Parqueting-Reflection Principle and Boundary Value Problems in Some Circular Polygons

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Abstract

This dissertation is an investigation of the theory of the parqueting-reflection principle and its applications to basic boundary value problems in some circular polygons.

The parqueting-reflection principle is applied to solve several boundary value problems for particular domains whose boundaries are composed of circular arcs. It provides heuristic ideas and procedures for constructing harmonic Green functions and harmonic Neumann functions, which play important roles in dealing with Dirichlet and Neumann boundary value problems for the Poisson equation. The parqueting-reflection principle also contributes a method to solve Schwarz boundary value problems for the homogeneous and inhomogeneous Cauchy-Riemann equations. The parqueting-reflection principle has been verified to successfully solve these boundary value problems for many planar domains. However, this principle has not yet been well explained or rigorously justified in theory. This dissertation dedicates to building a fundamental theory for the parqueting-reflection principle and exploring new domains in which the principle can be applied.

The main works of this dissertation are listed below.

We first discuss circle reflections in the extended complex plane and employ some matrix techniques in dealing with circle reflections. These matrix tools bring some convenience for the discussions and the computations. Some results on consecutive circle reflections are also prepared for further discussions.

We next introduce the definition of parqueting-reflection domains, in which the parqueting-reflection principle is supposed to be applicable. We prove that the parqueting-reflection principle succeeds in constructing the harmonic Green and Neumann functions for finite parqueting-reflection domains. We also obtain some properties of the normal derivatives of harmonic Green and Neumann functions on the boundary of the domains.

We then fully overview basic boundary value problems in disks and half-planes and unify the harmonic Green and Neumann functions, the Schwarz integral formulas, the Poisson integral formulas, and their boundary behaviors for disks and half-planes. On the basis of these discussions and by means of the parqueting-reflection principle, we generally solve the Schwarz problems for the Cauchy-Riemann equations and the Dirichlet problems for the Poisson equation in finite parqueting-reflection bounded domains.

The last two parts of this dissertation are about the applications of the parqueting-reflection principle to basic boundary value problems in a class of circular digons and a circular rectangle. The circular digons with the intersection angles π/n for some positive integer n are verified to be finite parqueting-reflection domains. We then solve the Dirichlet problem, Neumann problem, and Schwarz problem for this class of circular digons by means of the parqueting-reflection principle. We also verify that a circular rectangle is an infinite parqueting-reflection domain. We succeed in constructing the harmonic Green function and then solving the Dirichlet problem in this circular rectangle.

Contents

Chapter 1

Preliminaries

This chapter is an overall review of function theory about the differential operators of first and second order in a single complex variable, especially about the Cauchy-Riemann operator and the Laplacian operator. Some functional tools, integral representation formulas and integral operators are prepared for later discussions on several boundary value problems.

1.1 Differential operators, function classes and integral representation formulas

We denote the set of all complex numbers by $\mathbb{C}$. A complex number $z \in \mathbb{C}$ is usually written as $z = x + iy$, where $x, y \in \mathbb{R}$ and i the imaginary unit. For the complex number $z = x + iy$, x is called the real part and y is called the imaginary part, they are denoted by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively. The absolute value of z is defined by $|z| = \sqrt{x^2 + y^2}$, the complex conjugate of z is denoted by $\overline{z} = x - iy$.

We call $D \subset \mathbb{C}$ a regular domain if it is a connected, bounded open subset with piecewise smooth boundary ∂D. Let $\overline{D} = D \cup \partial D$ denote the closure of D. Throughout the whole thesis, we assume that domains are regular unless otherwise stated.

A complex-valued function in variable z is denoted by $w(z) = u(x, y) + iv(x, y)$, where $u(x, y)$ and $v(x, y)$ are two real-valued functions in variables x and y.

The complex differential operators of first order, $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \overline{z}}$, are defined respectively by :

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \ \frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

For a function $w(z)$ with differentiable real and imaginary parts,

$$\begin{aligned}
\frac{\partial w}{\partial z} &= \frac{1}{2}\Big[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\Big], \\
\frac{\partial w}{\partial \overline{z}} &= \frac{1}{2}\Big[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\Big].
\end{aligned}$$

The complex derivatives $\frac{\partial w}{\partial z}$ and $\frac{\partial w}{\partial \overline{z}}$ are also called the Wirtinger derivatives of w. We often use notations $w_z := \frac{\partial w}{\partial z}$ and $w_{\overline{z}} := \frac{\partial w}{\partial \overline{z}}$ for convenience, in the same manner that f_x and f_y denote the partial derivatives for a real-valued function $f(x,y)$ in the variables x and y. We also write the operators $\frac{\partial}{\partial z}$ by ∂_z, and $\frac{\partial}{\partial \overline{z}}$ by $\partial_{\overline{z}}$ for convenience.

Let D be a domain. We say that $w(z) \in C^n(D;\mathbb{C})$ if $u(x,y), v(x,y) \in C^n(D;\mathbb{R})$, i.e. both $u(x,y)$ and $v(x,y)$ have continuous partial derivatives up to order n in the domain D. In the case of $n = 0$, $C^0(D;\mathbb{C})$ and $C^0(D;\mathbb{R})$, usually denoted by $C(D;\mathbb{C})$ and $C(D;\mathbb{R})$ respectively, represent the complex and real valued continuous functions in D respectively.

One of the fundamental theorems of calculus in two variables is the Green theorem, or equivalently, the Gauss divergence theorem in two variables. See e.g. [29, Sect. 13.3.1, Prop. 1].

Theorem 1.1.1 (Green Theorem). *Let D be a bounded domain in $\mathbb{R}^2$ with a positively oriented, piecewise smooth boundary ∂D. Let $P(x,y), Q(x,y) \in C^1(D;\mathbb{R}) \cap C(\overline{D};\mathbb{R})$. Then*

$$\int_{\partial D} P\mathrm{d}x + Q\mathrm{d}y = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) \mathrm{d}x\mathrm{d}y.$$

Using complex derivatives, Green theorem can be easily restated in complex forms.

Theorem 1.1.2 ([2, Gauss Theorems])**.** *Let D be a bounded domain in $\mathbb{C}$ with a positively oriented, piecewise smooth boundary ∂D. Let $w(z) = u(x,y) + iv(x,y) \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then*

$$\int_{\partial D} w(z)\mathrm{d}z = 2i \int_D w_{\overline{z}}(z)\mathrm{d}x\mathrm{d}y,$$
$$\int_{\partial D} w(z)\mathrm{d}\overline{z} = -2i \int_D w_z(z)\mathrm{d}x\mathrm{d}y.$$

From now on, all the boundary integrals throughout this thesis are carried out along the positive orientation of the boundary curves unless otherwise specified.

A class of functions that is closely connected to the first-order differential operator with respect to $\overline{z}$ is the class of holomorphic functions. A holomorphic function is a complex-valued function that is, at every point of its domain, complex differentiable in a neighborhood of the point. For one complex variable, $w(z)$ is holomorphic in D if $w_{\overline{z}} = 0$ for all $z \in D$. The equation $w_{\overline{z}} = 0$ can also be described in real form. A function $w(z) = u(x,y) + iv(x,y)$ is holomorphic in the domain D if and only if the **Cauchy-Riemann equations**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied in D. We call $\partial_{\overline{z}}$ the Cauchy-Riemann operator.

For the complex-valued functions in one variable, the class of holomorphic functions is equivalent to the class of analytic functions, i.e. functions that can be locally given by convergent power series. The class of holomorphic functions on D is usually denoted by $\mathcal{H}(D)$ or $\mathcal{A}(D)$.

Holomorphic functions play a core role in complex analysis. Among plentiful results about holomorphic functions, Cauchy theorem and Cauchy integral formula are two fundamental ones.

Theorem 1.1.3 (**Cauchy Theorem**). *Let D be an bounded domain in $\mathbb{C}$ with a piecewise smooth boundary ∂D. Let $w(z)$ be a holomorphic function on $\overline{D}$. Then*

$$\int_{\partial D} w(z)\mathrm{d}z = 0.$$

Theorem 1.1.4 (**Cauchy Integral Formula**). *Let D be an open bounded domain in $\mathbb{C}$ with a piecewise smooth boundary ∂D. Let $w(z)$ be a holomorphic function on $\overline{D}$. Then for any $z \in D$*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{w(\zeta)}{\zeta - z}\mathrm{d}\zeta.$$

It is natural to consider analogously those functions which are related to $\frac{\partial}{\partial z}$, the conjugate operator of $\frac{\partial}{\partial \overline{z}}$. A function $w(z)$ that satisfies the condition $\frac{\partial w}{\partial z} = 0$ for every point in D is called an anti-holomorphic (or anti-analytic) function in D. We get an anti-holomorphic version of Cauchy-Riemann equations, namely

$$\frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}.$$

Anti-holomorphic functions are closely related to holomorphic functions. Via comparing the two versions of Cauchy-Riemann equations, one can show that $w(z)$ is anti-holomorphic in D if and only if $w(\overline{z})$ is holomorphic in D^c, where D^c is the conjugate domain of D, i.e. the reflection of D at the real axis. Any anti-holomorphic function can be obtained in this manner from a holomorphic function, and vice versa. Since a holomorphic function can be expanded locally as a power series in the variable z, an anti-holomorphic function therefore can be expanded as a power series in $\overline{z}$ in a neighborhood of each point in its domain. We therefore have a criterion for anti-holomorphic functions that is, a function $w(z)$ is anti-holomorphic in D if and only if $\overline{w(z)}$ is holomorphic in D.

The Cauchy integral formula can be generalized for the functions with continuous first-order derivatives. It results in the Cauchy-Pompeiu formulas, which derive from Pompeiu's work, see e.g. [27], Section I.4.1.

Theorem 1.1.5 (**Cauchy-Pompeiu Formulas**). *Let D be a regular domain. Let $w(z) \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then for any $z \in D$ we have the representation formulas:*

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z}, \tag{1.1}$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{\pi}\int_{D} w_{\zeta}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\overline{\zeta} - \overline{z}}, \tag{1.2}$$

where $\mathrm{d}\sigma_\zeta = \mathrm{d}\xi\mathrm{d}\eta$ denotes the area element with respect to the variable $\zeta = \xi + i\eta$.

Remark 1.1.6. Gauss theorems imply that

$$\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z} = 0, \tag{1.3}$$

$$-\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\overline{\zeta} - \overline{z}} = 0, \tag{1.4}$$

if $z \notin D$.

We call $\frac{1}{\zeta - z}$ the kernel function of $\partial_{\overline{z}}$ and $\frac{1}{\overline{\zeta} - \overline{z}}$ the kernel function of ∂_z.

Applying Cauchy-Pompeiu formulas to $w_z(z)$ and $w_{\overline{z}}(z)$ produces four representation formulas of second order for $w(z) \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$. They are

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\mathrm{d}\zeta \\ &+ \frac{1}{\pi}\int_D w_{\overline{\zeta}\overline{\zeta}}(\zeta)\frac{\overline{\zeta} - \overline{z}}{\zeta - z}\mathrm{d}\sigma_\zeta, \end{aligned} \tag{1.5}$$

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\log|\zeta - z|^2\mathrm{d}\overline{\zeta} \\ &+ \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2\mathrm{d}\sigma_\zeta, \end{aligned} \tag{1.6}$$

$$\begin{aligned} w(z) =& -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z}} + \frac{1}{2\pi i}\int_{\partial D} w_{\zeta}(\zeta)\frac{\zeta - z}{\overline{\zeta} - \overline{z}}\mathrm{d}\overline{\zeta} \\ &+ \frac{1}{\pi}\int_D w_{\zeta\zeta}(\zeta)\frac{\zeta - z}{\overline{\zeta} - \overline{z}}\mathrm{d}\sigma_\zeta, \end{aligned} \tag{1.7}$$

$$\begin{aligned} w(z) =& -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z}} - \frac{1}{2\pi i}\int_{\partial D} w_{\zeta}(\zeta)\log|\zeta - z|^2\mathrm{d}\zeta \\ &+ \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2\mathrm{d}\sigma_\zeta. \end{aligned} \tag{1.8}$$

From these second-order representation formulas we obtain the kernel functions $\frac{\overline{\zeta} - \overline{z}}{\zeta - z}$, $\log|\zeta - z|^2$ and $\frac{\zeta - z}{\overline{\zeta} - \overline{z}}$ for the operators $\partial_{\overline{z}}^2$, $\partial_z\partial_{\overline{z}}$ and ∂_z^2 respectively.

Via applying the iterating procedure of Cauchy-Pompeiu formulas, higher order representation formulas of Cauchy-Pompeiu type can be developed. Kernel functions with respect to high order differential operators are thus obtained, see [8].

Among the second-order differential operators ∂_z^2, $\partial_z\partial_{\overline{z}}$ and $\partial_{\overline{z}}^2$, we are mostly interested in $\partial_z\partial_{\overline{z}}$ in this thesis. Note that

$$4\partial_z\partial_{\overline{z}} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

is just the 2-dimensional Laplacian operator. Therefore, $\partial_z\partial_{\overline{z}}$ is also called the complex Laplacian operator.

We recall that a real-valued function $u(x, y)$ is harmonic in D if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for any $x + iy \in D$. Intrinsic connection between harmonic functions and holomorphic functions is well known in complex analysis. The real part and imaginary part of any holomorphic function are harmonic functions. For every harmonic function $u(x, y)$, there exists a conjugate harmonic function $v(x, y)$ such that $u(x, y) + iv(x, y)$ is a holomorphic function.

Here we recall two basic properties of harmonic functions.

Theorem 1.1.7 (Mean Value Theorem). *If $u : D \longrightarrow \mathbb{R}$ is a harmonic function and $\overline{B}(a, r)$ is a closed disk contained in D, then*

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta}) \mathrm{d}\theta.$$

Mean value property is a key property of harmonic functions. The converse of Mean Value Theorem is also true, namely, if $u : D \longrightarrow \mathbb{R}$ is a continuous function which has the mean value property then u is harmonic in D.

Theorem 1.1.8 (Maximum Principle). *Suppose that $u : D \longrightarrow \mathbb{R}$ is a harmonic function. If there is a point $z_0 \in D$ such that $u(z_0) \geq u(z)$ for all z in a neighborhood of z_0, then u is a constant function.*

We also call a complex-valued function $w(z)$ complex harmonic, or harmonic in short, if $w_{z\overline{z}} = 0$. Obviously, a complex harmonic function is a complex-valued funtion with harmonic real and imaginary parts.

Formulas (1.6) and (1.8) show two integral representation formulas for $\partial_z \partial_{\overline{z}}$. Hereafter two modified versions are deduced. They are developed to deal with Dirichlet and Neumann boundary value problems for Laplacian operator, see e.g. [7].

Employing the outward normal derivative ∂_{ν_ζ} and the arc length parameter s for the boundary curve ∂D, we have

$$\partial_{\nu_\zeta} \mathrm{d}s = -i\partial_\zeta \mathrm{d}\zeta + i\partial_{\overline{\zeta}} \mathrm{d}\overline{\zeta} \quad \text{for} \quad \zeta = \zeta(s) \in \partial D.$$

Adding up (1.6) and (1.8) gives

$$\begin{aligned} w(z) =& \frac{1}{4\pi} \int_{\partial D} w(\zeta) \partial_{\nu_\zeta} \log|\zeta - z|^2 \mathrm{d}s_\zeta - \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_\zeta} w(\zeta) \log|\zeta - z|^2 \mathrm{d}s_\zeta \\ &+ \frac{1}{\pi} \int_D w_{\zeta\overline{\zeta}}(\zeta) \log|\zeta - z|^2 \mathrm{d}\sigma_\zeta. \end{aligned} \tag{1.9}$$

Suppose that $h(z, \zeta)$ is a harmonic function in the variable ζ for every $z \in D$. Via Gauss theorem, we obtain that

$$\begin{aligned} \frac{1}{\pi} \int_D w_{\zeta\overline{\zeta}}(\zeta) h(z, \zeta) \mathrm{d}\sigma_\zeta &= \frac{1}{\pi} \int_D \left[\partial_\zeta \big(w_{\overline{\zeta}}(\zeta) h(z, \zeta)\big) - \partial_{\overline{\zeta}} \big(w(\zeta) h_\zeta(z, \zeta)\big)\right] \mathrm{d}\sigma_\zeta \\ &= -\frac{1}{2\pi i} \int_{\partial D} w_{\overline{\zeta}}(\zeta) h(z, \zeta) \mathrm{d}\overline{\zeta} + w(\zeta) h_\zeta(z, \zeta) \mathrm{d}\zeta, \end{aligned}$$

and a parallel version

$$\frac{1}{\pi}\int_D w_{\zeta\bar{\zeta}}(\zeta)h(z,\zeta)\mathrm{d}\sigma_\zeta = \frac{1}{\pi}\int_D \left[\partial_{\bar{\zeta}}\big(w_\zeta(\zeta)h(z,\zeta)\big) - \partial_\zeta\big(w(\zeta)h_{\bar{\zeta}}(z,\zeta)\big)\right]\mathrm{d}\sigma_\zeta$$
$$= \frac{1}{2\pi i}\int_{\partial D} w_\zeta(\zeta)h(z,\zeta)\mathrm{d}\zeta + w(\zeta)h_{\bar{\zeta}}(z,\zeta)\mathrm{d}\bar{\zeta}.$$

Adding these two formulas produces

$$\frac{1}{4\pi}\int_{\partial D}\big(w(\zeta)\partial_{\nu_\zeta}h(z,\zeta) - \partial_{\nu_\zeta}w(\zeta)h(z,\zeta)\big)\mathrm{d}s_\zeta + \frac{1}{\pi}\int_D w_{\zeta\bar{\zeta}}(\zeta)h(z,\zeta)\mathrm{d}\sigma_\zeta = 0. \tag{1.10}$$

Then from formulas (1.9) and (1.10) we obtain that

$$\begin{aligned} w(z) =& \frac{1}{4\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta}\big(\log|\zeta - z|^2 + h(z,\zeta)\big)\mathrm{d}s_\zeta \\ &- \frac{1}{4\pi}\int_{\partial D}\partial_{\nu_\zeta}w(\zeta)\big(\log|\zeta - z|^2 + h(z,\zeta)\big)\mathrm{d}s_\zeta \\ &+ \frac{1}{\pi}\int_D w_{\zeta\bar{\zeta}}(\zeta)\big(\log|\zeta - z|^2 + h(z,\zeta)\big)\mathrm{d}\sigma_\zeta \end{aligned} \tag{1.11}$$

for every $z \in D$ and for any harmonic function $h(z,\zeta)$ in variable ζ.

Given a harmonic function $h(z,\zeta)$,

$$\frac{1}{\pi}(\log|\zeta - z|^2 + h(z,\zeta))$$

is a fundamental solution for the complex Laplacian operator $\partial_z\partial_{\bar{z}}$. The choice of $h(z,\zeta)$ can be adjusted to meet some boundary conditions. Hereafter we introduce two types of adjustments.

Let D be a domain in the complex plane.

Definition 1.1.9. A real-valued function $G_1(z,\zeta), z,\zeta \in D, z \neq \zeta$, is called the **harmonic Green function** of D, if for any $z \in D$ it satisfies the properties:

(G1) $G_1(z,\zeta)$ is harmonic for any $\zeta \in D \setminus \{z\}$;

(G2) $G_1(z,\zeta) + \log|\zeta - z|^2$ is harmonic for any $\zeta \in D$;

(G3) $\lim\limits_{\zeta \to \partial D} G_1(z,\zeta) = 0$.

Remark 1.1.10. Note that this definition has a slight difference with the classical definition of Green functions. We call $G(z,\zeta)$ the Green function of D if $G(z,\zeta)$ satisfies (G1) , (G3) and the condition that $G(z,\zeta) + \log|\zeta - z|$ is harmonic instead of (G2). The relation $G_1(z,\zeta) = 2G(z,\zeta)$ holds. Also note that $-\frac{1}{\pi}G_1(z,\zeta)$ is a fundamental solution for $\partial_z\partial_{\bar{z}}$.

Theorem 1.1.11 (See e.g. [4, Thm. 9])**.** *The harmonic Green function $G_1(z,\zeta)$ of a domain D satisfies the additional properties:*

(i) $G_1(z,\zeta) > 0$, *for* $z, \zeta \in D$*;*

(ii) $G_1(z,\zeta)$ *is symmetric, i.e.* $G_1(z,\zeta) = G_1(\zeta, z)$*;*

(iii) $G_1(z,\zeta)$ *is unique if it exists.*

Let D be a regular domain in $\mathbb{C}$, $G_1(z,\zeta)$ its harmonic Green function, and $w(z) \in C^2(D,\mathbb{C}) \cap C(\overline{D},\mathbb{C})$. Via formula (1.11) we obtain a modified version of the second order representation formulas of Cauchy-Pompeiu type for $w(z)$, that is the **Green representation formula:**

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} w(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_D w_{\zeta\bar{\zeta}}(\zeta) G_1(z,\zeta)\mathrm{d}\sigma_\zeta. \tag{1.12}$$

Harmonic Green functions play an essential role in solving the Dirichlet boundary value problem for the Poisson equation. The following result shows an important property of harmonic Green functions, see e.g. [25, Thm. I. 21].

Theorem 1.1.12. *Let* D *be a bounded domain with piecewise smooth boundary* C *and* $G_1(z,\zeta)$ *its harmonic Green function; let* $\gamma \in C(D;\mathbb{C})$*. Then*

$$u(z) := -\frac{1}{4\pi}\int_C \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}s_\zeta \tag{1.13}$$

is a harmonic function on D*, where* ν *denotes the outward normal vector on* C *and* s *the arc length parameter of* C*. If* z_0 *is a smooth point in* C*, then holds the boundary property*

$$\lim_{z\to z_0} u(z) = \gamma(z_0). \tag{1.14}$$

Another important property of Green functions is that they are conformally invariant, i.e. conformal mappings preserve Green functions.

Theorem 1.1.13 ([16, Thm. 10.5.3])**.** *Let* G *and* Ω *be regions such that there is a one-to-one analytic function* $w = f(z)$ *of* D *onto* Ω*; let* $a \in G$ *and* $\alpha = f(a)$*. If* $g(z,a)$ *and* $\gamma(w,\alpha)$ *are the Green functions for* D *and* Ω *with singularities* a *and* α *respectively, then* $g(z,a) = \gamma(f(z), f(a))$*.*

Although the Green functions mentioned in the above theorem are in the classical sense as shown in Remark 1.1.10, this conclusion also works for harmonic Green functions defined by Definition 1.1.9. We usually consider conformal mappings from a domain onto the unit disk $\mathbb{D}$, since the harmonic Green function for $\mathbb{D}$ is well known. In this situation, we use the following result to show that harmonic Green functions are conformally invariant.

Theorem 1.1.14 ([4, Thm. 9])**.** *If* ϕ *is a conformal mapping from* D *onto the unit disk* $\mathbb{D}$*, then the harmonic Green function of* D *is*

$$G_1(z,\zeta) = \log\left|\frac{1-\overline{\phi(z)}\phi(\zeta)}{\phi(\zeta)-\phi(z)}\right|^2.$$

The existence of conformal mappings from a simply connected domain onto the unit disk is guaranteed by the famous Riemann mapping theorem. See e.g. [16, Thm. 7.4.2].

Theorem 1.1.15 (**Riemann Mapping Theorem**). *Let D be a simply connected domain which is not the whole plane and let $a \in D$. Then there exists a unique analytic function $f : D \longrightarrow \mathbb{D}$ having the properties:*

(i) $f(a) = 0$ *and* $f'(a) \neq 0$;

(ii) f *is bijective.*

Conformal mappings could be used to obtain Green functions for particular domains. However, it is generally difficult or too complicated to construct a conformal mapping from a given domain onto the unit disc. Obtaining harmonic Green functions by conformal mapping method is therefore limited. Thereby we need to develop new methods to compute the explicit expressions of Green functions. We will explore this topic in Chapter 3.

Next part is about harmonic Neumann functions, which provide another type of fundamental solutions for $\partial_z \partial_{\overline{z}}$.

Definition 1.1.16. A real-valued function $N_1(z, \zeta)$, $z, \zeta \in \overline{D}, z \neq \zeta$, is called a **harmonic Neumann function** of D, if for any $z \in D$ it has the properties:

(N1) $N_1(z, \zeta)$ is harmonic for $\zeta \in D \setminus \{z\}$ and continuously differentiable for $\zeta \in \overline{D} \setminus \{z\}$;

(N2) $N_1(z, \zeta) + \log |\zeta - z|^2$ is harmonic for $\zeta \in D$;

(N3) For $\zeta = \zeta(s) \in \partial D$, the density function $\delta(s) := \partial_{\nu_\zeta} N_1(z, \zeta)$ is a real-valued, piecewise constant function of s and has finite mass $\int_{\partial D} \delta(s) \mathrm{d}s$, where ∂_{ν_ζ} denotes the outward normal derivative on ∂D and s is the arc length parameter for ∂D.

Remark 1.1.17. Harmonic Neumann functions are also called harmonic Green functions of the second kind. Note that $-\frac{1}{\pi} N_1(z, \zeta)$ is a fundamental solution for $\partial_z \partial_{\overline{z}}$. For a domain, the harmonic Neumann function is not uniquely determined by the conditions (N1), (N2) and (N3). There might exist harmonic Neumann functions with different density functions. For a fixed density function, two harmonic Neumann functions differ by a constant. In some circumstance, we thus require an extra condition to obtain a unique harmonic Neumann function for the domain, that is

(N4) $\int_{\partial D} \delta(s) N_1(z, \zeta) \mathrm{d}s_\zeta = 0$ (normalization condition).

From formula (1.11) we obtain the **Neumann representation formula**:

$$\begin{aligned} w(z) = &- \frac{1}{4\pi} \int_{\partial D} w(\zeta) \partial_{\nu_\zeta} N_1(z, \zeta) \mathrm{d}s_\zeta + \frac{1}{4\pi} \int_{\partial D} \partial_{\nu_\zeta} w(\zeta) N_1(z, \zeta) \mathrm{d}s_\zeta \\ &- \frac{1}{\pi} \int_D w_{\zeta\bar{\zeta}}(\zeta) N_1(z, \zeta) \mathrm{d}\sigma_\zeta. \end{aligned} \tag{1.15}$$

Harmonic Neumann functions provide fundamental solutions for $\partial_z \partial_{\overline{z}}$ and help to solve Neumann boundary value problems for the Poisson equations. More discussions on harmonic Neumann functions will be carried on in the later chapters.

1.2 Some integral operators

We start with the notion of Hölder continuity.

Definition 1.2.1. A function f of one real or complex variable z is said to satisfy a Hölder condition or to be Hölder continuous in D if there exists $H > 0$ and $0 < \alpha \leq 1$ such that

$$|f(z_1) - f(z_2)| \leq H|z_1 - z_2|^\alpha$$

for all $z_1, z_2 \in D$. Particularly, in the case of $\alpha = 1$, f is called Lipschitz continuous in D.

For $\alpha \in (0, 1]$, let $H_\alpha(f) = H(f; D, \alpha)$ denote the infimum of those constants H which satisfy the inequality in the definition. It is obvious that

$$H_\alpha(f) = H(f; D, \alpha) = \sup_{z_1, z_2 \in D} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}.$$

The set of Hölder continuous functions with respect to α on D is denoted by $H^\alpha(D)$. Specifically, $H^\alpha(D; \mathbb{C})$ means the class of complex-valued Hölder continuous functions in D, and $H^\alpha(D; \mathbb{R})$ the class of real-valued ones. Obviously, if D is a bounded set, then a Hölder continuity with exponent α implies Hölder continuity with any exponent $\beta \leq \alpha$, i.e. $H^\alpha(D) \subset H^\beta(D)$ for any $\beta \leq \alpha$.

Let Γ be a positively oriented, rectifiable curve in the complex plane and φ integrable along Γ. Then

$$\phi(z) := \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z} \tag{1.16}$$

is an analytic function in $\mathbb{C}_\infty \backslash \Gamma$ vanishing at ∞, where $\mathbb{C}_\infty$ denotes the extended complex plane $\mathbb{C} \cup \{\infty\}$.

In general ϕ does not exists for points in Γ. We need the notion of Cauchy principal value of the integral $\phi(z)$ when z is in Γ.

Definition 1.2.2. Let Γ be a smooth curve in $\mathbb{C}$. For a fixed point $c \in \Gamma$, let $\varphi(\cdot; c)$ be an integrable function on $\Gamma \setminus \{c\}$ having a singularity at $\zeta = c$. Denote

$$\Gamma_\epsilon := \Gamma \setminus \{\zeta : |\zeta - c| < \epsilon\}$$

for $\epsilon > 0$. If

$$\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} \varphi(\zeta; c) \mathrm{d}\zeta$$

exists, then this value is called the Cauchy principal value of the singular integral, written as

$$C.P. \int_\Gamma \varphi(\zeta; c) \mathrm{d}\zeta,$$

or shortly as

$$\int_\Gamma \varphi(\zeta; c) \mathrm{d}\zeta.$$

Similarly, if $D \subset \mathbb{C}$ is a domain and $\varphi(\cdot; c)$ a function in $D \setminus \{c\}$ for some point $c \in D$ such that

$$\int_{D_\epsilon} \varphi(z; c) \mathrm{d}\sigma_z, \quad D_\epsilon := D \setminus \{z : |z - c| < \epsilon\},$$

exists for any small enough positive number ϵ, then

$$\int_D \varphi(z; c) \mathrm{d}\sigma_z := \lim_{\epsilon \to 0} \int_{D_\epsilon} \varphi(z; c) \mathrm{d}\sigma_z$$

is called the Cauchy principal value of the singular integral if the limits exists.

Theorem 1.2.3 ([4, Thm. 1])**.** *Let Γ be a simply closed piecewise smooth curve in $\mathbb{C}$ and $\varphi \in H^\alpha(\Gamma)$, then*

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z}$$

exists as Cauchy principle integral on Γ.

Theorem 1.2.4 (Plemelj-Sokhotski). *Let Γ be a smooth simply closed curve, D^+ the bounded domain with $\partial D^+ = \Gamma$, and $D^- = \mathbb{C}_\infty \setminus (D^+ \cup \Gamma)$. If $\varphi \in H^\alpha(\Gamma)$, then*

$$\phi(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(\zeta) \frac{\mathrm{d}\zeta}{\zeta - z}$$

has boundary values

$$\phi^+(\tau) := \lim_{\substack{z \to \tau \\ z \in D^+}} \phi(z), \quad \phi^-(\tau) := \lim_{\substack{z \to \tau \\ z \in D^-}} \phi(z)$$

for $\tau \in \Gamma$. Moreover,

$$\phi^+(\tau) = \frac{1}{2}\varphi(\tau) + \phi(\tau), \quad \phi^-(\tau) = -\frac{1}{2}\varphi(\tau) + \phi(\tau), \tag{1.17}$$

where $\phi(\tau)$ is understood as Cauchy principal value.

Theorem 1.2.5 (Plemelj-Privalov). *$\phi^+(\tau), \phi^-(\tau) \in H^\alpha(\Gamma)$ if $\varphi(\tau) \in H^\alpha(\Gamma)$.*

For proofs of the above two theorems, see e.g. Thm. 1.4 and Thm. 1.5 in [4]. The result of Plemelj-Sokhotski can be generalized to domains with piecewise smooth curves.

Theorem 1.2.6 (See e.g. [20, Thm. 2.5.1])**.** *Let Γ be positively oriented, piecewise smooth, simply closed curve and $\varphi(\tau) \in H(\Gamma)$, then, for $\tau \in \Gamma$, $\phi(z)$ has boundary values $\phi^+(\tau)$ and $\phi^-(\tau)$, and the Cauchy principal value $\phi(\tau)$ exists . Moreover,*

$$\phi^+(\tau) = \left(1 - \frac{\theta_\tau}{2}\right) \varphi(\tau) + \phi(\tau), \quad \phi^-(\tau) = -\frac{\theta_\tau}{2} \varphi(\tau) + \phi(\tau), \tag{1.18}$$

where θ_τ is the angle spanned by the two one-sided tangents of Γ at τ towards the positive side of Γ.

Let $C_0^1(D;\mathbb{C})$ denote the class of complex-valued functions in D which are continuously differentiable and have compact support in D. Let $L_p(D;\mathbb{C})$ denote the class of complex-valued functions in D which have finite L^p-norm.

Definition 1.2.7. Let $f(z), F(z) \in L_1(D;\mathbb{C})$. $f(z)$ is called generalized (distributional) derivative of $F(z)$ with respect to $\partial_{\overline{z}}$ if for all $\varphi \in C_0^1(D;\mathbb{C})$

$$\int_D F(z)\varphi_{\overline{z}}\mathrm{d}\sigma + \int_D f(z)\varphi(z)\mathrm{d}\sigma = 0.$$

This derivative is denoted by $\partial_{\overline{z}}F = f$.

The generalized derivatives with respect to z are defined in the same manner, we denote $\partial_z F = f$ if

$$\int_D F(z)\varphi_z\mathrm{d}\sigma + \int_D f(z)\varphi(z)\mathrm{d}\sigma = 0.$$

If a function is differentiable in the classical sense, then it is also differentiable in the distributional sense and both derivatives coincide (see e.g. [4], Page 82).

Definition 1.2.8. The integral operator given by

$$Tf := -\frac{1}{\pi}\int_D f(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z}, \quad z \in \mathbb{C},$$

for $f(z) \in L_1(D;\mathbb{C})$ is called the Pompeiu operator.

The Pompeiu operator is investigated in detail in Vekua's book [27], it is closely related to the theory of generalized analytic functions. Essential differential properties of T are stated bellow.

Theorem 1.2.9 (See e.g. [4, Thm. 26])**.** *If $f(z) \in L_1(D;\mathbb{C})$, then for all $\varphi \in C_0^1(D;\mathbb{C})$*

$$\int_D (Tf)(z)\varphi_{\overline{z}}(z)\mathrm{d}\sigma + \int_D f(z)\varphi(z)\mathrm{d}\sigma = 0.$$

This theorem implies that for $z \in D$

$$\partial_{\overline{z}}(Tf) = f \tag{1.19}$$

in distributional sense, hence the Pompeiu operator T can be viewed as the right inverse operator of $\partial_{\overline{z}}$. Besides, for $z \in \mathbb{C} \setminus \overline{D}$, Tf is analytic and its derivative is

$$\partial_z(Tf) = -\frac{1}{\pi}\int_D f(\zeta)\frac{\mathrm{d}\sigma_\zeta}{(\zeta - z)^2}.$$

Define the Π operator by

$$\Pi f := -\frac{1}{\pi}\int_D f(\zeta)\frac{\mathrm{d}\sigma_\zeta}{(\zeta - z)^2},\ z \in D,$$

where the integral on the right-hand side is understood as a Cauchy principal value. Π is a bounded linear operator on $H^\alpha(\overline{D})$. The derivative of Tf with respect to z is

$$\partial_z(Tf) = \Pi f \tag{1.20}$$

in distributional sense. This is a deep result from [15].

Next part is about Schwarz operators.

Given a harmonic function, its conjugate harmonic function can locally be calculated. Generally, the conjugate harmonic function is not a single-valued function, but it is true when the domain under consideration is simply connected. Analogously, we can consider the conjugate Green function. Let $G(z,\zeta)$ denote the Green function of domain D, see Remark 1.1.10. Denote $z = x + iy$. Let $H(z,\zeta)$ be the conjugate harmonic function for $G(z,\zeta)$ with respect to the variable z. Locally, namely in a neighborhood of a point $z_0 \in D$, $H(z,\zeta)$ is determined via the Cauchy-Riemann equations, given by

$$H(z,\zeta) = \int_{z_0}^{z} \left(-\frac{\partial G}{\partial y}\mathrm{d}x + \frac{\partial G}{\partial x}\mathrm{d}y \right) + \text{const.}$$

The function $H(z,\zeta)$ is single-valued when D is simply connected. The function

$$M(z,\zeta) := G(z,\zeta) + iH(z,\zeta)$$

is called the complex Green function for the domain D. This term was introduced by S. G. Mikhlin [21]. It is analytic in z everywhere except at the point $z = \zeta$ where it has a logarithmic singularity. For more details see [18, p. 209] and [4, p. 32].

For any function $u(x,y)$ which is harmonic in D and continuous in $\overline{D}$,

$$u(x,y) = \frac{1}{2\pi} \int\limits_{\partial D} \frac{\partial G(z,\zeta)}{\partial \boldsymbol{n}_\zeta} u(\zeta)\mathrm{d}s_\zeta,\ z \in D, \tag{1.21}$$

where s is the arc length parameter and $\boldsymbol{n}$ the interior normal vector on ∂D. A conjugate harmonic function to $u(x,y)$ is

$$v(x,y) = \frac{1}{2\pi} \int\limits_{\partial D} \frac{\partial H(z,\zeta)}{\partial \boldsymbol{n}_\zeta} u(\zeta)\mathrm{d}s_\zeta,\ z \in D. \tag{1.22}$$

The function

$$w(z) := u(x,y) + iv(x,y) = \frac{1}{2\pi} \int\limits_{\partial D} \frac{\partial M(z,\zeta)}{\partial \boldsymbol{n}_\zeta} u(\zeta)\mathrm{d}s_\zeta,\ z \in D, \tag{1.23}$$

is analytic in D.

Definition 1.2.10. An operator

$$S : C(\partial D; \mathbb{R}) \longrightarrow \mathcal{A}(D) \cap C(\overline{D}; \mathbb{C})$$

from the space of real-valued continuous functions on ∂D into the space of analytic functions in D which are also continuous on the closure $\overline{D}$ satisfying

$$\mathrm{Re}(Sf) = f \text{ on } \partial D$$

is called the Schwarz operator.

If D has a Green function, then S is given by

$$(Sf)(z) = \frac{1}{2\pi} \int\limits_{\partial D} \frac{\partial M(z,\zeta)}{\partial \boldsymbol{n}_\zeta} f(\zeta) \mathrm{d}s_\zeta. \tag{1.24}$$

Moreover, it is clear that S is determined only up to an additive imaginary constant. This constant can be fixed by requiring

$$\mathrm{Im}((Sf)(z_0)) = 0$$

for some given point $z_0 \in D$.

Chapter 2

Generalized Circle Reflections

In the extended complex plane, reflections through straight lines and inversions through circles are well-known. We call a reflection through a straight line or an inversion through a circle a generalized circle reflection, or shortly a circle reflection in many circumstances. In this chapter, we are going to review some preliminary knowledge about circle reflections and also develop some techniques for circle reflections. We will see the expressions of circle reflections in matrix form, and some results on consecutive circle reflections. These preparation serve for further discussions in the later chapters.

2.1 Generalized circles

Let $\mathbb{C}_\infty$ denote the extended complex plane. It is the union of the complex plane and an extra point, the point at infinity, denoted by the symbol ∞, namely $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. In $\mathbb{C}_\infty$, we adopt the convention that $0 \cdot \infty = 1$, $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. The extended complex plane can also be equivalently identified as the complex projective line $\mathbb{C}\mathrm{P}^1$, whose points can be expressed as homogeneous coordinates, e.g. $[z : w]$, where $z, w \in \mathbb{C}$, and z, w cannot both be 0. If $w \neq 0$, the point $[z : w]$ corresponds to the complex number z/w, while $[z : w]$ corresponds to ∞ if $w = 0$. Employing this model for $\mathbb{C}_\infty$ can bring us convenience for discussion in some situations.

Every straight line l in the complex plane can be expressed as a subset in $\mathbb{C}$ of the form

$$\{z \in \mathbb{C} \,\big|\, \bar{b}z + b\bar{z} + c = 0, b \in \mathbb{C} \setminus \{0\}, c \in \mathbb{R}\}.$$

The straight line l can be embedded into the extended complex plane. It corresponds to $l \cup \{\infty\}$ in $\mathbb{C}_\infty$. We call $l \cup \{\infty\}$ an extended straight line.

A circle in the complex plane is determined by an equation

$$az\bar{z} + \bar{b}z + b\bar{z} + c = 0,$$

where $a \in \mathbb{R} \setminus \{0\}, c \in \mathbb{R}, b \in \mathbb{C}$ and $ac - b\bar{b} < 0$. By rewriting the equation as

$$\left| z + \frac{b}{a} \right| = \frac{\sqrt{b\bar{b} - ac}}{|a|},$$

we see that the center of the circle is $-\frac{b}{a}$ and the radius $\frac{\sqrt{b\bar{b}-ac}}{|a|}$.

We call an extended straight line or a circle in $\mathbb{C}$ a generalized circle in $\mathbb{C}_\infty$. There is a natural understanding of generalized circles on 2-sphere S^2, another equivalent model of $\mathbb{C}_\infty$. The usual stereographic projection maps S^2 minus the North Pole onto $\mathbb{C}$ and is naturally extended to a bijection $S^2 \longrightarrow \mathbb{C}_\infty$ (the North pole is mapped to ∞). Under the extended stereographic projection, circles on S^2 are mapped to generalized circles in $\mathbb{C}_\infty$. Specifically speaking, circles on S^2 that pass through the North Pole are mapped to extended lines in $\mathbb{C}_\infty$, and circles on S^2 that do not pass through the North Pole are mapped to circles in $\mathbb{C}$.

Actually, we have uniform expressions for generalized circles in $\mathbb{C}_\infty$ by using homogeneous coordinates. Every generalized circle is a subset in $\mathbb{C}_\infty$ of the form

$$\{[z:w] \in \mathbb{C}_\infty \big| \, az\overline{z} + \overline{b}z\overline{w} + b\overline{z}w + cw\overline{w} = 0, \, a, c \in \mathbb{R}, \, b \in \mathbb{C}, \, ac - b\overline{b} < 0\}.$$

If $a = 0$, it is an extended straight line; otherwise, it is a circle. We immediately see that a generalized circle in $\mathbb{C}_\infty$ is determined by a matrix

$$\begin{pmatrix} a & \overline{b} \\ b & c \end{pmatrix}, \text{ where } a, \; c \in \mathbb{R}, \; b \in \mathbb{C}, \text{ and } ac - b\overline{b} < 0,$$

a 2×2 Hermitian matrix with a negative determinant. We call it a matrix of the generalized circle. It is easy to check that the matrix of a generalized circle is unique up to a nonzero real scalar multiple. Let

$$\mathrm{H}^- := \{A \in \mathrm{GL}_2(\mathbb{C}) \,|\, A = A^*, \, \det(A) < 0\}$$

be the set of all the 2×2 Hermitian matrices with negative determinants, where A^* means the conjugate transpose of A. We define an equivalence relation, denoted by $\overset{\mathrm{H}^-}{\sim}$, on H^-:

$$A \overset{\mathrm{H}^-}{\sim} B \text{ if and only if } \exists \lambda \in \mathbb{R} \backslash \{0\} \text{ s.t. } A = \lambda B. \tag{2.1}$$

It is obvious that there exists a one-to-one correspondence between the collection of all generalized circles in $\mathbb{C}_\infty$ and the set of all the equivalence classes in H^-.

2.2 Reflections through generalized circles

The section is about reflections at generalized circles and their matrix forms.

Reflections at straight lines are basic and well-known. A reflection at a straight line in the plane maps a point to an image point that lies on the perpendicular line through the point, and these two points have equal distances from the straight line. We restate below this content in complex form. The reflection at the straight line

$$l := \{z \in \mathbb{C} \,|\, \overline{b}z + b\overline{z} + c = 0, b \in \mathbb{C} \setminus \{0\}, c \in \mathbb{R}\}$$

is a transformation in $\mathbb{C}_\infty$ defined by

$$R_l(z) = \begin{cases} -\frac{b\overline{z}+c}{\overline{b}}, & \text{if } z \in \mathbb{C}, \\ \infty, & \text{if } z = \infty. \end{cases}$$

An inversion at a circle is the parallel version of a reflection at a straight line. Specifically speaking, the inversion at the circle

$$C := \{z \in \mathbb{C} \,\big|\, az\overline{z} + \overline{b}z + b\overline{z} + c = 0, a \in \mathbb{R} \setminus \{0\}, c \in \mathbb{R},\, b \in \mathbb{C},\, ac - b\overline{b} < 0\}$$

is a mapping $I_C : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ defined by

$$I_C(z) = \begin{cases} -\frac{b\overline{z}+c}{a\overline{z}+\overline{b}}, & \text{if } z \in \mathbb{C} \setminus \{-\frac{b}{a}\}, \\ \infty, & \text{if } z = -\frac{b}{a}, \\ -\frac{b}{a}, & \text{if } z = \infty. \end{cases}$$

An inversion fixes the points in the circle, and switches the inside domain and outside domain of the circle. A point and its image lie on the same straight line through the center of the reflecting circle.

We can unify reflections at lines and inversions at circles by adopting the term reflections at generalized circles. Let A be the matrix of a generalized circle $\{[z : w] \in \mathbb{C}_\infty \big|\, az\overline{z} + \overline{b}z\overline{w} + b\overline{z}w + cw\overline{w} = 0\}$. A transformation on $\mathbb{C}_\infty$ given by

$$\begin{array}{rccc} R_A : & \mathbb{C}_\infty & \longrightarrow & \mathbb{C}_\infty, \\ & [z : w] & \mapsto & [-b\overline{z} - c\overline{w} : a\overline{z} + \overline{b}\overline{w}] \end{array}$$

is called the reflection at the generalized circle A. If $a = 0$, the generalized circle is an extended straight line, the transformation R_A is just the reflection at the straight line. If $a \neq 0$, the generalized circle is a circle in $\mathbb{C}$, the transformation R_A is an inversion at this circle.

For convenience, we employ some operations for homogeneous coordinates. The multiplication of a homogeneous coordinate and an invertible matrix is defined naturally as the usual matrix multiplication, namely,

$$[z : w] \begin{pmatrix} a & b \\ c & d \end{pmatrix} \stackrel{\text{def}}{=\!=} [az + cw : bz + dw].$$

The conjugate of a homogeneous coordinate $\overline{[z : w]}$ can be naturally defined by $[\overline{z} : \overline{w}]$.

With the help of above operations for homogeneous coordinates, a reflection at a generalized circle $A = \left(\begin{smallmatrix} a & \overline{b} \\ b & c \end{smallmatrix}\right)$, denoted by R_A, can be written as

$$R_A([z : w]) = [-b\overline{z} - c\overline{w} : a\overline{z} + \overline{b}\overline{w}] = \overline{[z : w]}AP, \tag{2.2}$$

where $P = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Such expression will bring us some convenience for further discussions.

We show below some fundamental results about reflections at generalized circles.

Lemma 2.2.1. *For any $A \in \mathrm{H}^-$, the reflection R_A is a self-inverse transformation, i.e. R_A^2 is the identity transformation in $\mathbb{C}_\infty$.*

Proof. Let $[z : w] \in \mathbb{C}_\infty$, $A = \left(\begin{smallmatrix} a & \bar{b} \\ b & c \end{smallmatrix}\right) \in \mathrm{H}^-$. Then

$$R_A^2([z : w]) = \overline{\overline{[z : w]AP}AP} = [z : w]AP\overline{AP}.$$

$$AP\overline{AP} = \begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} b\bar{b} - ac & 0 \\ 0 & b\bar{b} - ac \end{pmatrix}.$$

It follows that $R_A^2([z : w]) = [z : w]$. Therefore R_A^2 is the identity transformation in $\mathbb{C}_\infty$. □

Theorem 2.2.2. *Reflections in the extended complex plane map generalized circles onto generalized circles. More specifically, the reflection at the generalized circle A maps the generalized circle B onto the generalized circle $AB^{-1}A$.*

Proof. Let A and B be two matrices associated with two generalized circles. It means that $A, B \in \mathrm{H}^-$. Suppose $[z : w]$ is a point in the circle B and its image under the reflection at the circle A is $[u : v]$. Because $[z : w]$ satisfies the equality

$$[z : w]B \begin{bmatrix} \bar{z} \\ \bar{w} \end{bmatrix} = 0,$$

substituting $[z : w]$ by $\overline{[u : v]AP}$, where $P = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, gives

$$[u : v]AP\overline{B}\overline{P^t A^t} \begin{bmatrix} \bar{u} \\ \bar{v} \end{bmatrix} = 0.$$

It is easy to verify that $P\overline{B}\overline{P^t} \overset{\mathrm{H}^-}{\sim} B^{-1}$, thus

$$AP\overline{B}\overline{P^t A^t} \overset{\mathrm{H}^-}{\sim} AB^{-1}A^* \overset{\mathrm{H}^-}{\sim} AB^{-1}A.$$

It is easy to verify that $AB^{-1}A \in \mathrm{H}^-$ since $A, B \in \mathrm{H}^-$. It follows that $AB^{-1}A$ is a matrix corresponding to a generalized circle. So the point $[u : v]$ is in the generalized circle $AB^{-1}A$. Hence reflecting the circle B at the circle A gives the circle $AB^{-1}A$. □

Theorem 2.2.3 (Angle Theorem, [14, Thm. 5.1.5])**.** *A reflection through any generalized circle preserves the magnitude of angles between curves but reverses their direction.*

Theorems 2.2.2 and 2.2.3 give us fundamental properties of reflections. A reflection at a generalized circle maps generalized circles onto generalized circles and preserves the magnitude of angles.

2.3 Inversive transformations and inversive group

In this section we introduce a class of transformations generated by generalized circle reflections. It includes generalized circle reflections, the well-known Möbius transformations, and their compositions. We will see some connections between generalized circle reflections and Möbius transformations and also their algebraic properties. For convenience, we often use the term 'reflections' in short, instead of reflections at generalized circles in $\mathbb{C}_\infty$.

Definition 2.3.1. A transformation $T : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ is called an inversive transformation if it is a composition of some reflections.

From the definition we know that inversive transformations are those functions in $\mathbb{C}_\infty$ which are generated by reflections.

Since every reflection preserves the magnitude of angles and maps generalized circles onto generalized circles, the same is true for any composition of reflections. We therefore have the following result.

Theorem 2.3.2 ([14, Thm. 5.3.2])**.** *Inversive transformations preserve the magnitude of angles, and map generalized circles onto generalized circles.*

We consider a natural group structure for the set of all inversive transformations. It is obvious that a composition of two inversive transformations is still an inversive transformation. Every inversive transformation has an inverse since each reflection is a self-inverse transformation by Lemma 2.2.1. This lemma also implies that the identity mapping is an inversive transformation. Therefore we have the following conclusion.

Theorem 2.3.3 ([14, Thm. 5.3.3])**.** *The set of inversive transformations forms a group under the operation of compositions of transformations.*

We call this group the inversive group of $\mathbb{C}_\infty$. It includes all reflections and their compositions.

Let us review some knowledge about Möbius transformations.

Definition 2.3.4. A Möbius transformation is a function $M : \mathbb{C}_\infty \longrightarrow \mathbb{C}_\infty$ of the form

$$M(z) = \frac{az+b}{cz+d},$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$. We adopt the convention that

$$M(\infty) = \infty, \quad \text{if } c = 0,$$

$$M\left(-\frac{d}{c}\right) = \infty, \ M(\infty) = \frac{a}{c}, \quad \text{if } c \neq 0.$$

We can also use homogeneous coordinates to express Möbius transformations as

$$M([z:w]) = [z:w]\begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

where $\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{C})$ is called a matrix associated with M. A matrix in $\mathrm{GL}_2(\mathbb{C})$ determines a unique Möbius transformation, but the inverse is not true. Matrices associated with the same Möbius transformation can differ up to a nonzero scalar multiple in $\mathbb{C}^*$. In other words, the family of all Möbius transformations is identified with

$$\mathrm{PSL}_2(\mathbb{C}) := \mathrm{GL}_2(\mathbb{C})/\mathbb{C}^* = \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{Id}\}.$$

It is easy to check that

$$M_A \circ M_B = M_{BA} \quad \text{and} \quad M_A^{-1} = M_{A^{-1}}, \quad \forall A, B \in \mathrm{GL}_2(\mathbb{C}).$$

These facts induce the next result.

Proposition 2.3.5. *All Möbius transformations in $\mathbb{C}_\infty$ form a group under compositions of transformations. It is isomorphic to* $\mathrm{PSL}_2(\mathbb{C})$ *via $M_A \longrightarrow A^{-1}$.*

The set of all Möbius transformations is called the Möbius group.

The next part shows some connections between reflections and Möbius transformations.

Theorem 2.3.6. *A composition of two reflections is a Möbius transformation. More precisely, $R_B \circ R_A = M_{AB^{-1}}$ for $A, B \in \mathrm{H}^-$.*

Proof. Let R_A, R_B be two reflections at two generalized circles and $A, B \in \mathrm{H}^-$ be respective associated matrices. Then for every point $[z:w] \in \mathbb{C}_\infty$,

$$R_B \circ R_A([z:w]) = \overline{\overline{[z:w]AP}BP} = [z:w]AP\overline{BP}.$$

Since the matrices A, B and P are all invertible, it follows that

$$AP\overline{BP} \overset{\mathrm{H}^-}{\sim} AB^{-1} \in \mathrm{GL}_2(\mathbb{C}).$$

Note that the relation '$\overset{\mathrm{H}^-}{\sim}$' on H^- can be lifted to $\mathrm{GL}_2(\mathbb{C})$. Therefore $R_B \circ R_A = M_{AB^{-1}}$ is a Möbius transformation. $\square$

Corollary 2.3.7. *If two generalized circles A and B intersect at a right angle, then the corresponding reflections R_A and R_B are commutative, i.e.*

$$R_A \circ R_B = R_B \circ R_A.$$

Proof. Because generalized circles A and B intersect at a right angle and the magnitude of angles is preserved by reflection, so the image circle of B under the reflection at circle A must be B itself. Then we have $AB^{-1}A \overset{\mathrm{H}^-}{\sim} B$ via Theorem 2.2.2. It follows that $AB^{-1} \overset{\mathrm{H}^-}{\sim} BA^{-1}$. Therefore from Theorem 2.3.6 we know that

$$R_A \circ R_B = M_{BA^{-1}} = M_{AB^{-1}} = R_B \circ R_A.$$

□

It is natural to ask if the inverse proposition is true, i.e., if any Möbius transformation is a composition of two reflections. The answer is negative in general. For instance, the dilation $z \mapsto 2z$ can not be a composition of two reflections. Otherwise, there exist two matrices $A, B \in \mathrm{H}^-$ such that

$$AB^{-1} \overset{\mathrm{H}^-}{\sim} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

But we can verify that such A and B do not exist. Although not every Möbius transformation can be decomposed into two reflections, we still have a weaker conclusion.

Theorem 2.3.8 ([14, Thm. 5.3.4])**.** *Every Möbius transformation is a composition of reflections.*

Therefore Möbius mappings are inversive transformations and the Möbius group is thus a subgroup of the inversive group.

Theorem 2.3.9 ([14, Thm. 5.3.5])**.** *Möbius transformations preserve the magnitude and orientation of angles, and map generalized circles to generalized circles.*

From this theorem we know that Möbius transformations are conformal mappings on $\mathbb{C}_\infty$ which send generalized circles to generalized circles. Since Möbius transformations preserve the orientation of angles but reflections reverse the orientation, we could make the claim in Theorem 2.3.8 more precisely, that every Möbius transformation must be a composition of an even number of reflections.

2.4 Consecutive reflections

In this section we consider a family of generalized circles generated by reflecting two given generalized circles consecutively and investigate some properties of the family of relative reflections.

Let A_0 and A_1 be two generalized circles in $\mathbb{C}_\infty$. We operate consecutive reflections starting with the reflections at circle A_0 and at circle A_1. Let A_{k+1} be the image circle of A_{k-1} under the reflection at circle A_k; we also proceeds reflections in the opposite direction, namely, reflecting A_{k+1} at circle A_k generates the circle A_{k-1}. Then we obtain a family of consecutive generalized circles, denoted by $\{A_k \,|\, k \in \mathbb{Z}\}$. We adopt the convention that A_k not only represents a generalized circle but also a matrix associated with it.

Lemma 2.4.1. *Let $\{A_k \mid k \in \mathbb{Z}\}$ be a family of consecutive generalized circles generated by A_0 and A_1. Then for any $k \in \mathbb{Z}$,*

$$A_k = A_0(A_0^{-1}A_1)^k.$$

Proof. We apply induction on k to prove the conclusion. The formula holds for $k = 0, 1$. Since A_{k+1} is the image of A_{k-1} under the reflection at A_k, via Theorem 2.2.2 we see that

$$\begin{aligned} A_{k+1} &= A_k A_{k-1}^{-1} A_k \\ &= A_0(A_0^{-1}A_1)^k (A_0(A_0^{-1}A_1)^{k-1})^{-1} A_0(A_0^{-1}A_1)^k \\ &= A_0(A_0^{-1}A_1)^{k+1} \end{aligned}$$

holds for $k > 1$. For $k < 0$, the circle A_k is obtained by reflecting A_{k+2} at the circle A_{k+1}. Applying Theorem 2.2.2 again, we have

$$\begin{aligned} A_k &= A_{k+1} A_{k+2}^{-1} A_{k+1} \\ &= A_0(A_0^{-1}A_1)^{k+1} (A_0(A_0^{-1}A_1)^{k+2})^{-1} A_0(A_0^{-1}A_1)^{k+1} \\ &= A_0(A_0^{-1}A_1)^k \end{aligned}$$

for $k < 0$. So $A_k = A_0(A_0^{-1}A_1)^k$ holds for all $k \in \mathbb{Z}$. $\square$

Theorem 2.2.2 and Lemma 2.4.1 imply the next result.

Corollary 2.4.2. *Let $\{A_k \mid k \in \mathbb{Z}\}$ be a family of consecutive generalized circles generated by A_0 and A_1. Then $A_{k+l} = A_k A_{k-l}^{-1} A_k$ for any $k, l \in \mathbb{Z}$. It means that reflecting the generalized circle A_{k-l} at A_k gives the generalized circle A_{k+l}.*

Theorem 2.4.3. *Let $\{A_k \mid k \in \mathbb{Z}\}$ be a family of consecutive generalized circles generated by A_0 and A_1. Let R_{A_k} denote the reflection at the circle A_k, $k \in \mathbb{Z}$. Then $R_{A_k} \circ R_{A_j} \circ R_{A_i} = R_{A_{i-j+k}}$ for all $i, j, k \in \mathbb{Z}$.*

Proof. Let $[z : w] \in \mathbb{C}_\infty$. The transformation $R_{A_k} \circ R_{A_j} \circ R_{A_i}$ maps the point $[z : w]$ to the point

$$\overline{\overline{[z : w]}\overline{A_i P} A_j P \overline{A_k P}}.$$

Note that

$$\overline{A_i P} A_j P \overline{A_k P} \overset{\mathrm{H}^-}{\sim} \overline{A_i A_j^{-1} A_k P}.$$

Corollary 2.4.2 guarantees that

$$\begin{aligned} A_i A_j^{-1} A_k &= A_0(A_0 A_1)^i (A_0(A_0 A_1)^j)^{-1} A_0(A_0 A_1)^k \\ &= A_0(A_0 A_1)^{i-j+k} \\ &= A_{i-j+k}. \end{aligned}$$

Then we have

$$R_{A_k} \circ R_{A_j} \circ R_{A_i}([z : w]) = \overline{[z : w] A_{i-j+k} P}.$$

The transformation $R_{A_k} \circ R_{A_j} \circ R_{A_i}$ therefore coincides with the reflection $R_{A_{i-j+k}}$. $\square$

Theorem 2.4.3 ensures that a composition of three reflections from a family of consecutive reflections is still a reflection in the family.

Corollary 2.4.4. *Let $\{A_k \mid k \in \mathbb{Z}\}$ be a family of consecutive generalized circles generated by A_0 and A_1. Let R_{A_k} denote the reflection at the circle A_k and $\{z_k \mid k \in \mathbb{Z}\}$ a family of points in $\mathbb{C}_\infty$ such that $z_k = R_{A_k}(z_{k-1})$ for all $k \in \mathbb{Z}$. Then $R_{A_0}(z_k) = z_{-k-1}$ and $R_{A_1}(z_k) = z_{-k+1}$ for all $k \in \mathbb{Z}$.*

Proof. Theorem 2.4.3 implies that

$$R_{A_{-1}} \circ R_{A_{-2}} \circ \cdots \circ R_{A_{-k}} \circ R_{A_0} \circ R_{A_k} \circ \cdots \circ R_{A_2} \circ R_{A_1} = R_{A_0}.$$

Then we see that

$$\begin{aligned} R_{A_0}(z_k) &= R_{A_0} \circ R_{A_k} \cdots \circ R_{A_2} \circ R_{A_1}(z_0) \\ &= R_{A_{-k}} \cdots \circ R_{A_{-2}} \circ R_{A_{-1}} \circ R_{A_0}(z_0) \\ &= z_{-k-1}. \end{aligned}$$

The relation $R_{A_1}(z_k) = z_{-k+1}$ can be verified similarly. $\square$

Theorem 2.4.5. *Let $\{A_k \mid k \in \mathbb{Z}\}$ be a family of consecutive generalized circles generated by A_0 and A_1. Let R_{A_k} denote the reflection at the circle A_k. Then*

$$R_{A_l} \circ R_{A_k} = M_{(A_0 A_1^{-1})^{l-k}} = M^{l-k}_{A_0 A_1^{-1}}.$$

Proof. This theorem is an immediate result from Theorem 2.3.6 and Lemma 2.4.1. Just note that

$$A_k A_l^{-1} = A_0 (A_0^{-1} A_1)^k (A_0^{-1} A_1)^{-l} A_0^{-1} = (A_0 A_1^{-1})^{l-k}.$$

$\square$

The next two results follow from Theorem 2.4.3 and Theorem 2.4.5.

Corollary 2.4.6.

$$\prod_{i=0}^{n} R_{A_i} := R_{A_n} \circ \cdots \circ R_{A_1} \circ R_{A_0} = \begin{cases} M^{\frac{n+1}{2}}_{A_0 A_1^{-1}}, & n \text{ is odd,} \\ R_{A_{\frac{n}{2}}}, & n \text{ is even.} \end{cases}$$

Corollary 2.4.7. $R_{A_n} = M^n_{A_0 A_1^{-1}} \circ R_{A_0} = R_{A_0} \circ M^n_{A_1 A_0^{-1}}, \forall n \in \mathbb{Z}.$

We are interested in the group generated by R_{A_0} and R_{A_1}. The above two corollaries imply the next conclusion.

Corollary 2.4.8. *The inversive group generated by R_{A_0} and R_{A_1} is equal to*

$$\{M^n_{A_0 A_1^{-1}},\ M^n_{A_0 A_1^{-1}} \circ R_{A_0} \mid n \in \mathbb{Z}\}.$$

Chapter 3

Parqueting-Reflection Principle

The parqueting-reflection principle is introduced to solve several boundary value problems, e.g. Schwarz problems for the homogeneous and inhomogeneous Cauchy-Riemann equations, Dirichlet problems and Neumann problems for the Laplace and the Poisson equation, in particular domains whose boundaries are composed of circular arcs. Research works on the parqueting-reflection principle are mainly contributed by H. Begehr and his students. The parqueting-reflection principle has been verified feasible for many domains, for instance, discs, half-planes, disc sectors, strip domains, half-strip domains, cones, concentric rings, hyperbolic strips, et al., see e.g. [1, 5, 6, 9, 10, 11, 12, 13, 17, 19, 23, 26, 28].

Harmonic Green functions and harmonic Neumann functions play important roles in dealing with Dirichlet and Neumann boundary value problems for the Poisson equation. The parqueting-reflection principle provides ideas and procedures on how to construct the harmonic Green and Neumann functions for particular domains. If a domain can provide a parqueting of the extended complex plane via reflections, a point in the domain generates a family of reflection images corresponding to the parqueting. These reflection images are used to construct candidate functions for the harmonic Green and Neumann functions and verify that these candidates are the required ones for the specific domain. Although the parqueting-reflection principle has been verified successful in obtaining harmonic Green and Neumann functions for many particular domains, we are not satisfied with the achievement. There might be a hidden theory that makes the parqueting-reflection principle playing the role. We are interested in investigating a theory behind the parqueting-reflection principle and exploring new domains to which the principle can be applied.

In this chapter, we first introduce the notions of parqueting domains and parqueting-reflection domains. Then we prove that the parqueting reflection principle succeeds in giving harmonic Green and Neumann functions for finite parqueting-reflection domains. We also obtain some results on normal derivatives of harmonic Green and Neumann functions for finite parqueting-reflection domains.

3.1 Parqueting of $\mathbb{C}_\infty$ via circle reflections

We call a curve in $\mathbb{C}_\infty$ a circular arc if it is a segment of a generalized circle. From now on, we focus on domains in $\mathbb{C}_\infty$ whose boundaries are composed of a finite number of circular arcs. Particularly, we pay attention to circular polygons, which are simply connected domains bounded by a finite number of connected circular arcs. In this thesis, all half-planes and discs are viewed as circular polygons, they are just those domains bounded by one generalized circle. We also adopt the terms circular digons, circular triangles, circular rectangles, circular pentagons, and so on.

We have already discussed generalized circle reflections in the previous chapter. We also need to consider reflections at boundary arcs of domains. When we talk about a reflection at a boundary arc of a domain, it means the reflection at the circle defined by that arc.

Now we introduce the concept of parqueting of the extended complex plane provided by reflections.

Definition 3.1.1. Let D and D' be two domains in $\mathbb{C}_\infty$. We say that D' is **reflecting-congruent** to D if there exist a finite sequence of domains $(D_0, D_1, \cdots, D_n)$, $n \in \mathbb{N}$ such that $D_0 = D, D_n = D'$ and D_{k+1} is the image of D_k under the reflection through a boundary arc of D_k for $k = 0, \cdots, n-1$. Every domain is considered to be reflecting-congruent to itself.

From the discussions in Chapter 2, we know that a composition of an even number of reflections is a Möbius transformation, and a composition of an odd number of reflections is a composition of the complex conjugate and a Möbius transformation. If D' is reflecting-congruent to D through an even number of reflections, then there exists a Möbius transformation $\Phi : D \to D'$,

$$\Phi(z) = \frac{az+b}{cz+d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

If D' is reflecting-congruent to D through an odd number of reflections, then there exists an inversive transformation $\widehat{\Phi} : D \to D'$,

$$\widehat{\Phi}(z) = \frac{\alpha\overline{z}+\beta}{\gamma\overline{z}+\delta}, \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}).$$

Definition 3.1.2. Let D be a domain in $\mathbb{C}_\infty$ whose boundary is composed of a finite number of circular arcs. We say that D provides a parqueting of $\mathbb{C}_\infty$ via reflections if there exists a family of domains $\{D_i \,|\, i \in I\}$, where I is an index set, such that

- D_i is reflecting-congruent to D for all $i \in I$;
- $D_i \cap D_j = \emptyset$ for any two different indices $i, j \in I$;
- $\mathbb{C}_\infty = \bigcup_{i \in I} \overline{D_i}$.

Such a domain is called a **parqueting domain** in $\mathbb{C}_\infty$.

There are plenty of parqueting domains in $\mathbb{C}_\infty$. Every disc, half-plane or half-disk is a parqueting domain. It is easy to know that equilateral triangles, rectangles, hexagons, half-hexagons, strip domains, half-strip domains, hyperbolic strips, ring domains, and cones with an angle $\frac{2\pi}{n}$ for a positive integer n are all parqueting domains.

Definition 3.1.3. Let D be a domain in $\mathbb{C}_\infty$ whose boundary is composed of finitely many circular arcs, say $\partial D = \cup_{k\in I} C_k$, where I is a finite index set. Let R_k denote the circle reflection at C_k, $k \in I$. The group generated by $\{R_k\}_{k\in I}$ is called the inversive group of D, denoted by $\mathrm{Inv}(D) := \langle R_k | k \in I\rangle$.

Lemma 3.1.4. *Let* $\mathrm{M}(D) := \{T \in \mathrm{Inv}(D)|\ T$ *is a composition of an even number of reflections* $\}$. *Then* $\mathrm{M}(D)$ *is a normal subgroup of* $\mathrm{Inv}(D)$*, the index of* $\mathrm{M}(D)$ *in* $\mathrm{Inv}(D)$ *is 2.*

Proof. We define a mapping

$$\phi : \mathrm{Inv}(D) \longrightarrow \{1, -1\},$$
$$T \mapsto \begin{cases} 1, & T \in \mathrm{M}(D), \\ -1, & \text{otherwise.} \end{cases}$$

It is easy to check that ϕ is a surjective group homomorphism. Then $\mathrm{M}(D) = \ker(\phi)$ is a normal subgroup of $\mathrm{Inv}(D)$. The isomorphism

$$\mathrm{Inv}(D)/\mathrm{M}(D) \simeq \{-1, 1\}$$

implies that the index of $\mathrm{M}(D)$ in $\mathrm{Inv}(D)$ is 2. □

Remark 3.1.5. Every element in $\mathrm{M}(D)$ is a Möbius transformation, Every element in $\mathrm{Inv}(D) \setminus \mathrm{M}(D)$ is a composition of an odd number of reflections. There is a one-to-one correspondence between $\mathrm{M}(D)$ and $\mathrm{Inv}(D) \setminus \mathrm{M}(D)$. For every circular arc C of ∂D, its corresponding reflection R_C satisfies that

$$R_C \mathrm{M}(D) = \mathrm{M}(D) R_C = \mathrm{Inv}(D) \setminus \mathrm{M}(D).$$

Definition 3.1.6. We call D a **parqueting-reflection domain** if

- $\mathbb{C}_\infty = \bigcup\limits_{T\in \mathrm{Inv}(D)} \overline{T(D)}$;
- $T(D) \cap T'(D) = \emptyset$ for all T, $T' \in \mathrm{Inv}(D)$, $T \neq T'$.

Furthermore, we call D a finite parqueting-reflection domain if $\mathrm{Inv}(D)$ is a finite group, otherwise, it is called an infinite parqueting-reflection domain.

Remark 3.1.7. Suppose D is a parqueting-reflection domain. Let $z \in D$, $P := \{\Phi(z)|\Phi \in \mathrm{M}(D)\}$, and $\widehat{P} := \{\widehat{\Phi}(z)|\widehat{\Phi} \in \mathrm{Inv}(D) \setminus \mathrm{M}(D)\}$. $P \cup \widehat{P}$ is the set of reflection images generated by z corresponding to the parqueting. Let C be a circular arc of ∂D and R_C its reflection. R_C induces a one-to-one correspondence between P and $\widehat{P}$. If $z \in C$, then $\Phi(z) = (\Phi \circ R_C)(z)$ for all $\Phi \in \mathrm{M}(D)$.

There are many examples of parqueting-reflection domains. Disks, half-planes, half-disks and cones or disk sectors with angles $\frac{\pi}{n}$ for all positive integer n are finite parqueting-reflection domains. In Chapter 5, we will verify that circular digons with angles $\frac{\pi}{n}$ for all positive integer n are also finite parqueting-reflection domains. It is easy to see that equilateral triangles, rectangles, strip domains, half-strip domains, hyperbolic strips and ring domains are infinite parqueting-reflection domains. We will see an infinite parqueting-reflection circular rectangle in Chapter 6. Note that some parqueting domains are not parqueting-reflection domains. For instance, cones or disk sectors with angles $\frac{2\pi}{n}$ where n is a positive odd integer are not parqueting-reflection domains.

It is natural to ask which circular polygons are parqueting-reflection domains and what criterion can determine parqueting-reflection domains. These questions are still open.

3.2 Parqueting-reflection principle

Now we are ready to fully explain the parqueting-reflection principle. We show below how to construct harmonic Green functions and harmonic Neumann functions via the parqueting-reflection principle. We also prove that the parqueting-reflection principle succeeds in giving harmonic Green and Neumann functions for finite parqueting-parqueting domains.

Let D be a parqueting-reflection domain in the extended complex planes. Let I be an index set, $\{\Phi_i | i \in I\} = \mathrm{M}(D)$ the family of Möbius transformations in $\mathrm{Inv}(D)$ and $\{\widehat{\Phi}_i | i \in I\} = \mathrm{Inv}(D) \setminus \mathrm{M}(D)$ the family of orientation-reversing inversive transformations in $\mathrm{Inv}(D)$. Denote $D_i := \Phi_i(D)$, $\widehat{D}_i := \widehat{\Phi}_i(D)$, $i \in I$. Then D provides a parqueting of $\mathbb{C}_\infty$:

$$\mathbb{C}_\infty = \bigcup_{i \in I} (\overline{D_i} \cup \overline{\widehat{D}_i}).$$

Let $z \in D$. z generates two families of reflection images

$$P = \{z_i := \Phi_i(z) | i \in I\} \quad \text{and} \quad \widehat{P} = \{\widehat{z}_i := \widehat{\Phi}_i(z) | i \in I\}$$

with respect to the parqueting. We know z_i is a linear fractional function in z of the form

$$z_i = \Phi_i(z) = \frac{a_i z + b_i}{c_i z + d_i}, \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \tag{3.1}$$

and $\widehat{z}_i$ is a linear fractional function in $\overline{z}$ of the form

$$\widehat{z}_i = \widehat{\Phi}_i(z) = \frac{\alpha_i \overline{z} + \beta_i}{\gamma_i \overline{z} + \delta_i}, \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}). \tag{3.2}$$

It could happen that z_i or $\widehat{z}_i$ is ∞ when the corresponding denominator is 0 for some point $z \in D$. Suppose that the expressions of z_i and $\widehat{z}_i$ are given by the forms of (3.1) and (3.2) respectively. We denote the denominators of z_i and $\widehat{z}_i$ by

$$\mathrm{Den}(z_i) := c_i z + d_i \quad \text{and} \quad \mathrm{Den}(\widehat{z}_i) := \gamma_i \overline{z} + \delta_i$$

respectively. These notations will be involved in later discussions.

Suppose D is a finite parqueting-reflection domain and it provides a finite parqueting of $\mathbb{C}_\infty$, namely, $\mathbb{C}_\infty = \bigcup_{i\in I}(\overline{D_i} \cup \overline{\widehat{D}_i})$, where I is a finite index set. Take a point $z \in \overline{D}$. z generates two finite families of reflection images $\{z_i = \Phi_i(z) | i \in I\}$ and $\{\widehat{z}_i = \widehat{\Phi}_i(z) | i \in I\}$. From these reflection images we define two functions

$$\widetilde{F}(z,\zeta) := \prod_{i\in I} \frac{\zeta - \widehat{z}_i}{\zeta - z_i}, \tag{3.3}$$

$$\widetilde{Q}(z,\zeta) := \prod_{i\in I} (\zeta - \widehat{z}_i)(\zeta - z_i). \tag{3.4}$$

For a finite parqueting-reflection domain D, $\widetilde{F}(z,\zeta)$ is used to construct the harmonic Green function which satisfies the conditions (G1)-(G3) of Definition 1.1.9, and $\widetilde{Q}(z,\zeta)$ is used to construct a harmonic Neumann function which meets the conditions (N1)-(N3) of Definition 1.1.16.

Lemma 3.2.1. *For $\zeta \in D$ and $z \in \partial D$, $\widetilde{F}(z,\zeta) = 1$ holds.*

Proof. Suppose C is a boundary arc of ∂D and R_C the reflection at C. From Remark 3.1.5 we see that

$$\{\widehat{\Phi}_i | i \in I\} = \{\Phi_i \circ R_C | i \in I\}.$$

Note that $R_C(z) = z$ when $z \in C$. Via a proper ordering of $\{\widehat{z}_i = \widehat{\Phi}_i(z) | i \in I\}$, we can make sure that

$$\widehat{z}_i = \widehat{\Phi}_i(z) = \Phi_i \circ R_C(z) = \Phi_i(z) = z_i, \ \forall i \in I,$$

when $z \in C$. Then

$$\widetilde{F}(z,\zeta) = \prod_{i\in I} \frac{\zeta - \widehat{z}_i}{\zeta - z_i} = 1.$$

for $\zeta \in D$, $z \in C$. The above conclusion holds for an arbitrary boundary arc, therefore $\widetilde{F}(z,\zeta) = 1$ for $\zeta \in D$, $z \in \partial D$. □

Lemma 3.2.2. *Suppose that D is a finite parqueting-reflection domain with a connected boundary. Let $\mathbb{C}_\infty = \bigcup_{i\in I}(\overline{D_i} \cup \overline{\widehat{D}_i})$. Let $z_i \in \overline{D_i}$ and $\widehat{z}_i \in \overline{\widehat{D}_i}$ be the reflection images generated by a point z in $\overline{D}$ corresponding to the parqueting. Then $V(z) := \prod_{i\in I} \left|\frac{\mathrm{Den}(\widehat{z}_i)}{\mathrm{Den}(z_i)}\right|$ is constant on ∂D.*

Proof. The domain D provides a finite parqueting of the extended complex plane; I is a finite index set. Since ∂D consists of a finite number of boundary arcs, each domain D_i or $\widehat{D}_i$ contains the same number of boundary arcs. Let $A(D)$ denote the set of boundary arcs of all domains for the parqueting. $A(D)$ is a finite set. If $z \in \partial D$, then the reflection points z_i and $\widehat{z}_i$ are located on $A(D)$ for all $i \in I$.

We first show that $V(z)$ is piecewise constant on ∂D. Suppose that C is a boundary arc of ∂D. If $z \in C$, by reordering the domains $\{\widehat{D}_i\}_{i\in I}$, we can make sure that $\widehat{z_i} = z_i \in A_i$, where A_i is a boundary arc in $A(D)$. Suppose the equation for A_i is

$$\alpha_i z\overline{z} + \overline{\beta_i} z + \beta_i \overline{z} + \gamma_i = 0.$$

We have

$$\widehat{z_i} = -\frac{\beta_i \overline{z_i} + \gamma_i}{\alpha_i \overline{z_i} + \overline{\beta_i}}$$

for $\widehat{z_i} = z_i \in A_i$. Suppose

$$z_i = \frac{a_i z + b_i}{c_i z + d_i}.$$

Then

$$\widehat{z_i} = -\frac{\beta_i \overline{(a_i z + b_i)} + \gamma_i \overline{(c_i z + d_i)}}{\alpha_i \overline{(a_i z + b_i)} + \overline{\beta_i}\,\overline{(c_i z + d_i)}}.$$

We see that

$$|\mathrm{Den}(\widehat{z_i})| = \left|\alpha_i \overline{(a_i z + b_i)} + \overline{\beta_i}\,\overline{(c_i z + d_i)}\right| = |\mathrm{Den}(z_i)||\alpha_i \overline{z_i} + \overline{\beta_i}|.$$

It follows that

$$V(z) = \prod_{i\in I} \left|\frac{\mathrm{Den}(\widehat{z_i})}{\mathrm{Den}(z_i)}\right| = \prod_{i\in I} \frac{|\mathrm{Den}(z_i)||\alpha_i \overline{z_i} + \overline{\beta_i}|}{|\mathrm{Den}(z_i)|} = \prod_{i\in I} |\alpha_i \overline{z_i} + \overline{\beta_i}|.$$

Note that

$$|\alpha_i \overline{z_i} + \overline{\beta_i}| = \begin{cases} |\beta_i|, & \alpha_i = 0, \\ \frac{\sqrt{\beta_i\overline{\beta_i} - \alpha_i\gamma_i}}{|\alpha_i|}, & \alpha_i \neq 0, \end{cases}$$

when $z_i \in A_i$. If $z \in C$, then $|\alpha_i \overline{z_i} + \overline{\beta_i}|$ is a constant which only depends on the arc A_i for any $i \in I$. Then $V(z)$ is a constant on C when $z \in C$ and thus piecewise constant for $z \in \partial D$.

Every two adjacent boundary arcs of ∂D at least meet at a common point, then constants of $V(z)$ on two adjacent arcs must coincide. Because D has a path-connected boundary, those piecewise constants for all boundary arcs of ∂D must be the same. Therefore $V(z)$ is constant on ∂D. $\square$

Suppose D is a finite parqueting-reflection domain with a path-connected boundary. Lemma 3.2.2 shows that $V(z) = \prod_{i\in I} \left|\frac{\mathrm{Den}(\widehat{z_i})}{\mathrm{Den}(z_i)}\right|$ is constant on ∂D. Denote this constant by $V(\partial D)$. Let

$$F(z,\zeta) := \frac{1}{V(\partial D)} \widetilde{F}(z,\zeta) \prod_{i\in I} \frac{\mathrm{Den}(\widehat{z_i})}{\mathrm{Den}(z_i)} = \frac{1}{V(\partial D)} \prod_{i\in I} \frac{\zeta - \widehat{z_i}}{\zeta - z_i} \frac{\mathrm{Den}(\widehat{z_i})}{\mathrm{Den}(z_i)}. \tag{3.5}$$

We use this function to construct the harmonic Green function for D.

Theorem 3.2.3. *Let D be a finite parqueting-reflection domain with a path-connected boundary. Then $G_1(z,\zeta) := \log|F(z,\zeta)|^2$ is the harmonic Green function of D.*

Proof. Let $\{z_i \in \overline{D_i} | i \in I\}$ and $\{\widehat{z_i} \in \widehat{\overline{D_i}} | i \in I\}$ be the reflection images generated by $z \in \overline{D}$ with respect to the parqueting provided by D. Let $D_{i_0} = D$ and $z_{i_0} = z$. $\log|(\zeta - z_{i_0})\mathrm{Den}(z_{i_0})|^2 = \log|\zeta - z|^2$ as a function in the variable z is harmonic in D except for one point $z = \zeta$. Suppose that

$$z_i = \frac{a_i z + b_i}{c_i z + d_i}, \quad \widehat{z_i} = \frac{\alpha_i \overline{z} + \beta_i}{\gamma_i \overline{z} + \delta_i}.$$

For all $i \in I \setminus \{i_0\}$, $\log|(\zeta - z_i)\mathrm{Den}(z_i)|^2 = \log|(c_i z + d_i)\zeta - (a_i z + b_i)|^2$ as a function in the variable z is harmonic in D since $\zeta \neq z_i$ for $z, \zeta \in D$. For all $i \in I$, $\log|(\zeta - \widehat{z_i})\mathrm{Den}(\widehat{z_i})|^2 = \log|(\gamma_i \overline{z} + \delta_i)\zeta - (\alpha_i \overline{z} + \beta_i)|^2$ as a function in the variable z is harmonic in D since $\zeta \neq \widehat{z_i}$ for $z, \zeta \in D$. Then it follows that $\log|F(z,\zeta)|^2$ as a function in the variable z is harmonic in D except for one pole $z = \zeta$ and $\log|F(z,\zeta)|^2 + \log|\zeta - z|^2$ is harmonic in D.

Lemma 3.2.1 and Lemma 3.2.2 imply that $|F(z,\zeta)| = 1$ for $\zeta \in D$, $z \in \partial D$. It follows that $\log|F(z,\zeta)|^2 = 0$ if $z \in \partial D$. Therefore $G_1(z,\zeta)$ is the harmonic Green function of D. $\square$

Theorem 3.2.3 shows that we can obtain the harmonic Green function $G_1(z,\zeta)$ for a finite parqueting-reflection domain with a path-connected boundary via parqueting-reflection principle. For a finite parqueting-reflection domain D, the explicit expression of the harmonic Green function is given by

$$G_1(z,\zeta) = \log\left|\frac{1}{V(\partial D)} \prod_{i \in I} \frac{\zeta - \widehat{z_i}}{\zeta - z_i} \frac{\mathrm{Den}(\widehat{z_i})}{\mathrm{Den}(z_i)}\right|^2. \tag{3.6}$$

We next investigate the normal derivatives of the harmonic Green functions for finite parqueting-reflection domains.

Lemma 3.2.4. *Let C be a generalized circle in the extended complex plane, whose equation is given by $a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0$, where $a, c \in \mathbb{R}, b \in \mathbb{C}$ and $ac - b\overline{b} < 0$. Let $z \in \mathbb{C}$ and $\widehat{z}$ the image of z under the reflection at C. Let $\nu_\zeta := \frac{a\zeta + b}{\sqrt{b\overline{b} - ac}}$ denote the outward normal vector for $\zeta \in C$. Then the outward normal derivative of the function $\log\left|\frac{\zeta - \widehat{z}}{\zeta - z}\right|^2$ on C is*

$$\partial_{\nu_\zeta} \log\left|\frac{\zeta - \widehat{z}}{\zeta - z}\right|^2 = \frac{2}{\sqrt{b\overline{b} - ac}}\left(a - \frac{a\zeta + b}{\zeta - z} - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}}\right).$$

Moreover, $\partial_{\nu_\zeta} \log\left|\frac{\zeta - \widehat{z}}{\zeta - z}\right|^2 = 0$ if $z \in C \setminus \{\zeta\}$.

Proof.

$$\partial_{\nu_\zeta} \log\left|\frac{\zeta - \widehat{z}}{\zeta - z}\right|^2 = 2\mathrm{Re}\left(\frac{a\zeta + b}{\sqrt{b\overline{b} - ac}} \partial_\zeta \log\left|\frac{\zeta - \widehat{z}}{\zeta - z}\right|^2\right)$$

$$= \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re} \left(\frac{a\zeta + b}{\zeta - \widehat{z}} - \frac{a\zeta + b}{\zeta - z} \right).$$

Note that

$$(a\zeta + b)(a\overline{\zeta} + \overline{b}) = b\overline{b} - ac,$$
$$(a\widehat{z} + b)(a\overline{z} + \overline{b}) = b\overline{b} - ac.$$

Then

$$\begin{aligned}
\frac{a\zeta + b}{\zeta - \widehat{z}} =& \frac{a(a\zeta + b)}{(a\zeta + b) - (a\widehat{z} + b)} \\
=& \frac{a(a\zeta + b)(a\overline{\zeta} + \overline{b})(a\overline{z} + \overline{b})}{[(a\zeta + b) - (a\widehat{z} + b)](a\overline{\zeta} + \overline{b})(a\overline{z} + \overline{b})} \\
=& \frac{a(b\overline{b} - ac)(a\overline{z} + \overline{b})}{(b\overline{b} - ac)(a\overline{z} + \overline{b}) - (b\overline{b} - ac)(a\overline{\zeta} + \overline{b})} \\
=& \frac{a\overline{z} + \overline{b}}{\overline{z} - \overline{\zeta}} \\
=& a - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}}.
\end{aligned}$$

We therefore see that

$$\partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}}{\zeta - z} \right|^2 = \frac{2}{\sqrt{b\overline{b} - ac}} \left(a - \frac{a\zeta + b}{\zeta - z} - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}} \right).$$

If $z \in C$, then $\widehat{z} = z$. Therefore

$$\partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}}{\zeta - z} \right|^2 = \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re} \left(\frac{a\zeta + b}{\zeta - \widehat{z}} - \frac{a\zeta + b}{\zeta - z} \right) = 0$$

for $z \in C \setminus \{\zeta\}$. $\square$

Proposition 3.2.5. *Suppose that D is a finite parqueting-reflection domain with a path-connected boundary. Let C be a circular arc of ∂D, whose equation is given by $a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0$, where $a, c \in \mathbb{R}, b \in \mathbb{C}, ac - b\overline{b} < 0$. Then*

$$\partial_{\nu_\zeta} G_1(z, \zeta) = \frac{-2}{\sqrt{b\overline{b} - ac}} \sum_{i \in I} \left(\frac{a\zeta + b}{\zeta - z_i} + \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z_i}} - a \right), \tag{3.7}$$

for $\zeta \in C$. Moreover,

$$\partial_{\nu_\zeta} G_1(z, \zeta) = \frac{2}{\sqrt{b\overline{b} - ac}} \left(a - \frac{a\zeta + b}{\zeta - z} - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}} \right), \quad \textit{for } z, \zeta \in C; \tag{3.8}$$

$$\partial_{\nu_\zeta} G_1(z, \zeta) = 0, \quad \textit{for } \zeta \in C, z \in \partial D \setminus \{\zeta\}. \tag{3.9}$$

Proof. From Theorem 3.2.3 we know that the harmonic Green function for D is

$$G_1(z,\zeta) = \log \left| \frac{1}{V(\partial D)} \prod_{i\in I} \frac{\zeta - \widehat{z}_i}{\zeta - z_i} \frac{\mathrm{Den}(\widehat{z}_i)}{\mathrm{Den}(z_i)} \right|^2 .$$

Let R_C denote the circle reflection at C. From Remark 3.1.7 we know that R_C induces a one-to-one correspondence between $\{z_i\}_{i\in I}$ and $\{\widehat{z}_i\}_{i\in I}$. Without loss of generality, we assume that $\widehat{z}_i = R_C(z_i)$ for all $i \in I$. Then Lemma 3.2.4 implies that

$$\partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_i}{\zeta - z_i} \right|^2 = \frac{2}{\sqrt{b\bar{b} - ac}} \left(a - \frac{a\zeta + b}{\zeta - z_i} - \frac{a\bar{\zeta} + \bar{b}}{\bar{\zeta} - \overline{z_i}} \right)$$

for all $i \in I$. Thus for $\zeta \in C$,

$$\partial_{\nu_\zeta} G_1(z,\zeta) = \sum_{i\in I} \partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_i}{\zeta - z_i} \right|^2 = \frac{-2}{\sqrt{b\bar{b} - ac}} \sum_{i\in I} \left(\frac{a\zeta + b}{\zeta - z_i} + \frac{a\bar{\zeta} + \bar{b}}{\bar{\zeta} - \overline{z_i}} - a \right).$$

Assume that $z_{i_0} = z \in D$ and $\widehat{z}_{i_0}$ is its image under the reflection at C. When z tends to C, $\widehat{z}_{i_0}$ and z_{i_0} coincide, they are located on C. Actually, via reordering the set $\{\widehat{z}_i | i \in I\}$ properly, we can make sure that $\widehat{z}_i = z_i$ for all $i \in I$ when $z \in C$. Then for $\zeta, z \in C$,

$$\begin{aligned}
\partial_{\nu_\zeta} G_1(z,\zeta) =& \partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_{i_0}}{\zeta - z_{i_0}} \right|^2 + \sum_{i\in I\setminus\{i_0\}} \partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_i}{\zeta - z_i} \right|^2 \\
=& \partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_{i_0}}{\zeta - z_{i_0}} \right|^2 + \frac{2}{\sqrt{b\bar{b} - ac}} \sum_{i\in I\setminus\{i_0\}} \mathrm{Re} \left(\frac{a\zeta + b}{\zeta - \widehat{z}_i} - \frac{a\zeta + b}{\zeta - z_i} \right) \\
=& \partial_{\nu_\zeta} \log \left| \frac{\zeta - \widehat{z}_{i_0}}{\zeta - z_{i_0}} \right|^2 \\
=& \frac{2}{\sqrt{b\bar{b} - ac}} \left(a - \frac{a\zeta + b}{\zeta - z} - \frac{a\bar{\zeta} + \bar{b}}{\bar{\zeta} - \bar{z}} \right).
\end{aligned}$$

If z is in the boundary ∂D but does not coincide with the point ζ, $\widehat{z}_i = z_i$ for all $i \in I$, and there is no singular term in

$$\sum_{i\in I} \mathrm{Re} \left(\frac{a\zeta + b}{\zeta - \widehat{z}_i} - \frac{a\zeta + b}{\zeta - z_i} \right),$$

namely, $\zeta \neq z_i, \widehat{z}_i$ for all $i \in I$. Therefore

$$\partial_{\nu_\zeta} G_1(z,\zeta) = \frac{2}{\sqrt{b\bar{b} - ac}} \sum_{i\in I} \mathrm{Re} \left(\frac{a\zeta + b}{\zeta - \widehat{z}_i} - \frac{a\zeta + b}{\zeta - z_i} \right) = 0.$$

□

The last part of this section is about the construction of harmonic Neumann functions via parqueting-reflection principle for finite parqueting-reflection domains.

Lemma 3.2.6. *Under the notations and assumptions as in Lemma 3.2.4, we have*

$$\partial_{\nu_\zeta} \log|(\zeta - z)(\zeta - \widehat{z})|^2 = \frac{2a}{\sqrt{b\overline{b} - ac}}, \quad \textit{for } \zeta \in C, z \in \mathbb{C} \setminus C; \tag{3.10}$$

$$\partial_{\nu_z} \log|(\zeta - z)(\zeta - \widehat{z})|^2 = 0, \quad \textit{for } z \in C, \zeta \in \mathbb{C} \setminus C; \tag{3.11}$$

$$\partial_{\nu_z} \log|(\zeta - z)(\zeta - \widehat{z})|^2 = \frac{2}{\sqrt{b\overline{b} - ac}} \left(a - \frac{a\zeta + b}{\zeta - z} - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}} \right), \quad \textit{for } z, \zeta \in C. \tag{3.12}$$

Proof. Since $\widehat{z}$ is the image of z under the reflection at C, z and $\widehat{z}$ satisfy the equation $a\widehat{z}\overline{z} + \overline{b}\widehat{z} + b\overline{z} + c = 0$, i.e.

$$\widehat{z} = -\frac{b\overline{z} + c}{a\overline{z} + \overline{b}}.$$

Then for $\zeta \in C$ and $z \in \mathbb{C} \setminus C$,

$$\begin{aligned}
\partial_{\nu_\zeta} \log|(\zeta - z)(\zeta - \widehat{z})|^2 =& 2\mathrm{Re}\left(\frac{a\zeta + b}{\sqrt{b\overline{b} - ac}} \partial_\zeta \log|(\zeta - z)(\zeta - \widehat{z})|^2 \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(\frac{a\zeta + b}{\zeta - z} + \frac{a\zeta + b}{\zeta - \widehat{z}} \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(\frac{a\zeta + b}{\zeta - z} + \frac{(a\zeta + b)(a\overline{z} + \overline{b})}{(a\zeta + b)\overline{z} + \overline{b}\zeta + c} \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(\frac{a\zeta + b}{\zeta - z} + \frac{a\overline{z} + \overline{b}}{\overline{z} + \frac{\overline{b}\zeta + c}{a\zeta + b}} \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(\frac{a\zeta + b}{\zeta - z} + \frac{a\overline{z} + \overline{b}}{\overline{z} - \overline{\zeta}} \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(\frac{a\zeta + b}{\zeta - z} - \frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z}} + a \right) \\
=& \frac{2a}{\sqrt{b\overline{b} - ac}}.
\end{aligned}$$

Because $(az + b)(a\overline{z} + \overline{b}) = b\overline{b} - ac$ and $\widehat{z} = z$ hold when $z \in C$, then

$$\begin{aligned}
\partial_{\nu_z} \log|(\zeta - z)(\zeta - \widehat{z})|^2 =& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(-\frac{az + b}{\zeta - z} + \frac{az + b}{\overline{\zeta} - \overline{\widehat{z}}} \frac{b\overline{b} - ac}{(az + b)^2} \right) \\
=& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(-\frac{az + b}{\zeta - z} + \frac{a\overline{z} + \overline{b}}{\overline{\zeta} - \overline{z}} \right) \\
=& 0
\end{aligned}$$

for $z \in C$ and $\zeta \in \mathbb{C} \setminus C$. Furthermore, if ζ is also in C, then

$$\frac{a\overline{z} + \overline{b}}{\overline{\zeta} - \overline{z}} = \frac{-a\frac{\overline{b}z+c}{az+b} + \overline{b}}{-\frac{\overline{b}\zeta+c}{a\zeta+b} + \frac{\overline{b}z+c}{az+b}} = \frac{a\zeta + b}{z - \zeta} = -a - \frac{az+b}{\zeta - z}.$$

In this case, the normal derivative with respect to z can be written as

$$\begin{aligned}\partial_{\nu_z} \log |(\zeta - z)(\zeta - \widehat{z})|^2 =& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(-\frac{az+b}{\zeta - z} + \frac{a\overline{z} + \overline{b}}{\overline{\zeta} - \overline{z}}\right) \\ =& \frac{2}{\sqrt{b\overline{b} - ac}} \mathrm{Re}\left(-\frac{az+b}{\zeta - z} - \frac{az+b}{\zeta - z} - a\right) \\ =& \frac{2}{\sqrt{b\overline{b} - ac}} \left(a - \frac{a\zeta+b}{\zeta - z} - \frac{a\overline{\zeta}+\overline{b}}{\overline{\zeta} - \overline{z}}\right).\end{aligned}$$

□

Let

$$Q(z,\zeta) := \widetilde{Q}(z,\zeta) \prod_{i \in I} \mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i) = \prod_{i \in I} (\zeta - z_i)(\zeta - \widehat{z}_i)\mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i).$$

This function is used to construct a harmonic Neumann function of D.

Theorem 3.2.7. *Suppose that D is a finite parqueting-reflection domain. Let $N_1(z,\zeta) := -\log |Q(z,\zeta)|^2$. Then $N_1(z,\zeta)$ is a harmonic Neumann function of D.*

Proof. It is obvious that $Q(z,\zeta) = \prod_{i \in I} (\zeta - z_i)(\zeta - \widehat{z}_i)\mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i)$, as a polynomial in the variable ζ, has only one zero $\zeta = z$ in D. It is easy to check that $-\log |Q(z,\zeta)|^2$ is harmonic in D except for the point $\zeta = z$ and

$$-\log |Q(z,\zeta)|^2 + \log |\zeta - z|^2$$

is harmonic in D.

Let C be a boundary arc of D and its equation given by

$$a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0,\ a, c \in \mathbb{R}, b \in \mathbb{C}, ac - b\overline{b} < 0.$$

Since D is a parqueting-reflection domain, according to Remark 3.1.7, the circle reflection at C, denoted by R_C, induces a one-to-one correspondence between $\{z_i | i \in I\}$ and $\{\widehat{z}_i | i \in I\}$. Via reordering the set $\{\widehat{z}_i | i \in I\}$, we can assume that $\widehat{z}_i = R_C(z_i)$ for all $i \in I$. Then via Lemma 3.2.6 we see that

$$\partial_{\nu_\zeta} \log |(\zeta - z_i)(\zeta - \widehat{z}_i)\mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i)|^2 = \frac{2a}{\sqrt{b\overline{b} - ac}}$$

for $\zeta \in C$. Note that the terms $\mathrm{Den}(z_i)$ and $\mathrm{Den}(\widehat{z}_i)$ on the left-hand side of the above equality do not affect the result of the normal derivative because they are functions in the variable z. Hence

$$\begin{aligned}
\partial_{\nu_\zeta} N_1(z,\zeta) &= -\partial_{\nu_\zeta}\left(\log\left(\prod_{i\in I}|(\zeta - z_i)(\zeta - \widehat{z}_i)\mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i)|^2\right)\right) \\
&= -\sum_{i\in I}\partial_{\nu_\zeta}\left(\log|(\zeta - z_i)(\zeta - \widehat{z}_i)|^2\right) \\
&= -\frac{2a}{\sqrt{b\bar{b} - ac}} \cdot \#I, \qquad (3.13)
\end{aligned}$$

where $\#I$ denotes the cardinality of I.

I is a finite set, since D provides a finite covering of $\mathbb{C}_\infty$. This means that the normal derivative $\partial_{\nu_\zeta} N_1(z,\zeta)$ is a constant on every boundary arc of ∂D. Thus it is piecewise constant on ∂D.

Therefore we have verified that

$$N_1(z,\zeta) = -\log\left|\prod_{i\in I}(\zeta - z_i)(\zeta - \widehat{z}_i)\mathrm{Den}(z_i)\mathrm{Den}(\widehat{z}_i)\right|^2 \qquad (3.14)$$

is a harmonic Neumann function of D. $\square$

From Theorem 1.1.11 we know that the harmonic Green function $G_1(z,\zeta)$ is symmetric in the variables z and ζ. We expect that harmonic Neumann functions would also possess this symmetry. We show a conjecture below.

Conjecture. *The functions* $\prod_{i\in I}|(\zeta - z_i)\mathrm{Den}(z_i)|$ *and* $\prod_{i\in I}|(\zeta - \widehat{z}_i)\mathrm{Den}(\widehat{z}_i)|$ *are both symmetric in the variables z and ζ.*

If this conjecture is true, then $|Q(z,\zeta)|$ is symmetric in the variables z and ζ, so is $N_1(z,\zeta)$. Since harmonic Neumann functions have been determined symmetric for many domains, we believe that this conjecture holds for parqueting-reflection domains.

3.3 Invariant property of harmonic Green functions

We know that harmonic Green functions are conformally invariant from Theorem 1.1.14. We show below that they are also invariant under circle reflections. It thus means that harmonic Green functions are invariant under inversive transformations.

Lemma 3.3.1. *Let D and Ω be domains in $\mathbb{C}_\infty$ such that there is a circle reflection R sending D onto Ω. If $h(w)$ is a harmonic function in Ω, then the function $h(R(z))$ is harmonic in D.*

Proof. Let $g(z) := h(R(z))$. Then with $w = R(z)$

$$\partial_z g(z) = \partial_w h(w)\partial_z R(z) + \partial_{\overline{w}} h(w)\partial_z \overline{R(z)},$$

$$
\begin{aligned}
\partial^2_{\overline{z}z} g(z) =& \partial^2_{ww} h(w)\partial_{\overline{z}} R(z)\partial_z R(z) + \partial^2_{w\overline{w}} h(w)\partial_{\overline{z}}\overline{R(z)}\partial_z R(z) \\
&+ \partial_w h(w)\partial^2_{z\overline{z}} R(z) + \partial^2_{\overline{w}w} h(w)\partial_{\overline{z}} R(z)\partial_z \overline{R(z)} \\
&+ \partial^2_{\overline{w}\overline{w}} h(w)\partial_{\overline{z}}\overline{R(z)}\partial_z \overline{R(z)} + \partial_{\overline{w}} h(w)\partial^2_{z\overline{z}} \overline{R(z)}.
\end{aligned}
\tag{3.15}
$$

Note that $\partial_z R(z)$ and $\partial_{\overline{z}}\overline{R(z)}$ are both 0, since $R(z)$ is of the form

$$
R(z) = -\frac{b\overline{z} + c}{a\overline{z} + \overline{b}}.
$$

Since $h(w)$ is harmonic in Ω, $\partial^2_{w\overline{w}} h(w) = 0$. Then the right hand side of the equality (3.15) is 0. Therefore $g(z)$ is harmornic in D. □

Theorem 3.3.2. *Let D and Ω be domains in $\mathbb{C}_\infty$ such that there is a circle reflection R sending D onto Ω. Let $H_1(w,\omega)$ be the harmonic Green function for Ω. Then $G_1(z,\zeta) := H_1(R(z), R(\zeta))$ is the harmonic Green function for D.*

Proof. For a fixed point $\zeta \in D$, let $\omega := R(\zeta) \in \Omega$. Since $H_1(w,\omega)$ is harmonic in $\Omega\setminus\{\omega\}$ as a function in the variable w, Lemma 3.3.1 implies that $G_1(z,\zeta) = H_1(R(z), R(\zeta))$ is harmonic in D except for the point ζ. Since $H_1(w,\omega) + \log|w-\omega|^2$ is harmonic in Ω, from Lemma 3.3.1 we also know that

$$
H_1(R(z), R(\zeta)) + \log|R(z) - R(\zeta)|^2 = G_1(z,\zeta) + \log|R(z) - R(\zeta)|^2
$$

is harmonic in D. Assume that the circle reflection R is given by

$$
R(z) = -\frac{b\overline{z} + c}{a\overline{z} + \overline{b}},
$$

then

$$
\begin{aligned}
\log|R(z) - R(\zeta)|^2 =& \log\left|\frac{b\overline{z} + c}{a\overline{z} + \overline{b}} - \frac{b\overline{\zeta} + c}{a\overline{\zeta} - \overline{b}}\right|^2 \\
=& \log\left|\frac{ac - b\overline{b}}{(az + b)(a\zeta + b)}\right|^2 + \log|z-\zeta|^2.
\end{aligned}
$$

Note that the first term of the right-hand side in the above equality is harmonic in D, hence $G_1(z,\zeta) + \log|z-\zeta|^2$ is a harmonic function on D. The last step is to verify that $G_1(z,\zeta)$ vanishes when z tends to the boundary ∂D. This is guaranteed by the facts that $H_1(w,\omega)$ vanishes when w tends to the $\partial\Omega$ and R sends the boundary ∂D onto the boundary $\partial\Omega$. □

Chapter 4

Application of Parqueting-Reflection Principle to Basic Boundary Value Problems

This chapter is an investigation of the application of parqueting-reflection principle to some basic boundary value problems in parqueting-reflection domains. First we discuss basic boundary value problems in disks and half-planes. We overview some well-known results about Schwarz operators, Pompeiu operators, Poisson integral formulas, harmonic Green functions, Poisson kernels and harmonic Neumann functions for disks and half-planes respectively. With the help of Schwarz and Pompeiu operators, we solve the Schwarz problems for the inhomogeneous Cauchy-Riemann equation in disks and half-planes. With the help of harmonic Green functions, Poisson integral formulas and Neumann functions, we solve the Dirichlet and Neumann problems for the Poisson equation in disks and half-planes. On the basis of the discussions for disks and half-planes, we unify the expressions for Schwarz operators, Poisson integral formulas, harmonic Green functions and harmonic Neumann functions of disks and half-planes. At the end, we generally solve the Schwarz problems for the inhomogeneous Cauchy-Riemann equation and the Dirichlet problems for the Poisson equation in finite parqueting-reflection bounded domains.

4.1 Basic boundary value problems in disks

Let $D(a,r) := \{z \,|\, |z-a| < r\}$, $a \in \mathbb{C}$, $r \in \mathbb{R}_{>0}$ be a disk in the complex plane, and let Γ denote its boundary circle. $D(a,r)$ is a parqueting-reflection domain. Let $z \in D(a,r)$ and $z_{\rm re}$ its reflection image under the circle reflection at Γ. It is obvious that $(z_{\rm re}-a)(\overline{z}-\overline{a}) = r^2$, i.e.

$$z_{\rm re} = a + \frac{r^2}{\overline{z}-\overline{a}}.$$

Applying the Cauchy-Pompeiu formula, we have

$$w(z) = \frac{1}{2\pi i}\int_{\Gamma} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D(a,r)} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_{\zeta}}{\zeta - z}, \tag{4.1}$$

$$0 = \frac{1}{2\pi i}\int_{\Gamma} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z_{\mathrm{re}}} - \frac{1}{\pi}\int_{D(a,r)} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_{\zeta}}{\zeta - z_{\mathrm{re}}} \tag{4.2}$$

for $w(z) \in C^1(D(a,r);\mathbb{C}) \cap C(\overline{D(a,r)};\mathbb{C})$. Inserting the formula of z_{re}, using the relation $(\zeta - a)(\overline{\zeta} - \overline{a}) = r^2$ for $\zeta \in \Gamma$, and taking the complex conjugate for the formula (4.2) lead to

$$0 = -\frac{1}{2\pi i}\int_{\Gamma} \overline{w(\zeta)}\frac{z-a}{\zeta - z}\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{a}} - \frac{1}{\pi}\int_{D(a,r)} \overline{w_{\overline{\zeta}}(\zeta)}\frac{z-a}{r^2 - (z-a)(\overline{\zeta} - \overline{a})}\mathrm{d}\sigma_{\zeta}. \tag{4.3}$$

Note that

$$\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta - a}} = -\frac{\mathrm{d}\zeta}{\zeta - a}$$

for $\zeta \in \Gamma$, then adding formula (4.3) to formula (4.1) gives

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\int_{\Gamma} \mathrm{Re}(w(\zeta))\frac{\zeta + z - 2a}{\zeta - z}\frac{\mathrm{d}\zeta}{\zeta - a} + \frac{1}{2\pi}\int_{\Gamma} \mathrm{Im}(w(\zeta))\frac{\mathrm{d}\zeta}{\zeta - a} \\ &- \frac{1}{\pi}\int_{D(a,r)}\left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} + \frac{(z-a)\overline{w_{\overline{\zeta}}(\zeta)}}{r^2 - (z-a)(\overline{\zeta} - \overline{a})}\right)\mathrm{d}\sigma_{\zeta}. \end{aligned} \tag{4.4}$$

Let s be the arc length parameter of Γ, then

$$i\frac{\mathrm{d}s}{r} = \frac{\mathrm{d}\zeta}{\zeta - a}.$$

Formula (4.4) can also be rewritten as

$$\begin{aligned} w(z) =& \frac{1}{2\pi r}\int_{\Gamma} \mathrm{Re}(w(\zeta))\frac{\zeta + z - 2a}{\zeta - z}\mathrm{d}s_{\zeta} + \frac{i}{2\pi r}\int_{\Gamma} \mathrm{Im}(w(\zeta))\mathrm{d}s_{\zeta} \\ &- \frac{1}{\pi}\int_{D(a,r)}\left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} + \frac{(z-a)\overline{w_{\overline{\zeta}}(\zeta)}}{r^2 - (z-a)(\overline{\zeta} - \overline{a})}\right)\mathrm{d}\sigma_{\zeta}. \end{aligned} \tag{4.5}$$

Formulas (4.4) and (4.5) are called the Cauchy-Schwarz-Pompeiu formulas of $D(a,r)$.

Applying (4.5) for $z = a$ and taking the imaginary part of both sides implies that

$$i \cdot \mathrm{Im}(w(a)) = \frac{i}{2\pi r}\int_{\Gamma} \mathrm{Im}(w(\zeta))\mathrm{d}s_{\zeta} - \frac{1}{2\pi}\int_{D(a,r)}\left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - a} - \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta} - \overline{a}}\right)\mathrm{d}\sigma_{\zeta} \tag{4.6}$$

Subtracting $i \cdot \mathrm{Im}(w(a))$ from formula (4.5) induces

$$\begin{aligned} w(z) =& \frac{1}{2\pi r}\int_{\Gamma} \mathrm{Re}(w(\zeta))\frac{\zeta+z-2a}{\zeta-z}\mathrm{d}s_\zeta + i\cdot \mathrm{Im}(w(a)) \\ &- \frac{1}{\pi}\int_{D(a,r)}\left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta-z} + \frac{(z-a)\overline{w_{\overline{\zeta}}(\zeta)}}{r^2-(z-a)(\overline{\zeta}-\overline{a})}\right)\mathrm{d}\sigma_\zeta \\ &+ \frac{1}{2\pi}\int_{D(a,r)}\left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta-a} - \frac{\overline{w_{\overline{\zeta}}(\zeta)}}{\overline{\zeta}-\overline{a}}\right)\mathrm{d}\sigma_\zeta. \end{aligned} \tag{4.7}$$

Before dealing with the Schwarz problems for the Cauchy-Riemann equation in disks, we first review the Schwarz operators for disks and their essential properties.

The Schwarz operator for the unit disk is well-known, see e.g. [2], [22]. By means of conformal mappings, the formula for the unit disk can be generalized to any simply connected domain. So it is easy to determine the Schwarz operator for the disk $D(a,r)$, which is given by

$$S\varphi(z) := \frac{1}{2\pi r}\int_\Gamma \varphi(\zeta)\frac{\zeta+z-2a}{r(\zeta-z)}\mathrm{d}s_\zeta = \frac{1}{2\pi i}\int_\Gamma \varphi(\zeta)\frac{\zeta+z-2a}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-a}. \tag{4.8}$$

For $\varphi(z)\in C(\Gamma;\mathbb{R})$, $S\varphi(z)$ provides an analytic function in $D(a,r)$ with the property

$$\mathrm{Re}(S\varphi) = \varphi \text{ on } \Gamma \tag{4.9}$$

in the sense of

$$\lim_{z\to\Gamma}\mathrm{Re}(S\varphi(z)) = \varphi(z).$$

The kernel function

$$\frac{\zeta+z-2a}{\zeta-z}$$

is called the Schwarz kernel of the disk $D(a,r)$.

The Schwarz operator S helps to solve the Schwarz problem for analytic functions. We immediately obtain the following result.

Theorem 4.1.1. *The Schwarz problem for analytic functions in $D(a,r)$*

$$\partial_{\overline{z}}w = 0 \textit{ in } D(a,r),\quad \mathrm{Re}(w)=\gamma \textit{ on } \Gamma,\quad \mathrm{Im}(w(a)) = c$$

is uniquely solvable for $\gamma\in C(\Gamma;\mathbb{R})$ and $c\in\mathbb{R}$; the solution is provided by

$$w(z) = \frac{1}{2\pi i}\int_\Gamma \gamma(\zeta)\frac{\zeta+z-2a}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-a} + ic. \tag{4.10}$$

Theorem 4.1.2. *The Schwarz problem for the inhomogeneous Cauchy-Riemann equation in $D(a,r)$*

$$\partial_{\overline{z}}w = f \textit{ in } D(a,r),\quad \mathrm{Re}(w)=\gamma \textit{ on } \Gamma,\quad \mathrm{Im}(w(a)) = c$$

is uniquely solvable for $f \in L_p(D(a,r);\mathbb{C})$, $p > 2$, $\gamma \in C(\Gamma;\mathbb{R})$ and $c \in \mathbb{R}$, the solution is provided by

$$\begin{aligned} w(z) =&\frac{1}{2\pi i}\int_\Gamma \gamma(\zeta)\frac{\zeta+z-2a}{\zeta-z}\frac{\mathrm{d}\zeta}{\zeta-a}+ic \\ &-\frac{1}{2\pi}\int_{D(a,r)}\left(\frac{f(\zeta)}{\zeta-a}\frac{\zeta+z-2a}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{a}}\frac{r^2+(z-a)(\overline{\zeta}-\overline{a})}{r^2-(z-a)(\overline{\zeta}-\overline{a})}\right)\mathrm{d}\sigma_\zeta. \end{aligned} \tag{4.11}$$

Proof. Let $u(z)$ denote the third term on the right-hand side of (4.11).

$$\begin{aligned} u(z) =&-\frac{1}{2\pi}\int_{D(a,r)}\left(\frac{f(\zeta)}{\zeta-a}\frac{\zeta+z-2a}{\zeta-z}+\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{a}}\frac{r^2+(z-a)(\overline{\zeta}-\overline{a})}{r^2-(z-a)(\overline{\zeta}-\overline{a})}\right)\mathrm{d}\sigma_\zeta \\ =&-\frac{1}{\pi}\int_{D(a,r)}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\sigma_\zeta-\frac{1}{\pi}\int_{D(a,r)}\frac{(z-a)\overline{f(\zeta)}}{r^2-(z-a)(\overline{\zeta}-\overline{a})}\mathrm{d}\sigma_\zeta \\ &+\frac{1}{2\pi}\int_{D(a,r)}\left(\frac{f(\zeta)}{\zeta-a}-\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{a}}\right)\mathrm{d}\sigma_\zeta. \end{aligned} \tag{4.12}$$

The first term on the right-hand side of (4.12) is $T(f(z))$, where T is the Pompeiu operator. Recall that $\partial_{\overline{z}}(Tf(z)) = f(z)$. Note that the second term on the right-hand side of (4.12) is

$$-\frac{1}{\pi}\int_{D(a,r)}\frac{(z-a)\overline{f(\zeta)}}{r^2-(z-a)(\overline{\zeta}-\overline{a})}\mathrm{d}\sigma_\zeta = \frac{1}{\pi}\int_{D(a,r)}\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{z_{\mathrm{re}}}}\mathrm{d}\sigma_\zeta.$$

It is holomorphic in $D(a,r)$. Then $\partial_{\overline{z}}u = f$. Thus $\partial_{\overline{z}}w = \partial_{\overline{z}}u = f$ follows from Theorem 4.1.1. When z tends to the boundary Γ, z_{re} coincides with z. It implies that

$$\lim_{\substack{z\to\Gamma \\ z\in D(a,r)}} \mathrm{Re}(u(z)) = 0.$$

Then $\mathrm{Re}(w) = \gamma$ on Γ. It is obvious that $\mathrm{Im}(w(a)) = c$. Therefore, $w(z)$ given by (4.11) is a solution to the Schwarz problem. Uniqueness of solution is ensured by the fact that the boundary problem

$$\partial_{\overline{z}}w = 0 \text{ in } D(a,r), \quad \mathrm{Re}(w) = 0 \text{ on } \Gamma, \quad \mathrm{Im}(w(a)) = 0$$

has only the trivial solution. □

The harmonic Green and Neumann functions of $D(a,r)$ can be obtained via applying the parqueting-reflection principle (Formulas (3.6) and (3.14)). Let

$$\begin{aligned} G_1(z,\zeta) :=& \log\left|\frac{(\zeta-a)(\overline{z}-\overline{a})-r^2}{r(\zeta-z)}\right|^2, \\ N_1(z,\zeta) :=& -\log\left|(\zeta-z)\big((\zeta-a)(\overline{z}-\overline{a})-r^2\big)\right|^2. \end{aligned}$$

We can verify directly that $G_1(z,\zeta)$ is the harmonic Green function of $D(a,r)$ and $N_1(z,\zeta)$ a harmonic Neumann function. From Formula (3.7) we obtain that

$$\partial_{\nu_\zeta} G_1(z,\zeta) = \frac{-2}{r}\left(\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1\right).$$

The Poisson kernel for the disk $D(a,r)$ is given by

$$p(z,\zeta) := \frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1.$$

We see that the Poisson kernel is the real part of the Schwarz kernel.

Applying Theorem 1.1.12 for $D = D(a,r)$ solves the Dirichlet problem for harmonic functions in $D(a,r)$.

Theorem 4.1.3. *The Dirichlet problem for harmonic functions on $D(a,r)$*

$$w_{z\overline{z}} = 0 \text{ in } D(a,r), \quad w = \gamma \text{ on } \Gamma$$

is uniquely solvable for $\gamma \in C(\Gamma;\mathbb{R})$, the solution is provided by

$$w(z) = \frac{1}{2\pi r}\int_\Gamma \gamma(\zeta)\left(\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1\right)\mathrm{d}s_\zeta. \tag{4.13}$$

Formula (4.13) is called the Poisson integral for disks. It is a generalization of the well-known Poisson integral for the unit disk, see e.g. [3].

Theorem 4.1.4. *The Dirichlet problem for the Poisson equation in $D(a,r)$*

$$w_{z\overline{z}} = f \text{ in } D(a,r), \quad w = \gamma \text{ on } \Gamma$$

is uniquely solvable for $f \in L_p(D(a,r);\mathbb{C})$, $p > 2$ and $\gamma \in C(\Gamma;\mathbb{R})$, the solution is provided by

$$\begin{aligned} w(z) =& \frac{1}{2\pi r}\int_\Gamma \gamma(\zeta)\Big(\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1\Big)\mathrm{d}s_\zeta \\ &-\frac{1}{\pi}\int_{D(a,r)} f(\zeta)\log\left|\frac{(\zeta-a)(\overline{z}-\overline{a})-r^2}{r(\zeta-z)}\right|^2 \mathrm{d}\sigma_\zeta. \end{aligned} \tag{4.14}$$

Proof. This theorem results from Theorem 4.1.3 and the properties of harmonic Green functions. □

We can compute the normal derivatives of $N_1(z,\zeta)$ directly. We see that

$$\partial_{\nu_\zeta} N_1(z,\zeta) = -\frac{2}{r}, \quad \text{for } \zeta\in\Gamma, z\in D(a,r), \tag{4.15}$$

$$\partial_{\nu_z} N_1(z,\zeta) = -\frac{2}{r}, \quad \text{for } z\in\Gamma, \zeta\in D(a,r), \tag{4.16}$$

$$\partial_{\nu_z} N_1(z,\zeta) = -\frac{2}{r} + \frac{2}{r}\left(\frac{\zeta-a}{\zeta-z} + \frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}} - 1\right), \quad \text{for } z,\zeta\in\Gamma. \tag{4.17}$$

These results are used to solve the Neumann boundary problem for the Poisson equation in $D(a,r)$.

Theorem 4.1.5. *The Neumann boundary problem for the Poisson equation in $D(a,r)$*

$$w_{z\overline{z}} = f \text{ in } D(a,r), \quad \partial_{\nu_z} w = \gamma \text{ on } \Gamma$$

is solvable for $f\in L_p(D(a,r);\mathbb{C})$, $p>2$ and $\gamma\in C(\Gamma;\mathbb{R})$ if and only if

$$\int_\Gamma \gamma(\zeta)\mathrm{d}s_\zeta = 4\int_{D(a,r)} f(\zeta)\mathrm{d}\sigma_\zeta.$$

The solutions are of the form

$$\begin{aligned} w(z) =& c - \frac{1}{4\pi}\int_\Gamma \gamma(\zeta)\log\left|(\zeta-z)\big((\zeta-a)(\overline{z}-\overline{a})-r^2\big)\right|^2 \mathrm{d}s_\zeta \\ &+ \frac{1}{\pi}\int_{D(a,r)} f(\zeta)\log\left|(\zeta-z)\big((\zeta-a)(\overline{z}-\overline{a})-r^2\big)\right|^2 \mathrm{d}\sigma_\zeta, \end{aligned} \tag{4.18}$$

where c is an arbitrary constant in $\mathbb{C}$.

Proof. Since $-\frac{1}{\pi}N_1(z,\zeta)$ is a fundamental solution to the Laplacian operator and the boundary integral is a harmonic function, (4.18) provides a solution to the Poisson equation. We only need to check the boundary behavior of $w(z)$. For $z\in\Gamma$,

$$\begin{aligned} \partial_{\nu_z} w(z) =& \frac{1}{4\pi}\int_\Gamma \gamma(\zeta)\partial_{\nu_z}N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D(a,r)} f(\zeta)\partial_{\nu_z}N_1(z,\zeta)\mathrm{d}\sigma_\zeta \\ =& \frac{1}{2\pi r}\int_\Gamma \gamma(\zeta)\left(\frac{\zeta-a}{\zeta-z} + \frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}} - 1\right)\mathrm{d}s_\zeta \\ &- \frac{1}{2\pi r}\left(\int_\Gamma \gamma(\zeta)\mathrm{d}s_\zeta - 4\int_{D(a,r)} f(\zeta)\mathrm{d}\sigma_\zeta\right) \\ =& \gamma(z) - \frac{1}{2\pi r}\left(\int_\Gamma \gamma(\zeta)\mathrm{d}s_\zeta - 4\int_{D(a,r)} f(\zeta)\mathrm{d}\sigma_\zeta\right). \end{aligned} \tag{4.19}$$

The last equality is guaranteed by the Poisson integral (4.13). Therefore $\partial_{\nu_z} w = \gamma$ on Γ if and only if

$$\int_\Gamma \gamma(\zeta)\mathrm{d}s_\zeta = 4\int_{D(a,r)} f(\zeta)\mathrm{d}\sigma_\zeta.$$

□

4.2 Basic boundary value problems in half-planes

This section is an investigation on the Schwarz problem, Dirichlet problem and Neumann problem in half-planes. For the case of the upper half-plane, they have already been well discussed, see e.g. [17]. We will deal with these topics for general half-planes. The expressions for the Schwarz kernel, the harmonic Green function, the harmonic Neumann function and the Poisson kernel will be given and these functions serve to solve the corresponding boundary problems.

A straight line in the complex plane can be given by the equation:

$$\overline{b}z + b\overline{z} + c = 0, \text{ where } b \in \mathbb{C} \setminus \{0\}, c \in \mathbb{R}.$$

Let L be the straight line determined by the above equation. Let $z_0 := -\frac{c}{2\overline{b}} \in L$. For $z \in L$, we have $\mathrm{Re}(\overline{b}(z - z_0)) = 0$. The straight line L has a parameterization:

$$\zeta(s) = -\frac{c}{2\overline{b}} + i\frac{b}{|b|}s, \ s \in \mathbb{R}.$$

For $\zeta(s) \in L$ we have

$$\overline{b}\mathrm{d}\zeta - b\mathrm{d}\overline{\zeta} = 2i|b|\mathrm{d}s, \quad \overline{b}\mathrm{d}\zeta + b\mathrm{d}\overline{\zeta} = 0.$$

Let $\Omega := \{z \in \mathbb{C} | \, \overline{b}z + b\overline{z} + c < 0\}$ be the domain bounded by L. We see that $\mathrm{Re}(\overline{b}(z - z_0)) < 0$ for $z \in \Omega$ and $b/|b|$ is the outward normal vector on L. Let $z \in \Omega$ and z_{re} its image under the reflection at L. It is easy to check that

$$z_{\mathrm{re}} = -\frac{b\overline{z} + c}{\overline{b}}.$$

According to the parqueting-reflection principle, we can obtain the harmonic Green function for Ω:

$$G_1(z, \zeta) = \log \left| \frac{\overline{b}\zeta + b\overline{z} + c}{b(\zeta - z)} \right|^2.$$

We see that

$$-\frac{1}{2}\partial_{\nu_\zeta} G_1(z, \zeta) = -\mathrm{Re}\left(\frac{b}{|b|}\partial_\zeta G_1(z, \zeta)\right) = \frac{1}{|b|}\left(\frac{b}{\zeta - z} + \frac{\overline{b}}{\overline{\zeta} - \overline{z}}\right)$$

for $\zeta \in L$.

Given $\gamma \in C(L; \mathbb{R})$ and $\gamma(z) = O(|z|^{-\alpha})$ for $\alpha > 0$, the formula

$$u(z) = \frac{1}{2\pi i}\int_L \gamma(\zeta)\left(\frac{b}{\zeta - z} + \frac{\overline{b}}{\overline{\zeta - z}}\right)\frac{\mathrm{d}\zeta}{b} \tag{4.20}$$

provides a harmonic function in the half-plane Ω with the boundary data γ. Formula (4.20) is called the Poisson integral formula for the half-planes. On the basis of the harmonic Green function $G_1(z, \zeta)$ and the Poisson formula (4.20) for Ω, the explicit solution to the Dirichlet problem for the Poisson equation can be obtained.

Theorem 4.2.1. *Let $f \in L_p(\Omega;\mathbb{C})$, $p > 2$, $\gamma \in C(L;\mathbb{C})$ and $\gamma(z) = O(|z|^{-\alpha})$ for $\alpha > 0$. The Dirichlet problem for the Poisson equation in Ω*

$$w_{z\overline{z}} = f \text{ in } \Omega, \quad w = \gamma \text{ on } L$$

has a unique solution

$$w(z) = \frac{1}{2\pi i}\int_L \gamma(\zeta)\left(\frac{b}{\zeta - z} + \frac{\overline{b}}{\overline{\zeta - z}}\right)\frac{\mathrm{d}\zeta}{b} - \frac{1}{\pi}\int_\Omega f(\zeta)\log\left|\frac{\overline{b}\zeta + b\overline{z} + c}{b(\zeta - z)}\right|^2 \mathrm{d}\sigma_\zeta. \tag{4.21}$$

Via the parqueting-reflection principle, we also obtain a harmonic Neumann function of Ω, namely,

$$N_1(z,\zeta) = -\log\left|(\zeta - z)(\overline{b}\zeta + b\overline{z} + c)\right|^2.$$

We can verify that

$$\partial_{\nu_\zeta} N_1(z,\zeta) = 0, \quad \text{for } \zeta \in L, z \in \Omega, \tag{4.22}$$

$$\partial_{\nu_z} N_1(z,\zeta) = 0, \quad \text{for } z \in L, \zeta \in \Omega, \tag{4.23}$$

$$\partial_{\nu_z} N_1(z,\zeta) = \frac{2}{|b|}\left(\frac{b}{\zeta - z} + \frac{\overline{b}}{\overline{\zeta} - \overline{z}}\right), \quad \text{for } z,\zeta \in L. \tag{4.24}$$

Theorem 4.2.2. *Let $f \in L_p(\Omega;\mathbb{C})$, $p > 2$, $\gamma \in C(L;\mathbb{C})$ and $\gamma(z) = O(|z|^{-\alpha})$ for $\alpha > 0$. The Neumann boundary problem for the Poisson equation in the half-plane Ω*

$$w_{z\overline{z}} = f \text{ in } \Omega, \quad \partial_{\nu_z} w = \gamma \text{ on } L$$

is solvable, the solutions are of the form

$$\begin{aligned} w(z) = &c_0 - \frac{1}{4\pi i}\int_L \gamma(\zeta)\log\left|(\zeta - z)(\overline{b}\zeta + b\overline{z} + c)\right|^2 \frac{|b|\mathrm{d}\zeta}{b} \\ &+ \frac{1}{\pi}\int_\Omega f(\zeta)\log\left|(\zeta - z)(\overline{b}\zeta + b\overline{z} + c)\right|^2 \mathrm{d}\sigma_\zeta, \end{aligned} \tag{4.25}$$

where c_0 is an arbitrary constant in $\mathbb{C}$.

Proof. (4.25) is given by

$$w(z) = c_0 + \frac{1}{4\pi i}\int_L \gamma(\zeta)N_1(z,\zeta)\frac{|b|\mathrm{d}\zeta}{b} - \frac{1}{\pi}\int_\Omega f(\zeta)N_1(z,\zeta)\mathrm{d}\sigma_\zeta.$$

Let

$$u(z) := \frac{1}{4\pi i}\int_L \gamma(\zeta)N_1(z,\zeta)\frac{|b|\mathrm{d}\zeta}{b},$$

$$v(z) := \frac{1}{\pi}\int_\Omega f(\zeta)N_1(z,\zeta)\mathrm{d}\sigma_\zeta.$$

It is obvious that $u(z)$ is a (complex valued) harmonic function in Ω, i.e. $u_{z\overline{z}} = 0$. Since $-\frac{1}{\pi}N_1(z,\zeta)$ is a fundamental solution to the Laplacian operator $\partial_z\partial_{\overline{z}}$, it follows that $v_{z\overline{z}} = f$ in Ω. So we have $w_{z\overline{z}} = f$. Formulas (4.23) and (4.24) are used to verify the normal derivative of $w(z)$ on the boundary.

$$\begin{aligned}
\partial_{\nu_z} w(z) &= \frac{1}{4\pi i}\int_L \gamma(\zeta)\partial_{\nu_z}N_1(z,\zeta)\frac{|b|\mathrm{d}\zeta}{b} - \frac{1}{\pi}\int_\Omega f(\zeta)\partial_{\nu_z}N_1(z,\zeta)\mathrm{d}\sigma_\zeta \\
&= \frac{1}{2\pi i}\int_L \gamma(\zeta)\left(\frac{b}{\zeta - z} + \frac{\overline{b}}{\overline{\zeta - z}}\right)\frac{\mathrm{d}\zeta}{b} \\
&= \gamma(z).
\end{aligned}$$

The last equality is guaranteed by the Poisson integral formula (4.20). □

Suppose that $w(z) \in C^1(\Omega;\mathbb{C})$, $w(z) = O(|z|^{-\alpha})$ for $z \in L$, $\alpha > 0$ and $w_{\overline{z}} \in L_p(\Omega;\mathbb{C})$, $p > 2$. Consider the **Cauchy-Pompeiu representation formula for half-planes:**

$$\frac{1}{2\pi i}\int_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_\Omega w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z} = \begin{cases} w(z), & z \in \Omega, \\ 0, & z \notin \Omega. \end{cases} \tag{4.26}$$

When $z \in \Omega$, its reflection image $z_{\text{re}} \notin \Omega$. Applying the Cauchy-Pompeiu formula for z and z_{re} gives

$$w(z) = \frac{1}{2\pi i}\int_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_\Omega w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z}, \tag{4.27}$$

$$0 = \frac{1}{2\pi i}\int_L w(\zeta)\frac{\mathrm{d}\zeta}{\zeta + \frac{b\overline{z}+c}{\overline{b}}} - \frac{1}{\pi}\int_\Omega w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta + \frac{b\overline{z}+c}{\overline{b}}}. \tag{4.28}$$

Taking the complex conjugate for both sides of (4.28) and subtracting it from (4.27) produces

$$\begin{aligned}
w(z) =& \frac{1}{2\pi i}\int_L \left(\frac{w(\zeta)}{\zeta - z}\mathrm{d}\zeta + \frac{b\overline{w(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\mathrm{d}\overline{\zeta}\right) - \frac{1}{\pi}\int_\Omega \left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} - \frac{b\overline{w_{\overline{\zeta}}(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\right)\mathrm{d}\sigma_\zeta \\
=& \frac{1}{2\pi i}\int_L \mathrm{Re}(w(\zeta))\left(\frac{\mathrm{d}\zeta}{\zeta - z} + \frac{b\mathrm{d}\overline{\zeta}}{b\overline{\zeta} + \overline{b}z + c}\right) \\
&+ \frac{1}{2\pi}\int_L \mathrm{Im}(w(\zeta))\left(\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{b\mathrm{d}\overline{\zeta}}{b\overline{\zeta} + \overline{b}z + c}\right) \\
&- \frac{1}{\pi}\int_\Omega \left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} - \frac{b\overline{w_{\overline{\zeta}}(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\right)\mathrm{d}\sigma_\zeta.
\end{aligned} \tag{4.29}$$

Note that for $\zeta \in L$

$$\frac{\mathrm{d}\zeta}{\zeta - z} + \frac{b\mathrm{d}\overline{\zeta}}{b\overline{\zeta} + \overline{b}z + c} = \frac{\mathrm{d}\zeta}{\zeta - z} + \frac{b\mathrm{d}\overline{\zeta}}{\overline{b}(z - \zeta)} = \frac{2\mathrm{d}\zeta}{\zeta - z},$$

$$\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{b\mathrm{d}\overline{\zeta}}{b\overline{\zeta} + \overline{b}z + c} = \frac{\overline{b}\mathrm{d}\zeta + b\mathrm{d}\overline{\zeta}}{\overline{b}(\zeta - z)} = 0.$$

Applying these two relations to (4.29) produces that, for $z \in \Omega$,

$$w(z) = \frac{1}{\pi i}\int_L \mathrm{Re}(w(\zeta))\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_\Omega \left(\frac{w_{\overline{\zeta}}(\zeta)}{\zeta - z} - \frac{b\overline{w_{\overline{\zeta}}(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\right)\mathrm{d}\sigma_\zeta. \tag{4.30}$$

For $w(z) \in C^1(\Omega;\mathbb{C})$, $w(z) = O(|z|^{-\alpha})$ for $z \in L$, $\alpha > 0$ and $w_{\overline{z}} \in L_p(\Omega;\mathbb{C})$, $p > 2$, Formula (4.30) holds. It is called the **Cauchy-Schwarz-Pompeiu formula for half-planes**. It helps to solve the Schwarz problem for the inhomogeneous Cauchy-Riemann equation in a half-plane.

For $\varphi \in C(L;\mathbb{R})$ and $\varphi(z) = O(|z|^{-\alpha})$, $\alpha > 0$, the Schwarz operator

$$S\varphi := \frac{1}{\pi i}\int_L \frac{\varphi(\zeta)}{\zeta - z}\mathrm{d}\zeta \tag{4.31}$$

provides a holomorphic function in Ω which satisfies the boundary behavior $\mathrm{Re}(S\varphi) = \varphi$. This formula is called the Schwarz integral for half-planes.

Theorem 4.2.3. *Let $f \in L_p(\Omega,\mathbb{C})$, $p > 2$, $\gamma \in C(L;\mathbb{R})$ and $\gamma(z) = O(|z|^{-\alpha})$, $\alpha > 0$. The Schwarz problem in Ω*

$$\partial_{\overline{z}} w = f \ in\ \Omega, \quad \mathrm{Re}(w) = \gamma \ on\ L$$

is solvable, the solutions are of the form

$$w(z) = \frac{1}{\pi i}\int_L \frac{\gamma(\zeta)}{\zeta - z}d\zeta - \frac{1}{\pi}\int_\Omega \left(\frac{f(\zeta)}{\zeta - z} - \frac{b\overline{f(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\right)\mathrm{d}\sigma_\zeta + ic_0, \tag{4.32}$$

where c_0 is a real constant.

Proof.

$$S\gamma = \frac{1}{\pi i}\int_L \frac{\gamma(\zeta)}{\zeta - z}d\zeta$$

is holomorphic in Ω and $\mathrm{Re}(S\gamma) = \gamma$ on L. It is easy to check that

$$\frac{1}{\pi}\int_\Omega \frac{b\overline{f(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\mathrm{d}\sigma_\zeta$$

is holomorphic in Ω. From the properties of the Pompeiu operator we know that

$$\partial_{\overline{z}}\left(-\frac{1}{\pi}\int_\Omega \frac{f(\zeta)}{\zeta - z}\mathrm{d}\sigma_\zeta\right) = f.$$

Note that $\frac{\overline{b}z + c}{b} = -\overline{z}$ for $z \in L$, thus

$$\mathrm{Re}\left(\int_\Omega \left(\frac{f(\zeta)}{\zeta - z} - \frac{b\overline{f(\zeta)}}{b\overline{\zeta} + \overline{b}z + c}\right)\mathrm{d}\sigma_\zeta\right) = 0$$

for $z \in L$. Therefore a function of the form (4.20) is a solution to the Schwarz problem. Moreover, every solution is of this form, since the Schwarz problem

$$\partial_{\bar{z}} w = 0 \text{ in } \Omega, \quad \mathrm{Re}(w) = 0 \text{ on } L$$

has only purely imaginary constant solutions. $\square$

4.3 Uniform expressions for disks and half-planes

It has been mentioned in Chapter 2 that circles and straight lines in the complex plane can be viewed as generalized circles in the extended complex planes. Therefore a disk or a half-plane is actually a domain in $\mathbb{C}_\infty$ which is bounded by one generalized circle. In this section, on the basis of the previous discussions about disks and half-planes, we are going to demonstrate uniform expressions of harmonic Green functions, harmonic Neumann functions, Poisson integrals and Schwarz integrals for disks and half-planes.

Let $D := \{z \in \mathbb{C} |\ az\bar{z} + \bar{b}z + b\bar{z} + c < 0\}$, where a is a non-negative real number, $b \in \mathbb{C}$, $c \in \mathbb{R}$ and $ac - b\bar{b} < 0$. D is a domain bounded by a generalized circle C in $\mathbb{C}_\infty$. The associated matrix of C is

$$\begin{pmatrix} a & \bar{b} \\ b & c \end{pmatrix} \in \mathrm{H}^-.$$

Denote by $\widehat{D}$ the image of D under the reflection at C. Then D and $\widehat{D}$ provide a covering of $\mathbb{C}_\infty$, i.e. $\mathbb{C}_\infty = \overline{D} \cup \overline{\widehat{D}}$. Obviously, D is a simply connected finite parqueting-reflection domain. We can apply the parqueting-reflection principle for constructing the harmonic Green and Neumann functions.

Let z be a point in D and $z_{\mathrm{re}} \in \widehat{D}$ the reflection of z. It is already known that

$$z_{\mathrm{re}} = -\frac{b\bar{z} + c}{a\bar{z} + \bar{b}}.$$

On the basis of $\frac{\zeta - z_{\mathrm{re}}}{\zeta - z}$, we construct the function

$$F(z, \zeta) := \frac{a\bar{z}\zeta + b\bar{z} + \bar{b}\zeta + c}{\sqrt{b\bar{b} - ac}(\zeta - z)}.$$

We see that

$$|F(z, \zeta)| = \left| \frac{\zeta(a\bar{z} + \bar{b}) + b\bar{z} + c}{(\zeta - z)} \right| = \left| \frac{\zeta - z_{\mathrm{re}}}{\zeta - z} \right| \frac{|a\bar{z} + \bar{b}|}{\sqrt{b\bar{b} - ac}}.$$

Since $\hat{z} = z$ when z is on the boundary C, it follows that

$$|F(z, \zeta)| = 1 \text{ for } z \in C.$$

The harmonic Green function of D is

$$G_1(z, \zeta) = \log |F(z, \zeta)|^2.$$

In the case of $a > 0$, $G_1(z,\zeta)$ is actually the harmonic Green function for the disk $D(-\frac{b}{a}, \frac{\sqrt{b\bar{b}-ac}}{a})$, while in the other case of $a = 0$, it is the harmonic Green function for the half-plane $\{\bar{b}z + b\bar{z} + c < 0\}$.

According to Lemma 3.2.4,

$$\begin{aligned}\partial_{\nu_\zeta} G_1(z,\zeta) =&\partial_{\nu_\zeta} \log \left| \frac{\zeta - z_{\mathrm{re}}}{\zeta - z} \frac{a\bar{z} + \bar{b}}{\sqrt{b\bar{b} - ac}} \right|^2 \\ =&\partial_{\nu_\zeta} \log \left| \frac{\zeta - z_{\mathrm{re}}}{\zeta - z} \right|^2 \\ =&\frac{-2}{\sqrt{b\bar{b} - ac}} \left(\frac{a\zeta + b}{\zeta - z} + \frac{a\bar{\zeta} + \bar{b}}{\bar{\zeta} - \bar{z}} - a \right).\end{aligned}$$

The Poisson integral

$$u(z) = \frac{1}{2\pi i} \int_C \gamma(\zeta) \left(\frac{a\zeta + b}{\zeta - z} + \frac{a\bar{\zeta} + \bar{b}}{\overline{\zeta - z}} - a \right) \frac{\mathrm{d}\zeta}{a\zeta + b} \tag{4.33}$$

provides a harmonic function in D with the boundary data $u = \gamma$ on C if γ satisfies proper conditions. This formula unifies (4.13) and (4.20).

Via the parqueting-reflection principle, a harmonic Neumann function of D is given by

$$N_1(z,\zeta) = -\log |(\zeta - z)(a\zeta\bar{z} + \bar{b}\zeta + b\bar{z} + c)|^2.$$

From (3.13) we know that

$$\partial_{\nu_\zeta} N_1(z,\zeta) = -\frac{2a}{\sqrt{b\bar{b} - ac}}, \quad \text{for } \zeta \in C, z \in D. \tag{4.34}$$

Since $N_1(z,\zeta)$ is symmetric in ζ and z, it follows that

$$\partial_{\nu_z} N_1(z,\zeta) = -\frac{2a}{\sqrt{b\bar{b} - ac}}, \quad \text{for } z \in C, \zeta \in D. \tag{4.35}$$

In the case of $z, \zeta \in C$, we can verify that

$$\partial_{\nu_z} N_1(z,\zeta) = -\frac{2a}{\sqrt{b\bar{b} - ac}} + \frac{2a}{\sqrt{b\bar{b} - ac}} \left(\frac{a\zeta + b}{\zeta - z} + \frac{a\bar{\zeta} + \bar{b}}{\overline{\zeta - z}} - a \right). \tag{4.36}$$

Unifying Schwarz operators for disks and half-planes, we obtain the Schwarz operator for D

$$S\gamma = \frac{1}{2\pi i} \int_{\partial D} \gamma(\zeta) \frac{a\zeta + az + 2b}{\zeta - z} \frac{\mathrm{d}\zeta}{a\zeta + b}. \tag{4.37}$$

We require that $\gamma(z) \in C(\partial D; \mathbb{R})$ if $a \neq 0$, and $\gamma(z) \in C(\partial D; \mathbb{R})$, $\gamma(z) = O(|z|^{-\alpha})$, $\alpha > 0$ if $a = 0$. The function $S\gamma$ is holomorphic in D and satisfies the boundary behavior

$$\mathrm{Re}(S\gamma) = \gamma \quad \text{on } \partial D.$$

The above conclusions about disks and half-planes are fundamental. More general results regarding parqueting-reflection domains are based on these conclusions. Particularly, the Poisson integrals and Schwarz integrals of disks and half-planes play important roles. We will see their contribution in the proof of two theorems in the following section. These two theorems ensure the feasibility of the parqueting-reflection principle in solving the Dirichlet and Schwarz boundary value problems for parqueting-reflection domains.

4.4 Basic boundary value problems for finite parqueting-reflection bounded domains

Suppose D is a finite parqueting-reflection domain, it provides a finite parqueting of $\mathbb{C}_\infty$, namely, $\mathbb{C}_\infty = \bigcup\limits_{k\in I}(\overline{D_k} \cup \overline{\widehat{D}_k})$, where I is a finite set. Let $z \in D$. z generates two families of finite reflection images $\{z_k | k \in I\}$ and $\{\widehat{z}_k | k \in I\}$.

From discussions in Chapter 3, we know that $G_1(z,\zeta) = \log |F(z,\zeta)|^2$ is the harmonic Green function of D, where $F(z,\zeta)$ is given by (3.5). We use this harmonic Green function to solve the Dirichlet problem for the Poisson equation in D.

Theorem 4.4.1. *Suppose D is a finite parqueting-reflection bounded domain and $G_1(z,\zeta)$ its harmonic Green function. Let $f \in L_p(D;\mathbb{C})$, $p > 2$, and $\gamma \in C(\partial D;\mathbb{C})$. Then the Dirichlet problem*

$$w_{z\bar{z}} = f \text{ in } D, \quad w = \gamma \text{ on } \partial D$$

has a unique solution

$$w(z) = -\frac{1}{4\pi}\int_{\partial D} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_D f(\zeta)G_1(z,\zeta)\mathrm{d}\sigma_\zeta, \tag{4.38}$$

where s is the arc length parameter on ∂D.

Proof. Via the maximum principle of harmonic functions we know that the Dirichlet problem

$$w_{z\bar{z}} = 0 \text{ in } D, \quad w = 0 \text{ on } \partial D$$

has only the trivial solution. Then the Dirichlet problem for the Poisson equation in D has a unique solution if it is solvable.

Since $-\frac{1}{\pi}G_1(z,\zeta)$ is a fundamental solution to $\partial_z\partial_{\bar{z}}$, we have

$$\partial_z\partial_{\bar{z}}\left(-\frac{1}{\pi}\int_D f(\zeta)G_1(z,\zeta)\mathrm{d}\sigma_\zeta\right) = f.$$

From Theorem 1.1.12 we know that

$$-\frac{1}{4\pi}\int_{\partial D}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\mathrm{d}s_\zeta$$

provides a harmonic function in D with the boundary value γ on ∂D. Therefore

$$w(z) = -\frac{1}{4\pi}\int_{\partial D}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_D f(\zeta)G_1(z,\zeta)\mathrm{d}\sigma_\zeta$$

is a solution to the Dirichlet problem.

We could also verify the boundary data of w via the the results from Proposition 3.2.5 and the Poisson integral formula (4.33). Let C be a circular arc of ∂D and its equation is given by $a\zeta\overline{\zeta}+\overline{b}\zeta+b\overline{\zeta}+c=0$, where $a,c\in\mathbb{R}, b\in\mathbb{C}, ac-b\overline{b}<0$. Then

$$\begin{aligned}
&\lim_{\substack{z\to C\\ z\in D}}\left(-\frac{1}{4\pi}\int_{\partial D}\gamma(\zeta)\partial_{\nu_\zeta}G_1(z,\zeta)\mathrm{d}s_\zeta\right)\\
=&\lim_{\substack{z\to C\\ z\in D}}\frac{1}{2\pi i}\int_C\gamma(\zeta)\left(\frac{a\zeta+b}{\zeta-z}+\frac{a\overline{\zeta}+\overline{b}}{\overline{\zeta-z}}-a\right)\frac{\mathrm{d}\zeta}{a\zeta+b}\\
=&\gamma(z).
\end{aligned}$$

□

Parqueting-reflection principle can also be used to solve the Schwarz problem for the inhomogeneous Cauchy-Riemann equation in D:

$$\partial_{\overline{z}}w=f \text{ in } D,\quad \mathrm{Re}(w)=\gamma \text{ on } \partial D,$$

where $f\in C(D,\mathbb{C})$, $\gamma\in C(\partial D;\mathbb{R})$.

Denote

$$\begin{aligned}
\Phi_\gamma(z_k):=&\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z_k},\\
\overline{\Phi_\gamma(\widehat{z}_k)}:=&-\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{\widehat{z}_k}},\\
T_f(z_k):=&-\frac{1}{\pi}\int_D\frac{f(\zeta)}{\zeta-z_k}\mathrm{d}\sigma_\zeta,\\
\overline{T_f(\widehat{z}_k)}:=&-\frac{1}{\pi}\int_D\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{\widehat{z}_k}}\mathrm{d}\sigma_\zeta.
\end{aligned}$$

Suppose that D provides a parqueting of the extended complex plane,

$$\mathbb{C}_\infty=\bigcup_{k\in I}(\overline{D_k}\cup\overline{\widehat{D}_k}).$$

Let $\{z_k\}_{k\in I}$ and $\{\widehat{z}_k\}_{k\in I}$ be the reflection images corresponding to the parqueting. Let $D_{k_0}=D$ and $z_{k_0}=z\in D$. We know that $z_k\notin D$ for all $k\in I\setminus\{k_0\}$ and $\widehat{z}_k\notin D$ for all $k\in I$. We also know that z_k is a linear fraction in z and $\widehat{z}_k$ is a linear fraction in $\overline{z}$. Via Cauchy integral formula we see that

$$\Phi_\gamma(z_{k_0})=\frac{1}{2\pi i}\int_{\partial D}\gamma(\zeta)\frac{\mathrm{d}\zeta}{\zeta-z}$$

is holomorphic in D, i.e.

$$\partial_{\overline{z}}(\Phi_\gamma(z_{k_0}))=0,\quad \text{for } z\in D, \tag{4.39}$$

and

$$\Phi_\gamma(z_k) = 0, \quad \forall k \in I \setminus \{k_0\}, \tag{4.40}$$
$$\overline{\Phi_\gamma(\widehat{z}_k)} = 0, \quad \forall k \in I. \tag{4.41}$$

From the properties of the T operator mentioned in Chapter 1, we see that

$$\partial_{\bar z}(T_f(z_{k_0})) = \partial_{\bar z}\left(-\frac{1}{\pi}\int_D \frac{f(\zeta)}{\zeta - z}\mathrm{d}\sigma_\zeta\right) = f, \tag{4.42}$$
$$\partial_{\bar z}(T_f(z_k)) = 0, \quad \forall k \in I \setminus \{k_0\}, \tag{4.43}$$
$$\partial_{\bar z}(\overline{T_f(\widehat{z}_k)}) = 0, \quad \forall k \in I. \tag{4.44}$$

Construct a function

$$\begin{aligned} w(z) :&= \sum_{k\in I}(\Phi_\gamma(z_k) - \overline{\Phi_\gamma(\widehat{z}_k)} + T_f(z_k) - \overline{T_f(\widehat{z}_k)}) \\ &= \frac{1}{2\pi i}\sum_{k\in I}\int_{\partial D}\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta - z_k} + \frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{\widehat{z}_k}}\right) - \frac{1}{\pi}\sum_{k\in I}\int_D\left(\frac{f(\zeta)}{\zeta - z_k} - \frac{\overline{f(\zeta)}}{\overline{\zeta} - \overline{\widehat{z}_k}}\right)\mathrm{d}\sigma_\zeta. \end{aligned}$$

We show below that $w(z)$ provides a solution to the Schwarz problem in D.

Theorem 4.4.2. *Suppose D is a finite parqueting-reflection bounded domain, it provides a finite parqueting of the extended complex plane, $\mathbb{C}_\infty = \bigcup_{k\in I}(\overline{D_k} \cup \overline{\widehat{D}_k})$. A point $z \in D$ generates two families of reflection images $\{z_k | k \in I\}$ and $\{\widehat{z}_k | k \in I\}$. Then for $f \in L_p(D, \mathbb{C})$, $p > 2$ and $\gamma \in C(\partial D; \mathbb{R})$, the Schwarz problem*

$$\partial_{\bar z} w = f \ \text{ in } D, \quad \mathrm{Re}(w) = \gamma \ \text{ on } \partial D$$

is solvable, the solutions are of the form

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\sum_{k\in I}\int_{\partial D}\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta - z_k} + \frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{\widehat{z}_k}}\right) \\ &- \frac{1}{\pi}\sum_{k\in I}\int_D\left(\frac{f(\zeta)}{\zeta - z_k} - \frac{\overline{f(\zeta)}}{\overline{\zeta} - \overline{\widehat{z}_k}}\right)\mathrm{d}\sigma_\zeta + ic_0, \end{aligned} \tag{4.45}$$

where c_0 is a real constant.

Proof. Suppose that $D_{k_0} = D$ and $z_{k_0} = z$. From Formulas (4.39)–(4.44) we immediately see that

$$\partial_{\bar z} w = \sum_{k\in I}\partial_{\bar z}(\Phi_\gamma(z_k) - \overline{\Phi_\gamma(\widehat{z}_k)} + T_f(z_k) - \overline{T_f(\widehat{z}_k)}) = \partial_{\bar z}(T_f(z_{k_0})) = f.$$

We next verify that $w(z)$ satisfies the boundary condition.

Suppose C is a boundary arc of ∂D and $a\zeta\overline{\zeta}+\overline{b}\zeta+b\overline{\zeta}+c=0$ gives the equation of C, where $a,c\in\mathbb{R}, b\in\mathbb{C}, ac-b\overline{b}<0$. Via a proper ordering of $\{\widehat{z}_k|k\in I\}$, we can ensure that $\widehat{z}_k=z_k$ for all k when z tends to C. Then we see that

$$
\begin{aligned}
&\lim_{z\to C}\mathrm{Re}\left(-\frac{1}{\pi}\sum_{k\in I}\int_D\left(\frac{f(\zeta)}{\zeta-z_k}-\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{\widehat{z}_k}}\right)\mathrm{d}\sigma_\zeta\right)\\
=&\mathrm{Re}\left(-\frac{1}{\pi}\sum_{k\in I}\int_D\left(\frac{f(\zeta)}{\zeta-z_k}-\frac{\overline{f(\zeta)}}{\overline{\zeta}-\overline{z_k}}\right)\mathrm{d}\sigma_\zeta\right)\\
=&0.
\end{aligned}
$$

In the case of $\zeta\in\partial D\setminus C$,

$$
\begin{aligned}
&\lim_{z\to C}\mathrm{Re}\left(\frac{1}{2\pi i}\int\limits_{\partial D\setminus C}\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta-z_k}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{\widehat{z}_k}}\right)\right)\\
=&\mathrm{Re}\left(\frac{1}{2\pi i}\int\limits_{\partial D\setminus C}\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta-z_k}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{z_k}}\right)\right)=0.
\end{aligned}
$$

In the case of $\zeta\in C$, since $\widehat{z}_k=z_k\notin C$ for $k\in I\setminus\{k_0\}$ when z tends to C, we also have

$$
\lim_{z\to C}\mathrm{Re}\left(\frac{1}{2\pi i}\int_C\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta-z_k}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{\widehat{z}_k}}\right)\right)=0,\ \forall k\in I\setminus\{k_0\}.
$$

We only need to investigate the boundary integral on C for the terms z_{k_0} and $\widehat{z}_{k_0}$. For $z,\zeta\in C$, we have $a\zeta\overline{\zeta}+\overline{b}\zeta+b\overline{\zeta}+c=0$, $az\widehat{z}+\overline{b}z+b\widehat{z}+c=0$ and $\widehat{z}=z$. Then

$$
\begin{aligned}
\frac{\mathrm{d}\zeta}{\zeta-z}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{\widehat{z}}}&=\frac{a\zeta+b}{\zeta-z}\frac{\mathrm{d}\zeta}{a\zeta+b}+\frac{a\overline{\zeta}+\overline{b}}{\overline{\zeta}-\overline{z}}\frac{\mathrm{d}\overline{\zeta}}{a\overline{\zeta}+\overline{b}}\\
&=\left(\frac{a\zeta+b}{\zeta-z}-\frac{a\overline{\zeta}+\overline{b}}{\overline{\zeta}-\overline{z}}\right)\frac{\mathrm{d}\zeta}{a\zeta+b}\\
&=\left(\frac{a\zeta+b}{\zeta-z}-\frac{az+b}{z-\zeta}\right)\frac{\mathrm{d}\zeta}{a\zeta+b}\\
&=\frac{a\zeta+az+2b}{\zeta-z}\frac{\mathrm{d}\zeta}{a\zeta+b}.
\end{aligned}
$$

Then from the Schwarz integral formula (4.37) we know that

$$
\begin{aligned}
&\lim_{z\to C}\mathrm{Re}\left(\frac{1}{2\pi i}\int_C\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta-z_{k_0}}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{\widehat{z}_{k_0}}}\right)\right)\\
=&\mathrm{Re}\left(\frac{1}{2\pi i}\int_C\gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta-z}+\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta}-\overline{z}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{Re}\left(\frac{1}{2\pi i}\int_C \gamma(\zeta)\frac{a\zeta + az + 2b}{\zeta - z}\frac{\mathrm{d}\zeta}{a\zeta + b}\right) \\
&= \gamma.
\end{aligned}
$$

Therefore we see that

$$\lim_{z\to C}\mathrm{Re}(w(z)) = \lim_{z\to C}\mathrm{Re}\left(\frac{1}{2\pi i}\int_C \gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta - z_{k_0}} + \frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{\widehat{z}_{k_0}}}\right)\right) = \gamma(z).$$

This boundary behavior works for any boundary arc of ∂D, thus the boundary condition

$$\lim_{z\to\partial D}\mathrm{Re}(w) = \gamma$$

holds.

Since the boundary problem

$$\partial_{\overline{z}} u = 0 \ \text{ in } D, \quad \mathrm{Re}(u) = 0 \ \text{ on } \partial D$$

has only trivial solutions, namely, $u(z)$ must be a pure imaginary constant. Therefore every solution to the Schwarz problem differs from $w(z)$ by a purely imaginary constant. □

From Theorem 3.2.7 we see that the parqueting-reflection principle can be used to construct harmonic Neumann functions for finite parqueting-reflection domains. Harmonic Neumann functions help to solve the Neumann boundary value problems for the Poisson equation. Although we do not have a general conclusion for the Neumann problems in finite parqueting-reflection domains, we still see many successful examples that harmonic Neumann functions produce explicit solutions to the Neumann problems in particular domains. We will see in the next chapter that the parqueting-reflection principle also works for solving the Neumann problems for a class of circular digons.

Chapter 5

Boundary Value Problems in a Class of Circular Digons

A circular digon in $\mathbb{C}_\infty$ is a domain whose boundary is composed of two circular arcs of $\mathbb{C}_\infty$ with two intersection points, it is a circular polygon with two circular arcs and two vertices. Figure 5.1 demonstrates all the four cases of circular digon domains. They are lune domains, circular segment domains, lens domains and cones, respectively corresponding to Figures 5.1a, 5.1b, 5.1c and 5.1d. In the last case, as seen in Figure 5.1d, the two boundary rays of a cone are considered to intersect at the corner point and at infinity.

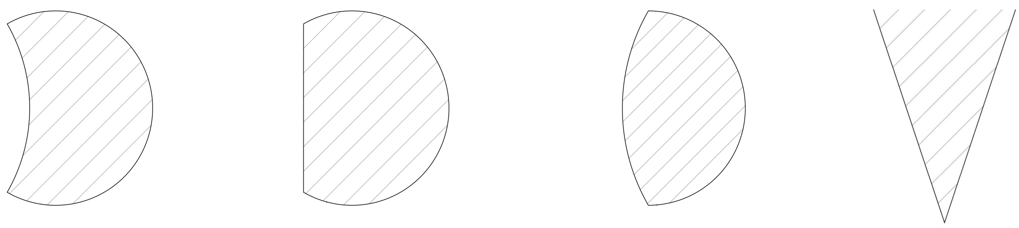
We focus on a class of circular digons. Let

$$D_{\alpha,\theta} := \{z \in \mathbb{C} \mid z\bar{z} - 1 < 0, z\bar{z}\sin(\alpha-\theta) + z\sin\theta + \bar{z}\sin\theta - \sin(\alpha+\theta) > 0\},$$

where $0 < \alpha < \pi$, $0 < \theta \leq \pi$. $D_{\alpha,\theta}$ is a circular digon with the two corner points $v_\pm := e^{\pm i\alpha}$ and two boundary arcs

$$\begin{aligned} C_0 &:= \partial D_{\alpha,\theta} \cap \{z\bar{z}\sin(\alpha-\theta) + z\sin\theta + \bar{z}\sin\theta - \sin(\alpha+\theta) = 0\}, \\ C_1 &:= \partial D_{\alpha,\theta} \cap \{z\bar{z} = 1\}. \end{aligned}$$

C_0 and C_1 intersect at the corner points v_+ and v_- with the intersection angle θ. C_1 is a part of the unit circle $\{|z| = 1\}$. If $\alpha = \theta$, $\{z\sin\theta + \bar{z}\sin\theta - \sin(\alpha+\theta) = 0\}$ is a straight



(a) (b) (c) (d)

Figure 5.1: Four cases of circular digons

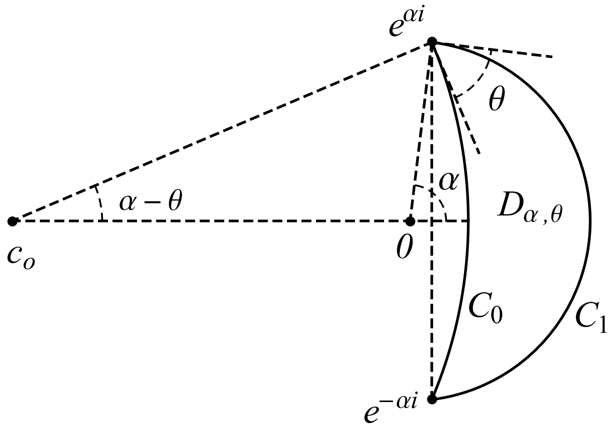


Figure 5.2: $D_{\alpha,\theta}$ in the case of $\alpha > \theta$

line and C_0 is the straight line segment connecting v_+ and v_-. In this case, $D_{\alpha,\theta}$ looks like Figure 5.1b. If $\alpha \neq \theta$, C_0 is a part of the circle whose center is $c_0 := \sin\theta / \sin(\theta - \alpha)$ and radius is $r_0 := \sin\alpha / |\sin(\theta - \alpha)|$. If $\alpha > \theta$, the domain $D_{\alpha,\theta}$ is a lune domain, while for $\alpha < \theta$, $D_{\alpha,\theta}$ is a lens domain. Figure 5.2 depicts the details of $D_{\alpha,\theta}$ in the case of $\alpha > \theta$.

In this chapter, we show that $D_{\alpha,\theta}$ is a finite parqueting-reflection domain when $\theta = \frac{\pi}{n}$, $n \in \mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of positive integers. We obtain the harmonic Green and Neumann functions for $D_{\alpha,\theta}$ via parqueting-reflection principle. With the help of harmonic Green and Neumann functions, we solve the Dirichlet and Neumann boundary value problems for the Poisson equation in $D_{\alpha,\theta}$. We also solve the Schwarz boundary value problem for the inhomogeneous Cauchy-Riemann equation in $D_{\alpha,\theta}$ via parqueting-reflection principle. Parts of the results are published in [19].

5.1 Parqueting of $\mathbb{C}_\infty$ provided by $D_{\alpha,\theta}$

In this section, we are going to investigate the parqueting of the extended complex plane provided by $D_{\alpha,\theta}$ in the case of $\theta = \frac{\pi}{n}$, $n \in \mathbb{N}^*$.

Let $D_0 := D_{\alpha,\theta}$. Reflecting D_0 at C_1 generates a new circular digon, denoted by D_1, and $\partial D_1 = C_1 \cup C_2$, where C_2 is the image of C_0 under the circle reflection at C_1. Carrying out consecutive reflections in anticlockwise direction produces a family of circular digons $\{D_k\}_{k\in\mathbb{N}}$ and a family of circular arcs $\{C_k\}_{k\in\mathbb{N}}$. (Reflecting D_k at C_{k+1} gives D_{k+1}.) The circular digon D_k is bounded by C_{k-1} and C_k, namely $\partial D_k = C_{k-1} \cup C_k$. This family of circular digons share the common corner points $v_\pm$. Since circle reflections preserve angles, every two adjacent circular arcs C_{k-1} and C_k intersect at $v_\pm$ with the same angle θ. In the case of $\theta = \pi/n$, after operating $2n - 1$ steps of circle reflections, C_{2n} coincides with C_0 and D_{2n} turns out to be D_0. More generally,

$C_{2n+k} = C_k$ and $D_{2n+k} = D_k$ for all $k \in \mathbb{N}$. The circular arcs $C_0, C_1, \cdots, C_{2n-1}$ thus divide the extended complex plane into $2n$ domains, which are just $D_0, D_1, \cdots, D_{2n-1}$. Therefore the family of domains $\{D_0, D_1, \cdots, D_{2n-1}\}$ provides a parqueting of $\mathbb{C}_\infty$, namely,

$$\mathbb{C}_\infty = \bigcup_{k=0}^{2n-1} \overline{D_k},\ D_j \cap D_k = \emptyset \text{ for } j \neq k.$$

So we have verified that $D_{\alpha,\theta}$ is a finite parqueting domain.

The following lemma provides the associated matrices for the circular arcs $\{C_k\}_{k\in\mathbb{N}}$.

Lemma 5.1.1. *Let A_k be an associated matrix of C_k. Then*

$$A_k \overset{\mathrm{H}^-}{\sim} \begin{pmatrix} -\sin(\alpha+(k-1)\theta) & \sin(k-1)\theta \\ \sin(k-1)\theta & \sin(\alpha-(k-1)\theta) \end{pmatrix}, \tag{5.1}$$

where "$\overset{\mathrm{H}^-}{\sim}$" is the equivalence relation on H^- *defined by* (2.1).

Proof. We proceed with the proof by induction on k. It is easy to see that

$$A_0 = \begin{pmatrix} \sin(\alpha-\theta) & \sin\theta \\ \sin\theta & -\sin(\alpha+\theta) \end{pmatrix} \overset{\mathrm{H}^-}{\sim} \begin{pmatrix} -\sin(\alpha-\theta) & \sin(-\theta) \\ \sin(-\theta) & \sin(\alpha+\theta) \end{pmatrix}$$

and

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \overset{\mathrm{H}^-}{\sim} \begin{pmatrix} -\sin\alpha & 0 \\ 0 & \sin\alpha \end{pmatrix}.$$

So A_0 and A_1 both satisfy formula (5.1). Suppose that

$$A_{k-2} = \begin{pmatrix} -\sin(\alpha+(k-3)\theta) & \sin(k-3)\theta \\ \sin(k-3)\theta & \sin(\alpha-(k-3)\theta) \end{pmatrix},$$

and

$$A_{k-1} = \begin{pmatrix} -\sin(\alpha+(k-2)\theta) & \sin(k-2)\theta \\ \sin(k-2)\theta & \sin(\alpha-(k-2)\theta) \end{pmatrix}.$$

Because C_k is obtained by reflecting C_{k-2} at C_{k-1}, via Theorem 2.2.2 we validate that

$$\begin{aligned} A_k &= A_{k-1}A_{k-2}^{-1}A_{k-1} \\ &\overset{\mathrm{H}^-}{\sim} \begin{pmatrix} -\sin(\alpha+(k-1)\theta) & \sin(k-1)\theta \\ \sin(k-1)\theta & \sin(\alpha-(k-1)\theta) \end{pmatrix}. \end{aligned}$$

$\square$

Let R_k denote the circle reflection at C_k, $k \in \mathbb{Z}$. Let $\mathrm{Inv}(D_{\alpha,\theta})$ be the group generated by R_0 and R_1, the two circle reflections respectively at the two boundary arcs C_0 and C_1 of the circular digon $D_{\alpha,\theta}$. Let $M := R_1 \circ R_0$. From Corollary 2.4.8 we see that

$$\mathrm{Inv}(D_{\alpha,\theta}) =< R_0, R_1 >= \{M^k,\ M^k \circ R_0 |\ k \in \mathbb{Z}\}.$$

Particularly, we see a result below for the case $\theta = \frac{\pi}{n}$, $n \in \mathbb{N}^*$.

Lemma 5.1.2. *If $\theta = \frac{\pi}{n}$ for some $n \in \mathbb{N}^*$, then*

$$\mathrm{Inv}(D_{\alpha,\theta}) =< R_0, R_1 >= \{M^k,\ M^k \circ R_0 |\ k = 0, 1, \cdots, n-1\},$$

where $M = R_1 \circ R_0$.

Proof. The matrix associated with C_0 is

$$A_0 := \begin{pmatrix} \sin(\alpha-\theta) & \sin(\theta) \\ \sin(\theta) & -\sin(\alpha+\theta) \end{pmatrix},$$

and the matrix associated with C_1 is

$$A_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$M = R_1 \circ R_0$ is a Möbius transformation, we also denote the matrix of M by M. Then Theorem 2.3.6 implies that

$$M = A_0 A_1^{-1} = \begin{pmatrix} \sin(\alpha+\theta) & \sin(\theta) \\ -\sin(\theta) & -\sin(\alpha-\theta) \end{pmatrix}.$$

It is easy to determine the eigenvalues of M, which are $\lambda_1 = \sin(\alpha)e^{i\theta}$ and $\lambda_2 = \sin(\alpha)e^{-i\theta}$. Then we see that

$$\begin{aligned} \begin{pmatrix} \sin(\alpha)e^{i\theta} & 0 \\ 0 & \sin(\alpha)e^{-i\theta} \end{pmatrix}^n &= \begin{pmatrix} (\sin(\alpha))^n e^{in\theta} & 0 \\ 0 & (\sin(\alpha))^n e^{-in\theta} \end{pmatrix} \\ &= -(\sin(\alpha))^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

if $\theta = \frac{\pi}{n}$. It follows that $M^n = \mathrm{Id}$. The Möbius transformation M generates a cyclic group of order n. Therefore

$$\mathrm{Inv}(D_{\alpha,\theta}) =< R_0, R_1 >= \{M^k,\ M^k \circ R_0 |\ k = 0, 1, \cdots, n-1\}.$$

□

We already know that the family of domains $\{D_0, D_1, \cdots, D_{2n-1}\}$ provides a parqueting of $\mathbb{C}_\infty$ if $\theta = \frac{\pi}{n}$, $n \in \mathbb{N}^*$. Note that $D_k = R_k(D_{k-1})$. Theorem 2.4.5 implies that

$$M^k(D_0) = \left(\prod_{j=1}^{2k} R_j\right)(D_0) = D_{2k},$$

$$(M^k \circ R_0)(D_0) = \left(\prod_{j=1}^{2k-1} R_j\right)(D_0) = D_{2k-1},$$

for $k = 0, 1, \cdots, n-1$. Note that $D_{-1} = D_{2n-1}$ and $D_0 = D_{\alpha,\theta}$. Then

$$\begin{aligned}\{g(D_{\alpha,\theta})|g \in \mathrm{Inv}(D_{\alpha,\theta})\} &= \{M^k(D_0), (M^k \circ R_0)(D_0)|k = 0, 1, \cdots, n-1\} \\ &= \{D_0, D_1, \cdots, D_{2n-1}\}.\end{aligned}$$

We have thus verified that $D_{\alpha,\theta}$ is a finite parqueting-reflection domain.

Let $z_0 = z \in D_0$. The family of reflections $\{R_k|k \in \mathbb{N}\}$ produces a family of reflection images $\{z_k|k \in \mathbb{N}\}$, $z_k = (R_k \circ R_{k-1} \circ \cdots \circ R_1)(z_0) \in D_k$. Particularly, $z_1 = R_1(z_0) = 1/\bar{z} \in D_1$. Via Theorem 2.4.3, Lemma 5.1.1 and Formula (2.2), we deduce that

$$R_{k+1}([1:\bar{z}]) = \overline{[1:\bar{z}]A_{k+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = [-z\sin(\alpha - k\theta) - \sin k\theta : z\sin k\theta - \sin(\alpha + k\theta)],$$

$$R_{k+1}([z:1]) = \overline{[z:1]A_{k+1}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} = [\bar{z}\sin k\theta + \sin(\alpha - k\theta) : \bar{z}\sin(\alpha + k\theta) - \sin k\theta].$$

Therefore, we have

$$z_{2k} = R_{k+1}(z_1) = \frac{-z\sin(\alpha - k\theta) - \sin k\theta}{z\sin k\theta - \sin(\alpha + k\theta)} \tag{5.2}$$

and

$$z_{2k+1} = R_{k+1}(z_0) = \frac{\bar{z}\sin k\theta + \sin(\alpha - k\theta)}{\bar{z}\sin(\alpha + k\theta) - \sin k\theta}. \tag{5.3}$$

Remark 5.1.3. In the case of $\theta = \pi/n$, the relation $z_{2n+k} = z_k$ holds. Besides, $z_k = z_{2n-k-1}$ holds when z is in the circle C_0, while $z_{2k} = z_{2k+1}$ holds when z lies on the circle C_1.

5.2 Schwarz problem for inhomogeneous Cauchy-Riemann equation in $D_{\alpha,\theta}$

Since the circular digon $D_{\alpha,\theta}$ is a finite parqueting-reflection domain, applying Theorem 4.4.2 to $D_{\alpha,\theta}$, we solve the Schwarz problem for the inhomogeneous Cauchy-Riemann equation in $D_{\alpha,\theta}$.

Theorem 5.2.1. *Given $f \in L_p(D_{\alpha,\theta}; \mathbb{C})$, $p > 2$ and $\gamma \in C(\partial D_{\alpha,\theta}; \mathbb{R})$, the Schwarz problem*

$$\partial_{\bar{z}} w = f \quad in\ D_{\alpha,\theta}, \quad \mathrm{Re}(w) = \gamma \quad on\ \partial D_{\alpha,\theta}$$

is solvable and the solutions are of the form

$$\begin{aligned} w(z) =& \frac{1}{2\pi i}\sum_{k=0}^{n-1}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\left(\frac{\mathrm{d}\zeta}{\zeta - z_{2k}} + \frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z_{2k+1}}}\right) \\ &- \frac{1}{\pi}\sum_{k=0}^{n-1}\int_{D_{\alpha,\theta}}\left(\frac{f(\zeta)}{\zeta - z_{2k}} - \frac{\overline{f(\zeta)}}{\overline{\zeta} - \overline{z_{2k+1}}}\right)\mathrm{d}\sigma_\zeta + ic, \end{aligned} \tag{5.4}$$

where c is a real constant.

We show below the derivation of the Cauchy-Schwarz-Pompeiu integral formula for $D_{\alpha,\theta}$ by using parqueting-reflection principle.

According to the Cauchy-Pompeiu formula, for a function $w(z) \in \mathrm{C}^1(D_{\alpha,\theta}, \mathbb{C}) \cap \mathrm{C}(\overline{D_{\alpha,\theta}}, \mathbb{C})$ the formula

$$\frac{1}{2\pi i}\int_{\partial D_{\alpha,\theta}} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z} - \frac{1}{\pi}\int_{D_{\alpha,\theta}} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z} = \begin{cases} w(z), & \text{for } z \in D_{\alpha,\theta}, \\ 0, & \text{for } z \notin D_{\alpha,\theta} \end{cases} \tag{5.5}$$

holds. If $z \in D_{\alpha,\theta}$, among the reflection images $z_0, \cdots, z_{2n-1}$, only z_0 is located in $D_{\alpha,\theta}$, all the other reflection images are outside $D_{\alpha,\theta}$. Thus we have

$$\frac{1}{2\pi i}\int_{\partial D_{\alpha,\theta}} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z_{2k}} - \frac{1}{\pi}\int_{D_{\alpha,\theta}} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z_{2k}} = \begin{cases} w(z), & k = 0, \\ 0, & k = 1, \cdots, n-1, \end{cases} \tag{5.6}$$

$$\frac{1}{2\pi i}\int_{\partial D_{\alpha,\theta}} w(\zeta)\frac{\mathrm{d}\zeta}{\zeta - z_{2k+1}} - \frac{1}{\pi}\int_{D_{\alpha,\theta}} w_{\overline{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z_{2k+1}} = 0, \quad \text{for } k = 0, \cdots, n-1. \tag{5.7}$$

Taking the complex conjugate of the formulas for z_{2k+1} produces

$$-\frac{1}{2\pi i}\int_{\partial D_{\alpha,\theta}} \overline{w(\zeta)}\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z_{2k+1}}} - \frac{1}{\pi}\int_{D_{\alpha,\theta}} \overline{w_{\overline{\zeta}}(\zeta)}\frac{\mathrm{d}\sigma_\zeta}{\overline{\zeta} - \overline{z_{2k+1}}} = 0, \quad \text{for } k = 0, \cdots, n-1. \tag{5.8}$$

Lemma 5.2.2. *Let $C := \{\zeta \in \mathbb{C} |\, a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0\}$ be a generalized circle, where $0 \le a \in \mathbb{R}$, $b \in \mathbb{C}$, $c \in \mathbb{R}$ and $ac - b\overline{b} < 0$. Let s be the arc length parameter of C, and let R denote the reflection at C. Let $z \in \mathbb{C}$ and $z_{\mathrm{re}} = R(z)$. Then for $\zeta \in C \setminus \{z\}$, we have*

$$\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z_{\mathrm{re}}}} = \frac{az+b}{\zeta - z}\frac{\mathrm{d}\zeta}{a\zeta + b} = i\frac{az+b}{\zeta - z}\frac{\mathrm{d}s}{\sqrt{b\overline{b} - ac}}.$$

Proof. We have

$$z_{\mathrm{re}} = R(z) = -\frac{b\overline{z} + c}{a\overline{z} + \overline{b}}$$

via the definition of circle reflections. If $a\zeta\overline{\zeta} + \overline{b}\zeta + b\overline{\zeta} + c = 0$, then

$$\frac{\mathrm{d}\overline{\zeta}}{a\overline{\zeta} + \overline{b}} = -\frac{\mathrm{d}\zeta}{a\zeta + b}$$

holds. Therefore

$$\frac{\mathrm{d}\overline{\zeta}}{\overline{\zeta} - \overline{z_{\mathrm{re}}}} = -\frac{a\overline{\zeta} + \overline{b}}{\overline{\zeta} - \overline{z_{\mathrm{re}}}}\frac{\mathrm{d}\zeta}{a\zeta + b} = \frac{a\overline{\zeta} + \overline{b}}{\frac{\overline{b}\zeta + c}{a\zeta + b} - \frac{\overline{b}z + c}{az + b}}\frac{\mathrm{d}\zeta}{a\zeta + b} = \frac{az+b}{\zeta - z}\frac{\mathrm{d}\zeta}{a\zeta + b}. \tag{5.9}$$

If $a > 0$, C is a circle, and

$$\frac{\mathrm{d}\zeta}{a\zeta + b} = i\frac{\mathrm{d}s}{\sqrt{b\overline{b} - ac}}$$

holds. Actually, this relation is also true for the case of $a = 0$. In this case, C is a straight line, and we have

$$\frac{\mathrm{d}\zeta}{b} = i\frac{\mathrm{d}s}{|b|}.$$

The second equality in the conclusion is thus guaranteed. □

Let R_0 denote the circle reflection at C_0 and R_1 denote the circle reflection at C_1. Via Corollary 2.4.4 we see that the reflection images satisfy the relations $R_0(z_k) = z_{-k-1}$ and $R_1(z_k) = z_{-k+1}$ for $k \in \mathbb{Z}$. Note that $z_k = z_{2n+k}$ holds. It follows that

$$R_0(z_{2k+1}) = z_{2n-2k-2}, \quad R_1(z_{2k+1}) = z_{2n-2k}, \quad k = 0, \cdots, n-1.$$

If $\zeta \in C_0$, i.e., $\zeta\bar{\zeta}\sin(\alpha-\theta) + \zeta\sin\theta + \bar{\zeta}\sin\theta - \sin(\alpha+\theta) = 0$, applying Lemma 5.2.2 to the the family of formulas (5.8) we see that

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_0}\overline{w(\zeta)}\frac{\sin(\alpha-\theta)z_{2n-2k-2}+\sin\theta}{\zeta - z_{2n-2k-2}}\frac{\mathrm{d}\zeta}{\sin(\alpha-\theta)\zeta+\sin\theta}\\ &+\frac{1}{2\pi i}\int_{C_1}\overline{w(\zeta)}\frac{z_{2n-2k}}{\zeta - z_{2n-2k}}\frac{\mathrm{d}\zeta}{\zeta} + \frac{1}{\pi}\int_{D_{\alpha,\theta}}\overline{w_{\bar{\zeta}}(\zeta)}\frac{\mathrm{d}\sigma_\zeta}{\bar{\zeta} - \overline{z_{2k+1}}} = 0\end{aligned} \tag{5.10}$$

for $k = 0, \cdots, n-1$. Rewriting the formulas (5.6) produces

$$\begin{aligned}&\frac{1}{2\pi i}\int_{C_0} w(\zeta)\frac{\sin(\alpha-\theta)\zeta+\sin\theta}{\zeta - z_{2k}}\frac{\mathrm{d}\zeta}{\sin(\alpha-\theta)\zeta+\sin\theta}\\ &+\frac{1}{2\pi i}\int_{C_1} w(\zeta)\frac{\zeta}{\zeta - z_{2k}}\frac{\mathrm{d}\zeta}{\zeta} - \frac{1}{\pi}\int_{D_{\alpha,\theta}} w_{\bar{\zeta}}(\zeta)\frac{\mathrm{d}\sigma_\zeta}{\zeta - z_{2k}} = \begin{cases} w(z), & k = 0,\\ 0, & k = 1,\cdots,n-1.\end{cases}\end{aligned} \tag{5.11}$$

Adding these two sets of formulas (5.10) and (5.11) produces the Cauchy-Schwarz-Pompeiu representation formula for $D_{\alpha,\theta}$:

$$\begin{aligned}w(z) =&\frac{1}{2\pi\sin\alpha}\int_{C_0}\mathrm{Re}(w(\zeta))\sum_{k=0}^{n-1}\frac{\sin(\alpha-\theta)(\zeta+z_{2k})+2\sin\theta}{\zeta - z_{2k}}\mathrm{d}s_\zeta\\ &+\frac{1}{2\pi}\int_{C_1}\mathrm{Re}(w(\zeta))\sum_{k=0}^{n-1}\frac{\zeta+z_{2k}}{\zeta - z_{2k}}\mathrm{d}s_\zeta + i\frac{n\sin(\alpha-\theta)}{2\pi\sin\alpha}\int_{C_0}\mathrm{Im}(w(\zeta))\mathrm{d}s_\zeta\\ &+i\frac{n}{2\pi}\int_{C_1}\mathrm{Im}(w(\zeta))\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}}\sum_{k=0}^{n-1}\left(\frac{w_{\bar{\zeta}}(\zeta)}{\zeta - z_{2k}} - \frac{\overline{w_{\bar{\zeta}}(\zeta)}}{\bar{\zeta} - \overline{z_{2k+1}}}\right)\mathrm{d}\sigma_\zeta.\end{aligned} \tag{5.12}$$

5.3 Dirichlet problem for Poisson equation in $D_{\alpha,\theta}$

In this section we apply the parqueting-reflection principle for constructing the harmonic Green function for $D_{\alpha,\theta}$ and then solve the Dirichlet problem for the Poisson equation in $D_{\alpha,\theta}$. Although the harmonic Green functions for finite parqueting-reflection domains have already been generally investigated in preceding chapters, it is still worthy to show the details for this particular domain $D_{\alpha,\theta}$ here.

Let

$$\widetilde{F}(z,\zeta) := \prod_{k=0}^{n-1} \frac{\zeta - z_{2k+1}}{\zeta - z_{2k}} = F(z,\zeta) \prod_{k=0}^{n-1} \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\bar{z}\sin(\alpha + k\theta) - \sin k\theta},$$

where

$$\begin{aligned} F(z,\zeta) &= \prod_{k=0}^{n-1} \frac{\bar{z}\zeta \sin(\alpha + k\theta) - (\bar{z} + \zeta)\sin k\theta - \sin(\alpha - k\theta)}{z\zeta \sin k\theta + z \sin(\alpha - k\theta) - \zeta \sin(\alpha + k\theta) + \sin k\theta} \\ &= \frac{\bar{z}\zeta - 1}{\zeta - z} \prod_{k=1}^{n-1} \frac{\bar{z}\zeta \sin(\alpha + k\theta) - (\bar{z} + \zeta)\sin k\theta - \sin(\alpha - k\theta)}{z\zeta \sin k\theta + z \sin(\alpha - k\theta) - \zeta \sin(\alpha + k\theta) + \sin k\theta}. \end{aligned}$$

It is obvious that $\log |F(z,\zeta)|^2$ as a function in the variable z is harmonic in $D_{\alpha,\theta} \setminus \{\zeta\}$, and $\log |F(z,\zeta)|^2 + \log |\zeta - z|^2$ is harmonic in $D_{\alpha,\theta}$. Before showing that $\log |F(z,\zeta)|^2$ vanishes on the boundary $\partial D_{\alpha,\theta}$, we firstly investigate some properties of $F(z,\zeta)$.

Lemma 5.3.1. *If* $\theta = \frac{\pi}{n}$, *then* $\prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\bar{z}\sin(\alpha + k\theta) - \sin k\theta} \right| = 1$ *hold for* $z \in \partial D_{\alpha,\theta}$.

Proof. In the case of $z \in C_0$, $z\bar{z}\sin(\alpha - \theta) + z\sin\theta + \bar{z}\sin\theta - \sin(\alpha + \theta) = 0$ and $|z\sin(\alpha - \theta) + \sin\theta| = \sin\alpha$ hold. Substituting $\bar{z}$ by $\frac{-z\sin\theta + \sin(\alpha+\theta)}{z\sin(\alpha-\theta)+\sin\theta}$, we see that

$$\begin{aligned} \prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\bar{z}\sin(\alpha + k\theta) - \sin k\theta} \right| &= \prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\frac{-z\sin\theta + \sin(\alpha+\theta)}{z\sin(\alpha-\theta)+\sin\theta}\sin(\alpha + k\theta) - \sin k\theta} \right| \\ &= \prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{z\sin(k+1)\theta - \sin(\alpha + (k+1)\theta)} \frac{z\sin(\alpha - \theta) + \sin\theta}{\sin\alpha} \right| \\ &= \left| \frac{-\sin\alpha}{z\sin n\theta - \sin(\alpha + n\theta)} \right| \\ &\overset{n\theta=\pi}{=\!=} \left| \frac{-\sin\alpha}{z\sin\pi - \sin(\alpha + \pi)} \right| \\ &= 1. \end{aligned}$$

In the other case of $z \in C_1$, the relation $z\bar{z} = 1$ implies that

$$\prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\bar{z}\sin(\alpha + k\theta) - \sin k\theta} \right| = \prod_{k=0}^{n-1} \left| \frac{z \sin k\theta - \sin(\alpha + k\theta)}{\sin(\alpha + k\theta) - z\sin k\theta} \right| = 1.$$

The conclusion holds for both cases. $\square$

From Remark 5.1.3 we know that $\lim_{z \to \partial D_{\alpha,\theta}} \widetilde{F}(z,\zeta) = 1$. Then Lemma 5.3.1 ensures that $\lim_{z \to \partial D_{\alpha,\theta}} |F(z,\zeta)| = 1$. Thus, $\log |F(z,\zeta)|^2$ vanishes on the boundary. Therefore the harmonic Green function of $D_{\alpha,\theta}$ is obtained, as stated in the following theorem.

Theorem 5.3.2. *The harmonic Green function of $D_{\alpha,\theta}$ is $G_1(z,\zeta) = \log|F(z,\zeta)|^2$.*

We next discuss the Poisson kernel for $D_{\alpha,\theta}$.

Let $p(z,\zeta) := -\frac{1}{2}\partial_{\nu_\zeta} G_1(z,\zeta)$, where $z \in D_{\alpha,\theta}$ and $\zeta \in \partial D_{\alpha,\theta}$. It is called the Poisson kernel of $D_{\alpha,\theta}$. We can calculate the expression of $p(z,\zeta)$ directly. First, we obtain that

$$\begin{aligned}\partial_\zeta G_1(z,\zeta) = &-\sum_{k=0}^{n-1} \frac{z\sin k\theta - \sin(\alpha + k\theta)}{z\zeta \sin k\theta + z\sin(\alpha - k\theta) - \zeta\sin(\alpha + k\theta) + \sin k\theta} \\ &+ \sum_{k=0}^{n-1} \frac{\bar{z}\sin(\alpha + k\theta) - \sin k\theta}{\bar{z}\zeta\sin(\alpha + k\theta) - \bar{z}\sin k\theta - \zeta\sin k\theta - \sin(\alpha - k\theta)}.\end{aligned}$$

On the boundary arc $C_0 \subset \{\zeta \in \mathbb{C} \,|\, \zeta\bar{\zeta}\sin(\alpha-\theta) + \zeta\sin\theta + \bar{\zeta}\sin\theta - \sin(\alpha+\theta) = 0\}$, the outward normal vector is

$$\nu_\zeta = -\frac{\zeta\sin(\alpha-\theta) + \sin\theta}{\sin\alpha},$$

and the outward normal derivative is

$$\partial_{\nu_\zeta} = -\left(\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_\zeta + \frac{\bar{\zeta}\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_{\bar{\zeta}}\right).$$

Especially, C_0 is a line segment when $\alpha = \theta$. In this case we have $\nu_\zeta = -1$ and $\partial_{\nu_\zeta} = -\partial_\zeta - \partial_{\bar{\zeta}}$.

We can check that

$$\begin{aligned}&\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\frac{z\sin k\theta - \sin(\alpha+k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta} \\ =&\frac{\sin(\alpha-\theta)}{\sin\alpha} - \frac{z\sin(\alpha-(k+1)\theta) + \sin(k+1)\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}.\end{aligned}$$

Replacing ζ by $\frac{-\bar{\zeta}\sin\theta + \sin(\alpha+\theta)}{\bar{\zeta}\sin(\alpha-\theta)+\sin\theta}$ in

$$\frac{\bar{z}\sin(\alpha+k\theta) - \sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta) - (\bar{z}+\zeta)\sin k\theta - \sin(\alpha-k\theta)}$$

gives

$$\begin{aligned}&-\frac{\bar{z}\sin(\alpha+k\theta) - \sin k\theta}{\bar{z}\bar{\zeta}\sin(k+1)\theta - \bar{z}\sin(\alpha+(k+1)\theta) + \bar{\zeta}\sin(\alpha-(k+1)\theta) + \sin(k+1)\theta} \\ &\times\frac{\bar{\zeta}\sin(\alpha-\theta)+\sin\theta}{\sin\alpha},\end{aligned}$$

which implies that

$$\begin{aligned}
&\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta)-(\bar{z}+\zeta)\sin k\theta-\sin(\alpha-k\theta)}\\
=&-\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\bar{\zeta}\sin{(k+1)\theta}-\bar{z}\sin(\alpha+(k+1)\theta)+\bar{\zeta}\sin(\alpha-(k+1)\theta)+\sin{(k+1)\theta}}\\
&\times\frac{|\zeta\sin(\alpha-\theta)+\sin\theta|^2}{\sin^2\alpha}\\
=&-\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\bar{\zeta}\sin{(k+1)\theta}-\bar{z}\sin(\alpha+(k+1)\theta)+\bar{\zeta}\sin(\alpha-(k+1)\theta)+\sin{(k+1)\theta}}.
\end{aligned}$$

Let $l:=n-k-1$. Since $n\theta=\pi$, we have

$$\begin{aligned}
&\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta)-(\bar{z}+\zeta)\sin k\theta-\sin(\alpha-k\theta)}\\
=&\frac{\bar{z}\sin(\alpha-(l+1)\theta)+\sin{(l+1)\theta}}{\bar{z}\bar{\zeta}\sin l\theta+\bar{z}\sin(\alpha-l\theta)-\bar{\zeta}\sin(\alpha+l\theta)+\sin l\theta}.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
&\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_\zeta G_1(z,\zeta)\\
=&-\frac{n\sin(\alpha-\theta)}{\sin\alpha}+2\sum_{k=0}^{n-1}\mathrm{Re}\left(\frac{z\sin(\alpha-(k+1)\theta)+\sin{(k+1)\theta}}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right).
\end{aligned}$$

Thus in the case of $\zeta\in C_0$, the Poisson kernel of $D_{\alpha,\theta}$ is

$$\begin{aligned}
&p(z,\zeta)\\
=&-\frac{1}{2}\partial_{\nu_\zeta}G_1(z,\zeta)\\
=&\mathrm{Re}\left(\frac{\zeta\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_\zeta G_1(z,\zeta)\right)\\
=&-\frac{n\sin(\alpha-\theta)}{\sin\alpha}+2\sum_{k=0}^{n-1}\mathrm{Re}\left(\frac{z\sin(\alpha-(k+1)\theta)+\sin{(k+1)\theta}}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right).
\end{aligned}$$

On the other boundary arc C_1, the outward normal derivative is $\partial_{\nu_\zeta} = \zeta\partial_\zeta + \bar{\zeta}\partial_{\bar{\zeta}}$. Then

$$\begin{aligned}
&\zeta\partial_\zeta G_1(z,\zeta)\\
=&-\sum_{k=0}^{n-1}\frac{z\zeta\sin k\theta-\zeta\sin(\alpha+k\theta)}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\\
&+\sum_{k=0}^{n-1}\frac{\bar{z}\zeta\sin(\alpha+k\theta)-\zeta\sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta)-\bar{z}\sin k\theta-\zeta\sin k\theta-\sin(\alpha-k\theta)}\\
\overset{\zeta\bar{\zeta}=1}{=\!=}&\sum_{k=0}^{n-1}\left(-1+\frac{z\sin(\alpha-k\theta)+\sin k\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right)\\
&+\sum_{k=0}^{n-1}\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\sin(\alpha+k\theta)-\bar{z}\bar{\zeta}\sin k\theta-\sin k\theta-\bar{\zeta}\sin(\alpha-k\theta)}\\
\overset{\substack{l:=n-k\\ n\theta=\pi}}{=\!=}&-n+\sum_{k=0}^{n-1}\frac{z\sin(\alpha-k\theta)+\sin k\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\\
&+\sum_{l=1}^{n}\frac{\bar{z}\sin(\alpha-l\theta)+\sin l\theta}{\bar{z}\bar{\zeta}\sin l\theta+\bar{z}\sin(\alpha-l\theta)-\bar{\zeta}\sin(\alpha+l\theta)+\sin l\theta}\\
=&-n+2\sum_{k=0}^{n-1}\operatorname{Re}\left(\frac{z\sin(\alpha-k\theta)+\sin k\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right).
\end{aligned}$$

Thus in the case of $\zeta\in C_1$, the Poisson kernel of $D_{\alpha,\theta}$ is

$$\begin{aligned}
&p(z,\zeta)\\
=&-\operatorname{Re}(\zeta\partial_\zeta G_1(z,\zeta))\\
=&n-2\sum_{k=0}^{n-1}\operatorname{Re}\left(\frac{z\sin(\alpha-k\theta)+\sin k\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right).
\end{aligned}$$

Remark 5.3.3. The definition of $p(z,\zeta)$ can be extended to $\overline{D_{\alpha,\theta}}\times\partial D_{\alpha,\theta}$. When $z\in\partial D_{\alpha,\theta}$, we have $\partial_\zeta G_1(z,\zeta)=0$ because $G_1(z,\zeta)$ vanishes on the boundary. Therefore $p(z,\zeta)=0$ holds when $z,\zeta\in\partial D_{\alpha,\theta}$ and $z\neq\zeta$.

Further investigation about the boundary behavior of the Poisson kernel $p(z,\zeta)$ is necessary. Let $g_0(z,\zeta)$ be the harmonic Green function for the domain

$$\{\zeta\in\mathbb{C}\,|\,\zeta\bar{\zeta}\sin(\alpha-\theta)+\zeta\sin\theta+\bar{\zeta}\sin\theta-\sin(\alpha+\theta)>0\},$$

and $p_0(z,\zeta)$ be the corresponding Poisson kernel. It is easy to check that

$$g_0(z,\zeta)=\log\left|\frac{\bar{z}\zeta\sin(\alpha-\theta)+\bar{z}\sin\theta+\zeta\sin\theta-\sin(\alpha+\theta)}{(z-\zeta)\sin\alpha}\right|^2,$$

$$p_0(z,\zeta) = -\frac{1}{2}\partial_{\nu_\zeta} g_0(z,\zeta) = -\frac{\sin(\alpha-\theta)}{\sin\alpha} + 2\,\mathrm{Re}\left(\frac{z\sin(\alpha-\theta)+\sin\theta}{(z-\zeta)\sin\alpha}\right).$$

Denote the harmonic Green function and the Poisson kernel for the unit disk $\{\zeta \in \mathbb{C} \,\big|\, |\zeta| \le 1\}$ respectively by $g_1(z,\zeta)$ and $p_1(z,\zeta)$. It is well known that

$$g_1(z,\zeta) = \log\left|\frac{\bar{z}\zeta - 1}{z-\zeta}\right|^2,$$

$$p_1(z,\zeta) = -\frac{1}{2}\partial_{\nu_\zeta} g_1(z,\zeta) = \frac{\zeta}{\zeta - z} + \frac{\overline{\zeta}}{\overline{\zeta} - \overline{z}} - 1 = 1 - 2\mathrm{Re}\left(\frac{z}{z-\zeta}\right).$$

With these notations, a boundary property of $p(z,\zeta)$ is shown in the next lemma.

Lemma 5.3.4.

$$\begin{cases} \lim\limits_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} p(z,\zeta) = \lim\limits_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} p_0(z,\zeta), & \textit{if } \zeta \in C_0,\\ \lim\limits_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} p(z,\zeta) = \lim\limits_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} p_1(z,\zeta), & \textit{if } \zeta \in C_1 \end{cases}$$

Proof. If z tends to C_0, replacing z by $\frac{-\bar{z}\sin\theta+\sin(\alpha+\theta)}{\bar{z}\sin(\alpha-\theta)+\sin\theta}$ gives

$$\begin{aligned} &\partial_\zeta G_1(z,\zeta) - \partial_\zeta g_0(z,\zeta)\\ =&-\sum_{k=1}^{n-1}\frac{\bar{z}\sin(\alpha+(k-1)\theta) - \sin(k-1)\theta}{\bar{z}\zeta\sin(\alpha+(k-1)\theta) - \bar{z}\sin(k-1)\theta - \zeta\sin(k-1)\theta - \sin(\alpha-(k-1)\theta)}\\ &+\sum_{k=0}^{n-2}\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta) - \bar{z}\sin k\theta - \zeta\sin k\theta - \sin(\alpha-k\theta)}\\ =&0. \end{aligned}$$

Hence $\partial_{\nu_\zeta} G_1(z,\zeta) - \partial_{\nu_\zeta} g_0(z,\zeta) = 0$ when z tends to C_0. Likewise, if z tends to C_1, then via substituting $\bar{z}$ by $1/z$, we see that

$$\begin{aligned} &\partial_\zeta G_1(z,\zeta) - \partial_\zeta g_1(z,\zeta)\\ =&-\sum_{k=1}^{n-1}\frac{z\sin k\theta - \sin(\alpha+k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\\ &+\sum_{k=1}^{n-1}\frac{\bar{z}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\zeta\sin(\alpha+k\theta) - \bar{z}\sin k\theta - \zeta\sin k\theta - \sin(\alpha-k\theta)}\\ =&0. \end{aligned}$$

Hence $\partial_{\nu_\zeta} G_1(z,\zeta) - \partial_{\nu_\zeta} g_1(z,\zeta) = 0$ follows when z goes to C_1.

By the above discussion, we deduce that, for $\zeta \in C_0$

$$\lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} p(z,\zeta) = \lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} p_0(z,\zeta),$$

while for $\zeta \in C_1$

$$\lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} p(z,\zeta) = \lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} p_1(z,\zeta).$$

$\square$

Then the following result is obtained from Remark 5.3.3 and Lemma 5.3.4.

Theorem 5.3.5. *Given $\gamma \in \mathrm{C}(\partial D_{\alpha,\theta};\mathbb{C})$, the boundary behavior*

$$\lim_{\substack{z\to \tilde{z}\\ z\in D_{\alpha,\theta}}} \frac{1}{2\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)p(z,\zeta)\mathrm{d}s_\zeta = \gamma(\tilde{z})$$

holds for any $\tilde{z} \in \partial D_{\alpha,\theta}$.

With the properties of Green function and the result of Theorem 5.3.5, the Green representation formula succeeds in giving the solution to the Dirichlet boundary problem for the Poisson equation in the domain $D_{\alpha,\theta}$, as shown in the next theorem.

Theorem 5.3.6. *The Dirichlet problem $w_{z\bar{z}} = f$ in $D_{\alpha,\theta}$, $w = \gamma$ on $\partial D_{\alpha,\theta}$ for $f \in \mathrm{L}_p(D_{\alpha,\theta};\mathbb{C})$, $p > 2$, $\gamma \in \mathrm{C}(\partial D_{\alpha,\theta},\mathbb{C})$ has a unique solution, which is provided by*

$$w(z) = -\frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\partial_{\nu_\zeta} G_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)G_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

5.4 Neumann problem for Poisson equation in $D_{\alpha,\theta}$

To construct a harmonic Neumann function for $D_{\alpha,\theta}$, we start with a function given by

$$\widetilde{Q}(z,\zeta) := \prod_{k=0}^{n-1}(\zeta - z_{2k})(\zeta - z_{2k+1}).$$

Via multiplying $\widetilde{Q}(z,\zeta)$ by the product of all the denominators appearing in the z_k terms, we obtain a new function

$$\begin{aligned} Q(z,\zeta) := &\prod_{k=0}^{n-1}(\bar{z}\zeta\sin(\alpha+k\theta) - (\bar{z}+\zeta)\sin k\theta - \sin(\alpha-k\theta)) \\ &\times\prod_{k=0}^{n-1}(z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta). \end{aligned}$$

From Theorem 3.2.7 we know that $N_1(z,\zeta) := -\log|Q(z,\zeta)|^2$ is a harmonic Neumann function for $D_{\alpha,\theta}$.

We see below some properties of the normal derivatives of $N_1(z,\zeta)$ on the boundary $\partial D_{\alpha,\theta}$.

Lemma 5.4.1.

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} \frac{2n\sin(\alpha-\theta)}{\sin\alpha}, & \text{for } z \in C_0, \zeta \in \overline{D_{\alpha,\theta}} \setminus \{z\}, \\ -2n, & \text{for } z \in C_1, \zeta \in \overline{D_{\alpha,\theta}} \setminus \{z\}. \end{cases}$$

Proof. The outward normal derivative can be calculated straightforwardly. Note that

$$\begin{aligned} \partial_z N_1(z,\zeta) = & -\sum_{k=0}^{n-1} \frac{\zeta\sin k\theta + sin(\alpha-k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta} \\ & -\sum_{k=0}^{n-1} \frac{\bar\zeta\sin(\alpha+k\theta) - \sin k\theta}{z\bar\zeta\sin(\alpha+k\theta) - (z+\bar\zeta)\sin k\theta - \sin(\alpha-k\theta)}. \end{aligned}$$

In the case of $z \in C_1$, $z\bar z = 1$ holds and we have

$$\begin{aligned} & z\partial_z N_1(z,\zeta) \\ = & -\sum_{k=0}^{n-1} \frac{z\zeta\sin k\theta + z\sin(\alpha-k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta} \\ & -\sum_{k=0}^{n-1} \frac{z\bar\zeta\sin(\alpha+k\theta) - z\sin k\theta}{z\bar\zeta\sin(\alpha+k\theta) - (z+\bar\zeta)\sin k\theta - \sin(\alpha-k\theta)} \\ = & -\sum_{k=0}^{n-1}\left(1 + \frac{\zeta\sin(\alpha+k\theta) - \sin k\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\right) \\ & +\sum_{k=0}^{n-1} \frac{\bar\zeta\sin(\alpha+k\theta) - \sin k\theta}{\bar z\bar\zeta\sin(k\theta) + \bar z\sin(\alpha-k\theta) - \bar\zeta\sin(\alpha+k\theta) + \sin k\theta} \\ = & -n - 2i\sum_{k=0}^{n-1} \mathrm{Im}\left(\frac{\zeta\sin(\alpha+k\theta) - \sin k\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\right). \end{aligned}$$

Thereby for $z \in C_1$ and $\zeta \in \overline{D_{\alpha,\theta}} \setminus \{z\}$, the normal derivative of $N_1(z,\zeta)$ with respect to z is

$$\partial_{\nu_z} N_1(z,\zeta) = z\partial_z N_1(z,\zeta) + \bar z\partial_{\bar z} N_1(z,\zeta) = 2\mathrm{Re}(z\partial_z N_1(z,\zeta)) = -2n.$$

In the other case of $z \in C_0$, the operator of outward normal derivative is

$$\partial_{\nu_z} = -\left(\frac{z\sin(\alpha-\theta) + \sin\theta}{\sin\alpha}\partial_z + \frac{\bar z\sin(\alpha-\theta) + \sin\theta}{\sin\alpha}\partial_{\bar z}\right).$$

Note that

$$\begin{aligned} & \frac{z\sin(\alpha-\theta) + \sin\theta}{\sin\alpha} \frac{\zeta\sin k\theta + \sin(\alpha-k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta} \\ = & \frac{\sin(\alpha-\theta)}{\sin\alpha} + \frac{\zeta\sin(\alpha+(k-1)\theta) - \sin\,(k-1)\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}. \end{aligned}$$

Replacing z by $\frac{-\bar{z}\sin\theta+\sin(\alpha+\theta)}{\bar{z}\sin(\alpha-\theta)+\sin\theta}$ in

$$\frac{\bar{\zeta}\sin(\alpha+k\theta)-\sin k\theta}{z\bar{\zeta}\sin(\alpha+k\theta)-(z+\bar{\zeta})\sin k\theta-\sin(\alpha-k\theta)}$$

gives

$$-\frac{\bar{\zeta}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\bar{\zeta}\sin(k+1)\theta+\bar{z}\sin(\alpha-(k+1)\theta)-\bar{\zeta}\sin(\alpha+(k+1)\theta)+\sin(k+1)\theta}$$
$$\times\frac{\bar{z}\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}.$$

Then we have

$$\begin{aligned}&\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\frac{\bar{\zeta}\sin(\alpha+k\theta)-\sin k\theta}{z\bar{\zeta}\sin(\alpha+k\theta)-(z+\bar{\zeta})\sin k\theta-\sin(\alpha-k\theta)}\\&=-\frac{\bar{\zeta}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\bar{\zeta}\sin(k+1)\theta+\bar{z}\sin(\alpha-(k+1)\theta)-\bar{\zeta}\sin(\alpha+(k+1)\theta)+\sin(k+1)\theta}.\end{aligned}$$

Hence

$$\begin{aligned}&\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_z N_1(z,\zeta)\\&=-\frac{n\sin(\alpha-\theta)}{\sin\alpha}-\sum_{k=0}^{n-1}\frac{\zeta\sin(\alpha+(k-1)\theta)-\sin(k-1)\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\\&\quad+\sum_{k=0}^{n-1}\frac{\bar{\zeta}\sin(\alpha+k\theta)-\sin k\theta}{\bar{z}\bar{\zeta}\sin(k+1)\theta+\bar{z}\sin(\alpha-(k+1)\theta)-\bar{\zeta}\sin(\alpha+(k+1)\theta)+\sin(k+1)\theta}\\&=-\frac{n\sin(\alpha-\theta)}{\sin\alpha}-2i\sum_{k=0}^{n-1}\operatorname{Im}\left(\frac{\zeta\sin(\alpha+(k-1)\theta)-\sin(k-1)\theta}{z\zeta\sin k\theta+z\sin(\alpha-k\theta)-\zeta\sin(\alpha+k\theta)+\sin k\theta}\right)\end{aligned}$$

follows, from which we know that, for $z\in C_0$ and $\zeta\in\overline{D_{\alpha,\theta}}\setminus\{z\}$,

$$\partial_{\nu_z}N_1(z,\zeta)=-2\operatorname{Re}\left(\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_z N_1(z,\zeta)\right)=\frac{2n\sin(\alpha-\theta)}{\sin\alpha}.$$

□

Remark 5.4.2. Let $\sigma(s):=\partial_{\nu_z}N_1(z,\zeta)$, $z=z(s)\in\partial D_{\alpha,\theta}$ and $\zeta\in\overline{D_{\alpha,\theta}}\setminus\{z\}$, where s is the arc length parameter. Then restating the result of Lemma 5.4.1 gives

$$\sigma(s)=\begin{cases}\frac{2n\sin(\alpha-\theta)}{\sin\alpha}, & \text{for } z(s)\in C_0,\\ -2n, & \text{for } z(s)\in C_1.\end{cases}$$

Formula (3.13) implies that

$$\partial_{\nu_\zeta}N_1(z,\zeta)=\begin{cases}\frac{2n\sin(\alpha-\theta)}{\sin\alpha}, & \text{for } \zeta\in C_0,\\ -2n, & \text{for } \zeta\in C_1.\end{cases}$$

Therefore $\sigma(s) = \partial_{\nu_\zeta} N_1(z,\zeta)$ holds for $\zeta = \zeta(s) \in \partial D_{\alpha,\theta}$. Moreover, we have

$$\begin{aligned}
-\frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \sigma(s)\mathrm{d}s =& -\frac{1}{4\pi}\left(\frac{2n\sin(\alpha-\theta)}{\sin\alpha}\int_{C_0}\mathrm{d}s - 2n\int_{C_1}\mathrm{d}s\right)\\
=& -\frac{1}{4\pi}(4n(\alpha-\theta) - 4n\alpha)\\
=& 1.
\end{aligned}$$

Remark 5.4.3. We are not sure if our Neumann function $N_1(z,\zeta)$ satisfies the normalization condition (N4) in Remark 1.1.17. However, we conjecture that the integral $\int_{\partial D_{\alpha,\theta}} \sigma(s) N_1(z,\zeta)\mathrm{d}s_\zeta$ is constant for $z \in D_{\alpha,\theta}$. If this conjecture can be verified, suppose that the constant is K, then $N_1(z,\zeta) - K$ will be the unique harmonic Neumann function of $D_{\alpha,\theta}$ satisfying the normalization condition.

Lemma 5.4.4. *For $\zeta \in C_0$,*

$$\lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} \left\{\mathrm{Re}\left(-\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_z N_1(z,\zeta)\right) - p_0(z,\zeta)\right\} = \frac{n\sin(\alpha-\theta)}{\sin\alpha};$$

while for $\zeta \in C_1$,

$$\lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} \{\mathrm{Re}(z\partial_z N_1(z,\zeta)) - p_1(z,\zeta)\} = -n.$$

Proof. For $\zeta \in C_0$,

$$-\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_z N_1(z,\zeta) = 2\frac{z\sin(\alpha-\theta)+\sin\theta}{(z-\zeta)\sin\alpha} + T_0(z,\zeta),$$

where

$$\begin{aligned}
T_0(z,\zeta) =& \sum_{k=1}^{n-1} \frac{\zeta\sin k\theta + sin(\alpha-k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\\
&+ \sum_{k=0}^{n-2} \frac{\bar\zeta\sin(\alpha+k\theta) - \sin k\theta}{z\bar\zeta\sin(\alpha+k\theta) - (z+\bar\zeta)\sin k\theta - \sin(\alpha-k\theta)}.
\end{aligned}$$

From the proof of Lemma 5.4.1, we see that $T_0(z,\zeta)$ tends to

$$\frac{(n-1)\sin(\alpha-\theta)}{\sin\alpha} - 2i\sum_{k=1}^{n-1}\mathrm{Im}\left(\frac{\zeta\sin(\alpha+(k-1)\theta) - \sin(k-1)\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\right)$$

when z tends to C_0. Hence

$$\begin{aligned}
&\lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} \mathrm{Re}\left(-\frac{z\sin(\alpha-\theta)+\sin\theta}{\sin\alpha}\partial_z N_1(z,\zeta)\right)\\
&= \lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} 2\mathrm{Re}\frac{z\sin(\alpha-\theta)+\sin\theta}{(z-\zeta)\sin\alpha} + \frac{(n-1)\sin(\alpha-\theta)}{\sin\alpha}\\
&= \lim_{\substack{z\to C_0\\ z\in D_{\alpha,\theta}}} p_0(z,\zeta) + \frac{n\sin(\alpha-\theta)}{\sin\alpha}.
\end{aligned}$$

For $\zeta\in C_1$,

$$z\partial_z N_1(z,\zeta) = -\frac{2z}{z-\zeta} - T_1(z,\zeta),$$

where

$$\begin{aligned}
T_1(z,\zeta) =& \sum_{k=1}^{n-1} \frac{\zeta\sin k\theta + sin(\alpha-k\theta)}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\\
&+ \sum_{k=1}^{n-1} \frac{\bar\zeta\sin(\alpha+k\theta) - \sin k\theta}{z\bar\zeta\sin(\alpha+k\theta) - (z+\bar\zeta)\sin k\theta - \sin(\alpha-k\theta)}.
\end{aligned}$$

When z tends to C_1, $T_1(z,\zeta)$ turns out to be

$$(n-1) + 2i\sum_{k=1}^{n-1} \mathrm{Im}\left(\frac{\zeta\sin(\alpha+k\theta)-\sin k\theta}{z\zeta\sin k\theta + z\sin(\alpha-k\theta) - \zeta\sin(\alpha+k\theta) + \sin k\theta}\right),$$

which is also shown in the proof of Lemma 5.4.1. Therefore for $\zeta\in C_1$,

$$\lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} \mathrm{Re}(z\partial_z N_1(z,\zeta)) = -\lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} 2\mathrm{Re}\frac{z}{z-\zeta} - (n-1) = \lim_{\substack{z\to C_1\\ z\in D_{\alpha,\theta}}} p_1(z,\zeta) - n.$$

□

Now we are ready to solve the Neumann boundary problem for the Poisson equation in $D_{\alpha,\theta}$.

Theorem 5.4.5. *For $f\in \mathrm{L}_p(D_{\alpha,\theta};\mathbb{C})$, $p>2$ and $\gamma\in \mathrm{C}(\partial D_{\alpha,\theta},\mathbb{C})$, the Neumann boundary problem*

$$w_{z\bar z} = f \ \text{ in } D_{\alpha,\theta}, \quad \partial_{\nu_z} w = \gamma \ \text{ on } \partial D_{\alpha,\theta} \text{ except for the two corner points,}$$

is solvable if and only if

$$\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\mathrm{d}s_\zeta = 4\int_{D_{\alpha,\theta}} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

The solutions are of the form

$$w(z) = c + \frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

where c is an arbitrary constant in $\mathbb{C}$.

Proof. We know that $-\frac{1}{\pi}N_1(z,\zeta)$ is a fundamental solution to the Poisson equation, and the boundary integral $\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)N_1(z,\zeta)\mathrm{d}s_\zeta$ is harmonic in $D_{\alpha,\theta}$. It immediately implies that $w_{z\bar{z}} = f$. The normal derivative of $w(z)$ is

$$\partial_{\nu_z} w(z) = \frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\partial_{\nu_z}N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)\partial_{\nu_z}N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta.$$

On the basis of Lemma 5.4.1 and Lemma 5.4.4, if $\tilde{z} \in C_0$,

$$\lim_{\substack{z\to\tilde{z}\\ z\in D_{\alpha,\theta}}} \partial_{\nu_z} w(z) = \gamma(\tilde{z}) + \frac{2n\sin(\alpha-\theta)}{\sin\alpha}\left(\frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\right),$$

if $\tilde{z} \in C_1$,

$$\lim_{\substack{z\to\tilde{z}\\ z\in D_{\alpha,\theta}}} \partial_{\nu_z} w(z) = \gamma(\tilde{z}) - 2n\left(\frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta\right).$$

Then $\partial_{\nu_z} w(z) = \gamma(z)$ on $\partial D_{\alpha,\theta}$ if and only if

$$\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)\mathrm{d}s_\zeta = 4\int_{D_{\alpha,\theta}} f(\zeta)\mathrm{d}\xi\mathrm{d}\eta. \tag{5.13}$$

Hence the function $w(z)$ solves the Neumann problem if and only if the solubility condition (5.13) is satisfied.

Suppose $\phi(z)$ also solves the Neumann problem, then $\phi(z) - w(z)$ is harmonic in $D_{\alpha,\theta}$ and its normal derivative vanishes on $\partial D_{\alpha,\theta}$. It implies that $\phi(z) - w(z)$ must be a constant.

To sum up, if the Neumann problem is solvable, all the solutions are of the form

$$w(z) = c + \frac{1}{4\pi}\int_{\partial D_{\alpha,\theta}} \gamma(\zeta)N_1(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_{D_{\alpha,\theta}} f(\zeta)N_1(z,\zeta)\mathrm{d}\xi\mathrm{d}\eta,$$

where c is a complex number. □

Chapter 6

Boundary Value Problems in a Circular Rectangle

We have shown that the parqueting-reflection principle can be used to construct harmonic Green and Neumann functions for finite parqueting-reflection domains and also help to solve Schwarz Problem and Dirichlet problem for finite parqueting-reflection domains. Although it is still unclear if the parqueting-reflection principle generally works for infinite parqueting domains, we have seen some successful examples, see e.g. [5, 10, 23, 26]. In this chapter, we show an example of infinite parqueting-reflection domains and verify that the parqueting-reflection principle method is feasible for this example.

The four circles $|z-1| = \sqrt{2}$, $|z+1| = \sqrt{2}$, $|z-\sqrt{3}i| = \sqrt{2}$ and $|z+\sqrt{3}i| = \sqrt{2}$ in the complex plane bound a circular rectangle

$$R := \{z \in \mathbb{C} \,|\, |z \pm 1| < \sqrt{2} < |z \pm \sqrt{3}i|\},$$

see Figure 6.1. We are going to show that R is a parqueting-reflection domain, apply the parqueting-reflection principle for constructing the harmonic Green function of R and then solve the Dirichlet problem for the Poisson equation in R.

6.1 A family of circles $\{|z - m_k i| = r_k, m_k^2 = r_k^2 + 1\}_{k\in\mathbb{Z}}$

Lemma 6.1.1. *The image of the circle $C_1 := \{z \in \mathbb{C} \,|\, |z - m_1 i| = r_1, m_1^2 = r_1^2 + 1\}$ under the reflection at another circle $C_2 := \{z \in \mathbb{C} \,|\, |z - m_2 i| = r_2, m_2^2 = r_2^2 + 1\}$ is the circle $C_3 := \{z \in \mathbb{C} \,|\, |z - m_3 i| = r_3, m_3^2 = r_3^2 + 1\}$ with the relations*

$$m_3 = \frac{\alpha_1 m_2 + \beta_1}{\beta_1 m_2 + \alpha_1}, \quad r_3 = \frac{r_1 r_2^2}{|\beta_1 m_2 + \alpha_1|},$$

where $\alpha_1 = m_1 m_2 - 1, \beta_1 = m_1 - m_2$.

Proof. By Lemma 2.2.2 we see that the matrix associated with C_3 is

$$\begin{pmatrix} 1 & im_2 \\ -im_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & im_1 \\ -im_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & im_2 \\ -im_2 & 1 \end{pmatrix}$$

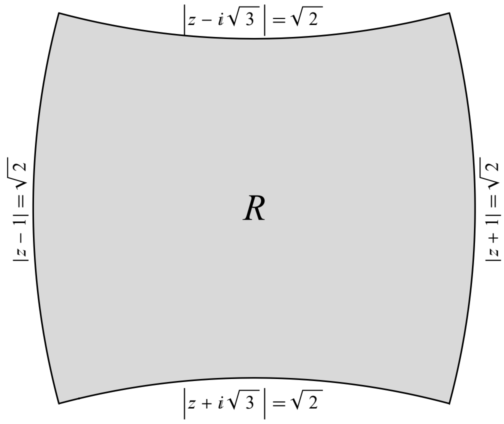


Figure 6.1: Circular rectangle R

$$\overset{\mathrm{H}^-}{\sim} \begin{pmatrix} (m_1-m_2)m_2+(m_1m_2-1) & ((m_1m_2-1)m_2+(m_1-m_2))i \\ -((m_1m_2-1)m_2+(m_1-m_2))i & (m_1-m_2)m_2+(m_1m_2-1) \end{pmatrix}.$$

Here "$\overset{\mathrm{H}^-}{\sim}$" means the equivalence relation on H^-, as introduced in Section 2.1. Let $\alpha_1 := m_1m_2-1$, $\beta_1 := m_1-m_2$, and $m_3 := \frac{\alpha_1 m_2+\beta_1}{\beta_1 m_2+\alpha_1}$. The center of C_3 is

$$\frac{(m_1m_2-1)m_2+(m_1-m_2)}{(m_1-m_2)m_2+(m_1m_2-1)}i = m_3 i,$$

and the radius is

$$r_3 = \sqrt{m_3^2-1} = \frac{\sqrt{m_1^2-1}(m_2^2-1)}{|(m_1-m_2)m_2+(m_1m_2-1)|} = \frac{r_1r_2^2}{|\beta_1m_2+\alpha_1|}.$$

□

By this lemma, we see that reflecting the circle $|z+\sqrt{3}i| = \sqrt{2}$ at the circle $|z-\sqrt{3}i| = \sqrt{2}$ results in the circle $|z-\frac{3\sqrt{3}}{5}i| = \frac{\sqrt{2}}{5}$. Let $m_1 := \sqrt{3}$, $r_1 := \sqrt{2}$, $m_2 := \frac{3\sqrt{3}}{5}$, and $r_2 := \frac{\sqrt{2}}{5}$. By operating consecutive reflections in the upward direction of R, a family of circles $\{|z-m_ki| = r_k,\ m_k^2 = r_k^2+1\}_{k\in\mathbb{N}^*}$ is produced. More specifically, reflecting the circle $|z-m_ki| = r_k$ at the circle $|z-m_{k+1}i| = r_{k+1}$ results in the circle $|z-m_{k+2}i| = r_{k+2}$. Lemma 6.1.1 ensures that

$$m_k^2 = r_k^2+1,\ m_{k+2} = \frac{\alpha_k m_{k+1}+\beta_k}{\beta_k m_{k+1}+\alpha_k},$$

where $\alpha_k = m_km_{k+1}-1$, $\beta_k = m_k-m_{k+1}$, $k \in \mathbb{N}^*$.

Remark 6.1.2. Actually, we can verify that $\frac{\beta_k}{\alpha_k}$ is constant, namely,

$$q := \frac{\beta_k}{\alpha_k} = \frac{\beta_1}{\alpha_1} = \frac{\sqrt{3}}{2} \text{ for } k \in \mathbb{N}^*.$$

It follows from

$$\frac{\alpha_{k+1}}{\beta_{k+1}} = \frac{m_{k+1}m_{k+2}-1}{m_{k+1}-m_{k+2}} = \frac{m_{k+1}\frac{\alpha_k m_{k+1}+\beta_k}{\beta_k m_{k+1}+\alpha_k}-1}{m_{k+1}-\frac{\alpha_k m_{k+1}+\beta_k}{\beta_k m_{k+1}+\alpha_k}} = \frac{\alpha_k(m_{k+1}^2-1)}{\beta_k(m_{k+1}^2-1)} = \frac{\alpha_k}{\beta_k}.$$

We thus have an iterating formula for m_k, namely, for $k \in \mathbb{N}^*$,

$$m_{k+1} = \frac{m_k + q}{qm_k + 1}. \tag{6.1}$$

Lemma 6.1.3. *The sequence $(m_k)_{k\in\mathbb{N}^*}$ decreases monotonically to the limit 1. Moreover,*

$$0 < m_{k+1} - 1 < (1-q)^k(m_1 - 1),$$

where $q = \frac{\sqrt{3}}{2}$, for all $k \in \mathbb{N}^*$.

Proof. Remark 6.1.2 immediately implies that

$$m_{k+1} - 1 = \frac{m_k + q}{qm_k + 1} - 1 = \frac{(1-q)(m_k-1)}{qm_k+1}.$$

It is easy to see that $m_k - 1 > 0$ for all $k \in \mathbb{N}^*$ by induction. Then the estimate

$$m_{k+1} - 1 = \frac{(1-q)(m_k-1)}{qm_k+1} < (1-q)(m_k-1)$$

follows. It shows that the sequence $(m_k - 1)_{k\in\mathbb{N}^*}$ is monotonically decreasing with limit 0, which implies the conclusion. $\square$

We consider analogously the consecutive circle reflections in the downward direction. Reflecting the circle $|z-\sqrt{3}i| = \sqrt{2}$ at the circle $|z+\sqrt{3}i| = \sqrt{2}$ results in the circle $|z + \frac{3\sqrt{3}}{5}i| = \frac{\sqrt{2}}{5}$. Let $m_{-1} := -\sqrt{3}$, $r_{-1} := \sqrt{2}$, $m_{-2} := -\frac{3\sqrt{3}}{5}$, and $r_{-2} := \frac{\sqrt{2}}{5}$. Consecutive downward reflections produce a sequence of circles $\{|z - m_{-k}i| = r_{-k}\}_{k\in\mathbb{N}^*}$ with the relations

$$m_{-k}^2 = r_{-k}^2 + 1,$$
$$m_{-(k+2)} = \frac{(m_{-k}m_{-(k+1)}-1)m_{-(k+1)} + (m_{-k}-m_{-(k+1)})}{(m_{-k}-m_{-(k+1)})m_{-(k+1)} + (m_{-k}m_{-(k+1)}-1)}.$$

The iterating formula implies that

$$\frac{m_{-k}-m_{-(k+1)}}{1-m_{-k}m_{-(k+1)}} = \frac{m_{-1}-m_{-2}}{1-m_{-1}m_{-2}} = \frac{\sqrt{3}}{2} = q, \ \forall k \in \mathbb{N}^*.$$

Then we have

$$m_{-(k+1)} = \frac{-m_{-k} + q}{qm_{-k} - 1}, \text{ or equivalently, } m_{-k} = \frac{m_{-(k+1)} + q}{qm_{-(k+1)} + 1}$$

for $k \in \mathbb{N}^*$. Moreover, we can verify the relations

$$m_{-k} = -m_k, \forall k \in \mathbb{N}^* \tag{6.2}$$

by mathematical induction.

Actually, from the iterating formula (6.1) and the first term m_1, an explicit formula for m_k can be derived.

Lemma 6.1.4. *For $k \in \mathbb{N}^*$, $m_k = \frac{(2+\sqrt{3})^{2k-1}+1}{(2+\sqrt{3})^{2k-1}-1}$ and $m_{-k} = \frac{(2+\sqrt{3})^{2k-1}+1}{1-(2+\sqrt{3})^{2k-1}}$.*

Proof. The iterating formula (6.1) can be rewritten in the form of homogenous coordinates as

$$[m_{k+1} : 1] = [m_k : 1] \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}.$$

Then we have

$$[m_k : 1] = [m_1 : 1] \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}^{k-1}.$$

By the Jordan normal decomposition

$$\begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} 1-q & 0 \\ 0 & 1+q \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix},$$

the calculation for the matrix power can be easily achieved, namely,

$$\begin{aligned} \begin{pmatrix} 1 & q \\ q & 1 \end{pmatrix}^{k-1} &= \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \begin{pmatrix} (1-q)^{k-1} & 0 \\ 0 & (1+q)^{k-1} \end{pmatrix} \begin{pmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} (1+q)^{k-1} + (1-q)^{k-1} & (1+q)^{k-1} - (1-q)^{k-1} \\ (1+q)^{k-1} - (1-q)^{k-1} & (1+q)^{k-1} + (1-q)^{k-1} \end{pmatrix}. \end{aligned}$$

Then we have

$$m_k = \frac{[(\frac{1+q}{1-q})^{k-1} + 1]m_1 + [(\frac{1+q}{1-q})^{k-1} - 1]}{[(\frac{1+q}{1-q})^{k-1} - 1]m_1 + [(\frac{1+q}{1-q})^{k-1} + 1]}.$$

Inserting $q = \frac{\sqrt{3}}{2}$ and $m_1 = \sqrt{3}$ results in the formula

$$m_k = \frac{(2+\sqrt{3})^{2k-1} + 1}{(2+\sqrt{3})^{2k-1} - 1}. \tag{6.3}$$

From the relation $m_{-k} = -m_k$ we immediately get the formula

$$m_{-k} = \frac{(2+\sqrt{3})^{2k-1} + 1}{1 - (2+\sqrt{3})^{2k-1}}. \tag{6.4}$$

□

Because the circle $|z-1| = \sqrt{2}$ is both perpendicular to $|z-m_1 i| = r_1$ and $|z-m_{-1}i| = r_{-1}$, so it is perpendicular to $|z-m_k i| = r_k$ for each $k \in \mathbb{Z}\backslash\{0\}$. Analogously, $|z+1| = \sqrt{2}$ is perpendicular to $|z - m_k i| = r_k$ for each $k \in \mathbb{Z} \setminus \{0\}$. These facts are based on angle-preserving property of circle reflections, and they will be applied in the following sections.

6.2 Parqueting of $\mathbb{C}_\infty$ provided by R

Let

$$
\begin{aligned}
C_0^l &:= \{z \in \mathbb{C} \,|\, |z-1| = \sqrt{2}\} \cap \overline{R},\\
C_0^r &:= \{z \in \mathbb{C} \,|\, |z+1| = \sqrt{2}\} \cap \overline{R},\\
C_1 &:= \{z \in \mathbb{C} \,|\, |z-\sqrt{3}i| = \sqrt{2}\} \cap \overline{R},\\
C_{-1} &:= \{z \in \mathbb{C} \,|\, |z+\sqrt{3}i| = \sqrt{2}\} \cap \overline{R},
\end{aligned}
$$

denote respectively the four boundary arcs of R, namely, $\partial R = C_0^l \cup C_0^r \cup C_{-1} \cup C_1$. Let $D_0 = R$. Reflecting D_0 at its upper boundary C_1 produces a new circular rectangle, denoted by D_1. Suppose that C_0^l, C_0^r and C_{-1} are reflected respectively to C_1^l, C_1^r and C_2, then $\partial D_1 = C_1^l \cup C_1^r \cup C_1 \cup C_2$. Operating consecutive reflections in the upward direction results in a family of circular rectangles $\{D_k\}_{k\in\mathbb{N}^*}$. Let $C_k^l \cup C_k^r \cup C_k \cup C_{k+1}$ denote the boundary of D_k. We next repeat the procedure of consecutive reflections in the downward direction. Reflecting D_0 at its lower boundary C_{-1} produces a circular rectangle, say D_{-1}, $\partial D_{-1} = C_{-1}^l \cup C_{-1}^r \cup C_{-1} \cup C_{-2}$. Consecutive reflections in the downward direction generate a family of circular rectangles, say D_{-k}, $k \in \mathbb{N}^*$. Let $C_{-k}^l \cup C_{-k}^r \cup C_{-k} \cup C_{-k-1}$ be the boundary of D_{-k}. With the above notations, C_k is a circular arc of the circle $|z-m_k i| = r_k$ for $k \in \mathbb{Z}\setminus\{0\}$. On the basis of the orthogonality for the boundary arcs of R, the two circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$ are mapped respectively to themselves under the reflection at circle $|z-m_k i| = r_k$ for each $k \in \mathbb{Z}\backslash\{0\}$. This implies that C_k^l is a circular arc of the circle $|z-1| = \sqrt{2}$ while C_k^r is a circular arc of the circle $|z+1| = \sqrt{2}$ for each $k \in \mathbb{Z}$. we therefore see that

$$
\begin{aligned}
D_k &= \{z \in \mathbb{C} \,|\, |z \pm 1| < \sqrt{2},\ |z - m_k i| < r_k,\ |z - m_{k+1} i| > r_{k+1}\},\\
D_{-k} &= \{z \in \mathbb{C} \,|\, |z \pm 1| < \sqrt{2},\ |z + m_k i| < r_k,\ |z + m_{k+1} i| > r_{k+1}\},
\end{aligned}
$$

for $k \in \mathbb{N}^*$.

From Lemma 6.1.3 and the fact $m_{-k} = -m_k$, we know that the sequence $(m_k)_{k\in\mathbb{N}^*}$ decreases monotonically to the limit 1 and $(m_{-k})_{k\in\mathbb{N}^*}$ increases monotonically to the limit -1. It then implies that the sequences $(r_k)_{k\in\mathbb{N}^*}$ and $(r_{-k})_{k\in\mathbb{N}^*}$ are both decreasing with limit 0. When k tends towards $+\infty$, the centers of circles $|z - m_k i| = r_k$ converge to the point i and theirs radii decrease to 0, in the meantime, the centers of circles $|z+m_k i| = r_k$ converge to the point $-i$. Note that i and $-i$ are just the two intersection points of the two circles $|z\pm 1| = \sqrt{2}$. Thereby we see that a family of domains $\{D_k\}_{k\in\mathbb{Z}}$ provides a parqueting of the lens domain

$$L_0 := \{z \in \mathbb{C} \,|\, |z \pm 1| < \sqrt{2}\},$$

namely,

$$\overline{L_0} = \bigcup_{k \in \mathbb{Z}} \overline{D_k}.$$

The orthogonality of the two circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$ guarantees a parqueting of the extended complex plane provided by L_0. Let

$$\begin{aligned} L_1 &:= \{z \in \mathbb{C} \,|\, |z-1| < \sqrt{2},\ |z+1| > \sqrt{2}\}, \\ L_2 &:= \{z \in \mathbb{C} \,|\, |z-1| > \sqrt{2},\ |z+1| > \sqrt{2}\} \cup \{\infty\}, \\ L_3 &:= \{z \in \mathbb{C} \,|\, |z-1| > \sqrt{2},\ |z+1| < \sqrt{2}\}. \end{aligned}$$

We see that L_1 is the reflection of L_0 at the circle $|z+1| = \sqrt{2}$, L_2 is the reflection of L_1 at the circle $|z-1| = \sqrt{2}$, and L_3 is the reflection of L_2 at the circle $|z+1| = \sqrt{2}$. Actually L_3 can also be produced by reflecting L_0 at the circle $|z-1| = \sqrt{2}$. Then the four domains L_0, L_1, L_2 and L_3 build a parqueting of the extended complex plane, namely,

$$\mathbb{C}_\infty = \overline{L_0} \cup \overline{L_1} \cup \overline{L_2} \cup \overline{L_3}.$$

Based on the above investigation, a parqueting for $\mathbb{C}_\infty$ can be achieved by reflections starting from the initial domain R. Let D_k^l denote the reflection of D_k at the circle $|z-1| = \sqrt{2}$, D_k^r denote the reflection of D_k at the circle $|z+1| = \sqrt{2}$, and D_k' be the reflection of D_k^r at the circle $|z-1| = \sqrt{2}$. All these domains are determined, namely,

$$\begin{aligned} D_0^l &= \{z \in \mathbb{C} \,|\, |z+1| < \sqrt{2} < |z-1|,\ |z+\sqrt{3}i| > \sqrt{2},\ |z-\sqrt{3}i| > \sqrt{2}\}, \\ D_0^r &= \{z \in \mathbb{C} \,|\, |z-1| < \sqrt{2} < |z+1|,\ |z+\sqrt{3}i| > \sqrt{2},\ |z-\sqrt{3}i| > \sqrt{2}\}, \\ D_0' &= \{z \in \mathbb{C} \,|\, |z \pm 1| > \sqrt{2},\ |z+\sqrt{3}i| > \sqrt{2},\ |z-\sqrt{3}i| > \sqrt{2}\} \cup \{\infty\}, \end{aligned}$$

and for $k \in \mathbb{N}^*$

$$\begin{aligned} D_k^l &= \{z \in \mathbb{C} \,|\, |z+1| < \sqrt{2} < |z-1|,\ |z-m_k i| < r_k,\ |z-m_{k+1} i| > r_{k+1}\}, \\ D_{-k}^l &= \{z \in \mathbb{C} \,|\, |z+1| < \sqrt{2} < |z-1|,\ |z+m_k i| < r_k,\ |z+m_{k+1} i| > r_{k+1}\}, \\ D_k^r &= \{z \in \mathbb{C} \,|\, |z-1| < \sqrt{2} < |z+1|,\ |z-m_k i| < r_k,\ |z-m_{k+1} i| > r_{k+1}\}, \\ D_{-k}^r &= \{z \in \mathbb{C} \,|\, |z-1| < \sqrt{2} < |z+1|,\ |z+m_k i| < r_k,\ |z+m_{k+1} i| > r_{k+1}\}, \\ D_k' &= \{z \in \mathbb{C} \,|\, |z \pm 1| > \sqrt{2},\ |z-m_k i| < r_k,\ |z-m_{k+1} i| > r_{k+1}\}, \\ D_{-k}' &= \{z \in \mathbb{C} \,|\, |z \pm 1| > \sqrt{2},\ |z+m_k i| < r_k,\ |z+m_{k+1} i| > r_{k+1}\}. \end{aligned}$$

Therefore the circular rectangle R generates a parqueting of the extended complex plane, namely

$$\mathbb{C}_\infty = \bigcup_{k \in \mathbb{Z}} (\overline{D_k} \cup \overline{D_k^l} \cup \overline{D_k^r} \cup \overline{D_k'}).$$

Figure 6.2 demonstrates this parqueting.

Let T_l, T_r, T_u and T_d denote the reflections respectively at the boundary arcs C_l, C_r, C_1 and C_{-1} of the circular rectangle R. Denote $\mathrm{Inv}(R) := \langle T_l, T_r, T_u, T_d \rangle$ the inversive

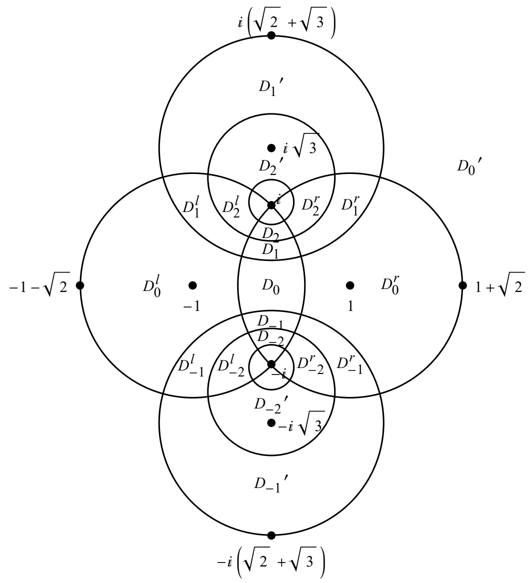


Figure 6.2: Parqueting generated by R

group generated by T_l, T_r, T_u and T_d. Let $M_1 := T_u \circ T_d$, $M_2 := T_r \circ T_l$. Since the two circles $|z \pm i\sqrt{3}| = \sqrt{2}$ lie outside each other, from Corollary 2.4.8 we know that M_1 generates an infinite cyclic group. Since the two circles $|z \pm 1| = \sqrt{2}$ are orthogonal, then Lemma 5.1.2 implies that $M_2^2 = \mathrm{Id}$. Because the two circle $|z \pm i\sqrt{3}| = \sqrt{2}$ are orthogonal to the other two $|z \pm 1| = \sqrt{2}$, Corollary 2.3.7 implies that

$$T_u \circ T_l = T_l \circ T_u, \quad T_u \circ T_r = T_r \circ T_u, \quad T_d \circ T_l = T_l \circ T_d, \quad T_d \circ T_r = T_r \circ T_d.$$

Then we can determine $\mathrm{Inv}(R)$, namely,

$$\mathrm{Inv}(R) = \left\{ \begin{matrix} M_1^k, & M_1^k \circ T_u, \\ T_l \circ M_1^k, & T_l \circ M_1^k \circ T_u, \\ T_r \circ M_1^k, & T_r \circ M_1^k \circ T_u, \\ M_2 \circ M_1^k, & M_2 \circ M_1^k \circ T_u \end{matrix} \right\}_{k \in \mathbb{Z}}.$$

It is easy to check that

$$\begin{array}{ll} M_1^k(R) = D_{2k}, & M_1^k \circ T_u(R) = D_{2k+1}, \\ T_l \circ M_1^k(R) = D_{2k}^l, & T_l \circ M_1^k \circ T_u(R) = D_{2k+1}^l, \\ T_r \circ M_1^k(R) = D_{2k}^r, & T_r \circ M_1^k \circ T_u(R) = D_{2k+1}^r, \\ M_2 \circ M_1^k(R) = D_{2k}^{'}, & M_2 \circ M_1^k(R) \circ T_u(R) = D_{2k+1}^{'}. \end{array}$$

Therefore we conclude that R is an infinite parqueting-reflection domain.

6.3 Reflection images

Let $z_0 = z \in D_0$. The reflection of z_0 at the circle $|z - m_1 i| = r_1$ is

$$z_1 := \frac{m_1 i \overline{z_0} - 1}{\overline{z_0} + m_1 i} \in D_1.$$

Let z_k denote the reflection of z_{k-1} at the circle $|z - m_k i| = r_k$ for $k \in \mathbb{N}^*$. We have $z_k \in D_k$ and the relation

$$z_k = \frac{m_k i \overline{z_{k-1}} - 1}{\overline{z_{k-1}} + m_k i}.$$

Then

$$z_{k+1} = \frac{m_{k+1} i \overline{z_k} - 1}{\overline{z_k} + m_{k+1} i} = \frac{\alpha_k z_{k-1} + \beta_k i}{-\beta_k i z_{k-1} + \alpha_k} = \frac{z_{k-1} + qi}{-qi z_{k-1} + 1}.$$

Denote that

$$A := \begin{pmatrix} 1 & -qi \\ qi & 1 \end{pmatrix}.$$

From the iteration for z_k we see that

$$[z_{2k} : 1] = [z_0 : 1] A^k,$$
$$[z_{2k+1} : 1] = [z_1 : 1] A^k = [\overline{z_0} : 1] \begin{pmatrix} m_1 i & 1 \\ -1 & m_1 i \end{pmatrix} A^k.$$

By determining the Jordan normal form of A, it is easy to compute the matrix powers of A, that is

$$A^k = \frac{1}{2} \begin{pmatrix} (1+q)^k + (1-q)^k & -i[(1+q)^k - (1-q)^k] \\ i[(1+q)^k - (1-q)^k] & (1+q)^k + (1-q)^k \end{pmatrix}.$$

Then we obtain that

$$z_{2k} = \frac{i[(\frac{1+q}{1-q})^k + 1] z_0 - [(\frac{1+q}{1-q})^k - 1]}{[(\frac{1+q}{1-q})^k - 1] z_0 + i[(\frac{1+q}{1-q})^k + 1]},$$
$$z_{2k+1} = \frac{i[(m_1+1)(\frac{1+q}{1-q})^k + (m_1 - 1)] \overline{z_0} - [(m_1+1)(\frac{1+q}{1-q})^k - (m_1 - 1)]}{[(m_1+1)(\frac{1+q}{1-q})^k - (m_1 - 1)] \overline{z_0} + i[(m_1+1)(\frac{1+q}{1-q})^k + (m_1 - 1)]}.$$

Inserting the values of m_1 and q results in

$$z_{2k} = \frac{i[(2+\sqrt{3})^{2k} + 1] z - [(2+\sqrt{3})^{2k} - 1]}{[(2+\sqrt{3})^{2k} - 1] z + i[(2+\sqrt{3})^{2k} + 1]}, \tag{6.5}$$

$$z_{2k+1} = \frac{i[(2+\sqrt{3})^{2k+1} + 1] \overline{z} - [(2+\sqrt{3})^{2k+1} - 1]}{[(2+\sqrt{3})^{2k+1} - 1] \overline{z} + i[(2+\sqrt{3})^{2k+1} + 1]}. \tag{6.6}$$

Let

$$\gamma_k := \frac{(2+\sqrt{3})^k + 1}{(2+\sqrt{3})^k - 1}, \ k \in \mathbb{Z},$$

with the convention that $\gamma_0 = \infty$. Then

$$z_{2k} = \frac{i\gamma_{2k} z - 1}{z + i\gamma_{2k}}, \tag{6.7}$$

$$z_{2k+1} = \frac{i\gamma_{2k+1}\overline{z} - 1}{\overline{z} + i\gamma_{2k+1}}, \tag{6.8}$$

for $k \in \mathbb{N}$.

Remark 6.3.1. From the formulas of m_k and γ_k, we deduce that

$$m_k = \gamma_{2k-1}, \quad m_{-k} = \gamma_{-(2k-1)}, \quad \forall k \in \mathbb{N}^*; \tag{6.9}$$

$$\gamma_{-k} = -\gamma_k, \quad \forall k \in \mathbb{Z} \setminus \{0\}; \tag{6.10}$$

$$\gamma_{k+l} = \frac{\gamma_k \gamma_l + 1}{\gamma_k + \gamma_l}, \quad \forall k, l \in \mathbb{Z}. \tag{6.11}$$

The last relation can be verified directly via the expression of γ_k.

Analogously, we deal with the reflection points in the downward direction. Let $z_{-(k+1)}$ be the reflection of z_{-k} at the circle $|z + m_{k+1} i| = r_{k+1}$ for $k \in \mathbb{N}$. Then

$$z_{-(k+1)} = \frac{-im_{k+1}\overline{z_{-k}} - 1}{\overline{z_{-k}} - im_{k+1}}, \quad \text{i.e.} \quad z_{-k} = \frac{im_{k+1}\overline{z_{-(k+1)}} - 1}{\overline{z_{-(k+1)}} + im_{k+1}}.$$

We thus have

$$z_{-(k-1)} = \frac{z_{-(k+1)} + iq}{-iqz_{-(k+1)} + 1}.$$

Then

$$[z_0 : 1] = [z_{-2k} : 1]A^k,$$
$$[z_{-1} : 1] = [z_{-(2k+1)} : 1]A^k.$$

By the formula of A^k, we have

$$z_0 = \frac{i\gamma_{2k} z_{-2k} - 1}{z_{-2k} + i\gamma_{2k}}, \quad z_{-1} = \frac{i\gamma_{2k} z_{-(2k+1)} - 1}{z_{-(2k+1)} + i\gamma_{2k}}.$$

The above two formulas and Remark 6.3.1 imply that

$$z_{-2k} = \frac{i\gamma_{2k} z_0 + 1}{-z_0 + i\gamma_{2k}} = \frac{i\gamma_{-2k} z - 1}{z + i\gamma_{-2k}}, \tag{6.12}$$

$$z_{-(2k+1)} = \frac{i\gamma_{2k} z_{-1} + 1}{-z_{-1} + i\gamma_{2k}} = \frac{i\gamma_{2k+1}\overline{z_0} + 1}{-\overline{z_0} + i\gamma_{2k+1}} = \frac{i\gamma_{-(2k+1)}\overline{z} - 1}{\overline{z} + i\gamma_{-(2k+1)}}. \tag{6.13}$$

We can unify the expressions for z_k and z_{-k}, namely,

$$z_{2k} = \frac{i\gamma_{2k} z - 1}{z + i\gamma_{2k}}, \quad \forall k \in \mathbb{Z}; \tag{6.14}$$

$$z_{2k+1} = \frac{i\gamma_{2k+1}\overline{z} - 1}{\overline{z} + i\gamma_{2k+1}}, \quad \forall k \in \mathbb{Z}. \tag{6.15}$$

The image of z_k under the reflection at the circle $|z-1| = \sqrt{2}$ is

$$z_k^l := \frac{\overline{z_k} + 1}{\overline{z_k} - 1} \in D_k^l.$$

Then the formulas for z_k^l are given by

$$z_{2k}^l = \frac{-\overline{\epsilon_{2k}}\overline{z} + \epsilon_{2k}}{\epsilon_{2k}\overline{z} + \overline{\epsilon_{2k}}}, \quad \forall k \in \mathbb{Z}, \tag{6.16}$$

$$z_{2k+1}^l = \frac{-\overline{\epsilon_{2k+1}}z + \epsilon_{2k+1}}{\epsilon_{2k+1}z + \overline{\epsilon_{2k+1}}}, \quad \forall k \in \mathbb{Z}, \tag{6.17}$$

where $\epsilon_k = 1 + i\gamma_k$. In particular, $z_0^l = \frac{\overline{z}+1}{\overline{z}-1}$.

The image of z_k under the reflection at the circle $|z+1| = \sqrt{2}$ is

$$z_k^r := \frac{-\overline{z_k} + 1}{\overline{z_k} + 1} \in D_k^r.$$

The formulas for z_k^r are given by

$$z_{2k}^r = \frac{\epsilon_{2k}\overline{z} + \overline{\epsilon_{2k}}}{\overline{\epsilon_{2k}}\overline{z} - \epsilon_{2k}}, \quad \forall k \in \mathbb{Z}; \tag{6.18}$$

$$z_{2k+1}^r = \frac{\epsilon_{2k+1}z + \overline{\epsilon_{2k+1}}}{\overline{\epsilon_{2k+1}}z - \epsilon_{2k+1}}, \quad \forall k \in \mathbb{Z}. \tag{6.19}$$

In particular, $z_0^r = \frac{-\overline{z}+1}{\overline{z}+1}$.

Reflecting z_k^r at the circle $|z-1| = \sqrt{2}$ gives

$$z_k' := -\frac{1}{z_k} \in D_k'.$$

The formulas for z_k' are given by

$$z_{2k}' = \frac{z + i\gamma_{2k}}{-i\gamma_{2k}z + 1}, \quad \forall k \in \mathbb{Z}, \tag{6.20}$$

$$z_{2k+1}' = \frac{\overline{z} + i\gamma_{2k+1}}{-i\gamma_{2k+1}\overline{z} + 1}, \quad \forall k \in \mathbb{Z}. \tag{6.21}$$

In particular, $z_0' = -\frac{1}{z}$.

Moreover, we have the relations

$$z_k^l z_k^r = -1 = z_k z_k', \quad \forall k \in \mathbb{Z}. \tag{6.22}$$

Lemma 6.3.2. *Let $w_k \in \{z_k, z_k^l, z_k^r, z_k'\}$, $k \in \mathbb{Z}$. Then for $z \in R$*

$$|w_k - i| < 3(1-q)^{\frac{k}{2}}, \quad |w_{-k} + i| < 3(1-q)^{\frac{k}{2}}.$$

for $k \in \mathbb{N}^*$.

Proof. When $z \in R$, w_k is located inside the circle $|z - m_k i| = r_k$, i.e. $|w_k - m_k i| \leq r_k$ for $k \in \mathbb{Z} \setminus \{0\}$. Lemma 6.1.3 implies that

$$\begin{aligned}
|w_k - i| &\leq |w_k - m_k i| + (m_k - 1) \\
&\leq \sqrt{m_k^2 - 1} + (m_k - 1) \\
&\leq \sqrt{m_k - 1}(\sqrt{m_1 + 1} + \sqrt{m_1 - 1}) \\
&< 3(1 - q)^{\frac{k}{2}}
\end{aligned}$$

for $k \in \mathbb{N}^*$. Analogously, Lemma 6.1.3 and the relation $m_{-k} = -m_k$ imply that

$$\begin{aligned}
|w_{-k} + i| &\leq |w_{-k} - m_{-k} i| + |1 + m_{-k}| \\
&\leq \sqrt{m_k^2 - 1} + (m_k - 1) \\
&< 3(1 - q)^{\frac{k}{2}}
\end{aligned}$$

for $k \in \mathbb{N}^*$. □

Lemma 6.3.3. *i) In the case of $z \in C_0^l$, we have*

$$z_k = z_k^l, z_k' = z_k^r \in \{|z - 1| = \sqrt{2}\}$$

for $k \in \mathbb{Z}$. In the case of $z \in C_0^r$, we have

$$z_k = z_k^r, z_k' = z_k^l \in \{|z + 1| = \sqrt{2}\}$$

for $k \in \mathbb{Z}$.

ii) Let $\{w_k\}_{k \in \mathbb{Z}} \in \big\{\{z_k\}_{k \in \mathbb{Z}}, \{z_k^l\}_{k \in \mathbb{Z}}, \{z_k^r\}_{k \in \mathbb{Z}}, \{z_k'\}_{k \in \mathbb{Z}}\big\}$. Then, in the case of $z \in C_1$, we have

$$\begin{gathered}
w_{2k} = w_{2k+1} \in \{|z - m_{2k+1} i| = r_{2k+1}\}, \\
w_{-(2k+1)} = w_{-(2k+2)} \in \{|z + m_{2k+2} i| = r_{2k+2}\}
\end{gathered}$$

for $k \in \mathbb{N}$, while in the case of $z \in C_{-1}$, we have

$$\begin{gathered}
w_{2k+1} = w_{2k+2} \in \{|z - m_{2k+2} i| = r_{2k+2}\}, \\
w_{-2k} = w_{-(2k+1)} \in \{|z + m_{2k+1} i| = r_{2k+1}\}
\end{gathered}$$

for $k \in \mathbb{N}$.

Proof. i) Denote the two circles $\{|z \pm 1| = \sqrt{2}\}$ respectively by $C^{\pm}$ and let $T^{\pm}$ be the circle reflections corresponding to $C^{\pm}$ respectively. According to the parqueting provided by R, C_k^l is a circular arc of C^- and C_k^r is a circular arc of C^+ for $k \in \mathbb{Z}$. Obviously $z_k \in C_k^l$ when $z \in C_0^l$. It follows that $z_k^l = T^-(z_k) = z_k \in C^-$ when $z \in C_0^l$. Let A_k be the image of C_k^l under the reflection T^+. Since the two circles $C^{\pm}$ are orthogonal, T^+ maps the circle C^- onto itself. Then the circular arc $A_k = T^+(C_k^l)$ is a part of the circle C^-. When $z \in C_0^l$, $z_k^r = T^+(z_k)$ is located on C^- and $z_k' = T^-(z_k^r) = z_k^r \in C^-$. The claim for the case of $z \in C_0^l$ is thus verified. The discussion for the case of $z \in C_0^r$ is similar.

ii) Let C^k denote the circle $\{|z-m_k| = r_k\}$ and T_k the corresponding circle reflection, $k \in \mathbb{Z} \setminus \{0\}$. In the case of $z \in C_1$, $z_{2k} \in C^{2k+1}$ and $z_{-(2k+1)} \in C^{-(2k+2)}$ for $k \in \mathbb{N}$. Then $z_{2k+1} = T_{2k+1}(z_{2k}) = z_{2k} \in C^{2k+1}$ and $z_{-(2k+2)} = T_{-(2k+2)}(z_{-(2k+1)}) = z_{-(2k+1)} \in C^{-(2k+2)}$ follow. In the case of $z \in C^{-1}$, $z_{2k+1} \in C^{2k+2}$ and $z_{-2k} \in C^{-(2k+1)}$ for $k \in \mathbb{N}$. In the case of $z \in C_{-1}$, from the parqueting provided by R we see that $z_{2k+2} = T_{2k+2}(z_{2k+1}) = z_{2k+1} \in C^{2k+2}$ and $z_{-(2k+1)} = T_{-(2k+1)}(z_{-2k}) = z_{-2k} \in C^{-(2k+1)}$ for $k \in \mathbb{N}$.

Note that the two circle $C^{\pm}$ are both orthogonal to C^k for any $k \in \mathbb{Z} \setminus \{0\}$. It implies that z_k, z_k^l, z_k^r and z_k' are located on the same circle if $z \in C_1$ or $z \in C_{-1}$. The relations for $\{z_k\}_{k\in\mathbb{Z}}$ also work for $\{z_k^l\}_{k\in\mathbb{Z}}$, $\{z_k^r\}_{k\in\mathbb{Z}}$ and $\{z_k'\}_{k\in\mathbb{Z}}$. Therefore the verification for part ii) is complete.

□

6.4 Harmonic Green function of R

According to the parqueting-reflection principle, we construct a formal function

$$\widetilde{F}(z,\zeta) := \prod_{k\in\mathbb{Z}} \frac{\zeta - z_{2k+1}}{\zeta - z_{2k}} \frac{\zeta - z'_{2k+1}}{\zeta - z'_{2k}} \frac{\zeta - z^l_{2k}}{\zeta - z^l_{2k+1}} \frac{\zeta - z^r_{2k}}{\zeta - z^r_{2k+1}}.$$

As a function in the variable ζ, $\widetilde{F}(z,\zeta)$ has the poles

$$z_{2k}, z'_{2k}, z^l_{2k+1}, z^r_{2k+1}, \quad k \in \mathbb{Z},$$

and the zeros

$$z_{2k+1}, z'_{2k+1}, z^l_{2k}, z^r_{2k}, \quad k \in \mathbb{Z}.$$

The next lemma ensures that $\widetilde{F}(z,\zeta)$ is well-defined.

Lemma 6.4.1. *The infinite product $\widetilde{F}(z,\zeta)$ converges absolutely for $z \neq \zeta$, $z, \zeta \in \overline{R}$.*

Proof. It is sufficient to show that the infinite product $\prod_{k\in\mathbb{Z}} \frac{\zeta - w_{2k+1}}{\zeta - w_{2k}}$ is absolutely convergent for any $\{w_k\}_{k\in\mathbb{Z}} \in \{\{z_k\}_{k\in\mathbb{Z}}, \{z'_k\}_{k\in\mathbb{Z}}, \{z^l_k\}_{k\in\mathbb{Z}}, \{z^r_k\}_{k\in\mathbb{Z}}\}$. We know that the infinite product $\prod_{k\in\mathbb{Z}} \frac{\zeta - w_{2k+1}}{\zeta - w_{2k}}$ is absolutely convergent if and only if the series $\sum_{k\in\mathbb{Z}}(\frac{\zeta - w_{2k+1}}{\zeta - w_{2k}} - 1)$ is absolutely convergent, see [24]. Then it is sufficient to prove that $\sum_{k\in\mathbb{N}} |\frac{\zeta - w_{2k+1}}{\zeta - w_{2k}} - 1|$ and $\sum_{k\in\mathbb{N}} |\frac{\zeta - w_{-2k-1}}{\zeta - w_{-2k}} - 1|$ are both convergent. Using the estimates in Lemma 6.3.2 we have

$$|w_{2k} - w_{2k+1}| \leq |w_{2k} - i| + |w_{2k+1} - i| < 6(1-q)^k$$

and

$$|\zeta - w_{2k}| \geq |\zeta - i| - |w_{2k} - i| > \frac{|\zeta - i|}{2}$$

when k is large enough. We thus see that

$$\left|\frac{\zeta - w_{2k+1}}{\zeta - w_{2k}} - 1\right| = \frac{|w_{2k} - w_{2k+1}|}{|\zeta - w_{2k}|} \leq \frac{12}{|\zeta - i|}(1-q)^k.$$

It follows that the series $\sum_{k\in\mathbb{N}} |\frac{\zeta - w_{2k+1}}{\zeta - w_{2k}} - 1|$ converges. Applying again the estimate for w_{-k} in lemma 6.3.2, we obtain that

$$\left|\frac{\zeta - w_{-2k-1}}{\zeta - w_{-2k}} - 1\right| = \frac{|w_{-2k} - w_{-2k-1}|}{|\zeta - w_{-2k}|} \leq \frac{12}{|\zeta + i|}(1-q)^k.$$

It immediately implies convergence of the series $\sum_{k\in\mathbb{N}} \left|\frac{\zeta - w_{-2k-1}}{\zeta - w_{-2k}} - 1\right|$. □

Lemma 6.3.3 immediately leads to the following results.

Lemma 6.4.2. *For $z \in \partial R, \zeta \in \overline{R}\backslash\{z\}$ the function $\widetilde{F}(z,\zeta)$ has unit modulus, i.e.*

$$|\widetilde{F}(z,\zeta)| = 1 \ \ \text{for}\, z \in \partial R, \zeta \in \overline{R}\backslash\{z\}.$$

In Section 6.3 we have seen that all the reflection points are linear fractions in the variable z or $\overline{z}$. For some particular terms, denominator of the corresponding linear fraction may turn out to be zero, which means the reflection point is infinity. This can only happen for the reflection point z_0' in the domain D_0', since D_0' is unbounded while the other domains for the parqueting are bounded. Actually, from the formula $z_0' = -\frac{1}{z}$ we see that $z_0' = \infty$ when $z = 0$. On this basis, we somehow need to deal with the denominators in linear fractions of the reflection points. Therefore we modify the function $\widetilde{F}(z,\zeta)$ by multiplying with the formal product

$$V(z) := \prod_{k\in\mathbb{Z}} \frac{\overline{z} + i\gamma_{2k+1}}{z + i\gamma_{2k}} \frac{-i\gamma_{2k+1}\overline{z} + 1}{-i\gamma_{2k}z + 1} \frac{\epsilon_{2k}\overline{z} + \overline{\epsilon_{2k}}}{\epsilon_{2k+1}z + \overline{\epsilon_{2k+1}}} \frac{\overline{\epsilon_{2k}}\overline{z} - \epsilon_{2k}}{\overline{\epsilon_{2k+1}}z - \epsilon_{2k+1}},$$

whose terms appear in the denominators of z_k, z_k^l, z_k^r and z_k'.

Remark 6.4.3. Note that $\gamma_0 = \infty$ does not cause problems for the clarity of the expression of $V(z)$. The factor for $k = 0$,

$$\frac{\epsilon_0\overline{z} + \overline{\epsilon_0}}{z + i\gamma_0} \frac{\overline{\epsilon_0}\overline{z} - \epsilon_0}{-i\gamma_0 z + 1} = \frac{(1 + i\infty)\overline{z} + (1 - i\infty)}{z + i\infty} \frac{(1 - i\infty)\overline{z} - (1 + i\infty)}{-i\infty z + 1}$$

should be interpreted as $\frac{(\overline{z}-1)(\overline{z}+1)}{z}$.

Considering the function $V(z)$ makes sense due to the following conclusion.

Lemma 6.4.4. *The infinite product $V(z)$ converges absolutely and the function has unitary modulus on the boundary of R, i.e.*

$$|V(z)| = 1 \,\text{for}\, z \in \partial R.$$

Proof. In the case of $z \in C_0^l$, we have $(z-1)(\overline{z}-1)=2$. Let

$$A_k^l(z) := \frac{\overline{z}+i\gamma_{2k+1}}{z+i\gamma_{2k}} \frac{\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}}}{\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}}}, \quad B_k^l(z) := \frac{-i\gamma_{2k+1}\overline{z}+1}{-i\gamma_{2k}z+1} \frac{\overline{\epsilon_{2k}}\overline{z}-\epsilon_{2k}}{\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1}}.$$

Inserting $\overline{z}=\frac{z+1}{z-1}$ into $A_k^l(z)$ and $B_k^l(z)$ results in

$$|A_k^l(z)| = \left| \frac{\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}}}{(z-1)(z+i\gamma_{2k})} \frac{2(z+i\gamma_{2k})}{(z-1)(\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}})} \right| = 1,$$
$$|B_k^l(z)| = \left| \frac{\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1}}{(z-1)(-i\gamma_{2k}z+1)} \frac{2(-i\gamma_{2k}z+1)}{(z-1)(\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1})} \right| = 1.$$

Thus $|A_k^l(z)B_k^l(z)|=1$ for $z \in C_0^l, k \in \mathbb{Z}$. Therefore for $z \in C_0^l$

$$|V(z)| = \prod_{k\in\mathbb{Z}} |A_k^l(z)B_k^l(z)| = 1.$$

In the case of $z \in C_0^r$, we have $(z+1)(\overline{z}+1)=2$. The verification is similar to the first case. Let

$$A_k^r(z) := \frac{\overline{z}+i\gamma_{2k+1}}{z+i\gamma_{2k}} \frac{\overline{\epsilon_{2k}}\overline{z}-\epsilon_{2k}}{\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1}}, \quad B_k^r(z) := \frac{-i\gamma_{2k+1}\overline{z}+1}{-i\gamma_{2k}z+1} \frac{\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}}}{\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}}}.$$

Inserting $\overline{z}=\frac{-z+1}{z+1}$ into $A_k^r(z)$ and $B_k^r(z)$ one can deduce that

$$|A_k^r(z)| = 1 = |B_k^r(z)|.$$

Hence

$$|V(z)| = \prod_{k\in\mathbb{Z}} |A_k^r(z)B_k^r(z)| = 1$$

holds for $z \in C_0^r$.

In the case of $z \in C_1$, the relation $(z-i\gamma_1)(\overline{z}+i\gamma_1)=2$ holds. Let

$$A_k(z) := \frac{\overline{z}+i\gamma_{2k+1}}{z+i\gamma_{2k}} \frac{-i\gamma_{2k+1}\overline{z}+1}{-i\gamma_{2k}z+1}, \quad B_k(z) := \frac{\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}}}{\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}}} \frac{\overline{\epsilon_{2k}}\overline{z}-\epsilon_{2k}}{\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1}}.$$

Inserting $z=\frac{i\gamma_1\overline{z}-1}{\overline{z}+i\gamma_1}$ into $A_k(z)$ results in

$$A_k(z) = \frac{(\overline{z}+i\gamma_{2k+1})(\overline{z}+i\gamma_1)}{i(\gamma_1+\gamma_{2k})\overline{z}-(1+\gamma_1\gamma_{2k})} \frac{(-i\gamma_{2k+1}\overline{z}+1)(\overline{z}+i\gamma_1)}{(1+\gamma_1\gamma_{2k})\overline{z}+i(\gamma_1+\gamma_{2k})}.$$

Remark 6.3.1 ensures that $\gamma_{2k+1}=\frac{\gamma_1\gamma_{2k}+1}{\gamma_1+\gamma_{2k}}$ holds. Then

$$|A_k(z)| = \frac{|z-i\gamma_1|^2}{|\gamma_1+\gamma_{2k}|^2}$$

follows. Inserting $\overline{z}=\frac{i\gamma_1 z+1}{-z+i\gamma_1}$ into $B_k(z)$ produces

$$\begin{aligned}|B_k(z)|=&\left|\frac{[(\gamma_1\gamma_{2k}+1)-i(\gamma_1+\gamma_{2k})]z-[(\gamma_1\gamma_{2k}+1)+i(\gamma_1+\gamma_{2k})]}{(\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}})(-z+i\gamma_1)}\right|\\ &\times\left|\frac{[(\gamma_1\gamma_{2k}+1)+i(\gamma_1+\gamma_{2k})]z+[(\gamma_1\gamma_{2k}+1)-i(\gamma_1+\gamma_{2k})]}{(\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1})(-z+i\gamma_1)}\right|\\ =&\frac{|\gamma_1+\gamma_{2k}|^2}{|z-i\gamma_1|^2}.\end{aligned}$$

Therefore

$$|V(z)|=\prod_{k\in\mathbb{Z}}|A_k(z)B_k(z)|=1$$

holds for $z\in C_1$.

In the case of $z\in C_{-1}$, the relation $(z+i\gamma_1)(\overline{z}-i\gamma_1)=2$ holds.We rewrite $V(z)$ as

$$V(z)=\prod_{k\in\mathbb{Z}}A_k'(z)B_k'(z),$$

where

$$A_k'(z):=\frac{\overline{z}+i\gamma_{2k-1}}{z+i\gamma_{2k}}\frac{-i\gamma_{2k-1}\overline{z}+1}{-i\gamma_{2k}z+1},\quad B_k'(z):=\frac{\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}}}{\epsilon_{2k-1}z+\overline{\epsilon_{2k-1}}}\frac{\overline{\epsilon_{2k}}\overline{z}-\epsilon_{2k}}{\overline{\epsilon_{2k-1}}z-\epsilon_{2k-1}}.$$

Via inserting $z=\frac{i\gamma_1\overline{z}+1}{-\overline{z}+i\gamma_1}$ into $A_k'(z)$ and $\overline{z}=\frac{i\gamma_1 z-1}{z+i\gamma_1}$ into $B_k'(z)$, one can verify that

$$|A_k'(z)B_k'(z)|=1$$

for $z\in C_{-1},k\in\mathbb{Z}$. Therefore $|V(z)|=1$ also holds for $z\in C_{-1}$. □

We modify the function $\widetilde{F}(z,\zeta)$ by multiplying with $V(z)$. Let

$$F(z,\zeta):=\widetilde{F}(z,\zeta)V(z).$$

The function $F(z,\zeta)$ is well-defined and has the expression:

$$\begin{aligned}F(z,\zeta)=&\prod_{k\in\mathbb{Z}}\frac{\zeta(\overline{z}+i\gamma_{2k+1})-(i\gamma_{2k+1}\overline{z}-1)}{\zeta(z+i\gamma_{2k})-(i\gamma_{2k}z-1)}\frac{\zeta(-i\gamma_{2k+1}\overline{z}+1)-(\overline{z}+i\gamma_{2k+1})}{\zeta(-i\gamma_{2k}z+1)-(z+i\gamma_{2k})}\\ &\times\frac{\zeta(\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}})-(-\overline{\epsilon_{2k}}\overline{z}+\epsilon_{2k})}{\zeta(\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}})-(-\overline{\epsilon_{2k+1}}z+\epsilon_{2k+1})}\\ &\times\frac{\zeta(\overline{\epsilon_{2k}}\overline{z}-\epsilon_{2k})-(\epsilon_{2k}\overline{z}+\overline{\epsilon_{2k}})}{\zeta(\overline{\epsilon_{2k+1}}z-\epsilon_{2k+1})-(\epsilon_{2k+1}z+\overline{\epsilon_{2k+1}})}\\ =&\prod_{k\in\mathbb{Z}}\frac{(\zeta\overline{z}+1)+i\gamma_{2k+1}(\zeta-\overline{z})}{(\zeta z+1)+i\gamma_{2k}(\zeta-z)}\frac{(\zeta-\overline{z})-i\gamma_{2k+1}(\zeta\overline{z}+1)}{(\zeta-z)-i\gamma_{2k}(\zeta z+1)}\\ &\times\frac{\epsilon_{2k}(\zeta\overline{z}-1)+\overline{\epsilon_{2k}}(\zeta+\overline{z})}{\epsilon_{2k+1}(\zeta z-1)+\overline{\epsilon_{2k+1}}(\zeta+z)}\frac{\overline{\epsilon_{2k}}(\zeta\overline{z}-1)-\epsilon_{2k}(\zeta+\overline{z})}{\overline{\epsilon_{2k+1}}(\zeta z-1)-\epsilon_{2k+1}(\zeta+z)}.\end{aligned}$$

Remark 6.4.5. From the constructions of $\widetilde{F}(z,\zeta)$ and $F(z,\zeta)$, we see immediately that $F(z,\zeta)$ has poles at the points $\zeta = z_{2k}, z'_{2k}, z^l_{2k+1}, z^r_{2k+1}$ and has zeros at the points $\zeta = z_{2k+1}, z'_{2k+1}, z^l_{2k}, z^r_{2k}$, $k \in \mathbb{Z}$. Moreover, one can verify directly that the function $F(z,\zeta)$ is symmetric in z and ζ via using the relation $\gamma_{-k} = -\gamma_k$.

Theorem 6.4.6. *The harmonic Green function of R is $G_1(z,\zeta) = \log|F(z,\zeta)|^2$.*

Proof. For $z \in R$, from the parqueting generated by R we have seen that all the reflection points except $z_0 = z$ are outside the domain R. Then Remark 6.4.5 implies that $F(z,\zeta)$ as a function in the variable z has only one pole $z = \zeta$ for $\zeta \in R$. From the expression of $F(z,\zeta)$ we thus see that $\log|(z-\zeta)F(z,\zeta)|^2$ is harmonic in R. In addition, Lemmas 6.4.2 and 6.4.4 ensure that $G_1(z,\zeta) = 0$ for $z \in \partial R$. Therefore $G_1(z,\zeta)$ is the harmonic Green function of R. □

6.5 Poisson kernel of R

Let v_1, v_2, v_3 and v_4 denote the four corner points of R. The Poisson kernel for the domain R is defined by

$$p(z,\zeta) := -\frac{1}{2}\partial_{\nu_\zeta} G_1(z,\zeta)$$

for $z \in R, \zeta \in \partial R \backslash \{v_1, v_2, v_3, v_4\}$, where ν_ζ denotes the outward normal vector and ∂_{ν_ζ} is the outward normal derivative operator on ∂R. Note that

$$\partial_{\nu_\zeta} = \nu_\zeta \partial_\zeta + \overline{\nu_\zeta}\partial_{\overline{\zeta}} = 2\,\mathrm{Re}(\nu_\zeta \partial_\zeta).$$

This formula is used to calculate the normal derivative.

Since the boundary ∂R consists of four circular arcs C_0^l, C_0^r, C_{-1} and C_1, the outward normal derivative has to be calculated respectively for the four parts. The formula

$$\begin{aligned}\partial_\zeta G_1(z,\zeta) =& \partial_\zeta \log|F(z,\zeta)|^2 = \partial_\zeta \log|\widetilde{F}(z,\zeta)|^2 \\ =& \sum_{k\in\mathbb{Z}} \Big(\frac{1}{\zeta - z_{2k+1}} - \frac{1}{\zeta - z_{2k}} + \frac{1}{\zeta - z'_{2k+1}} - \frac{1}{\zeta - z'_{2k}} \\ &+ \frac{1}{\zeta - z^l_{2k}} - \frac{1}{\zeta - z^l_{2k+1}} + \frac{1}{\zeta - z^r_{2k}} - \frac{1}{\zeta - z^r_{2k+1}} \Big)\end{aligned}$$

will be used.

Remark 6.5.1. It is easy to check that all the series $\sum_{k\in\mathbb{Z}} \big(\frac{1}{\zeta - z_{2k+1}} - \frac{1}{\zeta - z_{2k}}\big)$, $\sum_{k\in\mathbb{Z}} \big(\frac{1}{\zeta - z'_{2k+1}} - \frac{1}{\zeta - z'_{2k}}\big)$, $\sum_{k\in\mathbb{Z}} \big(\frac{1}{\zeta - z^l_{2k}} - \frac{1}{\zeta - z^l_{2k+1}}\big)$ and $\sum_{k\in\mathbb{Z}} \big(\frac{1}{\zeta - z^r_{2k}} - \frac{1}{\zeta - z^r_{2k+1}}\big)$ converge absolutely via applying Lemma 6.3.2. Therefore $\partial_\zeta G_1(z,\zeta)$ converges absolutely.

We discuss below the Poisson kernel respectively for the four boundary arcs.

Case 1. $\zeta \in C_0^l$.

In this case, the relation $(\zeta - 1)(\overline{\zeta} - 1) = 2$ holds, the outward normal vector is $\nu_\zeta = \frac{\zeta - 1}{\sqrt{2}}$ and the outward normal derivative operator is $\partial_{\nu_\zeta} = \sqrt{2}\mathrm{Re}((\zeta-1)\partial_\zeta)$. Hence $p(z,\zeta) = -\frac{1}{\sqrt{2}}\mathrm{Re}((\zeta-1)\partial_\zeta G_1(z,\zeta))$.

Lemma 6.5.2. *Let* $\{w_k\}_{k\in\mathbb{Z}} \in \big\{\{z_k\}_{k\in\mathbb{Z}}, \{z_k^l\}_{k\in\mathbb{Z}}, \{z_k^r\}_{k\in\mathbb{Z}}, \{z_k'\}_{k\in\mathbb{Z}}\big\}$. *Then*

$$\big||w_k-1|^2-2\big| < \frac{192}{(2+\sqrt{3})^{|k|}-1}\big||z-1|^2-2\big|$$

holds for $z \in R$, $k \in \mathbb{Z}$.

Proof. Applying the formula

$$z_{2k} = \frac{i\gamma_{2k}z-1}{z+i\gamma_{2k}}$$

gives

$$\big||z_{2k}-1|^2-2\big| = \frac{r_{2k}^2-1}{|z+i\gamma_{2k}|^2}\big||z-1|^2-2\big|.$$

Analogously, by inserting

$$z_{2k+1} = \frac{i\gamma_{2k+1}\overline{z}-1}{\overline{z}+i\gamma_{2k+1}},$$

we have

$$\big||z_{2k+1}-1|^2-2\big| = \frac{r_{2k+1}^2-1}{|z-i\gamma_{2k+1}|^2}\big||z-1|^2-2\big|.$$

In general, we have

$$\big||z_k-1|^2-2\big| = \frac{r_k^2-1}{|z+(-1)^k i\gamma_k|^2}\big||z-1|^2-2\big|.$$

The formula

$$\gamma_k = \frac{(2+\sqrt{3})^k+1}{(2+\sqrt{3})^k-1}, \quad k \in \mathbb{Z}$$

implies that

$$\gamma_k^2-1 = (|\gamma_k|+1)(|\gamma_k|-1) \le (\gamma_1+1)\frac{2}{(2+\sqrt{3})^{|k|}-1} < \frac{6}{(2+\sqrt{3})^{|k|}-1}.$$

For $z \in R$, the estimate

$$|z \pm i\gamma_k| \ge |z \pm i\gamma_1| - (\gamma_1 - |\gamma_k|) > \sqrt{2} - (\sqrt{3}-1) > \frac{1}{2}$$

holds. Then we have

$$\big||z_k-1|^2-2\big| < \frac{24}{(2+\sqrt{3})^{|k|}-1}\big||z-1|^2-2\big|$$

for $z \in R$, $k \in \mathbb{Z}$.

Applying the relation $z_k' = -\frac{1}{z_k}$, we see that

$$\big||z_k'-1|^2-2\big| = \frac{\big||z_k-1|^2-2\big|}{|z_k|^2} = \frac{\gamma_k^2-1}{|1-(-1)^k i\gamma_k z|^2}\big||z_k-1|^2-2\big|.$$

With the estimate

$$\begin{aligned}
|1-(-1)^k i\gamma_k z| &= |\gamma_k|\left|z+i\frac{(-1)^k}{\gamma_k}\right| \\
&\geq |z+(-1)^k i\gamma_1| - (\gamma_1 - |\frac{1}{\gamma_k}|) \\
&> \sqrt{2} - (\sqrt{3} - \frac{1}{\sqrt{3}}) \\
&> \frac{1}{4},
\end{aligned}$$

the inequality

$$\big||z_k' - 1|^2 - 2\big| < \frac{96}{(2+\sqrt{3})^{|k|} - 1}\big||z-1|^2 - 2\big|$$

follows.

Applying the relation

$$z_k^l = \frac{\overline{z_k}+1}{\overline{z_k}-1}$$

results in

$$\big||z_k^l - 1|^2 - 2\big| = \frac{2\big||z_k - 1|^2 - 2\big|}{|z_k - 1|^2}.$$

By the fact that each z_k is located inside the circle $|z+1| = \sqrt{2}$, the estimate $|z_k - 1| \geq 2 - |z_k + 1| > 2 - \sqrt{2} > \frac{1}{2}$ follows. Thus we have

$$\big||z_k^l - 1|^2 - 2\big| < \frac{192}{(2+\sqrt{3})^{|k|} - 1}\big||z-1|^2 - 2\big|.$$

Analogously, by applying the relation

$$z_k^r = \frac{-\overline{z_k}+1}{\overline{z_k}+1}$$

and the estimate $|z_k + 1| \geq 2 - |z_k - 1| > 2 - \sqrt{2} > \frac{1}{2}$, we show that

$$\big||z_k^r - 1|^2 - 2\big| = \frac{2\big||z_k - 1|^2 - 2\big|}{|z_k + 1|^2} < \frac{192}{(2+\sqrt{3})^{|k|} - 1}\big||z-1|^2 - 2\big|.$$

To sum up, we have shown that

$$\big||w_k - 1|^2 - 2\big| < \frac{192}{(2+\sqrt{3})^{|k|} - 1}\big||z-1|^2 - 2\big|$$

for $z \in R$, $k \in \mathbb{Z}$. $\square$

Lemma 6.5.3. *On the boundary part C_0^l except for the two corner points, the Poisson kernel satisfies that*

$$p(z,\zeta) = \frac{1}{\sqrt{2}}\left(\frac{\zeta-1}{\zeta-z} + \overline{\frac{\zeta-1}{\zeta-z}} - 1\right) + O(2-|z-1|^2) \tag{6.23}$$

for $z \in R$, $\zeta \in C_0^l$.

Proof. On the basis of the absolute convergence of $\partial_\zeta G_1(z,\zeta)$, we can rewrite $(\zeta - 1)\partial_\zeta G_1(z,\zeta)$ as

$$\begin{aligned}
&-(\zeta-1)\Big(\frac{\zeta-1}{\zeta-z_0} - \frac{\zeta-1}{\zeta-z_0^l}\Big) + (\zeta-1)\sum_{k\in\mathbb{Z}^*}(-1)^{k-1}\Big(\frac{1}{\zeta-z_k} - \frac{1}{\zeta-z_k^l}\Big)\\
&+(\zeta-1)\sum_{k\in\mathbb{Z}}(-1)^{k-1}\Big(\frac{1}{\zeta-z_k^l} - \frac{1}{\zeta-z_k^r}\Big),
\end{aligned}$$

where $\mathbb{Z}^*$ denotes the set of nonzero integers. Since the relation $(\zeta-1)(\overline{\zeta}-1) = 2$, i.e. $\zeta\overline{\zeta} - \zeta - \overline{\zeta} - 1 = 0$, holds for $\zeta \in C_0^l$, substituting z_k^l by $\frac{\overline{z_k}+1}{\overline{z_k}-1}$ gives

$$\begin{aligned}
\frac{\zeta-1}{\zeta-z_k} - \frac{\zeta-1}{\zeta-z_k^l} &= \frac{\zeta-1}{\zeta-z_k} - \frac{\zeta-1}{\zeta-\frac{\overline{z_k}+1}{\overline{z_k}-1}}\\
&= \frac{\zeta-1}{\zeta-z_k} - \frac{(\zeta-1)(\overline{z_k}-1)}{(\zeta-1)(\overline{z_k}-1)-2}\\
&= \frac{\zeta-1}{\zeta-z_k} - \frac{(\overline{\zeta}-1)(\zeta-1)(\overline{z_k}-1)}{(\overline{\zeta}-1)(\zeta-1)(\overline{z_k}-1)-2(\overline{\zeta}-1)}\\
&= \frac{\zeta-1}{\zeta-z_k} + \frac{\overline{z_k-1}}{\overline{\zeta-z_k}}\\
&= \frac{\zeta\overline{\zeta}-\zeta-\overline{\zeta}+z+\overline{z}-z\overline{z}}{|\zeta-z_k|^2}\\
&= \frac{2-|z_k-1|^2}{|\zeta-z_k|^2}.
\end{aligned}$$

Especially, we have

$$\frac{\zeta-1}{\zeta-z_0} - \frac{\zeta-1}{\zeta-z_0^l} = \frac{2-|z-1|^2}{|\zeta-z|^2} = \frac{\zeta-1}{\zeta-z} + \overline{\frac{\zeta-1}{\zeta-z}} - 1.$$

Lemma 6.3.2 ensures that $|\zeta - z_k|$ has the limit $|\zeta - i|$ when k tends to ∞, while it has the limit $|\zeta + i|$ when k tends to $-\infty$. Then we see from Lemma 6.5.2 that

$$(\zeta-1)\sum_{k\in\mathbb{Z}^*}\Big(\frac{1}{\zeta-z_k} - \frac{1}{\zeta-z_k^l}\Big) = O(2-|z-1|^2).$$

Analogously, via substituting z_k^r by $\frac{\overline{z_k'}+1}{\overline{z_k'}-1}$ we can show that

$$\frac{\zeta-1}{\zeta-z_k'}-\frac{\zeta-1}{\zeta-z_k^r}=\frac{2-|z_k'-1|^2}{|\zeta-z_k'|^2}.$$

Again Lemma 6.3.2 and Lemma 6.5.2 guarantee that

$$(\zeta-1)\sum_{k\in\mathbb{Z}}\left(\frac{1}{\zeta-z_k'}-\frac{1}{\zeta-z_k^r}\right)=O(2-|z-1|^2).$$

Therefore on C_0^l except for the two corner points

$$\begin{aligned}p(z,\zeta)&=-\frac{1}{\sqrt{2}}\mathrm{Re}\big((\zeta-1)\partial_\zeta G_1(z,\zeta)\big)\\&=\frac{1}{\sqrt{2}}\left(\frac{\zeta-1}{\zeta-z}+\overline{\frac{\zeta-1}{\zeta-z}}-1\right)+O(2-|z-1|^2).\end{aligned}$$

$\square$

Case 2. $\zeta\in C_0^r$**.**

In this case, the relation $(\zeta+1)(\overline{\zeta}+1)=2$ holds, the outward normal vector is $\nu_\zeta=\frac{\zeta+1}{\sqrt{2}}$, the outward normal derivative operator is $\partial_{\nu_\zeta}=\sqrt{2}\mathrm{Re}((\zeta+1)\partial_\zeta)$ and the Poisson kernel is $p(z,\zeta)=-\frac{1}{\sqrt{2}}\mathrm{Re}((\zeta-1)\partial_\zeta G_1(z,\zeta))$. Since the computation in this case is analogous to the case of C_0^l, we just list the following two parallel lemmas omitting the proofs.

Lemma 6.5.4. *Let* $\{w_k\}_{k\in\mathbb{Z}}\in\big\{\{z_k\}_{k\in\mathbb{Z}},\{z_k^l\}_{k\in\mathbb{Z}},\{z_k^r\}_{k\in\mathbb{Z}},\{z_k'\}_{k\in\mathbb{Z}}\big\}$*. Then*

$$\big||w_k+1|^2-2\big|<\frac{192}{(2+\sqrt{3})^{|k|}-1}\big||z+1|^2-2\big|$$

holds for $z\in R$, $k\in\mathbb{Z}$*.*

Lemma 6.5.5. *On the boundary part* C_0^r *except for the two corner points, the Poisson kernel satisfies that*

$$p(z,\zeta)=\frac{1}{\sqrt{2}}\left(\frac{\zeta+1}{\zeta-z}+\overline{\frac{\zeta+1}{\zeta-z}}-1\right)+O(2-|z+1|^2)\tag{6.24}$$

for $z\in R$ *and* $\zeta\in C_0^r$ *except for the two corner points.*

Case 3. $\zeta\in C_1$**.**

In this case, we have the relation $(\zeta-\sqrt{3}i)(\overline{\zeta}+\sqrt{3}i)=2$. The outward normal vector is $\nu_\zeta=-\frac{\zeta-\sqrt{3}i}{\sqrt{2}}$, the outward normal derivative operator is $\partial_{\nu_\zeta}=-\sqrt{2}\mathrm{Re}((\zeta-\sqrt{3}i)\partial_\zeta)$ and the Poisson kernel is $p(z,\zeta)=\frac{1}{\sqrt{2}}\mathrm{Re}((\zeta-\sqrt{3}i)\partial_\zeta G_1(z,\zeta))$.

Lemma 6.5.6. *Let* $\{w_k\}_{k\in\mathbb{Z}} \in \big\{\{z_k\}_{k\in\mathbb{Z}}, \{z_k^l\}_{k\in\mathbb{Z}}, \{z_k^r\}_{k\in\mathbb{Z}}, \{z_k'\}_{k\in\mathbb{Z}}\big\}$. *Then*

$$\sum_{k\in\mathbb{Z}} \left| \frac{1}{\zeta - w_{2k+1}} - \frac{1}{\zeta - w_{2k}} \right| = O(|z - m_1 i|^2 - r_1^2)$$

holds for $\zeta \in R, z \in R\backslash\{\zeta\}$.

Proof. Lemma 6.3.2 implies that $|\zeta - w_k|$ tends towards $|\zeta \pm i|$ when k goes to $\pm\infty$. So we only need to show that

$$\sum_{k\in\mathbb{Z}} |w_{2k+1} - w_{2k}| = O(|z - m_1 i|^2 - r_1^2). \tag{6.25}$$

Applying the formulas $z_k' = \frac{-1}{z_k}$, $z_k^l = \frac{\overline{z_k}+1}{\overline{z_k}-1}$ and $z_k^r = \frac{-\overline{z_k}+1}{\overline{z_k}+1}$, we obtain that

$$\begin{aligned}
|z'_{2k+1} - z'_{2k}| &= \frac{|z_{2k+1} - z_{2k}|}{|z_{2k}||z_{2k+1}|}, \\
|z^l_{2k+1} - z^l_{2k}| &= \frac{2|z_{2k+1} - z_{2k}|}{|z_{2k} - 1||z_{2k+1} - 1|}, \\
|z^r_{2k+1} - z^r_{2k}| &= \frac{2|z_{2k+1} - z_{2k}|}{|z_{2k} + 1||z_{2k+1} + 1|}.
\end{aligned}$$

By the fact that $|z_k|$ tends towards 1 when k goes to $\pm\infty$ and $|z_k \pm 1| > \frac{1}{2}$ which is seen from the proof of Lemma 6.5.2, we only need to verify the estimate (6.25) for the case $w_k = z_k$.

Inserting the formulas

$$z_{2k} = \frac{i\gamma_{2k} z - 1}{z + i\gamma_{2k}} \quad \text{and} \quad z_{2k+1} = \frac{i\gamma_{2k+1}\overline{z} - 1}{\overline{z} + i\gamma_{2k+1}}$$

results in

$$\begin{aligned}
&|z_{2k+1} - z_{2k}| \\
&\quad = \frac{|i(\gamma_{2k+1} - \gamma_{2k})z\overline{z} + (\gamma_{2k+1}\gamma_{2k} - 1)z - (\gamma_{2k+1}\gamma_{2k} - 1)\overline{z} + i(\gamma_{2k+1} - \gamma_{2k})|}{|\overline{z} + i\gamma_{2k+1}||z + i\gamma_{2k}|}.
\end{aligned}$$

Note that

$$\frac{\gamma_{2k+1}\gamma_{2k} - 1}{\gamma_{2k+1} - \gamma_{2k}} = -\frac{\gamma_{2k+1}\gamma_{-2k} + 1}{\gamma_{2k+1} + \gamma_{-2k}} = -\gamma_1 = -m_1$$

follows from Remark 6.3.1. Then

$$\begin{aligned}
|z_{2k+1} - z_{2k}| &= \frac{|\gamma_{2k+1} - \gamma_{2k}|}{|\overline{z} + i\gamma_{2k+1}||z + i\gamma_{2k}|} |z\overline{z} + im_1 z - im_1\overline{z} + 1| \\
&= \frac{|\gamma_{2k+1} - \gamma_{2k}|}{|\overline{z} + i\gamma_{2k+1}||z + i\gamma_{2k}|} ||z - m_1 i|^2 - r_1^2|.
\end{aligned}$$

From the formula of γ_k we see that

$$|\gamma_{2k+1}-\gamma_{2k}| = \left|\frac{2}{(2+\sqrt{3})^{2k+1}-1} - \frac{2}{(2+\sqrt{3})^{2k}-1}\right| < \frac{4}{(2+\sqrt{3})^{2|k|}-1}.$$

In the proof of Lemma 6.5.2 we have already shown that $|z \pm i\gamma_k| > \frac{1}{2}$ for $z \in R$. Hence the series

$$\sum_{k\in\mathbb{Z}} \frac{|\gamma_{2k+1}-\gamma_{2k}|}{|\overline{z}+i\gamma_{2k+1}||z+i\gamma_{2k}|}$$

converges. It follows that

$$\sum_{k\in\mathbb{Z}} |z_{2k+1}-z_{2k}| = O(|z-m_1 i|^2 - r_1^2).$$

□

Lemma 6.5.7. *On the boundary part C_1 except for the two corner points, the Poisson kernel satisfies that*

$$p(z,\zeta) = -\frac{1}{\sqrt{2}}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z} + \overline{\frac{\zeta-\sqrt{3}i}{\zeta-z}} - 1\right) + O(|z-\sqrt{3}i|^2 - 2) \tag{6.26}$$

for $z \in R$ and $\zeta \in C_1$ except for the two corner points.

Proof. We can rewrite $(\zeta-\sqrt{3}i)\partial_\zeta G_1(z,\zeta)$ as

$$\begin{aligned}
&\frac{\zeta-\sqrt{3}i}{\zeta-z_1} - \frac{\zeta-\sqrt{3}i}{\zeta-z}\\
&+\sum_{k\in\mathbb{Z}^*}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z_{2k+1}} - \frac{\zeta-\sqrt{3}i}{\zeta-z_{2k}}\right) + \sum_{k\in\mathbb{Z}}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z'_{2k+1}} - \frac{\zeta-\sqrt{3}i}{\zeta-z'_{2k}}\right)\\
&-\sum_{k\in\mathbb{Z}}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z^l_{2k+1}} - \frac{\zeta-\sqrt{3}i}{\zeta-z^l_{2k}}\right) - \sum_{k\in\mathbb{Z}}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z^r_{2k+1}} - \frac{\zeta-\sqrt{3}i}{\zeta-z^r_{2k}}\right).
\end{aligned}$$

Lemma 6.5.6 implies that each of the four sums in above expression is at most a positive constant multiple of $|z-m_1 i|^2 - r_1^2$ for $\zeta \in C_1, z \in R$. For the first term, by using $(\zeta-\sqrt{3}i)(\overline{\zeta}+\sqrt{3}i) = 2$ and $z_1 = \frac{\sqrt{3}i\overline{z}-1}{\overline{z}+\sqrt{3}i} = \sqrt{3}i + \frac{2}{\overline{z}+\sqrt{3}i}$,

$$\begin{aligned}
\frac{\zeta-\sqrt{3}i}{\zeta-z_1} - \frac{\zeta-\sqrt{3}i}{\zeta-z} &= \frac{(\zeta-\sqrt{3}i)(\overline{z}+\sqrt{3}i)}{(\zeta-\sqrt{3}i)(\overline{z}+\sqrt{3}i)-2} - \frac{\zeta-\sqrt{3}i}{\zeta-z}\\
&= \frac{\overline{z}+\sqrt{3}i}{(\overline{z}+\sqrt{3}i)-(\overline{\zeta}+\sqrt{3}i)} - \frac{\zeta-\sqrt{3}i}{\zeta-z}\\
&= 1 - \left(\frac{\overline{\zeta}+\sqrt{3}i}{\overline{\zeta}-\overline{z}} + \frac{\zeta-\sqrt{3}i}{\zeta-z}\right)
\end{aligned}$$

is seen. Thus we have

$$\begin{aligned} p(z,\zeta) &= \frac{1}{\sqrt{2}}\mathrm{Re}((\zeta-\sqrt{3}i)\partial_\zeta G_1(z,\zeta)) \\ &= -\frac{1}{\sqrt{2}}\left(\frac{\zeta-\sqrt{3}i}{\zeta-z}+\frac{\overline{\zeta-\sqrt{3}i}}{\overline{\zeta-z}}-1\right)+O(|z-\sqrt{3}i|^2-2) \end{aligned}$$

$\square$

Case 4. $\zeta \in C_{-1}$**.**

In this case, the relation $(\zeta+\sqrt{3}i)(\overline{\zeta}-\sqrt{3}i)=2$ holds, the outward normal vector is $\nu_\zeta = -\frac{\zeta+\sqrt{3}i}{\sqrt{2}}$, the outward normal derivative operator is $\partial_{\nu_\zeta} = -\sqrt{2}\mathrm{Re}((\zeta+\sqrt{3}i)\partial_\zeta)$ and the Poisson kernel is $p(z,\zeta)=\frac{1}{\sqrt{2}}\mathrm{Re}((\zeta+\sqrt{3}i)\partial_\zeta G_1(z,\zeta))$.

Lemma 6.5.8. *Let* $\{w_k\}_{k\in\mathbb{Z}} \in \big\{\{z_k\}_{k\in\mathbb{Z}}, \{z_k^l\}_{k\in\mathbb{Z}}, \{z_k^r\}_{k\in\mathbb{Z}}, \{z_k'\}_{k\in\mathbb{Z}}\big\}$. *Then*

$$\sum_{k\in\mathbb{Z}}\left|\frac{1}{\zeta-w_{2k}}-\frac{1}{\zeta-w_{2k-1}}\right| = O(|z+m_1 i|^2-r_1^2)$$

holds for $\zeta\in R, z\in R\backslash\{\zeta\}$.

Proof. The verification is analogous to the proof of Lemma 6.5.6, One only needs to show that

$$\sum_{k\in\mathbb{Z}}|z_{2k-1}-z_{2k}| = O(|z+m_1 i|^2-r_1^2).$$

Combining the relations

$$\begin{aligned} &|z_{2k-1}-z_{2k}| \\ &\quad= \frac{|i(\gamma_{2k-1}-\gamma_{2k})z\overline{z}+(\gamma_{2k-1}\gamma_{2k}-1)z-(\gamma_{2k-1}\gamma_{2k}-1)\overline{z}+i(\gamma_{2k-1}-\gamma_{2k})|}{|\overline{z}+i\gamma_{2k-1}||z+i\gamma_{2k}|} \end{aligned}$$

and

$$\frac{\gamma_{2k-1}\gamma_{2k}-1}{\gamma_{2k-1}-\gamma_{2k}} = \frac{\gamma_{1-2k}\gamma_{2k}+1}{\gamma_{1-2k}+\gamma_{2k}} = \gamma_1 = m_1$$

results in

$$\begin{aligned} |z_{2k-1}-z_{2k}| &= \frac{|\gamma_{2k-1}-\gamma_{2k}|}{|\overline{z}+i\gamma_{2k-1}||z+i\gamma_{2k}|}|z\overline{z}-im_1 z+im_1\overline{z}+1| \\ &= \frac{|\gamma_{2k-1}-\gamma_{2k}|}{|\overline{z}+i\gamma_{2k-1}||z+i\gamma_{2k}|}||z+m_1 i|^2-r_1^2|. \end{aligned}$$

It follows that

$$\sum_{k\in\mathbb{Z}}|z_{2k-1}-z_{2k}| = O(|z+m_1 i|^2-r_1^2).$$

$\square$

Lemma 6.5.9. *On the boundary part C_{-1} except for the two corner points, the Poisson kernel satisfies that*

$$p(z,\zeta) = -\frac{1}{\sqrt{2}}\left(\frac{\zeta+\sqrt{3}i}{\zeta-z} + \frac{\overline{\zeta+\sqrt{3}i}}{\overline{\zeta-z}} - 1\right) + O(|z+\sqrt{3}i|^2-2) \tag{6.27}$$

for $z \in R$ and $\zeta \in C_{-1}$ except for the two corner points.

Proof. We can rewrite $(\zeta+\sqrt{3}i)\partial_\zeta G_1(z,\zeta)$ as

$$\begin{aligned}
&\frac{\zeta+\sqrt{3}i}{\zeta-z_{-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z} \\
&+\sum_{k\in\mathbb{Z}^*}\left(\frac{\zeta+\sqrt{3}i}{\zeta-z_{2k-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z_{2k}}\right) + \sum_{k\in\mathbb{Z}}\left(\frac{\zeta+\sqrt{3}i}{\zeta+z'_{2k-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z'_{2k}}\right) \\
&-\sum_{k\in\mathbb{Z}}\left(\frac{\zeta+\sqrt{3}i}{\zeta-z^l_{2k-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z^l_{2k}}\right) - \sum_{k\in\mathbb{Z}}\left(\frac{\zeta+\sqrt{3}i}{\zeta-z^r_{2k-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z^r_{2k}}\right)
\end{aligned}$$

Lemma 6.5.8 implies that each of the four sums in above expression is at most a positive constant multiple of $|z+m_1 i|^2 - r_1^2$ for $\zeta \in C_{-1}, z \in R$. For the first term, by using $(\zeta+\sqrt{3}i)(\overline{\zeta}-\sqrt{3}i) = 2$ and $z_{-1} = \frac{\sqrt{3}i\overline{z}+1}{-\overline{z}+\sqrt{3}i} = -\sqrt{3}i + \frac{2}{\overline{z}-\sqrt{3}i}$,

$$\frac{\zeta+\sqrt{3}i}{\zeta-z_{-1}} - \frac{\zeta+\sqrt{3}i}{\zeta-z} = 1 - \left(\frac{\overline{\zeta}-\sqrt{3}i}{\overline{\zeta}-\overline{z}} + \frac{\zeta+\sqrt{3}i}{\zeta-z}\right)$$

is deduced. Thus

$$\begin{aligned}
p(z,\zeta) &= \frac{1}{\sqrt{2}}\mathrm{Re}((\zeta+\sqrt{3}i)\partial_\zeta G_1(z,\zeta)) \\
&= -\frac{1}{\sqrt{2}}\left(\frac{\zeta+\sqrt{3}i}{\zeta-z} + \frac{\overline{\zeta+\sqrt{3}i}}{\overline{\zeta-z}} - 1\right) + O(|z+\sqrt{3}i|^2-2).
\end{aligned}$$

□

6.6 Dirichlet problem in R

On the basis of previous results on the harmonic Green function and the boundary properties of the Poisson kernel for R, the Dirichlet problem for the Poisson equation in R can be solved.

Theorem 6.6.1. *The Dirichlet problem for the Poisson equation in R:*

$$w_{z\overline{z}} = f \text{ in } R, \quad w = \gamma \text{ on } \partial R,$$

where $f \in L_p(R;\mathbb{C})$, $p > 2$, $\gamma \in C(\partial R, \mathbb{C})$, is uniquely solvable. The solution is provided by

$$w(z) = \frac{1}{2\pi}\int_{\partial R} \gamma(\zeta)p(z,\zeta)\mathrm{d}s_\zeta - \frac{1}{\pi}\int_R f(\zeta)G_1(z,\zeta)\mathrm{d}\sigma_\zeta,$$

where $p(z,\zeta)$ and $G_1(z,\zeta)$ are respectively the Poisson kernel and harmonic Green function of R.

Proof. We only need to verify the boundary behavior of the solution. Denote the circle $C(a,r) := \{|z-a| = r\}$, $a \in \mathbb{C}$, $r > 0$ and $D(a,r)$ the corresponding disk. From Theorem 4.1.3 we know the Poisson integral formula for the disk $D(a,r)$ and also the boundary property

$$\lim_{\substack{z\to\tilde{z}\\ z\in D(a,r)}} \frac{1}{2\pi r}\int_{C(a,r)} \gamma(\zeta)\left(\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1\right)\mathrm{d}s_\zeta = \gamma(\tilde{z})$$

for $\gamma \in C(C(a,r);\mathbb{C})$, $\tilde{z} \in C(a,r)$. Note that the Poisson kernel for $D(a,r)$ satisfies the boundary behavior

$$\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1 = 0, \text{ for } z,\zeta \in C(a,r) \text{ and } z \neq \zeta.$$

It follows that for a circular arc $\Gamma \subset C(a,r)$,

$$\lim_{\substack{z\to\tilde{z}\in\Gamma\\ z\in D(a,r)}} \frac{1}{2\pi r}\int_{\Gamma} \gamma(\zeta)\left(\frac{\zeta-a}{\zeta-z}+\frac{\overline{\zeta}-\overline{a}}{\overline{\zeta}-\overline{z}}-1\right)\mathrm{d}s_\zeta = \gamma(\tilde{z}). \tag{6.28}$$

Since the harmonic Green function $G_1(z,\zeta)$ vanishes on the boundary, we know that

$$\lim_{\substack{z\to\partial R\\ z\in R}} p(z,\zeta) = -\frac{1}{2}\lim_{\substack{z\to\partial R\\ z\in R}} \partial_{\nu_\zeta}G_1(z,\zeta) = 0.$$

This property can also be verified by the boundary behavior of $p(z,\zeta)$ given by the relations (6.23), (6.24), (6.26) and (6.27).

We keep the notations C_0^l, C_0^r, C_1 and C_{-1} as the boundary arcs of our circular rectangle R. Note that $C_0^l \subset C(1,\sqrt{2})$, $C_0^r \subset C(-1,\sqrt{2})$, $C_1 \subset C(\sqrt{3}i,\sqrt{2})$ and $C_{-1} \subset C(-\sqrt{3}i,\sqrt{2})$. The relations (6.23) and (6.28) imply that

$$\begin{aligned}
&\lim_{\substack{z\to\tilde{z}\in C_0^l\\ z\in R}} \frac{1}{2\pi}\int_{\partial R}\gamma(\zeta)p(z,\zeta)\mathrm{d}s_\zeta\\
&= \lim_{\substack{z\to\tilde{z}\in C_0^l\\ z\in R}} \frac{1}{2\pi}\int_{\partial R}\gamma(\zeta)\left(\frac{1}{\sqrt{2}}\left(\frac{\zeta-1}{\zeta-z}+\frac{\overline{\zeta}-1}{\overline{\zeta}-\overline{z}}-1\right)+O(2-|z-1|^2)\right)\mathrm{d}s_\zeta\\
&= \lim_{\substack{z\to\tilde{z}\in C_0^l\\ z\in R}} \frac{1}{2\sqrt{2}\pi}\int_{\partial R}\gamma(\zeta)\left(\frac{\zeta-1}{\zeta-z}+\frac{\overline{\zeta}-1}{\overline{\zeta}-\overline{z}}-1\right)\mathrm{d}s_\zeta
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\substack{z \to \tilde{z} \in C_0^l \\ z \in R}} \frac{1}{2\sqrt{2}\pi} \int_{C_0^l} \gamma(\zeta) \left(\frac{\zeta - 1}{\zeta - z} + \frac{\overline{\zeta} - 1}{\overline{\zeta} - \overline{z}} - 1 \right) \mathrm{d}s_\zeta \\
&= \gamma(\tilde{z}).
\end{aligned}
$$

Analogously, applying the relations (6.24), (6.26), (6.27) and (6.28) respectively for the boundary arcs C_0^r, C_1 and C_{-1}, we obtain that

$$
\begin{aligned}
&\lim_{\substack{z \to \tilde{z} \in C_0^r \\ z \in R}} \frac{1}{2\pi} \int_{\partial R} \gamma(\zeta) p(z, \zeta) \mathrm{d}s_\zeta \\
&= \lim_{\substack{z \to \tilde{z} \in C_0^r \\ z \in R}} \frac{1}{2\sqrt{2}\pi} \int_{C_0^r} \gamma(\zeta) \left(\frac{\zeta + 1}{\zeta - z} + \frac{\overline{\zeta} + 1}{\overline{\zeta} - \overline{z}} - 1 \right) \mathrm{d}s_\zeta \\
&= \gamma(\tilde{z}),
\end{aligned}
$$

$$
\begin{aligned}
&\lim_{\substack{z \to \tilde{z} \in C_1 \\ z \in R}} \frac{1}{2\pi} \int_{\partial R} \gamma(\zeta) p(z, \zeta) \mathrm{d}s_\zeta \\
&= \lim_{\substack{z \to \tilde{z} \in C_1 \\ z \in R}} \frac{-1}{2\sqrt{2}\pi} \int_{C_1} \gamma(\zeta) \left(\frac{\zeta - \sqrt{3}i}{\zeta - z} + \frac{\overline{\zeta - \sqrt{3}i}}{\overline{\zeta - z}} - 1 \right) \mathrm{d}s_\zeta \\
&= \gamma(\tilde{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
&\lim_{\substack{z \to \tilde{z} \in C_{-1} \\ z \in R}} \frac{1}{2\pi} \int_{\partial R} \gamma(\zeta) p(z, \zeta) \mathrm{d}s_\zeta \\
&= \lim_{\substack{z \to \tilde{z} \in C_{-1} \\ z \in R}} \frac{-1}{2\sqrt{2}\pi} \int_{C_{-1}} \gamma(\zeta) \left(\frac{\zeta + \sqrt{3}i}{\zeta - z} + \frac{\overline{\zeta + \sqrt{3}i}}{\overline{\zeta - z}} - 1 \right) \mathrm{d}s_\zeta \\
&= \gamma(\tilde{z}).
\end{aligned}
$$

Therefore, the boundary behavior of the solution for each boundary arc has been verified.

□

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List of Symbols

$\mathbb{C}$	complex numbers, 1
$\mathbb{R}$	real numbers, 1
i	imaginary unit, 1
$\mathrm{Re}(z)$	real part of z, 1
$\mathrm{Im}(z)$	imaginary part of z, 1
$\lvert z\rvert$	modulus of z, 1
$\bar{z}$	conjugate of z, 1
D	a domain in $\mathbb{C}$, 1
∂D	boundary of D, 1
$\overline{D}$	closure of D, 1
$\frac{\partial w}{\partial z}$ (w_z)	complex derivative of $w(z)$ with respect to z, 2
$\frac{\partial w}{\partial \bar{z}}$ $(w_{\bar{z}})$	complex derivative of $w(z)$ with respect to $\bar{z}$, 2
$\partial_{\bar{z}}$	first order differential operator with respect to $\bar{z}$, 2
∂_z	first order differential operator with respect to z, 2
$C^n(D;\mathbb{C})$	complex-valued functions with continuous derivatives up to the nth order in a domain D, 2
$C(D;\mathbb{C})$ $(C^0(D;\mathbb{C}))$	continuous complex-valued functions in a domain D, 2
$\mathcal{H}(D)$	holomorphic functions in a domain D, 3
$\mathcal{A}(D)$	analytic functions in a domain D, 3
D^c	conjugate domain of a domain D, 3
$\mathrm{d}\sigma_\zeta$	area element (with respect to ζ), 3
$\mathrm{d}\zeta$	differential with respect to the variable ζ, 3
$w_{\zeta\zeta}$	second order derivative of $w(\zeta)$ with respect to ζ, 4
$w_{\zeta\bar{\zeta}}$	second order derivative of $w(\zeta)$ with respect to ζ and $\bar{\zeta}$, 4
$w_{\overline{\zeta\zeta}}$	second order derivative of $w(\zeta)$ with respect to $\bar{\zeta}$, 4
∂_z^2	second order differential operator with respect to z, 4
$\partial_z\partial_{\bar{z}}$	second order differential operator with respect to z and $\bar{z}$, 4
$\partial_{\bar{z}}^2$	second order differential operator with respect to $\bar{z}$, 4
Δ	Laplacian operator, 5
$\mathrm{d}\theta$	differential of argument, 5
s	arc length parameter, 5
$\mathrm{d}s_\zeta$	differential of arc length with respect to $\zeta(s)$, 5

∂_{ν_ζ}	outward normal derivative with respect to ζ, 5
$G_1(z,\zeta)$	harmonic Green function, 6
$G(z,\zeta)$	(classical) Green function, 6
$\mathbb{D}$	(open) unit disk in the complex plane, 7
$N_1(z,\zeta)$	harmonic Neumann function, 8
$\delta(s)$	normal derivative of harmonic Neumann function with respect to the parameter of arc length, 8
$H_\alpha(f)$ $(H(f;D,\alpha))$	Hölder constant of a function f in D with respect to the index α, 9
$H^\alpha(D;\mathbb{C})$	complex-valued functions satisfying Hölder condition in a domain D with respect to the index α, 9
$H^\alpha(D;\mathbb{R})$	real-valued functions satisfying Hölder condition in a domain D with respect to the index α, 9
$\mathbb{C}_\infty$	extended complex plane, 9
$C.P.$	Cauchy principal value, 9
$C_0^1(D;\mathbb{C})$	complex-valued functions with continuous first order derivatives and compact support in a domain D, 11
$L_p(D;\mathbb{C})$	complex-valued functions in D with finite L^p-norm, 11
T	Pompeiu operator, 11
Π	an integral operator, 11
$\frac{\partial}{\partial \boldsymbol{n}_\zeta}$	interior normal derivative with respect to the variable ζ, 12
S	Schwarz operator, 12
$\mathbb{C}\mathrm{P}^1$	complex projective line, 15
$[z:w]$	a homogeneous coordinate of complex projective line, 15
S^2	2-sphere, 16
A^*	conjugate transpose of a matrix A, 16
H^-	2×2 Hermitian matrices with negative determinants, 16
$\overset{\mathrm{H}^-}{\sim}$	an equivalent relation defined by H^-, 16
$\mathrm{GL}_2(\mathbb{C})$	general linear group of degree 2 over $\mathbb{C}$, 16
R_A	reflection at the generalized circle determined by a matrix A, 17
A^t	transpose of a matrix A, 18
B^{-1}	inverse of a matrix B, 18
Id	identity matrix, 20
$\mathrm{SL}_2(\mathbb{C})$	special linear group of degree 2 over $\mathbb{C}$, 20
$\mathrm{PSL}_2(\mathbb{C})$	projective special linear group of degree 2 over $\mathbb{C}$, 20
$\mathbb{C}^*$	nonzero complex numbers, 20
$\mathbb{N}$	natural numbers (including zero), 26
I	an index set, 26
$\mathrm{Inv}(D)$	the group generated by reflections at the boundary arcs of a domain D, 27
$\mathrm{M}(D)$	Möbius transformations in $\mathrm{Inv}(D)$, 27
$\#I$	cardinality of I, 36
$\mathbb{R}_{>0}$	positive real numbers, 39
$D(a,r)$	open disk of center a and radius r in $\mathbb{C}$, 39

z_{re}	reflection image of z, 39
$O(\|z\|^{-\alpha})$	a function whose absolute value is at most a constant multiple of $\|z\|^{-\alpha}$ as $z \to \infty$ for a given positive index α, 45
Ω	a half-plane, 45
$D_{\alpha,\theta}$	a circular digon, 57
$\mathbb{N}^*$	positive integers, 58
R	a circular rectangle, 75
$\mathbb{Z}^*$	nonzero integers, 93

Zusammenfassung

Diese Arbeit beschäftigt sich mit dem Spiegelparkettierungsprinzip und seinen Anwendungen in grundlegenden Randwertproblemen in einigen kreisförmigen Polygonen.

Das Spiegelparkettierungsprinzip wird verwendet, um mehrere Randwertprobleme für bestimmte Bereiche zu lösen, deren Grenzen aus Kreisbögen bestehen. Es liefert Ideen und Verfahren zur Konstruktion harmonischer Green-Funktionen und harmonischer Neumann-Funktionen, die bei der Behandlung von Dirichlet- und Neumann-Randwertproblemen für die Poisson-Gleichungen eine wichtige Rolle spielen. Das Spiegelparkettierungsprinzip trägt auch eine Methode zur Lösung des Schwarzschen Randwertproblems für die homogenen und inhomogenen Cauchy-Riemann-Gleichungen bei. Das Spiegelparkettierungsprinzip hat sich bei der Lösung dieser Randwertproblemen für viele spezielle Bereiche bewährt. Die Theorie dahinter und für welche Gebiete dieses Prinzip generell funktionieren kann, ist jedoch noch ein Mythos. Daher sind wir daran interessiert, die verborgene Theorie hinter dem Spiegelparkettierungsprinzip zu untersuchen und neue Bereiche zu erkunden, für die das Prinzip funktionieren kann.

Wir diskutieren zunächst Kreispiegelungen in der erweiterten komplexen Ebene und entwickeln auch einige Techniken im Umgang mit Kreispiegelungen. Wir verwenden Matrixwerkzeuge und sehen, dass die Formulierung mit Matrizen eine gewisse Bequemlichkeit für die Diskussionen bieten. Einige Ergebnisse zu konsekutiven Kreispiegelungen werden für weitere Untersuchungen aufbereitet.

Als nächstes führen wir die Definition von Spiegelparkettierungsgebieten ein, einer Klasse von Gebieten, für die Spiegelparkettierungen durchführbar sind. Wir beweisen, dass es dem Spiegelparkettierungsprinzip im Allgemeinen gelingt, die harmonischen Green- und Neumann-Funktionen für endliche Spiegelparkettierungsgebiete zu konstruieren. Wir erhalten auch einige Ergebnisse über die Normalableitungen von harmonischen Green- und Neumann-Funktionen an Gebietsrändern.

Wir untersuchen auch grundlegende Randwertprobleme in Kreisen und Halbebenen. Wir besprechen die Schwarzsche und die Poissonsche Integralformeln und ihr Randverhalten auf verallgemeinerten Kreisen. Mit Hilfe der Schwarzschen Integralformeln und den Eigenschaften des Pompeiu-Operators lösen wir das Schwarzsche Randwertproblem für die Cauchy-Riemann-Gleichungen in endlichen spiegelparkettbeschränkten Gebieten. Mit Hilfe harmonischer Green-Funktionen und Poisson-Integralformeln lösen wir auch die Dirichlet-Probleme für die Poisson-Gleichungen in endlichen spiegelparkettbeschränkten Gebieten.

Die letzten beiden Teile dieser Arbeit beschäftigen sich mit der Anwendung des Spiegelparkettierungsprinzips für grundlegende Randwertprobleme einiger kreisförmiger Polygone. Kreisdigonen, deren Randbögen sich an den beiden Eckpunkten mit einem Schnittwinkel $\frac{\pi}{n}$ für eine beliebige positive ganze Zahl n schneiden, werden als endliche Spiegelparkettierungsgebiete verifiziert. Mit Hilfe des Spiegelparkettierungsprinzips werden das Dirichlet-Problem, das Neumann-Problem und das Schwarz-Problem für diese Klasse von Kreisdigonen gelöst. Wir verifizieren auch, dass ein bestimmtes kreisförmiges Rechteck ein unendliches Spiegelparkettierungsgebiet ist. Es gelingt uns, die harmonische Green-Funktion zu konstruieren und dann das Dirichlet-Problem im Kreisrechteck zu lösen. Da wir mit Erfolg Beispiele für die Anwendung des Spiegelparkettierungsprinzips zur Lösung einiger Randwertproblem in unendlichen Spiegelparkettierungsgebieten behandeln, ist es lohnenswert, die Theorie des Spiegelparkettierungsprinzips für den Fall von unendliche Spiegelungen weiter zu untersuchen.