The passage from the integral to the rational group ring in algebraic K-theory

von

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Contents

1 Introduction

Let G be a group, R a ring and define RG to be the group algebra of G over R. The algebraic K-theory group is defined as

$$K_0RG \coloneqq \mathbb{Z}\{\text{isomorphism classes of f.g. projective } RG\text{-modules}\} / \equiv$$

where $\equiv$ is the equivalence relation generated by $[A \oplus B] \equiv [A] + [B]$.

For R being the ring of integers, the subgroup of $K_0\mathbb{Z}G$ generated by the free modules is always a split summand isomorphic to $\mathbb{Z}$. The *reduced* K-theory group $\widetilde{K_0}\mathbb{Z}G$ is defined as the quotient of $K_0\mathbb{Z}G$ by this summand.

The group $\widetilde{K_0}\mathbb{Z}G$ is an important invariant of G appearing in a variety of geometric problems, the most notable as the group containing Wall's finiteness obstruction. Wall [Wal65] showed that for every finitely dominated CW-complex X with fundamental group $G = \pi_1(X)$, there exists an element $w(X)$ in $\widetilde{K_0}\mathbb{Z}G$ which is trivial iff X is actually finite. Moreover, every element of $\widetilde{K_0}\mathbb{Z}G$ is realized in this way from some finitely dominated CW-complex.[1]

However, $\widetilde{K_0}\mathbb{Z}G$ tends to be very hard to compute in general. One of the few structural things that can be said about $\widetilde{K_0}\mathbb{Z}G$ is a theorem due to Swan that for G being a finite group $\widetilde{K_0}\mathbb{Z}G$ is finite.

Changing the base ring to the rational numbers, we can define $\widetilde{K_0}\mathbb{Q}G$ in a similar manner. As before, we obtain a splitting $K_0\mathbb{Q}G \cong \mathbb{Z} \oplus \widetilde{K_0}\mathbb{Q}G$. For G a finite group, $K_0\mathbb{Q}G$ is inherently easier to compute than its integral counterpart. Since in this case $\mathbb{Q}G$ is a finite dimensional semisimple algebra over $\mathbb{Q}$, it splits as a product of matrix algebras over division algebras over $\mathbb{Q}$, one for each irreducible $\mathbb{Q}$-representation of G. This means that $K_0\mathbb{Q}G \cong \mathbb{Z}^{r_\mathbb{Q}}$, where $r_\mathbb{Q}$ is the number of isomorphism classes of irreducible $\mathbb{Q}$-representations of G. In particular, $\widetilde{K_0}\mathbb{Q}G$ is a free abelian group of rank $r_\mathbb{Q} - 1$.

We can thus see that for G being a finite group, the map $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ defined via $[P] \mapsto [P \otimes \mathbb{Q}]$, for P being a f.g. projective $\mathbb{Z}G$-module, is an isomorphism on the summands corresponding to free modules over $\mathbb{Z}G$ and $\mathbb{Q}G$ respectively, and trivial on the quotients

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G,$$

since there it is a homomorphism from a finite to a free abelian group. Swan also showed the following slightly stronger result.

Theorem 1.1 (Swan, [Swa60])**.** *Suppose G is a finite group and P a finitely generated projective $\mathbb{Z}G$-module. Then $P \otimes \mathbb{Q}$ is free.*

The statement that $\widetilde{K_0}\mathbb{Q}G$ is free does not generalize to arbitrary groups. In fact, Kropholler, Moselle [KM91], and Leary [Lea00] constructed specific examples of groups which have 2-torsion elements in $K_0\mathbb{Q}G$. This means we cannot expect to find a straightforward generalization of Swans theorem to infinite groups.

[1] See also [Var89]

Bass [Bas76] investigated this question and formulated what is now known as the strong Bass conjecture for $K_0\mathbb{Z}G$. For this, let $r\colon K_0\mathbb{Z}G \to HH_0(\mathbb{Z}G)$ denote the Hattori-Stallings trace map and define $r_P(g)$ as the coefficient in the sum

$$r(P) = \sum_{[g]\in\mathrm{Conj}(G)} r_P(g)$$

under the isomorphism $HH_0(RG) = \bigoplus_{\mathrm{Conj}(G)} R$. A discussion of the terms involved can be found in Section 8.

Conjecture 1.2 (Strong Bass Conjecture for $K_0\mathbb{Z}G$, [Bas76]). The function $r_P(g)$ is 0 for $g \neq 1$.

Lück, Reich [LR05], Section 3.1.3, show that the strong Bass conjecture for $K_0\mathbb{Z}G$ follows from the stronger claim that the map

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

vanishes rationally, and they also show that this holds true if G satisfies the Farrell-Jones conjecture, see below for a definition. In Remark 3.13 they ask whether this is true integrally.

Conjecture 1.3 (Integral $\widetilde{K_0}\mathbb{Z}G$-to-$\widetilde{K_0}\mathbb{Q}G$-conjecture). The map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is trivial.

Part of this thesis will show the following.

Theorem 1.4 (See Section 10). *The Integral $\widetilde{K_0}\mathbb{Z}G$-to-$\widetilde{K_0}\mathbb{Q}G$-Conjecture is false. A counterexample is given by the group $G := QD_{32} *_{Q_{16}} QD_{32}$, where QD_{32} is the quasi-dihedral group of order 32, and Q_{16} is the generalized quaternion group of order 16.*

The other half of this thesis is an investigation into how much the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ can fail to be trivial under the assumption that G satisfies the Farrell-Jones conjecture. The Farrell-Jones conjecture states that the assembly map

$$\mathrm{colim}_{(G/H)\in\mathrm{Or}G_{\mathrm{VCyc}}} \mathbf{K}\mathbb{Z}H \to \mathbf{K}\mathbb{Z}G$$

is a weak equivalence of spectra. Here $\mathbf{K}\mathbb{Z}G$ refers to the non-connective K-theory spectrum of $\mathbb{Z}G$, and the colimit in question is a homotopy colimit over the category $\mathrm{Or}G_{\mathrm{VCyc}}$, which is the full subcategory of the orbit category of G spanned by the objects G/H with virtually cyclic isotropy group H. The Farrell-Jones conjecture has been shown to be true for a wide class of groups by the work of Bartels, Lück, Reich [BLR07], Bartels, Bestvina [BB18] Kammeyer, Lück, Rüping [KLR16], and Wegner [Weg15] among many others.

Theorem 1.5 (See Section 7). *Suppose G is a group satisfying the Farrell-Jones conjecture. Then*

$$\mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G)$$

is a 2-torsion subgroup of $\widetilde{K_0}\mathbb{Q}G$.[2]

[2] The abelian group $\widetilde{K_0}\mathbb{Q}G$ can have torsion elements away from the prime 2 as well. An example of a group G with 3-torsion in $\widetilde{K_0}\mathbb{Q}G$ is constructed in Example 11.7.

In fact, we can reduce the question of the non-vanishing of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ to the existence of finite subgroups of G admitting certain types of quaternionic representations. Concretely, for finite H let $s(H)$ be the number of irreducible $\mathbb{Q}$-representations that have even global Schur index but odd local Schur index at every prime dividing the order of H. The number $s(H)$ is zero for a wide class of groups, including all abelian, symmetric, alternating, and dihedral groups, as well as all groups admitting an abelian normal subgroup of odd index. The smallest group such that $s(H) \neq 0$ is the generalized quaternion group Q_{16} of order 16 with $s(Q_{16}) = 1$.

Theorem 1.6 (See Section 7)**.** *Suppose G is a group satisfying the Farrell-Jones conjecture. If there exists a model of $E(G;\mathrm{Fin})$ such that for all 0- and 1-cells of $E(G;\mathrm{Fin})$ their isotropy groups H have $s(H) = 0$, then the map*

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

is trivial.[3]

This gives a certain limitation on the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$. As a consequence, we see that $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ vanishes if either $K_0\mathbb{Q}G$ is a free group or $s(H) = 0$ for all finite subgroups H of G, the latter being the case, for example, for G abelian or torsion free.

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[3] Theorem 1.6 is not optimal in the sense that the existence of subgroups H with $s > 0$ alone does not guarantee that the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is non-trivial. See Example 11.6.

2 Preliminaries

Throughout this thesis we will denote the *non-connective algebraic K-theory spectrum* of a non-commutative unital ring R by $\mathbf{K}R$, see e.g. [Wei13], Chapter IV. Its homotopy groups $K_nR \coloneqq \pi_n\mathbf{K}R$ are the algebraic K-theory groups of R.

Sections 3 to 5 are used to phrase and setup the Farrell-Jones conjecture and discuss how to deal with functors on the orbit category $\mathrm{Or}G$. Section 6 is concerned with primarily classical results about lower K-theory groups of finite groups G. The proofs of the main theorems will be found in section 7. The claimed counterexample to the integral $K_0\mathbb{Z}G$-to-$K_0\mathbb{Q}G$ conjecture is discussed in Section 10. Appendix A discusses tools on how to compute negative K-theory for finite groups, using the computer language *GAP*.

We will use the language of ∞-categories in our proofs. The author remarks that this choice is due to convenience, not necessity. The reader not familiar with the topic shall be assured that all arguments can be phrased using the notions of model categories and t-structures on a triangulated category, only that many formal arguments become harder to formulate (e.g. the existence of the object-wise t-structure on a functor category or exactness of many functors involved). A model for the notion of ∞-categories is given by the notion of quasi-categories developed by Joyal and Lurie, which are simplicial sets fulfilling a certain lifting property. The standard reference is [Lur12]. We further included the main results used about stable ∞-categories in Appendix B. We want to remark that most of our results will be phrased in a model independent way, treating the notion of ∞-categories as a black box. The terms *limit* and *colimit* will always be interpreted in an ∞-categorical way. In situations where our ∞-category $\mathcal{C}$ arises from a model category $\mathcal{M}$, limits and colimits in $\mathcal{C}$ are modelled by homotopy limits and homotopy colimits in $\mathcal{M}$. If C is a 1-category, then the nerve $N(C)$ is an ∞-category in which limits model ordinary 1-categorical limits and similarly for colimits. We will often omit the notation for the nerve and just refer to the ∞-category $N(C)$ simply as C when the context is clear. Given two ∞-categories $\mathcal{C}$ and $\mathcal{D}$, there is the ∞-category of functors $\mathrm{Fun}(\mathcal{C},\mathcal{D})$ from $\mathcal{C}$ to $\mathcal{D}$. Functors $A\colon \mathcal{C}\to\mathcal{D}$ are sometimes written as $A(-)$ to highlight that the value of A is dependent on the input. Natural transformations between functors will be depicted with a double arrow, like $A \implies B$.

The ∞-category of *spaces*, sometimes also referred to as *homotopy types* or *anima*, will be denoted as Spc and is characterized via a universal property as the free cocomplete ∞-category generated by a single object (the point) similar to how the category of sets is generated under coproducts by a single object (the set with a single element). It is modelled, for example, by the simplicial category of Kan complexes or the relative category of CW-complexes and weak equivalences being homotopy equivalences. The undercategory $\mathrm{Spc}_{\mathrm{pt}/}$ is called the ∞-category of *pointed spaces* and will be denoted as Spc_*. We have a natural functor $(-)_+\colon \mathrm{Spc}\to\mathrm{Spc}_*$ that adds a disjoint basepoint. The ∞-category of spectra will be denoted as Sp and is characterized again via a universal property as the stabilization of Spc, or equivalently, as the free cocomplete stable ∞-category generated by a single object (the sphere spectrum $\mathbb{S}$). It is modelled, for example, by the relative category of Ω-spectra and weak equivalences being maps that induce isomorphisms on all homotopy groups. Spectra will be denoted by bold-faced letters, e.g. $\mathbf{A}, \mathbf{K}R$ or $\mathbf{Wh}(R;G)$.

Since Sp is stable, the suspension $\mathbf{A} \mapsto \Sigma\mathbf{A}$ defined as the pushout

$$\begin{tikzcd}
\mathbf{A} \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma\mathbf{A}
\end{tikzcd}$$

produces an auto-equivalence of Sp with itself. The suspension functor $\mathrm{Spc}_* \to \mathrm{Sp}$ will be denoted as Σ^∞, the composition $\Sigma^\infty \circ (-)_+$ will be denoted as Σ^∞_+. The homotopy groups of a spectrum $\mathbf{A}$ are denoted as $\pi_n(\mathbf{A})$. Further, denote the 1-category of abelian groups by Ab. Taking homotopy groups yields functors $\pi_n\colon \mathrm{Sp} \to \mathrm{Ab}$. Spectra $\mathbf{A}$ with the property that $\pi_n\mathbf{A} = 0$ for $n < 0$ will be called connective, and spectra $\mathbf{A}$ with the property that $\pi_n\mathbf{A} = 0$ for $n > 0$ will be called coconnective. The functor π_0 becomes an equivalence when restricted to the intersection of the full subcategories of connective and coconnective spectra (essentially a consequence of the Brown representability theorem). Its inverse will be denoted by $\mathbf{H}$, or the *Eilenberg-Maclane functor*. The inclusion of the full subcategory of connective spectra into Sp admits a right adjoint which will be called $\tau_{\geq 0}$, and we define for any $a \in \mathbb{Z}$ the functor $\tau_{\geq a}$ as $\Sigma^a\tau_{\geq 0}\Sigma^{-a}$. Similarly, the inclusion of the full subcategory of coconnective spectra into Sp admits a left adjoint which will be called $\tau_{\leq 0}$, and $\tau_{\leq a}$ is defined as $\Sigma^a\tau_{\leq 0}\Sigma^{-a}$ in the same way. For $a, b \in \mathbb{Z}$ the compositions $\tau_{\geq a}\tau_{\leq b}$ and $\tau_{\leq b}\tau_{\geq a}$ are naturally isomorphic and will denoted as $\mathbf{A} \mapsto \mathbf{A}[a, b]$. This type of structure defines a t-structure on Sp, more on this in Appendix B.

For a fixed ∞-category $\mathcal{C}$, for two given objects $c, c' \in \mathcal{C}$ the mapping space from c to c' will be denoted $\mathrm{Map}_{\mathcal{C}}(c, c')$. The subscript is omitted in the case of $\mathcal{C}$ being the ∞-category of spaces. $\mathrm{Map}_{\mathcal{C}}$ is a bi-functor into the category Spc, contravariant in the left and covariant in the right variable. We also use the notation $[c, c'] := \pi_0\mathrm{Map}_{\mathcal{C}}(c, c')$. Note that $[c, c']$ is just the Hom-set of the 1-category given by the homotopy category of $\mathcal{C}$. If $\mathcal{C}$ is moreover a stable ∞-category, the space $\mathrm{Map}_{\mathcal{C}}(c, c')$ is naturally the zero-th space of a spectrum $\mathbf{map}_{\mathcal{C}}(c, c')$, which is called the mapping spectrum from c to c'. Again $\mathbf{map}_{\mathcal{C}}$ is naturally a functor in both variables. We, similarly, omit the subscript in the case of $\mathcal{C}$ being the ∞-category of spectra. Since

$$[c, c'] = \pi_0\mathrm{Map}_{\mathcal{C}}(c, c') \cong \pi_0\mathbf{map}_{\mathcal{C}}(c, c')$$

is the zero-th homotopy group of a spectrum, the set $[c, c']$ comes naturally with the structure of an abelian group. The mapping space for two functors $F, G\colon \mathcal{D} \to \mathcal{C}$ in the functor category will also be denoted as $\mathrm{Nat}_{\mathcal{C}}(F, G)$ and is called the *space of natural transformations* from F to G. If $\mathcal{C}$ is stable, the corresponding mapping spectrum is also written as $\mathbf{nat}_{\mathcal{C}}(F, G)$.

If G is a group, the category $\mathcal{M}$ of topological spaces with left G-action admits the structure of a closed simplicial model category [DK84]. The associated ∞-category is the ∞-category of G-spaces, G-Spc. Similarly, the category of pointed G-spaces, $G\text{-}\mathrm{Spc}_*$ is defined as the over category $G\text{-}\mathrm{Spc}_{\mathrm{pt}/}$ where pt is the one-point space with trivial G-action. Let H be a subgroup of G. We can think of the left G-set G/H as a discrete G-space. We can also interpret the n-sphere S^n as well as the n-disc D^n as G-spaces by equipping them with the trivial G-action. If X is an object of $\mathcal{M}$, i.e. a topological space with continuous left G-action, a G-CW-structure on X refers to a sequence of G-spaces $(X^{(k)})_{k\geq 0}$, maps $\iota_k : X^{(k)} \to X^{(k+1)}$ such that there exist pushout squares in the category $\mathcal{M}$,

$$\begin{tikzcd}
\coprod_{i\in I_k} G/H_i \times S^n \arrow[r, "\phi_k"] \arrow[d] & X^{(k)} \arrow[d, "\iota_k"] \\
\coprod_{i\in I_k} G/H_i \times D^{n+1} \arrow[r] & X^{(k+1)},
\end{tikzcd}$$

and an equivariant homeomorphism $X \cong \mathrm{colim}_k X^{(k)}$, where the colimit in question is over the tower given by the maps $\iota_k \colon X^{(k)} \to X^{(k+1)}$. The maps

$$\phi_k \colon \coprod_{i\in I_k} G/H_i \times S^n \to X^{(k)}$$

are called *attaching maps*. The indexing sets I_k can be arbitrary sets. The space $X^{(k)}$ is also called the *k-skeleton* of X. We also refer to X together with a fixed choice G-CW-structure as a G-CW-complex. By [DK84], Theorem 2.2, the cofibrant objects in $\mathcal{M}$ are exactly retracts of G-CW-spaces. Moreover, every object of the ∞-category G-Spc can be represented by a G-CW-space. The maps $G/H_i \times S^n \to G/H_i \times D^{n+1}$ are cofibrations in the model category $\mathcal{M}$. This means that for a given G-CW-complex X, the squares

$$\begin{tikzcd}
\coprod_{i\in I_k} G/H_i \times S^n \arrow[r, "\phi_k"] \arrow[d] & X^{(k)} \arrow[d, "\iota_k"] \\
\coprod_{i\in I_k} G/H_i \times D^{n+1} \arrow[r] & X^{(k+1)},
\end{tikzcd}$$

are also pushout squares in the ∞-category G-Spc. If X is an object of G-Spc we will define a G-CW-structure on X to be a G-CW-structure on any representing object of X in $\mathcal{M}$. We will also refer to objects of G-Spc from now on as G-spaces. Note that often we will leave it implicit that a given object of G-Spc, i.e. a G-space, is technically speaking only represented by a topological space with G-action up to weak equivalence.

3 G-homology theories and functors on the orbit category

Define the *orbit category* $\mathrm{Or}G$ as the full subcategory of the 1-category of G-sets spanned by the G-sets with transitive action. Equivalently, each object S of $\mathrm{Or}G$ is G-equivariantly isomorphic to a set of left cosets G/H, acted on by the left, where H is the isotropy group of a chosen element s of S. It is an elementary fact that each map in $\mathrm{Or}G$ can be decomposed into a composition of maps given by inclusions $\iota : G/H \to G/H', kH \mapsto kH'$ for $H \subset H'$ and conjugations $c_g : G/H \to G/(g^{-1}Hg), kH \mapsto kHg = (kg)(g^{-1}Hg)$. If we have a G-space X, we can think of the assignment $G/H \mapsto X^H$ as a functor

$$X^- \colon \mathrm{Or}G^{op} \to \mathrm{Spc}.$$

Elmendorf's Theorem states that mapping X to X^- produces an equivalence of ∞-categories $G\text{-}\mathrm{Spc} \simeq \mathrm{Fun}(\mathrm{Or}G^{op}, \mathrm{Spc})$, see e.g. [DK84] Theorem 3.1. Note that it further refines to an equivalence $G\text{-}\mathrm{Spc}_* \simeq \mathrm{Fun}(\mathrm{Or}G^{op}, \mathrm{Spc}_*)$ for pointed G-spaces. Throughout this section, if $\mathbf{A}$ is a functor $\mathrm{Or}G \to \mathrm{Sp}$, we will write $\mathbf{A}(H)$ for the value of $\mathbf{A}$ at the orbit G/H.

Definition 3.1 (Orbit smash product)**.** Let X be a pointed G-space and $\mathbf{A}$ be a functor $\mathrm{Or}G \to \mathrm{Sp}$. Define the *orbit smash product* as the coend (see Section B.4)

$$X \wedge_{\mathrm{Or}G} \mathbf{A} := \int^{G/H} X^H \wedge \mathbf{A}(H).$$

It is straightforward to show that $X \wedge_{\mathrm{Or}G} \mathbf{A} = \Sigma^\infty X \otimes_{\mathrm{Or}G} \mathbf{A}$ where $\otimes_{\mathrm{Or}G}$ is defined in section B.4. Since Σ^∞ preserves colimits, it is straightforward to see that the functors $A_* := \pi_*(- \wedge_{\mathrm{Or}G} \mathbf{A})$ define a G-equivariant homology theory on CW-complexes in the sense of Lück [Lüc19], Definition 2.1. Moreover, we can equip $\mathrm{Fun}(\mathrm{Or}G, \mathrm{Sp})$ with the object-wise t-structure defined in Section B.33. Since Σ^∞ also preserves connectivity, we get from Lemma B.42 that if $\mathbf{A}$ is object-wise connective, then $X \wedge_{\mathrm{Or}G} \mathbf{A}$ is connective and further, if X is m-connected, then so is $X \wedge_{\mathrm{Or}G} \mathbf{A}$. As a special case we deduce the following useful lemma.

Lemma 3.2. Let X be a pointed G-CW-complex and $\mathbf{A}$ a functor $\mathrm{Or}G \to \mathrm{Sp}_{\geq 0}$ with values in connective spectra. Denote by $X^{(k)}$ the k-skeleton of X. Then the homomorphism $\pi_n(X^{(k)} \wedge_{\mathrm{Or}G} \mathbf{A}) \to \pi_n(X \wedge_{\mathrm{Or}G} \mathbf{A})$ is an isomorphism for $n < k$ and an epimorphism for $n = k$.

Proof. Taking the cofiber of $X^{(k)} \to X^{(k+1)}$ gives a cofiber sequence

$$X^{(k)} \to X^{(k+1)} \to \bigvee_{i \in I} G/H_i \times S^{k+1}$$

Applying the exact functor $- \wedge_{\mathrm{Or}G} \mathbf{A}$ thus gives the fiber sequence

$$X^{(k)} \wedge_{\mathrm{Or}G} \mathbf{A} \to X^{(k+1)} \wedge_{\mathrm{Or}G} \mathbf{A} \to \bigvee_{i \in I} (G/H_i \times S^{k+1}) \wedge_{\mathrm{Or}G} \mathbf{A}.$$

Since $\mathbf{A}$ is object-wise connective, smashing preserves connectivity (see Lemma B.42), and thus $\pi_n((G/H_i \times S^{k+1}) \wedge_{\mathrm{Or}G} \mathbf{A}) = 0$ for $n < k+1$. The claimed statements now follow by induction for X being finite dimensional G-CW from the long exact sequence in homotopy groups of the above fiber sequence. For general X we have $X = \mathrm{colim}_k X^{(k)}$. The functor $- \wedge_{\mathrm{Or}G} \mathbf{A}$ preserves colimits thus $\pi_n(X \wedge_{\mathrm{Or}G} \mathbf{A}) \cong \pi_n(\mathrm{colim}_k(X^{(k)} \wedge_{\mathrm{Or}G} \mathbf{A})) \cong \pi_n(X^{(k)} \wedge_{\mathrm{Or}G} \mathbf{A})$ for $k > n$, thus reducing the lemma to the finite case. $\square$

Lemma 3.3. Suppose $\mathbf{A}$ is a functor $\mathrm{Or}G \to \mathrm{Sp}$ and X a G-CW-space that admits a 1-dimensional model of the form

$$\begin{tikzcd} \coprod_{i\in I} G/H_i \times S^0 \arrow[r, "{(f,g)}"] \arrow[d] & \coprod_{j\in J} G/K_i \arrow[d] \\ \coprod_{i\in I} G/H_i \times D^1 \arrow[r] & X \end{tikzcd}$$

Then there is a fiber sequence

$$\bigvee_{i\in I} \mathbf{A}(H_i) \xrightarrow{f-g} \bigvee_{j\in J} \mathbf{A}(K_j) \to X_+ \wedge_{\mathrm{Or}G} \mathbf{A}.$$

Proof. The functor $(-)_+ \wedge_{\mathrm{Or}G} \mathbf{A}$ is an exact functor from the category of G-spaces to Sp. It thus sends the pushout square

$$\begin{tikzcd} \coprod_{i\in I} G/H_i \times S^0 \arrow[r, "{(f,g)}"] \arrow[d] & \coprod_{j\in J} G/K_i \arrow[d] \\ \coprod_{i\in I} G/H_i \times D^1 \arrow[r] & X \end{tikzcd}$$

to a pushout square of spectra. We have equivalences $(G/H)_+ \wedge_{\mathrm{Or}G} \mathbf{A} \simeq \mathbf{A}(H)$, $(G/H \times S^0)_+ \wedge_{\mathrm{Or}G} \mathbf{A} \simeq \mathbf{A}(H) \vee \mathbf{A}(H)$ and $(G/H \times D^1)_+ \wedge_{\mathrm{Or}G} \mathbf{A} \simeq \mathbf{A}(H)$. This means we have the pushout square

$$\begin{tikzcd} \bigvee_{i\in I} \mathbf{A}(H_i) \vee \mathbf{A}(H_i) \arrow[r, "{(f,g)}"] \arrow[d, "{(id,id)}"] & \bigvee_{j\in J} \mathbf{A}(K_j) \arrow[d] \\ \bigvee_{i\in I} \mathbf{A}(H_i) \arrow[r] & X_+ \wedge_{\mathrm{Or}G} \mathbf{A}. \end{tikzcd}$$

This is equivalent to the fiber sequence

$$\bigvee_{i\in I} \mathbf{A}(H_i) \vee \mathbf{A}(H_i) \xrightarrow{\left(\begin{smallmatrix} \mathrm{id} & \mathrm{id} \\ -f & -g \end{smallmatrix}\right)} \bigvee_{i\in I} \mathbf{A}(H_i) \vee \bigvee_{j\in J} \mathbf{A}(K_j) \to X_+ \wedge_{\mathrm{Or}G} \mathbf{A}.$$

Elementary row and column reduction now yields the desired fiber sequence. $\square$

4 Assembly and the Farrell-Jones conjecture

Definition 4.1. Let G be a group. A *family of subgroups* is a set of subgroups $\mathcal{F}$ that is closed under subgroups and conjugation.

Example 4.2. The following four examples of families will be relevant.

- The trivial family Triv consisting of only the subgroup $\{1\}$.
- The family All consisting of all subgroups.
- The family Fin consisting of all finite subgroups.
- The family VCyc consisting of all virtually cyclic subgroups. A group H is virtually cyclic if it contains a cyclic subgroup of finite index.

If $\mathbf{A}$ is a functor $\mathrm{Or}G$ to Sp then

$$\mathrm{colim}_{\mathrm{Or}G}\mathbf{A} \simeq \mathbf{A}(G/G)$$

since $G/G \cong \mathrm{pt}$ is a terminal object in the category $\mathrm{Or}G$. The property of $\mathbf{A}$ satisfying assembly states that this still holds true when the domain, over which the colimit is taken, is suitably restricted.

Definition 4.3. Let $\mathcal{F}$ be a family of subgroups for the group G and $\mathbf{A}$ a functor $\mathrm{Or}G$ to Sp. Denote by $\mathrm{Or}G_{\mathcal{F}}$ the full subcategory of $\mathrm{Or}G$ spanned by the objects G/H with $H \in \mathcal{F}$. Then the inclusion $\mathrm{Or}G_{\mathcal{F}} \subset \mathrm{Or}G$ induces a natural map

$$\mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}}\mathbf{A} \to \mathrm{colim}_{\mathrm{Or}G}\mathbf{A} \simeq \mathbf{A}(G/G).$$

We say $\mathbf{A}$ *satisfies assembly for* $\mathcal{F}$ if this map is an equivalence.

Lemma 4.4. Assume we have a fiber sequence of functors $\mathrm{Or}G \to \mathrm{Sp}$,

$$\mathbf{A} \implies \mathbf{B} \implies \mathbf{C}.$$

If any two of them satisfy assembly for a family $\mathcal{F}$, then so does the third.

Proof. Both $\mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}}$ as well as $\mathrm{colim}_{\mathrm{Or}G}$ are exact functors from the ∞-category $\mathrm{Fun}(\mathrm{Or}G, \mathrm{Sp})$ to Sp giving the diagram

$$\begin{tikzcd}
\mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}}\mathbf{A} \arrow[r] \arrow[d] & \mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}}\mathbf{B} \arrow[r] \arrow[d] & \mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}}\mathbf{C} \arrow[d] \\
\mathrm{colim}_{\mathrm{Or}G}\mathbf{A} \arrow[r] & \mathrm{colim}_{\mathrm{Or}G}\mathbf{B} \arrow[r] & \mathrm{colim}_{\mathrm{Or}G}\mathbf{C}
\end{tikzcd}$$

with rows being fiber sequences. The statement now follows from the 5-lemma. $\square$

We are concerned with one particular type of functor on the orbit category - the functor that associates G/H to the algebraic K-theory spectrum of its group algebra over a fixed base ring R, $\mathbf{K}RH$. However, note that algebraic K-theory is a priori only functorial in ring homomorphisms. This models the morphisms $\mathbf{K}RH \to \mathbf{K}RH'$ corresponding to inclusions $H \subset H'$. We also need functoriality with respect to conjugation morphisms $c_g\colon G/H \to G/(g^{-1}Hg)$. These can give a priori different ring homomorphisms $RH \to Rg^{-1}Hg$ depending on the choice of representative g, meaning there is no good functor $\mathrm{Or}G \to \mathrm{Rings}$. This issue has been addressed by James Davis and Wolfgang Lück, and we will summarize the main results necessary for this thesis here.

Lemma 4.5 (Davis, Lück [DL98] Chapter 2 and Lemma 2.4)**.** Let G be a group and R be a ring. There exists a functor of 1-categories

$$\mathbf{K}^{alg}R(-)\colon \mathrm{Grpds} \to \Omega\text{-}\mathrm{Sp}$$

where Grpds is the category of groupoids and functors between them and Ω-Sp is the 1-category of Ω-Spectra. If $\mathcal{G}$ is a groupoid, the spectrum $\mathbf{K}^{alg}R(\mathcal{G})$ is defined as the non-connective K-theory spectrum of the additive category

$$((P(R\mathcal{G}_{\oplus})^{\cong})^{gp}$$

where $\mathcal{G}_{\oplus}$ is the free symmetric monoidal category generated by $\mathcal{G}$, $R\mathcal{C}$ is the free R-linear category generated by $\mathcal{C}$, $P(-)$ is idempotent completion, $(-)^{\cong}$ is the underlying groupoid, and $(-)^{gp}$ refers to the group completion of a symmetric monoidal R-category. It has the properties

1. If $F_i\colon \mathcal{G}_0 \to \mathcal{G}_1$ for $i = 0, 1$ are functors of groupoids and $T : F_0 \to F_1$ is a natural transformation between them, then the induced maps of spectra

$$\mathbf{K}^{alg}R(F_i)\colon \mathbf{K}^{alg}R(\mathcal{G}_0) \to \mathbf{K}^{alg}R(\mathcal{G}_1)$$

   are homotopy equivalent.

2. Let $\mathcal{G}$ be a groupoid. Suppose that $\mathcal{G}$ is connected, i.e. there is a morphism between any two objects. For an object $x \in \mathrm{Ob}(\mathcal{G})$, let $\mathcal{G}_x$ be the full subgroupoid with precisely one object, namely x. Then the inclusion $i_x : \mathcal{G}_x \to \mathcal{G}$ induces a homotopy equivalence

$$\mathbf{K}^{alg}R(i_x)\colon \mathbf{K}^{alg}R(\mathcal{G}_x) \to \mathbf{K}^{alg}R(\mathcal{G})$$

   and $\mathbf{K}^{alg}R(\mathcal{G}_x)$ is isomorphic to the non-connective algebraic K-theory spectrum associated to the group ring $R\mathrm{aut}_{\mathcal{G}}(x)$.

Theorem 4.6. *Let G be a group and R be a ring. There exists a functor*

$$\mathbf{K}R(-)\colon \mathrm{Or}G \to \mathrm{Sp}$$

with the properties

- $\mathbf{K}R(G/H) \simeq \mathbf{K}RH$ *where* $\mathbf{K}RH$ *is the non-connective algebraic K-theory spectrum of the group ring RG.*

- *If $H \subset H'$ giving the canonical map $G/H \to G/H'$, the induced map $\mathbf{K}RH \to \mathbf{K}RH'$ corresponds to the map induced by the ring homomorphism $RH \to RH'$.*

- *The action of the conjugation morphism $c_g\colon G/\{1\} \to G/\{1\}$ on $\mathbf{K}R(G/\{1\}) \simeq \mathbf{K}R$ is homotopic to the identity.*

- *Let $g \in G$. The action of the conjugation morphism $c_g\colon G/H \to G/(g^{-1}Hg)$ on the homotopy groups $K_nRH \to K_nR(g^{-1}Hg)$ is induced by the ring homomorphism $RH \to R(g^{-1}Hg), h \mapsto g^{-1}hg$ and independent of the chosen representative g.*

Remark 4.7. The action in the last point of c_g on K_0 can be understood in the following way. It sends a f.g. projective RH-module P to the $R(g^{-1}Hg)$-module P^g with same underlying R-module as P and the scalar multiplication of an element $h' \in g^{-1}Hg$ on $x \in P$ given by $h * x \coloneqq gh'g^{-1} \cdot x$.

Proof. The ∞-category of spectra is a localization of the nerve of the 1-category of Ω-spectra. Let $L : N(\Omega\text{-Sp}) \to \mathrm{Sp}$ be the corresponding localization functor. There is also a natural functor $G \int (-) : \mathrm{Or}G \to \mathrm{Grpds}$ which sends the G-set G/H to the groupoid $G \int G/H$ with object set G/H and a morphism from gH to $g'H$ for each element $g'' \in G$ such that $g''gH = g'H$. Define $\mathbf{K}R(-)$ as the composite

$$\mathbf{K}R(-) \coloneqq L \circ \mathbf{K}^{alg}R \circ G \textstyle\int (-)$$

with the functor $\mathbf{K}^{alg}R$ being given by Lemma 4.5.

For a subgroup H of G write BH for the groupoid with a single object and automorphism group H. The statement that $\mathbf{K}R(G/H) \simeq \mathbf{K}RH$ follows from Lemma 4.5 (2), since the automorphism group of the object H in $G \int G/H$ is exactly H. This means we have the inclusion functor $BH \to G \int G/H$, which produces the claimed equivalence in the first point.

If $H \subset H'$ are two subgroups of G, we have a functor $BH \to BH'$, which fits into the commutative square

$$\begin{tikzcd} BH \arrow[r] \arrow[d] & G \textstyle\int G/H \arrow[d] \\ BH' \arrow[r] & G \textstyle\int G/H'. \end{tikzcd}$$

The functor $\mathbf{K}^{alg}R(BH) \to \mathbf{K}^{alg}R(BH')$ is equivalent to the classical map $\mathbf{K}RH \to \mathbf{K}RH'$ induced by the the ring homomorphism $RH \to RH'$, proving the second claim.

For the third claim we use Lemma 4.5 (1). The groupoid $G \int G/\{1\}$ is contractible since every object is in fact both terminal and initial, which means that all endofunctors of $G \int G/\{1\}$ are connected via natural transformations. In particular the functor coming from the conjugation morphism c_g and the identity realize to homotopic maps in K-theory.

For the last point, we have a commutative square of groupoids

$$\begin{tikzcd} BH \arrow[r, "\mathrm{inc}_H"] \arrow[d, "g^{-1}(-)g"'] & G \textstyle\int G/H \arrow[d, "c_g \circ \mathrm{trans}_{g^{-1}}"] \\ B(g^{-1}Hg) \arrow[r, "\mathrm{inc}_{g^{-1}Hg}"] & G \textstyle\int G/(g^{-1}Hg) \end{tikzcd}$$

where the functor $g^{-1}(-)g : BH \to B(g^{-1}Hg)$ is given via the group homomorphism that is the conjugation $h \mapsto g^{-1}hg$ (it is trivial on objects as both groupoids have only a single object), and

$$\mathrm{trans}_{g^{-1}} : G \int G/H \to G \int G/H$$

is the functor that acts on objects as

$$g'H \mapsto g^{-1}g'H,$$

and sends the morphism

$$g'H \xrightarrow{k} kg'H$$

to

$$g^{-1}g'H \xrightarrow{g^{-1}kg} g^{-1}kg'H.$$

Note that $\mathrm{trans}_{g^{-1}}$ is an auto-equivalence of $G \int G/H$ that sends the object H to $g^{-1}H$. The group homomorphism $c_g \colon H \to g^{-1}Hg$ induces the claimed map in K_n, thus showing the last point. If g'' is another element in G such that $c_g = c_{g''} : G/H \to G/g^{-1}Hg$ represent the same map in $\mathrm{Or}G$, which is exactly the case when $g(g'')^{-1} \in H$, it is elementary to show that $\mathrm{trans}_{g^{-1}}$ and $\mathrm{trans}_{(g'')^{-1}}$ are naturally equivalent functors, thus inducing homotopic maps on K-theory spectra and therefore the same map in K_n. □

Definition 4.8 (Farrell-Jones conjecture)**.** A group G has the *Farrell-Jones property* if the functor for non-connective algebraic K-theory $\mathbf{K}R\colon \mathrm{Or}G \to \mathrm{Sp}$ satisfies assembly for the family of virtually cyclic subgroups and any ring R. We will also sometimes refer to this as saying that G *satisfies the Farrell-Jones conjecture.*

Remark 4.9. The Farrell-Jones conjecture is known to hold for a wide range of groups. A recent summary of results can be found in [RV18], Theorem 27.

In order to compute the colimits involved in the assembly maps, a useful tool for geometric arguments is the notion of classifying spaces for a family of subgroups $\mathcal{F}$ of G.

Definition 4.10. Let $\mathcal{F}$ be a family of subgroups of G. If X is a G-CW-space with

$$X^H \simeq \begin{cases} \mathrm{pt} & \text{if } H \in \mathcal{F} \\ \emptyset & \text{if } H \notin \mathcal{F}, \end{cases}$$

we call X a *classifying space for the family* $\mathcal{F}$. We will write $E(G;\mathcal{F})$ or sometimes $E\mathcal{F}$ for such a G-CW-space X.

Remark 4.11. Our choice of fixed notation for classifying spaces of families is justified, since they exist and are unique up to G-equivariant homotopy, see [Lüc05] Theorem 1.9.

Example 4.12. The universal cover EG of BG with its free G-action is a classifying space for the trivial family Triv consisting only of the single subgroup $\{1\}$. The point with trivial G-action is a classifying space for the family of all subgroups.

Theorem 4.13 (See also [MNN19], Proposition A.2)**.** *Let $i_{\mathcal{F}}$ be the inclusion of the category* $\mathrm{Or}G_{\mathcal{F}}$ *into the category of G-spaces. Then*

$$\mathrm{colim}(i_{\mathcal{F}}) = E(G;\mathcal{F})$$

is a model for a classifying space for $\mathcal{F}$.

Proof. Let H be a subgroup of G. Under the equivalence $G\text{-Spc} \simeq \mathrm{Fun}(\mathrm{Or}G, \mathrm{Sp})$ given by Elmendorf's theorem the operation of taking H-fixed points corresponds to evaluation at G/H. Since colimits of functors are computed objectwise this means that taking H-fixed points commutes with colimits. Now, if $H \notin \mathcal{F}$, then

$$\mathrm{colim}(i_{\mathcal{F}})^H \simeq \mathrm{colim}_{G/K \in \mathrm{Or}G_{\mathcal{F}}}((G/K)^H) = \emptyset.$$

Now suppose H in $\mathcal{F}$. Then

$$\mathrm{colim}(i_{\mathcal{F}})^H \simeq \mathrm{colim}_{G/K \in \mathrm{Or}G_{\mathcal{F}}}(\mathrm{Map}_{\mathrm{Or}G_{\mathcal{F}}}(G/H, G/K))$$

is the colimit over a corepresentable functor and thus contractible, see Example B.40. □

The following lemma now explains why classifying spaces of families are such a useful tool for understanding assembly maps.

Lemma 4.14. Let $\mathcal{F}$ be a family of subgroups of G and $\mathbf{A}$ a functor $\mathrm{Or}G \to \mathrm{Sp}$. There is a natural equivalence

$$\mathrm{colim}_{\mathrm{Or}G_{\mathcal{F}}} \mathbf{A} \to E(G;\mathcal{F})_+ \wedge_{\mathrm{Or}G} \mathbf{A}.$$

Proof. Since $\wedge_{\mathrm{Or}G}$ commutes with colimits we have

$$\begin{aligned} E(G;\mathcal{F})_+ \wedge_{\mathrm{Or}G} \mathbf{A} &\simeq (\mathrm{colim}_{G/K \in \mathrm{Or}G_{\mathcal{F}}}(G/K))_+ \wedge_{\mathrm{Or}G} \mathbf{A} \\ \simeq \mathrm{colim}_{G/K \in \mathrm{Or}G_{\mathcal{F}}}(G/K_+ \wedge_{\mathrm{Or}G} \mathbf{A}) &\simeq \mathrm{colim}_{G/K \in \mathrm{Or}G_{\mathcal{F}}}(\mathbf{A}(K)). \end{aligned}$$

□

We will also need the following two results on assembly in K-theory.

Theorem 4.15 (See [Bar03])**.** *The map*

$$E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{K}R \to E\mathrm{VCyc}_+ \wedge_{OrG} \mathbf{K}R$$

is split injective and is so naturally with respect to the ring R and the group G.

Lemma 4.16 (See [LR05], Proposition 2.14)**.** The map

$$E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{K}\mathbb{Q} \to E\mathrm{VCyc}_+ \wedge_{\mathrm{Or}G} \mathbf{K}\mathbb{Q}$$

is an equivalence.

Remark 4.17. A consequence of Lemma 4.16 is that if G satisfies the Farrell-Jones conjecture, the functor $\mathbf{K}\mathbb{Q}-$ actually satisfies finite assembly. Since $\mathbf{K}\mathbb{Q}H$ is connective for finite groups H, this implies that $K_0\mathbb{Q}-$ satisfies assembly too, in the sense that

$$K_0\mathbb{Q}G = \mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{Fin}}} K_0\mathbb{Q}H$$

with the colimit in question being relative to the 1-category of abelian groups. This is because we have

$$\begin{aligned} \mathbf{H}K_0\mathbb{Q}G &\simeq \tau_{\leq 0}\mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{Fin}}} \mathbf{K}\mathbb{Q}H \\ &\simeq \mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{Fin}}} \tau_{\leq 0}\mathbf{K}\mathbb{Q}H \simeq \mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{Fin}}} \mathbf{H}K_0\mathbb{Q}H, \end{aligned}$$

since the Postnikov truncation $\tau_{\leq 0}$ commutes with colimits.

5 The Whitehead spectrum $\mathbf{Wh}(R;G)$ and the spectrum $\mathbf{SC}(G)$

We remind the reader that for any ring R, $\widetilde{K_0}R$ is defined as the cokernel of the natural homomorphism $K_0\mathbb{Z} \to K_0R$. For a group ring RG the group K_0RG naturally has K_0R as a split summand, with the split given via the augmentation map $RG \to R$ that sends all $g \in G$ to 1. If the base ring R is such that every projective module is stably free, such as when R is a PID or a local ring, it follows that $K_0R \cong K_0\mathbb{Z} \cong \mathbb{Z}$, and we have $K_0RG \cong \mathbb{Z} \oplus \widetilde{K_0}RG$.

We are ultimately interested in understanding the map

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G.$$

Since our tool of choice, the Farrell-Jones conjecture, gives us a priori the full map on spectra $\mathbf{K}\mathbb{Z}G \to \mathbf{K}\mathbb{Q}G$, we would like to split off the superfluous data in a sensible way. This is where the Whitehead spectrum comes into play.

Definition 5.1. Given a group G and a ring R, we define the *Whitehead spectrum* $\mathbf{Wh}(R;G)$ to be the cofiber of the assembly map

$$BG_+ \wedge \mathbf{K}R \to \mathbf{K}R[G] \to \mathbf{Wh}(R;G),$$

corresponding to the trivial family Triv.

Example 5.2. Let R be a ring. Then

- $\pi_{-i}\mathbf{Wh}(R;G) = K_{-i}RG$ for $i > 0$, if R is regular noetherian (see [Wei13] III, Definition 4.1.),
- $\pi_0\mathbf{Wh}(R;G) = \widetilde{K_0}RG$, if R is in addition a local ring or a PID (see [Wei13] II, §2),
- and furthermore, if R is a field or the ring of integers, then

$$\pi_1\mathbf{Wh}(R;G) = K_1(RG)/\left\{rg | r \in R^\times, g \in G\right\}.$$

In the particular case of R being the integers we have:

- $\pi_{-1}\mathbf{Wh}(\mathbb{Z};G) = K_{-1}\mathbb{Z}G$,
- $\pi_0\mathbf{Wh}(\mathbb{Z};G) = \widetilde{K_0}\mathbb{Z}G$,
- $\pi_1\mathbf{Wh}(\mathbb{Z};G) = \mathrm{Wh}(G) = K_1\mathbb{Z}G/(\{\pm 1\} \otimes G)$, with $\mathrm{Wh}(G)$ being the Whitehead group of G.

More generally, we can do the following construction. Let EG be a universal cover of BG. Note that EG is equivalently a classifying space for the trivial family. The functor

$$(B(-)_+ \wedge \mathbf{K}R)(G/H) := (G/H \times EG)_+ \wedge_{\mathrm{Or}G} \mathbf{K}R$$

from $\mathrm{Or}G$ to spectra comes with a natural transformation

$$\theta \colon B(-)_+ \wedge \mathbf{K}R \to \mathbf{K}R(-),$$

induced from the projection $G/H \times EG \to G/H$, to the functor $\mathbf{K}R$. Let us explain why the notation $B(-)_+ \wedge \mathbf{K}R$ makes sense. The value at G/G is given as

$$EG_+ \wedge_{\mathrm{Or}G} \mathbf{K}R \simeq \mathrm{colim}_{BG}\mathbf{K}R \simeq BG_+ \wedge \mathbf{K}R$$

where we used that the subcategory of $\mathrm{Or}G$ generated by the single object $G/\{1\}$ is a BG and that the action of G on the value $\mathbf{K}R(G/\{1\}) = \mathbf{K}R$ is homotopically trivial. This means that the natural transformation θ becomes the assembly map

$$BG_+ \wedge \mathbf{K}R \to \mathbf{K}RG$$

when evaluated at G/G. If G/H is an arbitrary object of $\mathrm{Or}G$, then $G/H \times E(G;\mathrm{Triv}) \simeq \mathrm{Ind}_H^G(E(H;\mathrm{Triv}))$. The functor Ind_H^G is the left Kan extension induced by the functor $\mathrm{Or}H^{op} \to \mathrm{Or}G^{op}$ under the equivalence of Elmendorfs theorem G-Spc $\simeq \mathrm{Fun}(\mathrm{Or}G^{op}, \mathrm{Spc})$. With this understood, we can use Theorem B.43 to get

$$(G/H \times EG)_+ \wedge_{\mathrm{Or}G} \mathbf{K}R = \mathrm{Ind}_H^G(E(H;\mathrm{Triv})_+) \wedge_{\mathrm{Or}G} \mathbf{K}R \simeq E(H;\mathrm{Triv})_+ \wedge_{\mathrm{Or}H} \mathbf{K}R \simeq BH_+ \wedge \mathbf{K}R.$$

and see that θ has as component on the object G/H the assembly map

$$BH_+ \wedge \mathbf{K}R \to \mathbf{K}RH.$$

This allows us to define the functor $\mathbf{Wh}(R;-)\colon \mathrm{Or}G \to \mathrm{Sp}$ as the cofiber of this natural transformation.

Remark 5.3. More generally, if $\mathcal{F}$ is a family and $\mathbf{A}\colon \mathrm{Or}G \to \mathrm{Sp}$ is a functor, we can define

$$\mathbf{A}_{\mathcal{F}}(G/H) := (G/H \times E\mathcal{F})_+ \wedge_{\mathrm{Or}G} \mathbf{A}$$

and get via the projection $G/H \times E\mathcal{F} \to G/H$ a natural transformation

$$\mathbf{A}_{\mathcal{F}} \to \mathbf{A}.$$

The functor $\mathbf{A}_{\mathcal{F}}$ can be shown to satisfy $\mathcal{F}$-assembly, and we can think of it as a universal approximation of $\mathbf{A}$ from the left by a functor that satisfies $\mathcal{F}$-assembly. This construction appears for example in [DQR11], Lemma 4.1.

The following is an essential lemma that is a consequence of Theorem 4.16 and Lemma 4.15.

Lemma 5.4. The map

$$E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(R;-) \to E\mathrm{VCyc}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(R;-)$$

is split injective and is so naturally with respect to the ring R and the group G.

Proof. We have the commutative diagram

$$\begin{tikzcd}
BG_+ \wedge \mathbf{K}R \arrow[r, "="] \arrow[d] & BG_+ \wedge \mathbf{K}R \arrow[d] \\
E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{K}R \arrow[r] \arrow[d] & E\mathrm{VCyc}_+ \wedge_{\mathrm{Or}G} \mathbf{K}R \arrow[d] \\
E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(R;-) \arrow[r] & E\mathrm{VCyc}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(R;-)
\end{tikzcd}$$

with the columns being fiber sequences, so a natural split in the middle map induces one on the bottom. Hence, the statement follows from Theorem 4.15. □

An immediate consequence of Lemma 4.4 is that if G satisfies the Farrell-Jones conjecture, the functor $\mathbf{Wh}(R;-)$ on $\mathrm{Or}G$ satisfies assembly for the family VCyc. moreover, Theorem 4.16 implies that for $R = \mathbb{Q}$, the functor $\mathbf{Wh}(\mathbb{Q};-)$ then also satisfies assembly for the family Fin.

Corollary 5.5. If G satisfies the Farrell-Jones conjecture, then the image of the map

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

agrees with the image of the map

$$\pi_0(E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(\mathbb{Z},-)) \to \pi_0(E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(\mathbb{Q},-)) \cong \widetilde{K_0}\mathbb{Q}G. \tag{$\star$}$$

In particular if ($\star$) vanishes p-locally for some prime p, then so does the map

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G.$$

Proof. By Lemma 5.4 the group $\pi_0(E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(\mathbb{Z},-))$ is a split summand of $\widetilde{K_0}\mathbb{Z}G$, similarly for $\mathbb{Q}$, so by naturality of the split with respect to ring homomorphisms, the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ splits as a sum of two maps, the second of which has to be trivial, since $\pi_0(E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{Wh}(\mathbb{Q},-)) \cong \widetilde{K_0}\mathbb{Q}G$. To get the second statement, apply the functor $\mathbb{Z}_{(p)} \otimes -$ and use exactness. □

Definition 5.6. Define the *spectrum of singular characters* $\mathbf{SC}(G)$ as the cofiber

$$\mathbf{Wh}(\mathbb{Z};G) \to \mathbf{Wh}(\mathbb{Q};G) \to \mathbf{SC}(G).$$

Write in short $\mathrm{SC}(G) := \pi_0\mathbf{SC}(G)$.

Note that we always have a long exact sequence

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Wh}(G) \arrow[r] & K_1(\mathbb{Q}G)/\{rg \mid r \in \mathbb{Q}^\times, g \in G\} \arrow[r] & \pi_1\mathbf{SC}(G) \arrow[dll, rounded corners] \\
& \widetilde{K_0}\mathbb{Z}G \arrow[r] & \widetilde{K_0}\mathbb{Q}G \arrow[r] & \mathrm{SC}(G) \arrow[dll, rounded corners] \\
& K_{-1}\mathbb{Z}G \arrow[r] & \cdots &
\end{tikzcd}$$

From this we see that the vanishing of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is equivalent to the injectivity of $\widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G)$. We will give a concrete description of the group $\mathrm{SC}(G)$ for finite groups in the next section.

6 Lower K-theory of finite groups

In the following section assume G is finite. We will write $\mathbb{Z}_p$ and $\mathbb{Q}_p$ for the p-adic integers as well as p-adic rationals for a prime p. We will be concerned with the groups $K_0\mathbb{Z}_pG, K_0\mathbb{Q}G, K_0\mathbb{Q}_pG$, as well as $K_{-1}\mathbb{Z}G$.

6.1 Wedderburn decomposition and Schur indices

Suppose in the following that k is a subfield of $\mathbb{C}$. It is a standard fact in representation theory that the group ring kG is semisimple. In particular, it decomposes uniquely as

$$kG \cong \prod_{I \in \mathrm{Irr}_k(G)} M_{n_I \times n_I}(D_I)$$

where $\mathrm{Irr}_k(G)$ is the set of isomorphism classes of irreducible k-representations of G and D_I are division algebras given by

$$D_I = \hom_G(I, I),$$

and $n_I = \langle kG, I \rangle$ is the multiplicity of I appearing in the regular representation kG. This is known as the *Wedderburn decomposition* of kG. The K-theory of a division algebra D is $\mathbb{Z}$ in degree 0 since every left D-module is in fact a D-vectorspace and its negative K-theory vanishes since it is trivially regular noetherian. Using that K-theory commutes with products as well as invariance of K-theory under Morita-equivalence, we get the formula $K_0kG \cong \mathbb{Z}^{r_k(G)}$ where $r_k(G)$ is the number of irreducible k-representations of G and the irreducible representations form a set of generators for this group. We also get that $K_{-n}kG = 0$ for all $n > 0$.

Definition 6.1 (Schur index, see also [Die06] 9.3. and [Ser77] 12.2.)**.** Let I be an irreducible k-representation of G. Then D_I is a division algebra over its center K_I of degree $m(I)^2$ with $m(I) = [D_I, E_I]$ for E_I a maximal field contained in D_I. We call $m(I)$ the *Schur index* of I.

Suppose ρ is an automorphism of $\mathbb{C}$ fixing k. This induces a ring isomorphism $\rho\colon \mathbb{C}[G] \to \mathbb{C}[G]$. For any $\mathbb{C}[G]$-module V define $\rho(V) := \rho^*(V)$. It is not difficult to show that if J is an irreducible $\mathbb{C}$-representation of G, then $\rho(J)$ is irreducible as well. In other words, ρ induces a permutation on the set $\mathrm{Irr}_{\mathbb{C}}(G)$ of isomorphism classes of irreducible $\mathbb{C}$-representations of G and $\mathrm{Irr}_{\mathbb{C}}(G)$ thus has a natural action of $\mathrm{Gal}(\mathbb{C}/k)$.

The following is a consequence of [Ser77] 12.2.:

Lemma 6.2. Let J be an irreducible $\mathbb{C}$-representation. Then, there exists an irreducible k-representation I, unique up to isomorphism, and a number $m(I) > 0$, such that

$$I \otimes \mathbb{C} \cong (\oplus_\rho \rho(J))^{m(I)}$$

with the $\rho(J)$ being all the distinct translates of J under the action of the Galois group $\mathrm{Gal}(\mathbb{C}/k)$, in other words the direct sum is over the orbit of the action of the Galois group $\mathrm{Gal}(\mathbb{C}/k)$ on $\mathrm{Irr}_{\mathbb{C}}(G)$. Conversely, if I is an irreducible k-representation, $I \otimes \mathbb{C}$ splits into a sum of irreducible $\mathbb{C}$-representations as above.

Now fix an embedding of $\mathbb{Q}_p \subset \mathbb{C}$ for a prime p. Assume I is an irreducible $\mathbb{Q}$-representation and suppose $I \otimes \mathbb{Q}_p \cong K_1^{m_1} \oplus \cdots \oplus K_{n_I}^{m_k}$ with the K_i being irreducible $\mathbb{Q}_p$-representations. We can apply the previous lemma to both I in the case of $k = \mathbb{Q}$ as well as K_i in the case $k = \mathbb{Q}_p$. We see that

$$K_i \otimes \mathbb{C} \cong (J_{i1} \oplus \cdots \oplus J_{ij_i})^{m(K_i)}$$

with j_i being the number of distinct $\mathbb{C}$-representations that appear in $K_i \otimes \mathbb{C}$ and

$$I \otimes \mathbb{C} \cong \left(\bigoplus_i J_{i1} \oplus \cdots \oplus J_{ij_i} \right)^{m(I)},$$

where the J_{ij}'s are all Galois translates under the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ and thus representations of the same dimension, which implies that $m(I) = m_i \cdot m(K_i)$. The set of J_{ij}'s forms a transitive $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$-set. Thinking of the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{Q}_p)$ as a subgroup of $\mathrm{Gal}(\mathbb{C}/\mathbb{Q})$ we can understand that the sets of J_{ij}'s for fixed i are exactly the orbits under $\mathrm{Gal}(\mathbb{C}/\mathbb{Q}_p)$. This implies that the numbers j_i are all the same since they are exactly the sizes of those orbits, and thus by a dimension argument we must have that m_i as well as $m(K_i)$ are numbers independent of i. It follows that

$$I \otimes \mathbb{Q}_p \cong (K_1 \oplus \cdots \oplus K_{n_I})^m$$

with $m(I) = m \cdot m(K_i)$ and the K_i are distinct and irreducible $\mathbb{Q}_p$-representations. A similar result holds for $\mathbb{R}$ instead of $\mathbb{Q}_p$.

Definition 6.3 (Local Schur index)**.** Let I be an irreducible $\mathbb{Q}$-representation of G. The *local Schur index* $m_p(I)$ of I at the prime p is defined to be the Schur index of any of the irreducible components of $I \otimes \mathbb{Q}_p$ and is independent of this choice by the argument given above. The *local Schur index at infinity* $m_\infty(I)$ is similarly defined as the Schur index of any of the irreducible components of $I \otimes \mathbb{R}$. [4]

Corollary 6.4. The map $K_0\mathbb{Q}G \to K_0\mathbb{Q}_pG$ is acting on the basis of irreducible $\mathbb{Q}$-representations by

$$[I] \mapsto \frac{m(I)}{m_p(I)}([K_1] + \cdots + [K_{n_I}]).$$

where the K_i are representatives of the irreducible $\mathbb{Q}_p$-representations that appear in $I \otimes \mathbb{Q}_p$ and n_I depends on I. If I and J are distinct irreducible $\mathbb{Q}$-representations, the irreducible components appearing in their individual p-completions are pairwise non-isomorphic. In other words, the map

$$K_0\mathbb{Q}G \to K_0\mathbb{Q}_pG$$

splits as

$$\bigoplus_{I \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathbb{Z} \to \bigoplus_{I \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathbb{Z}\left\{K \in \mathrm{Irr}_{\mathbb{Q}_p}(G) | K \text{ appears as a summand in } I \otimes \mathbb{Q}_p\right\}$$

where we consider irreducible $\mathbb{Q}$- or $\mathbb{Q}_p$-representations up to isomorphism.

[4] As a good summary of what is currently known about rational and local Schur indices we recommend [Ung19].

The following is an essential theorem about the behaviour of the global Schur index of a rational representation with respect to its local indices and follows from [Pie82], Corollary 18.6.

Theorem 6.5. *Let I be a rational irreducible representation of a finite group G. Then*

$$m(I) = \operatorname{lcm}\{m_p(I) | p \text{ prime}\} \cup \{m_\infty\}$$

Corollary 6.6. Assume I as before. The greatest common divisor of the ratios

$$\frac{m(I)}{m_p(I)}$$

for p prime is at most 2. It is 2 exactly when $m(I)$ is even, but $m_p(I)$ is odd for all primes dividing the order of G.

Proof. We have

$$\gcd\left\{\frac{m(I)}{m_p(I)}\middle| p \text{ prime}\right\} = \frac{m(I)}{\operatorname{lcm}\{m_p(I)|p \text{ prime}\}}$$

The local Schur index at ∞ is at most 2, which is the only index missing. This proves that the greatest common divisor of the ratios $\dfrac{m(I)}{m_p(I)}$ is at most 2. In order for the ratio to be exactly 2, we must have $m(I) = 2 \cdot \operatorname{lcm}\{m_p(I)|p \text{ prime}\}$, so we know that $m(I)$ must be even. However, $m(I) = \operatorname{lcm}\{m_p(I)|p \text{ prime}\} \cup \{m_\infty\}$, so this can only happen if all $m_p(I)$ are odd, and $m_\infty(I) = 2$. The converse is clear. ◻

6.2 The map $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ for finite G

The following theorem is due to Swan.

Theorem 6.7 ([SE70], Theorem 3.8.)**.** *The group $\widetilde{K_0}\mathbb{Z}G$ is finite.*

A straightforward consequence is the following theorem, which is the center piece of this thesis:

Theorem 6.8. *Let G be finite. The homomorphism $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ induced by the ring homomorphism $\mathbb{Z} \to \mathbb{Q}$ is zero.*

Proof. By the arguments from the previous section we know that $K_0(\mathbb{Q}G) \cong \mathbb{Z}^{r_\mathbb{Q}}$ is free. The group $\widetilde{K_0}\mathbb{Q}G$ is given as the kernel of the map $K_0\mathbb{Q}G \to K_0\mathbb{Q} \cong \mathbb{Z}$ induced by the trivial group homomorphism $G \to 1$ and thus as a subgroup of a free abelian group also free. Since every homomomorphism from a finite abelian group into a free abelian group vanishes, the claim follows. ◻

6.3 Localization squares for finite groups

The negative K-theory groups of $\mathbb{Z}G$ for G a finite group have been first computed by Carter in [Car80a]. We will repeat the essential points. First, we need the following lemma, which is a consequence of a theorem due to Karoubi, see e.g. [Wei13], Prop. V.7.5.

Lemma 6.9. Let G be a finite group and P the set of primes dividing the order of G. Then the following square is a pullback square of spectra:

$$\begin{tikzcd} \mathbf{K}\mathbb{Z}G \arrow[r] \arrow[d] & \bigvee_{p\in P} \mathbf{K}\mathbb{Z}_p G \arrow[d] \\ \mathbf{K}\mathbb{Z}[P^{-1}]G \arrow[r] & \bigvee_{p\in P} \mathbf{K}\mathbb{Q}_p G, \end{tikzcd}$$

where the maps appearing are induced by the corresponding inclusions of the involved rings.

Corollary 6.10. Fix the same assumptions as in the previous lemma. Then

$$\begin{tikzcd} \mathbf{Wh}(\mathbb{Z};G) \arrow[r] \arrow[d] & \bigvee_{p\in P} \mathbf{Wh}(\mathbb{Z}_p;G) \arrow[d] \\ \mathbf{Wh}(\mathbb{Z}[P^{-1}];G) \arrow[r] & \bigvee_{p\in P} \mathbf{Wh}(\mathbb{Q}_p;G) \end{tikzcd}$$

is a pullback square of spectra as well.

Proof. The square in question is given as the levelwise cofibers of the square

$$\begin{tikzcd} BG_+ \wedge \mathbf{K}\mathbb{Z} \arrow[r] \arrow[d] & \bigvee_{p\in P} BG_+ \wedge \mathbf{K}\mathbb{Z}_p \arrow[d] \\ BG_+ \wedge \mathbf{K}\mathbb{Z}[P^{-1}] \arrow[r] & \bigvee_{p\in P} BG_+ \wedge \mathbf{K}\mathbb{Q}_p \end{tikzcd}$$

and the square in Lemma 6.9. Both are pushouts via the previous lemma, hence the claim follows. $\square$

The following is a straightforward consequence of the fact that any idempotent in the rational group algebra $\mathbb{Q}G$ is already defined over $\mathbb{Z}[P^{-1}]G$.

Lemma 6.11. The map $K_0\mathbb{Z}[P^{-1}]G \to K_0\mathbb{Q}G$ is an isomorphism.

Carter [Car80b] shows that the terms in low degree of the induced Mayer-Vietoris sequence split into the following form:

Lemma 6.12. The following sequence is exact:

$$0 \to K_0\mathbb{Z} \to K_0\mathbb{Q}G \oplus \bigoplus_{p\in P} K_0\mathbb{Z}_p G \to \bigoplus_{p\in P} K_0\mathbb{Q}_p G \to K_{-1}\mathbb{Z}G \to 0.$$

Moreover, the groups $K_{-i}\mathbb{Z}G$ vanish for $i > 1$.

Using the fact that $\mathbb{Q}_pG$ is a finite dimensional semisimple algebra over the field $\mathbb{Q}_p$, we get that $K_{-i}\mathbb{Q}_pG = 0$ for $i > 0$. This together with the previous lemma also implies that $K_{-i}\mathbb{Z}_pG = 0$ and $K_{-i}\mathbb{Z}[P^{-1}]G = 0$ for $i > 0$. The group $K_0\mathbb{Z}_pG$ is free and of rank $r_{\mathbb{F}_p}$ where $r_{\mathbb{F}_p}$ is the number of irreducible $\mathbb{F}_p$-representations by Serre [Ser77], Chapter 14, Corollary 3 and Chapter 16, Corollary 1.

6.4 The singular character group $\mathrm{SC}(G)$ for finite G

Write Conj(G) for the set of conjugacy classes of G. Assume I is a k-representation of G for G finite and k a field of characteristic 0. The *character* of I is defined as the function

$$\begin{aligned} \chi_I \colon \mathrm{Conj(G)} &\to k \\ g &\mapsto \mathrm{tr}(g\cdot \colon I \to I). \end{aligned}$$

This is well-defined since the trace of an endomorphism is invariant under conjugation. Let $Cl(G;k) := \mathrm{Fun}(\mathrm{Conj(G)}, k)$ be the k-vectorspace of class functions of G with values in k. It is a standard fact from representation theory that the association $I \mapsto \chi_I$ gives an injection $K_0kG \hookrightarrow Cl(G;k)$. We call a class function in the image of this inclusion a *k-valued virtual character* of G.[5]

The character of the regular representation kG is given by

$$\chi_{kG}(g) = \begin{cases} |G| & \text{if } g = 1 \\ 0 & \text{else.} \end{cases}$$

In general, for I any k-representation, the value of the character χ_I at 1 is $\chi_I(1) = \dim(I)$. It follows that we have a commutative square

$$\begin{tikzcd} K_0kG \arrow[r] \arrow[d, hook, "\chi"] & K_0k \arrow[d, hook, "\dim"] \\ Cl(G;k) \arrow[r, "\mathrm{ev}_1"] & k. \end{tikzcd}$$

From this we can deduce that

$$\widetilde{K_0}kG = \ker(K_0kG \to K_0k) \hookrightarrow \mathrm{Fun}(\mathrm{Conj}(G) \setminus \{[1]\}, k),$$

in other words, we can interpret the reduced K-theory group as the set of k-valued virtual characters defined on non-trivial conjugacy classes.

Fix a prime p. An element g of G is called singular with respect to p if p divides the order of g. Write $\mathrm{Conj}_p(G)$ for the set of p-singular conjugacy classes of G. The following theorem can be found in Serre [Ser77], Chapter 16, Theorem 34 and 36:

[5] Berman's theorem actually shows that the character χ_I for a k-linear representation I is a well-defined function on *k-conjugacy classes* of G and the irreducible representations form an orthogonal basis of the space $\mathrm{Fun}(k\text{-}\mathrm{Conj}(G), k)$ with respect to the scalar product given by $\langle \chi_1, \chi_2 \rangle := \frac{1}{|G|}\sum_{g \in G} \chi_1(g)\chi_2(g^{-1})$. In particular the number r_k is equal to the number of k-conjugacy classes of G. See [CR81], Theorem 21.5.

Theorem 6.13. *The map $K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG$ is split injective, and the image consists of all virtual representations with characters vanishing on p-singular elements of G.*

Definition 6.14. As a consequence, we can identify the cokernel of $K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG$ with the set of virtual characters defined on $\mathrm{Conj}_p(G)$. Note that such a character always takes values in $\mathbb{Q}(\zeta_n)$ where n is the order of the group G and ζ_n is an n-th root of unity. We write $\mathrm{SC}_p(G)$ for the subgroup of $\mathrm{Fun}(\mathrm{Conj}_p(G), \mathbb{Q}(\zeta_n))$ spanned by those characters and we call them *p-singular virtual characters* of G. The image of $K_0\mathbb{C}G \to \mathrm{Fun}(\mathrm{Conj}_p(G), \mathbb{Q}(\zeta_n))$ will be denoted as $\mathrm{SC}_p^{\mathbb{C}}(G)$ and its characters will be called *p-singular complex virtual characters.* The characters in $Cl(G; \mathbb{Q}(\zeta_n))$ that lie in the image coming from $K_0\mathbb{Z}_pG$, equivalently those that vanish on p-singular elements, will be called *p-regular virtual characters* and the subgroup of $Cl(G; \mathbb{Q}(\zeta_n))$ spanned by them will be denoted $\mathrm{RegC}_p(G)$. The subgroup defined by all complex virtual characters that vanish on p-singular elements will be denoted $\mathrm{RegC}_p^{\mathbb{C}}(G)$ and called *p-regular complex virtual characters.*

Note that p-singular virtual characters are automatically also p-singular complex virtual characters and similarly for regular, i.e. we have inclusions $\mathrm{SC}_p(G) \hookrightarrow \mathrm{SC}_p^{\mathbb{C}}(G)$ and $\mathrm{RegC}_p(G) \hookrightarrow \mathrm{RegC}_p^{\mathbb{C}}(G)$. Of course, all four groups such defined are finitely generated and free abelian groups.

Lemma 6.15. Let G be finite and n be its order. The group $\mathrm{SC}(G)$ of singular characters of G from Definition 5.6 is isomorphic to the cokernel of the map

$$\bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Q}_pG.$$

As a consequence of the previous remark, this can be identified with the subgroup of the group of functions

$$\mathrm{Fun}\left(\coprod_{p \text{ prime}, p|n} \mathrm{Conj}_p(G), \mathbb{Q}(\zeta_n)\right)$$

consisting of tuples (χ_p) where each χ_p is the restriction of a $\mathbb{Q}_p$-valued virtual character of G to the set of p-singular elements of G. In other words, we have an isomorphism

$$\mathrm{SC}(G) \cong \bigoplus_{p \text{ prime}, p|n} \mathrm{SC}_p(G).$$

In analogy, define

$$\mathrm{SC}^{\mathbb{C}}(G) := \bigoplus_{p \text{ prime}, p|n} \mathrm{SC}_p^{\mathbb{C}}(G).$$

Remark 6.16. Note that it follows that $\mathrm{SC}(G)$ is finitely generated free of rank

$$r_{\mathrm{SC}} = \sum_{p| \ |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}),$$

since it is isomorphic to the sum of the cokernels of the split injective maps

$$K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG$$

between free abelian groups of rank $r_{\mathbb{F}_p}$ and $r_{\mathbb{Q}_p}$, respectively.

Remark 6.17. If G is a finite p-group, by the above lemma we have $\mathrm{SC}(G) \cong \mathrm{cok}(K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG)$. Moreover, since every non-trivial element of G is p-singular, by Theorem 6.13 the image of $K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG$ is given by those virtual representations I for which their character χ_I vanishes away from 1, which means that $K_0\mathbb{Z}_pG$ is generated by the free modules. Hence we have an isomorphism $\mathrm{SC}(G) \cong \widetilde{K_0}\mathbb{Q}_pG$.

Proof. Recall the pullback square

$$\begin{tikzcd}
\mathbf{Wh}(\mathbb{Z};G) \arrow[r] \arrow[d] & \bigvee_{p\in P} \mathbf{Wh}(\mathbb{Z}_p;G) \arrow[d] \\
\mathbf{Wh}(\mathbb{Z}[P^{-1}];G) \arrow[r] & \bigvee_{p\in P} \mathbf{Wh}(\mathbb{Q}_p;G)
\end{tikzcd}$$

of Corollary 6.10. Denote by C the common vertical cofiber, i.e.

$$C := \mathrm{cof}\Big(\mathbf{Wh}(\mathbb{Z};G) \to \mathbf{Wh}(\mathbb{Z}[P^{-1}];G)\Big) \simeq \mathrm{cof}\left(\bigvee_{p\in P} \mathbf{Wh}(\mathbb{Z}_p;G) \to \bigvee_{p\in P} \mathbf{Wh}(\mathbb{Q}_p;G)\right).$$

Since $\mathbf{Wh}(\mathbb{Z}_p;G)$ and $\mathbf{Wh}(\mathbb{Q}_p;G)$ are connective, so is C, and we have

$$\pi_0 C = \mathrm{cok}\left(\bigoplus_{p \text{ prime}, p|n} \widetilde{K_0}\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p|n} \widetilde{K_0}\mathbb{Q}_pG\right).$$

Note that the summand of $K_0\mathbb{Q}_pG$ corresponding to free $\mathbb{Q}_pG$-modules lies in the image of $K_0\mathbb{Z}_pG \to K_0\mathbb{Q}_pG$ hence the cokernel does not change when going to unreduced K-theory, therefore

$$\pi_0 C \cong \mathrm{cok}\left(\bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Z}_pG \to \bigoplus_{p \text{ prime}, p|n} K_0\mathbb{Q}_pG\right).$$

There is a natural map $C \to \mathbf{SC}(G)$ induced by the map $\mathbf{Wh}(\mathbb{Z}[P^{-1}];G) \to \mathbf{Wh}(\mathbb{Q};G)$, which by the Lemma 6.11 is an isomorphism in π_0. The 5-lemma thus implies that $\pi_0 C \cong \pi_0\mathbf{SC}(G) = \mathrm{SC}(G)$. □

Lemma 6.18. The map $\mathrm{RegC}_p(G) \hookrightarrow \mathrm{RegC}_p^{\mathbb{C}}(G)$ is split injective.

Proof. The group $\mathrm{RegC}_p(G)$ can be identified with $K_0\mathbb{Z}_pG$ and $\mathrm{RegC}_p^{\mathbb{C}}(G)$ can be identified with K_0AG, where A is the ring of integers of some finite extension of $\mathbb{Q}_p$ containing all n-th roots of unity where n is the order of G. The statement becomes that $K_0\mathbb{Z}_pG \to K_0AG$ is split injective, which is proven in [Ser77], Section 14.6. □

Corollary 6.19. The commuting square

$$\begin{tikzcd}
K_0\mathbb{Q}_pG \arrow[r] \arrow[d] & \mathrm{SC}_p(G) \arrow[d] \\
K_0\mathbb{C}G \arrow[r] & \mathrm{SC}_p^{\mathbb{C}}(G)
\end{tikzcd}$$

induces an isomorphism

$$\mathrm{tors}(\mathrm{cok}(K_0\mathbb{Q}_pG \to K_0\mathbb{C}G)) \xrightarrow{\cong} \mathrm{tors}(\mathrm{cok}(\mathrm{SC}_p(G) \to \mathrm{SC}_p^{\mathbb{C}}(G))).$$

Proof. We have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{RegC}_p(G) \arrow[r] \arrow[d] & K_0\mathbb{Q}_pG \arrow[r] \arrow[d] & \mathrm{SC}_p(G) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{RegC}^{\mathbb{C}}(G) \arrow[r] & K_0\mathbb{C}G \arrow[r] & \mathrm{SC}_p^{\mathbb{C}}(G) \arrow[r] & 0
\end{tikzcd}$$

with short exact rows. The bottom row is split exact and by Lemma 6.18 the left-hand map is split injective, which means the resulting sequence

$$0 \to \mathrm{cok}(\mathrm{RegC}_p(G) \to \mathrm{RegC}^{\mathbb{C}}(G)) \to \mathrm{cok}(K_0\mathbb{Q}_pG \to K_0\mathbb{C}G) \to \mathrm{cok}(\mathrm{SC}_p(G) \to \mathrm{SC}_p^{\mathbb{C}}(G)) \to 0$$

is split exact. The left term is free, from which the claim follows. $\square$

Remark 6.20. Using Lemma 6.2, we have an isomorphism

$$\mathrm{tors}(\mathrm{cok}(K_0\mathbb{Q}_pG \to K_0\mathbb{C}G)) \cong \bigoplus_{K \in \mathrm{Irr}_{\mathbb{Q}_p}} \mathbb{Z}/m(K)$$

where $m(K)$ is the Schur index of K. Similarly for $\mathbb{Q}$, we have the isomorphism

$$\mathrm{tors}(\mathrm{cok}(K_0\mathbb{Q}G \to K_0\mathbb{C}G)) \cong \bigoplus_{I \in \mathrm{Irr}_{\mathbb{Q}}} \mathbb{Z}/m(I).$$

6.5 Negative K-theory of finite groups

Let us now discuss the group $K_{-1}\mathbb{Z}G$ for finite G.

Lemma 6.21. Let G be finite. There is a natural short exact sequence

$$0 \to \widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G) \to K_{-1}\mathbb{Z}G \to 0,$$

which is a free resolution of the abelian group $K_{-1}\mathbb{Z}G$, and the map $\widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G)$ simply sends a rational representation I to the corresponding singular character $(\chi_p)_{p \text{ prime}}$ defined as $\chi_p(g) := \chi_I(g)$, where χ_I is the character of I.

Proof. The long exact sequence of the fiber sequence

$$\mathbf{Wh}(\mathbb{Z}; G) \to \mathbf{Wh}(\mathbb{Q}; G) \to \mathbf{SC}(G)$$

gives the exact sequence

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G) \to K_{-1}\mathbb{Z}G \to 0.$$

Theorem 6.8 is precisely the claim that this sequence splits off to the left, giving the claimed short exact sequence. The group $\widetilde{K_0}\mathbb{Q}G$ is free since it is isomorphic to $\ker(K_0\mathbb{Q}G \to K_0\mathbb{Q})$, which is, as a subgroup of the free abelian group $K_0\mathbb{Q}G$, again free and $\mathrm{SC}(G)$ is free as discussed in Remark 6.16. Lastly, the claim that $\widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G)$ sends a representation to the singular character $(\chi_p)_{p \text{ prime}}$ follows from the fact that $\widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G)$ factors as

$$\widetilde{K_0}\mathbb{Q}G \to \bigoplus_{p| \ |G|} K_0\mathbb{Q}_pG \to \mathrm{SC}(G),$$

where the first map is induced by the ring homorphisms $\mathbb{Q} \to \mathbb{Q}_p$ and the second by Lemma 6.15. ◻

Theorem 6.22 (Carter, [Car80a]). *Let G be finite. The group $K_{-1}\mathbb{Z}G$ has the form*

$$K_{-1}\mathbb{Z}G = \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$$

where

$$r = 1 - r_{\mathbb{Q}} + \sum_{p| \ |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p})$$

and s is equal to the number of irreducible $\mathbb{Q}$-representations I with even Schur index $m(I)$ but odd local Schur index $m_p(I)$ at every prime p dividing the order of G.

Proof. We will use the short exact sequence from Lemma 6.21, given as

$$0 \to \widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G) \to K_{-1}\mathbb{Z}G \to 0.$$

The formula for the free rank of $K_{-1}\mathbb{Z}G$ now follows directly from the fact that $\widetilde{K_0}\mathbb{Q}G$ has rank $r_{\mathbb{Q}} - 1$ and $\mathrm{SC}(G)$ has rank $\sum_{p| \ |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p})$ by Remark 6.16.

In order to see why the formula for the torsion holds true, take the commutative diagram with short exact rows

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{K_0}\mathbb{Q}G \arrow[r] \arrow[d] & \mathrm{SC}(G) \arrow[r] \arrow[d] & K_{-1}\mathbb{Z}G \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widetilde{K_0}\mathbb{C}G \arrow[r] & \mathrm{SC}^{\mathbb{C}}(G) \arrow[r] & Z \arrow[r] & 0,
\end{tikzcd}$$

where Z is simply the cokernel of the map $\widetilde{K_0}\mathbb{C}G \to \mathrm{SC}^{\mathbb{C}}(G)$.[6] The bottom row is split exact, which implies that Z is free and thus $\mathrm{tors}(K_{-1}\mathbb{Z}G)$ is contained in the kernel of $K_{-1}\mathbb{Z}G \to Z$. The left and middle maps are injective maps between free abelian groups. Using the snake lemma we get an isomorphism

$$\mathrm{tors}(K_{-1}\mathbb{Z}G) \cong \mathrm{tors}(\ker(\mathrm{cok}(\widetilde{K_0}\mathbb{Q}G \to \widetilde{K_0}\mathbb{C}G) \to \mathrm{cok}(\mathrm{SC}(G) \to \mathrm{SC}^{\mathbb{C}}(G)))).$$

Note that

$$\mathrm{cok}(\mathrm{SC}(G) \to \mathrm{SC}^{\mathbb{C}}(G)) \cong \bigoplus_{p| \ |G|} \mathrm{cok}(\mathrm{SC}_p(G) \to \mathrm{SC}_p^{\mathbb{C}}(G)).$$

[6] Z can be interpreted as $K_{-1}\mathbb{Z}[\zeta_n]G$, where n is the order of G.

By Corollary 6.19 and Remark 6.20, this leads to an isomorphism

$$\operatorname{tors}(K_{-1}\mathbb{Z}G) \cong \ker\left(\bigoplus_{I\in\operatorname{Irr}_{\mathbb{Q}}} \mathbb{Z}/m(I) \to \bigoplus_{p|\ |G|} \bigoplus_{K\in\operatorname{Irr}_{\mathbb{Q}_p}} \mathbb{Z}/m(K)\right).$$

Corollary 6.4 states that the map $\widetilde{K_0}\mathbb{Q}G \to \widetilde{K_0}\mathbb{Q}_pG$ splits as

$$\bigoplus_{I\in\operatorname{Irr}_{\mathbb{Q}}(G)} \mathbb{Z} \to \bigoplus_{I\in\operatorname{Irr}_{\mathbb{Q}}(G)} \mathbb{Z}\left\{K \in \operatorname{Irr}_{\mathbb{Q}_p}(G) | K \text{ appears as a summand in } I\otimes\mathbb{Q}_p\right\}$$

and it acts on an irreducible $\mathbb{Q}$-representation I as

$$[I] \mapsto \frac{m(I)}{m_p(I)}([K_1] + \cdots + [K_{n_{I_p}}]),$$

with the K_i being the irreducible $\mathbb{Q}_p$-representations that appear in $I\otimes\mathbb{Q}_p$. So I contributes to a summand in the torsion of $K_{-1}\mathbb{Z}G$ equal to

$$\ker\left(\mathbb{Z}/m(I) \to \bigoplus_{p|\ |G|} (\mathbb{Z}/m_p(I))^{n_{I_p}}\right) \cong \mathbb{Z}/d_I,$$

where the maps $(\mathbb{Z}/m(I) \to \mathbb{Z}/m_p(I)$ are reduction mod $m_p(I)$ and d_I is equal to the gcd of the ratios $\dfrac{m(I)}{m_p(I)}$ for p dividing the order of G, which by Corollary 6.6 is at most 2, which is the case iff I has even Schur index $m(I)$ but $m_p(I)$ is odd at all primes dividing the order of G. □

Define the *Bockstein morphism* $\beta_n\colon \mathbf{H}\mathbb{Z}/n \to \Sigma\mathbf{H}\mathbb{Z}$ as the boundary morphism to the fiber sequence of spectra

$$\mathbf{H}\mathbb{Z} \xrightarrow{n\cdot} \mathbf{H}\mathbb{Z} \to \mathbf{H}\mathbb{Z}/n.$$

The following theorem is now a consequence of Lemma 6.21 and Theorem 6.22.

Theorem 6.23. *Let G be finite and let s be the number of irreducible $\mathbb{Q}$-representations with even Schur index but odd local Schur index at every prime p dividing the order of G. The map of spectra $\mathbf{Wh}(\mathbb{Z};G)[-1,0] \to \mathbf{Wh}(\mathbb{Q};G)[-1,0]$ factorizes as*

$$\mathbf{Wh}(\mathbb{Z};G)[-1,0] \xrightarrow{p} \Sigma^{-1}\mathbf{H}(\mathbb{Z}/2)^s \xrightarrow{(\beta_2)^s} \mathbf{H}\mathbb{Z}^s \xrightarrow{i} \mathbf{H}\widetilde{K_0}\mathbb{Q}G \cong \mathbf{Wh}(\mathbb{Q};G)[-1,0]$$

where the map p is given by the Postnikov truncation of $\mathbf{Wh}(\mathbb{Z};G)[-1,0]$ followed by the projection onto the torsion summand of $K_{-1}\mathbb{Z}G$ and the map $i\colon \mathbf{H}\mathbb{Z}^s \to \mathbf{H}\widetilde{K_0}\mathbb{Q}G$ is induced by the inclusion of all linear combinations of the irreducible $\mathbb{Q}$-representations that contribute to s.

We want to stress the importance of this theorem to the reader in regard to the analysis of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$. The first major obstruction for generalizing the triviality of $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ from finite to arbitrary groups lies in the fact that while $\mathbf{Wh}(\mathbb{Z};G)[-1,0] \to \mathbf{Wh}(\mathbb{Q};G)[-1,0]$ for finite groups G is trivial on homotopy groups, it is not the trivial map of spectra, unless $s(G) = 0$.

Proof. The map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is zero for G being finite by Theorem 6.8 and $\mathbf{Wh}(\mathbb{Q};G)[-1,0]$ is actually concentrated in degree 0 since the negative K-theory of $\mathbb{Q}G$ vanishes. This means we can apply Lemma B.28 to see that

$$\mathbf{Wh}(\mathbb{Z};G)[-1,0] \to \mathbf{Wh}(\mathbb{Q};G)[-1,0]$$

factors through a unique map

$$\Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}G \to \mathbf{H}\widetilde{K_0}\mathbb{Q}G.$$

It corresponds under Lemma B.27 to the short exact sequence

$$0 \to \widetilde{K_0}\mathbb{Q}G \to \mathrm{SC}(G) \to K_{-1}\mathbb{Z}G \to 0$$

from Lemma 6.21. Now $K_{-1}\mathbb{Z}G = \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$ by Theorem 6.22. The abelian group $\widetilde{K_0}\mathbb{Q}G$ is f.g. free and maps of degree 1 of the form $\Sigma^{-1}\mathbf{H}\mathbb{Z} \to \mathbf{H}\mathbb{Z}$ are necessarily zero, since $\mathrm{Ext}^1_{\mathbb{Z}}(\mathbb{Z},\mathbb{Z}) = 0$. This means the map $\Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}G \to \mathbf{H}\widetilde{K_0}\mathbb{Q}G$ further factors through the 2-torsion, i.e. as

$$\Sigma^{-1}\mathbf{H}(\mathbb{Z}/2)^s \to \mathbf{H}\widetilde{K_0}\mathbb{Q}G.$$

The generators of $\widetilde{K_0}\mathbb{Q}G$ are given by the isomorphism classes of non-trivial irreducible $\mathbb{Q}$-representations of G. By the proof of Theorem 6.22, each of these contributes to a single $\mathbb{Z}/2$-summand in $K_{-1}\mathbb{Z}G$ iff it has even global Schur index but odd local Schur index at every prime p dividing the order of G. Each of these corresponds to a single Bockstein morphism β_2. In other words, the map $\Sigma^{-1}\mathbf{H}(\mathbb{Z}/2)^s \to \mathbf{H}\widetilde{K_0}\mathbb{Q}G$ factors further through i,

$$\Sigma^{-1}\mathbf{H}(\mathbb{Z}/2)^s \xrightarrow{(\beta_2)^s} \mathbf{H}\mathbb{Z}^s \xrightarrow{i} \mathbf{H}\widetilde{K_0}\mathbb{Q}G,$$

with i being the inclusion of the subgroup of $\widetilde{K_0}\mathbb{Q}G$ generated by all the irreducible $\mathbb{Q}$-representations that contribute to s. □

Remark 6.24. Most finite groups one encounters have $s(G) = 0$, this includes abelian, symmetric, alternating, and dihedral groups as well as all groups having an abelian subgroup of odd index [Car80a]. The most prominent examples of groups with $s > 0$ can be found in Magurn [Mag13], they are the finite subgroups

$$\mathrm{Dic}_n, \tilde{O}, \tilde{I}$$

of $SU(2)$, called dicyclic group of order n, binary octahedral and binary icosahedral group, respectively. These are defined as the inverse images under the double covering $SU(2) \to SO(3)$ of the groups

$$C_{2n}, O, I$$

with C_{2n} the cyclic group of order $2n$, O being the octahedral group and I the icosahedral group, thought of as the symmetries of the regular $2n$-gon, cube and icosahedron, respectively. The smallest group with $s > 0$ is the group $Q_{16} = \mathrm{Dic}_4$ with

$$K_{-1}\mathbb{Z}Q_{16} = \mathbb{Z}/2.$$

The dicyclic groups have $s = 0$ exactly for all n that are of the form $n = p^k$ for p a prime that is 3 mod 4 or $n = 2$. The lower K-theory of the binary octahedral and the binary icosahedral group are

$$\begin{aligned} K_{-1}\mathbb{Z}\tilde{O} &= \mathbb{Z} \oplus \mathbb{Z}/2 \\ K_{-1}\mathbb{Z}\tilde{I} &= \mathbb{Z}^2 \oplus \mathbb{Z}/2. \end{aligned}$$

We included a discussion and computation of all groups with non-trivial s for small orders in Appendix A.4.

Remark 6.25. Further computations of negative K-theory can be found in Lafont, Magurn, Ortiz [LMO09] for the groups D_n, $D_n \times C_2$ and $A_5 \times C_2$, as well as in Magurn [Mag13] for the groups C_n, Dic_n, $\tilde{T}$, $\tilde{O}$ and $\tilde{I}$.

7 The map $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ for infinite groups

The following section is concerned with proving the two main theorems of this thesis.

Theorem 7.1. *Suppose G is a group satisfying the Farrell-Jones conjecture. Then*

$$\mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G)$$

is a 2-torsion subgroup of $\widetilde{K_0}\mathbb{Q}G$.

Theorem 7.2. *Suppose G is a group satisfying the Farrell-Jones conjecture and there exists a model of $E(G;\mathrm{Fin})$ such that for all 0- and 1-cells of $E(G;\mathrm{Fin})$ their isotropy groups H have $s(H) = 0$. Then the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is trivial.*

Example 7.3. Assume G satisfies the Farrell-Jones conjecture. Then

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

is trivial if we know that $s(H) = 0$ for all finite subgroups H of G. In particular,

- if G is abelian or
- if G is torsion-free.

Now, throughout the following section assume that G satisfies the Farrell-Jones conjecture and that $E\mathrm{Fin}$ is a fixed model for the classifying space of finite subgroups together with a chosen CW-structure $(E\mathrm{Fin}^{(k)})_{k\in\mathbb{N}}$. Before we begin with the proofs, we need a few more arguments.

If D is a 1-category, then the category of functors $D \to \mathrm{Ab} \subset \mathrm{Sp}$ with values in the heart Ab of Sp is again a 1-category and thus a natural transformation $\eta\colon \mathbf{A} \implies \mathbf{B}$ between two functors $\mathbf{A}, \mathbf{B}\colon D \to \mathrm{Sp}$ with values in the heart is the zero map in the category $\mathrm{Fun}(D, \mathrm{Sp})$ if and only if its value on all the components $\eta_d\colon \mathbf{A}(d) \to \mathbf{B}(d)$ is the zero homomorphism for all $d \in D$. Note that here it is essential that the category of functors with values in the heart of Sp is again a 1-category, it is not true in general that a natural transformation between two functors with values in spectra is zero if all its components are zero maps.

Now, since the map $\widetilde{K_0}\mathbb{Z}H \to \widetilde{K_0}\mathbb{Q}H$ vanishes for all finite subgroups H, the natural transformation $\mathbf{H}\widetilde{K_0}\mathbb{Z}- \implies \mathbf{H}\widetilde{K_0}\mathbb{Q}-$ becomes the zero map when restricted to the subcategory $\mathrm{Or}G_{\mathrm{Fin}}$. Furthermore, $\mathbf{Wh}(\mathbb{Q};-)[-1,0]$ is as a functor on $\mathrm{Or}G_{\mathrm{Fin}}$ concentrated in degree 0 since the negative K-theory of the group algebras $\mathbb{Q}H$ vanishes for H being finite. By using the object-wise t-structure on $\mathrm{Fun}(\mathrm{Or}G_{\mathrm{Fin}}, \mathrm{Sp})$ (see Definition B.33), we are now in a position to apply Lemma B.28 with $\mathcal{C} = \mathrm{Fun}(\mathrm{Or}G_{\mathrm{Fin}}, \mathrm{Sp})$, and the map f in question being the natural transformation $\mathbf{Wh}(\mathbb{Z};-)[-1,0] \implies \mathbf{Wh}(\mathbb{Q};-)[-1,0]$. Lemma B.28 states that the natural transformation of functors

$$\mathbf{Wh}(\mathbb{Z};-)[-1,0] \implies \mathbf{Wh}(\mathbb{Q};-)[-1,0]$$

descends to a unique natural transformation of functors $\mathrm{Or}G_{\mathrm{Fin}} \to \mathrm{Sp}$,

$$\Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}- \implies \mathbf{H}\widetilde{K_0}\mathbb{Q}- \,.$$

Proposition 7.4. If G satisfies the Farrell-Jones conjecture, the image of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ agrees with the image of

$$\pi_1 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \to \widetilde{K_0}\mathbb{Q}G,$$

as well as with the image of the map

$$\pi_1 E\mathrm{Fin}_+^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \to \widetilde{K_0}\mathbb{Q}G$$

induced by the inclusion $E\mathrm{Fin}^{(1)} \subset E\mathrm{Fin}$.

Proof. The fiber sequence

$$\mathbf{H}\widetilde{K_0}\mathbb{Z}- \implies \mathbf{Wh}(\mathbb{Z}; G)[-1,0] \implies \Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}-$$

of functors leads to the exact sequence

$$\pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} (\mathbf{Wh}(\mathbb{Z};-)[-1,0]) \to \pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}- \to \pi_{-1} E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}\widetilde{K_0}\mathbb{Z}- \,.$$

Since $\mathbf{H}\widetilde{K_0}\mathbb{Z}-$ is a connective functor and smashing with a G-space preserves connectivity, the group $\pi_{-1} E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}\widetilde{K_0}\mathbb{Z}-$ vanishes, which means that the map

$$\pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} (\mathbf{Wh}(\mathbb{Z};-)[-1,0]) \to \pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z}-$$

is an epimorphism.

As discussed before, we have a commuting triangle of natural transformations of functors $\mathrm{Or}G_{\mathrm{Fin}} \to \mathrm{Sp}$,

$$\begin{tikzcd} \mathbf{Wh}(\mathbb{Z};-)[-1,0] \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \mathbf{H}\widetilde{K_0}\mathbb{Q}- \\ \Sigma^{-1}\mathbf{H}K_{-1}\mathbb{Z} \arrow[ur, Rightarrow] & \end{tikzcd}$$

Taking colimits over $\mathrm{Or}G_{\mathrm{Fin}}$, we get the triangle

$$\begin{tikzcd} \pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} (\mathbf{Wh}(\mathbb{Z};-)[-1,0]) \arrow[r] \arrow[d, two heads] & \widetilde{K_0}\mathbb{Q}G \\ \pi_1 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z}, \arrow[ur] & \end{tikzcd}$$

which together with Lemma 5.5 proves the first statement.

For the second statement we use Lemma 3.2 to get that

$$\pi_1 E\mathrm{Fin}_+^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \to \pi_1 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z}$$

is an epimorphism. This allows us to reduce further to the image of the composition

$$\pi_1 E\mathrm{Fin}_+^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \twoheadrightarrow \pi_1 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \to \pi_0 E\mathrm{Fin}_+ \wedge_{\mathrm{Or}G} \mathbf{H}\widetilde{K_0}\mathbb{Q} \cong \widetilde{K_0}\mathbb{Q}G.$$

$\square$

Write

$$(f,g)\colon \coprod_{i\in I} G/H_i \times S^0 \to \coprod_{j\in J} G/K_j$$

for the degree 0 attaching map of $E\mathrm{Fin}$ with the H_i and K_j being finite subgroups of G. For a functor $F\colon \mathrm{Or}G \to \mathrm{Ab}$ define

$$\ker^F := \ker(F(f) - F(g))\colon \bigoplus_{i\in I} F(G/H_i) \to \bigoplus_{j\in J} F(G/K_j).$$

Lemma 7.5. There is an exact sequence

$$0 \to \ker^{\widetilde{K_0}\mathbb{Q}} \to \ker^{\mathrm{SC}} \to \ker^{K_{-1}\mathbb{Z}} \to \operatorname{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G) \to 0$$

and the map $\ker^{K_{-1}\mathbb{Z}} \to \operatorname{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G)$ is the connecting map induced from the snake lemma applied to the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}(H_i) \arrow[r] \arrow[d] & \bigoplus_{i\in I} \mathrm{SC}(H_i) \arrow[r] \arrow[d] & \bigoplus_{i\in I} K_{-1}\mathbb{Z}(H_i) \arrow[r] \arrow[d, "f-g"] & 0 \\
0 \arrow[r] & \bigoplus_{j\in J} \widetilde{K_0}\mathbb{Q}(K_j) \arrow[r] & \bigoplus_{j\in J} \mathrm{SC}(K_j) \arrow[r] & \bigoplus_{j\in J} K_{-1}\mathbb{Z}(K_j) \arrow[r] & 0.
\end{tikzcd}$$

Moreover, let p be a prime number. If the following two conditions are true, the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ vanishes p-locally.

1. The attaching map in $K_{-1}\mathbb{Z}(-)$

$$f-g\colon \bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i \to \bigoplus_{j\in J} K_{-1}\mathbb{Z}K_j$$

is a split surjection into its image p-locally.

2. The exact sequence

$$0 \to \bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}H_i \to \bigoplus_{i\in I} SC(H_i) \to \bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i \to 0$$

is split exact p-locally.

Proof. We can already reduce the image of

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

to that of the map

$$\pi_1 E\mathrm{Fin}_+^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \to \widetilde{K_0}\mathbb{Q}G$$

thanks to Corollary 7.4.

By Lemma 3.3, there is the following commutative diagram of spectra

$$\begin{tikzcd}
\Sigma^{-1} E\mathrm{Fin}_+^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \arrow[r] \arrow[d] & \mathbf{H}\widetilde{K_0}\mathbb{Q}G \arrow[d] \\
\mathbf{H}(\bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i) \arrow[r] \arrow[d, "f-g"] & \Sigma\mathbf{H}(\bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}H_i) \arrow[d] \\
\mathbf{H}(\bigoplus_{j\in J} K_{-1}\mathbb{Z}K_j) \arrow[r] & \Sigma\mathbf{H}(\bigoplus_{j\in J} \widetilde{K_0}\mathbb{Q}K_j)
\end{tikzcd} \tag{$*$}$$

with the columns being fiber sequences.

By Lemma B.29, the map induced on π_0 on the fibers is equivalent to the map induced by the snake lemma of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}H_i \arrow[r] \arrow[d] & \bigoplus_{i\in I} \mathrm{SC}(H_i) \arrow[r] \arrow[d] & \bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}K_j \arrow[r] & \bigoplus_{i\in I} \mathrm{SC}(K_j) \arrow[r] & \bigoplus_{i\in I} K_{-1}\mathbb{Z}K_j \arrow[r] & 0
\end{tikzcd}$$

with exact rows.

This means we get the claimed exact sequence

$$0 \to \ker^{\widetilde{K_0}\mathbb{Q}} \to \ker^{\mathrm{SC}} \to \ker^{K_{-1}\mathbb{Z}} \to \mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G) \to 0.$$

For the second claim we use Remark B.31, which states that condition (1) and (2) imply the vanishing of the map induced by the snake lemma. □

Remark 7.6. Since the 1-skeleton $E\mathrm{Fin}^{(1)}$ is a 1-dimensional G-CW-complex, the spectrum

$$E\mathrm{Fin}^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z}$$

is concentrated in degree 0 and 1. Its homotopy type is entirely determined by the k-invariant

$$\mathbf{H}\pi_0(E\mathrm{Fin}^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z}) \to \Sigma^2\mathbf{H}\pi_1(E\mathrm{Fin}^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z}).$$

Note that $\pi_0 E\mathrm{Fin}^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \cong K_{-1}\mathbb{Z}G$ and $\pi_1 E\mathrm{Fin}^{(1)} \wedge_{\mathrm{Or}G} \mathbf{H}K_{-1}\mathbb{Z} \cong \ker^{K_{-1}\mathbb{Z}}$. By Lemma C.16 this k-invariant corresponds to a homomorphism

$$K_{-1}\mathbb{Z}G/2 \to \ker^{K_{-1}\mathbb{Z}}.$$

We therefore see that this k-invariant vanishes in particular if $K_{-1}\mathbb{Z}G$ is 2-torsion free.

Proof of Theorem 7.1 and Theorem 7.2. By Lemma 7.5 the only thing that is left to show is that the conditions:

1. The attaching map in $K_{-1}\mathbb{Z}(-)$

$$f-g\colon \bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i \to \bigoplus_{j\in J} K_{-1}\mathbb{Z}K_j$$

is a split surjection onto its image p-locally.

2. The exact sequence

$$0 \to \bigoplus_{i\in I} \widetilde{K_0}\mathbb{Q}H_i \to \bigoplus_{i\in I} \mathrm{SC}(H_i) \to \bigoplus_{i\in I} K_{-1}\mathbb{Z}H_i \to 0$$

is split exact p-locally.

hold for all odd primes to prove Theorem 7.1. In order to prove Theorem 7.2 we need to show that this also holds for the prime 2 if $E\mathrm{Fin}$ has a G-CW-model such that for all 0 and 1-cells the isotropy groups H have $s(H)=0$. Both of these statements are now clear since, if H is a finite group, $K_{-1}\mathbb{Z}H$ has at most 2-torsion, which vanishes exactly if $s(H)=0$, and $0\to\widetilde{K_0}\mathbb{Q}H\to \mathrm{SC}(H)\to K_{-1}\mathbb{Z}H\to 0$ is split exact since in this case $K_{-1}\mathbb{Z}H$ is free. □

8 The Hattori-Stallings trace and a conjecture of Bass

Bass [Bas76] already investigated a generalization of the fact that $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ vanishes for finite groups. He proposed what is now known as the *Strong Bass Conjecture*. We will show in this section, that the strong Bass conjecture is a consequence of the Farrell-Jones conjecture as a consequence of Theorem 7.1. We remark that this result is already known, see [LR05], Proposition 87. We give a different proof in this section.

Definition 8.1. Let R be a ring. The zeroth Hochschild homology group of R is defined as

$$HH_0(R) := R/[R,R]$$

where $[R,R]$ is the ideal generated by all commutators $rs - sr$ for r, s in R.

Example 8.2. Suppose R is a commutative ring and G is a group. Then $HH_0(RG)$ has the form

$$HH_0(RG) = \bigoplus_{\mathrm{Conj}(G)} R.$$

This can be seen by observing that the ideal $[RG, RG]$ is equal to the sub-module

$$\left\{ \sum_{g \in G} a_g g \,\middle|\, \text{for all } [g] \in \mathrm{Conj}(G) : \sum_{g' \in [g]} a_{g'} = 0 \right\}.$$

Let P be a projective f.g. module over R. Then there exists a complement P' such that $P \oplus P' \cong R^n$ for some n. If ϕ is an endomorphism of P we can extend ϕ as the matrix

$$\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

to an endomorphism of $P \oplus P' \cong R^n$. Define $\mathrm{tr}(\phi)$ as the trace of the corresponding $n \times n$ matrix over R, as an element in $HH_0(R) = R/[R,R]$. One can show that $\mathrm{tr}(\phi)$ is independent of the choice of splitting $P \oplus P' \cong R^n$. Denote by $r(P)$ the trace of id_P. The assignment

$$P \mapsto r(P)$$

induces a well-defined homomorphism

$$r \colon K_0R \to HH_0(R)$$

called the *Hattori-Stallings rank* [Hat65], [Sta65]. Note that $r(R^n) = n \cdot 1$.

Now, if we are in the case of example 8.2, define $r_P(g)$ as the coefficient in the sum

$$r(P) = \sum_{[g] \in \mathrm{Conj}(G)} r_P(g)$$

under the isomorphism $HH_0(RG) = \bigoplus_{\mathrm{Conj}(G)} R$. We can now phrase the following conjecture.

Conjecture 8.3 (Strong Bass Conjecture for $K_0\mathbb{Z}G$, [Bas76])**.** The function $r_P(g)$ is 0 for $g \neq 1$.

Theorem 8.4. *Suppose G satisfies the Farrell-Jones conjecture. Then G satisfies the Strong Bass Conjecture for $K_0\mathbb{Z}G$.*

Proof. We have a commutative square

$$\begin{tikzcd}
K_0\mathbb{Z}G \arrow[r] \arrow[d] & K_0\mathbb{Q}G \arrow[d] \\
HH_0(\mathbb{Z}G) \arrow[r] & HH_0(\mathbb{Q}G),
\end{tikzcd}$$

where $HH_0(\mathbb{Z}G) \cong \bigoplus_{\mathrm{Conj}(G)} \mathbb{Z}$ and $HH_0(\mathbb{Q}G) \cong \bigoplus_{\mathrm{Conj}(G)} \mathbb{Q}$. The map $r\colon K_0\mathbb{Q}G \to HH_0(\mathbb{Q}G)$ maps into a rational vectorspace, thus killing all torsion elements in $K_0\mathbb{Q}G$. Theorem 7.1 states that if P is a f.g. projective $\mathbb{Z}G$ module, then $[P \otimes \mathbb{Q}] = n[\mathbb{Q}G] + x$ for some $n \in \mathbb{Z}$ and x being a 2-torsion element in $K_0\mathbb{Q}G$. Therefore,

$$r(P \otimes \mathbb{Q}) = nr(\mathbb{Q}G) + r(x) = n \cdot 1.$$

The map $HH_0(\mathbb{Z}G) \to HH_0(\mathbb{Q}G)$ is injective, hence $r(P) = r(P \otimes \mathbb{Q}) = n \cdot 1$, which implies that $r_P(g) = 0$ for $g \neq 1$. $\square$

9 Virtually cyclic groups

A group G is called virtually cyclic if it contains a cyclic subgroup of finite index. Virtually cyclic groups can be classified into three families of groups.

Lemma 9.1 (See [Hem04], Lemma 11.4.)**.** A group G is virtually cyclic if it is of one of the three forms

- G is finite.
- G is finite-by-infinite cyclic. This means that there is an exact sequence of groups

$$1 \to H \to G \to C_\infty \to 1$$

  with H being finite, and C_∞ an infinite cyclic group. We will call G of type VC1.
- G is finite-by-infinite dihedral. This means that there is an exact sequence of groups

$$1 \to H \to G \to D_\infty \to 1$$

  with H being finite, and D_∞ an infinite dihedral group. We will call G of type VC2.

9.1 Virtually cyclic groups of type 1

In the following fix a group G of type VC1 and write H for the unique maximal finite subgroup. Write $\pi\colon G \to G/H \cong C_\infty$ for the canonical projection.

Lemma 9.2. Let G be of type VC1. Then a model of the classifying space $E(G;\mathrm{Fin})$ is given by $\mathbb{R}$ with the action lifted from the translation action of $C_\infty = G/H$.

Proof. First assume that $G = C_\infty$. It is easy to see that $\mathbb{R}$ with the translation action is an $E(G;\mathrm{Fin})$ as any non-trivial subgroup K of C_∞ acts freely, thus $\mathbb{R}^K = \emptyset$. If $K = \{1\}$ then $\mathbb{R}^K = \mathbb{R} \simeq \mathrm{pt}$.

Now let G be a general group of type VC1. If K is a finite subgroup it is contained in H and therefore acts trivially on $\mathbb{R}$, which means that as before $\mathbb{R}^K = \mathbb{R} \simeq \mathrm{pt}$. If K is an infinite subgroup then $\pi(K)$ is a non-trivial subgroup of C_∞ and thus again we have $\mathbb{R}^K = \emptyset$. $\square$

A C_∞-CW-structure of $\mathbb{R}$ with the translation action can be described with the following pushout square.

$$\begin{array}{ccc} C_\infty/1 \times S^0 & \xrightarrow{(id,t\cdot)} & C_\infty/1 \\ \downarrow & & \downarrow \\ C_\infty/1 \times D^1 & \longrightarrow & \mathbb{R} \end{array}$$

This generalizes for G being of type VC1 in the following way. Let $\tilde{t} \in G$ be a choice of lift of the generator t in C_∞. Then the following is a pushout square of G-spaces.

$$\begin{array}{ccc} G/H \times S^0 & \xrightarrow{(id, \tilde{t}\cdot)} & G/H \\ \downarrow & & \downarrow \\ G/H \times D^1 & \longrightarrow & \mathbb{R} \end{array}$$

Applying Lemma 3.3 now states that if F is any functor $\mathrm{Or}G \to \mathrm{Sp}$, then there is a fiber sequence

$$F(H) \xrightarrow{1-\tilde{t}} F(H) \to E(G; \mathrm{Fin})_+ \wedge F.$$

The functors $K_0\mathbb{Q}-$ and $K_{-1}\mathbb{Z}-$ satisfy finite assembly (see Remark 4.17 as well as Corollary 9.11). We thus have the exact sequences

$$K_0\mathbb{Q}H \xrightarrow{1-t} K_0\mathbb{Q}H \to K_0\mathbb{Q}G \to 0$$

and

$$K_{-1}\mathbb{Z}H \xrightarrow{1-t} K_{-1}\mathbb{Z}H \to K_{-1}\mathbb{Z}G \to 0.$$

Theorem 9.3. *Let G be a group of type $VC1$. Then $K_0\mathbb{Q}G$ is a finitely generated and free abelian group.*

Proof of Theorem 9.3. Assume G is finite-by-infinite cyclic, given by a short exact sequence

$$1 \to H \to G \xrightarrow{\pi} C_\infty \to 1$$

with H finite, C_∞ an infinite cyclic group and let t be a generator of C_∞ and $\tilde{t}$ any lift of t in G.

Since $K\mathbb{Q}-$ satisfies finite assembly, as remarked above, we have the exact sequence

$$K_0\mathbb{Q}H \xrightarrow{1-t} K_0\mathbb{Q}H \to K_0\mathbb{Q}G \to 0.$$

What is left to understand is the action of t on $K_0\mathbb{Q}H$. The endomorphisms of the object G/H in the category $\mathrm{Or}G$ are equal to $N(H)/H = G/H = C_\infty = \langle t \rangle$. By Theorem 4.5 this action of t on $K_0\mathbb{Q}H$ sends a representation $V = (V, \rho)$ to the representation $V_t := (V, \rho(\tilde{t}(-)\tilde{t}^{-1}))$. Let $\mathrm{Irr}_\mathbb{Q}(G)$ be the set of isomorphism classes of irreducible representations of H over $\mathbb{Q}$. If V is irreducible, then so is V_t, hence

$$K_0\mathbb{Q}G \cong \mathrm{cok}(1-t) \cong (K_0\mathbb{Q}H)_{C_\infty} = \mathbb{Z}[\mathrm{Irr}_\mathbb{Q}(G)]_{C_\infty} \cong \mathbb{Z}[\mathrm{Irr}_\mathbb{Q}(G)/\equiv]$$

with $\equiv$ being the equivalence relation generated by $V \equiv V_t$. Hence $K_0\mathbb{Q}G$, is free generated by the finite set of C_∞-equivalence classes of rational irreducible representations of H. □

Corollary 9.4. The map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is trivial for G of type VC1.

Proof. A virtually cyclic group trivially satisfies the Farrell-Jones conjecture. Theorem 7.1 implies that the image of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is 2-torsion, which has to be trivial, assuming Theorem 9.3. □

9.2 Virtually cyclic groups of type 2

In the following fix a group G of type VC2. Write $\pi\colon G \to G/H \cong D_\infty$ for the canonical projection. Let H be the kernel of π.

Lemma 9.5. A group G is of type VC2 iff $G \cong K_1 *_H K_2$, where K_1 and K_2 are two finite groups that both contain H as an index 2 subgroup.

Proof. Write G as an extension

$$1 \to H \to G \xrightarrow{\pi} D_\infty \to 1$$

with H finite. Write $D_\infty = \left\langle a, b | a^2 = b^2 = 1 \right\rangle$. We can define two finite subgroups of G as $G^a := \pi^{-1}(\langle a \rangle)$ and $G^b := \pi^{-1}(\langle b \rangle)$. Both contain H as an index 2 subgroup. We have a natural homomorphism

$$G^a *_H G^b \to G$$

induced by the two inclusions. We claim that:

- $G^a *_H G^b \to G$ is an epimorphism. We prove this by showing that the images of G^a and G^b generate G as a group. For a given g in G, $\pi(g)$ is an alternating word in a and b. Proceed by induction on the word length. If $\pi(g)$ has word length 1, g is either in G^a or G^b, and so we are done. Now assume without loss of generality $\pi(g) = aba \cdots ba$, with word length n. Pick any g_0 with $\pi(g_0) = a$, in particular $g_0 \in G^a$. Then $\pi(g_0 g)$ has word length $n-1$, therefore, arguing by induction, it is generated from G^a and G^b. But then $g = g_0^{-1} g_0 g$ is also generated from G^a and G^b.

- $G^a *_H G^b \to G$ is a monomorphism. So, assume we have a string $g_1 g_2 \cdots g_n$ of elements representing an element s in $G^a *_H G^b$ and assume without loss of generality $g_1 \in G^a$, $g_2 \in G^b$ and so on. If $g_1 g_2 \cdots g_n = 1$ in G, then in particular $\pi(g_1 g_2 \cdots g_n) = 1$. This implies that all $g_i \in H$. Since $g_1 g_2 \cdots g_n = 1$ is thus an equation that holds entirely in H, we have that $s = 1$ in $G^a *_H G^b$.

For the converse, assume $G \cong K_1 *_H K_2$ with K_1, K_2 finite and H an index 2 subgroup in both of them. Then H is necessarily normal in them, giving the short exact sequences

$$1 \to H \to K_i \to C_2 \to 1,$$

which, put together, give the short exact sequence

$$1 \to H \to K_1 *_H K_2 \to C_2 * C_2 \to 1,$$

hence G is of type VC2. □

We can equip $\mathbb{R}$ with an action of $D_\infty = \left\langle a, b | a^2 = b^2 = 1 \right\rangle$ by sending a to the reflection around 0 and b to the reflection around $1/2$. If G is any group of type VC2, we can equip $\mathbb{R}$ with a G-action via the projection $G \to D_\infty$.

Lemma 9.6. Let G be virtually cyclic of type 2. The G-space $\mathbb{R}$ with action lifted from the projection $G \to D_\infty$ is a model for $E(G;\mathrm{Fin})$.

Proof. First assume $G = D_\infty$. Any finite subgroup K of D_∞ is either conjugate to $C_2^a = \langle a \rangle$ or $C_2^b = \langle b \rangle$. Without loss of generality, assume K is contained in C_2^a. If $K = \{1\}$, then $\mathbb{R}^K = \mathbb{R} \simeq \mathrm{pt}$. For $K = C_2^a$ we have $\mathbb{R}^K = \{0\} = \mathrm{pt}$. Now assume K is an infinite subgroup of D_∞. Then $K \cap C_\infty \neq \emptyset$ with $C_\infty = \langle ab \rangle$. This means that there exists an element of K which acts via a translation on $\mathbb{R}$, which implies that $\mathbb{R}^K = \emptyset$.

Now let G be a general virtually cyclic group of type 2 written in the form $G = K_1 *_H K_2$, with projection $\pi\colon G \to D_\infty$. If K is finite, it is conjugate to either a subgroup of K_1 or K_2. Assume without loss of generality that it is contained in K_1. This means $\pi(K)$ is contained in C_2^a and we are in the previous special case. If K is infinite, then so is $\pi(K)$, hence again we reduce to $G = D_\infty$. □

Lemma 9.7. Suppose $G = K_1 *_H K_2$ is of type VC2. Then there is a pushout square

$$\begin{array}{ccc} G/H \times S^1 & \longrightarrow & G/K_1 \sqcup G/K_2 \\ \downarrow & & \downarrow \\ G/H \times D^1 & \longrightarrow & E\mathrm{Fin} \end{array}$$

of G-spaces, giving $E\mathrm{Fin}$ a 1-dimensional G-CW-structure.

Proof. By Lemma 9.6 the space $\mathbb{R}$ with the action lifted from the projection $\pi\colon G \to D_\infty$ is a model for $E\mathrm{Fin}$. This means we may as well assume that $G = D_\infty = \left\langle a, b | a^2 = b^2 = 1 \right\rangle$, i.e. $H = \{1\}$, $K_1 = \langle a \rangle$, $K_2 = \langle b \rangle$. Now it is an elementary exercise to see that $\mathbb{R}$ indeed fits into a pushout square of the shape

$$\begin{array}{ccc} D_\infty \times S^1 & \longrightarrow & D_\infty/\langle a \rangle \sqcup D_\infty/\langle b \rangle \\ \downarrow & & \downarrow \\ D_\infty \times D^1 & \longrightarrow & \mathbb{R}. \end{array}$$

□

Write ι_i for the inclusions $H \hookrightarrow K_i$. The functors $K_0\mathbb{Q}-$ and $K_{-1}\mathbb{Z}-$ satisfy finite assembly (see Remark 4.17 as well as Corollary 9.11) therefore as a consequence of the pushout square from Lemma 9.7 we get the exact sequences

$$\widetilde{K_0}\mathbb{Q}H \xrightarrow{(\iota_1,-\iota_2)} \widetilde{K_0}\mathbb{Q}K_1 \oplus \widetilde{K_0}\mathbb{Q}K_2 \to \widetilde{K_0}\mathbb{Q}G \to 0$$
$$K_{-1}\mathbb{Z}H \xrightarrow{(\iota_1,-\iota_2)} K_{-1}\mathbb{Z}K_1 \oplus K_{-1}\mathbb{Z}K_2 \to K_{-1}\mathbb{Z}G \to 0$$

and we have a long exact sequence from Lemma 7.5

$$0 \to \ker^{\widetilde{K_0}\mathbb{Q}} \to \ker^{\mathrm{SC}} \to \ker^{K_{-1}\mathbb{Z}} \to \mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G) \to 0$$

with

$$\begin{aligned}
\ker^{\widetilde{K_0}\mathbb{Q}} &\cong \ker\left(\widetilde{K_0}\mathbb{Q}H \xrightarrow{(\iota_1,-\iota_2)} \widetilde{K_0}\mathbb{Q}K_1 \oplus \widetilde{K_0}\mathbb{Q}K_2\right)\\
\ker^{\mathrm{SC}} &\cong \ker\left(\mathrm{SC}(H) \xrightarrow{(\iota_1,-\iota_2)} \mathrm{SC}(K_1) \oplus \mathrm{SC}(K_2)\right)\\
\ker^{K_{-1}\mathbb{Z}} &\cong \ker\left(K_{-1}\mathbb{Z}H \xrightarrow{(\iota_1,-\iota_2)} K_{-1}\mathbb{Z}K_1 \oplus K_{-1}\mathbb{Z}K_2\right).
\end{aligned}$$

The following corollary is now a special case of Theorem 7.2.

Corollary 9.8. Suppose $G = K_1 *_H K_2$ is of type VC2. Suppose the s-ranks of H, K_1, and K_2 are zero. Then the map

$$\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$$

is trivial.

Remark 9.9. We will construct an example of a group G of type VC2, for which the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is non-trivial, in section 10.

9.3 Negative K-theory of virtually cyclic groups

The following theorem due to Farrell, Jones extends Carter's results to virtually cyclic groups.

Theorem 9.10 ([FJ95], Theorem 2.1.)**.** *Let G be a virtually infinite cyclic group. Then*

(a) $K_n\mathbb{Z}G = 0$ *for all integers* $n \leq -2$.

(b) $K_{-1}\mathbb{Z}G$ *is generated by the images of* $K_{-1}\mathbb{Z}F$ *under the maps induced by the inclusions* $F \subset G$ *where* F *varies over representatives of the conjugacy classes of finite subgroups of* G.

(c) $K_{-1}\mathbb{Z}G$ *is a finitely generated abelian group.*

This has a few implications for groups that satisfy the Farrell Jones conjecture.

Corollary 9.11. Let G be a group satisfying the Farrell Jones conjecture. Then

- $K_n\mathbb{Z}G = 0$ for all integers $n \leq -2$.
- The functor $K_{-1}\mathbb{Z}-$ satisfies finite assembly in the sense that

$$K_{-1}\mathbb{Z}G \cong \mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{Fin}}} K_{-1}\mathbb{Z}H.$$

Proof. The first statement is clear, since Theorem 9.10 (a) implies that the functor $\mathbf{K}\mathbb{Z}-$ is -1-connective, when restricted to the category $\mathrm{Or}G_{\mathrm{VCyc}}$ and thus

$$\mathbf{K}\mathbb{Z}G \simeq \mathrm{colim}_{G/H \in \mathrm{Or}G_{\mathrm{VCyc}}} \mathbf{K}\mathbb{Z}H$$

is -1-connective as well.

For the second statement, the Farrell Jones conjecture implies that

$$\begin{aligned}\mathbf{H}K_{-1}\mathbb{Z}G &\simeq \tau_{\leq -1}\mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{VCyc}}}\mathbf{K}\mathbb{Z}H\\ &\simeq \mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{VCyc}}}\tau_{\leq -1}\mathbf{K}\mathbb{Z}H \simeq \mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{VCyc}}}\mathbf{H}K_{-1}\mathbb{Z}H,\end{aligned}$$

since the Postnikov truncation $\tau_{\leq -1}$ commutes with colimits and the functor $\mathbf{K}\mathbb{Z}-$ is -1-connective. Hence we need to show that

$$\mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{Fin}}}\mathbf{H}K_{-1}\mathbb{Z}H \to \mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{VCyc}}}\mathbf{H}K_{-1}\mathbb{Z}H$$

induced by the inclusion $\mathrm{Or}G_{\mathrm{Fin}} \subset \mathrm{Or}G_{\mathrm{VCyc}}$ is an isomorphism. Theorem 4.15 already states that it is injective. For surjectivity, let $[x]$ be an element in $\mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{VCyc}}}\mathbf{H}K_{-1}\mathbb{Z}H$ represented by some x in $K_{-1}\mathbb{Z}H'$ for a given virtually cyclic subgroup H'. If H' is finite we are done. Otherwise, Theorem 9.10 (b) implies the existence of a finite subgroup H'' of H' and an element $y \in K_{-1}\mathbb{Z}H''$ that maps to x under the map induced by the inclusion $H'' \subset H'$. Thus y represents an element $[y]$ in $\mathrm{colim}_{G/H\in \mathrm{Or}G_{\mathrm{Fin}}}\mathbf{H}K_{-1}\mathbb{Z}H$ which is mapped to $[x]$. □

10 A counterexample to the integral $K_0\mathbb{Z}G$-to-$K_0\mathbb{Q}G$ conjecture

The following section is concerned with an example of a group G with the property that $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is non-trivial.

For the construction take Q_{16} contained in the semidihedral group QD_{32}. We will show that the group $G := QD_{32} *_{Q_{16}} QD_{32}$ has the property that $\widetilde{K_0}\mathbb{Z}G$ maps onto a summand $\mathbb{Z}/2$ sitting inside $\widetilde{K_0}\mathbb{Q}G$. The group G is not special in this regard. The reason for choosing it is that G contains the group Q_{16} as a maximal finite normal subgroup. The group Q_{16} is the smallest group with torsion in negative K-theory, hence we do not expect counterexamples of groups which are particularly simpler than G.

10.1 The group Q_{16}

We let a presentation of Q_{16} be given as

$$Q_{16} = \left\langle r, s | r^8 = 1, r^4 = s^2, srs^{-1} = r^7 \right\rangle .$$

It has the following conjugacy classes:

Class	$\{1\}$	$\{s^2\}$	$\{r^2, r^6\}$	$\{s, r^2s, r^4s, r^6s\}$	$\{rs, r^3s, r^5s, r^7s\}$	$\{r, r^7\}$	$\{r^3, r^5\}$
Order	1	2	4	4	4	8	8
Size	1	1	2	4	4	2	2

Since Q_{16} is a 2-group, by Remark A.3 the group $K_{-1}\mathbb{Z}Q_{16}$ must be torsion. Using the "wedderga" package in GAP, we can check the Schur indices appearing in the Wedderburn decomposition of $\mathbb{Q}Q_{16}$:

```
gap> G := QuaternionGroup(16);
gap> WedderburnDecompositionWithDivAlgParts( GroupRing( Rationals, G ) );
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ], [ 1, rec(
Center := NF(8,[ 1, 7 ]), DivAlg := true, LocalIndices := [ [ infinity, 2 ] ], SchurIndex := 2 ) ]
]
```

A few comments on how to read this output are needed. As described in Section 6.1, the group algebra $\mathbb{Q}G$ splits as

$$\mathbb{Q}G \cong \prod_{I \in \mathrm{Irr}_{\mathbb{Q}}(G)} M_{n_I \times n_I}(D_I),$$

with the D_I being finite dimensional division algebras over $\mathbb{Q}$. The function WedderburnDecompositionWithDivAlgParts returns a list containing information about each part $M_{n_I \times n_I}(D_I)$ appearing in the Wedderburn decomposition. First, we have 6 entries corresponding to the 6 irreducible representations of Q_{16}. The first number in each of the entries refers to the number n_I. Next to it is information about D_I. In our case the first 5 entries happen to have $D_I = \mathbb{Q}$. For the last entry, its division algebra D is non-commutative, which is signalled by `DivAlg := true`. The center A of D is a finite field extension of $\mathbb{Q}$ and described as `NF(8,[ 1, 7 ])`. This notation means that A is a sub-field of the cyclotomic field extension $\mathbb{Q}(\zeta_8)$ being fixed by the subgroup $\{\underline{1}, \underline{7}\}$ of the Galois group $(\mathbb{Z}/8)^\times = \{\underline{1}, \underline{3}, \underline{5}, \underline{7}\}$. It is not difficult to see that

$A = \mathbb{Q}(\sqrt{2})$, using the code:

```
gap> A := NF( 8, [ 1 , 7 ] );
NF(8,[ 1, 7 ])
gap> Dimension(A);
2
gap> Sqrt(2) in A;
true
```

This means we have the decomposition

$$\mathbb{Q}Q_{16} \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times M_{2\times 2}(\mathbb{Q}) \times D.$$

The entry `SchurIndex` gives the global Schur index of the representation I and is displayed only when it is bigger than 1. `LocalIndices` gives a list of all primes at which the local Schur index of I is not equal to 1, together with the real Schur index for the value `infinity`.

In our case we can see that $\mathbb{Q}Q_{16}$ has a single irreducible rational representation α contributing to $s(Q_{16})$, with endomorphism algebra D, hence $K_{-1}\mathbb{Z}Q_{16} = \mathbb{Z}/2$. This representation is concretely given by the action of Q_{16} on the quaternion algebra $\mathbb{H}_{\mathbb{Q}(\sqrt{2})} := \mathbb{Q}(\sqrt{2})\left\langle i, j | i^4 = j^4 = -1, ij = -ji\right\rangle$ over the field $\mathbb{Q}(\sqrt{2})$, realized by

$$\begin{cases} r \mapsto (\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}) \\ s \mapsto j. \end{cases}$$

acting via left multiplication on $\mathbb{H}_{\mathbb{Q}(\sqrt{2})}$. Note that α can also be characterized as the unique faithful irreducible $\mathbb{Q}$-representation of Q_{16}.

10.2 The group QD_{32}

A presentation of QD_{32} is given as

$$QD_{32} = \left\langle a, b | a^{16} = 1, b^2 = 1, bab = a^7 \right\rangle.$$

It is easy to see that every element of QD_{32} can be represented in the form $a^n b^i$ for $n = 0, \ldots, 15$ and $i = 0, 1$, from which it follows that QD_{32} has in fact 32 elements. The inclusion $Q_{16} \to QD_{32}$ can be realized by sending $r \mapsto a^2, s \mapsto ab$ as seen by the calculations

$$(a^2)^4 = a^8 = a(a^7 b)b = a(ba)b = (ab)^2$$

as well as

$$(ab)a^2 = a(a^{2*7})b = (a^2)^7(ab).$$

The image of this homomorphism consists of all $a^n b^i$ for which $n + i$ is even, of which there are exactly 16 elements from which it follows that it actually is an inclusion.

The conjugacy classes are given as follows

Class	$\{1\}$	$\{a^8\}$	$\{a^{2n}b\}$	$\{a^4, a^{12}\}$	$\{a^{2n+1}b\}$	$\{a^2, a^{14}\}$	$\{a^6, a^{10}\}$
Order	1	2	2	4	4	8	8
Size	1	1	8	2	8	2	2
Class	$\{a, a^7\}$	$\{a^3, a^5\}$	$\{a^9, a^{15}\}$	$\{a^{11}, a^{13}\}$			
Order	16	16	16	16			
Size	2	2	2	2			

Similarly to before, QD_{32} is a 2-group, so $K_{-1}\mathbb{Z}QD_{32}$ is torsion.[7]. Doing the same computation of the Schur indices appearing in the Wedderburn decomposition of QD_{32} we get:

```
gap> G := SmallGroup(32,19);
<pc group of size 32 with 5 generators>
gap> WedderburnDecompositionWithDivAlgParts( GroupRing( Rationals, G ) );
[ [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 1, Rationals ], [ 2, Rationals ], [ 2, NF(8,[
1, 7 ]) ], [ 2, NF(16,[ 1, 7 ]) ] ]
```

Similarly to before, this means we have the Wedderburn decomposition

$$\mathbb{QD}_{32} \cong \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times M_{2\times 2}(\mathbb{Q}) \times M_{2\times 2}(A_1) \times M_{2\times 2}(A_2),$$

with A_1 being the sub-field of $\mathbb{Q}(\zeta_8)$ fixed by $\{\underline{1}, \underline{7}\} \subset (\mathbb{Z}/8)^\times$ and A_2 being the sub-field of $\mathbb{Q}(\zeta_{16})$ fixed by $\{\underline{1}, \underline{7}\} \subset (\mathbb{Z}/16)^\times$. From this we can see that no irreducible rational representations contribute to s. Hence $K_{-1}\mathbb{Z}QD_{32} = 0$.

10.3 The group $QD_{32} *_{Q_{16}} QD_{32}$

The group we want to consider is the group

$$G := QD_{32} *_{Q_{16}} QD_{32}.$$

A concrete presentation is given by

$$G = \left\langle a, b, a', b' | a^{16} = 1, b^2 = 1, aba^{-1} = a^7, a'^{16} = 1, b'^2 = 1, a'b'a'^{-1} = a'^7, a^2 = a'^2, ab = a'b' \right\rangle.$$

The group G is virtually cyclic of type 2, which means that we can apply the formulas from Section 9.2. We have the long exact sequence

$$0 \to \ker^{\widetilde{K_0}\mathbb{Q}} \to \ker^{\mathrm{SC}} \to \ker^{K_{-1}\mathbb{Z}} \to \mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G) \to 0.$$

The previous calculations show that $K_{-1}\mathbb{Z}Q_{16} = \mathbb{Z}/2$ and $K_{-1}\mathbb{Z}QD_{32} = 0$, which gives $\ker^{K_{-1}\mathbb{Z}} = \mathbb{Z}/2$. We claim that the map $\ker^{K_{-1}\mathbb{Z}} \to \mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G)$ is injective, which is equivalent to the map $\ker^{\mathrm{SC}} \to \ker^{K_{-1}\mathbb{Z}}$ being trivial. Since Q_{16} and QD_{32} are 2-groups, by Remark 6.17, we have isomorphisms $\mathrm{SC}(Q_{16}) \cong \widetilde{K_0}\mathbb{Q}_2Q_{16}$ and similarly $\mathrm{SC}(QD_{32}) \cong \widetilde{K_0}\mathbb{Q}_2QD_{32}$. This means that

$$\ker^{\mathrm{SC}} \cong \ker(\widetilde{K_0}\mathbb{Q}_2Q_{16} \xrightarrow{(\iota_1, -\iota_2)} \widetilde{K_0}\mathbb{Q}_2QD_{32} \oplus \widetilde{K_0}\mathbb{Q}_2QD_{32}) = \ker(\widetilde{K_0}\mathbb{Q}_2Q_{16} \to \widetilde{K_0}\mathbb{Q}_2QD_{32}).$$

[7] See Remark A.3

By Corollary 6.4 the map $\widetilde{K_0}\mathbb{Q}Q_{16} \to \widetilde{K_0}\mathbb{Q}_2 Q_{16}$ splits as

$$\bigoplus_{I \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathbb{Z} \to \bigoplus_{I \in \mathrm{Irr}_{\mathbb{Q}}(G)} \mathbb{Z}\{K \in \mathrm{Irr}_{\mathbb{Q}_2}(Q_{16}) | K \text{ appears as a summand in } I \otimes \mathbb{Q}_2\}$$

Now, since $K_{-1}\mathbb{Z}Q_{16} = \mathbb{Z}/2$ there is a unique irreducible $\mathbb{Q}_2$-representation β of Q_{16} such that $\alpha \otimes_{\mathbb{Q}} \mathbb{Q}_2 = 2\beta$. The negative K-theory group $K_{-1}\mathbb{Z}Q_{16}$ is generated by the image of β. Neither α nor β can lie in the kernels of ι_1 and ι_2, respectively, since their inductions to QD_{32} are neither the trivial nor regular representations (by looking at their dimensions), which shows the claim.

In summary, we have just shown that for the group G,

$$\mathrm{im}(\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G) \cong \mathbb{Z}/2.$$

11 Other examples

Example 11.1. Free products and pushouts of abelian groups, such as

$$SL_2(\mathbb{Z}) \cong C_4 *_{C_2} C_6,$$

necessarily have that $s(H) = 0$ for all finite subgroups H, therefore $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is trivial.

Example 11.2. A particular special case of the previous example is the infinite dihedral group $D_\infty = C_2 * C_2$. It follows that all products D_∞^n for $n \geq 0$ have the property that $s(H) = 0$ for all finite subgroups H as well, since all finite subgroups are of the form C_2, and the same holds for all subgroups of D_∞^n. The groups $G_n := \ker(D_\infty^n \to C_2)$ have been studied Kropholler, Moselle [KM91]. For odd n with $n \geq 3$ the group G_n has a single element in $K_0\mathbb{Q}G_n$ of order 2 given by the Euler class.

We actually have the stronger condition that negative K-theory of all finite subgroups of G_n vanishes, since $K_{-1}\mathbb{Z}C_2 = 0$. Using finite assembly of negative K-theory (Corollary 9.11) we can conclude that the K-theory groups $K_{-1}\mathbb{Z}G_n$ also vanish.

Example 11.3. This is due to Leary [Lea00]. Take the group $D_6 *_{C_3} D_6$. It is virtually cyclic of type 2. Then the rational K-theory computes as $K_0\mathbb{Q}G = \mathbb{Z}^4 \oplus \mathbb{Z}/2$, hence $\mathbb{Q}G$ has a torsion projective class. The map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$, however, is trivial, since the groups C_3 and D_6 contain no torsion in negative K-theory. Using finite assembly on negative K-theory and the fact that $K_{-1}\mathbb{Z}C_3 = 0$ and $K_{-1}\mathbb{Z}D_6 = 0$ we see that $K_{-1}\mathbb{Z}G = 0$.

Example 11.4. Take $C_6 \subset D_{12}$ and define $G := D_{12} *_{C_6} D_{12}$. A computation of the pushout

$$\begin{tikzcd}
K_0\mathbb{Q}C_6 \arrow[r] \arrow[d] & K_0\mathbb{Q}D_{12} \arrow[d] \\
K_0\mathbb{Q}D_{12} \arrow[r] & K_0\mathbb{Q}G
\end{tikzcd}$$

shows $K_0\mathbb{Q}G = \mathbb{Z}^8 \oplus (\mathbb{Z}/2)^2$, hence similarly to the example before $\mathbb{Q}G$ has torsion in K_0. The negative K-theory of C_6 and D_{12} is $K_{-1}\mathbb{Z}C_6 = \mathbb{Z}$ and $K_{-1}\mathbb{Z}D_{12} = \mathbb{Z}$ and the map given by the inclusion is multiplication by two. From this we can conclude that $K_{-1}\mathbb{Z}G = \mathbb{Z} \oplus \mathbb{Z}/2$. This example shows that the class of groups H with $K_{-1}\mathbb{Z}H$ being free is not closed under pushouts of groups. Of course, the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ is trivial since C_6 and D_{12} contain no torsion in negative K-theory.

Example 11.5. Take C_{14} contained in D_{28}. There are two irreducible $\mathbb{Q}$-representations over C_{14} with $\mathrm{Ind}_{C_{14}}^{D_{28}}(I) = 2 \cdot J$, and the map $K_{-1}\mathbb{Z}C_{14}$ to $K_{-1}ZD_{2*14}$ is not injective. Define

$$G = D_{14} *_{C_{14}} D_{14}.$$

This is virtually cyclic of type 2, and has $\pi_0 \mathrm{colim}_{\mathrm{Or}G_{\mathrm{Fin}}} \mathbf{Wh}(\mathbb{Z};-) = \mathbb{Z}$ and $K_0\mathbb{Q}G = \mathbb{Z}^8 \oplus (\mathbb{Z}/2)^2$. Since the image of the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ agrees with that of the map $\pi_0 \mathrm{colim}_{\mathrm{Or}G_{\mathrm{Fin}}} \mathbf{Wh}(\mathbb{Z};-) \to$

$\widetilde{K_0}\mathbb{Q}G$ and we know that the image is 2-torsion it seemed plausible that this group was a potential counterexample.[8] Theorem 7.2 however denies that, since neither C_{14} nor D_{28} have $s > 0$.

Example 11.6. The braid group $B_4(\mathbb{S}^2)$ was analyzed by Guaschi, Juan-Pineda, and Millán-López in [GJM18]. The group $B_4(\mathbb{S}^2)$ can be realized as an amalgamated product

$$B_4(\mathbb{S}^2) \cong Q_{16} *_{Q_8} \tilde{T}$$

with $\tilde{T} = Q_8 \rtimes C_3$ being the binary tetrahedral group. Theorem 7.2 does not rule out that $\widetilde{K_0}\mathbb{Z}B_4(\mathbb{S}^2) \to \widetilde{K_0}\mathbb{Q}B_4(\mathbb{S}^2)$ is non-trivial since $s(Q_{16}) = 1$, however explicit computation shows that the abelian group $\pi_0\mathrm{colim}_{\mathrm{Or}B_4(\mathbb{S}^2)_{\mathrm{Fin}}}\mathbf{Wh}(\mathbb{Z};-)$ is trivial. It corresponds to the group denoted B_n for the value of $n = 0$ in [GJM18] Section 4.2., which is a subgroup of $K_{-1}\mathbb{Z}Q_8 = 0$ and therefore zero. Hence $\widetilde{K_0}\mathbb{Z}B_4(\mathbb{S}^2) \to \widetilde{K_0}\mathbb{Q}B_4(\mathbb{S}^2)$ is in fact the zero map. This example shows that Theorem 7.2 is not optimal by itself and case by case analysis has to be done for groups that do not fulfil the criterion given.

Example 11.7. Take the extra-special group 3_-^{1+2}. It has the SmallGroup library ID $(27, 4)$ and is concretely given by the presentation

$$\left\langle a, b | a^9 = b^3 = 1, bab^{-1} = a^4 \right\rangle.$$

It has a unique irreducible faithful $\mathbb{Q}$-representation α of dimension 6 with character

$$\chi_\alpha(g) = \begin{cases} 6 & g = 1 \\ -3 & g = a^3, a^6 \\ 0 & \text{else.} \end{cases}$$

The subgroup C_9 spanned by a has a unique irreducible $\mathbb{Q}$-representation β of dimension 6 as well, with character

$$\chi_\beta(g) = \begin{cases} 6 & g = 1 \\ -3 & \mathrm{Order}(g) = 3 \\ 0 & \mathrm{Order}(g) = 9. \end{cases}$$

By looking at their characters it can easily be checked that $\mathrm{Ind}_{C_9}^{3_-^{1+2}}(\beta) = 3\alpha$. Now define

$$G := 3_-^{1+2} *_{C_9} (3_-^{1+2})'$$

where $(3_-^{1+2})'$ is another copy of 3_-^{1+2} and let α' be the corresponding unique irreducible faithful $\mathbb{Q}$-representation of dimension 6. Then

$$\gamma := [\mathrm{Ind}_{3_-^{1+2}}^G \alpha] - [\mathrm{Ind}_{(3_-^{1+2})'}^G \alpha']$$

[8] At the early stages of our work on understanding the map $\widetilde{K_0}\mathbb{Z}G \to \widetilde{K_0}\mathbb{Q}G$ we could quickly establish the result that the image is 2-torsion and that it can be reduced to coming from $\pi_0\mathrm{colim}_{\mathrm{Or}G_{\mathrm{Fin}}}\mathbf{Wh}(\mathbb{Z};-)$. However, it was at that time not clear that the condition $s(H) = 0$ for all finite subgroups H is strong enough to imply the vanishing of the image.

is an element of order 3 in $K_0\mathbb{Q}G$. In order to see that γ is non-trivial we note that we have a pushout square (see e.g. Waldhausen [Wal78])

$$\begin{tikzcd} K_0\mathbb{Q}C_9 \arrow[r] \arrow[d] & K_0\mathbb{Q}3_-^{1+2} \arrow[d] \\ K_0\mathbb{Q}3_-^{1+2} \arrow[r] & K_0\mathbb{Q}G, \end{tikzcd}$$

hence $[\mathrm{Ind}_{3_-^{1+2}}^G \alpha] - [\mathrm{Ind}_{(3_-^{1+2})'}^G \alpha']$ could only vanish if there was an element x in $K_0\mathbb{Q}C_9$ such that $\mathrm{Ind}_{C_9}^{3_-^{1+2}} x = [\alpha]$, which can be ruled out by looking the images of the three irreducible $\mathbb{Q}$-representations of C_9 under $\mathrm{Ind}_{C_9}^{3_-^{1+2}}$. This example shows that groups can have torsion in $K_0\mathbb{Q}G$ that is not 2-torsion. All computations presented have been done using *GAP* and the "wedderga" package.

A Computing $K_{-1}\mathbb{Z}G$ for finite groups using GAP

In this section, we want to give a description of how to compute the negative K-theory groups $K_{-1}\mathbb{Z}G$ using the computer algebra system *GAP*, [20]. The code used can be found online at `https://github.com/georglehner/negativektheory`. In the following, fix a finite group G and let n be the order of G. As described in Section 6, we have

$$K_{-1}\mathbb{Z}G \cong \mathbb{Z}^{r(G)} \oplus \mathbb{Z}/2^{s(G)}$$

with the sum going over all prime numbers p dividing the order of G. The two coefficients compute as

$$r(G) = 1 - r_{\mathbb{Q}} + \sum_{p|n} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}),$$

where r_k is defined as the number of irreducible representations of G over the field k and $s(G)$ is equal to the number of isomorphism classes of irreducible $\mathbb{Q}$-representations of G with even global Schur index, but odd local Schur index at every prime p dividing the order of G.

A.1 The free rank $r(G)$

As written before, to compute $r(G)$ we need to compute the number of irreducible representations of G over the fields $\mathbb{Q}, \mathbb{F}_p$ and $\mathbb{Q}_p$ as we have the formula

$$r(G) = 1 - \mathbb{Q} + + \sum_{p|\ |G|} (r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}).$$

An essential tool for this is Berman's theorem. For the following, fix a field k of characteristic p. We call an element g of G *p-singular* if p divides the order of g and *p-regular* otherwise. Note that if p does not divide the order of G, all elements of G are p-regular. Let m be the exponent of G, i.e. the least common multiple over all orders of elements of G. Denote by ζ_m a primitive m-th root of unity. The Galois group $\mathrm{Gal}(k(\zeta_m) : k)$ can by identified with a subgroup T_m of $(\mathbb{Z}/m)^\times$, since any $\phi \in \mathrm{Gal}(k(\zeta_m) : k)$ is uniquely defined by its image of ζ_m, and so $\phi(\zeta_m) = \zeta_m^t$ for a unique $t \in T_m$. We say two elements g_1 and g_2 of G are *k-conjugate* if there exists $h \in G$ and a $t \in T_m$ such that $g_1^t = hg_2h^{-1}$.

Theorem A.1 (Berman, [Ber56], see also [Rei64])**.** *The number of irreducible k-representations of G is equal to the number of k-conjugacy classes of p-regular elements of G.*

For the field $\mathbb{Q}$, the Galois group $\mathrm{Gal}(\mathbb{Q}(\zeta_m) : \mathbb{Q})$ is always equal to the entire group $(\mathbb{Z}/m)^\times$. The number $r_{\mathbb{Q}}$ is thus equal to the number of conjugacy classes of cyclic subgroups of G.

In the case of the field $\mathbb{Q}_p$ a number t is in $t \in T_m$ iff t is not divisible by p and t is congruent to a power of p mod μ, see [GS13] Chapter IV, Section 4.

The following lemma will also be useful.

Lemma A.2 (Magurn [Mag13], see Lemma 1.)**.** For each prime p dividing n, $r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}$ is equal to the number of $\mathbb{Q}_p$-conjugacy classes of p-singular elements in G.

Remark A.3. In particular, if the group G is a p-group, all non-trivial elements of G are p-singular, hence the number $r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}$ is simply the number of $\mathbb{Q}_p$-conjugacy classes of non-trivial elements. In this case the exponent m is a power of p, hence the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p)$ is actually the entire group $(\mathbb{Z}/m)^\times$. This means that $\mathbb{Q}$-conjugacy and $\mathbb{Q}_p$-conjugacy agree and thus

$$r(G) = 0.$$

We will use the pre-existing functionality of *GAP* to compute the set of conjugacy classes of G. For the following, we chose the function ConjugacyClassesByRandomSearch, but any of the predefined methods should work.

Since k-conjugacy is a coarser equivalence relation than conjugacy, it induces an equivalence relation on the set *cc* of conjugacy classes of G. In order to simplify code, we have written a routine MyPartitionSet that takes a set together with an equivalence relation as input and gives the set of equivalence classes as output.

```
MyPartitionSet := function( S, rel )
# This function exists to make the code more readable
# It takes a set S and an equivalence relation on elements of S as input
# rel should be given as a function in two variables
# The output is a list of rel-equivalence classes of elements of S
# This ONLY WORKS if rel is an equivalence relation
# MyPartitionSet does not check if rel is an equivalence relation

local Sprime, classes, s, allobjectsequivtos;
Sprime := S;
classes := [];
while not IsEmpty(Sprime) do
      s:= First(Sprime);
      allobjectsequivtos := Filtered ( Sprime , sprime -> rel(s,sprime) );
      Add(classes, allobjectsequivtos);
      SubtractSet(Sprime, allobjectsequivtos);
od;
return classes;
end;
```

The set of $\mathbb{Q}$-conjugacy classes can then be computed with the following function.

```
RationalConjugacyClasses := function(cc)
# This function takes a set of conjugacyclasses as input
# The output is a list of sets of Q-conjugate conjugacy classes of G
# Two elements are rationally conjugate iff
# the cyclic subgroups they generate are conjugate
# Note: this function has the same functionality as RationalClasses

local orders, m, galoisgroup, ccsortedbyorder, classesoforderi, kconjugate,
```

```
rationalconjugacyclasses, c, t, allconjugatestoc;
orders := Set(List(cc, c->Order(First(c)) ));
# The set of all appearing orders of elements of G
m := Lcm(orders);
# This is the exponent of G
galoisgroup := PrimeResidues(m);

rationalconjugacyclasses := [];
ccsortedbyorder := List( orders, i -> Filtered( cc , c -> Order(First(c)) = i ) );
# ccSortedByOrder is a list with entries
# the set of all conjugacy classes of given order i.
for classesoforderi in ccsortedbyorder do
    UniteSet(rationalconjugacyclasses, MyPartitionSet(classesoforderi,
         {c,d} -> ForAny( galoisgroup , t -> First(c)^t in d ) ));
# We now partition all classes of order i into the set of equivalence classes
# of conjugacy classes with respect to k conjugacy
od;
return rationalconjugacyclasses;
end;
```

For the terms $r_{\mathbb{Q}_p} - r_{\mathbb{F}_p}$ we can simplify their computation using Lemma A.2 hence we need to compute the number of $\mathbb{Q}_p$-conjugacy classes of singular elements of G. Using Berman's theorem, the only thing left to understand is the Galois group $\mathrm{Gal}(\mathbb{Q}_p(\zeta_m) : \mathbb{Q}_p)$. For this we need to compute T_m as a subset of $(\mathbb{Z}/m)^\times$. We have that $t \in T_m$ iff t is not divisible by p and t is congruent to a power of p mod μ.

In *GAP* this can be realized with the following function.

```
GaloisGroupOfQpZetamOverQp := function(m, p)
  # Gives the Galois group as a subset of prime residues of m
local q, mu, PowersOfPModmu, ZmUnits;
q := p^Length(Filtered(FactorsInt(m) , x -> x = p));
mu := BestQuoInt(m,q);
PowersOfPModmu := List( [1..OrderMod(p,mu)], i -> PowerModInt(p,i,mu) );
return Filtered( PrimeResidues(m), i -> i mod mu in PowersOfPModmu);
end;
```

Using this group we then define the equivalence relation of $\mathbb{Q}_p$-conjugacy on the set of conjugacy classes of G.

```
QpConjugate := function(p, c, d, m )
# QpConjugate is an equivalence relation of conjugacy classes of G
# The inputs are a prime p, two conjugacy classes c and d of G
# as given by the function ConjugacyClasses
# and the exponent m of the group G
# The output is either true or false
```

```
local Tm;

Tm := GaloisGroupOfQpZetamOverQp(m,p);
# Gives the Galois group as a subset of prime residues of m
return ForAny( Tm, t->First(c)^t in d) ;
# First(c) is any representative of the conjugacy class c.
end;
```

With this defined, we now compute the set of singular $\mathbb{Q}_p$-conjugacy classes.

```
SingularQpConjugacyClasses := function(cc,p)
# This function takes a set of conjugacyclasses and a prime p as input
# The output is a list of sets of Qp-conjugate conjugacy
# classes of G with p-singular order
# An element g is p-singular if p divides its order

local orders, singularorders, m, ccsortedbyorder, classesoforderi,
singularqpconjugacyclasses;
orders := Set(List(cc, c->Order(First(c)) ));
# The set of all appearing orders of elements of G
m := Lcm(orders);
# This is the exponent of G

singularorders := Filtered(List(cc, c->Order(First(c)) ), i -> i mod p = 0 );
# We will only look at orders that are multiples of p
# First(c) picks a representative element of the conjugacy class c

singularqpconjugacyclasses := [];
ccsortedbyorder := List( singularorders,
      i -> Filtered( cc , c -> Order(First(c)) = i ) );
# ccSortedByOrder is a list with entries the set of
# all conjugacy classes of given order i.
# We only care about orders divisible by p, hence SingularOrders
for classesoforderi in ccsortedbyorder do
     UniteSet(singularqpconjugacyclasses, MyPartitionSet(classesoforderi,
          {c,d} -> QpConjugate(p,c,d,m) ) );
# We now partition all classes of order i into the set of equivalence classes
# of conjugacy classes with respect to Qp conjugacy
od;
return singularqpconjugacyclasses;
end;
```

Using the formula

$$r(G) = 1 - r_{\mathbb{Q}} + \sum_{p|n} \#\left\{\text{singular } \mathbb{Q}_p\text{-conjugacy classes}\right\},$$

we can now compute $r(G)$ using the function:

```
rOfGroup := function(G)
local primes, rQ, rsingQp, cc;
cc := ConjugacyClassesByRandomSearch(G);
primes := PrimeDivisors(Order(G));
rQ := Length( RationalConjugacyClasses(cc) );
rsingQp := List( primes, p -> Length(SingularQpConjugacyClasses(cc,p)) );
return 1-rQ + Sum(rsingQp);
end;
```

A.2 The $\mathbb{Z}/2$ rank $s(G)$

The number $s(G)$ is computed as the number of irreducible rational representations of G that have global even Schur index but odd Schur index at every prime p dividing the order of G. This can be computed using the Wedderga package in *GAP* [20]. The main function we will use is WedderburnDecompositionInfo, which computes the Wedderburn decomposition of G as a list in which each entry corresponds to a numerical description of the cyclotomic algebras appearing in the Wedderburn decomposition. The global Schur index can be computed with the function SchurIndex and the local Schur indices with LocalIndicesOfCyclotomicAlgebra, which gives a list of pairs (p, m_p) contain to the finitely many places p at which the local Schur index m_p is non-trivial. Note that $p = \infty$ corresponding to the real Schur index can also appear in this list, which we have to remove when computing the relevant indices for $s(G)$. The following function checks if a cyclotomic algebra A described in the format given by WedderburnDecompositionInfo gives a contribution to the number $s(G)$.

```
# s is the number of algebras A appearing in the Wedderburn decomposition of QG
# that have even Schur index, but odd Schur index
# at every prime dividing the order of G
# AlgebraContributesToS checks this condition for a single algebra
# The Algebra must be in the form of an output of the function
# WedderburnDecompositionInfo of the package wedderga
# primes is a list of primes to be checked against

AlgebraContributesToS := function(A,primes)
local RelevantIndices;
if IsEvenInt(SchurIndex(A)) then
RelevantIndices := Filtered( LocalIndicesOfCyclotomicAlgebra(A) , l -> l[1] in primes );
return ForAll(RelevantIndices, l -> IsOddInt( l[2] ) );
else return false; fi;
```

```
end;
```

With AlgebraContributesToS, it is now a simple matter of counting all the summands in the Wedderburn decomposition which contribute.

```
sOfGroup := function(G)
local n, primes, ww, s;
n := Order(G);
primes := PrimeDivisors(n);
ww := WedderburnDecompositionInfo( GroupRing( Rationals, G ) );
s := Number( ww, A -> AlgebraContributesToS(A,primes) );
return s;
end;
```

A.3 Wrapping things up

For user-friendliness we can put both rOfGroup and sOfGroup into the simple function KMinusOne which returns the values of r and s in one record.

```
KMinusOne := function(G)
return rec(r := rOfGroup(G), s := sOfGroup(G) );
end;
```

As an example, take the binary icosahedral group $\tilde{I}$, also identifiable as $SL(2,\mathbb{F}_5)$. The ID of this group in the small groups library is given as $(120,5)$.

```
gap> G := SmallGroup(120,5);
Group([ (1,2,4,8)(3,6,9,5)(7,12,13,17)(10,14,11,15)(16,20,21,24)(18,22,19,23),
 (1,3,7)(2,5,10)(4,9,13)(6,11,8), (12,16,20)(14,18,22)(15,19,23)(17,21,24) ])
gap> StructureDescription(G);
"SL(2,5)"
gap> KMinusOne(G);
rec( r := 2, s := 1 )
```

From this result we can read off that $K_{-1}\mathbb{Z}[SL(2,\mathbb{F}_5)] = \mathbb{Z}^2 \oplus \mathbb{Z}/2$, which agrees with the results of [Mag13].

A.4 All groups of order ≤ 100 with non-trivial torsion in $K_{-1}\mathbb{Z}G$

We will include a computation of all finite groups H with $s(H) > 0$ of order less than 100. The relevant tables are Tables 1, 2, 3 and 4. However, a few comments are needed. As mentioned in Remark 6.24, the binary polyhedral groups

$$\mathrm{Dic}_n, \tilde{O}, \tilde{I}$$

play a special role as examples of groups with $s > 0$. These are finite subgroups of $SU(2)$, called dicyclic group of order n for $n \geq 4$, binary octahedral and binary icosahedral group, respectively.

They are defined as the inverse images under the double covering $SU(2) \to SO(3)$ of the groups

$$C_{2n}, O, I$$

with C_{2n} the cyclic group of order $2n$, O being the octahedral group and I the icosahedral group, thought of as the symmetries of the regular $2n$-gon, cube and icosahedron, respectively. Magurn [Mag13] computes the negative K-theory groups of all of these. The smallest group with $s > 0$ is the group $Q_{16} = \mathrm{Dic}_4$. Note that if $n = 2^k$ is a power of 2, the group Dic_n is also called generalized quaternion group $Q_{2^{k+2}}$ of order $4 \cdot 2^k = 2^{k+2}$.

In order to compute a table of all groups with non-trivial s, we use the function sOfGroup constructed in the previous section as well as the functionality provided by the SmallGroup library in *GAP*.

The SmallGroups library allows one to address all groups of reasonably small order by two parameters, the order of the group n as well as an index i. The idea is to fix an order n, and then compute s for each group of order n by going through all possible indices i. We will then print only groups with non-trivial s. In addition, if the group G is of the form $\mathrm{Dic}_n, \tilde{O}$, $\tilde{I}$, or a product of these and another group, we address the group as such in the column "Structure". If it is not of this form, we include a printout of the functionality StructureDescription of *GAP*. Note that StructureDescription(G) does not specify the group G up to isomorphism, but only serves to give a quick idea of the type of group which G is.

If G has a quotient G/N with $s(G/N) > 0$, then each irreducible G/N-representation (I, ρ) that contributes to $s(G/N)$ gives a G-representation $(I, \rho \circ \pi)$ with $\pi\colon G \to G/N$, which contributes to s. In the column "Quotients" we have included all smaller groups with $s > 0$ which appear as quotients. Most groups which are not binary polyhedral groups which appear in our list are, in fact, explained by having binary polyhedral groups as quotients, i.e. the group ring $\mathbb{Z}G$ fails the Eichler condition.[9]

However, not all groups in our list have their irreducible representations contributing to s lifted from binary polyhedral quotients. The smallest example is the group $C_8.C_2^2$ of order 32 with SmallGroup library ID $(32, 44)$. It has a unique irreducible representation contributing to s induced from the subgroup Q_{16}. We included in Tables 5 and 6 all groups of order ≤ 180 which do not have a quotient with non-trivial s.

[9] The Eichler condition on the group ring $\mathbb{Z}G$ is the condition that $\mathbb{Z}G \otimes \mathbb{R} \cong \mathbb{R}G$ contains no copies of the quaternions $\mathbb{H}$ in its Wedderburn decomposition, see [SE70], Chapter 9., also [Nic20].

$n = 16$	Index	Structure	$s(G)$	Quotients
	9	$Q_{16} = \mathrm{Dic}_4$	1	
$n = 20$	Index	Structure	$s(G)$	Quotients
	1	Dic_5	1	
$n = 24$	Index	Structure	$s(G)$	Quotients
	4	Dic_6	1	
$n = 28$	Index	Structure	$s(G)$	Quotients
$n = 32$	Index	Structure	$s(G)$	Quotients
	10	Q8 : C4	1	Q_{16}
	14	C8 : C4	1	Q_{16}
	20	$Q_{32} = \mathrm{Dic}_8$	1	
	41	$C_2 \times Q_{16}$	2	
	44	(C2 x Q8) : C2 = $C_8.C_2^2$	1	
$n = 36$	Index	Structure	$s(G)$	Quotients
$n = 40$	Index	Structure	$s(G)$	Quotients
	1	C5 : C8	1	Dic_5
	4	Dic_{10}	1	
	7	$C_2 \times \mathrm{Dic}_5$	2	
$n = 44$	Index	Structure	$s(G)$	Quotients
$n = 48$	Index	Structure	$s(G)$	Quotients
	8	Dic_{12}	2	
	12	(C3 : C4) : C4	1	Dic_6
	13	C12 : C4	1	Dic_6
	18	C3 : Q16	1	Q_{16}
	27	$C_3 \times Q_{16}$	1	
	28	$\tilde{O}$	1	
	34	$C_2 \times \mathrm{Dic}_6$	2	
$n = 52$	Index	Structure	$s(G)$	Quotients
	1	Dic_{13}	1	
$n = 56$	Index	Structure	$s(G)$	Quotients
	3	Dic_{14}	1	
$n = 60$	Index	Structure	$s(G)$	Quotients
	2	$C_3 \times \mathrm{Dic}_5$	1	
	3	Dic_{15}	2	

Table 1: All groups H with $s(H) > 0$ of order $n < 64$.

n = 64	Index	Structure	s(G)	Quotients
	7	Q8 : C8	1	Q_{16}
	9	(C4 : C4) : C4	1	Q_{16}
	13	(C2 x C2) . ((C4 x C2) : C2)	1	Q_{16}
	14	(C2 x C2) . ((C4 x C2) : C2)	2	2 copies of Q_{16}
	16	C8 : C8	1	Q_{16}
	21	(C4 : C4) : C4	1	Q_{16}
	39	Q16 : C4	1	Q_{32}
	43	C2 . ((C8 x C2) : C2)	1	
	47	C16 : C4	2	Q_{16}, Q_{32}
	48	C16 : C4	1	Q_{16}
	49	C4 . D16 = C8 . (C4 x C2)	1	Q_{16}
	54	$Q_{64} = \mathrm{Dic}_{16}$	1	
	96	C2 x (Q8 : C4)	2	
	107	C2 x (C8 : C4)	2	
	120	C4 x Q16	2	
	129	(C2 x QD16) : C2	1	$C_8.C_2^2$
	132	(C2 x Q16) : C2	3	$C_8.C_2^2, C_2 \times Q_{16}$
	133	((C2 x Q8) : C2) : C2	1	$C_8.C_2^2$
	142	(Q8 : C4) : C2	1	$C_8.C_2^2$
	143	C4 : Q16	3	$C_8.C_2^2, C_2 \times Q_{16}$
	145	(C4 x Q8) : C2	1	$C_8.C_2^2$
	148	(Q8 : C4) : C2	2	$C_2 \times Q_{16}$
	149	((C8 x C2) : C2) : C2	1	$C_8.C_2^2$
	151	(Q8 : C4) : C2	2	2 copies of $C_2 \times Q_{16}$
	154	(C2 . ((C4 x C2) : C2)	1	
	155	(C8 : C4) : C2	1	$C_8.C_2^2$
	156	Q8 : Q8	1	$C_8.C_2^2$
	158	Q8 : Q8	2	$C_2 \times Q_{16}$
	160	(C2 x C2) . (C2 x D8)	1	$C_8.C_2^2$
	161	((C8 x C2) : C2) : C2	1	$C_8.C_2^2$
	164	(Q8 : C4) : C2	1	$C_8.C_2^2$
	165	(Q8 : C4) : C2	2	$C_2 \times Q_{16}$
	166	(Q8 : C4) : C2	1	$C_8.C_2^2$
	168	(C2 x C2) . (C2 x D8)	2	$C_2 \times Q_{16}$
	175	C4 : Q16	4	2 copies of $C_2 \times Q_{16}$
	178	(C4 : Q8) : C2	2	$C_8.C_2^2$
	181	C8 : Q8	2	$C_2 \times Q_{16}$
	188	$C_2 \times Q_{32}$	2	
	191	QD32 : C2	1	
	252	$C_2 \times C_2 \times Q_{16}$	4	
	255	C2 x ((C2 x Q8) : C2)	2	
	259	(C2 x Q16) : C2	1	

Table 2: All groups H with $s(H) > 0$ of order $n = 64$.

$n = 68$	Index	Structure	$s(G)$	Quotients
	1	Dic_{17}	1	
$n = 72$	Index	Structure	$s(G)$	Quotients
	4	Dic_{18}	2	
	24	(C3 x C3) : Q8	2	2 copies of Dic_6
	26	$C_3 \times \mathrm{Dic}_6$	1	
	31	(C3 x C3) : Q8	4	4 copies of Dic_6
$n = 76$	Index	Structure	$s(G)$	Quotients
$n = 80$	Index	Structure	$s(G)$	Quotients
	1	C5 : C16	1	Dic_5
	8	Dic_{20}	2	
	9	C2 x (C5 : C8)	2	
	10	(C5 : C8) : C2	2	2 copies of Dic_5
	11	$C_4 x \mathrm{Dic}_5$	2	
	12	(C5 : C4) : C4	1	Dic_{10}
	13	C20 : C4	3	2 copies of Dic_5 and 1 copy of Dic_{10}
	18	C5 : Q16	1	Q_{16}
	19	(C10 x C2) : C4	2	2 copies of Dic_5
	27	$C_5 \times Q_{16}$	1	
	33	(C5 : C8) : C2	1	
	35	$C_2 \times \mathrm{Dic}_{10}$	2	
	40	(C4 x D10) : C2	1	
	43	$C_2 \times C_2 \times \mathrm{Dic}_5$	4	
$n = 84$	Index	Structure	$s(G)$	Quotients
	5	Dic_{21}	1	
$n = 88$	Index	Structure	$s(G)$	Quotients
	3	Dic_{22}	1	
$n = 92$	Index	Structure	$s(G)$	Quotients
$n = 96$	Index	Structure	$s(G)$	Quotients
	8	Dic_{24}	2	
	11	C3 : (C4 : C8)	1	Dic_6
	14	C3 : (C8 : C4)	2	Q_{16}, Dic_6
	15	C3 : (C8 : C4)	1	Dic_6
	17	C3 : (Q8 : C4)	1	Q_{16}
	21	C3 : (C4 : C8)	1	Dic_6
	23	C3 : (Q8 : C4)	2	$Q_{16}, \mathrm{Dic}_{12}$

Table 3: All groups H with $s(H) > 0$ of order $68 \leq n \leq 100$, part 1.

$n = 96$	Index	Structure	$s(G)$	Quotients
	24	C3 : (C8 : C4)	1	Dic_6
	25	C3 : (C8 : C4)	3	$Q_{16}, \mathrm{Dic}_6, \mathrm{Dic}_{12}$
	26	C3 : (C4 . D8 = C4 . (C4 x C2))	1	Dic_6
	29	C3 : (C4 . D8 = C4 . (C4 x C2))	1	Dic_6
	31	C3 : (C2 . ((C4 x C2) : C2)	1	
	36	C3 : Q32	1	Q_{32}
	38	C3 : ((C4 x C2) : C4)	1	Dic_6
	42	C3 : (Q8 : C4)	1	Q_{16}
	53	C3 x (Q8 : C4)	1	Q_{16}
	57	C3 x (C8 : C4)	1	Q_{16}
	63	$C_3 \times Q_{32}$	1	
	66	SL(2,3) : C4	1	$\tilde{O}$
	75	$C_4 \times \mathrm{Dic}_6$	2	
	76	C3 : (C4 : Q8)	4	4 copies of Dic_6
	77	C3 : ((C2 x C2) . (C2 x C2 x C2))	2	2 copies of Dic_6
	85	C3 : ((C2 x Q8) : C2)	2	2 copies of Dic_6
	95	C3 : (C4 : Q8)	2	2 copies of Dic_6
	97	C3 : ((C2 x C2) . (C2 x C2 x C2))	2	2 copies of Dic_6
	112	$C_2 \times \mathrm{Dic}_{12}$	4	
	116	C3 : ((C2 x Q8) : C2)	1	$C_8.C_2^2$
	119	C3 : ((C8 x C2) : C2)	1	
	122	C3 : ((C2 x Q8) : C2)	1	$C_8.C_2^2$
	124	$Q_{16} \times S_3$	3	
	130	C2 x ((C3 : C4) : C4)	2	2 copies of Dic_6
	131	C3 : ((C2 x Q8) : C2)	2	2 copies of Dic_6
	132	C2 x (C12 : C4)	2	2 copies of Dic_6
	150	C2 x (C3 : Q16)	2	2 copies of Q_{16}
	158	C3 : ((C2 x Q8) : C2)	1	$C_8.C_2^2$
	181	$C_6 \times Q_{16}$	2	
	185	A4 : Q8	1	Dic_6
	188	$C_2 \times \tilde{O}$	2	
	190	(C2 x SL(2,3)) : C2	1	
	191	(C2 . S4 = SL(2,3) . C2) : C2	2	
	205	$C_2 \times C_2 \times \mathrm{Dic}_6$	4	
	217	C3 : ((C2 x Q8) : C2)	1	
$n = 100$	Index	Structure	$s(G)$	Quotients
	1	Dic_{25}	2	
	6	$C_5 \times \mathrm{Dic}_5$	1	
	7	(C5 x C5) : C4	6	6 copies of Dic_5
	10	(C5 x C5) : C4	1	Dic_5

Table 4: All groups H with $s(H) > 0$ of order $68 \leq n \leq 100$, part 2.

n	Index	Structure	s
16	9	Q_{16}	1
20	1	Dic_5	1
24	4	Dic_6	1
32	20	Q_{32}	1
32	44	$C_8.C_2^2$	1
40	4	Dic_{10}	1
48	28	$\tilde{O}$	1
52	1	Dic_{13}	1
56	3	Dic_{14}	1
64	54	Q_{64}	1
64	154	(C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2)) : C2	1
64	191	QD32 : C2	1
64	259	(C2 x Q16) : C2	1
68	1	Dic_{17}	1
80	33	$C_2^2.F_5$	1
80	40	(C4 x D10) : C2	1
84	5	Dic_{21}	1
88	3	Dic_{22}	1
96	31	C3 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2))	1
96	119	C3 : ((C8 x C2) : C2)	1
96	190	(C2 x SL(2,3)) : C2	1
96	191	(C2 . S4 = SL(2,3) . C2) : C2	2
96	217	C3 : ((C2 x Q8) : C2)	1
104	4	Dic_{26}	1
116	1	Dic_{29}	1
120	5	$\tilde{I}$	1
120	8	(C3 : C4) x D10	1
128	66	Q16 : C8	1
128	72	(C2 x Q16) : C4	1
128	74	C2 . (((C8 x C2) : C2) : C2) = (C8 x C2) . (C4 x C2)	1
128	82	(C2 x C2) . ((C8 x C2) : C2) = (C8 x C2) . (C4 x C2)	1
128	90	C2 . ((C4 : C8) : C2) = (C4 x C4) . (C4 x C2)	2
128	137	((C2 x C2) . ((C4 x C2) : C2) = (C4 x C2) . (C4 x C2)) : C2	1
128	143	C2 . (((C2 x C2 x C2) : C4) : C2) = (C4 x C4) . (C4 x C2)	2
128	152	C2 . ((C16 x C2) : C2) = C16 . (C4 x C2)	1
128	163	Q_{128}	1
128	634	((C8 x C2) : C4) : C2	1
128	637	((Q8 : C4) : C2) : C2	1
128	879	C2 x (C2 . ((C8 x C2) : C2) = C8 . (C4 x C2))	2
128	912	(Q16 : C4) : C2	1
128	927	((C2 x Q16) : C2) : C2	1
128	946	(C2 . ((C8 x C2) : C2) = C8 . (C4 x C2)) : C2	2
128	954	(C2 . ((C8 x C2) : C2) = C8 . (C4 x C2)) : C2	1
128	971	(C2 . ((C8 x C2) : C2) = C8 . (C4 x C2)) : C2	1

62 Table 5: All groups H with $s(H) > 0$ of order $n \leq 180$ with no quotient with $s > 0$, part 1.

n	Index	Structure	s
128	996	QD64 : C2	1
128	2025	(((C2 x Q8) : C2) : C2) : C2	2
128	2149	(C2 x Q32) : C2	1
128	2318	(C2 x ((C2 x Q8) : C2)) : C2	1
132	3	Dic_{33}	1
136	3	C17 : C8	1
136	4	Dic_{34}	1
144	15	C9 : QD16	1
144	43	Q8 x D18	1
144	59	(C3 x C3) : QD16	1
144	118	(C3 x C3) : QD16	1
144	138	(C3 x C3) : ((C4 x C2) : C2)	1
148	1	Dic_{37}	1
152	3	Dic_{38}	1
156	1	C13 : C12	1
160	31	C5 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2))	1
160	80	C5 : (C2 . ((C4 x C2) : C2) = (C2 x C2) . (C4 x C2))	1
160	133	C5 : ((C8 x C2) : C2)	1
160	225	C5 : ((C2 x Q8) : C2)	1
164	1	Dic_{41}	1
168	7	C7 : (C3 x Q8)	1
168	12	(C3 : C4) x D14	1
168	15	C3 : ((C14 x C2) : C2)	1

Table 6: All groups H with $s(H) > 0$ of order $n \leq 180$ with no quotient with $s > 0$, part 2.

A.5 Methodology

All computations have been done on a Microsoft Surface Pro 6 with Windows 10 as operating system. The software used was GAP 4.11.0 [20].

For the computation of groups with non-trivial s the following code was used.

```
output := OutputTextFile( "output.txt" , true );

SearchAllGroupsForS := function(n)
local j, s, G;
for j in [1..Length(AllSmallGroups(n))] do
G := SmallGroup(n,j);
s := sOfGroup(G);
if s>0 then AppendTo(output, "␣&␣",j,"␣&␣",StructureDescription(G),"␣&␣",s,
      "␣&␣\\","\\","\n"); fi;
od;
end;

for k in Filtered([4..100], i -> GcdInt(i,4)=4) do
# k is the order of the groups searched.
# Note that we only have s>0 for groups of order divisible by 4

AppendTo(output, "\\","hline␣\n","$n␣=␣", k,"$␣&␣Index␣␣&␣␣Structure␣␣&␣$s(G)$
␣␣␣␣␣␣&␣Quotients␣\\","\\␣\n","\\","hline␣\n");
SearchAllGroupsForS(k);
od;
```

For the computation of those groups with non-trivial s without a smaller quotient with $s > 0$ the following code was used.

```
ElementGrps := []
#ElementGrps is a list of the SmallGroupLibrary IDs
#of those groups with non-trivial s
#without quotients with that property
#Our algorithm checks all quotients of each new group with s>0
#against groups that already appear in ElementGrps
#This is done by computing NormalSubgroups(G)
#and for each normal subgroup N compute
#the ID of the quotient G/N and compare this new ID
#with all IDs already in ElementGrps
#If no match is found, add the ID of G to ElementGrps

SearchAllBuildingBlocksForS := function(n)
local j, s, G;
for j in [1..Length(AllSmallGroups(n))] do
G := SmallGroup(n,j);
s := sOfGroup(G);
if s>0 then
  if not ForAny( NormalSubgroups(G),
      N -> ForAny( ElementGrps,
         id -> id = IdSmallGroup( FactorGroup( G, N )) ) )
      then Add(ElementGrps, [n,j]); fi;
  fi;
od;
end;

for k in Filtered([4..180], i -> GcdInt(i,4)=4) do
# k is the order of the groups searched.
# Note that we only have s>0 for groups of order divisible by 4
SearchAllBuildingBlocksForS(k);
od;

Print( ElementGrps );
```

We want to remark that the computations for orders above 100, especially at order 128, tended to be quite slow. The actual computation has been done in steps for fixed intervals of orders rather than all at once.

B Stable ∞-categories and t-structures

The notion of a t-structure on a stable ∞-category has been quite useful in phrasing and proving the main results of this thesis. The following section collects results used and proves many of them for completeness, however, the author claims no originality. The standard reference will be Chapter 1 in Lurie, [Lur17]. An analogy to keep in mind is that stable ∞-categories behave much like a higher algebraic analogue of abelian categories in the 1-categorical world. In the following, by a commutative square in an ∞-category $\mathcal{C}$, we mean a functor $\Delta^1 \times \Delta^1 \to \mathcal{C}$. Similarly, a commuting triangle is a functor $\Delta^2 \to \mathcal{C}$.

Definition B.1. Let $\mathcal{C}$ be an ∞-category. A *zero object* 0 is an object of $\mathcal{C}$ that is both terminal and initial. We call $\mathcal{C}$ *pointed* if it contains a zero object.

Definition B.2. Let $\mathcal{C}$ be pointed. If $f\colon X \to Y$ is a map in $\mathcal{C}$, we say that f is a zero map if it factors through a zero object. Suppose we have a triangle

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr] & Y \arrow[d, "g"] \\ & Z. \end{tikzcd}$$

We say that this triangle is a cofiber sequence (fiber sequence) if gf is a zero map and the resulting square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a pushout (pullback).

Definition B.3. Let $\mathcal{C}$ be an ∞-category. We call $\mathcal{C}$ *stable* if it contains a zero object 0, it is closed under the formation of cofibers and fibers of maps, and a triangle of the shape

$$X \to Y \to Z$$

is a cofiber sequence iff it is a fiber sequence.

Example B.4. The ∞-category of spectra Sp is stable.

Example B.5. Let R be a ring. The derived ∞-category $D(R)$ represented by chain complexes over R localized by weak equivalences is stable.

There are many consequences following from stability. For one $\mathcal{C}$ is closed under finite limits and colimits, any square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$

is a pullback square iff it is a pushout square. Moreover, finite products and coproducts agree and the sets $[X,Y] \coloneqq \pi_0 \mathrm{Map}_{\mathcal{C}}(X,Y)$ naturally form abelian groups. Given an object X, define ΣX via the pushout square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma X \end{tikzcd}$$

and ΩX via the pullback square

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd}$$

Then Σ and Ω form a pair of autoequivalences of $\mathcal{C}$ with $\Sigma^{-1} \simeq \Omega$. If

$$X \to Y \to Z$$

is a fiber sequence, we get naturally induced shifted fiber sequences

$$Y \to Z \to \Sigma X$$

and

$$\Omega Z \to X \to Y.$$

For fixed objects X, Y in $\mathcal{C}$ the sequence of spaces $\mathrm{Map}_{\mathcal{C}}(X, \Sigma^n Y)$ naturally form a spectrum $\mathrm{map}_{\mathcal{C}}(X,Y)$ which lifts to an enrichment of $\mathcal{C}$ over the symmetric monoidal ∞-category of spectra [GH15].

The homotopy category of $\mathcal{C}$ naturally inherits the structure of a triangulated category with shift functors being represented by Σ and distinguished triangles given by sequences of the form

$$X \to Y \to Z \to \Sigma X$$

where $X \to Y \to Z$ is a fiber sequence and $Z \to \Sigma X$ is the connecting map coming from the shifted sequence $Y \to Z \to \Sigma X$.

The following proposition is essential.

Proposition B.6 ([Lur17], Proposition 1.1.3.1.)**.** Let $\mathcal{C}$ be a stable ∞-category and K any simplicial set. Then

$$\mathrm{Fun}(K, \mathcal{C})$$

is again a stable ∞-category.

Let F by a functor between stable ∞-categories $\mathcal{C} \to \mathcal{D}$. We say that F is *exact* if it preserves zero objects and fiber sequences.

Proposition B.7 ([Lur17], Proposition 1.1.4.1.)**.** A functor $F \colon \mathcal{C} \to \mathcal{D}$ between stable ∞-categories is exact iff it preserves finite limits iff it preserves finite colimits.

Example B.8. In the following let $\mathcal{C}$ be a stable ∞-category and K a simplicial set.

- Any functor that is a right adjoint preserves limits and is thus exact. Similarly, any left adjoint preserves colimits and is thus exact as well.
- The functor $\lim\colon \mathrm{Fun}(K,\mathcal{C}) \to \mathcal{C}$ is an exact functor. Similarly, colim is exact as well.
- Assume $\mathcal{C}$ is symmetric monoidal and the tensor product $\otimes$ preserves colimits in both variables. Suppose F is a functor $K^{op} \to \mathcal{C}$. Then the coend construction

$$\int^{k\in K} F(k) \otimes -\colon \mathrm{Fun}(K,\mathcal{C}) \to \mathcal{C}$$

  is an exact functor.

At first, it might seem odd that left exactness implies exactness, as a similar statement does not hold for abelian categories. If R is a ring and M an R-module, then $\mathrm{Hom}_R(M,-)$ is a left exact functor $\mathrm{Mod}_R \to \mathrm{Ab}$, but exactness on the right only holds up to the right derived functors of $\mathrm{Hom}_R(M,-)$, namely $\mathrm{Ext}^i(M,-)$. The reason why this works in the stable setting is that the natural lift of the functor $\mathrm{Hom}_R(M,-)$ to a functor on the derived categories $D(R) \to D(\mathbb{Z})$ already contains all Ext-groups as the negative homotopy groups of its image.

In order to continue our analogy with abelian categories, we can think of a fiber sequence

$$X \to Y \to Z$$

as an analogue of an exact sequence

$$0 \to A \to B \to C \to 0$$

in the case of an abelian category. However, there is one crucial difference: The boundary map $\delta\colon Z \to \Sigma X$ contains information about the sequence $X \to Y \to Z$ which does not exist in the 1-categorical world. As an example, if we assume

$$0 \to A \to B \to C \to 0$$

to be a short exact sequence of abelian groups, we get an induced fiber sequence of Eilenberg-Maclane spectra

$$\mathbf{H}A \to \mathbf{H}B \to \mathbf{H}C$$

The boundary map $\mathbf{H}C \to \Sigma\mathbf{H}A$ essentially encodes all information about the corresponding short exact sequence. In particular, it is zero if and only if the short exact sequence

$$0 \to A \to B \to C \to 0$$

splits. We will make this precise in the following part.

Definition B.9. We call two fiber sequences $X_i \to Y_i \to Z_i, (i = 1, 2)$ with boundary maps $\delta_i \colon Z_i \to \Sigma X_i$ equivalent if they fit into a commuting diagram

$$\begin{tikzcd}
X_1 \arrow[r] \arrow[d, "f_X"] & Y_1 \arrow[r] \arrow[d, "f_Y"] & Z_1 \arrow[r, "\delta_1"] \arrow[d, "f_Z"] & \Sigma X_1 \arrow[d, "\Sigma f_X"] \\
X_2 \arrow[r] & Y_2 \arrow[r] & Z_2 \arrow[r, "\delta_2"] & \Sigma X_2
\end{tikzcd}$$

with all three vertical maps f_X, f_Y and f_Z being equivalences.

Remark B.10. Any two of the vertical maps being equivalences implies that the third one is. This is a consequence of the 5-lemma, since a map is an equivalence iff for all objects C the induced map under $\mathrm{Map}_{\mathcal{C}}(C, -)$ is a weak equivalence of spaces, and $\mathrm{Map}_{\mathcal{C}}(C, -)$ maps fiber sequences to fiber sequences of spaces.

Definition B.11. A fiber sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

is called *split* if one of the following equivalent conditions is satisfied:

1. f has a right inverse,
2. g has a left inverse,
3. There is an equivalence of fiber sequences

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "="] & Y \arrow[r, "g"] \arrow[d, "\simeq"] & Z \arrow[d, "="] \\
X \arrow[r] & X \vee Z \arrow[r] & Z
\end{tikzcd}$$

4. The boundary map $\delta \colon Z \to \Sigma X$ is zero

Proof. (3) clearly implies (1) and (2).

(2) $\Rightarrow$ (4) : Assume that g has a left inverse $z \colon Z \to Y$, i.e. $g \circ z \simeq \mathrm{id}_Z$. The map $\delta \in [Z, \Sigma X]$ is the image of id_Z under δ_*, which is part of the exact sequence

$$[Z, Y] \xrightarrow{g_*} [Z, Z] \xrightarrow{\delta_*} [Z, \Sigma X].$$

Since the left morphism has a left inverse given by z_* it is surjective, therefore, by exactness $\delta_* = 0$, which implies that $\delta = \delta_*(\mathrm{id}_Z) = 0$.

(4) $\Rightarrow$ (2) Assume that $\delta = 0$. The exact sequence

$$[Z, Y] \xrightarrow{g_*} [Z, Z] \xrightarrow{\delta_* = 0} [Z, \Sigma X]$$

implies that g_* is surjective. Pick a map $z \colon Z \to Y$ in the preimage of id_Z. Then this z is a left inverse to g.

(2) $\Rightarrow$ (3): Denote the left inverse of g as z. We have the exact sequence

$$[Y, \Omega Z] \xrightarrow{\delta_*} [Y, X] \xrightarrow{f_*} [Y, Y] \xrightarrow{g_*} [Y, Z].$$

The element $\mathrm{id}_Y - z \circ g \in [Y, Y]$ lies in the kernel of g_*, hence in the image of f_*. Choose an element x in its preimage. We claim x is a right inverse to f. Applying f_* to $x \circ f - \mathrm{id}_X$ yields

$$f_*(x \circ f - \mathrm{id}_X) = f \circ x \circ f - f = (\mathrm{id}_Y - z \circ g) \circ f - f = f - z \circ g \circ f - f = 0$$

and since $\delta = 0$ because of the previous point, f_* is injective, hence we can conclude $x \circ f \simeq \mathrm{id}_X$.

The two maps x and z together give a map $Y \to X \vee Z$, which fits into the commuting diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "="] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d, "="] \\ X \arrow[r] & X \vee Z \arrow[r] & Z \end{tikzcd}$$

Using Remark B.10, the middle map is an equivalence and thus we have an equivalence of fiber sequences. (1) $\Rightarrow$ (3) follows analogously. $\square$

Remark B.12. Given a split fiber sequence $X \to Y \to Z$, the choices of splittings in terms of either right inverses for f or left inverses for g[10] are not necessarily unique. We can, however, measure their non-uniqueness. Since $\delta = 0$, we have the short exact sequence

$$0 \to [Z, X] \to [Z, Y] \xrightarrow{g_*} [Z, Z] \to 0.$$

Any two choices of left inverses of g thus differ by an element in $[Z, X]$. As a special case, we see that if $[Z, X] = 0$ we have a *unique* splitting.

The following lemma can be quite useful. Moreover, it highlights another key difference of the higher categorical to the 1-categorical world - that commutative squares involve the datum of their choices of 2-cells which verify the commutativity and that even if all maps involved are zero maps, the square can be non-trivial.

Lemma B.13. Let $\mathcal{C}$ be a stable ∞-category and two objects $X, Y \in \mathcal{C}$. There is a bijection between the set of commutative squares, i.e. functors $\Delta^1 \times \Delta^1 \to \mathcal{C}$ of the form

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f'"] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma Y \end{tikzcd}$$

and $[\Sigma X, \Sigma Y] \cong [X, Y]$ induced by taking vertical cofibers.

[10] Left inverses of g and right inverses of f determine each other.

Proof. An inverse is given by completing a map $g\colon \Sigma X \to \Sigma Y$ to the commutative square (unique up to contractible choice),

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \Sigma Y \arrow[d, "="] \\
\Sigma X \arrow[r, "g"] & \Sigma Y
\end{tikzcd}$$

and taking the square one gets on vertical fibers. $\square$

Note that by [Lur17] Lemma 1.1.2.10, the inverse of g corresponds to the flipped square

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"] & 0 \arrow[d] \\
0 \arrow[r] & \Sigma Y.
\end{tikzcd}$$

B.1 t-structures

While we thought of a stable ∞-category as analogous to abelian categories during the previous sections, there was no direct relationship between the two notions. This, however, changes when we add the additional datum of a *t-structure* on a stable ∞-category $\mathcal{C}$. The existence of a t-structure defines an abelian subcategory of $\mathcal{C}$ called the *heart* $\mathcal{C}^{\heartsuit}$ of the t-structure as well as natural functors $\pi_n\colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ which behave like the homotopy groups of spectra or the homology groups of a chain complex. The notion of t-structures was originally developed for triangulated categories in [BD82].

Definition B.14. Let $\mathcal{D}$ be a triangulated category. A t-structure on $\mathcal{D}$ is a pair of full subcategories $\mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq 0}$, both closed under isomorphisms, such that the following three conditions hold. Here $\mathcal{D}_{\geq n} := \Sigma^n \mathcal{D}_{\geq 0}$ and $\mathcal{D}_{\leq n} := \Sigma^n \mathcal{D}_{\leq 0}$ are defined as the essential images under the functors Σ^n for all $n \in \mathbb{Z}$.

- If $X \in \mathcal{D}_{\geq 0}$, $Y \in \mathcal{D}_{\leq -1}$ then $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$.
- $\mathcal{D}_{\geq 1} \subset \mathcal{D}_{\geq 0}$, $\mathcal{D}_{\leq 1} \subset \mathcal{D}_{\leq 0}$
- For all objects X in $\mathcal{D}$ we have a distinguished triangle

  $$X' \to X \to X'' \to \Sigma X'$$

  with $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq -1}$.

Remark B.15. Either of $\mathcal{D}_{\geq 0}$ or $\mathcal{D}_{\leq 0}$ determine the other. Take, for example, $X \in \mathcal{D}$ with the condition $\mathrm{Hom}(X, Y) = 0$ for all $Y \in \mathcal{D}_{\leq -1}$. Take a distinguished triangle

$$X' \to X \to X'' \to \Sigma X'$$

as given by the third property of Definition B.14 with $X' \in \mathcal{D}_{\geq 0}$ and $X'' \in \mathcal{D}_{\leq -1}$. Since by assumption the map $X \to X''$ is the zero morphism, this triangle implies that $X' \cong X \oplus \Sigma^{-1} X''$. But any map $X' \to \Sigma^{-1} X''$ is necessarily trivial, hence $X \cong X'$ is already in $\mathcal{D}_{\geq 0}$. We thus see that $\mathcal{D}_{\geq 0}$ is equal to the full subcategory spanned by those objects X such that $\mathrm{Hom}_{\mathcal{D}}(X, Y) = 0$ for all $Y \in \mathcal{D}_{\leq -1}$. Similarly $\mathcal{D}_{\leq 0}$ can be determined the other way around.

Definition B.16. A *t-structure* on a stable ∞-category $\mathcal{C}$ is a t-structure on $h\mathcal{C}$. Denote $\mathcal{C}_{\geq n}, \mathcal{C}_{\leq n}$ the full subcategory defined by $h\mathcal{C}_{\geq n}$, $h\mathcal{C}_{\leq n}$.

Proposition B.17. The inclusions $\mathcal{C}_{\leq n} \to \mathcal{C}$ have left adjoints, denoted by $\tau_{\leq n}$. The inclusions $\mathcal{C}_{\geq n} \to \mathcal{C}$ have right adjoints, denoted by $\tau_{\geq n}$.

Proof. The following is just a sketch. Let us just look at the case for $\mathcal{C}_{\leq -1} \to \mathcal{C}$, the rest follows similarly. By [Lur12] Proposition 5.2.7.8., we need to show that for all objects X in $\mathcal{C}$ there exists a morphism $f\colon X \to X''$ with $X'' \in \mathcal{C}_{\leq -1}$ with $X'' \in \mathcal{C}_{\leq -1}$ such that for each $Y \in \mathcal{C}_{\leq -1}$ we have that

$$\mathrm{Map}_{\mathcal{C}}(X'', Y) \xrightarrow{f^*} \mathrm{Map}_{\mathcal{C}}(X, Y)$$

is a weak equivalence.

So let

$$X' \to X \xrightarrow{f} X'' \to \Sigma X'$$

be the distinguished triangle given by X, with $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in \mathcal{C}_{\leq -1}$, which gives us the morphism f we are looking for. By the long exact sequence in homotopy groups of mapping spaces, we only need to show that

$$\pi_n \mathrm{Map}(X', Y) = 0$$

for all $Y \in \mathcal{C}_{\leq -1}$ and $n \geq 0$. Since $X' \in \mathcal{C}_{\geq 0}$ we also have $\Sigma^n X' \in C_{\geq 0}$ which gives

$$\pi_n \mathrm{Map}_{\mathcal{C}}(X', Y) \cong \pi_0 \mathrm{Map}_{\mathcal{C}}(\Sigma^n X', Y) \cong \mathrm{Hom}_{h\mathcal{C}}(\Sigma^n X', Y) \simeq 0.$$

□

Corollary B.18. $\mathcal{C}_{\geq n}$ is stable under limits in $\mathcal{C}$. $\mathcal{C}_{\leq n}$ is stable under colimits in $\mathcal{C}$.

Remark B.19. An immediate consequence from the proof before is that

- $\tau_{\leq n}$ is equivalent to the identity on $\mathcal{C}_{\leq m}$ if $m \leq n$.
- $\tau_{\leq n}$ maps into $\mathcal{C}_{\leq n} \subset \mathcal{C}_{\leq m}$ if $m \geq n$.

To see why this holds for $\tau_{\geq n}$ observe that we have the distinguished triangle

$$\tau_{\geq n} X \to X \xrightarrow{f} \tau_{\leq n-1} X \to \Sigma \tau_{\geq n} X,$$

which means that $\tau_{\geq n} X = \mathrm{fib}(f)$ is in $\mathcal{C}_{\leq m}$ if X and $\tau_{\leq n-1} X$ are in $\mathcal{C}_{\leq m}$ since $\mathcal{C}_{\leq m}$ is stable under limits.

Furthermore, this means that there is the commutative diagram

$$\begin{tikzcd}
\mathcal{C}_{\geq n} \arrow[r] \arrow[d, "\tau_{\leq m}"] & \mathcal{C} \arrow[d, "\tau_{\leq m}"] \\
\mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m} \arrow[r] & \mathcal{C}_{\leq m}
\end{tikzcd}$$

which induces a natural transformation $\tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$ by [Lur12] Section 7.3.1.

Proposition B.20. This is a natural equivalence of functors $\mathcal{C} \to \mathcal{C}_{\geq n} \cap \mathcal{C}_{\leq m}$

Proof. We need to show that $\phi \colon \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$ is an isomorphism in $h\mathcal{C}$. If $m < n$, we are done as both sides are 0. Assume $m \geq n$. By Yoneda's lemma ϕ is an isomorphism iff for all Y in $C_{\geq n} \cap \mathcal{C}_{\leq m}$ the induced map $\phi^* \colon \pi_0 \mathrm{map}_{\mathcal{C}}(\tau_{\geq n} \tau_{\leq m} X, Y) \to \pi_0 \mathrm{map}_{\mathcal{C}}(\tau_{\leq m} \tau_{\geq n} X, Y)$ is a bijection. Using the distinguished triangles associated to X and $\tau_{\leq m} X$ we have two long exact sequences and commuting maps f_i in between them as in the diagram

$$\begin{tikzcd}
\pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\leq n-1}\tau_{\leq m}X, Y) \arrow[r, "f_0"] \arrow[d] & \pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\leq n-1}X, Y) \arrow[d] \\
\pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\leq m}X, Y) \arrow[r, "f_1"] \arrow[d] & \pi_0\mathrm{map}_{\mathcal{C}}(X, Y) \arrow[d] \\
\pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\geq n}\tau_{\leq m}X, Y) \arrow[r, "f_2"] \arrow[d] & \pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\geq n}X, Y) \arrow[d] \\
\pi_{-1}\mathrm{map}_{\mathcal{C}}(\tau_{\leq n-1}\tau_{\leq m}X, Y) \arrow[r, "f_3"] \arrow[d] & \pi_{-1}\mathrm{map}_{\mathcal{C}}(\tau_{\leq n-1}X, Y) \arrow[d] \\
\pi_{-1}\mathrm{map}_{\mathcal{C}}(\tau_{\leq m}X, Y) \arrow[r, "f_4"] & \pi_{-1}\mathrm{map}_{\mathcal{C}}(X, Y).
\end{tikzcd}$$

The maps f_0 and f_3 are isomorphisms since for $m \geq n$ the natural transformation $\tau_{\leq n-1} \to \tau_{\leq n-1}\tau_{\leq m}$ is an equivalence.

We have the long exact sequence

$$\pi_1\mathrm{map}_{\mathcal{C}}(\tau_{\geq m+1}X, Y) \to \pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\leq m}X, Y) \xrightarrow{f_1} \pi_0\mathrm{map}_{\mathcal{C}}(X, Y) \to$$
$$\to \pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\geq m+1}X, Y) \to \pi_{-1}\mathrm{map}_{\mathcal{C}}(\tau_{\leq m}X, Y) \xrightarrow{f_4} \pi_{-1}\mathrm{map}_{\mathcal{C}}(X, Y)$$

Since Y is in $\mathcal{C}_{\leq m}$ the blue terms vanish and thus f_1 is an isomorphism and f_4 injective. The 5-lemma implies now that f_2 is an isomorphism. The natural transformation $\tau_{\leq m}\tau_{\geq n} \to \tau_{\geq n}$ gives the isomorphism

$$\pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\geq n}X, Y) \xrightarrow{\cong} \pi_0\mathrm{map}_{\mathcal{C}}(\tau_{\leq m}\tau_{\geq n}X, Y)$$

since, again, Y is in $\mathcal{C}_{\leq m}$. The composition of this isomorphism with the inverse of f_2 gives ϕ^*, which is thus an isomorphism. □

Definition B.21. The *heart* of a stable ∞-category $\mathcal{C}$ with a t-structure is the full subcategory $\mathcal{C}^\heartsuit := C_{\geq 0} \cap \mathcal{C}_{\leq 0}$

The heart of a stable ∞-category is always an abelian category, in the following sense: For any X, Y in $\mathcal{C}^\heartsuit$ the homotopy groups

$$\pi_n\mathrm{Map}_{\mathcal{C}}(X, Y) = 0$$

for $n > 0$. This means that the ∞-category $\mathcal{C}^\heartsuit$ is actually discrete, or said differently $Nh\mathcal{C}^\heartsuit \simeq \mathcal{C}^\heartsuit$. When we say that $C^\heartsuit$ is abelian, we, technically speaking, mean that $hC^\heartsuit$ is abelian (as a 1-category).

Let us recap what this means for $hC^\heartsuit$. We have the four axioms

1. $hC^\heartsuit$ has a zero object,

2. $hC^{\heartsuit}$ has all biproducts,

3. $hC^{\heartsuit}$ has all kernels and cokernels, and

4. in $hC^{\heartsuit}$, every monomorphism is the kernel of some map, as well as dually, every epimorphism is the cokernel of some map.

The main takeaway from the following proof is that for a given morphism $[f]\colon X \to Y$ in $hC^{\heartsuit}$ where f is a representative in $\mathcal{C}$ of the class $[f]$, the kernel and cokernel of $[f]$ are given by

$$\ker[f] = \tau_{\geq 0}\mathrm{fib}(f) \quad \text{and} \quad \mathrm{coker}[f] = \tau_{\leq 0}\mathrm{cof}(f).$$

Proof. Property (1) can be seen from the fact that $0 \in \mathcal{C}$ is contained in $\mathcal{C}^{\heartsuit}$.

For property (2), note that $\mathcal{C}$ has biproducts since it is stable. The subcategory $\mathcal{C}_{\geq 0}$ is closed under coproducts. Since all products $X \times Y$ are biproducts, it is closed under finite products. The subcategory $\mathcal{C}_{\leq 0}$ is closed under limits, hence in particular products. It follows that their intersection, $\mathcal{C}^{\heartsuit}$, is closed under finite products, which all happen to be biproducts.

For (3), we claim that the kernel of a morphism $[f]\colon X \to Y$ in $hC^{\heartsuit}$ is realized as $\tau_{\geq 0}\mathrm{fib}(f)$, where f is a representative in $\mathcal{C}$ of the class $[f]$, together with the natural map

$$\tau_{\geq 0}\mathrm{fib}(f) \to \mathrm{fib}(f) \to X.$$

In order to see that this satisfies the universal property of the kernel, let $[g]\colon Z \to X$ be a morphism with representative g such that $f \circ g \simeq 0$. This means that g has a lift $\tilde{g}$ to $\mathrm{fib}(f)$, unique up to homotopy, giving the homotopy commutative diagram

$$\begin{tikzcd} Z \arrow[d, "\tilde{g}"'] \arrow[dr, "g"] & & \\ \mathrm{fib}(f) \arrow[r] & X \arrow[r, "f"] & Y. \end{tikzcd}$$

Since Z is in $\mathcal{C}_{\geq 0}$, we have the weak equivalence

$$\mathrm{Map}_{\mathcal{C}}(Z, \mathrm{fib}(f)) \simeq \mathrm{Map}_{\mathcal{C}}(Z, \tau_{\geq 0}\mathrm{fib}(f))$$

giving a unique lift of $[\tilde{g}]$ in the homotopy category, giving the commutative diagram in $hC^{\heartsuit}$

$$\begin{tikzcd} & Z \arrow[dl] \arrow[d, "{[\tilde{g}]}"'] \arrow[dr, "{[g]}"] & & \\ \tau_{\geq 0}\mathrm{fib}(f) \arrow[r] & \mathrm{fib}(f) \arrow[r] & X \arrow[r, "{[f]}"] & Y \end{tikzcd}$$

which realizes the universal property of the kernel of $[f]$. The cokernel is realized dually as $\tau_{\leq 0}\mathrm{cof}(f)$.

For (4), assume that $[f]$ is an epimorphism in $h\mathcal{C}^{\heartsuit}$ for some representing map $f\colon X \to Y$ in $\mathcal{C}^{\heartsuit}$. We will try to show that the universal map $\tau_{\geq 0}\mathrm{fib}(f) \to \mathrm{fib}(f)$ is an equivalence, or equivalently,

$\mathrm{fib}(f) \in \mathcal{C}^{\heartsuit}$. The stability of $\mathcal{C}$ implies that $Y \simeq \mathrm{cof}(\mathrm{fib}(f) \to X)$, which together gives that $X \to Y$ realizes the cokernel of $\mathrm{fib}(f) \to X$. The dual statement follows analogously.

Now let us show that $\tau_{\geq 0}\mathrm{fib}(f) \to \mathrm{fib}(f)$ is an equivalence. The statement that $[f]$ is an epimorphism is equivalent to the statement that

$$\pi_0\mathrm{Map}_{\mathcal{C}}(Y,Z) \xrightarrow{-\circ f} \pi_0\mathrm{Map}_{\mathcal{C}}(X,Z)$$

is injective. We have the long exact sequence

$$\pi_1\mathrm{Map}_{\mathcal{C}}(X,Z) \to \pi_1\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z) \to \pi_0\mathrm{Map}_{\mathcal{C}}(Y,Z) \xrightarrow{-\circ f} \pi_0\mathrm{Map}_{\mathcal{C}}(X,Z) \to \pi_0\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z).$$

The left-most group $\pi_1\mathrm{Map}_{\mathcal{C}}(X,Z)$ is 0, since both X and Z are in the heart and $-\circ f$ is injective on π_0, so $\pi_1\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z)$ must be 0. Since all higher homotopy groups in this sequence vanish as well, it follows that $\pi_n\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z) = 0$ for all $n \geq 1$.

The cofiber of $f\colon X \to Y$ is equivalent by stability to $\Sigma\mathrm{fib}(f)$. Since X and Y are in $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\geq 0}$ is closed under colimits, $\Sigma\mathrm{fib}(f)$ is in $\mathcal{C}_{\geq 0}$, hence $\mathrm{fib}(f)$ lies in $\mathcal{C}_{\geq -1}$, and thus altogether

$$\mathrm{fib}(f) \in \mathcal{C}_{\geq -1} \cap \mathcal{C}_{\leq 0}.$$

The kernel $\tau_{\geq 0}\mathrm{fib}(f)$ is in $\mathcal{C}^{\heartsuit} \subset \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq -1}$ by definition. This means that by Yoneda's lemma it is enough to check that

$$\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z) \to \mathrm{Map}_{\mathcal{C}}(\tau_{\geq 0}\mathrm{fib}(f),Z) \tag{1}$$

is an weak equivalence for $Z \in \mathcal{C}_{\geq -1} \cap \mathcal{C}_{\leq 0}$. Any such Z fits into a distinguished triangle

$$Z_0 \to Z \to Z_{-1} \to \Sigma Z_0$$

with $Z_0 \in \mathcal{C}^{\heartsuit}$ and $Z_{-1} \in \Sigma^{-1}\mathcal{C}^{\heartsuit}$. Thus, using the 5-lemma, it further suffices to know that (1) is a weak equivalence for Z in $C^{\heartsuit}$ or in $\Sigma^{-1}\mathcal{C}^{\heartsuit}$.

The case $Z \in C^{\heartsuit}$ is true simply by the adjunction between $\tau_{\geq 0}$ and $\mathrm{inc}_{\geq 0}$, and since Z is, in particular, in $\mathcal{C}_{\geq 0}$.

For the case $Z \in \Sigma^{-1}C^{\heartsuit}$ both mapping spaces involved are contractible. Let us write $Z = \Sigma^{-1}W$ for some W in the heart of $\mathcal{C}$. Then

$$\pi_n\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),Z) \cong \pi_{n+1}\mathrm{Map}_{\mathcal{C}}(\mathrm{fib}(f),W) = 0$$

by the argument given earlier. $\mathrm{Map}_{\mathcal{C}}(\tau_{\geq 0}\mathrm{fib}(f),Z)$ is contractible by the definition of a t-structure. Thus (1) is a weak equivalence trivially. $\square$

Definition B.22. Let $a \leq b$ be two integers. We define the *truncation functors* $[a,b]\colon \mathcal{C} \to \mathcal{C}_{\geq a} \cap \mathcal{C}_{\leq b}$ (written as $X \mapsto X[a,b]$) to be either of the two naturally equivalent functors $\tau_{\geq a}\tau_{\leq b}$ or $\tau_{\leq b}\tau_{\geq a}$. We will write in a similar manner $[a,\infty)$ for $\tau_{\geq a}$, $(-\infty,a]$ for $\tau_{\leq a}$ and $[a]$ for $[a,a]$. The *n-th homotopy group functor* $\pi_n\colon \mathcal{C} \to \mathcal{C}^{\heartsuit}$ is defined as $\Sigma^{-n}[n] \simeq [0]\Sigma^{-n}$.

Given an object X, we always have a natural map

$$l_n \colon X \to X(-\infty, n]$$

realizing the n-th truncation of X. This map comes from the unit map $\mathrm{id}_{\mathcal{C}} \to \mathrm{inc}_{\leq n}\tau_{\leq n}$ of the adjunction $\mathrm{inc}_{\leq n} \dashv \tau_{\leq n}$. It is killing off homotopy groups of degrees higher than n in the sense that

$$\pi_m X \to \pi_m X(-\infty, n]$$

is an equivalence for $m \leq n$ (this follows from the fact that $\tau_{\leq n}$ and $\tau_{\geq m}$ commute up to natural equivalence and as we showed earlier, that $\tau_{\leq n}$ is the identity on $\mathcal{C}_{\leq m}$) and for $m > n$, the homotopy groups $\pi_m X(-\infty, n]$ are 0, as the intersection $\mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq m} = 0$.

Dually, we have a natural map

$$u_n \colon X[n, \infty) \to X$$

realizing the n-th connected cover of X. It is coming from the counit $\mathrm{inc}_{\geq n}\tau_{\geq n} \to \mathrm{id}_{\mathcal{C}}$ of the adjunction $\tau_{\geq n} \dashv \mathrm{inc}_{\geq n}$.

These two maps fit as we have seen before together into the distinguished triangle

$$X[n+1, \infty) \to X \to X(-\infty, n] \to \Sigma X[n+1, \infty)$$

This means $X[n+1, \infty)$ can be recovered as the fiber of the natural map $l_n \colon X \to X(-\infty, n]$ and, similarly, $X(-\infty, n]$ as the cofiber of $u_{n+1} \colon X[n+1, \infty) \to X$. Moreover, we get a natural connecting morphism

$$\delta_n \colon X(-\infty, n] \to \Sigma X[n+1, \infty)$$

simply by stability of $\mathcal{C}$.

Definition B.23. The following is terminology. Suppose we have a t-structure on $\mathcal{C}$. Let X be an object in $\mathcal{C}$. X is called m-connective if it is in $\mathcal{C}_{\geq m}$ and m-connected if it is in $\mathcal{C}_{\geq m+1}$. A map $f \colon E \to F$ is called m-connected if $\mathrm{Fib}(f)$ is m-connected. Dually, Y is called m-coconnective if $Y \in \mathcal{C}_{\leq m}$. Note that $\mathrm{Map}(X, Y) \simeq 0$ for X m-connected and Y m-coconnective and that the condition of the map f to be m-connected is, by the long exact sequence in homotopy groups of the fiber sequence

$$\mathrm{Fib}(f) \to E \xrightarrow{f} F,$$

equivalent to the statement that f induces isomorphisms on π_n for $n \leq m$ and an epimorphism on π_{m+1}.

Example B.24. Given any object X, the Postnikov truncation $X \to X(-\infty, m]$ is an m-connected map.

Remark B.25. If $X \in \mathcal{C}$ has that $\pi_m X = 0$ for all $m < 0$, it does not automatically follow that X is connective, like in the case of the category of spectra. As a trivial counterexample, consider the t-structure given by $\mathcal{C}_{\leq 0} = \mathcal{C}$ and $\mathcal{C}_{\geq 0} = 0$. For this t-structure, every object X satisfies $\pi_m X = 0$, since the heart is trivial, but only the zero object is connective.

B.2 Homological algebra in the setting of t-structures

The following section is concerned with the relationship between computations involving fiber sequences in $\mathcal{C}$ and homological algebra in the abelian category $\mathcal{C}^{\heartsuit}$. The following theorem is the central part of this section: Fiber sequences in $\mathcal{C}$ give rise to long exact sequences in $\mathcal{C}^{\heartsuit}$. We want to remark that the proof is not original and follows ideas from a blog post by Akhil Matthew, which can be found at the time of publication under the link `https://amathew.wordpress.com/2011/06/24/the-long-exact-sequence-in-cohomology-on-a-t-category/`.

Theorem B.26. *Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Let*

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

be a fiber sequence. Then there is an induced long exact sequence

$$\cdots \to \pi_{n+1}Z \to \pi_n X \xrightarrow{\pi_n f} \pi_n Y \xrightarrow{\pi_n g} \pi_n Z \to \pi_{n-1}X \to \cdots$$

where the maps $\pi_n Z \to \pi_{n-1}X$ come from π_n applied to the boundary map $Z \to \Sigma X$ which realizes the cofiber of $X \to Y$.

Proof. It suffices to show that

$$\pi_0 X \xrightarrow{\pi_0 f} \pi_0 Y \xrightarrow{\pi_0 g} \pi_0 Z$$

is exact at $\pi_0 Y$. The general case follows by the suspension equivalence and because $Y \to Z \to \Sigma X$ is again a fiber sequence.

We will proof this statement in steps: First, assume X, Y and Z are in $\mathcal{C}_{\geq 0}$. For any W in $C^{\heartsuit}$ we have the long exact sequence in homotopy groups of the mapping spaces

$$\pi_1\mathrm{Map}(X, W) \to \pi_0\mathrm{Map}(Z, W) \to \pi_0\mathrm{Map}(Y, W) \to \pi_0\mathrm{Map}(X, W).$$

The left hand group $\pi_1\mathrm{Map}(X, W)$ is isomorphic to $\pi_0\mathrm{Map}(X, \Sigma^{-1}W)$ and therefore 0. For X in $\mathcal{C}_{\geq 0}$ we have $\pi_0 X = \tau_{\leq 0}X$. The adjunction of $\tau_{\leq 0}$ with the inclusion of the subcategory $\mathcal{C}_{\leq 0}$ gives the weak equivalence of mapping spaces

$$\mathrm{Map}(X, W) \simeq \mathrm{Map}(\tau_{\leq 0}X, W) = \mathrm{Map}(\pi_0 X, W)$$

and, similarly, for Y and Z. Plugging those natural isomorphisms in the sequence above gives that the sequence

$$0 \to \pi_0\mathrm{Map}(\pi_0 Z, W) \to \pi_0\mathrm{Map}(\pi_0 Y, W) \to \pi_0\mathrm{Map}(\pi_0 X, W)$$

is exact for each W in $C^{\heartsuit}$, hence we can deduce that

$$\pi_0 X \to \pi_0 Y \to \pi_0 Z \to 0$$

was exact in $C^{\heartsuit}$.

We generalize further: It suffices that only X is in $\mathcal{C}_{\geq 0}$. As a first step, $\tau_{\leq -1}$ is a left adjoint, hence sends the cofiber sequence

$$X \to Y \to Z$$

to the cofiber sequence

$$\tau_{\leq -1}X \to \tau_{\leq -1}Y \to \tau_{\leq -1}Z$$

in $\mathcal{C}_{\leq -1}$. Since $\tau_{\leq -1}X \simeq 0$ the map $\tau_{\leq -1}g\colon \tau_{\leq -1}Y \to \tau_{\leq -1}Z$ is an equivalence.

Now take a look at the following diagram

$$\begin{tikzcd}
X \arrow[r, "\tau_{\geq 0}f"] \arrow[d, "="] & \tau_{\geq 0}Y \arrow[r] \arrow[d] & \mathrm{cof}(\tau_{\geq 0}f) \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & \tau_{\leq -1}Y \arrow[r, "="] & \tau_{\leq -1}Y
\end{tikzcd}$$

All the rows are cofiber sequences as well as the first and second column. From this follows that the last column is a cofiber sequence. Since $\tau_{\leq -1}Y \simeq \tau_{\leq -1}Z$ this implies that

$$\mathrm{cof}(\tau_{\geq 0}f) \simeq \tau_{\geq 0}Z.$$

Hence the sequence

$$X \to \tau_{\geq 0}Y \to \tau_{\geq 0}Z$$

is a cofiber sequence in $\mathcal{C}$, and we can apply the previous result.

Summarizing, we have proven the following: If

$$X \to Y \to Z$$

is a fiber sequence with X in $\mathcal{C}_{\geq 0}$, the sequence

$$\pi_0 X \to \pi_0 Y \to \pi_0 Z \to 0$$

is exact.

As a next step, we dualize to see if instead Z is in $\mathcal{C}_{\leq 0}$, the sequence

$$0 \to \pi_0 X \to \pi_0 Y \to \pi_0 Z$$

is exact.

As the last step, assume the general case. Write Q for the cofiber of the map $\tau_{\geq 0}X \to X \to Y$ to get the fiber sequence

$$\tau_{\geq 0}X \to Y \to Q$$

which falls into the previous case to give the exact sequence

$$\pi_0 X \to \pi_0 Y \to \pi_0 Q \to 0.$$

We have the commutative diagram

$$\begin{tikzcd}
\tau_{\geq 0}X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d, "="] & Q \arrow[d] \\
X \arrow[r, "f"] \arrow[d] & Y \arrow[r, "g"] \arrow[d] & Z \arrow[d] \\
\tau_{\leq -1}X \arrow[r] & 0 \arrow[r] & \Sigma\tau_{\leq -1}X
\end{tikzcd}$$

with all rows cofiber sequences, as well as the first and second column being cofiber sequences. Hence

$$Q \to Z \to \Sigma\tau_{\leq -1}X$$

is a cofiber sequence. The object $\Sigma\tau_{\leq -1}X$ lies in $\mathcal{C}_{\leq 0}$. Therefore we are in the previous case again and get the exact sequence

$$0 \to \pi_0 Q \to \pi_0 Z \to \pi_{-1}X.$$

The composition $Y \to Q \to Z$ is equal to g by construction, so these two exact sequences can be assembled together into the exact sequence

$$\pi_0 X \xrightarrow{\pi_0 f} \pi_0 Y \xrightarrow{\pi_0 g} \pi_0 Z \to \pi_{-1}X,$$

which finishes the proof. □

Next, we are concerned with degree 1 maps between objects in the heart $\mathcal{C}$.

Lemma B.27. Let $\mathcal{C}$ be a stable ∞-category with a t-structure and let A, C be two objects in the heart $\mathcal{C}^{\heartsuit}$. There is a natural isomorphism

$$\phi\colon [C, \Sigma A] \cong \mathrm{Ext}^1_{\mathcal{C}^{\heartsuit}}(C, A)$$

where

$$\phi(\beta\colon C \to \Sigma A) = (0 \to A \to \mathrm{fib}(\beta) \to C \to 0).$$

Proof. To show that ϕ is well-defined, we still have to show that $\mathrm{fib}(\beta)$ lies in the heart of $\mathcal{C}$. To do so, note that by Theorem B.26 we have the long exact sequence in homotopy groups

$$\cdots \to \pi_1(C) \to \pi_1(\Sigma A) \to \pi_0(\mathrm{fib}(\beta)) \to \pi_0(C) \to \pi_0(\Sigma A) \to \cdots.$$

Since A and C are in the heart, we have $\pi_1(C) = 0$, $\pi_0(\Sigma A) = 0$, which shows that $\mathrm{fib}(\beta)$ lies in the heart. Furthermore, $\pi_0(C) = C$ and $\pi_1(\Sigma A) = A$, which means we do, in fact, get the claimed exact sequence.

The inverse map $\psi\colon \mathrm{Ext}^1_{\mathcal{C}^{\heartsuit}}(C, A) \to [C, \Sigma A]$ is constructed as follows. A given exact sequence

$$0 \to A \to B \to C \to 0$$

in the heart produces a fiber sequence

$$A \to B \to C$$

in $\mathcal{C}$ which can be mapped to the boundary map $\delta\colon C \to \Sigma A$. It is clear that that the two processes are mutually inverse. □

The following lemma clarifies an argument used multiple times during the main part of the thesis.

Lemma B.28. Let $\mathcal{C}$ be a stable ∞-category with a t-structure and let A be an object of $\mathcal{C}$ concentrated in degrees -1 and 0 and B an object in the heart $\mathcal{C}^{\heartsuit}$. Let f be map $A \to B$ such that the composition $A[0] \to A \xrightarrow{f} B$ is zero. Then f factorizes through an up to homotopy unique map $\tilde{f}\colon A[-1] \to B$, i.e. we have a commutative triangle

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & B \\ A[-1] \arrow[ur, "\tilde{f}"] & \end{tikzcd}$$

Proof. We have the fiber sequence

$$A[0] \to A \to A[-1],$$

which implies the long exact sequence

$$[\Sigma A[0], B] \to [A[-1], B] \to [A, B] \to [A[0], B].$$

The abelian group $[\Sigma A[0], B]$ is zero since $\Sigma A[0]$ is 1-connected and B was assumed to be concentrated in degree 0, so if $f\colon A \to B$ is a map that becomes the zero map when precomposed with $A[0] \to A$, it factors through a map $\tilde{f}\colon A[-1] \to B$ which is unique up to homotopy. ◻

The last part of this section is concerned with a technical lemma about the relationship between the induced map on fibers coming from a square involving degree 1 maps and the well known connecting map from the snake lemma in homological algebra.

Lemma B.29. Let $\mathcal{C}$ be a stable ∞-category with a t-structure and let A_1, A_2, C_1 and C_2 be objects in the heart $\mathcal{C}^{\heartsuit}$ and suppose we have a commutative square

$$\begin{tikzcd} C_1 \arrow[r, "\beta_1"] \arrow[d, "f_C"] & \Sigma A_1 \arrow[d, "\Sigma f_A"] \\ C_2 \arrow[r, "\beta_2"] & \Sigma A_2 \end{tikzcd}$$

Write $\mathrm{fib}_A \coloneqq \mathrm{fib}(f_A)$, $\mathrm{cok}_A \coloneqq \pi_{-1}\mathrm{fib}_A$, $\mathrm{fib}_C \coloneqq \mathrm{fib}(f_C)$, and $\ker_C \coloneqq \pi_0\mathrm{fib}_C$. Then the induced map $\pi_0\mathrm{fib}_C \to \pi_0\Sigma\mathrm{fib}_A$ agrees with the map $\delta\colon \ker_C \to \mathrm{cok}_A$ induced by the snake lemma for the corresponding map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r] \arrow[d] & B_1 \arrow[r] \arrow[d] & C_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & 0 \end{tikzcd}$$

in the heart $\mathcal{C}^{\heartsuit}$.

Before we begin the proof of Lemma B.29, we want to establish some facts about the map induced by the well known snake lemma (see e.g. [Wei95], Lemma 1.3.2).

Lemma B.30. Suppose $\mathcal{A}$ is an abelian category and

$$\begin{tikzcd} 0 \arrow[r] & A_1 \arrow[r, "g_1"] \arrow[d, "f_A"] & B_1 \arrow[r, "\pi_1"] \arrow[d, "f_B"] & C_1 \arrow[r] \arrow[d, "f_C"] & 0 \\ 0 \arrow[r] & A_2 \arrow[r, "g_2"] & B_2 \arrow[r, "\pi_2"] & C_2 \arrow[r] & 0 \end{tikzcd}$$

is a diagram in $\mathcal{A}$ with exact rows. Write $\ker_C \coloneqq \ker(f_C)$ and $\operatorname{cok}_A \coloneqq \operatorname{cok}(f_A)$. Then:

1. There is an isomorphism

$$\theta\colon \frac{\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2)}{\operatorname{im}(A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2)} \cong \ker_C$$

   induced by the composition $B_1 \oplus A_2 \to B_1 \xrightarrow{\pi_1} C_1$, where the first map is the projection onto the first summand.

2. There is a natural map

$$\phi\colon \frac{\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2)}{\operatorname{im}(A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2)} \to \operatorname{cok}_A$$

   induced by the composition $B_1 \oplus A_2 \to A_2 \to \operatorname{cok}_A$, where the first map is projection onto the second summand.

3. The composition $\phi\theta^{-1}\colon \ker_C \to \operatorname{cok}_A$ is the natural connecting map from the snake lemma.

4. If π_1 and π_2 have sections $s_1\colon C_1 \to B_1$, $s_2\colon C_2 \to B_2$ such that

$$\begin{tikzcd} B_1 \arrow[d, "f_B"] & C_1 \arrow[l, "s_1"'] \arrow[d, "f_C"] \\ B_2 & C_2 \arrow[l, "s_2"'] \end{tikzcd}$$

   commutes then $\phi\theta^{-1} = 0$.

Proof. To show that θ is well-defined, we need to show:

- The composition

$$A_1 \xrightarrow{(g_1,f_A)} \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{\pi_1} C_1$$

  is the trivial map. This is simply because $\pi_1 g_1 = 0$. From this follows that the map

$$\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{\pi_1} C_1$$

  factors through $\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2)/\operatorname{im}(A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2)$.

- The composition
$$\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{\pi_1} C_1 \xrightarrow{f_C} C_2$$
is the trivial map. We have the following equalities of maps
$$\frac{\dfrac{\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{\pi_1} C_1 \xrightarrow{f_C} C_2}{\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{f_B} B_2 \xrightarrow{\pi_2} C_2}}{\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to A_2 \xrightarrow{g_2} B_2 \xrightarrow{\pi_2} C_2}$$
The claim now follows since $\pi_2 g_2 = 0$. From this follows that the map
$$\ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) \to B_1 \xrightarrow{\pi_1} C_1$$
maps into $\ker_C$.

To see that θ is an isomorphism, we show two things:

- $\ker(\theta) = 0$. This is because
$$\ker(B_1 \oplus A_2 \to B_1 \to C_1) = A_1 \oplus A_2$$
which implies
$$\ker(B_1 \oplus A_2 \to B_1 \to C_1) \cap \ker(B_1 \oplus A_2 \xrightarrow{(g_2,-f_B)} B_2) = \operatorname{im}(A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2)$$
Hence, $\ker(\theta) = 0$.
- $\operatorname{cok}(\theta) = 0$. A simple diagram chase using exactness at B_2 shows that
$$\operatorname{im}(\ker(B_1 \oplus A_2 \to B_2) \to B_1) = \ker(\pi_2 f_B)$$
Hence, since $B_1 \to C_1$ is an epimorphism,
$$\operatorname{im}(\ker(B_1 \oplus A_2 \to B_2) \to B_1 \to C_1) = \ker_C$$
From this follows that θ is an epimorphism.

We now need to show that ϕ is well-defined. To do so, we need to show that the composition
$$A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2 \to A_2 \to \operatorname{cok}_A$$
is trivial. This is clear, however, as the composition $A_1 \xrightarrow{(g_1,f_A)} B_1 \oplus A_2 \to A_2$ is just equal to f_A.

The next claim is that the map $\phi\theta^{-1}$ agrees with the map induced by the snake lemma. For simplicity, assume that $\mathcal{A}$ is the category of abelian groups.[11] The traditional way of defining the

[11] An element-free proof can be done, of course. Our proof is sufficient by the Freyd-Mitchell embedding theorem.

boundary map δ goes as follows. Assume $c \in \ker_C$. Using surjectivity of $B_1 \to C_1$, find a preimage $b_1 \in B$ of c. Since $\pi_2 f_B = f_C \pi_1$, the element $f(b_1)$ lies in the kernel of π_2; hence, there is a unique $a_2 \in A_2$ such that $g_2(a_2) = f(b_1)$. The image of δ of the element c is defined as the class of a_2 in the cokernel cok_A. The reason this agrees with $\phi\theta^{-1}$ is as follows. The class $[b_1, a_2]$ is just a preimage of c under the map θ and the assignment $[b_1, a_2] \to [a_2]$ is exactly what defines the map ϕ.

Lastly, assume π_1 and π_2 have commuting sections s_1 and s_2 respectively. Then the map $C_1 \xrightarrow{(s_1,0)} B_1 \oplus A_2$ descends to the inverse of θ. It is then clear that $\phi\theta^{-1} = 0$ since ϕ is induced by projection on the A_2 coordinate. □

Remark B.31. The last condition that π_1 and π_2 have commuting sections is in particular satisfied if π_1 and f_C are split surjective. To see this, let s_1 be a section of π_1 and s_C be a section of f_C. Then $s_2 := f_B \circ s_1 \circ s_C$ is a commuting choice of section for π_2.

More generally, the map $\phi\theta^{-1} = 0$ if π_1 is split surjective and f_C a split surjection onto its image. To see this, note that if we have a commutative diagram with fiber sequences as columns

$$\begin{tikzcd}
\mathrm{fib}_C \arrow[r] \arrow[d] & \Sigma \mathrm{fib}_A \arrow[d] \\
C_1 \arrow[r] \arrow[d] & \Sigma A_1 \arrow[d] \\
C_2 \arrow[r] & \Sigma A_2
\end{tikzcd}$$

and by Lemma B.29 the map $\phi\theta^{-1}$ is the map $\pi_0 \mathrm{fib}_C \to \pi_0 \Sigma \mathrm{fib}_A$, then this map does not change on π_0 by going to connective covers $\ker_C \to \Sigma \mathrm{fib}_A$. This fits into the diagram

$$\begin{tikzcd}
\ker_C \arrow[r] \arrow[d] & \Sigma \mathrm{fib}_A \arrow[d] \\
C_1 \arrow[r] \arrow[d] & \Sigma A_1 \arrow[d] \\
\mathrm{im}_C \arrow[r] & \Sigma A_2
\end{tikzcd}$$

where now $C_1 \to \mathrm{im}_C$ is a split surjection and we are back in the previous case.

We still need to introduce some new terminology. Suppose D is a commutative square

$$\begin{tikzcd}
X_1 \arrow[r, "g_1"] \arrow[d, "f_X"] & Y_1 \arrow[d, "f_Y"] \\
X_2 \arrow[r, "g_2"] & Y_2
\end{tikzcd}$$

in a stable ∞-category $\mathcal{C}$. Define the *iterated cofiber* of D as

$$\mathrm{cof}(D) := \mathrm{cof}(\mathrm{cof}(g_1) \to \mathrm{cof}(g_2)) \simeq \mathrm{cof}(\mathrm{cof}(f_X) \to \mathrm{cof}(f_Y))$$

Define $\square := \Delta^1 \times \Delta^1$. It is clear that taking iterated cofibers is functorial in the sense that it defines an exact functor

$$\mathrm{cof}\colon \mathrm{Fun}(\square, \mathcal{C}) \to \mathcal{C},$$

which is the left adjoint to the functor that sends an object $X \in \mathcal{C}$ to the square

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & X. \end{tikzcd}$$

The reason,, we are interested in this construction is that if we take the objects X_i and Y_i to be in the heart of a t-structure on $\mathcal{C}$, this allows us to model chain complexes of length ≤ 3 in $\mathcal{C}$.[12] The following lemma will make this precise.

Lemma B.32. Suppose D is a commutative square

$$\begin{tikzcd} A_1 \arrow[r, "g_1"] \arrow[d, "f_A"] & B_1 \arrow[d, "f_B"] \\ A_2 \arrow[r, "g_2"] & B_2 \end{tikzcd}$$

with values in $\mathcal{C}^{\heartsuit}$. Then:

1. The iterated cofiber cof(D) is concentrated in degrees 0,1 and 2. Moreover, we have

$$\pi_2\mathrm{cof}(D) = \ker(A_1 \xrightarrow{(g_1, f_A)} B_1 \oplus A_2)$$
$$\pi_1\mathrm{cof}(D) = \frac{\ker(B_1 \oplus A_2 \xrightarrow{(g_2, -f_B)} B_2)}{\mathrm{im}(A_1 \xrightarrow{(g_1, f_A)} B_1 \oplus A_2)}$$
$$\pi_0\mathrm{cof}(D) = \mathrm{cok}(B_1 \oplus A_2 \xrightarrow{(g_2, -f_B)} B_2)$$

2. If the square D has the form

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_1 \arrow[d, "f_B"] \\ 0 \arrow[r] & B_2, \end{tikzcd}$$

   then $\mathrm{cof}(D) = \mathrm{cof}(f_B)$.

3. If the square D has the form

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "f_A"] & 0 \arrow[d] \\ A_2 \arrow[r] & 0, \end{tikzcd}$$

   then $\mathrm{cof}(D) = \Sigma\mathrm{cof}(f_A)$.

[12] This construction generalizes to arbitrary length by defining iterated cofibers of n-cubes in a similar manner.

4. If the square D has the form
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & 0, \end{tikzcd}$$
then $\mathrm{cof}(D) = \Sigma(A_2 \oplus B_1)$.

Proof. Point (2) and (3) are trivial. For point (4), note that the space of morphisms $\mathrm{Map}_{\mathcal{C}}(0,0)$ is contractible, hence the square
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & 0 \end{tikzcd}$$
is trivially commutative and taking vertical cofibers realizes to the zero map $A_2 \to \Sigma B_1$ which implies that $\mathrm{cof}(D) = \mathrm{cof}(A_2 \xrightarrow{0} \Sigma B_1) = \Sigma(A_2 \oplus B_1)$.

Now assume D is of the shape
$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & B_1 \arrow[d] \\ A_2 \arrow[r] & B_2. \end{tikzcd}$$
Then we have the following fiber sequence of square diagrams,
$$\begin{tikzcd} 0 \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[rr] \arrow[dr] \arrow[dd] & & 0 \arrow[dr] \arrow[dd] & \\ & B_1 \arrow[rr] \arrow[dd, "f_B"] & & B_1 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\ A_2 \arrow[rr] \arrow[dr, "g_2"'] & & A_2 \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\ & B_2 \arrow[rr] & & 0 \arrow[rr] & & \Sigma B_2. \end{tikzcd}$$
Taking vertical cofibers of the right hand cube results in the square
$$\begin{tikzcd} A_2 \arrow[r] \arrow[d, "0"] & 0 \arrow[d] \\ \Sigma B_1 \arrow[r, "-f_B"] & \Sigma B_2 \end{tikzcd}$$
where the resulting square
$$\begin{tikzcd} A_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \Sigma B_2 \end{tikzcd}$$

classifies the map $g_2 \colon \Sigma A_2 \to \Sigma B_2$. This means taking further cofibers results in the map

$$\Sigma(A_2 \oplus B_1) \xrightarrow{(g_2,-f_B)} \Sigma B_2.$$

This means we have a fiber sequence

$$\mathrm{cof}(D) \to \Sigma(A_2 \oplus B_1) \xrightarrow{(g_2,-f_B)} \Sigma B_2$$

from which we can read off that $\mathrm{cof}(D)$ is concentrated in degree 0 and 1 with the homotopy groups

$$\pi_1\mathrm{cof}(D) = \ker(A_2 \oplus B_1 \xrightarrow{(g_2,-f_B)} B_2)$$
$$\pi_0\mathrm{cof}(D) = \mathrm{cok}(A_2 \oplus B_1 \xrightarrow{(g_2,-f_B)} B_2).$$

Now assume D is a general commutative square of the form

$$\begin{tikzcd}
A_1 \arrow[r, "g_1"] \arrow[d, "f_A"] & B_1 \arrow[d, "f_B"] \\
A_2 \arrow[r, "g_2"] & B_2.
\end{tikzcd}$$

We have the following fiber sequence of square diagrams,

$$\begin{tikzcd}
\Omega A_1 \arrow[rr] \arrow[dd] \arrow[dr] & & 0 \arrow[rr] \arrow[dd] \arrow[dr] & & A_1 \arrow[dd, "f_A" near end] \arrow[dr, "g_1"] & \\
& 0 \arrow[rr] \arrow[dd] & & B_1 \arrow[rr] \arrow[dd] & & B_1 \arrow[dd] \\
0 \arrow[rr] \arrow[dr] & & A_2 \arrow[rr] \arrow[dr] & & A_2 \arrow[dr] & \\
& 0 \arrow[rr] & & B_2 \arrow[rr] & & B_2,
\end{tikzcd}$$

which produces the following two exact sequences

$$0 \to \pi_2\mathrm{cof}(D) \to A_1 \to \ker(A_2 \oplus B_1 \xrightarrow{(g_2,-f_B)} B_2) \to \pi_1\mathrm{cof}(D) \to 0$$

as well as

$$0 \to \mathrm{cok}(A_2 \oplus B_1 \xrightarrow{(g_2,-f_B)} B_2) \to \pi_0\mathrm{cof}(D) \to 0,$$

which proves point (1). □

We are now ready to prove Lemma B.29. Assume now that we have a diagram with short exact rows,

$$\begin{tikzcd}
0 \arrow[r] & A_1 \arrow[r, "g_1"] \arrow[d, "f_A"] & B_1 \arrow[r] \arrow[d, "f_B"] & C_1 \arrow[r] \arrow[d, "f_C"] & 0 \\
0 \arrow[r] & A_2 \arrow[r, "g_2"] & B_2 \arrow[r] & C_2 \arrow[r] & 0,
\end{tikzcd}$$

and write $\mathrm{cof}_A := \mathrm{cof}(f_A)$, $\mathrm{cof}_B := \mathrm{cof}(f_B)$,$\mathrm{cof}_C := \mathrm{cof}(f_C)$, $\ker_C := \pi_0\mathrm{fib}_C$ and $\mathrm{cok}_A := \pi_{-1}\mathrm{fib}_A$.

Proof of Lemma B.29. We now come back to the claim that the map

$$\mathrm{fib}_C \to \Sigma \mathrm{fib}_A$$

induces the map described by the snake lemma in π_0. Note that $\mathrm{cof}_A = \Sigma\mathrm{fib}_A$, and similarly for C, so to proof that $\pi_0\mathrm{fib}_C \to \pi_0\Sigma\mathrm{fib}_A$ is the map induced by the snake lemma, it suffices to show the same thing for π_1 on the cofibers.

Take the square D

$$\begin{tikzcd} A_1 \arrow[r, "g_1"] \arrow[d, "f_A"] & B_1 \arrow[d, "f_B"] \\ A_2 \arrow[r, "g_2"] & B_2. \end{tikzcd}$$

There is a commuting cube

$$\begin{tikzcd} A_1 \arrow[rr] \arrow[dd] \arrow[dr] & & 0 \arrow[dd] \arrow[dr] & \\ & B_1 \arrow[rr] \arrow[dd] & & C_1 \arrow[dd] \\ A_2 \arrow[rr] \arrow[dr] & & 0 \arrow[dr] & \\ & B_2 \arrow[rr] & & C_2. \end{tikzcd}$$

Taking iterated cofibers yields a map $\mathrm{cof}(D) \to \mathrm{cof}_C$. It is clear that the map induced on π_1 of this is the map θ described in lemma B.30. Moreover, the map in π_0 is an isomorphism as well and $\pi_2(D) = 0$ since $A_1 \to B_1$ is injective, hence $\mathrm{cof}(D) \to \mathrm{cof}_C$ is an equivalence.

Now take the fiber sequence of commutative squares

$$\begin{tikzcd} 0 \arrow[rr] \arrow[dd] \arrow[dr] & & A_1 \arrow[rr] \arrow[dd] \arrow[dr] & & A_1 \arrow[dd] \arrow[dr] & \\ & B_1 \arrow[rr] \arrow[dd] & & B_1 \arrow[rr] \arrow[dd] & & 0 \arrow[dd] \\ 0 \arrow[rr] \arrow[dr] & & A_2 \arrow[rr] \arrow[dr] & & A_2 \arrow[dr] & \\ & B_2 \arrow[rr] & & B_2 \arrow[rr] & & 0. \end{tikzcd}$$

Taking iterated cofibers gives the fiber sequence

$$\mathrm{cof}_B \to \mathrm{cof}(D) \to \Sigma\mathrm{cof}_A$$

Here it is clear that the map $\mathrm{cof}(D) \to \Sigma\mathrm{cof}_A$ on π_1 becomes the map ϕ in lemma B.30. Taking all things together we see that the map

$$\mathrm{cof}_C \to \Sigma\mathrm{cof}_A \simeq \mathrm{cof}_C \to \mathrm{cof}(D) \to \Sigma\mathrm{cof}_A$$

gives the map $\phi\theta^{-1}$ on π_1, which by B.30 is the map induced by the snake lemma. $\square$

B.3 Functor categories and t-structures

Given a stable ∞-category $\mathcal{C}$ and a small ∞-category $\mathcal{D}$, we know that $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is again a stable ∞-category. If we have a t-structure on $\mathcal{C}$, we can put a natural t-structure on $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$:

Definition B.33. Suppose $\mathcal{C}$ is a stable ∞-category with t-structure and $\mathcal{D}$ a small ∞-category. The object-wise t-structure on $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ is defined via

$$\begin{aligned} \mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0} &:= \quad \mathrm{Fun}(\mathcal{D}, \mathcal{C}_{\leq 0}) \\ \mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\geq 0} &:= \quad \mathrm{Fun}(\mathcal{D}, \mathcal{C}_{\geq 0}). \end{aligned}$$

We view the category of functors $\mathcal{D} \to \mathcal{C}_{\leq 0}$ as the full subcategory of functors $\mathcal{D} \to \mathcal{C}$ with values in the subcategory $\mathcal{C}_{\leq 0}$ and similarly for ≥ 0.

Proof. We have to check three things:

- $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0}$ is closed under Ω. This is true since limits are computed object-wise and $\mathcal{C}_{\leq 0}$ is closed under limits. Similarly, $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\geq 0}$ is closed under Σ.
- Given X in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\geq 1}$ and Y in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0}$ the abelian group $\pi_0 \mathrm{Nat}(X, Y)$ is zero. The space $\mathrm{Nat}_{\mathcal{C}}(X, Y)$ is given as the end

  $$\mathrm{Nat}_{\mathcal{C}}(X, Y) \simeq \int_{d \in \mathcal{D}} \mathrm{Map}_{\mathcal{C}}(X(d), Y(d))$$

  by Lemma B.37. The spaces $\mathrm{Map}_{\mathcal{C}}(X(d), Y(d'))$ for $d, d' \in \mathcal{D}$ are all contractible by assumption, so the end in question is, as a limit of a functor into contractible objects, contractible as well[13], therefore $\mathrm{Nat}_{\mathcal{C}}(X, Y) \simeq \mathrm{pt}$ and thus $[X, Y] = \pi_0 \mathrm{Nat}(X, Y) = 0$.
- For any X in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$ there is a fiber sequence

  $$X_1 \to X \to X_0$$

  with X_1 in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\geq 1}$ and X_0 in $\mathrm{Fun}(\mathcal{D}, \mathcal{C})_{\leq 0}$. To see this, note that we have a fiber sequence $\tau_{\geq 1} \to \mathrm{id}_{\mathcal{C}} \to \tau_{\leq 0}$ of functors $\mathcal{C} \to \mathcal{C}$. Precomposing with X gives the fiber sequence

  $$\tau_{\geq 1} X \to X \to \tau_{\leq 0}, X$$

  which is our desired fiber sequence.

$\square$

Remark B.34. The heart of this t-structure is given as

$$\mathrm{Fun}(\mathcal{D}, \mathcal{C})^{\heartsuit} \simeq \mathrm{Fun}(\mathcal{D}, \mathcal{C}^{\heartsuit}) \simeq \mathrm{Fun}(h\mathcal{D}, \mathcal{C}^{\heartsuit}),$$

where right equivalence follows from $\mathcal{C}^{\heartsuit}$ being a 1-category.

[13] The statement that a limit of a functor $F \colon \mathcal{C}' \to \mathcal{C}$ into contractible objects is contractible follows from the observation that F is a terminal object in the ∞-category $\mathrm{Fun}(\mathcal{C}', \mathcal{C})$ and the functor $\lim \colon \mathrm{Fun}(\mathcal{C}', \mathcal{C}) \to \mathcal{C}$ preserves all limits and in particular terminal objects.

B.4 Ends and Coends

The earliest account of ends and coends in the setting of ∞-categories goes back to [Cra10]. In the following section we always assume that $\mathcal{C}$ is a complete and cocomplete ∞-category and that $\mathcal{D}$ is an (essentially) small ∞-category. For an ∞-category $\mathcal{D}$ there always exists the twisted arrow category $\mathrm{tw}(\mathcal{D})$, together with a natural functor $\mathrm{tw}(\mathcal{D}) \to \mathcal{D}^{op} \times \mathcal{D}$, see [Bar14] Section 2 or [Gla15] Definition 2.1. Note that if $\mathcal{D} = N(D)$ for D a 1-category, then $\mathrm{tw}(\mathcal{D}) \simeq N(\mathrm{tw}(D))$, where $\mathrm{tw}(D)$ is the 1-categorical twisted arrow category.

Definition B.35. Let F be a functor $\mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}$. Then the end of F, written as

$$\int_{d\in\mathcal{D}} F(d,d)$$

is defined as the limit over the composition

$$\mathrm{tw}(\mathcal{D}) \to \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

Dually, the coend

$$\int^{d\in\mathcal{D}} F(d,d)$$

is defined as the colimit of the composition

$$\mathrm{tw}(\mathcal{D}^{op})^{op} \to \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{F} \mathcal{C}.$$

Example B.36. If $F\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{C}$ is constant in the left variable, i.e. F factors through the projection as $F\colon \mathcal{D}^{op} \times \mathcal{D} \to \mathcal{D} \xrightarrow{F'} \mathcal{C}$, then

$$\int_{d\in D} F(d,d) \simeq \lim F'.$$

Dually,

$$\int^{d\in D} F(d,d) \simeq \mathrm{colim} F'.$$

Similarly in the right variable.

Proposition B.37 ([GHN20], Prop. 5.1.)**.** Let $A, B\colon \mathcal{D} \to \mathcal{C}$ be two functors. Then there is a natural equivalence

$$\mathrm{Nat}_{\mathcal{C}}(A,B) \simeq \int^{d\in\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(Ad, Bd),$$

where $\mathrm{Nat}_{\mathcal{C}}(A,B)$ is the space of natural transformations from A to B.

Theorem B.38 (Fubini's Theorem, [AL18] Proposition 3.5.)**.** *Let $F\colon \mathcal{D}^{op} \times \mathcal{D} \times \mathcal{E}^{op} \times \mathcal{E} \to \mathcal{C}$ be a functor. Then*

$$\int^{d\in\mathcal{D}}\int^{e\in\mathcal{E}} F(d,d,e,e) \simeq \int^{e\in\mathcal{E}}\int^{d\in\mathcal{D}} F(d,d,e,e).$$

The assumption that $\mathcal{C}$ is cocomplete implies that $\mathcal{C}$ is tensored over spaces, i.e. for any space X and object $c \in C$ there exists an object $X \otimes c$, which is defined as $\mathrm{colim}_X \underline{c}$, where $\underline{c}$ is the constant functor $X \to \mathcal{C}$ with value c. Dually, the assumption that $\mathcal{C}$ is complete implies that $\mathcal{C}$ is powered over spaces, i.e. there exists c^X defined as $\lim_X \underline{c}$. Note that for fixed X, the functor $c \mapsto X \otimes c$ is left adjoint to the functor $c \mapsto c^X$.

Lemma B.39 (Yoneda Lemma, [AL18] Proposition 3.5.)**.** Let G be a functor $\mathcal{D} \to \mathcal{C}$. There exists a natural equivalence
$$G(d) \simeq \int^{d' \in \mathcal{D}} \mathrm{Map}_{\mathcal{D}}(d', d) \otimes G(d').$$
Dually, let G be a functor $\mathcal{D}^{op} \to \mathcal{C}$. There exists a natural equivalence
$$G(d) \simeq \int_{d' \in \mathcal{D}} G(d')^{\mathrm{Map}_{\mathcal{D}}(d', d)}.$$

If $G\colon \mathcal{D}^{op} \to \mathrm{Spc}$ the Yoneda lemma has the more familiar form
$$\mathrm{Nat}_{\mathrm{Spc}}(\mathrm{Map}_{\mathcal{D}}(-, d), G) \simeq G(d).$$

Example B.40. Let $d \in \mathcal{D}$. The colimit over a corepresentable functor $\mathrm{Map}_{\mathcal{D}}(d, -)\colon \mathcal{D} \to \mathrm{Spc}$ is contractible. This follows directly from the Yoneda Lemma by
$$\begin{aligned} \mathrm{colim}_{\mathcal{D}} \mathrm{Map}_{\mathcal{D}}(d, -) &\simeq \int^{d' \in \mathcal{D}^{op}} \mathrm{Map}_{\mathcal{D}}(d, d') \otimes \underline{\mathrm{pt}} \\ &\simeq \int^{d' \in \mathcal{D}^{op}} \mathrm{Map}_{\mathcal{D}^{op}}(d', d) \otimes \underline{\mathrm{pt}} \simeq \mathrm{pt} \end{aligned}$$
where $\underline{\mathrm{pt}}$ is the constant functor $\mathcal{D}^{op} \to \mathrm{Spc}$ with value pt.

Let $p\colon \mathcal{D} \to \mathcal{D}'$ be a functor. The *left Kan extension functor* $\mathrm{Lan}_p\colon \mathrm{Fun}(\mathcal{D}, \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}', \mathcal{C})$ is defined as the left adjoint to the precomposition functor $\circ p\colon \mathrm{Fun}(\mathcal{D}', \mathcal{C}) \to \mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Lemma B.41. Let $F\colon \mathcal{D} \to \mathcal{C}$ and $p\colon \mathcal{D} \to \mathcal{D}'$ be functors. The left Kan extension of F along p is given as
$$\mathrm{Lan}_p F(d') \simeq \int^{d \in \mathcal{D}} \mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes F(d).$$

Proof. Let G be a functor $\mathcal{D}' \to \mathcal{C}$. There is the following sequence of natural equivalences
$$\begin{aligned} &\mathrm{Nat}_{\mathcal{C}}\left(\int^{d \in \mathcal{D}} \mathrm{Map}_{\mathcal{D}'}(p(d), -) \otimes F(d), G\right) \\ &\simeq \int_{d \in \mathcal{D}} \mathrm{Nat}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{D}'}(p(d), -) \otimes F(d), G) \\ &\simeq \int_{d \in \mathcal{D}} \int_{d' \in \mathcal{D}'} \mathrm{Map}_{\mathcal{C}}(\mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes F(d), G(d')) \\ &\simeq \int_{d \in \mathcal{D}} \int_{d' \in \mathcal{D}'} \mathrm{Map}_{\mathcal{C}}(F(d), G(d')^{\mathrm{Map}_{\mathcal{D}'}(p(d), d')}) \end{aligned}$$

$$\begin{aligned}
&\simeq \int_{d\in\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(F(d), \int_{d'\in\mathcal{D}'} G(d')^{\mathrm{Map}_{\mathcal{D}'}(p(d),d')}) \\
&\simeq \int_{d\in\mathcal{D}} \mathrm{Map}_{\mathcal{C}}(F(d), G(p(d))) \\
&\simeq \mathrm{Nat}_{\mathcal{C}}(F, G\circ p)
\end{aligned}$$

showing that the functor $F \mapsto \int^{d\in\mathcal{D}} \mathrm{Map}_{\mathcal{D}'}(p(d),-)\otimes F(d)$ is in fact the left adjoint to the precomposition by p. ◻

We will use coends in this thesis for one particular major case. Let $\mathcal{C}$ be a cocomplete symmetric monoidal ∞-category. If $F\colon \mathcal{D}^{op}\to\mathcal{C}$ and $G\colon\mathcal{D}\to\mathcal{C}$ then define

$$F\otimes_{\mathcal{D}} G = \int^{d\in\mathcal{D}} F(d)\otimes G(d).$$

Lemma B.42. Suppose that $\mathcal{C}$ is a symmetric monoidal ∞-category such that $\otimes$ preserves colimits in both variables. Then the functor

$$\otimes_{\mathcal{D}}\colon \mathrm{Fun}(\mathcal{D}^{op},\mathcal{C})\times\mathrm{Fun}(\mathcal{D},\mathcal{C})\to\mathcal{C}$$

preserves colimits in both variables. If, furthermore, $\mathcal{C}$ is equipped with a t-structure such that $\mathcal{C}_{\geq 0}\times\mathcal{C}_{\geq 0}$ maps to $\mathcal{C}_{\geq 0}$ under $\otimes$, then $\mathrm{Fun}(\mathcal{D}^{op},\mathcal{C}_{\geq 0})\times\mathrm{Fun}(\mathcal{D},\mathcal{C}_{\geq 0})$ maps to $\mathcal{C}_{\geq 0}$. As a consequence, if $F\colon \mathrm{Fun}(\mathcal{D}^{op},\mathcal{C})$ is object-wise m-connective and $G\colon\mathrm{Fun}(\mathcal{D},\mathcal{C})$ is object-wise n-connective, then $F\otimes_{\mathcal{D}} G$ is $m+n$-connective in $\mathcal{C}$.

Proof. Fix $G\colon\mathcal{D}\to\mathcal{C}$ and let $F=\mathrm{colim}_{i\in I}F_i$ be the colimit of a system of functors $\mathcal{D}^{op}\to\mathcal{C}$. Since colimits are computed object-wise, for each $d\in\mathcal{D}$ we have

$$F(d)\otimes G(d)\simeq(\mathrm{colim}_{i\in I}F_i(d))\otimes G(d)\simeq\mathrm{colim}_{i\in I}(F_i(d)\otimes G(d)).$$

The coend is obtained by taking the colimit over $\mathrm{tw}(\mathcal{D})$ of this functor, and by commutativity of colimits we get that $\otimes_{\mathcal{D}}$ preserves colimits in the left variable. The proof for the right side is analogous.

Now assume that $\mathcal{C}$ comes equipped with a t-structure and that $\otimes$ maps $\mathcal{C}_{\geq 0}\times\mathcal{C}_{\geq 0}$ to $\mathcal{C}_{\geq 0}$. Assume that $G\colon\mathcal{D}\to\mathcal{C}$ and $F\colon\mathcal{D}^{op}\to\mathcal{C}$ are object-wise connective. This implies that the bifunctor $F\otimes G\colon\mathcal{D}^{op}\times\mathcal{D}\to\mathcal{C}$ is object-wise connective. Since $\mathcal{C}_{\geq 0}$ is closed under colimits, the colimit under the precomposition with $\mathrm{tw}(\mathcal{D}^{op})^{op}\to\mathcal{D}^{op}\times\mathcal{D}$ lies in $\mathcal{C}_{\geq 0}$. ◻

The D-tensor product $\otimes_{\mathcal{D}}$ comes with a projection formula, similar to one in the context of genuine equivariant spectra.

Theorem B.43. *Suppose that $\mathcal{C}$ is a symmetric monoidal ∞-category such that $\otimes$ preserves colimits in both variables. Let p be a functor $\mathcal{D} \to \mathcal{D}'$, F a functor $\mathcal{D}^{op} \to \mathcal{C}$, G a functor $\mathcal{D}' \to \mathcal{C}$. There is a natural equivalence*

$$\mathrm{Lan}_p F \otimes_{\mathcal{D}'} G \simeq F \otimes_{\mathcal{D}} (G \circ p).$$

Proof. We have the following sequence of natural equivalences.

$$\begin{aligned}
\mathrm{Lan}_p F \otimes_{\mathcal{D}'} G &\simeq \int^{d' \in \mathcal{D}'} \mathrm{Lan}_p F(d') \otimes G(d') \\
&\simeq \int^{d' \in \mathcal{D}'} \left(\int^{d \in \mathcal{D}} \mathrm{Map}_{\mathcal{D}'}(p(d), d') F(d) \right) \otimes G(d') \\
&\simeq \int^{d' \in \mathcal{D}'} \int^{d \in \mathcal{D}} (\mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes F(d)) \otimes G(d') \\
&\simeq \int^{d \in \mathcal{D}} \int^{d' \in \mathcal{D}'} (\mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes F(d)) \otimes G(d') \\
&\simeq \int^{d \in \mathcal{D}} \int^{d' \in \mathcal{D}'} F(d) \otimes (\mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes G(d')) \\
&\simeq \int^{d \in \mathcal{D}} F(d) \otimes \int^{d' \in \mathcal{D}'} \mathrm{Map}_{\mathcal{D}'}(p(d), d') \otimes G(d') \\
&\simeq \int^{d \in \mathcal{D}} F(d) \otimes G(p(d)) \simeq F \otimes_{\mathcal{D}} (G \circ p).
\end{aligned}$$

□

C Moore Spectra

Definition C.1. Let A be an abelian group. We call a spectrum $\mathbf{X}$ a *Moore spectrum* for A if it satisfies

- $\mathbf{X}$ is connective
- $\pi_0\mathbf{X} = A$
- $H_i(\mathbf{X};\mathbb{Z}) = 0$ for $i > 0$.

We will often denote a choice of Moore spectrum for A by $\mathbf{M}A$.

The simplest example to have in mind is that the sphere spectrum $\mathbb{S}$ is a Moore spectrum for $\mathbb{Z}$. We claim that Moore spectra exist and are unique up to weak equivalence.

Proof. First assume that $A = \bigoplus_{i\in I}\mathbb{Z}$ is a free abelian group and that $\mathbf{X}$ is a Moore spectrum for A. Since $\pi_0(X) = \bigoplus_{i\in I}\mathbb{Z}$, choose a map $S \to X$ for each generator $i \in I$. These give a map $\bigvee_{i\in I} S \to X$, which is clearly an isomorphism on π_0, and thus on $H_i(-;\mathbb{Z})$ for all i. This means the fiber is connected and has trivial homology, which implies that it is contractible and therefore $\bigvee_{i\in I} S \simeq X$. This proves uniqueness and existence in this case.

Now assume A is a general abelian group. There is the canonical free resolution

$$\bigoplus_{\omega\in\Omega}\mathbb{Z} \to \bigoplus_{a\in A}\mathbb{Z} \to A.$$

Use the left map to define $\mathbf{M}A$ as the cofiber in

$$\bigvee_{\omega\in\Omega}\mathbb{S} \to \bigvee_{a\in A}\mathbb{S} \to \mathbf{M}A.$$

This proves existence. Let $\mathbf{X}$ be any other Moore spectrum for A. The condition $\pi_0(\mathbf{X}) = A$ gives us a map

$$\bigvee_{a\in A}\mathbb{S} \to X.$$

The fiber $\mathbf{F}$ of this map satisfies all the conditions for a Moore spectrum for $\bigoplus_{\omega\in\Omega}\mathbb{Z}$ and is thus equivalent to $\bigvee_{\omega\in\Omega}\mathbb{S}$ by the point before. Putting this together gives us a map of fiber sequences

$$\begin{tikzcd}
\bigvee_{\omega\in\Omega}\mathbb{S} \arrow[r] \arrow[d] & \bigvee_{a\in A}\mathbb{S} \arrow[r] \arrow[d, "="] & \mathbf{M}A \arrow[d] \\
\mathbf{F} \arrow[r] & \bigvee_{a\in A}\mathbb{S} \arrow[r] & \mathbf{X}.
\end{tikzcd}$$

Since the left and middle map are equivalences, we get that $\mathbf{M}A \simeq \mathbf{X}$. ∎

Remark C.2. The choice of free resolution doesn't really matter. Any free resolution of A,

$$\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z} \to A$$

will give a fiber sequence

$$\bigvee_J \mathbb{S} \to \bigvee_I \mathbb{S} \to \mathbf{M}A.$$

Remark C.3. And immediate consequence of the definition of a Moore spectrum is that

$$\mathbf{M}A \wedge \mathbf{H}\mathbb{Z} \simeq \mathbf{H}A.$$

The reverse holds as well: If $\mathbf{X}$ is a connective spectrum such that $\mathbf{X} \wedge \mathbf{H}\mathbb{Z} \simeq \mathbf{H}A$, then $\mathbf{X} \simeq \mathbf{M}A$.

Lemma C.4. Suppose A is an abelian group and $\mathbf{X}$ a spectrum. Then the natural maps $\mathbf{X} \to \mathbf{X}(-\infty, 1] \leftarrow \mathbf{X}[0, 1]$ induce isomorphisms

$$[\mathbf{M}A, \mathbf{X}] \xrightarrow{\cong} [\mathbf{M}A, \mathbf{X}(-\infty, 1]] \xleftarrow{\cong} [\mathbf{M}A, \mathbf{X}[0, 1]].$$

As a special case, if $\mathbf{X}$ is 2-connective, $[\mathbf{M}A, \mathbf{X}] = 0$.

Proof. We only need to prove the special case, as $\mathbf{M}A$ is connective, which implies the the right map is an isomorphism. Suppose that $\mathbf{Y}$ is 2-connected. Pick a free resolution of A to get a fiber sequence

$$\bigvee_J \mathbb{S} \to \bigvee_I \mathbb{S} \to \mathbf{M}A.$$

Then the long exact sequence from applying $[-, \mathbf{Y}]$ gives the desired result. $\square$

Theorem C.5 ([BR20], Lemma 8.4.8.)**.** *Let A and $\mathbf{X}$ be as before. We have short exact sequences*

$$0 \to \pi_k \mathbf{X} \otimes_{\mathbb{Z}} A \to \pi_k(\mathbf{X} \wedge \mathbf{M}A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{k-1}\mathbf{X}, A) \to 0$$

as well as

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \pi_{k+1}\mathbf{X}) \to \pi_k \mathbf{map}(\mathbf{M}A, \mathbf{X}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \pi_k \mathbf{X}) \to 0.$$

Proof. Pick a free resolution of A,

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{f} \bigoplus_I \mathbb{Z} \to A \to 0.$$

Taking the tensor product with an abelian group B, we get the long exact sequence

$$0 \to \mathrm{Tor}_1^{\mathbb{Z}}(B, A) \to \bigoplus_J B \xrightarrow{f \otimes B} \bigoplus_I B \to A \otimes B \to 0.$$

Dually, taking $\mathrm{Hom}_{\mathbb{Z}}(-, B)$, we get

$$0 \leftarrow \mathrm{Ext}^1_{\mathbb{Z}}(A, B) \leftarrow \prod_J B \xleftarrow{f^*} \prod_I B \leftarrow \mathrm{Hom}_{\mathbb{Z}}(A, B) \leftarrow 0.$$

From the definition of a Moore spectrum we have the fiber sequence

$$\bigvee_J \mathbb{S} \xrightarrow{f} \bigvee_I \mathbb{S} \to \mathbf{M}A.$$

Taking the smash product with $\mathbf{X}$ and then looking at the long exact sequence in homotopy groups, we have

$$\begin{tikzcd}
\cdots \arrow[r] & \bigoplus_I \pi_{k+1}(\mathbf{X}) \arrow[r] & \pi_{k+1}(\mathbf{X} \wedge \mathbf{M}A) \arrow[dll, rounded corners] \\
\bigoplus_J \pi_k(\mathbf{X}) \arrow[r, "f \otimes \pi_k(\mathbf{X})"] & \bigoplus_I \pi_k(\mathbf{X}) \arrow[r] & \pi_k(\mathbf{X} \wedge \mathbf{M}A) \arrow[dll, rounded corners] \\
\bigoplus_J \pi_{k-1}(\mathbf{X}) \arrow[r] & \cdots .
\end{tikzcd}$$

Using exactness at $\pi_k(\mathbf{X} \wedge \mathbf{M}A)$ and the fact that we know kernel and cokernel of $f \otimes \pi_k(\mathbf{X})$ from the previous long exact sequence, we get the short exact sequences

$$0 \to \pi_k \mathbf{X} \otimes_{\mathbb{Z}} A \to \pi_k(\mathbf{X} \wedge \mathbf{M}A) \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{k-1}\mathbf{X}, A) \to 0.$$

Similarly, we get the fiber sequence

$$\prod_J \mathbf{X} \xleftarrow{f^*} \prod_I \mathbf{X} \leftarrow \mathbf{map}(\mathbf{M}A, \mathbf{X}),$$

and the long exact sequence in homotopy groups takes the form

$$\begin{tikzcd}
\cdots & \prod_I \pi_{k-1}(\mathbf{X}) \arrow[l] & \pi_{k-1}(\mathbf{map}(\mathbf{M}A, \mathbf{X})) \arrow[l] \\
\prod_J \pi_k(\mathbf{X}) \arrow[urr, rounded corners] & \prod_I \pi_k(\mathbf{X}) \arrow[l, "f^*"'] & \pi_k(\mathbf{map}(\mathbf{M}A, \mathbf{X})) \arrow[l] \\
\prod_J \pi_{k+1}(\mathbf{X}) \arrow[urr, rounded corners] & \cdots . \arrow[l]
\end{tikzcd}$$

Using exactness at $\pi_k(\mathbf{map}(\mathbf{M}A, \mathbf{X}))$ and the fact that we know kernel and cokernel of f^*, we get the short exact sequences

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(A, \pi_{k+1}\mathbf{X}) \to \pi_k \mathbf{map}(\mathbf{M}A, \mathbf{X}) \to \mathrm{Hom}_{\mathbb{Z}}(A, \pi_k \mathbf{X}) \to 0.$$

$\square$

Corollary C.6. The homotopy groups of $\mathbf{M}A$ fit into short exact sequences

$$0 \to \pi_i\mathbb{S} \otimes_{\mathbb{Z}} A \to \pi_i\mathbf{M}A \to \mathrm{Tor}_1^{\mathbb{Z}}(\pi_{k-1}\mathbb{S}, A) \to 0.$$

Corollary C.7. Suppose A is finitely generated. Then $\pi_*\mathbf{M}A$ is finite for $* > 0$.

Corollary C.8. Let B be an abelian group. The B-homology of $\mathbf{M}A$ is given as

$$\pi_k(\mathbf{M}A \wedge \mathbf{H}B) = \begin{cases} A \otimes_{\mathbb{Z}} B, & k = 0 \\ \mathrm{Tor}^1_{\mathbb{Z}}(A, B), & k = 1 \\ 0, & \text{else} \end{cases}$$

The B-cohomology is given as

$$\pi_{-k}\mathbf{map}(\mathbf{M}A, \mathbf{H}B) = \begin{cases} \mathrm{hom}_{\mathbb{Z}}(A, B), & k = 0 \\ \mathrm{Ext}^1_{\mathbb{Z}}(A, B), & k = 1 \\ 0, & \text{else} \end{cases}$$

Corollary C.9. Suppose A is flat. Then

$$(\mathbf{X} \wedge \mathbf{M}A)[k] \simeq \Sigma^k\mathbf{H}(\pi_k(\mathbf{X}) \otimes A) \simeq \mathbf{X}[k] \wedge \mathbf{M}A$$

.

Proof. By Theorem C.5 we know that $\pi_k(\mathbf{X}\wedge\mathbf{M}A) = \pi_k\mathbf{X}\otimes A$ since A is flat and thus $\mathrm{Tor}^1_{\mathbb{Z}}(\pi_{k-1}\mathbf{X}, A)$ vanishes. This shows the first equivalence. The spectrum $\mathbf{X}[k]$ is $\Sigma^k\mathbf{H}\pi_k(\mathbf{X})$ and so Corollary C.8, using that $\mathrm{Tor}^1_{\mathbb{Z}}(\pi_k\mathbf{X}, A)$ vanishes since A is flat, gives us that $\mathbf{X}[k] \wedge \mathbf{M}A$ is concentrated in degree k with homotopy group $\pi_k\mathbf{X} \otimes A$, thus giving the second equivalence. □

Remark C.10. Since $\mathbb{Z}_{(p)}$ is flat, the p-localization of of a spectrum $\mathbf{X}$ given by $\mathbf{X}_{(p)} := \mathbf{X} \wedge \mathbb{S}_{(p)}$ preserves the t-structure on spectra, and we have that $\pi_n(\mathbf{X}_{(p)}) = \pi_n(\mathbf{X})_{(p)}$. Similarly for the rationalization given by $\mathbf{X}_{\mathbb{Q}} := \mathbf{X} \wedge \mathbf{M}\mathbb{Q}$.

Corollary C.11 ([RR98], Proposition 4.33)**.** Let $\mathbf{X}$ be a spectrum. For each $k \in \mathbb{Z}$ there exists a map

$$\alpha_k \colon \Sigma^k\mathbf{M}\pi_k\mathbf{X} \to \mathbf{X}$$

that is the identity on π_k for a fixed choice of isomorphism $\pi_0\mathbf{M}\pi_k\mathbf{X} \cong \pi_k\mathbf{X}$. Moreover, the set of all such maps is an $\mathrm{Ext}^1_{\mathbb{Z}}(\pi_k\mathbf{X}, \pi_{k+1}\mathbf{X})-$torsor.

Proof. For simplicity, set $k = 0$. Since $\mathbf{M}\pi_0\mathbf{X}$ is connective, we can also assume that $\mathbf{X}$ is connective. By theorem C.5 we have a surjection

$$[\mathbf{M}\pi_0\mathbf{X}, \mathbf{X}] \twoheadrightarrow \mathrm{hom}_{\mathbb{Z}}(\pi_0\mathbf{X}, \pi_0\mathbf{X}).$$

Pick a map α_0 in the preimage of $\mathrm{id}_{\pi_0\mathbf{X}}$. Moreover, since the kernel of this map is $\mathrm{Ext}^1_{\mathbb{Z}}(\pi_0\mathbf{X}, \pi_1\mathbf{X})$, we also have that the choice of α_0 is unique up to elements in $\mathrm{Ext}^1_{\mathbb{Z}}(\pi_0\mathbf{X}, \pi_1\mathbf{X})$. We claim that α_0 is the identity on π_0: The epimorphism

$$[\mathbf{M}\pi_0\mathbf{X}, \mathbf{X}] \twoheadrightarrow \hom_{\mathbb{Z}}(\pi_0\mathbf{X}, \pi_0\mathbf{X})$$

decomposes as the two morphisms

$$[\mathbf{M}\pi_0\mathbf{X}, \mathbf{X}] \overset{\tau_{\leq 0}\mathbf{X}\circ}{\twoheadrightarrow} [\mathbf{M}\pi_0\mathbf{X}, H\pi_0\mathbf{X}] \overset{\circ\tau_{\leq 0}\mathbf{M}\pi_0\mathbf{X}}{\longleftarrow} [H\pi_0\mathbf{X}, H\pi_0\mathbf{X}]$$

and so $\tau_{\leq 0}\mathbf{X} \circ \alpha_0 = \mathrm{id}_{H\pi_0\mathbf{X}} \circ \tau_{\leq 0}\mathbf{M}\pi_0\mathbf{X} = \tau_{\leq 0}\mathbf{M}\pi_0\mathbf{X}$. All of these maps are the identity on π_0, and so is α_0. □

Theorem C.12 ([RR98], Corollary 7.6)**.** *Let* $\mathbf{X}$ *be an* $\mathbf{H}\mathbb{Z}-$*module. Then there exists an equivalence*

$$\mathbf{X} \simeq \bigvee_{k\in\mathbb{Z}} \mathbf{X}[k].$$

Proof. By Corollary C.11, we have maps

$$\alpha_k \colon \Sigma^k\mathbf{M}\pi_k\mathbf{X} \to \mathbf{X}.$$

Applying $\mathbf{H}\mathbb{Z} \wedge -$ and then using the $\mathbf{H}\mathbb{Z}-$module structure on $\mathbf{X}$, we get a map

$$\mathbf{X}[k] \simeq \Sigma^k\mathbf{H}\pi_k\mathbf{X} \to \mathbf{H}\mathbb{Z} \wedge \mathbf{X} \to \mathbf{X}.$$

We remind the reader that the structure of an $\mathbf{H}\mathbb{Z}-$module structure on $\mathbf{X}$ implies that $\mathbf{X}$ is a split summand of $\mathbf{H}\mathbb{Z} \wedge \mathbf{X}$ via the maps

$$\mathbf{X} \xrightarrow{u} \mathbf{H}\mathbb{Z} \wedge \mathbf{X} \xrightarrow{\mu} \mathbf{X}$$

where u is induced from the Hurewicz map $h\colon \mathbb{S} \to \mathbf{H}\mathbb{Z}$ and μ is the action of $\mathbf{H}\mathbb{Z}$ on $\mathbf{X}$. The map h gives us the commutative square

$$\begin{tikzcd}
\Sigma^k\mathbf{M}\pi_k\mathbf{X} \arrow[r, "\alpha_k"] \arrow[d] & \mathbf{X} \arrow[d] \\
\Sigma^k H\pi_k\mathbf{X} \arrow[r, "H\mathbb{Z}\wedge\alpha_k"] & \mathbf{H}\mathbb{Z} \wedge \mathbf{X}.
\end{tikzcd}$$

It is now straightforward to see that the composition

$$\tilde{\alpha}_k \colon \Sigma^k\mathbf{H}\pi_k\mathbf{X} \xrightarrow{H\mathbb{Z}\wedge\alpha_k} \mathbf{H}\mathbb{Z} \wedge \mathbf{X} \xrightarrow{\mu} \mathbf{X}$$

is an isomorphism on π_k. Taking the wedge sum over all $\tilde{\alpha}_k$ gives us a map

$$\bigvee_{k\in\mathbb{Z}} \Sigma^k\mathbf{H}\pi_k\mathbf{X} \to \mathbf{X}$$

which is an isomorphism on each π_k and thus an equivalence. □

We remind the reader that for a given integer n we have a natural fiber sequence

$$\mathbf{M}A[n+1] \to \mathbf{M}A[0,n+1] \to \mathbf{M}A[0,n].$$

The n-th k-invariant is the associated boundary map of this sequence,

$$k\colon \mathbf{M}A[0,n] \to \Sigma\mathbf{M}A[n+1].$$

We also have a natural fiber sequence

$$\mathbf{M}A[n+1,\infty) \to \mathbf{M}A \to \mathbf{M}A[0,n].$$

We denote the associated boundary map by

$$\tilde{k}\colon \mathbf{M}A[0,n] \to \Sigma\mathbf{M}A[n+1,\infty).$$

Theorem C.13. *The k-invariant induces an isomorphism under precomposition*

$$k^*\colon \mathrm{End}(\pi_{n+1}\mathbf{M}A) \cong [\mathbf{M}A[0,n], \Sigma\mathbf{M}A[n+1]].$$

Proof. The fiber sequence

$$\mathbf{M}A[n+1,\infty) \to \mathbf{M}A \to \mathbf{M}A[0,n]$$

induces the long exact sequence

$$[\mathbf{M}A, \mathbf{M}A[n+1]] \to [\mathbf{M}A[n+1,\infty), \mathbf{M}A[n+1]] \xrightarrow{\tilde{k}^*} [\mathbf{M}A[0,n], \Sigma\mathbf{M}A[n+1]] \to [\mathbf{M}A, \Sigma\mathbf{M}A[n+1]].$$

By naturality of $\tau_{\geq n+1} \to \mathrm{id}$ we have the commutative square

$$\begin{tikzcd}
{[\mathbf{M}A[n+1,\infty), \mathbf{M}A[n+1]]} \arrow[r, "\tilde{k}^*"] \arrow[d, "\cong"] & {[\mathbf{M}A[0,n], \Sigma\mathbf{M}A[n+1]]} \arrow[d, "="] \\
{[\mathbf{M}A[n+1], \mathbf{M}A[n+1]]} \arrow[r, "k^*"] & {[\mathbf{M}A[0,n], \Sigma\mathbf{M}A[n+1]].}
\end{tikzcd}$$

The left-hand map is an isomorphism since the fiber of $\mathbf{M}A[n+1,\infty) \to \mathbf{M}A[n+1]$ is $n+2$-connective. The group $[\mathbf{M}A[n+1], \mathbf{M}A[n+1]]$ is isomorphic to $\mathrm{End}(\pi_{n+1}\mathbf{M}A)$.

Since the cohomology for $\mathbf{M}A$ vanishes in degree 2 and higher, we are done with our claim for $n > 0$. Now assume $n = 0$. The long exact sequence from before takes the shape

$$[\mathbf{M}A[0], \mathbf{M}A[1]] \xrightarrow{h^*} [\mathbf{M}A, \mathbf{M}A[1]] \to [\mathbf{M}A[1,\infty), \mathbf{M}A[1]] \xrightarrow{\tilde{k}^*} [\mathbf{M}A[0], \Sigma\mathbf{M}A[1]] \to 0.$$

We claim that the left map h^* is an isomorphism, from which it follows that $\tilde{k}^*$ is an isomorphism. Choose a resolution for A,

$$\bigoplus_J \mathbb{Z} \to \bigoplus_I \mathbb{Z} \to A.$$

Under the Hurewicz map we have a map of fiber sequences

$$\begin{tikzcd} \bigvee_J \mathbb{S} \arrow[r] \arrow[d] & \bigvee_I \mathbb{S} \arrow[r] \arrow[d] & \mathbf{M}A \arrow[d] \\ \bigvee_J H\mathbb{Z} \arrow[r] & \bigvee_I H\mathbb{Z} \arrow[r] & \mathbf{M}A[0]. \end{tikzcd}$$

Looking at the long exact sequence from taking $[-, \mathbf{M}A[1]]$ and using the 5-lemma together with the fact that $[\mathbf{H}\mathbb{Z}, \mathbf{M}A[1]] = \mathrm{Ext}_{\mathbb{Z}}(\mathbb{Z}, \pi_1\mathbf{M}A) = 0$, we conclude the claim that the Hurewicz map induces an isomorphism $[\mathbf{M}A[0], \mathbf{M}A[1]] \to [\mathbf{M}A, \mathbf{M}A[1]]$, with both groups being equal to $\mathrm{Ext}_{\mathbb{Z}}(\pi_0\mathbf{M}A, \pi_1\mathbf{M}A)$. ◻

Example C.14. From this we have a variety of examples at hand:

- The first k-invariant of the sphere spectrum $\mathbb{S}$ is a nontrivial map from $\mathbf{H}\mathbb{Z} \to \Sigma^2\mathbf{H}\mathbb{Z}/2$ and thus represents the integral Steenrod-square Sq^2.
- The first k-invariant of the Moore spectrum $\mathbf{M}\mathbb{Z}/2$, also written $\mathbb{S}/2$, is a nontrivial map from $\mathbf{H}\mathbb{Z}/2 \to \Sigma^2\mathbf{H}\mathbb{Z}/2$ and thus represents the mod 2 Steenrod-square sq^2.
- More generally, for a prime p there is a unique p-torsion subgroup $\mathbb{Z}/p$ in $\pi_{2(p-1)-1}\mathbb{S}$ and no p-torsion in $\pi_i\mathbb{S}$ for $i < 2(p-1)-1$. The first non-trivial k-invariant of the Moore spectrum $\mathbf{M}\mathbb{Z}/p = \mathbb{S}/p$ thus represents the mod p Steenrod-square $P^1 \colon \mathbf{H}\mathbb{Z}/p \to \Sigma^{2(p-1)}\mathbf{H}\mathbb{Z}/p$.
- The p-local sphere $\mathbb{S}_{(p)} = \mathbf{M}\mathbb{Z}_{(p)}$ represents the integral Steenrod-operation $\tilde{P}^1 \colon \mathbf{H}\mathbb{Z}_{(p)} \to \Sigma^{2(p-1)}\mathbf{H}\mathbb{Z}/p$.

Lemma C.15. Let $\mathbf{X}$ be connective. Then the zero-th k-invariant k_M of the Moore spectrum $\mathbf{M}\pi_0\mathbf{X}$ factors through the zero-th k-invariant $k \colon \mathbf{X}[0] \to \Sigma\mathbf{X}[1]$ and the order of k is at most 2.

Proof. By Corollary C.11, we have a map

$$\mathbf{M}\pi_0\mathbf{X} \to \mathbf{X}$$

which is an isomorphism on π_0. We have the natural commuting square

$$\begin{tikzcd} \mathbf{M}\pi_0\mathbf{X}[0,1] \arrow[r] \arrow[d] & \mathbf{X}[0,1] \arrow[d] \\ H\pi_0\mathbf{X} \arrow[r] & \mathbf{X}[0] \end{tikzcd}$$

where $H\pi_0\mathbf{X} \to \mathbf{X}[0]$ is an equivalence. Recall that $\pi_1\mathbf{M}\pi_0\mathbf{X} \cong \pi_0\mathbf{X} \otimes_{\mathbb{Z}} \mathbb{Z}/2$. Taking vertical fibers gives the commutative diagram

$$\begin{tikzcd} H\pi_0\mathbf{X} \arrow[r, "\simeq"] \arrow[d, "k_M"] & \mathbf{X}[0] \arrow[d, "k"] \\ \Sigma^2 H\pi_0\mathbf{X} \otimes_{\mathbb{Z}} \mathbb{Z}/2 \arrow[r] & \Sigma\mathbf{X}[1] \end{tikzcd}$$

which concludes the claim. ◻

If $\mathbf{X}$ is concentrated in degrees 0 and 1, the map $\pi_0\mathbf{X}/2 \to \pi_1(\mathbf{X})$ actually completely classifies the homotopy type of $\mathbf{X}$. We cite the following Lemma from [Str13], Lemma 6.5.

Lemma C.16. For any abelian groups A and B we have $H_1\mathbf{H}A = 0$ and $H_2\mathbf{H}A = A/2$ and $[\mathbf{H}A, \Sigma^2\mathbf{H}B] = H_2(\mathbf{H}A; B) = \mathrm{Hom}(A/2, B)$. Moreover, if $\mathbf{X}$ is the fiber of a map $k\colon \mathbf{H}A \to \Sigma^2\mathbf{H}B$ corresponding to a homomorphism $\sigma\colon A/2 \to B$, then the map $\eta\colon A = \pi_0(\mathbf{X}) \to \pi_1(\mathbf{X}) = B$ is just the composite of σ with the projection $A \to A/2$.

Here, η refers to the Hopf map $\mathbb{S}^1 \to \mathbb{S}$, the unique non-trivial map $\mathbb{S}^1 \to \mathbb{S}$ of order 2.

C.1 The rationals

In general, Eilenberg-Maclane spectra and Moore Spectra are very far apart from each other. This difference, however, disappears when one considers $\mathbf{M}\mathbb{Q}$.

Lemma C.17. For any spectrum $\mathbf{X}$, $\pi_*(\mathbf{X} \wedge \mathbf{M}\mathbb{Q}) = \pi_*(\mathbf{X}) \otimes \mathbb{Q}$.

Proof. This is simply Corollary C.9, since $\mathbb{Q}$ is flat. □

Corollary C.18. If a spectrum $\mathbf{X}$ has all torsion homotopy groups in each degree, then $\mathbf{X} \wedge \mathbf{M}\mathbb{Q} \simeq 0$.

Proof. Follows directly from the previous lemma, since for a torsion group A, $A \otimes \mathbb{Q} = 0$. □

Lemma C.19. There is an equivalence $\mathbf{M}\mathbb{Q} \simeq \mathbf{H}\mathbb{Q}$.

Proof. Smashing the fiber sequence $\mathbb{S}[1,\infty) \to \mathbb{S} \to \mathbf{H}\mathbb{Z}$ with $\mathbf{M}\mathbb{Q}$ gives that $\mathbf{M}\mathbb{Q} \simeq \mathbf{H}\mathbb{Z} \wedge \mathbf{M}\mathbb{Q}$ since $\mathbb{S}[1,\infty)$ has all finite homotopy groups. But by the defining properties of a Moore spectrum $\mathbf{H}\mathbb{Z} \wedge \mathbf{M}\mathbb{Q} \simeq \mathbf{H}\mathbb{Q}$. □

Corollary C.20. For any spectrum $\mathbf{X}$, $H_*(\mathbf{X}; \mathbb{Q}) = \pi_*(\mathbf{X}) \otimes \mathbb{Q}$.

Lemma C.21. Suppose $\mathbf{X}$ is a spectrum with $\pi_1(\mathbf{X})$ being a rational vector space. Then

$$[\mathbf{H}\mathbb{Q}, \mathbf{X}] = \mathrm{hom}(\mathbb{Q}, \pi_0(\mathbf{X}))$$

Proof. Since $\pi_1(\mathbf{X})$ is a rational vector space, the group $\mathrm{Ext}_{\mathbb{Z}}(\mathbb{Q}, \pi_1(\mathbf{X}))$ vanishes. The claim now follows from Theorem C.5 using $\mathbf{H}\mathbb{Q} \simeq \mathbf{M}\mathbb{Q}$. □

Corollary C.22. The rational Steenrod-algebra $\pi_{-*}\mathbf{map}(\mathbf{H}\mathbb{Q}, \mathbf{H}\mathbb{Q})$ is concentrated in degree 0.

References

[20] *GAP – Groups, Algorithms, and Programming, Version 4.11.0*. The GAP Group. 2020. URL: https://www.gap-system.org.

[AL18] Mathieu Anel and Damien Lejay. *Exponentiable Higher Toposes*. 2018. arXiv: 1802.10425 [math.CT].

[Bar03] Arthur C Bartels. "On the domain of the assembly map in algebraic K–theory". In: *Algebraic & Geometric Topology* 3.2 (Oct. 2003), pp. 1037–1050. ISSN: 1472-2747. DOI: 10.2140/agt.2003.3.1037. URL: http://dx.doi.org/10.2140/agt.2003.3.1037.

[Bar14] C. Barwick. *Spectral Mackey functors and equivariant algebraic K-theory (I)*. 2014. arXiv: 1404.0108 [math.AT].

[Bas76] H. Bass. "Euler characteristics and characters of discrete groups". In: *Inventiones mathematicae* 35 (1976), pp. 155–196.

[BB18] Arthur Bartels and Mladen Bestvina. *The Farrell-Jones Conjecture for mapping class groups*. 2018. arXiv: 1606.02844 [math.GT].

[BD82] J. Beĭlinson A. A.; Bernstein and P. Deligne. "Faisceaux pervers. (French) [Perverse sheaves]". In: *Analysis and topology on singular spaces* I, (Luminy, 1981) (1982), pp. 5–171.

[Ber56] S.D. Berman. "The number of irreducible representations of a finite group over an arbitrary field". In: *Dokl. Akad. Nauk SSSR* 106 (1956), pp. 767–769.

[BLR07] Arthur Bartels, Wolfgang Lück, and Holger Reich. "The K-theoretic Farrell–Jones conjecture for hyperbolic groups". In: *Inventiones mathematicae* 172.1 (Dec. 2007), pp. 29–70. ISSN: 1432-1297. DOI: 10.1007/s00222-007-0093-7. URL: http://dx.doi.org/10.1007/s00222-007-0093-7.

[BR20] David Barnes and Constanze Roitzheim. *Foundations of Stable Homotopy Theory*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2020. DOI: 10.1017/9781108636575.

[Car80a] David M. Carter. "Lower K-theory of finite groups". In: *Communications in Algebra* 8.20 (1980), pp. 1927–1937. DOI: 10.1080/00927878008822554. eprint: https://doi.org/10.1080/00927878008822554. URL: https://doi.org/10.1080/00927878008822554.

[Car80b] David W. Carter. "Localization in lower algebraic K-theory". In: *Communications in Algebra* 8.7 (1980), pp. 603–622. DOI: 10.1080/00927878008822478. eprint: https://doi.org/10.1080/00927878008822478. URL: https://doi.org/10.1080/00927878008822478.

[CR81] C.W. Curtis and I. Reiner. *Methods of Representation Theory*. Wiley-Interscience publication v. 2. Wiley, 1981. ISBN: 9780471189947. URL: https://books.google.de/books?id=Eoc%5C_AQAAIAAJ.

[Cra10] James Cranch. *Algebraic Theories and $(\infty, 1)$-Categories*. 2010. arXiv: 1011.3243 [math.AT].

[Die06] T.T. Dieck. *Transformation Groups and Representation Theory*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006. ISBN: 9783540385172. URL: https://books.google.de/books?id=IOd7CwAAQBAJ.

[DK84] W.G. Dwyer and D.M. Kan. “Singular functors and realization functors”. In: *Indagationes Mathematicae (Proceedings)* 87.2 (1984), pp. 147–153. ISSN: 1385-7258. DOI: https://doi.org/10.1016/1385-7258(84)90016-7. URL: https://www.sciencedirect.com/science/article/pii/1385725884900167.

[DL98] J. Davis and W. Lück. “Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory.” In: *K-theory* 15 (1998), pp. 201–252.

[DQR11] James F. Davis, Frank Quinn, and Holger Reich. “Algebraic K-theory over the infinite dihedral group: a controlled topology approach”. In: *Journal of Topology* 4.3 (July 2011), pp. 505–528. ISSN: 1753-8416. DOI: 10.1112/jtopol/jtr009. URL: http://dx.doi.org/10.1112/jtopol/jtr009.

[FJ95] F.T. Farrell and L.E. Jones. “The lower algebraic K-theory of virtually infinite cyclic groups”. In: *K-Theory 9* (1995).

[GH15] David Gepner and Rune Haugseng. “Enriched ∞-categories via non-symmetric ∞-operads”. In: *Advances in Mathematics* 279 (July 2015), pp. 575–716. ISSN: 0001-8708. DOI: 10.1016/j.aim.2015.02.007. URL: http://dx.doi.org/10.1016/j.aim.2015.02.007.

[GHN20] David Gepner, Rune Haugseng, and Thomas Nikolaus. *Lax colimits and free fibrations in ∞-categories*. 2020. arXiv: 1501.02161 [math.CT].

[GJM18] John Guaschi, Daniel Juan-Pineda, and Silvia Millán-López. *The lower algebraic K-theory of virtually cyclic subgroups of the braid groups of the sphere and of $\mathbb{Z}[B\_4(\mathbb{S}^2)]$*. 2018. arXiv: 1209.4791 [math.KT].

[Gla15] Saul Glasman. *A spectrum-level Hodge filtration on topological Hochschild homology*. 2015. arXiv: 1408.3065 [math.AT].

[GS13] M.J. Greenberg and J.P. Serre. *Local Fields*. Graduate Texts in Mathematics. Springer New York, 2013. ISBN: 9781475756739. URL: https://books.google.de/books?id=3LAJCAAAQBAJ.

[Hat65] Akira Hattori. “Rank Element of a Projective Module”. In: *Nagoya Mathematical Journal* 25 (1965), pp. 113–120. DOI: 10.1017/S002776300001148X.

[Hem04] J. Hempel. *3-Manifolds*. AMS Chelsea Publishing Series. AMS Chelsea Pub., American Mathematical Society, 2004. ISBN: 9780821836958. URL: https://books.google.de/books?id=eqcJBAAAQBAJ.

[KLR16] Holger Kammeyer, Wolfgang Lück, and Henrik Rüping. “The Farrell–Jones conjecture for arbitrary lattices in virtually connected Lie groups”. In: *Geometry & Topology* 20.3 (July 2016), pp. 1275–1287. ISSN: 1465-3060. DOI: 10.2140/gt.2016.20.1275. URL: http://dx.doi.org/10.2140/gt.2016.20.1275.

[KM91] P. H. Kropholler and B. Moselle. “A family of crystallographic groups with 2-torsion in K_0 of the rational group algebra”. In: *Proceedings of the Edinburgh Mathematical Society* 34.2 (1991), pp. 325–331. DOI: 10.1017/S0013091500007215.

[Lea00] Ian J. Leary. “A Torsion Projective Class for a Group Algebra”. In: *Bulletin of the London Mathematical Society* 32.1 (2000), pp. 75–77. DOI: https://doi.org/10.1112/S0024609399006566. eprint: https://londmathsoc.onlinelibrary.wiley.com/doi/pdf/10.1112/S0024609399006566. URL: https://londmathsoc.onlinelibrary.wiley.com/doi/abs/10.1112/S0024609399006566.

[LMO09] Jean-François Lafont, Bruce A. Magurn, and Ivonne J. Ortiz. “Lower algebraic K-theory of certain reflection groups”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 148.2 (Nov. 2009), pp. 193–226. ISSN: 1469-8064. DOI: 10.1017/s0305004109990363. URL: http://dx.doi.org/10.1017/S0305004109990363.

[LR05] W. Lück and H. Reich. “The Baum-Connes and the Farrell-Jones conjectures in K and L-theory”. In: *in Handbook of K-theory. Vol. 1, 2* (2005), pp. 703–842.

[Lüc05] Wolfgang Lück. “Survey on Classifying Spaces for Families of Subgroups”. In: *arXiv: Geometric Topology* (2005), pp. 269–322.

[Lüc19] Wolfgang Lück. *Assembly Maps*. 2019. arXiv: 1805.00226 [math.KT].

[Lur12] Jacob Lurie. *Higher Topos Theory*. 2012.

[Lur17] Jacob Lurie. *Higher Algebra*. 2017.

[Mag13] Bruce A. Magurn. “Negative K-Theory of Generalized Quaternion Groups and Binary Polyhedral Groups”. In: *Communications in Algebra* 41.11 (2013), pp. 4146–4160. DOI: 10.1080/00927872.2012.692005. eprint: https://doi.org/10.1080/00927872.2012.692005. URL: https://doi.org/10.1080/00927872.2012.692005.

[MNN19] Akhil Mathew, Niko Naumann, and Justin Noel. “Derived induction and restriction theory”. In: *Geometry & Topology* 23.2 (Apr. 2019), pp. 541–636. ISSN: 1465-3060. DOI: 10.2140/gt.2019.23.541. URL: http://dx.doi.org/10.2140/gt.2019.23.541.

[Nic20] John Nicholson. “A cancellation theorem for modules over integral group rings”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 171.2 (Nov. 2020), pp. 317–327. ISSN: 1469-8064. DOI: 10.1017/s0305004120000237. URL: http://dx.doi.org/10.1017/S0305004120000237.

[Pie82] R. S. Pierce. *Associative algebras*. Springer-Verlag, 1982.

[Rei64] Irving Reiner. “On the Number of Irreducible Modular Representations of a Finite Group”. In: *Proceedings of the American Mathematical Society* 15.5 (1964), pp. 810–812. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2034603.

[RR98] Y.B. Rudyak and J. Rudjak. *On Thom Spectra, Orientability, and Cobordism*. Monographs in Mathematics. Springer, 1998. ISBN: 9783540620433. URL: https://books.google.de/books?id=JYk0GKTNlBIC.

[RV18] Holger Reich and Marco Varisco. "Assembly maps, controlled algebra, and trace methods". In: *Space – Time – Matter* (Apr. 2018), pp. 1–50. DOI: 10.1515/9783110452150-001. URL: http://dx.doi.org/10.1515/9783110452150-001.

[SE70] R.G. Swan and E.G. Evans. *K-theory of Finite Groups and Orders*. Lecture notes in mathematics. Springer-Verlag, 1970. ISBN: 9780387049380. URL: https://books.google.co.ls/books?id=RjYZAQAAIAAJ.

[Ser77] J.P. Serre. *Linear Representations of Finite Groups*. Collection Méthodes. Mathématiques. Springer-Verlag, 1977. ISBN: 9783540901907. URL: https://books.google.de/books?id=sFqVQgAACAAJ.

[Sta65] John Stallings. "Centerless groups—an algebraic formulation of Gottlieb's theorem". In: *Topology* 4.2 (1965), pp. 129–134. ISSN: 0040-9383. DOI: https://doi.org/10.1016/0040-9383(65)90060-1. URL: https://www.sciencedirect.com/science/article/pii/0040938365900601.

[Str13] Neil Strickland. *An abelian embedding for Moore spectra*. 2013. arXiv: 1205.2247 [math.AT].

[Swa60] R.G. Swan. "Induced Representations and Projective Modules". In: *Annals of Mathematics* 71 (1960), p. 552.

[Ung19] W.R. Unger. "An algorithm for computing Schur indices of characters". In: *Journal of Symbolic Computation* 93 (July 2019), pp. 148–160. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2018.06.002. URL: http://dx.doi.org/10.1016/j.jsc.2018.06.002.

[Var89] K. Varadarajan. *The Finiteness Obstruction of C. T. C. Wall*. Wiley-Interscience and Canadian Mathematics Series of Monographs and Texts. Wiley, 1989. ISBN: 9780471623069. URL: https://books.google.de/books?id=hQ6oAAAAIAAJ.

[Wal65] C. T. C. Wall. "Finiteness Conditions for CW-Complexes". In: *Annals of Mathematics* 81.1 (1965), pp. 56–69. ISSN: 0003486X. URL: http://www.jstor.org/stable/1970382.

[Wal78] Friedhelm Waldhausen. "Algebraic K-Theory of Generalized Free Products, Part 1". In: *Annals of Mathematics* 108.2 (1978), pp. 135–204. ISSN: 0003486X. URL: http://www.jstor.org/stable/1971165.

[Weg15] Christian Wegner. "The Farrell–Jones conjecture for virtually solvable groups:" in: *Journal of Topology* 8.4 (Sept. 2015), pp. 975–1016. ISSN: 1753-8424. DOI: 10.1112/jtopol/jtv026. URL: http://dx.doi.org/10.1112/jtopol/jtv026.

[Wei13] C. Weibel. "The K-Book: An Introduction to Algebraic K-Theory". In: 2013.

[Wei95] C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995. ISBN: 9780521559874. URL: https://books.google.de/books?id=flm-dBXfZ%5C_gC.

Abstract

In this dissertation we investigate a conjecture proposed by W. Lück and H. Reich that asserts for an arbitrary group G that the map $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ given by rationalization is trivial in reduced K-theory or in other words, that any finitely generated projective $\mathbb{Z}G$-module is stably free after rationalization. We show that this statement is false and give a concrete counterexample of a virtually group G for which there exists a finitely generated projective $\mathbb{Z}G$-module P which is not stably free after rationalization, however $P \oplus P$ is. We also show that for groups G satisfying the Farrell-Jones conjecture the image in reduced K-theory is always 2-torsion and we give vanishing conditions in terms of the structure of the finite subgroups of G. We also give a comparison to the strong Bass conjecture and show, using our results, that it follows from the Farrell-Jones conjecture.

One of the necessities for constructing an infinite group G such that $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ is non-trivial in reduced K-theory is the existence of finite subgroups H which have torsion in the negative K-theory group $K_{-1}\mathbb{Z}H$. This goes back to results of D. Carter, who showed that $K_{-1}\mathbb{Z}H = \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$. We give an explicit algorithm for computing s for a finite group using the computer algebra system *GAP* and compute all such groups with non-trivial s for small orders.

Zusammenfassung

In dieser Dissertation untersuchen wir eine Vermutung, welche von W. Lück und H.Reich aufgestellt wurde und besagt, dass für eine Gruppe G, nicht notwendigerweise endlich, die Abbildung $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$, gegeben durch die Rationalisierung, in reduzierter K-Theorie verschwindet. Anders ausgedrückt soll jeder endlich erzeugte projektive $\mathbb{Z}G$-Modul nach Rationalisierung stabil frei sein. Wir zeigen, dass diese Vermutung falsch ist, und konstruieren eine virtuell zyklische Gruppe G mit der Eigenschaft, dass ein $\mathbb{Z}G$-Modul P existiert, welcher nicht stabil frei nach Rationalisierung ist, wobei allerdings $(P \oplus P) \otimes \mathbb{Q}$ stabil frei ist. Wir zeigen weiters, dass für Gruppen G, welche die Farrell-Jones Vermutung erfüllen gilt, dass das Bild der Abbildung in reduzierter K-Theorie immer eine 2-Torsionsgruppe ist, und wir geben Bedingungen für das Verschwinden ebenjener Abbildung an, in Bezug auf die Struktur der endlichen Untergruppen von G. Wir vergleichen ebenfalls mit der starken Bass Vermutung und zeigen, durch Verwendung unserer Ergebnisse, dass die Bass Vermutung aus der Farrell-Jones Vermutung folgt.

Eine der Notwendigkeiten für das Konstruieren einer unendlichen Gruppe G, für die die Abbildung $K_0\mathbb{Z}G \to K_0\mathbb{Q}G$ in reduzierter K-Theorie nicht trivial wirkt, ist die Existenz von endlichen Untergruppen H mit der Eigenschaft, dass die negative K-Theorie-Gruppe $K_{-1}\mathbb{Z}H$ nicht-triviale 2-Torsion besitzt. Diese Frage wurde ursprünglich von D.Carter behandelt, welcher zeigte, dass $K_{-1}\mathbb{Z}H = \mathbb{Z}^r \oplus (\mathbb{Z}/2)^s$. Wir geben einen expliziten Algorithmus an, welcher den Koeffizienten s für endliche Gruppen mit dem Computeralgebrasystem *GAP* berechnet und berechnen alle Gruppen mit nicht-trivialem s für kleine Ordnungen.

Selbstständigkeitserklärung

Ich erkläre gegenüber der Freien Universität Berlin, dass ich die vorliegende Dissertation selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht. Diese Dissertation wurde in gleicher oder ähnlicher Form noch in keinem früheren Promotionsverfahren eingereicht. Mit einer Prüfung meiner Arbeit durch ein Plagiatsprüfungsprogramm erkläre ich mich einverstanden.

Berlin, 20. September 2021 Georg Lehner