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Exceptional sequences of 8 line bundles on $(\mathbb{P}^1)^3$

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Abstract

We investigate maximal exceptional sequences of line bundles on $(\mathbb{P}^1)^r$, i.e., those consisting of 2^r elements. For $r = 3$ we show that they are always full, meaning that they generate the derived category. Everything is done in the discrete setup: Exceptional sequences of line bundles appear as special finite subsets s of the PICARD group $\mathbb{Z}^r$ of $(\mathbb{P}^1)^r$, and the question of generation is understood like a process of contamination of the whole $\mathbb{Z}^r$ out of an infectious seed s.

Keywords Exceptional sequences · Line bundles · Derived category · Line configurations

1 Introduction

The content of the paper can be understood in two different languages. While the version of Sect. 1.1 presents everything as a challenging self-contained, combinatorial task, like a game to play, the background motivation stems from the algebro geometric scenario explained in Sect. 1.2. The paper sticks to the first language. That is, beyond Sect. 2.1, no algebraic geometry will appear.

1.1 The combinatorial language

Let $s = \{s^0, \dots, s^{m-1}\} \subset \mathbb{Z}^3$ be an ordered subset such that for each $i < j$ there is an index $\nu = \nu(i, j) \in \{1, 2, 3\}$ satisfying $s^j_\nu - s^i_\nu = 1$. Then, beginning with s, we start a contamination procedure by declaring each affine line $\ell \subset \mathbb{Z}^3$ parallel to a coordinate

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axis to be infected if it contains at least two adjacent infected points. See Definitions 2.2 and 2.3 for more details. Then, Theorem 2.4 states that s consists of at most eight elements and, moreover, if $|s|$ equals this maximal number, then it contaminates the whole lattice $\mathbb{Z}^3$.

1.2 The algebro geometric language

To investigate the derived category of smooth, projective algebraic varieties X one tries to mimic the methods of linear algebra by working with semiorthogonal decompositions and, more special, with so-called exceptional sequences, cf. Definition 2.1. The latter can be understood as an analog to linearly independent sets. Moreover, by definition, full exceptional sequences generate the entire derived category $\mathcal{D}(X)$ so they correspond to bases of vector spaces in linear algebra.

However, this comparison has a flaw: While the cardinality of full exceptional sequences s, if they exist at all, is known to be the rank of the Grothendieck group $K_0(X)$, it is not clear whether exceptional sequences s with $|s| = \mathrm{rk}\, K_0(X)$ are automatically full. In [3], this problem was related to the existence of so-called phantom categories, i.e., non-trivial triangulated categories with vanishing Hochschild homology and trivial Grothendieck group. They may appear as orthogonal complements of those exceptional sequences. See [10] and the citations therein for examples and a general discussion of this subject.

A special situation for those questions appears with the class of smooth, projective toric varieties $X = \mathbb{TV}(\Sigma)$ for fans Σ in some real vector space $N_\mathbb{R} = \mathbb{R}^n$. Here, the rank of $K_0(X)$ equals the number of full-dimensional cones inside Σ. Alternatively, if Σ appears as the normal fan $\mathcal{N}(\Delta)$ of a smooth lattice polytope Δ in the dual space $M_\mathbb{R} = (\mathbb{R}^n)^*$, then $\mathrm{rk}\, K_0(X)$ equals the number of vertices of Δ. If one drops the requirement of line bundles and asks for general complexes of coherent sheaves as elements of full exceptional sequences instead, then their existence was guaranteed by Kawamata's papers [5–7]. On the other hand, if one insists on line bundles, then Efimov has shown in [4] that full exceptional sequences cannot exist for all smooth, projective toric varieties.

In the present paper, we do not address the question for which fans Σ those sequences exist at all. Instead, we consider the very special situation of $X = (\mathbb{P}^1)^r$ where Δ is the r-dimensional cube having 2^r vertices. Here, the existence is guaranteed by the trivial example $s = \{\mathcal{O}(a) \mid a \in \{0,1\}^r\}$. However, it is not trivial at all whether all other exceptional sequences consisting of 2^r line bundles generate $\mathrm{Pic}(\mathbb{P}^1)^r = \mathbb{Z}^r$ which is generating $\mathcal{D}(\mathbb{P}^1)^r$. This was also the main problem in [9, (4.2)]. For $r = 2$ the question is rather trivial and will be discussed in Sect. 3.2. Our main result is an affirmative answer for the case $r = 3$:

Theorem 1.1 *Every maximal exceptional sequence of line bundles on $(\mathbb{P}^1)^3$, i.e., consisting of eight elements, is full.*

This theorem appears later, as Theorem 2.4 in Sect. 2.4, in a slightly different form. There it is formulated in the combinatorial manner as it was already announced in Sect. 1.1—that is, making use of the language of contaminations which will be introduced in Subsections (2.2) and (2.3). It should be mentioned that even in this very special case of $r = 3$ we depend on the usage of computers.

1.3 Previous work

Related work into the direction of the above theorem is contained in the recent preprint [8] dealing with the case of toric FANO varieties of PICARD rank two and dimensions 3 or 4. This is going to be generalized to arbitrary smooth, projective toric varieties of PICARD rank two in [2]. The latter contains the case of $X = \mathbb{P}^d \times \mathbb{P}^e$, e.g., $X = (\mathbb{P}^1)^2$. The result of the present paper is the first step into the direction of a higher PICARD rank.

2 The basic setup

2.1 Basic definitions

Let us recall the basic definitions. We restrict ourselves to the case of line bundles.

Definition 2.1 1) A sequence $s = [\mathcal{L}_0, \ldots, \mathcal{L}_{m-1}]$ of line bundles on a smooth, projective variety X is called *exceptional* if there are no backward homomorphisms, i.e., for each $i < j$ we have $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{L}_j, \mathcal{L}_i[*]) = 0$. Here we denote by $\mathcal{D}(X) := \mathcal{D}^b(X)$ the bounded derived category, and $*$ refers to arbitrary shifts. In other words, $\mathrm{H}^k(X, \mathcal{L}_i \otimes \mathcal{L}_j^{-1}) = 0$ for all $k \in \mathbb{Z}$.

2) An exceptional sequence s is called *maximal* if $m = |s| = n := \mathrm{rk}\, K_0(X)$.

3) An exceptional sequence s is called *full* if the set $\{\mathcal{L}_0, \ldots, \mathcal{L}_{m-1}\}$ generates the derived category $\mathcal{D}(X)$. That is, the latter is the smallest triangulated category containing these sheaves.

Note that, in contrast to (1), the properties (2) and (3) of the previous definition do not depend on the particular ordering within the sequence s. For smooth, projective toric varieties $X = \mathbb{TV}(\Sigma)$, we can identify $\mathrm{Cl}(X) = \mathbb{Z}^r$ with $r = \#\Sigma(1) - \dim X$. Then, the classes $s^i = [\mathcal{L}_i]$ correspond to certain points in $\mathbb{Z}^r$, and the exceptionality condition asks for $\overrightarrow{s^i s^j} = s^j - s^i \in -\mathrm{Imm}(X)$ for $i < j$ where $\mathrm{Imm}(X) \subseteq \mathbb{Z}^r$ denotes the *immaculate locus*, i.e., the set of those classes of line bundles carrying no cohomology at all, including the 0-th one. See [1] for a discussion of this interesting region.

2.2 Products of projective spaces

Assume now that $X = \mathbb{P}^{d_1} \times \cdots \times \mathbb{P}^{d_r}$. In contrast to arbitrary smooth, projective toric varieties, this special case provides a very explicit and clear description of the immaculate locus. Since the invertible immaculate sheaves on $\mathbb{P}^d$ are $\mathcal{O}(-1), \mathcal{O}(-2), \ldots, \mathcal{O}(-d)$, we know that

$$-\mathrm{Imm}(X) = \{a \in \mathbb{Z}^r \mid 1 \leq a_\nu \leq d_\nu \text{ for some } \nu = 1, \ldots, r\}.$$

Thus, everything can be described in a purely combinatorial language. According to this, and besides a short motivating remark at the beginning of Sect. 2.3,

no algebraic geometry will appear beyond this point.

For any ordered subset $s \subseteq \mathbb{Z}^r$ we will encode the particular position of its elements by the upper index, i.e., $s = [s^0, \ldots, s^{m-1}]$. In contrast, the coordinates of a single $s^i = (s^i_1, \ldots, s^i_r) \in \mathbb{Z}^r$ are indicated by lower indices. Now, the content of Definition 2.1 can be rewritten as follows:

Definition 2.2 Assume that we are given a dimension vector $d = (d_1, \ldots, d_r) \in \mathbb{N}^r$ with $r \geq 1$ and $d_\nu \geq 1$ for $\nu = 1, \ldots, r$.

1) A sequence $s = [s^0, \ldots, s^{m-1}]$ in $\mathbb{Z}^r$ is called *d-exceptional* if, for each $i < j$, there is an index $\nu = \nu(i, j) \in \{1, \ldots, r\}$ such that

$$(s^j - s^i)_\nu = s^j_\nu - s^i_\nu \in \{1, 2, \ldots, d_\nu\}.$$

2) A d-exceptional sequence s is called *maximal* if $m = |s| = n(d) := \prod_{\nu=1}^r (d_\nu + 1)$.
3) A d-exceptional sequence s is called *full* if it d-contaminates the whole lattice $\mathbb{Z}^r$ where the latter notion will be explained in Definition 2.3.

2.3 The contamination procedure

The following definition of the d-contamination process arises from the KOSZUL complex on $\mathbb{P}^d$

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & \mathcal{O} & \longrightarrow & \mathcal{O}(1)^{d+1} & \longrightarrow & \mathcal{O}(2)^{\binom{d+1}{2}} & \longrightarrow & \ldots & \longrightarrow & \mathcal{O}(d+1) & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & & & \| & & \\
0 & \longrightarrow & \Lambda^0\mathcal{O}(1)^{d+1} & \longrightarrow & \Lambda^1\mathcal{O}(1)^{d+1} & \longrightarrow & \Lambda^2\mathcal{O}(1)^{d+1} & \longrightarrow & \ldots & \longrightarrow & \Lambda^{d+1}\mathcal{O}(1)^{d+1} & \longrightarrow & 0
\end{array}$$

showing that the successive sheaves $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d)$ generate $\mathcal{O}(d+1)$ and, eventually, the whole line $\operatorname{Pic}\mathbb{P}^d = \mathbb{Z}$.

Definition 2.3 Let $d \in \mathbb{N}^r$ and $S \subseteq \mathbb{Z}^r$ an arbitrary subset. Then, with e_ν denoting the ν-th canonical basis vector of $\mathbb{Z}^r$, the elements of the set

$$\operatorname{cont}(S) \;:=\; S \,\cup\, \Big(\bigcup_{\nu=1}^r \bigcup_{p,\, p+e_\nu, \ldots,\, p+d_\nu e_\nu \,\in\, S} (p + \mathbb{Z}\cdot e_\nu)\Big) \;\subseteq\; \mathbb{Z}^r$$

are called *directly d-contaminated* from S. This gives rise to the inductive spreading $\operatorname{cont}^0(S) := S$ and

$$\operatorname{cont}^k(S) := \operatorname{cont}\big(\operatorname{cont}^{k-1}(S)\big) \;\text{ for }\; k \geq 1$$

which contain the results of d-contamination in at most k steps. Finally, the elements of the overall union $\operatorname{cont}^\infty(S) := \bigcup_{k\geq 0} \operatorname{cont}^k(S)$ are simply called *d-contaminated*.

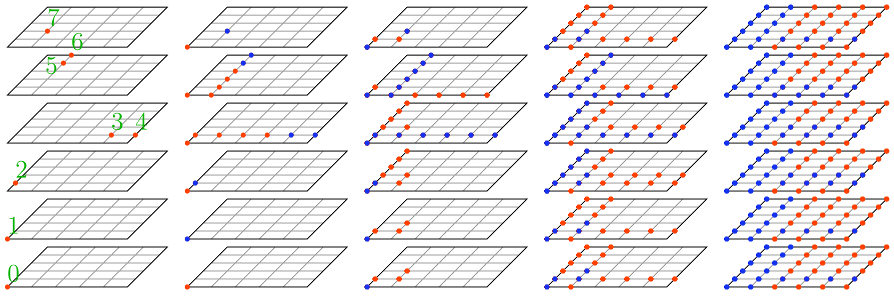


Fig. 1 An infection spreading, contaminating $\mathbb{Z}^3$ in four steps

2.4 The special case $(\mathbb{P}^1)^r$

Now, we focus on the special case $d = (1, \ldots, 1) \in \mathbb{N}^r$. The notion of d-exceptionality will simply be called *exceptionality* then. By Definition 2.2, it means that, for each $i < j$, there is an index $\nu = \nu(i, j) \in \{1, \ldots, r\}$ such that

$$(s^j - s^i)_\nu = s^j_\nu - s^i_\nu = 1.$$

The size of a maximal exceptional sequence is $n = 2^r$. The notion of d-contamination from Sect. 2.2 will simply be called *contamination*. The essential part of Definition 2.3, i.e.,

$$\operatorname{cont}(S) \;=\; S \,\cup\, \Big(\bigcup_{\nu=1}^r \bigcup_{p,\, p+e_\nu \in S} (p + \mathbb{Z} \cdot e_\nu) \Big) \;\subseteq\; \mathbb{Z}^r$$

does now just mean that two adjacent lattice points of $\mathbb{Z}^r$ infect the whole affine line they span, cf. Fig. 1. And here is our main result using the contamination language; it is equivalent to Theorem 1.1 from the introduction:

Theorem 2.4 *Let* $r = 3$. *Then, every* $(1, 1, 1)$*-exceptional sequence* $s \subset \mathbb{Z}^3$ *consists of at most eight elements. Moreover, if* $|s|$ *equals this maximal number, then* s *contaminates the whole lattice* $\mathbb{Z}^3$.

The proof is contained in Sect. 7.

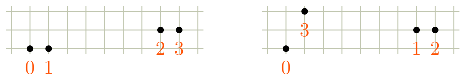
2.5 The width of exceptional sequences

We conclude this setup section with another basic notion. For any finite subset $S \subseteq \mathbb{Z}^r$ and $\nu \in \{1, \ldots, r\}$, we define the ν-width of S as

$$w_\nu(S) := \max_{s \in S} s_\nu - \min_{s \in S} s_\nu + 1.$$

That is, $w_\nu(S)$ is the smallest number $w \in \mathbb{N}$ such that S fits into w consecutive layers, i.e., integral hyperplanes $[x_\nu = \text{const}]$ in ν-direction. In [2] it was shown that (maximal) exceptional sequences for $X = \mathbb{P}^{d_1} \times \mathbb{P}^{d_2}$, i.e., in the case $r = 2$ of the

Fig. 2 Plane exceptional sequences



setting of Sect. 2.2, any exceptional sequence s satisfies either $w_1(s) \leq 2d_1 + 1$ or $w_2(s) \leq 2d_2 + 1$. This was an essential step for proving that maximal sequences are always full. However, as we already have seen in Fig. 1 and will see in Example 3.1, this simple kind of bounds does not stay true for $r \geq 3$.

3 Examples

3.1 The case of $r = 1$

There is not much to say in this case—maximal exceptional sequences consist of two adjacent points in $\mathbb{Z}^1$. And they contaminate the whole line immediately.

3.2 The case of $r = 2$

Up to a possible switch of the coordinates, there are two principal types of maximal exceptional configurations—they are shown in Fig. 2.

In both cases, the pair of points on the central horizontal line ℓ can be arbitrarily shifted along ℓ. In any case, the whole line ℓ becomes infected first, causing further "vertical contaminations" from the second pair of points.

In most cases, the numbering, i.e., the right "exceptional ordering" of the points (indicated by the red labels $0, \ldots, 3$) is unique for both types—but for a few shifts along ℓ, there remains a choice. This classification result follows from simple combinatorial arguments—or, alternatively, from the general theory developed later.

Note that the (unique) order of the right-hand example from Fig. 2 is horizontally lexicographic but not vertically lexicographic.

3.3 The case of $r = 3$

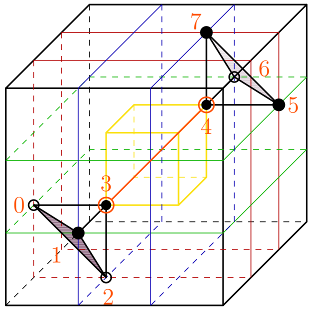
We are looking for configurations of 8 points in $\mathbb{Z}^3$. The standard full exceptional sequence corresponds to the 3-dimensional $(2 \times 2 \times 2)$-grid. For this the contamination of the whole lattice $\mathbb{Z}^3$ is immediate. However, there are more interesting and prettier examples.

Example 3.1 A quite symmetric maximal exceptional sequence is depicted in Fig. 3.

In the proposed numbering of the eight points of s the order of the three vertices on both of the tilted, regular triangles does not matter at all. The only mandatory ordering that matters is

$$\text{[triangle down left]} < \text{[lower-left red point]} <$$
$$< \text{[upper-right red point]} < \text{[triangle up right]},$$

Fig. 3 Symmetric $4 \times 4 \times 4$ grid



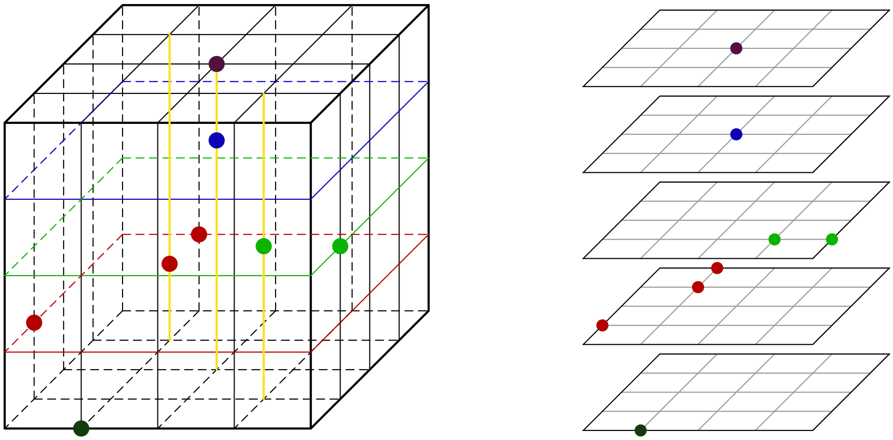


Fig. 4 Spanning the $5 \times 5 \times 5$ grid

that is, $\{s^0, s^1, s^2\} < s^3 < s^4 < \{s^5, s^6, s^7\}$. The contamination process starts with the three lines through the points s^3 and s^4, respectively. They infect all remaining six vertices of the central yellow cube.

Example 3.2 An even larger, but less symmetric example inside a $5 \times 5 \times 5$ grid is presented in Fig. 4. One possible numbering of the vertices is

$$(1, 0, 0)\ (0, 1, 1)\ (1, 3, 1)\ (1, 4, 1)\ (3, 1, 2)\ (4, 1, 2)\ (2, 2, 3)\ (2, 2, 4)$$

where the last coordinate indicates the height. The contamination process begins with the central vertical line and a red and a green line in the layers of height one and two, respectively.

The whole process is illustrated in Fig. 5. Here, as we have already done in Fig. 1, we have used bright red color to mark recently infected points—afterward, in the following steps, they will turn blue. Note that the present example, as it did Example 3.1, lacks empty inner layers in any direction.

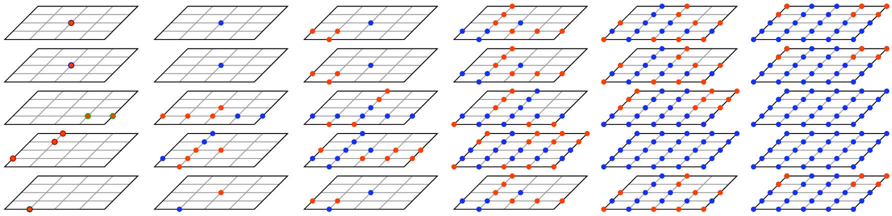


Fig. 5 Contaminating the whole $5 \times 5 \times 5$ grid (and $\mathbb{Z}^3$) in five steps

Example 3.3 According to Definition 2.3, the contamination process is an inductive procedure, i.e., it consists of a number of successive steps. In Fig. 1, we had depicted an example where the whole lattice $\mathbb{Z}^3$ becomes contaminated in four steps. Its starting maximal exceptional sequence consisted of the points

$$(0,0,0),\ (0,0,1),\ (0,1,2),\ (4,1,3),\ (5,1,3),\ (1,4,4),\ (1,5,4),\ (1,2,5).$$

While, in Example 3.2, we had already presented a configuration leading to a contamination process involving five steps, there are even longer examples: The sequences

$$[(0,0,0),(0,0,1),(0,1,4),(1,2,5),(1,4,4),(1,5,4),(4,1,3),(5,1,5)] \text{ and}$$
$$[(0,0,0),(0,1,3),(1,0,4),(1,4,3),(1,5,5),(2,2,1),(4,1,4),(5,1,4)]$$

require six or even seven steps, respectively.

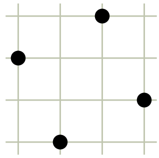
Example 3.4 Allowing empty layers, as we have already done in Example 3.3, leads to maximal exceptional sequences that can stretch arbitrarily far in all directions. For $n \in \mathbb{N}$ there is the maximal exceptional sequence

$$(0,0,0)\ (1,n,n)\ (1,n,n+1)\ (1,n+1,0)\ (2,1,0)$$
$$(2,n,1)\ (n,n+1,1)\ (n+1,n+1,1)$$

which stretches $n+1$ steps in every direction. That is, in the language of Sect. 2.5, $w_1(s) = w_2(s) = w_3(s) = n+2$. However, compare with Proposition 6.6.

Example 3.5 Finally, we would like to point out another aspect where the 3-dimensional situation is worse compared to the one from Sect. 3.2. Putting the right-hand Example from Fig. 2, i.e., $s = [(0,0), (7,1), (8,1), (1,2)]$, and its transpose $s' = [(0,0), (1,7), (1,8), (2,1)]$ into adjacent layers, the resulting 3-dimensional exceptional sequence does not allow any lexicographic, exceptional order.

Fig. 6 A maximal subset (not a sequence) of $\mathbb{Z}^2$ that is not full



4 Mapping to the cube

4.1 The labeling

Assume that $s \subset \mathbb{Z}^r$ is a maximal exceptional sequence, i.e., in particular, it consists of 2^r points. Then, we consider the map

$$\Phi : s \hookrightarrow \mathbb{Z}^r \twoheadrightarrow (\mathbb{Z}/2\mathbb{Z})^r.$$

Proposition 4.1 *The map Φ is bijective.*

Proof It suffices to check that Φ is injective. If $s^i, s^j \in s$ with $i < j$ but $\Phi(s^i) = \Phi(s^j)$, then $s^j - s^i \in 2 \cdot \mathbb{Z}^r$, i.e., $(s^j - s^i)_\nu \in 2\mathbb{Z}$ for all coordinates $\nu = 1, \ldots, r$. This contradicts the condition stated in Sect. 2.4 asking for $(s^j - s^i)_\nu = 1$. □

Since s comes with a total ordering, it induces a labeling of the vertices of the r-cube, i.e., of the elements of $(\mathbb{Z}/2\mathbb{Z})^r$, with the numbers $0, \ldots, 2^r - 1$. For example, if the lower-left vertex of the cube from Fig. 3 is taken as the origin, then the full exceptional sequence leads to the following ordering of the elements of $(\mathbb{Z}/2\mathbb{Z})^3$:

$$\Phi = (0, 1, 1),\ (1, 0, 1),\ (1, 1, 0),\ (1, 1, 1),\ (0, 0, 0),\ (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1).$$

Remark 4.2 The map Φ stays bijective if we replace the maximal exceptional sequence s by a subset $\{s^0, \ldots, s^{2^r-1}\}$ of $\mathbb{Z}^r$, having the less restrictive property that for each $i \neq j$ there is an index $\nu \in \{1, \ldots, r\}$ satisfying $|s^j_\nu - s^i_\nu| = 1$.

The subset of $\mathbb{Z}^2$ in Fig. 6 does not infect any other points of $\mathbb{Z}^2$, in particular, it is not full. This shows that the requirement that s is *ordered* is essential.

Remark 4.3 Let s be an r-dimensional maximal exceptional sequence. Then, the sequence $\tilde{s}$ defined by $\tilde{s}^i_k := -s^{2^r-1-i}_k$ is a maximal exceptional sequence, too. Moreover, $\tilde{s}$ is full if and only if s is full. The labeling of the vertices of the r-cube induced by $\tilde{s}$ is the reverse of the one induced by s.

4.2 Symmetries

We have symmetries governed by the group $(\mathbb{Z}/2\mathbb{Z})^r$: A shift of $s \subseteq \mathbb{Z}^r$ by the unit vector $e_\nu \in \mathbb{Z}^r$ for some $\nu = 1, \ldots, r$ does neither change the status of exceptionality, maximality, or fullness, i.e., infectivity. Within the cube $(\mathbb{Z}/2\mathbb{Z})^r$, this operation is still given by the map $\theta_\nu : x \mapsto x + e_\nu$, but it simply means the reflection along the ν-th hyperplane.

$$\begin{array}{|c|c|}\hline 3 & 2 \\ \hline 0 & 1 \\ \hline \end{array} \text{ meaning } \begin{array}{|c|c|}\hline s^3 & s^2 \\ \hline s^0 & s^1 \\ \hline \end{array} \qquad \text{and} \qquad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 0 & 3 \\ \hline \end{array} \text{ meaning } \begin{array}{|c|c|}\hline s^2 & s^1 \\ \hline s^0 & s^3 \\ \hline \end{array}$$

Fig. 7 Labelings for the full exceptional sequences of Fig. 2

$$\begin{array}{|c|c|}\hline 6 & 2 \\ \hline 4 & 5 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 0 & 3 \\ \hline 7 & 1 \\ \hline \end{array} \qquad \text{becomes normalized to} \qquad \begin{array}{|c|c|}\hline 7 & 1 \\ \hline 0 & 3 \\ \hline \end{array} \text{ and } \begin{array}{|c|c|}\hline 4 & 5 \\ \hline 6 & 2 \\ \hline \end{array}$$

Fig. 8 Labelings for the full exceptional sequence of Fig. 3

We may get rid of these symmetries by normalizing the labeling via putting the first element s^0 of s into the lower-left corner, i.e., asking for $\Phi(s^0) = 0 \in (\mathbb{Z}/2\mathbb{Z})^r$ or even $s^0 = 0$. For instance, the two plane exceptional sequences from Fig. 2 lead to the normalized labelings presented in Fig. 7. Doing the same for the full exceptional sequence from Fig. 3 yields the labeled cube encoded as two squares (lower/upper layer) in Fig. 8.

Finally, we should remark that, via permuting the coordinates, the symmetric group S_r acts, too. This action is visible on both the exceptional subsets of $\mathbb{Z}^r$ as well as on the labelings of $(\mathbb{Z}/2\mathbb{Z})^r$. Altogether we have $(2^r)!/(2^r \cdot r!) = (2^r - 1)!/r!$ (that is 840 for $r = 3$) cosets of possible labelings of $(\mathbb{Z}/2\mathbb{Z})^r$. It might be interesting to know which or how many of them can be induced from full exceptional sequences.

5 The configuration of layers

Let s be an r-dimensional exceptional sequence. In the present section we choose and fix some $\nu \in \{1, \ldots, r\}$ and denote by $\pi_\nu : \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}$ the associated coordinate. Moreover, we denote by $q_\nu : \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}^{r-1}$ the projection forgetting the chosen ν-th component. Accordingly, we obtain a decomposition of s into layers $L_c = s \cap \pi_\nu^{-1}(c)$. When it simplifies the notation, we may always assume, w.l.o.g., that $\nu = r$ is the last coordinate.

5.1 Reduction to $(r-1)$-dimensional sequences

We start with the key lemma for reducing the dimension from r to $r-1$. At this point, no maximality of s is assumed.

Lemma 5.1 *Let $C \subset \mathbb{Z}$ be such that no two elements of C are adjacent to each other, i.e., $1 \notin C - C$. Then, the map q_ν is injective when restricted to $\bigcup_{c\in C} L_c$. Moreover, its image $q_\nu(\bigcup_{c\in C} L_c) \subset \mathbb{Z}^{r-1}$ with the inherited total order is an $(r-1)$-dimensional exceptional sequence.*

Proof Consider $s^i, s^j \in \bigcup_{c\in C} L_c$ for some $i < j$. Since the ν-th coordinate of these elements belong to C, we know for sure that $(s^j - s^i)_\nu = s^j_\nu - s^i_\nu \neq 1$. Thus, by Sect. 2.4, there has to be another index $\mu \neq \nu$ such that $s^j_\mu - s^i_\mu = 1$, i.e., that $q_\nu(s^j)_\mu - q_\nu(s^i)_\mu = 1$ (if $\mu < \nu$) or $q_\nu(s^j)_{\mu-1} - q_\nu(s^i)_{\mu-1} = 1$ $(\mu > \nu)$. This implies both that $q_\nu(s^i) \neq q_\nu(s^j)$ and that these two elements satisfy the exceptionality condition in $\mathbb{Z}^{r-1}$. $\square$

Corollary 5.2 *Assume that $A, B \subset \mathbb{Z}$ are disjoint subsets fulfilling the assumptions of C from Lemma 5.1. Then, if $s \subset \mathbb{Z}^r$ is a maximal exceptional sequence being contained in $\bigcup_{c\in A\cup B} L_c$* (for instance, if $A \cup B = \mathbb{Z}$)*, both images $\overline{s}_A := q_\nu(\bigcup_{c\in A} L_c)$ and $\overline{s}_B := q_\nu(\bigcup_{c\in B} L_c)$ are maximal exceptional sequences in $\mathbb{Z}^{r-1}$.*

Proof On the one hand, it follows from Lemma 5.1 that $\overline{s}_A$ and $\overline{s}_B$ are exceptional in $\mathbb{Z}^{r-1}$, implying that $|\overline{s}_A|, |\overline{s}_B| \le 2^{r-1}$. On the other, we do also know that

$$|\overline{s}_A| + |\overline{s}_B| = |s| = 2^r.$$

Hence, $|\overline{s}_A| = |\overline{s}_B| = 2^{r-1}$, i.e., both $\overline{s}_A$ and $\overline{s}_B$ are maximal exceptional. □

The main example for Corollary 5.2 is $A = 2\mathbb{Z}$ and $B = 2\mathbb{Z} + 1$. Thus, we obtain two $(r-1)$-dimensional maximal exceptional sequences $\overline{s}_{\text{even}}$ and $\overline{s}_{\text{odd}}$ arising as the projections from the even and odd layers of s, respectively. Via the labeling map Φ established in Sect. 4.1, the sequences $\overline{s}$ even and $\overline{s}_{\text{odd}}$ correspond to facets, i.e., to $(r-1)$-dimensional slices of the cube $(\mathbb{Z}/2\mathbb{Z})^r$.

Note that a kind of an opposite implication is also true: If s_A and s_B are two $(r-1)$-dimensional maximal or even full exceptional sequences, then we obtain by

$$s \;:=\; \big(s_A \times \{0\}\big) \;\sqcup\; \big(s_B \times \{1\}\big)$$

an r-dimensional one of the same quality. Moreover, proceeding with s, we recover $s_A = \overline{s}_{\text{even}}$ and $s_B = \overline{s}_{\text{odd}}$ we have started with.

5.2 Thin sequences and maximal layers

Since we have still fixed a coordinate $\nu \in \{1, \ldots, r\}$, we will call $w(s) := w_\nu(s)$ from Sect. 2.5 simply the width of s. Then, for example, the maximal exceptional sequence s just built from the lower-dimensional s_A, s_B at the end of Sect. 5.1 has width 2. And both layers are maximal, meaning that they contain the maximal number of 2^{r-1} points.

Proposition 5.3 *Let s be a maximal exceptional sequence in $\mathbb{Z}^r$. Then, its width is bounded by $w(s) \le 3$ if and only if s contains a maximal layer.*

Moreover, if this is the case, then s is full, i.e., contaminating the whole lattice $\mathbb{Z}^r$—provided that we already know that maximality implies fullness in dimension $r-1$.

Proof Assume that $w(s) \le 3$ and denote by A_0, B_1, A_2 the three layers of s. Then, with $A = 2\mathbb{Z}$ and $B = 2\mathbb{Z} + 1$ as in Sect. 5.1, the layer B_1 equals the $(r-1)$-dimensional maximal exceptional sequence $\overline{s}_B$ from Corollary 5.2.

Conversely, if L_c is a full layer, then any $L_d \ne \emptyset$ with $|d - c| \ge 2$ will yield a contradiction via Lemma 5.1: Just take $C := \{c, d\}$.

Now, for the second part, let us assume that we have a maximal layer B_1 sitting in between A_0 and A_2. In particular, by the induction hypothesis, all points of the associated hyperplane $[x_\nu = 1] \subset \mathbb{Z}^r$ containing B_1 will be contaminated.

In particular, this means that for each $(a, 0) \in A_0$ or $(a, 2) \in A_2$, the points $(a, 1)$ are contaminated, In the second round, these pairs will infect the whole vertical lines $a \times \mathbb{Z}$. This, however, means that $\overline{s}_A \times \mathbb{Z}$ becomes contaminated.

Finally, as a third round, we proceed with the contamination procedure for $\overline{s}_A \subseteq \mathbb{Z}^{r-1}$ in each layer $\mathbb{Z}^{r-1} \times \{c\}$ with $c \in \mathbb{Z}$ simultaneously. □

Since we have seen in Sect. 3.2 that 2-dimensional maximal exceptional sequences are always full, the previous proposition implies the same fact for 3-dimensional maximal exceptional sequences of width (in *some* direction) at most 3.

5.3 Heavy layers stick together

Another immediate consequence of Lemma 5.1 is the observation that heavy layers L_c, i.e., those having a large size $\ell_c := |L_c|$ will always be close to each other:

Proposition 5.4 *Let $s \subset \mathbb{Z}^r$ be an r-dimensional exceptional sequence. Then, any two different layers L_a and L_b with $\ell_a + \ell_b > 2^{r-1}$ have to be neighbors, i.e., they satisfy $|b - a| = 1$.*

Proof Assume that L_a and L_b are not neighbors. Then, by Lemma 5.1 it follows that $q_\nu : (L_a \cup L_b) \hookrightarrow \mathbb{Z}^{r-1}$ is injective, and that its image is an exceptional sequence of dimension $r - 1$. Hence, $\ell_a + \ell_b \leq 2^{r-1}$. □

5.4 Lower bounds for the heaviest layers

In particular, for $r = 3$, the previous Proposition 5.4 has quite strong implications. The reason is the following statement ensuring the existence of a layer L with $|L| \geq 3$ in at least *one direction* $\nu \in \{1, \dots, r\}$:

Lemma 5.5 *Let $s \subset \mathbb{Z}^r$ be an r-dimensional maximal exceptional sequence. Then, there is a $\nu \in \{1, \dots, r\}$ such that at least one of the associated layers L satisfies $\ell = |L| \geq \frac{2^r - 1}{r}$. In particular, if $r = 3$, then there has to be a layer with $\ell \geq 3$.*

Proof In the ordered sequence s all elements after the initial one, i.e., after s^0, have to be in one of the following layers

$$\left(\{s_1^0 + 1\} \times \mathbb{Z} \times \cdots \times \mathbb{Z}\right), \left(\mathbb{Z} \times \{s_2^0 + 1\} \times \mathbb{Z} \times \cdots \times \mathbb{Z}\right),$$
$$\dots, \left(\mathbb{Z} \times \cdots \times \mathbb{Z} \times \{s_r^0 + 1\}\right).$$

Since there are $2^r - 1$ elements from $s \backslash \{s^0\}$ to be distributed to r possible layers, the result follows from the pigeon hole principle. □

Finally, let us focus on the case $r = 3$. While we already have treated the case of full layers, i.e., $\ell_c = 4$, in Proposition 5.3, we may assume that $\ell_c \leq 3$ for all $c \in \mathbb{Z}$ and all directions $\nu \in \{1, 2, 3\}$. But then, Lemma 5.5 guarantees the existence of at least one layer L_c (in some direction ν) satisfying $\ell_c = 3$. Moreover, by Proposition 5.4, any further layer (in the same direction ν) containing more than one point has to be adjacent to L_c.

6 Block structure

As in Sect. 5, we start with an r-dimensional exceptional sequence s and fix a coordinate $\nu \in \{1, \dots, r\}$. Recall that $\pi_\nu : \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}$ and $q_\nu : \mathbb{Z}^r \twoheadrightarrow \mathbb{Z}^{r-1}$ denote the associated projections. According to this, the non-empty layers $L_c = s \cap \pi_\nu^{-1}(c)$ of s, i.e., those with $\ell_c = |L_c| \geq 1$, can be arranged in several blocks. The following definition makes this precise:

Definition 6.1 A finite subset $B \subset \mathbb{Z}$ of consecutive numbers is called a segment for ν if $L_b \neq \emptyset$ for all $b \in B$ and if $L_{\min(B)-1} = L_{\max(B)+1} = \emptyset$. The preimage

$$L_B := s \cap \pi_\nu^{-1}(B) = \bigcup_{b \in B} L_b$$

is called the block associated with B. The layers $L_{\min(B)}$ and $L_{\max(B)}$ are called the boundary or outer layers of B.

6.1 Outer layers are lighter than their neighbors

The following proposition shows that the number of elements in a layer decreases toward the boundaries of the corresponding block.

Proposition 6.2 *Let s be an r-dimensional maximal exceptional sequence. Then, for any direction $\nu \in \{1, \dots, r\}$ and for each segment $B \subset \mathbb{Z}$, no outer layer of L_B contains more points than its neighboring "inner" layer. That is,*

$$\ell_{\min(B)} \leq \ell_{\min(B)+1} \quad \textit{and} \quad \ell_{\max(B)} \leq \ell_{\max(B)-1}.$$

In particular, the width of the block $w(L_B) = \max(B) - \min(B) + 1$ does always exceed one.

Note that blocks may have width $w(L_B) = 2$. Then, the neighbors $L_{\min(B)+1}$ and $L_{\max(B)-1}$ are obviously not inner layers at all—nevertheless, they were called so in the previous proposition. In this special case, it implies that $\ell_{\min(B)} = \ell_{\max(B)}$. Anyway, for all cases, this proposition is a direct consequence (just set $b := \min(B)$ or $b := \max(B)$) of the following lemma:

Lemma 6.3 *Let s be an r-dimensional maximal exceptional sequence. Then, for any direction $\nu \in \{1, \dots, r\}$ and any $b \in \mathbb{Z}$ we have $\ell_b \leq \ell_{b-1} + \ell_{b+1}$.*

Proof We start with the subsets

$$C := (b+1) + 2\mathbb{Z} \quad \text{and} \quad C' := b + 2\mathbb{Z} = C + 1.$$

They fit the assumptions of Corollary 5.2; hence $s_C := s \cap \pi_\nu^{-1}(C) = \bigcup_{c \in C} L_c$ consists of exactly 2^{r-1} elements—and so does $s_{C'}$ what, nevertheless, we do not need. Instead, we consider

$$C'' := \{b\} \cup C \backslash \{b \pm 1\},$$

i.e., $s_{C''} = L_b \cup s_C \backslash (L_{b-1} \cup L_{b+1})$. Furthermore, $C'' \subset \mathbb{Z}$ meets the requirements of Lemma 5.1, thus $|s_{C''}| \le 2^{r-1}$. Hence, $2^{r-1} \ge |s_{C''}| = |L_b \cup s_C \backslash (L_{b-1} \cup L_{b+1})| = \ell_b + |s_C| - \ell_{b-1} - \ell_{b+1} = \ell_b + 2^{r-1} - \ell_{b-1} - \ell_{b+1}$ and therefore $\ell_b \le \ell_{b-1} + \ell_{b+1}$. □

6.2 Blocks are balanced

In Corollary 5.2, we had seen that the even and the odd layers contribute the same number of points to a maximal exceptional sequence s, namely 2^{r-1} in each case. However, a similar statement is true for each block separately.

Proposition 6.4 *Let s be a maximal exceptional sequence. Then, within each block L_B, the alternating sums $\sum_{c \in B} (-1)^c \, \ell_c$ vanish.*

Note that this proposition excludes once again, like it does Proposition 6.2, the existence of blocks of width one.

Proof We do again exploit Corollary 5.2. We start with the standard pair $C := 2\mathbb{Z}$ and $C' := 2\mathbb{Z} + 1$. We then produce $D, D' \subset \mathbb{Z}$ out of $C, C' \subset \mathbb{Z}$ by switching $B \cap C$ with $B \cap C'$ inside these sets. More precisely, we set

$$D := C \cup (B \cap C') \backslash (B \cap C) \backslash \{\min(B) - 1,\ \max(B) + 1\}$$

and

$$D' := C' \cup (B \cap C) \backslash (B \cap C') \backslash \{\min(B) - 1,\ \max(B) + 1\}.$$

These sets are still disjoint, and the removal of the empty layers beyond B ensures the assumptions $1 \notin D - D$ and $1 \notin D' - D'$ without destroying the property $s \subseteq \bigcup_{c \in D \cup D'} L_c$. Thus, both s_C and s_D (and also $s_{C'}$ and $s_{D'}$) contain 2^{r-1} elements each. But this implies that $\sum_{c \in B \cap C} \ell_c = \sum_{c \in B \cap C'} \ell_c$. □

6.3 Empty layers

The presence of inner empty layers, i.e., those separating blocks in s, allows to manipulate s in different ways with various results. To demonstrate this, we assume that $L_0 = \emptyset$. As usual, this is meant for a fixed direction $\nu \in \{1, \ldots, r\}$. Recall from Definition 2.3 that $e_\nu \in \mathbb{Z}^r$ denotes the corresponding unit vector, i.e., $\pi_\nu(e_\nu) = 1$ and $q_\nu(e_\nu) = 0 \in \mathbb{Z}^{r-1}$.

6.3.1 Merging positive and negative layers

Fix a natural number $m \in \mathbb{N}$. Then, we obtain a new sequence $s(m)$ out of s by defining

$$s(m)^i := \begin{cases} s^i & \text{if } s^i_\nu > 0 \\ s^i + m \cdot e_\nu & \text{if } s^i_\nu < 0. \end{cases}$$

While $s(0) = s$, the new sequence $s(1)$ arises from s by simply erasing the 0-th (empty) layer. For larger $m \geq 2$, it will more and more happen that the former negative part s^- and the former positive part s^+ merge. In particular, a potential block structure of s will disappear—or reappear, in a different manner, for $m \gg 0$.

While it is clear that $s(m)$ stays a maximal exceptional sequence if s was one, it should be mentioned that it is *not* guaranteed at all (and almost always false) that exceptionality survives when including an empty layer in an exceptional sequence. Similarly, a potential fullness of s is bequeathed to $s(m)$—but the reverse conclusion does not work either.

6.3.2 Duplicating or reducing empty layers

A much easier situation arises when the definition of $s(m)$ is literally extended to negative $m < 0$. This means to amend the empty layer L_0 by additional $|m|$ empty ones. The reverse operation means to thin out sequences of consecutive empty layers such that at least one of them survives. This operation is a special type of those from Sect. 6.3.1.

Proposition 6.5 *Duplicating or reducing empty layers in a sequence s (without extinguishing them all) leads to a sequence s' sharing with s exactly the same properties with regard to exceptionality, maximality, and infectivity, i.e., fullness.*

Proof The property of $(s^j - s^i)_\mu$ being equal or not equal to 1 is equivalent to the same property of $(s'^j - s'^i)_\mu$ for every $\mu = 1, \dots, r$. Moreover, within the contamination procedure, all new points $(p, c, q) \in \mathbb{Z}^{\nu-1} \times \mathbb{Z} \times \mathbb{Z}^{r-\nu}$ arising in the empty layers L_c appear simultaneously in all layers, i.e., as a line $p \times \mathbb{Z} \times q$. $\square$

The previous observation means that, while empty layers do indeed matter, we can restrict ourselves to the case of isolated (but, maybe, still several) ones. This applies either for the classification of maximal exceptional sequences or for proving that maximality implies fullness.

6.4 Bounding the sequence

The previous results have the consequence that the problem of fullness of all maximal exceptional sequence can be reduced to maximal exceptional sequence in a bounded region.

Proposition 6.6 *All r-dimensional maximal exceptional sequence are full if and only all those contained in a cube with $3 \cdot 2^{r-1} - 1$ layers in each direction are full.*

Proof By Proposition 6.2, all blocks of a maximal exceptional sequence s have at least width 2 in any direction. By Proposition 6.5, we can exchange s by s' that has no empty blocks of width larger than 1 and still the same properties with regard to fullness. Hence, s' has at most 2^r non-empty layers and at most $2^{r-1} - 1$ empty layers in between in all ν-directions. Thus s' fits in a cube with $2^r + 2^{r-1} - 1$ layers in each direction. $\square$

7 Special issues for $r = 3$

In this concluding section, we assume that s is a 3-dimensional maximal exceptional sequence, i.e., with $r = 3$. Moreover, by Proposition 5.3, we may and will assume that s has no layer of size 4. Finally, according to Sect. 6.3.2, we suppose that all inner empty layers, in all directions, are isolated.

We are going to present upper bounds for the widths of s with respect to all three directions. Let $\nu \in \{1, 2, 3\}$—we look for all possible "load sequences" $(\ldots, \ell_{c-1}, \ell_c, \ell_{c+1}, \ldots)$ indicating the sizes of the consecutive layers, up to equivalence relations like shifts or reflections within each block.

7.1 Assuming a layer of size 3

For the fixed direction ν we suppose that we have a layer of size 3 within the maximal exceptional sequence s. We are going to distinguish a few cases—but the overall result will be that $w_\nu(s) \leq 6$. Moreover, if lacking empty layers, we can bound it by 5.

7.1.1 Two layers of size 3

If s contains, in the same direction ν, two layers of size 3, then, by Proposition 5.4, they have to be adjacent. The remaining two layers of width 1 cannot be isolated by Proposition 6.2. Thus, we obtain as possible load sequences $(1, 3, 3, 1)$ or $(3, 3, 1, 1)$, or $(3, 3, 0, 1, 1)$. In any case, we have $w_\nu(s) \leq 5$.

7.1.2 Two layers of size 2

They are supposed to exist additionally to the layer of size 3 we have anyway. Again by Proposition 5.4, we see that the load sequence has to look like $(\ldots, 2, 3, 2, \ldots)$, thus, it has to be $(2, 3, 2, 1)$ yielding $w_\nu(s) = 4$.

7.1.3 Exactly one layer of size 2

Here, we obtain $(1, 3, 2)$ as a necessary subsequence. The remaining part has to be $(1, 1)$—either separated from the big part by an empty layer or not. We obtain $w_\nu(s) =$ 5 or 6.

7.1.4 All layers are of size 1

This is meant up to the big layer of size 3 we have assumed anyway. Then, this case cannot occur—it contradicts Lemma 6.3.

7.2 No layer of size 3

By Proposition 6.6, the longest possible load sequence is

$$(1, 1, 0, 1, 1, 0, 1, 1, 0, 1, 1);$$

it has length 11. And this can indeed occur in a maximal exceptional sequence. On the other hand, if there is at least one layer of size 2 involved, then one of the subsequences $(1, 1, 0, 1, 1)$ has to be replaced by $(1, 2, 1)$ or even $(2, 2)$. Thus, the estimate for $w_\nu(s)$ drops from 11 to 9.

7.3 Computational evidence

We will conclude our proof of Theorem 2.4 (or, equivalently, Theorem 1.1) by strengthening the claim of Lemma 5.5. The starting point was that the seven elements of $\{s^1, s^2, \ldots, s^7\}$ are distributed on three affine planes having distance one to s^0. While the original claim of the lemma states that there is one plane containing three points of $s\backslash\{s^0\}$, we do even know that the distribution must be either $(3, 3, 3)$, $(3, 3, 2)$, $(3, 3, 1)$ or $(3, 2, 2)$—recall that there are no planes containing 4 points of s.

Assume first that the distribution is $(3, 3, 3)$, $(3, 3, 2)$ or $(3, 3, 1)$. This means that, w.l.o.g. for $\nu = 1, 2$, we may apply the scenario of Sect. 7.1 leading to $w_{1/2}(s) \leq 6$ and, for $\nu = 3$, the first one from Sect. 7.2 leading to $w_3(s) \leq 11$.

Similarly, the distribution $(3, 2, 2)$ leads to, w.l.o.g., $w_1(s) \leq 6$ and $w_{2/3}(s) \leq 9$.

Altogether this means that s fits either in a $(6 \times 6 \times 11)$- or in a $(6 \times 9 \times 9)$-grid. Both cases had been checked with a computer, using a rather straight algorithm. The result was that all possible maximal exceptional sequences had contaminated the whole space $\mathbb{Z}^3$ in at most seven steps.

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References

1. Altmann, K., Buczyński, J., Kastner, L., Winz, A.-L.: Immaculate line bundles on toric varieties. Pure Appl. Math. Q. **16**(4), 1147–1217 (2020)
2. Altmann, K., Witt, F.: The structure of exceptional sequences on toric varieties of Picard rank two. arXiv:2112.14637 [math.AG]

3. Böhning, C., von Bothmer, H.-C.G., Katzarkov, L., Sosna, P.: Determinantal Barlow surfaces and phantom categories. J. Europ. Math. Soc. (JEMS) **17**(7), 1569–1592 (2015)
4. Efimov, A.I.: Maximal lengths of exceptional collections of line bundles. J. Lond. Math. Soc. (2) **90**(2), 350–372 (2014)
5. Kawamata, Y.: Derived categories of toric varieties. Michigan Math. J. **54**(3), 517–535 (2006)
6. Kawamata, Y.: Derived categories of toric varieties. II. Michigan Math. J. **62**(2), 353–363 (2013)
7. Kawamata, Y.: Derived categories of toric varieties. III. Europ. J. Math. **2**(1), 196–207 (2016)
8. Lee, D.-W.: Classification of full exceptional collections on smooth toric Fano varieties with Picard rank two. arXiv:2005.09783v2 [math.AG]
9. Mironov, M.: Lefschetz exceptional collections in S_k-equivariant categories of $(\mathbb{P}^n)^k$. Europ. J. Math. **7**(3), 1182–1208 (2021)
10. Sosna, P.: Some remarks on phantom categories and motives. Bull. Belg. Math. Soc. Simon Stevin **27**(3), 337–352 (2020)

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