4\nu_v^{-2}\phi''\,|A|^2\left|\nabla\nu_v^{-1}\right|^2 \\
&= -2kg^2 + \frac{|A|^2}{\left(1-k\nu_v^{-2}\right)^2}\left(2\nu_v^{-2}\left(\frac{3}{v^2}\right) - 6\left|\nabla\nu_v^{-1}\right|^2\right) \\
&\quad -2\nabla\phi\cdot\nabla|A|^2 - \frac{8\,|A|^2\,\nu_v^{-2}k}{\left(1-k\nu_v^{-2}\right)^3}\left|\nabla\nu_v^{-1}\right|^2 - 2\phi\,|\nabla|A||^2 \\
&\leq -2kg^2 + \frac{2\,|A|^2\,\nu_v^{-2}}{\left(1-k\nu_v^{-2}\right)^2}\frac{3}{v^2} - 2\nabla\phi\cdot\nabla|A|^2 \\
&\quad -\frac{6}{\left(1-k\nu_v^{-2}\right)^3}\left|\nabla\nu_v^{-1}\right|^2|A|^2 - 2\phi\,|\nabla|A||^2
\end{aligned}
$$

Young's inequality can be used to simplify the second term.

$$
\frac{2\nu_v^{-2}\,|A|^2}{\left(1-k\nu_v^{-2}\right)^2}\frac{3}{v^2} \leq \varepsilon^2 g^2 + \frac{9}{v^4\varepsilon^2\left(1-k\nu_v^{-2}\right)^2}
$$

where we take $\varepsilon^2 \leq k$.

The third term can be estimated similarly:

$$
\begin{aligned}
-2\nabla\phi\cdot\nabla|A|^2 &= -\nabla\phi\cdot\nabla|A|^2 - 4|A|\nu_v^{-1}\phi'\nabla\nu_v^{-1}\cdot\nabla|A| \\
&= \phi^{-1}\nabla\phi\cdot\nabla g + \phi^{-1}\,|\nabla\phi|^2\,|A|^2 \\
&\quad -4|A|\nu_v^{-1}\phi'\nabla\nu_v^{-1}\cdot\nabla|A| \\
&\leq \frac{2\nu_v^{-1}\phi'}{\phi}\nabla\nu_v^{-1}\cdot\nabla g + \frac{4\nu_v^{-2}\,(\phi')^2}{\phi}\left|\nabla\nu_v^{-1}\right|^2|A|^2 \\
&\quad +\frac{2\nu_v^{-2}\,(\phi')^2}{\phi}\left|\nabla\nu_v^{-1}\right|^2|A|^2 + 2\phi\,|\nabla|A||^2 \\
&= \frac{2}{\nu_v^{-1}\left(1-k\nu_v^{-2}\right)}\nabla\nu_v^{-1}\cdot\nabla g + \frac{6}{\left(1-k\nu_v^{-2}\right)^3}\left|\nabla\nu_v^{-1}\right|^2|A|^2 + 2\phi\,|\nabla|A||^2
\end{aligned}
$$

Combining these results we get the following.

$$\left(\frac{\partial}{\partial t}-\Delta_M\right)g \leq \frac{2}{\nu_v^{-1}\left(1-k\nu_v^{-2}\right)}\nabla\nu_v^{-1}\cdot\nabla g-\left(2k-\varepsilon^2\right)g^2+\frac{9}{v^4\varepsilon^2\left(1-k\nu_v^{-2}\right)^2}$$
$$\leq \frac{2}{\nu_v^{-1}\left(1-k\nu_v^{-2}\right)}\nabla\nu_v^{-1}\cdot\nabla g-\left(2k-\varepsilon^2\right)g^2+\frac{9}{v_0^4\varepsilon^2\left(1-2k\right)^2}$$

where v_0 is a lower bound for v. Note we have (chapter 4)

$$\left|\frac{\nabla\nu_v^{-1}}{\nu_v^{-1}\left(1-k\nu_v^{-2}\right)}\right| = \left|\frac{\nu_v^{-1}\nabla\nu_v}{\left(1-k\nu_v^{-2}\right)}\right| \leq \left|\frac{\nu_u A_7^7}{\nu_v\left(1-k\nu_v^{-2}\right)}\right| \leq CA_7^7 \leq C|A|$$

The function g satisfies the inequality of the form required for f in Corollary 3.2(ii), since $\phi(\nu_v^{-2}) \geq 1$. This gives us that $\sup_{M_t}|A|^2$ will be bounded by a value depending on $\sup_{M_0}|A|$, $\sup_{M_0}\nu_v$ and v_0^{-1}.

□

8 Derivatives of curvature

The following is taken from [5] Chapter 3 and [7] section 3.4.

Theorem 8.1. *Suppose our initial hypersurface M_0 satisfies the conditions of Theorem 7.1 and in addition M_0 has a bound on $|\nabla^m A|$ for all $m \geq 0$, then our evolving hypersurfaces maintain a bound on $|\nabla^m A|$ for all $m \geq 0$ for all time.*

Proof. This is a proof for the $m = 1$ case, the proof for the higher order cases are contained in appendix A. Let

$$f_1 = |\nabla A|^2 \left(\Lambda_0 + |A|^2\right)$$

$$f_m = |\nabla^m A|^2 \left(\Lambda_0 + |\nabla^{m-1} A|^2\right)$$

We shall use

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |\nabla A|^2 \leq -2|\nabla^2 A|^2 + C(n)|\nabla A|^2 \, |A|^2$$

from [7]. We then calculate

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \leq -2|\nabla^2 A|^2 \left(\Lambda_0 + |A|^2\right) + C(n)|\nabla A|^2|A|^2 \left(\Lambda_0 + |A|^2\right)$$

$$-2|\nabla A|^4 + |A|^2|\nabla A|^2 - 2\nabla|\nabla A|^2 \cdot \nabla \, |A|^2$$

The last term can be estimated by

$$2\nabla|\nabla A|^2 \cdot \nabla \, |A|^2 \leq 8|A| \, |\nabla A| \, |\nabla|\nabla A|| \, |\nabla|A|| \leq 8|A| \, |\nabla A|^2 \, |\nabla^2 A|$$

$$\leq 2|\nabla^2 A| \left(\Lambda_0 + |A|^2\right) + \frac{8\,|A|^2}{\Lambda_0 + |A|^2}|\nabla A|^4$$

where we have used Kato's inequality

$$|\nabla|\nabla A|| \leq |\nabla^2 A|$$

and $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2$ to get the result.

This gives us

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \leq C(n)|\nabla A|^2|A|^2 \left(\Lambda_0 + |A|^2\right)$$

$$-\left(2 - \frac{8\,|A|^2}{\Lambda_0 + |A|^2}\right) |\nabla A|^4 + |A|^2|\nabla A|^2$$

Since $|A|^2 \le c_0$ we must also have that

$$\frac{|A|^2}{\Lambda_0 + |A|^2} \le \frac{c_0}{\Lambda_0 + c_0}$$

The inequality becomes

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \le -\left(2 - \frac{8c_0}{\Lambda_0 + c_0}\right) |\nabla A|^4 + C(n, c_0, \Lambda_0)|\nabla A|^2$$

Choosing Λ_0 so that $2 - \frac{8c_0}{\Lambda_0+c_0} = \frac{3}{2}$ and estimating

$$C(n, c_0, \Lambda_0)|\nabla A|^2 \le \frac{1}{2}|\nabla A|^4 + K$$

we get

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \le -|\nabla A|^4 + K$$

but since we have

$$|\nabla A|^4 \ge \frac{f^2}{(\Lambda_0 + c_0)^2}$$

we get

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f \le -\delta f^2 + K$$

where both δ and K depend only on n and c_0.

Thus we get that f and thus $|\nabla A|$ will become bounded at $t > \varepsilon$ even if it is initially unbounded. □

The higher derivative bounds follow the same argument. This is a strong induction argument which requires all lower derivatives of A to be bounded after some small time.

9 Long time existence & convergence

Theorem 9.1. *Under the conditions of Theorem 8.1 the short time solution of Theorem 5.1 can be extended to a smooth solution for all time.*

Proof. The conditions allow us to reapply all estimates of the previous chapters at time $\frac{T_o}{2}$ say to restart the flow using Theorem 5.1. The smooth transition between the solution starting at $t = 0$ and this new one is ensured by the uniqueness Theorem 5.2. □

Theorem 9.2. *Suppose that in addition to the conditions of the last theorem M_0 lies below an area-minimising hypersurface M^+ of the foliation in [3] which in turn is contained inside the set $\{v > u\}$, and that M^+ can be written as a normal graph over M_0. Suppose also that $\nu_v \geq \frac{1}{\sqrt{2}}$ on M_0. Then the flow M_t remains below M^+.*

Proof. We know from Corollary 6.9 that under the condition $\inf_{M_0} \nu_v = \frac{1}{\sqrt{2}}$, M_t can be written as normal graph over M_0, as can M^+ by our assumption. Thus we can apply the same arguments used in Theorem 5.2 to the function $f = \gamma - \gamma^+$ where γ, γ^+ denote the normal height functions for M_t and M^+ respectively. Note that γ^+ satisfies equation (14) in chapter 5, but with zero time derivative. f satisfies the assumptions of Theorem 3.1 since M^+, being an area minimising hypersurface from [3], has by [8] globally bounded geometry, i.e. all derivatives of γ^+ are globally bounded. Since $f \leq 0$ at time $t = 0$ this inequality is maintained and therefore M_t stays below M^+ □

Theorem 9.3. *Under the conditions of Theorem 9.2, in addition to the conditions stated in Theorem 9.1, the flow M_t has a smooth limit for $t \to \infty$.*

The limiting hypersurface M_∞ is smooth and starshaped.

Proof. Consider the function $h(\cdot, t)$ as defined in chapter 4. Taking an arbitrary u-value we get

$$h(u,t) - h(u,t_0) = \int_{\tau_0}^{\tau} \frac{\partial h}{\partial \tau}(u,\tau) d\tau = -\int_{\tau_0}^{\tau} \frac{H(u,\tau)}{\nu_v} d\tau.$$

from equation (11) in chapter 4.

Since $H \leq 0$ and the surface lies beneath M^+, we have that this is a bounded monotone function in t and thus must converge pointwise for all u values as $t \to \infty$. Let us call this limiting function $h(\cdot, \infty)$. Rotating this function around both axes gives us a limiting hypersurface M_∞.

We have (by Theorems 7.1 & 8.1) that there exists constants C_m such that each surface M_t has $||A||_{C^m} \leq C_m < \infty$ and thus we have (chapter 4) that $||h'||_{C^{m+1}} \leq \tilde{C}_m$ for all $m \geq 0$. We can, by standard arguments, conclude that $h(u,t)$ converges in C^∞ on compact subsets of the u-axis and thus the curvatures of the limiting hypersurface are also bounded. In particular this gives us that the mean curvature of the surfaces converge in C^∞ uniformly on compact subsets of the u-axis.

M_∞ can be shown to be starshaped using the same arguments in Corollary 6.9. □

Theorem 9.4. *Under the conditions described above, the limiting hypersurface is minimal, that is it has $H \equiv 0$.*

Proof. Writing

$$\int_{t_0}^{\infty} \frac{|H(u,t)|}{\nu_v} dt = h(u,\infty) - h(u,t_0)$$

and using the existence of the two minimal barrier surfaces, we have $h(u,\infty) - h(u,t_0)$ bounded for all u in any compact interval $[a,b]$ and so

$$\int_{t_0}^{\infty}\int_a^b |H(u,t)| du dt \le \int_a^b \int_{t_0}^{\infty} \frac{|H(u,t)|}{\nu_v} dt du = \int_a^b \int_{t_0}^{\infty} \frac{\partial h}{\partial t} dt du$$

$$= \int_a^b h(u,\infty) - h(u,t_0) du < C(a,b) < \infty$$

Since $\frac{\partial H}{\partial t} = \Delta H + H\,|A|^2$ and $|\Delta H| \le C|\nabla^2 A|$ we have a global bound on $|\frac{\partial H}{\partial t}|$ depending on the bounds on $|A|^2$ and $|\nabla^2 A|$. We thus get that

$$\frac{\partial}{\partial t}\int_a^b |H(u,t)| du \le \int_a^b \left|\frac{\partial}{\partial t} H(u,t)\right| du < C$$

with this constant depending on a, b and bounds on $|A|^2$ and $|\nabla^2 A|$.

These two statements tell us that

$$\lim_{t\to\infty} \int_a^b |H(u,t)| du = 0$$

by a standard calculus lemma, since the mean curvature converges uniformly on compact subsets we obtain

$$\int_a^b |H(u,\infty)| du = \int_a^b \lim_{t\to\infty} |H(u,t)| du = \lim_{t\to\infty} \int_a^b |H(u,t)| du = 0$$

for any interval (a,b). So $H_{M_\infty} \equiv 0$

□

Theorem 9.5. *Under the conditions described above the limiting surface M_∞ is area minimising.*

Proof. [16] tells us that the class of minimal surfaces which are asymptotic to the Simons cone is unique up to diffeomorphism, and thus our surface must be one of those surfaces, and thus is area minimising. □

A Higher derivatives of the curvature

The argument for the higher derivatives of $|A|^2$ is contained here. They follow the same argument.

$$f_m = |\nabla^m A|^2 \left(\Lambda_0 + |\nabla^{m-1} A|^2\right)$$

From [7] we have that

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + C(m,n) \sum_{i+j+k=m} |\nabla^m A||\nabla^i A||\nabla^j A||\nabla^k A|$$

In particular if we have $|\nabla^i A|^2 \leq c_0$ for all $i \leq m-1$ we get from Young's inequality

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |\nabla^m A|^2 \leq -2|\nabla^{m+1} A|^2 + C(m,n,c_0)\left(1 + |\nabla^m A|^2\right)$$

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) |\nabla^{m-1} A|^2 \leq -2|\nabla^m A|^2 + C(m,n,c_0)$$

Thus

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f_m \leq -2|\nabla^{m+1} A|^2 \left(\Lambda_0 + |\nabla^{m-1} A|^2\right) + C(m,n,c_0)\left(1 + |\nabla^m A|^2\right)\left(\Lambda_0 + |\nabla^{m-1} A|^2\right)$$

$$-2|\nabla^m A|^4 + C(m,n,c_0)|\nabla^m A|^2 - 2\nabla\left(|\nabla^m A|^2\right) \cdot \nabla\left(|\nabla^{m-1} A|^2\right)$$

Again the final term can be estimated with

$$2\nabla\left(|\nabla^m A|^2\right) \cdot \nabla\left(|\nabla^{m-1} A|^2\right) \leq 8|\nabla^{m-1} A|\, |\nabla^m A|\, |\nabla|\nabla^m A||\, |\nabla|\nabla^{m-1} A||$$

$$\leq 8|\nabla^{m-1} A||\nabla^m A|^2|\nabla^{m+1} A| \leq 2|\nabla^{m+1} A|^2 \left(\Lambda_0 + |\nabla^{m-1} A|^2\right) + 8\frac{|\nabla^{m-1} A|^2}{\Lambda_0 + |\nabla^{m-1} A|^2}|\nabla^m A|^4$$

again using Kato's and Young's inequality.

Using that $|\nabla^p A| \leq c_0$ for $p \leq m-1$ and gathering terms we get.

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f_m \leq C(m,n,c_0)\left(1 + |\nabla^m A|^2\right) - \left(2 - 8\frac{|\nabla^{m-1} A|^2}{\Lambda_0 + |\nabla^{m-1} A|^2}\right)|\nabla^m A|^4$$

Choosing Λ_0 so that the coefficient of $|\nabla^m A|^4$ is $-\frac{3}{2}$ and using Young's inequality

$$C(m,n,c_0)|\nabla^m A|^2 \leq \frac{1}{2}|\nabla^m A|^2 + K(m,n,c_0)$$

we get.

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) f_m \leq -|\nabla^m A|^4 + K \leq -\frac{f_m^2}{(\Lambda + c_0)^2} + K$$

Our strong induction argument assumes we have $|\nabla^i A|^2 \leq c_0$ for all $i \leq m-1$ for all time. Corollary 3.2 then tells us that $|\nabla^m A|$ will be bounded.

B A class of quadratic cones

Take $y \in \mathbb{R}^{p+1}$ and $z \in \mathbb{R}^{q+1}$. Let $u = |y|$, $v = |z|$ and $n = p+q+1$ with $n \geq 8$ or $n = 7$ & $|p-q| \leq 4$. Then the surface defined by

$$qu^2 = pv^2$$

is an n-dimensional minimal cone [14] with $pv^2 > qu^2$ foliated by minimal surfaces [16]. The long term existence result (Theorem 9.1) holds for all of these cones, and the remaining theorems in Chapter 9 hold for cones where $p \geq q$

A curve in the u, v plane can be rotated around both axes to give a bi-rotationally symmetric surface as shown in chapter 4. The main difference is the mean curvature

$$H = p\frac{\nu_u}{u} + q\frac{\nu_v}{v} + A_n^n$$

Most of the thesis follows through with requisite changes to indicies and dimension. Due to the change in the angle of the cone, the lower bound for v in Theorem 5.4 in the set $\{v > u\} \cap \mathbb{R}^8 \setminus (B_R(0))$ would now be $\frac{\sqrt{q}R}{\sqrt{p+q}}$. The major change is in Theorem 6.8. Firstly equation 17 of chapter 6 becomes

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) \nu_v = \nu_v \left(|A|^2 - q\frac{1}{v^2}\right)$$

and the proof of Theorem 6.8 is

Proof.

$$\begin{aligned}
\left(\frac{\partial}{\partial t} - \Delta_M\right) \nu_v &= \nu_v \left(p\frac{\nu_u^2}{u^2} + q\frac{\nu_v^2}{v^2} - q\frac{1}{v^2} + (A_n^n)^2\right) \\
&= \nu_v \left(\nu_u^2 \left(\frac{p}{u^2} - \frac{q}{v^2}\right) + (A_n^n)^2\right) \\
&\geq 0
\end{aligned}$$

because we are on the side of the cone where $pv^2 > qu^2$ □

Since the cone would have a different angle to the one in the main body of this thesis we now may require in addition that

$$\nu_v \geq \frac{\sqrt{p}}{\sqrt{p+q}}$$

initially, so Corollary 6.9 will now tell us that $\sqrt{q}\nu_v \geq \sqrt{p}|\nu_u|$ giving us along with $\sqrt{p}v > \sqrt{q}u$

$$\langle \mathbf{x}, \boldsymbol{\nu} \rangle = u\nu_u + v\nu_v \geq \frac{\sqrt{q}u\nu_u + \sqrt{p}v|\nu_u|}{q} \geq 0$$

and thus under this extra condition on M_0 we get that the evolving hypersurface stays star shaped.

Furthermore, part (ii) of Theorem 6.9 and subsequently Theorem 9.2 only hold in the case $p \geq q$.

C Additional gradient estimates

The following is an additional result with no bearing on the main body of the thesis. Due to its length the reader may choose not to read it. It contains an approach that was taken initially and abandoned, which gives enough to get a long term existence result, but not convergence.

My initial (and unsuccessful) approach was to proceed in analogy with the settings taken in [6]. There we can consider Euclidean space to be foliated by the level sets of $F = x_{n+1}$ with the vector field $V = e_{n+1}$ forming the normal vectors to these level sets. We can consider a non-compact surface bounded between two minimal surfaces with height given by $F|_{M_t}$ and gradient $\langle \boldsymbol{\nu}, V \rangle^{-1}$ and see that the surfaces will flow to one of the leaves of this foliation.

In our setting we again have Euclidean space foliated by minimal surfaces, expressed as some level sets of a function F as given in [3] with the vector field of normals $\boldsymbol{\omega} = \frac{DF}{|DF|}$. We take that height to be $F|_{M_t}$ and gradient $\langle \boldsymbol{\nu}, \boldsymbol{\omega} \rangle^{-1}$. We expect the height to be bounded by its initial maximum and expect the gradient to be well behaved. Unfortunately this approach did not work, although we did obtain some estimate proving that the graidient $\langle \boldsymbol{\nu}, V \rangle^{-1}$, $V = \frac{DF}{|DF|}$, where $F = v^2 - u^2$ grows at most linearly in time and so is bounded on every finite time interval, which is enough to prove long term existence, but not convergence.

The evolution of the height function can be easily calculated.

We define $F_2 = u^2 - v^2$

Lemma C.1. *We have:*
a) $\Delta_{\mathbb{R}^n} F_2 = 0$
b) $\Delta_M F_2 = 2\langle \boldsymbol{\nu}, \bar{\boldsymbol{\nu}}\rangle - 2H_M|x|\langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle$

Proof. a) is clear

b)

$$\begin{aligned}\Delta_M F_2 &= \Delta_{\mathbb{R}^n} F_2 - D^2 F_2(\boldsymbol{\nu}, \boldsymbol{\nu}) + \mathbf{H}_M \cdot \mathbf{D} F_2 \\ &= -2\bar{\delta}_{ij}\nu_i\nu_j - 2H\langle \boldsymbol{\nu}, \bar{x}\rangle \\ &= -2\langle \bar{\boldsymbol{\nu}}, \boldsymbol{\nu}\rangle - 2H\langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle |\mathbf{x}|\end{aligned}$$

□

Lemma C.2. *We further obtain:*
$\left(\frac{\partial}{\partial t} - \Delta_M\right) F_2 = -2\langle \bar{\boldsymbol{\nu}}, \boldsymbol{\nu}\rangle$

Proof.

$$\begin{aligned}\frac{\partial}{\partial t} F &= \mathbf{D} F \cdot \frac{\partial x}{\partial t} = -\mathbf{H}_M \cdot \mathbf{D} F \\ &= -H_M |DF| \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle\end{aligned}$$

□

Corollary C.3. *For a bi-rotationally-symmetric surface we get:*

$\left(\frac{\partial}{\partial t} - \Delta_M\right) F_2 = 2(\nu_v^2 - \nu_u^2)$

We define the gradient here to be

$$w = \langle \boldsymbol{\nu}, \boldsymbol{\omega}\rangle^{-1}$$

Lemma C.4.

$$\boldsymbol{\omega}(H) \geq \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x}, \boldsymbol{\nu}\rangle H}{|x|^2} - w^{-1}|A|^2$$

Proof.

$$\begin{aligned}\boldsymbol{\omega}(H) &= \boldsymbol{\omega}\left(g^{ij}h_{ij}\right) = -\boldsymbol{\omega}\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j\rangle h_{ij} + \boldsymbol{\omega}\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle \\ &= -\boldsymbol{\omega}\langle \boldsymbol{\tau}_i, \boldsymbol{\tau}_j\rangle h_{ij} - \boldsymbol{\omega}\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle \\ &= -2\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\boldsymbol{\omega}\right\rangle h_{ij} - \langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\omega}}\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle - \langle \mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle \\ &= -2\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\boldsymbol{\omega}\right\rangle h_{ij} - \langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\omega}\rangle - \langle \mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle\end{aligned}$$

Since $\omega = \frac{\bar{x}}{|x|}$ we can proceed by calculating these three quantities directly. Firstly:

$$(\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\omega})^\alpha = \left(\mathrm{D}_{\boldsymbol{\tau}_i}\frac{\bar{x}}{|x|}\right)^\alpha = \frac{\bar{\delta}^\alpha_\beta}{|x|}\tau_i^\beta - \frac{\bar{x}^\alpha x_\beta}{|x|^3}\tau_j^\beta = \frac{\bar{\delta}^\alpha_\beta}{|x|}\tau_i^\beta - \frac{\omega^\alpha\left\langle \mathbf{x}, \boldsymbol{\tau}_\mathbf{i}\right\rangle}{|x|^2} \tag{18}$$

$$\begin{aligned}(\mathrm{D}_{\boldsymbol{\tau}_i}\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\omega})^\alpha &= \tau_i^\gamma\frac{\partial}{\partial x^\gamma}\left(\frac{\bar{\delta}^\alpha_\beta}{|x|}\tau_i^\beta - \frac{\bar{x}^\alpha x_\beta}{|x|^3}\tau_i^\beta\right) \\ &= \tau_i^\gamma\left(\bar{\delta}^\alpha_\beta\frac{\frac{\partial}{\partial x^\gamma}\tau_i^\beta}{|x|} - \frac{\bar{\tau}_i^\alpha x_\gamma}{|x|^3} - \frac{\bar{\delta}^\alpha_\gamma x_\beta + \bar{x}^\alpha\delta^\gamma_\beta}{|x|^3}\tau_i^\beta - \frac{\bar{x}^\alpha x_\beta\frac{\partial}{\partial x_\gamma}\tau_i^\beta}{|x|^3} + 3\frac{\bar{x}^\alpha x_\beta x_\gamma}{|x|^5}\tau_i^\beta\right) \\ &= \bar{\delta}^\alpha_\beta\left(\frac{(\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i)^\beta}{|x|} - 2\frac{\tau_i^\beta\left\langle\boldsymbol{\tau}_i,\mathbf{x}\right\rangle}{|x|^3}\right) - \omega^\alpha\left(\frac{2p+1+\left\langle \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i,\mathbf{x}\right\rangle}{|x|^2} - \frac{3\Sigma_i\left\langle\mathbf{x},\boldsymbol{\tau}_\mathbf{i}\right\rangle^2}{|x|^4}\right)\end{aligned}$$

For $i \neq 2n-1$ we get $\left\langle\mathbf{x},\boldsymbol{\tau}_\mathbf{i}\right\rangle = 0$

$$\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i = \begin{cases} -\frac{1}{u}\hat{\mathbf{u}} & 1 \le i \le p \\ -\frac{1}{v}\hat{\mathbf{v}} & p+1 \le i \le 2p \\ -A_7^7\boldsymbol{\nu} & 2n-1 \end{cases} \tag{19}$$

Noting that $\sum_{i=1}^{2p}\left\langle \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i,\mathbf{x}\right\rangle = -2p$ we get

$$\begin{aligned}(\mathrm{D}_{\boldsymbol{\tau}_i}\mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\omega})^\alpha = &\frac{\frac{\hat{\mathbf{u}}}{u}^\alpha - \frac{\hat{\mathbf{v}}}{v}^\alpha + A_7^7 2\bar{\delta}^\alpha_\beta\nu^\beta}{|x|} - 2\bar{\delta}^\alpha_\beta\frac{\tau_{2n-1}^\beta\left\langle\mathbf{x},\boldsymbol{\tau}_{\mathbf{2n-1}}\right\rangle}{|x|^3} \\ &-\omega^\alpha\left(\frac{1+A_7^7\left\langle\boldsymbol{\nu},\mathbf{x}\right\rangle}{|x|^2} - \frac{3\left\langle\mathbf{x},\boldsymbol{\tau}_{\mathbf{2n-1}}\right\rangle^2}{|x|^4}\right)\end{aligned} \tag{20}$$

Now, $\left\langle\boldsymbol{\tau}_{2n-1}, \mathrm{D}_{\boldsymbol{\tau}_{2n-1}}\frac{\bar{x}}{|x|}\right\rangle = \frac{\nu_v^2-\nu_u^2}{|x|} + \frac{\nu_u^2v^2-\nu_v^2u^2}{|x|^3} = \frac{\nu_v^2v^2-\nu_u^2u^2}{|x|^3} = -\frac{w^{-1}\left\langle\mathbf{x},\boldsymbol{\nu}\right\rangle}{|x|^2}$
So we get:

$$\left\langle\boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\boldsymbol{\omega}\right\rangle = \begin{cases} \frac{1}{|x|} & 1 \le i = j \le p \\ -\frac{1}{|x|} & p+1 \le i = j \le 2p \\ -\frac{w^{-1}\left\langle\mathbf{x},\boldsymbol{\nu}\right\rangle}{|x|^2} & i = j = 2n-1 \\ 0 & \text{Otherwise} \end{cases}$$

To calculate the $\mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}$ term, we must recall the definition of the deformation of M_t created by $\boldsymbol{\omega}$, the first order approximation of this deformation is:

$$\boldsymbol{\omega} : \boldsymbol{\nu} \mapsto \frac{\left(\nu_u\left(1-\varepsilon\frac{u^2}{|x|^3}\right) - \varepsilon\nu_v\frac{uv}{|x|^3}\right)\hat{\mathbf{u}} + \left(\nu_v\left(1+\varepsilon\frac{v^2}{|x|^3}\right) + \varepsilon\nu_u\frac{uv}{|x|^3}\right)\hat{\mathbf{v}}}{\sqrt{1+2\varepsilon\left(\nu_v^2\frac{v^2}{|x|^3} - \nu_u^2\frac{u^2}{|x|^3}\right)}}$$

Thus

$$\mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu} = \left.\frac{d}{d\varepsilon}\boldsymbol{\nu}\right|_{\varepsilon=0} = -\left(\nu_v^2\frac{v^2}{|x|^3} - \nu_u^2\frac{u^2}{|x|^3}\right)\boldsymbol{\nu} - \left(\nu_u\frac{u^2}{|x|^3} + \nu_v\frac{uv}{|x|^3}\right)\hat{\mathbf{u}} + \left(\nu_v\frac{v^2}{|x|^3} + \nu_u\frac{uv}{|x|^3}\right)\hat{\mathbf{v}}$$

Thus we get the first term in the $\omega(H)$ equation:

$$\left\langle \boldsymbol{\tau}_i, \mathrm{D}_{\boldsymbol{\tau}_j}\boldsymbol{\omega}\right\rangle h_{ij} = \frac{p}{|x|}\left(\frac{\nu_u}{u} - \frac{\nu_v}{v}\right) - \frac{w^{-1}\langle \mathbf{x}, \boldsymbol{\nu}\rangle}{|x|^2} = \frac{p}{uv|x|}\langle \mathbf{x}, \boldsymbol{\tau}_{\mathbf{2n-1}}\rangle - \frac{w^{-1}\langle \mathbf{x}, \boldsymbol{\nu}\rangle}{|x|^2}$$

$$\begin{aligned}
-2\frac{p}{uv|x|}\langle \mathbf{x}, \boldsymbol{\tau}_{\mathbf{2n-1}}\rangle \quad + \quad 2p\frac{w^{-1}\nu_u\nu_v}{uv} &= \frac{2p}{|x|uv}\left(-u\nu_v + v\nu_u + \nu_u^2\nu_v u - \nu_u\nu_v^2 v\right) \\
&= \frac{2p}{|x|uv}\left(-\nu_v^3 u + \nu_u^3 v\right) = \frac{2p}{|x|}\left(-\frac{\nu_v^3}{v} + \frac{\nu_u^3}{u}\right) \\
&\leq \frac{p\nu_v^4 + \nu_u^4}{|x|^2}w^{-1} + p\left(\frac{\nu_v^2}{v^2} + \frac{\nu_u^2}{u^2}\right)
\end{aligned}$$

$$\begin{aligned}
\langle \boldsymbol{\nu}, \mathrm{D}_{\boldsymbol{\tau}_i}\mathrm{D}_{\boldsymbol{\tau}_i}X\rangle &= \boldsymbol{\nu}^\alpha\boldsymbol{\tau}_i^\gamma\frac{\partial}{\partial x^\gamma}\left(\frac{\bar{\delta}_{\alpha\beta}}{|x|}\boldsymbol{\tau}_i^\beta - \frac{\bar{x}_\alpha x_\beta}{|x|^3}\boldsymbol{\tau}_i^\beta\right) = \boldsymbol{\nu}^\alpha\boldsymbol{\tau}_i^\gamma\frac{\partial}{\partial x^\gamma}\left(\frac{\bar{\boldsymbol{\tau}}_i^\alpha}{|x|} - \frac{\bar{x}_\alpha\langle \mathbf{x}, \boldsymbol{\tau}_i\rangle}{|x|^3}\right) \\
&= \frac{\langle \bar{\boldsymbol{\nu}}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|} - \frac{\boldsymbol{\nu}^\alpha\boldsymbol{\tau}_i^\gamma x_\gamma\bar{\boldsymbol{\tau}}_i^\alpha}{|x|^3} - \frac{\boldsymbol{\nu}^\alpha\boldsymbol{\tau}_i^\gamma x_\gamma\bar{\boldsymbol{\tau}}_i^\alpha}{|x|^3} - w^{-1}\left(\frac{\boldsymbol{\tau}_i\langle \mathbf{x}, \boldsymbol{\tau}_i\rangle}{|x|^2} - \frac{3\langle \mathbf{x}, \boldsymbol{\tau}_i\rangle^2}{|x|^4}\right) \\
&= \frac{\langle \bar{\boldsymbol{\nu}}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|} - 2\frac{\langle \mathbf{x}, \boldsymbol{\tau}_{2n-1}\rangle\langle \boldsymbol{\nu}, \bar{\boldsymbol{\tau}}_{2n-1}\rangle}{|x|^3} \\
&\quad -w^{-1}\left(\frac{\langle \mathrm{D}_{\tau_i}\mathbf{x}, \boldsymbol{\tau}_i\rangle + \langle \mathbf{x}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|^2} - \frac{3\left(v\nu_u - u\nu_v\right)^2}{|x|^4}\right) \\
&= \frac{\langle \bar{\boldsymbol{\nu}}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|} + \frac{4\nu_u\nu_v\left(v\nu_u - u\nu_v\right)}{|x|^3} \\
&\quad -w^{-1}\left(\frac{2p + 1 - 2p - \langle \mathbf{x}, \boldsymbol{\nu}\rangle A_7^7}{|x|^2} - \frac{3\left(v\nu_u - u\nu_v\right)^2}{|x|^4}\right) \\
&= \frac{\langle \bar{\boldsymbol{\nu}}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|} - 2\left(\nu_v^2 - \nu_u^2\right)\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle}{|x|^3} + w^{-1}\left(\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle A_7^7}{|x|^2} - 3\frac{\left(v\nu_v + u\nu_u\right)^2}{|x|^4}\right) \\
&= \frac{\langle \bar{\boldsymbol{\nu}}, \mathrm{D}_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle}{|x|} - 2\left(\nu_v^2 - \nu_u^2\right)\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle}{|x|^3} + w^{-1}\left(\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle A_7^7}{|x|^2} - 3\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle^2}{|x|^4}\right)
\end{aligned}$$

$$\frac{2\nu_u\nu_v\left(v\nu_u - u\nu_v\right)}{|x|} = \frac{v\nu_v - u\nu_u - \left(\nu_v^2 - \nu_u^2\right)\left(v\nu_v + u\nu_u\right)}{|x|} = -w^{-1} - \left(\nu_v^2 - \nu_u^2\right)\frac{\langle \mathbf{x}, \boldsymbol{\nu}\rangle}{|x|}$$

$$\langle \mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle = \left\langle \mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}, p\frac{\hat{\mathbf{u}}}{u} + p\frac{\hat{\mathbf{v}}}{v} + A_7^7\boldsymbol{\nu}\right\rangle$$

So

$$\langle \mathrm{D}_{\boldsymbol{\omega}}\boldsymbol{\nu}, D_{\boldsymbol{\tau}_i}\boldsymbol{\tau}_i\rangle = \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} + \frac{u\nu_u + v\nu_v}{|x|^3}\left(-u\nu_u + v\nu_v\right) A_7^7$$

Thus

$$\begin{aligned}
\boldsymbol{\omega}(H) &= -\frac{\left(\nu_u^2 - \nu_v^2\right) A_7^7 + p\frac{\nu_u}{u} - p\frac{\nu_v}{v}}{|x|} + 2\left(\nu_v^2 - \nu_u^2\right)\frac{\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^3} \\
&\quad -w^{-1}\left(\frac{\langle \mathbf{x},\boldsymbol{\nu}\rangle A_7^7}{|x|^2} - 3\frac{\langle \mathbf{x},\boldsymbol{\nu}\rangle^2}{|x|^4}\right) \\
&\quad +\frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} - \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^2}A_7^7 + 2\frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^2}A_7^7 \\
&= \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} \\
&\quad +2p\frac{w^{-1}\nu_u\nu_v}{uv} + \frac{2\langle \mathbf{x},\boldsymbol{\nu}\rangle\left(\nu_v^2 - \nu_u^2\right)}{|x|^3} + \frac{3w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle^2}{|x|^4} \\
&= \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle^2}{|x|^4} \\
&\quad +2p\frac{w^{-1}\nu_u\nu_v}{uv} + \frac{2\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^5}\left(\left(\nu_v^2 - \nu_u^2\right)\left(u^2+v^2\right) + \left(u\nu_u - v\nu_v\right)\left(u\nu_u + v\nu_v\right)\right) \\
&= \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle^2}{|x|^4} \\
&\quad +2p\frac{w^{-1}\nu_u\nu_v}{uv} + \frac{2\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^5}\left(u\nu_v - v\nu_u\right)\left(v\nu_u + u\nu_v\right) \\
&\geq \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} - w^{-1}p\left(\frac{\nu_u^2}{u^2} + \frac{\nu_v^2}{v^2}\right) - \frac{2}{|x|^2} \\
&\geq \frac{\left(\nu_v^2 - \nu_u^2\right) H}{|x|} + \frac{w^{-1}\langle \mathbf{x},\boldsymbol{\nu}\rangle H}{|x|^2} - w^{-1}|A|^2 - \frac{2}{|x|^2}
\end{aligned}$$

Since $\left(\nu_u^2 - \nu_v^2\right)\left(\frac{\nu_u}{u} + \frac{\nu_v}{v}\right) - \frac{\nu_u}{u} + \frac{\nu_v}{v} = 2\nu_u\nu_v\left(\frac{\nu_u}{v} - \frac{\nu_v}{u}\right) = 2|x|w^{-1}\frac{\nu_u\nu_v}{uv}$ and $\left(\nu_v^2 - \nu_u^2\right)\left(u^2+v^2\right) + \left(u\nu_u - v\nu_v\right)\left(u\nu_u + v\nu_v\right) = v^2\nu_v^2 - u^2\nu_u + u^2\nu_v^2 - v^2\nu_u^2 - v^2\nu_v^2 + u^2\nu_u = \left(u\nu_v - v\nu_u\right)\left(u\nu_v + v\nu_u\right) = \left(u\nu_v - v\nu_u\right)\langle \boldsymbol{\tau}_{2n-1}, \boldsymbol{\omega}\rangle$

Finally, since $\frac{(\nu_v^2 - \nu_u^2)}{|x|} = \frac{D^2F(\boldsymbol{\nu},\boldsymbol{\nu})}{|DF|}$ and $\frac{\langle \mathbf{x},\boldsymbol{\nu}\rangle}{|x|^2} = \frac{D^2F(\boldsymbol{\nu},\boldsymbol{\omega})}{|DF|}$, combining lemma C.4 with Lemma 6.4 we get the following result.

□

Theorem C.5.

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) w \leq -w^{-1}|\nabla w|^2 + \frac{2}{|x|^2} \leq \frac{2}{|x|^2}$$

And so, by Theorem 3.1 we get the following corollary, giving a bound on the gradient.

Corollary C.6.

$$\sup_{M_t} w \leq \sup_{M_0} w + Ct$$

Proof. Let $w_1 = w - Ct$ Where $\sup_{M_0} \frac{2}{|x|^2} \leq C < \infty$ Thus

$$\left(\frac{\partial}{\partial t} - \Delta_M\right) w_1 \leq 0$$

□

We can now employ Theorem 7.1 to conclude curvature estimates on every finite time interval $[0, T]$ since they hold as long as we have a bound on w,w hich here depends on T. These in turn allow us to extend the solution according to the argument in Theorem 9.1. Therefore, Corollary D.6 implies longtime existence of our solution.

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