Research Article

Serik Altynbek and Heinrich Begehr*

A pair of rational double sequences

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Abstract: Double sequences appear in a natural way in cases of iteratively given sequences if the iteration allows to determine besides the successors from the predecessors also the predecessors from their followers. A particular pair of double sequences is considered which appears in a parqueting-reflection process of the complex plane. While one end of each sequence is a natural number sequence, the other consists of rational numbers. The natural numbers sequences are not yet listed in OEIS Wiki. Complex versions from the double sequences are provided.

Keywords: Two-sided sequences, natural numbers sequences, complex sequences

MSC 2010: 40B05, 40A05

1 Introduction

While sequences are known as mappings from the set of natural numbers, double (two-sided) sequences are based on the entire numbers. Although a double sequence can be rearranged in a sequence, such a rearrangement does not necessarily result in a convergent sequence even if the two ends of the double sequence are convergent (when the indices tend to $+\infty$ or $-\infty$). Of course, other combinations of the limit behavior are possible. Double sequences appear in a natural way in the cases of iteratively given sequences if the iteration recipe allows to determine besides the successors from the predecessors also the predecessors from their followers.

Repeatedly, iteratively given sequences appear when applying the parqueting-reflection principle to certain circular domains of the (complex) plane; see e.g. [1]. A pair of double sequences arises when repeatedly reflecting a certain circular rectangle at its boundary parts [2]. This pair of double sequences consists of sequences, the tails of which are, on one hand, natural numbers sequences (see [3]) and, on the other hand, made from rational numbers. Sequences can be extended to ones with complex numbers bearing similar properties as their origins.

2 Recurrence relations

With $a_0 = b_0 = 1$ for $k \in \mathbb{N}$, the relations

$$\begin{aligned} a_{2k} &= 3a_{2k-1} + b_{2k-1}, \quad & a_{2k+1} &= a_{2k} + b_{2k}, \\ b_{2k} &= a_{2k-1} + b_{2k-1}, \quad & b_{2k+1} &= a_{2k} + 3b_{2k} \end{aligned}$$

***Corresponding author: Heinrich Begehr,** Institute of Mathematics, FU Berlin, Arnimallee 3, 14195 Berlin, Germany, e-mail: begehrh@zedat.fu-berlin.de. https://orcid.org/0000-0003-0316-0897
Serik Altynbek, Kazakh University of Economics, Finance and International Trade, Zhubanov str. 7, 010005 Nur-Sultan, Kazakhstan, e-mail: serikaa@bk.ru

determine two real sequences. Obviously, both sets of equations may be solved for the lower indexed numbers as

$$2a_{2k-1} = a_{2k} - b_{2k}, \quad 2a_{2k} = 3a_{2k+1} - b_{2k+1},$$
$$2b_{2k-1} = 3b_{2k} - a_{2k}, \quad 2b_{2k} = b_{2k+1} - a_{2k+1}.$$

Hence, (a_k, b_k) is a pair of double sequences (a_k), (b_k), their first numbers being

$$\dots, a_{-4} = \frac{-5}{4}, \quad a_{-3} = \frac{-1}{2}, \quad a_{-2} = \frac{-1}{2}, \quad a_{-1} = 0,$$
$$a_0 = 1, \quad a_1 = 2, \quad a_2 = 10, \quad a_3 = 16, \quad a_4 = 76, \dots,$$
$$\dots, b_{-4} = \frac{1}{4}, \quad b_{-3} = 1, \quad b_{-2} = \frac{1}{2}, \quad b_{-1} = 1,$$
$$b_0 = 1, \quad b_1 = 4, \quad b_2 = 6, \quad b_3 = 28, \quad b_4 = 44, \dots.$$

3 Properties of the double sequences

In order to determine the particular explicit numbers in the sequences, some properties are investigated.

Lemma 3.1. *For any $k \in \mathbb{Z}$ the relations $3b_{2k}^2 - a_{2k}^2 = 2^{k+1}$ and $b_{2k+1}^2 - 3a_{2k+1} = 2^{2k+2}$ hold.*

Proof. For $k = 0$ and $k = 1$, the relations $3b_0^2 - a_0^2 = 2$ and $b_1^2 - 3a_1^2 = 2^2$ are true. By the recurrence relations, for $0 < k$,

$$\begin{aligned} 3b_{2k}^2 - a_{2k}^2 &= 3(a_{2k-1} + b_{2k-1})^2 - (3a_{2k-1} + b_{2k-1})^2 = 2(b_{2k-1}^2 - 3a_{2k-1}^2) \\ &= 2[(a_{2k-2} + 3b_{2k-2})^2 - 3(a_{2k-2} + b_{2k-2})^2] = 4(3b_{2k-2}^2 - a_{2k-2}^2), \\ b_{2k+1}^2 - 3a_{2k+1} &= (a_{2k} + 3b_{2k})^2 - 3(a_{2k} + b_{2k})^2 = 2(3b_{2k}^2 - a_{2k}^2) \\ &= 2[3(a_{2k-1} + b_{2k-1})^2 - (3a_{2k-1} + b_{2k-1})^2] = 4(b_{2k-1}^2 - 3a_{2k-1}^2) \end{aligned}$$

hold, and for $k < 0$,

$$\begin{aligned} 3b_{2k}^2 - a_{2k}^2 &= \frac{3}{4}(b_{2k+1} - a_{2k+1})^2 - \frac{1}{4}(3a_{2k+1} - b_{2k+1})^2 = \frac{1}{2}(b_{2k+1}^2 - 3a_{2k+1}^2) \\ &= \frac{1}{8}[(3b_{2k+2} - a_{2k+2})^2 - 3(a_{2k+2} - b_{2k+2})^2] = \frac{1}{4}(3b_{2k+2}^2 - a_{2k+2}^2), \\ b_{2k-1}^2 - 3a_{2k-1}^2 &= \frac{1}{4}(3b_{2k} - a_{2k})^2 - \frac{3}{4}(a_{2k - b_{2k}})^2 = \frac{1}{2}(3b_{2k}^2 - a_{2k}^2) \\ &= \frac{1}{8}[3(b_{2k+1} - a_{2k+1})^2 - (3a_{2k+1} - b_{2k+1})^2] = \frac{1}{4}(b_{2k+1}^2 - 3a_{2k+1}^2). \end{aligned}$$ □

Remark 3.2. Both formulas from the last lemma are unified as

$$3^{1+[\frac{k}{2}]-[\frac{k+1}{2}]} b_k^2 - 3^{[\frac{k+1}{2}]-[\frac{k}{2}]} a_k^2 = 2^{k+1}.$$

For convenience, the notation $m_1 = \sqrt{3}$ will be further used.

Lemma 3.3. *For $k \in \mathbb{Z}$, we have $m_1 b_{2k} \pm a_{2k} = (m_1 \pm 1)^{2k+1}$ and $b_{2k+1} \pm m_1 a_{2k+1} = (m_1 \pm 1)^{2k+2}$.*

Proof. For $k = 0$ and $k = 1$, the relations $m_1 b_0 \pm a_0 = m_1 \pm 1$ and $b_1 \pm m_1 a_1 = 4 \pm 2m_1 = (m_1 \pm 1)^2$ hold. For $0 < k$,

$$\begin{aligned} m_1 b_{2k} \pm a_{2k} &= m_1(a_{2n-1} + b_{2k-1}) \pm (3a_{2n-1} + b_{2n-1}) = (m_1 \pm 1)(b_{2k-1} \pm m_1 a_{2n-1}) \\ &= (m_1 \pm 1)[(a_{2k-2} + 3b_{2k-2}) \pm m_1(a_{2k-2} + b_{2k-2})] = (m_1 \pm 1)^2(m_1 b_{2n-2} \pm a_{2k-2}), \\ b_{2k+1} \pm m_1 a_{2k+1} &= a_{2k} + 3b_{2k} \pm m_1(a_{2k} + b_{2k}) = (m_1 \pm 1)(m_1 b_{2k} \pm a_{2k}) \\ &= (m_1 \pm 1)[m_1(a_{2k-1} + b_{2k-1}) \pm (3a_{2k-1} + b_{2k-1})] = (m_1 \pm 1)^2(b_{2k-1} \pm m_1 a_{2k-1}), \end{aligned}$$

and for $k < 0$, taking into account $(m_1 \pm 1)(m_1 \mp 1) = 2$,

$$
\begin{aligned}
m_1 b_{2k} \pm a_{2k} &= \frac{m_1}{2(b_{2k+1} - a_{2k+1})} \pm \frac{1}{2}(3a_{2k+1} - b_{2k+1}) \\
&= \frac{1}{2}[(m_1 \mp 1)b_{2k+1} - m_1(1 \mp m_1)a_{2k+1}] = \frac{m_1 \mp 1}{2}(b_{2k+1} \pm m_1 a_{2k+1}) \\
&= \frac{m_1 \mp 1}{2}[3b_{2k+2} - a_{2k+2} \pm m_1(a_{2k+2} - b_{2k+2})] \\
&= \frac{m_1 \mp 1}{4}[(3 \mp m_1)b_{2k+2} - (1 \mp m_1)a_{2k+2}] = \frac{(m_1 \mp 1)^2}{4}(m_1 b_{2k+2} \pm a_{2k+2}) \\
&= \left(\frac{(m_1 \mp 1)^2}{4}\right)^{-k}(m_1 b_0 \pm a_0) = \left(\frac{(m_1 \mp 1)^2}{4}\right)^{-k}(m_1 \pm 1) \\
&= \left(\frac{(m_1 \mp 1)^2}{4}\right)^{-k-1}\frac{m_1 \mp 1}{2} = \left(\frac{(m_1 \mp 1)}{2}\right)^{-2k-1} = (m_1 \pm 1)^{2k+1}, \\
b_{2k-1} \pm m_1 a_{2k-1} &= \frac{1}{2}[3b_{2k} - a_{2k} \pm m_1(a_{2k} - b_{2k})] = \frac{m_1 \mp 1}{2}(m_1 b_{2k} \pm a_{2k}) \\
&= \frac{m_1 \mp 1}{4}[m_1(b_{2k+1} - a_{2k+1}) \pm (3a_{2k+1} - b_{2k+1})] = \frac{(m_1 \mp 1)^2}{4}(b_{2k+1} \pm m_1 a_{2k+1}) \\
&= \left(\frac{(m_1 \mp 1)^2}{4}\right)^{-k}(b_{-1} \pm a_{-1}) = \left(\frac{(m_1 \mp 1)^2}{4}\right)^{-k} \\
&= \left(\frac{4 \mp 2m_1}{4}\right)^{-k} = \left(\frac{m_1 \mp 1}{2}\right)^{-2k} = (m_1 \pm 1)^{2k}. \qquad \square
\end{aligned}
$$

With the formulas from Lemma 3.3, the terms of the sequences are determined.

Theorem 3.4. *For $k \in \mathbb{Z}$, the double sequences (a_k) and (b_k) are given via*

$$
\begin{aligned}
2a_{2k} &= (m_1+1)^{2k+1} - (m_1-1)^{2k+1}, \quad & 2m_1 a_{2k+1} &= (m_1+1)^{2k+2} - (m_1-1)^{2k+2}, \\
2m_1 b_{2k} &= (m_1+1)^{2k+1} + (m_1-1)^{2k+1}, \quad & 2b_{2k+1} &= (m_1+1)^{2k+2} + (m_1-1)^{2k+2}.
\end{aligned}
$$

4 Complex version

The double sequence pair (a_k, b_k) of rational numbers handled in $\mathbb{Q}(\sqrt{3})$ has the counterpart in $\mathbb{Q}(\sqrt{3}, i)$.

For $k \in \mathbb{Z}$, define complex numbers as

$$c_{2k} = (-1)^k a_{2k}, \quad c_{2k+1} = (-1)^{k+1} i m_1 a_{2k+1}, \quad d_{2k} = (-1)^{k+1} i m_1 b_{2k}, \quad d_{2k+1} = (-1)^{k+1} b_{2k+1}.$$

Then the recursion relations for a_k and b_k are reflected into ones for c_k and d_k as

$$
\begin{aligned}
c_k &= -im_1 c_{k-1} + d_{k-1}, \quad & d_k &= -c_{k-1} - im_1 d_{k-1} \quad & &\text{with } c_0 = 1,\ d_0 = -im_1, \\
2c_{k-1} &= im_1 c_k + d_k, \quad & 2d_{k-1} &= im_1 d_k - c_k \quad & &\text{with } c_{-1} = 0,\ d_{-1} = 1.
\end{aligned}
$$

The first terms are

$$
\begin{aligned}
&\dots, c_{-4} = -\frac{5}{4}, \quad c_{-3} = \frac{1}{2}im_1, \quad c_{-2} = \frac{1}{2}, \quad c_{-1} = 0, \\
&c_0 = 1, \quad c_1 = -2im_1, \quad c_2 = 10, \quad c_3 = -16im_1, \quad c_4 = 76, \dots, \\
&\dots, d_{-4} = -\frac{3}{4}im_1, \quad d_{-3} = -1, \quad d_{-2} = \frac{1}{2}im_1, \quad d_{-1} = 1, \\
&d_0 = -im_1, \quad d_1 = -4, \quad d_2 = 6im_1, \quad d_3 = 28, \quad d_4 = -44im_1, \dots.
\end{aligned}
$$

Their properties are

$$
\begin{aligned}
&|d_k|^2 - |c_k|^2 = 2^{k+1}, \quad |c_k| + |d_k| = (m_1+1)^{k+1}, \quad |d_k| - |c_k| = (m_1-1)^{k+1}, \\
&c_k^2 + d_k^2 = (-2)^{k+1}, \quad c_k\overline{d_k} + \overline{c_k}d_k = 0, \quad -ic_k\overline{d_k} = i\overline{c_k}d_k = |c_k d_k|
\end{aligned}
$$

for $k \in \mathbb{Z}$.

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