Singular SPDEs and Fluctuations of Particle Systems

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Abstract

This thesis studies the large scale behaviour of biological processes in a random environment. We start by considering a system of branching random walks in which the branching rates are determined by a random spatial catalyst. In an appropriate setting we show that this process converges to a superBrownian motion in a space white noise potential. We study the asymptotic properties of this superprocess and prove that it survives with positive probability. We then consider scaling limits of a spatial Λ–Fleming–Viot model, relating it both to the process we just introduced and to a stochastic Fisher-KPP equation. Finally, we study the longtime behaviour of the Kardar–Parisi–Zhang equation on finite volume, proving asymptotic synchronization and a one force, one solution principle. Our analyses rely on techniques from singular stochastic partial differential equations for the parabolic Anderson model and the KPZ equation, and on the theory of superprocesses.

To Q

"However much I love to look at the sky or the sea, still I'm more fascinated by a grain of sand under a microscope. The world in a single drop. The great and incredible life I discover in it. How can I call the small small and the great great, when both are so boundless?"

– From *The unwomanly face of war*, by Svetlana Aleksievic

Above all, my gratitude for guiding me to the completion of this work goes to my supervisor Nicolas Perkowski. In discussions, he always showed me a path to the clear and simple nature of a problem. He shared with me his excitement for mathematics and encouraged me to follow any question that would capture my interest, so that I could find new, personal reasons to study and enjoy this subject. For all this, for all the patience and the effort I must have cost him, I am very grateful.

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How boring would these studies have been, without Helena and Ana's company, the Adlershof coffee breaks with Bernhard and Katherina, the wonderful days in Rio with the people of the IRTG, without Tal's great stories and Willem's questions – and his hugs? A piece of this thesis is also yours.

You may believe that a student in probability appreciates only large numbers – and if you turn page you will not be disappointed. But actually, I am most indebted to the few. The few colleagues that have accompanied me in the preparation of this thesis, the friends with whom I have discovered mathematics and those that have been on my side for much longer, my parents and my love. To all of you,

Thank you,

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Contents

Introduction

In this thesis we study stochastic processes that describe the evolution in space and time of idealized chemical or biological systems. These processes, much like the Gaussian distribution in the Central Limit Theorem, capture the mesoscopic behaviour of microscopic particle systems. Often the small scale probabilistic features of these particle systems are irrelevant and only their overall structure is important. This phenomenon, which motivates the importance of the Gaussian distribution and of Brownian motion, is called universality and will be a leitmotiv throughout our work.

A notable process that arises in such a way is the superBrownian motion, introduced by Dawson and Watanabe [Daw75, Wat68]. It describes the evolution of the density of a large number of particles that, independently of each other, perform a random walk and occasionally give birth to new particles, or die. It is a measure-valued process that solves, in an appropriate sense, the stochastic partial differential equation (SPDE)

$$(\partial_t - \Delta)\mu(t,x) = \sqrt{\mu(t,x)}\tilde{\xi}(t,x), \qquad \mu(0,x) = \mu_0(x), \qquad t \in (0,\infty), \;\; x \in \mathbf{R}^d, \tag{1}$$

where $\tilde{\xi}$ is space-time white noise, a Gaussian field with covariance

$$\mathbb{E}[\tilde{\xi}(t,x)\tilde{\xi}(s,y)] = \delta(t-s, x-y),$$

the latter being the Dirac δ function.

The main theme of this thesis is to take into account the additional effect of a random spatial environment in the branching mechanism that leads to the superBrownian motion (SBM). At a microscopic level, we imagine particles performing a random walk on $\mathbf{Z}^d$. In every point x on the lattice we fix a potential $\xi(x)$ such that

$$\{\xi(x)\}_{x\in\mathbf{Z}^d} \;\text{ is an i.i.d. sequence of random variables, }\; \xi(x) \sim \Phi,$$

for a given random variable Φ normalized via $\mathbb{E}\Phi = 0, \mathbb{E}\Phi^2 = 1$. A particle in position $X(t)$ at time $t \geq 0$ gives birth to a new particle at rate $\xi(X(t))_+$ or dies at rate $\xi(X(t))_-$. After branching, the old and the new particle follow the same rule independently of one another. This process is called a branching random walk in a random environment (BRWRE). In the first part of the thesis we determine scales at which the density of particles associated to these dynamics is well approximated by a process that does not depend, other than for one parameter, on the particular distribution of Φ. We then study the relationship between this process and the SBM.

In determining the large scale behaviour of this particle system we see two opposing forces. On the one hand there is an averaging effect over space, since the random variables are centered, normalized and independent. On the other hand, the randomness determines the existence of arbitrary high peaks of the potential, where particles can reproduce at an accordingly high rate. In fact, this model was particularly studied in relation to intermittency and localization [ZMRS87, GM90, GM98]. For example, some works [ABMY00, GKS13] show that for long times the moments of the BRWRE are far from Gaussian and the strength of intermittency depends on the moment generating function $t \mapsto \mathbb{E}e^{t\Phi}$. Such results also tell us that if we consider only the longtime behaviour of the BRWRE, its properties will still strongly depend on the particular distribution of the potential. Instead, we show that in a diffusive regime, that is on large scales both in space and time and with a small potential, the system is well approximated by a process that does not depend on the particular distribution of ξ. A similar approach was taken by Mytnik [Myt96], under the assumption that the environment is white also in time. If we tune the potential so that we are in the regime of the Central Limit Theorem, we prove that the empirical measure associated to the particle system converges to the solution of the SPDE

$$(\partial_t - \Delta)\mu(t,x) = \xi(x)\mu(t,x) + \sqrt{2\nu\mu(t,x)}\tilde{\xi}(t,x), \quad \mu(0,x) = \mu_0(x), \quad t \in (0,\infty), \ x \in \mathbf{R}^d, \quad (2)$$

where $\nu = \mathbb{E}\Phi_+$, ξ is now space white noise on $\mathbf{R}^d$ and $\tilde{\xi}$ is a space-time white noise independent of ξ. The problem we are confronted with in studying this convergence is that in dimensions $d > 1$ the SBM is very irregular, and one can make sense of the related SPDE only via its martingale problem. At the same time, if we average out the randomness of the fluctuations we are left with the parabolic Anderson model (PAM)

$$(\partial_t - \Delta)w(t,x) = \xi(x)w(t,x), \qquad w(0,x) = w_0(x), \qquad t \in (0,\infty), \ x \in \mathbf{R}^d. \quad (3)$$

This equation admits a known solution only in dimensions $d \leq 3$ and requires theories from singular stochastic PDEs [Hai14, GIP15] to be solved if $d = 2,3$ (we restrict to $d = 1,2$ in this work). In Chapter II we show how to combine these approaches. We also try to understand whether some of the interesting longtime properties are conserved in this passage to the limit. While we do not address the question of intermittency, we prove that the process is locally persistent – and indeed the local mass may explode more than exponentially fast. This is in stark contrast with the classical SBM and is a consequence of the particular spectral properties of the Anderson Hamiltonian.

A different approach to this problem is to consider a population consisting of two types, say $\mathfrak{a}$ and $\mathfrak{A}$ that are competing against each other. Here the random environment takes the role of a selection coefficient that favors one of the two types according to its sign. If the selection coefficient is sign changing these kind of models attempt to explain the coexistence of the two types in different regions of space. Already in early works by Wright [Wri43], spatial structure and heterogeneous selection play a key role in understanding genetic diversity and give rise to a mechanism called isolation by distance. This is supported by empirical studies on plants [PCFF03], bacteria [RT98] and animals [KP97] (see also [TBG+04, Hed06, SGK14]).

The starting point for our analysis is the class of spatial Λ–Fleming–Viot (SLFV) models, introduced by Barton, Etheridge and Veber [Eth08, BEV10]. In these models, the population is distributed over continuous space, whereas the reproductive events involve macroscopic regions of space – say balls of a certain radius – and are driven by a space-time Poisson point process. Since the SLFV combines features from discrete and continuous settings, we refer to it as a semidiscrete model.

We study two different scenarios. In the first one, we assume that type $\mathfrak{a}$ is rare compared to $\mathfrak{A}$. At the same time, we consider the selection coefficient s_n to scale to a spatial white noise ξ on the torus $\mathbf{T}^d$ and perform a diffusive scaling in space and time. Just as a small sub-population in the Wright-Fisher model is described by a branching process, we obtain that in the limit, the density of particles of type $\mathfrak{a}$ is described by the superBrownian motion in a random spatial environment of Equation (2). Similar scaling results were obtained by Chetwynd-Diggle and Etheridge [CDE18] without selection (see also [CDP00] for an analogous scaling regarding the voter model) and recently extended in [CP20] to certain critical values of the scaling parameters. For an SLFV with a selection coefficient that is white in time and correlated in space the scaling limit was obtained by [CK19] using a lookdown representation. The main difficulty in proving our convergence result is to treat the vanishing nonlinear terms that derive from the interaction between the two types of particles. To simplify this analysis we restrict to the compact domain $\mathbf{T}^d$ instead of the entire space. For our proof, we need to adapt the tools for singular stochastic PDEs to incorporate semidiscrete approximations. In this we rely on suitable two-scale regularity estimates.

In the second scenario, the selection coefficient s_n approximates a smooth random function $\overline{\xi}$, and we do not take the sparsity assumption. In this case, under diffusive scaling we obtain convergence of the relative particle density to a solution of the (in $d = 1$ stochastic) Fisher-KPP equation

$$\begin{aligned}(\partial_t - \nu_0\Delta)\mu(t,x) &= \overline{\xi}(x)\mu(1-\mu)(t,x) + \sqrt{\mu(1-\mu)(t,x)}\tilde{\xi}(t,x)1_{\{d=1\}}, \quad t \in (0,\infty), \quad x \in \mathbf{T}^d \\ \mu(0,x) &= \mu_0(x), \quad x \in \mathbf{T}^d\end{aligned} \tag{4}$$

for some $\nu_0 > 0$. As before $\tilde{\xi}$ is a space-time white noise independent of $\overline{\xi}$. The treatment of this second regime is apparently much simpler, as the solution is bounded between 0 and 1. The only difficulty is to prove convergence in a topology, in which one can pass the limit inside the nonlinearity. Unlike previous works [EVY20, BEK18] we can make use of the regularity estimates we mentioned and provide a concise argument for the tightness of the approximating sequence in a Sobolev space of positive regularity. In this setting we can study the longtime behavior of the equation.

The last problem considered in this thesis pertains to a different class of models. We will study the longtime behavior of SPDEs of the form

$$(\partial_t - \Delta)h(t,x) = |\nabla h|^2(t,x) + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad t \in (0,\infty), \quad x \in \mathbf{T}^d, \tag{5}$$

where η is some random noise. The most notable example is to be found in dimension $d = 1$ with η being space-time white noise. In this case we obtain the Kardar–Parisi–Zhang (KPZ) Equation [KPZ86]. The latter is the scaling limit of many microscopic growth models under weak asymmetry or intermediate disorder: see, inter alia, [BG97, HQ18, GP16, HS17]. Yet the KPZ equation is itself not scale invariant. It is conjectured to connect – by considering the solution on appropriately large scales – the microscopic models to a less understood object, called the KPZ fixed point: see [QS15] for an overview. These conjectures motivate the interest behind the longtime behavior of equations of type (5). Another motivation comes Burgers-like equations. These are toy models in fluid dynamics, and are formally linked to KPZ via $v = \nabla h$:

$$(\partial_t - \Delta)v(t,x) = \nabla|v|^2(t,x) + \nabla\eta(t,x), \quad v(0,x) = v_0(x), \quad t \in (0,\infty), \ x \in \mathbf{T}^d. \tag{6}$$

Wellposedness for the KPZ equation was a milestone obtained in works by Hairer [Hai13, Hai14] and Gubinelli, Imkeller and Perkowski [GIP15, GP17] that contributed to the development of the theory of singular SPDEs. Preceding these results there was no clear understanding of the quadratic nonlinearity in (5), yet the equation could be studied through the Cole-Hopf transform, by imposing that $u = \exp(h)$ solves the linear stochastic heat equation (SHE) with multiplicative noise, a step that can be made rigorous for smooth η but requires particular care and the introduction of renormalisation constants if η is space-time white noise:

$$(\partial_t - \Delta)u(t,x) = \eta(t,x)u(t,x), \qquad u(0,x) = u_0(x), \qquad t \in (0,\infty), \ x \in \mathbf{T}^d. \tag{7}$$

In addition to proving wellposedness for the KPZ equation, Hairer introduced the notion of local subcriticality [Hai14], which provides a formal condition on η under which Equations (5) and (7) are well posed. Recent works show that this condition is indeed sufficient [BHZ19, BCCH21, CH16]. Therefore it makes sense to investigate the longtime behaviour of KPZ-like equations of type (5) for arbitrary noise η, under the assumption that a solution map to the equation is given and satisfies some natural requirements.

For the KPZ Equation, unique ergodicity "modulo constants" – since the equation is translation invariant – was established by Hairer and Mattingly [HM18b] as a consequence of a strong Feller property that holds for a wide class of SPDEs. In addition, the invariant measure is known to be the Brownian bridge [FQ15] and in [GP18] the authors prove a spectral gap for Burgers equation, implying exponential convergence to the invariant measure, although restricting to initial conditions that are "near-stationary".
On the other hand, Sinai [Sin91] considered a noise of the form $\eta(t,x) = V(x)\partial_t\beta(t)$, for smooth $V \in C^\infty(\mathbf{T})$ and a Brownian motion β. The article shows that there exists a random function $\overline{v}(t,x)$ defined for all $t \in \mathbf{R}$ such that almost surely, independently of the initial conditions v_0 within a certain (random) class:

$$\lim_{t\to\infty} v(t,x) - \overline{v}(t,x) = 0,$$

for all $x \in \mathbf{T}^d$ and with v solving Equation (6). This property is referred to as *synchronization*. In addition, if one starts Burgers equation at time $-n$ with $v^{-n}(-n,x) = v_0(x)$:

$$\lim_{n\to\infty} v^{-n}(t,x) = \overline{v}(t,x), \qquad \forall t \in (-\infty,\infty), \ x \in \mathbf{T}^d.$$

The last property is called a *one force, one solution* principle (1F1S) and it implies that $\overline{v}$ is the unique (ergodic) solution to Equation (6) on $\mathbf{R}$. Results of this kind have subsequently been extended, most notably to the inviscid case [WKMS00] or to infinite volume, for example in [BCK14] and recently in [DGR19], all for specific noises.

In Chapter V we will prove synchronization and 1F1S on the torus for a large class of ergodic noises, including space-time white noise, providing deterministic exponential convergence rates in appropriate Hölder spaces. The proof relies on the theory of random dynamical systems and on a contraction principle, due to Birkhoff [Bir57], in the cone of positive functions endowed with a particular metric.

In the rest of the introduction we will give a more detailed review of our results.

Chapter II

This chapter is based on the joint work [PR19b] with Nicolas Perkowski.

Scaling limits of branching particle systems have been an active field of research since the early results by Dawson et al. and gave rise to the study of superprocesses, most prominently the so-called superBrownian motion (see [Eth00, DMS93, LG99] for excellent introductions). We follow this original setting and study the behavior of the BRWRE introduced above under diffusive scaling: spatial increments will be of order $\Delta x \simeq 1/n$, while temporal increments will be of order $\Delta t \simeq 1/n^2$. The particular nature of our problem requires us to couple the diffusive scaling with the scaling of the environment: this is done via an "averaging parameter" $\varrho \geq d/2$, while the noise is assumed to scale to space white noise, namely we fix a sequence of random potentials such that $\xi^n(x) \simeq n^{d/2}$.

The diffusive scaling of spatial branching processes in a random environment has already been studied, for example by Mytnik [Myt96]. As opposed to the current setting, the environment in Mytnik's work is white also in time. This has the advantage that the model is amenable to probabilistic martingale arguments, which are not available in the static noise case that we investigate here. Therefore, we replace some of the probabilistic tools with arguments of a more analytic flavor. Nonetheless, at a purely formal level our limiting process is very similar to the one obtained by Mytnik. Moreover, our approach to uniqueness is reminiscent of the conditional duality appearing in later works by Crişan [Cri04], Mytnik and Xiong [MX07]. Notwithstanding these resemblances, we shall see that some statistical properties of the two processes differ substantially.

At the heart of our study of the BRWRE lies the following observation. If $u(t,x)$ indicates the numbers of particles in position x at time t, then the conditional expectation given the realization of the random environment, $w(t,x) = \mathbb{E}[u(t,x)|\xi]$, solves a discrete version of the parabolic Anderson model (3). The PAM has been studied both in the discrete and in the continuous setting (see [Kön16] for an overview). In the latter case the SPDE is not solvable via Itô integration theory. In particular, in dimension $d = 2, 3$ the study of the continuous PAM requires special analytical and stochastic techniques in the spirit of rough paths [Lyo98], such as the theory of regularity structures [Hai14] or of paracontrolled distributions [GIP15]. In dimension $d = 1$ classical analytical techniques are sufficient. In dimension $d \geq 4$ no solution is expected to exist, because the equation is no longer *locally subcritical* in the sense of Hairer [Hai14]. The dependence of the subcriticality condition on the dimension is explained by the fact that white noise loses regularity as the dimension increases.

To apply the named theories in singular SPDE in dimension $d = 2,3$, we need to tame certain functionals of the white noise via a technique called *renormalisation*, with which we remove diverging singularities. In this work, we restrict to dimensions $d = 1,2$ as this simplifies several calculations. At the level of the 2-dimensional BRWRE, the renormalisation has the effect of slightly tilting the centered potential by considering instead an effective potential:

$$\xi^n_e(x) = \xi^n(x) - c_n, \qquad c_n \simeq \log(n).$$

So if we take the average over the environment, the system is slightly out of criticality, in the biological sense, namely births are less likely than deaths. This asymmetry is counter-intuitive at first. Yet the random environment has a strongly benign effect on the process, since it generates extremely favorable regions. These favorable regions are not seen upon averaging, and they have to be compensated for by subtracting the renormalisation

The special character of the noise and the analytic tools just highlighted will allow us, in a nutshell, to fix one realization of the environment – outside a null set – and derive the following scaling limits. For "averaging parameter" $\varrho > d/2$ a law of large numbers holds: the process converges to the continuous PAM. Instead, for $\varrho = d/2$ one captures fluctuations from the branching mechanism. The limiting process can be characterized via duality or a martingale problem and we call it *rough superBrownian motion (rSBM)*. In dimension $d = 1$, following analogous results for SBM [KS88, Rei89], the rSBM admits a density which in turn solves the SPDE (2). The solution is weak both in the probabilistic and in the analytic sense. This means that the product $\sqrt{\mu(t,x)}\tilde{\xi}(t,x)$ is interpreted via a stochastic integral in the sense of Walsh [Wal86] and the space-time white noise is constructed starting from the solution μ. At the same time, the product $\xi(x)\mu(t,x)$ is defined only upon testing with functions in the *random* domain of the Anderson Hamiltonian $\mathscr{H} = \Delta + \xi$, a random operator that was introduced by Fukushima-Nakao [FN77] in $d = 1$ and by Allez-Chouk [AC15] in $d = 2$, see also [Lab19] for $d = 3$. The crux of our analysis is to combine the martingale and the pathwise approach via a mild formulation of the martingale problem based on the Anderson Hamiltonian. A similar point of view was recently taken by Corwin-Tsai [CT20], and to a certain extent also in [GUZ20].

Coming back to the rSBM, we conclude this work with a proof of persistence of the process in dimension $d = 1,2$. More precisely we even show that with positive probability we have $\mu(t,K) \to \infty$ (in fact the mass may explode faster than exponentially) for all compact sets $K \subset \mathbf{R}^d$ with non-empty interior. This is opposed to what happens for the classical SBM, where persistence holds only in dimension $d \geq 3$, whereas in dimensions $d = 1,2$ the process dies out: see [Eth00, Section 2.7] and the references therein. Even more striking is the difference between our process and the SBM in random, white in time, environment: under the assumption of a heavy-tailed spatial correlation function Mytnik and Xiong [MX07] prove extinction in finite time in any dimension. Note also that in [Eth00, MX07] the process is started in the Lebesgue measure, whereas here we prove persistence if the initial value is a Dirac mass.

Chapter III

This chapter is based on the joint work [KR20] with Aleksander Klimek.

There are many approaches one can take to model a spatially structured population consisting of two competing types. For example, in the stepping stone models [Kim53] the population evolves in separated islands distributed on a lattice and interacts only with neighboring islands. Other approaches are based around the Wright–Malécot formula [BDE02, Mal48, Wri43], which was introduced to quantify the phenomenon of isolation by distance. These models suffer in part from inconsistencies in their assumptions: see [BEV13] for an overview of difficulties associated with modeling spatially distributed populations. Moreover, in dimension $d = 2$, Equation (4) has no known analogous that incorporates "genetic drift" (i.e. the space-time white noise term), essentially because of the irregularity of the noise in higher dimensions. The spatial Lambda-Fleming-Viot (SLFV) class of models, introduced in [Eth08] and formally constructed in [BEV10], has been proposed specifically to overcome these difficulties, and is at the basis of our work. As we already mentioned, in the SLFV the population is distributed over continuous space and at random times particles of type $\mathfrak{a}$ or $\mathfrak{A}$ reproduce, in an amount proportional to an intensity $\mathfrak{u}$ in balls B_n of radius $1/n$. In fact, the radii of the balls can be chosen themselves at random, leading to long-range diffusion, so that in scaling limits the Laplacian is replaced by some fractional Laplace operator (see e.g. [EVY20]). We are interested in the limit $n \to \infty$, by scaling time diffusively and the intensity parameter $\mathfrak{u}$ at the correct level to see fluctuations.

In the neutral SLFV there is no bias in the relative fitness of the populations at hand. Our work considers instead the case in which there is a bias, which is modeled by a sign changing selection coefficient $s_n(x), x \in \mathbf{T}^d$, so that $\mathfrak{a}$ is favored in the location x if $s_n(x) > 0$ and $\mathfrak{A}$ is favored in the opposite case. Instead of choosing a specific selection coefficient, we sample it from a probability distribution $\mathbb{P}$. We will consider the proportion $X_n(\omega, t, x)$, evaluated at time $t \geq 0$ and position $x \in \mathbf{T}^d$, of particles of type $\mathfrak{a}$ with respect to the total population, given the realization $s_n(\omega)$ of the selection coefficient. In all our scaling limits the effect of selection is *weak*, that is of lower order, with respect to neutral events.

In the first regime under study, we assume that type $\mathfrak{a}$ is rare compared to $\mathfrak{A}$. The rarity is described by considering an initial condition $X_n(\omega, 0, x)$ of order $n^{-\varrho}$ for certain values of $\varrho > 0$. In this scenario $\mathfrak{a}$ represents a mutation which tries to establish itself among the wild type $\mathfrak{A}$. We prove that the scaling limit of $Y_n = n^{\varrho} X_n$ is given by the superBrownian motion in a random spatial environment (2). By considering the linearization near zero of the Fisher-KPP Equation instead of the SLFV in a sparse regime, the reader can imagine that in performing this limit we have to treat vanishing nonlinear terms of the form $n^{-\varrho} Y_n^2$. These terms are vanishing, yet difficult to control. To treat them, previous articles make use duality, under the assumption that the selection coefficient has a fixed sign (see e.g. [EVY20]). In our setting, although we present a dual for the SLFV with sign-changing selection, we do not understand yet its behaviour under diffusive scaling and even less so if the selection becomes rough. For this reason we use a purely analytic argument to bound the square terms, which on the downside leads to some unnatural assumptions on the parameter ρ. Eventually, the scaling limit follows by an application of the Krein-Rutman theorem. At this point it is particularly important that the space is compact, while all other results in this chapter seem to extend from $\mathbf{T}^d$ to $\mathbf{R}^d$. The Krein-Rutman theorem is applied to a sequence of operators approximating the Anderson Hamiltonian:

$$\mathscr{H}_n = \mathscr{A}_n + \xi^n - c_n 1_{\{d=2\}},$$

where $\mathscr{A}_n(\varphi)(x) = n^2 \fint_{B_n(x)} \fint_{B_n(y)} \varphi(z) - \varphi(x) \mathrm{d}z \mathrm{d}y$ is a semidiscrete version of the Laplace operator. Understanding completely this limit is postponed to Chapter IV, but in the study of singular SPDEs the smoothing effect of the Laplacian is essential. Hence, as a first step towards the convergence of the operators, in Chapter III we establish the regularization properties of the approximate Laplacian $\mathscr{A}_n$, commonly known as Schauder estimates. Through a two-scale argument, we separate macroscopic scales in frequency space, at which $\mathscr{A}_n$ regularizes analogously to the Laplacian, and microscopic scales, which are small but see no regularization.

The regularizing properties we show turn out to be useful in the second scenario, where s_n is chosen to scale to a smooth random function $\overline{\xi}$ and they allow us to provide a streamlined proof of the scaling limit. In this second case, the intensity of the fluctuations is governed by a parameter $\eta \geq 0$ that is linked to the intensity of impacts in the SLFV. There exists a critical value $\eta_c(d) \geq 0$ such that the noise is of order $n^{-(\eta-\eta_c)}$. In dimension $d = 1$ we consider $\eta = \eta_c$, while in dimension $d = 2$ we choose $\eta > \eta_c$. In some models, again by taking into account dual processes, cf. [Eth08, FP17], one can prove that in $d = 2$ the deterministic limit holds also *at* the critical value $\eta = \eta_c$. This is linked to the fact that the stochastic Fisher-KPP equation has no analogous in dimension $d \geq 2$. In our analysis the convergence at the critical value remains open.

Eventually, we briefly study the longtime behavior of the limiting processes. Regarding the Fisher-KPP equation (4), many works study the longtime properties if selection has a definite sign, especially in relation to the existence and speed of traveling waves (see for example [MS95]). If the selection is sign changing the first question is whether there exists a *unique* nontrivial limit. In this case, if the noise is sufficiently strong, uniqueness follows as an application of a result by Henry [Hen81] (see also [BO86] in the case of definite sign but Dirichlet boundary conditions), which relies on a bifurcation argument. This argument depends on the particular form of the nonlinearity $\mu(1-\mu)$, which is concave and of first order in 0 and 1. Outside this setting it may happen that the limit is not unique (see for example [Sov18] for an overview and many references).

Chapter IV

This chapter is based on the joint work [KR20] with Aleksander Klimek and on[Ros20].

It is dedicated to two technical points, both instances of discretizations of the Anderson model. First, we consider the semidiscrete approximations

$$\mathscr{H}_n = \mathscr{A}_n + \xi^n - c_n 1_{\{d=2\}}$$

of the Anderson Hamiltonian we introduced previously. Then we study lattice discretizations of PAM (3) on a box with Dirichlet boundary conditions. The first analysis is essential to derive the scaling limits in Chapter III. The second analysis is used to deduce the longtime behaviour of the superBrownian motion in a random spatial environment (2). In analogy to the classical case of SDEs, where one has to take care that discretizations converge to the correct notion of stochastic integral, discretizations of singular SPDEs require some attention. In the frameworks of regularity structures or paracontrolled distributions the problem is usually reduced to a two-step analysis. First one has to show that certain, possibly renormalised, stochastic quantities converge (from the discrete to the continuous setting). Then one needs an appropriately robust analytic machinery to see that discrete solutions depend continuously on these stochastic quantities. Previous works (see [MW17, HM18a, EH19, CGP17, MP19] for a partial literature) do not consider the semidiscrete setting, so in this case our main point is to extend the available analytic tools for paracontrolled distributions in order to incorporate the two-scale regularity estimates we proved in Chapter III. Eventually we prove that the semidiscrete Anderson Hamiltonian converges in the resolvent sense to its continuous analogue.

As for the second setting, we fix an $L \in \mathbf{N}$ and consider PAM on a box:

$$\begin{aligned} (\partial_t - \Delta) w(t,x) = \xi(x) w(t,x), \qquad & w(0,x) = w_0(x), & t \in (0,\infty),\ \ x \in (0,L)^d \\ & w(t,x) = 0, & t \in (0,\infty),\ \ x \in \partial[0,L]^d, \end{aligned} \tag{8}$$

where ξ is space white noise. Singular SPDEs with Dirichlet boundary conditions have been studied using both regularity structures [GH19] and paracontrolled distributions [Cv19]. Since we introduce a boundary the most natural renormalisation procedure, which consists in taking Wick products of distributions, amounts to removing a space-dependent function and not just a constant. Intuitively, this becomes clear if we consider the solution X to the elliptic problem $(-\Delta + 1)X = \xi$ with Dirichlet boundary conditions. The process X is Gaussian, but its variance $\mathrm{Var}(X(x))$, $x \in [0,L]^d$, is spatially inhomogeneous. Therefore we will need to pay particular attention to the renormalisation procedure and make sure that it is sufficient to remove only a constant and that moreover this constant does not depend on the size of the box. The last points are crucial for the application in Chapter II. On the contrary, the analytic theory does not require much attention, since one can adapt the tools for discrete paracontrolled distributions [MP19, CGP17] to the method for Dirichlet boundary conditions introduced in [Cv19].

Chapter V

This chapter is based on [Ros19].

We will attempt to understand and extend the results by Sinai [Sin91] regarding the synchronization and 1F1S principle for KPZ-like equations of the form (5) via an application of the theory of random dynamical systems. The power of our approach lies in the possibility of treating any noise η which satisfies roughly the following two conditions:

i The noise η is ergodic.

ii The stochastic heat equation (7) is almost surely well-posed: there exists a unique, global in time solution for every $u_0 \in C(\mathbf{T}^d)$, the solution map being a linear, compact, strictly positive operator on $C(\mathbf{T}^d)$.

For example, η can be chosen to be space-time white noise or a noise that is fractional in time.

In the original work [Sin91], the solution u to (7) evaluated at time n is represented by $u(n,x) = A^n u_0(x)$ for a compact strictly positive operator A^n. The proof of the result makes use in turn of the explicit representation of the operator A^n via the Feynman-Kac formula. Such representation becomes more technical when the noise η is not smooth and requires some understanding of random polymers. Although in principle this path appears feasible also in the case of space-time white noise (see [CC18, DD16] for the constructions of the random polymers in this setting) it is quite technical. Instead, we follow a different road.

If η were a time-independent noise, the synchronization of the solution v to (6) would amount to the convergence, upon rescaling, of u to the random eigenfunction of A^1 associated to its largest eigenvalue: an instance of the Krein-Rutman Theorem. We will extend this argument to the non-static case with an application of the theory of random dynamical systems.

To be precise, we will introduce a particular distance d_H, called Hilbert's projective distance, on the cone of positive functions, such that for any positive operator A and strictly positive functions f, g:

$$d_H(Af, Ag) \leq \tau(A) d_H(f, g),$$

for some constant $\tau(A) \in [0,1]$. The existence of such a contraction constant is a result by Birkhoff [Bir57] (see [Bus73] for an overview), and this contraction method was already deployed by [AGD94] and later refined by [Hen97] in the study of random matrices. In fact, with an application of the ergodic theorem, these results give rise to an ergodic version of the Krein-Rutman theorem.

In this way we obtain synchronization and 1F1S with a deterministic exponential speed in the topology determined by the distance d_H. This topology is naturally linked to convergence in the space of continuous functions for solutions to the KPZ equation "modulo constants". With this we mean that we identify two functions $h, h' : [0,\infty) \times \mathbf{T}^d \to \mathbf{R}$ if there exists a $c \colon [0,\infty) \to \mathbf{R}$ such that $h(t,x) - h^{'}(t,x) = c(t), \forall t \geq 0, x \in \mathbf{T}^d$. In particular, we obtain synchronization and 1F1S for the gradient $v = \nabla_x h$, which satisfies Burgers equation. In an example with smooth noise we show that the constant that has to be subtracted for synchronization at the level of the KPZ equation can be chosen time-independent, a fact that we expect to hold in general. Of course, the constant will always depend on the initial conditions since this is the case also in the time-independent setting, but the explicit dependence is not given.

So far we showed synchronization for KPZ in the space of continuous functions. This is a very weak topology: in fact we would fall short of the result by Sinai, which proves synchronization pointwise for Burgers equation. Therefore we would like to lift the convergence to appropriate Hölder spaces, depending on the regularity of the driving noise.

We prove that this is possible and obtain the same deterministic exponential speed of convergence. On the downside, our arguments require some additional moment bounds, such as

$$\mathbb{E}\sup_{x\in\mathbf{T}}|h(t,x)|<\infty,$$

for $t>0$. In concrete examples we show how to obtain these bounds from a quantitative version of a strong maximum principle for (7). The case of space-time white noise requires particular attention. While a classical result by Mueller [Mue91] guarantees almost sure strict positivity for the solution to the SHE (7), this is not sufficient to bound the required expectation. Instead, the proof we present makes use of the pathwise solution theory to the equation through a bound obtained in [PR19a] (also the variational principle developed in [GP17, Section 7] appears sufficient). Although the study of convergence in Hölder spaces seems to be new, for different reasons moment bounds of the likes of the one above appeared already in the finite-dimensional case [AGD94].

As we already mentioned, the examples we treat are the original KPZ equation, namely the case of η being space-time white noise in $d=1$, and the case of $\eta(t,x)=V(x)\,\mathrm{d}\beta^H_t$ for β^H a fractional Brownian motion of Hurst parameter $H>\frac{1}{2}$ and $V\in C^\infty(\mathbf{T})$. In the latter case the solution is not Markovian and ergodic results are rare, see for example [MP08] for ergodicity of linear SPDEs with additive fractional noise.

Finally, there are several instances of applications of the theory of random dynamical systems to stochastic PDEs. Particularly related to our work is the study of order-preserving systems which admit some random attractor [AC98, FGS17, BS19]. The spirit of these results is similar to ours. Yet, although the linearity of Equation (7) on the one hand guarantees order preservation, on the other hand it does not allow the existence of a unique random attractor. In this sense, our essentially linear case appears to be a degenerate example of the synchronization addressed in the works above.

I

Notations & function spaces

The following definitions will hold throughout the thesis.

- $\mathbf{N} = \{1, 2, \dots\}$,
- $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$,
- $\mathbf{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$,
- $\mathbf{R}$ is the set of real numbers,
- $\mathbf{R}_+ = [0, \infty)$,
- $\imath = \sqrt{-1}$,
- $\mathbf{C}$ is the set of complex numbers,
- $|x|$, for $x \in \mathbf{C}^d, d \in \mathbf{N}$, denotes the Euclidean norm.
- $\mathbf{T}^d$ is the d-dimensional torus: $\mathbf{T}^d = \mathbf{R}^d / \mathbf{Z}^d$, for a dimension $d \in \mathbf{N}$,
- $C(X; Y)$, given two metric spaces X, Y, is the space for continuous functions from X to Y. If $Y = \mathbf{R}$ we write $C(X)$,
- $C_b(X)$ is the space of continuous and bounded real-valued functions on a metric space X,
- $\mathscr{B}(X)$ is the Borel σ-algebra of a topological space X,
- $\mathrm{supp}(\varphi)$, for a continuous function $\varphi \colon X \to \mathbf{R}$ on a topological space X, is the closed support:
$$\mathrm{supp}(\varphi) = \overline{\left\{ x \in X \;\middle|\; \varphi(x) \neq 0 \right\}},$$
- For a sufficiently smooth function φ on an open set $O \subseteq \mathbf{R}^d$ and $k \in \mathbf{N}_0^d$ denote with $|k| = k_1 + \cdots k_d$ and write the derivative:
$$\partial^k \varphi(x) = \frac{\mathrm{d}^{|k|}}{\mathrm{d}^{k_1} x_1 \cdots \mathrm{d}^{k_d} x_d} \varphi(x), \qquad \forall x \in O,$$

- For any set $\mathscr{X}$ and any two functions $f,g\colon \mathscr{X} \to \mathbf{R}$ we write:
$$f \lesssim g$$
if there exists a constant $C > 0$ such that $f(x) \leq Cg(x)$, $\forall x \in \mathscr{X}$. Similarly
$$f \simeq g$$
if $f \lesssim g$ and $f \gtrsim g$.

I.0.1 Function spaces

Below, let O denote an arbitrary open subset $O \subseteq \mathbf{R}^d$ or $O \subseteq \mathbf{T}^d$.

- $C^\infty(O)$ is the space of smooth functions, that is of functions $\varphi\colon O \to \mathbf{R}$ such that the partial derivatives $\partial^k \varphi$ exist for any $k \in \mathbf{N}_0^d$,
- $C_c^\infty(O)$, $C_c(O)$ are respectively the spaces of smooth and continuous functions with compact support in O,
- $\mathscr{S}(O)$ is the space of functions $\varphi \in C^\infty(O)$ such that for any $k \in \mathbf{N}_0^d, a \in \mathbf{N}$:
$$\|\varphi\|_{a,k,O} := \sup_{x \in O}(1+|x|)^a |\partial^k \varphi(x)| < \infty,$$
with the topology induced by the seminorms $\|\cdot\|_{a,k,O}$.
- $\mathscr{S}'(O)$ is the topological dual of $\mathscr{S}(O)$: the space of Schwartz distributions on O,
- $L^p(O)$, for $p \in [1,\infty]$, is the space of measurable functions $\varphi\colon O \to \mathbf{R}$ (up to modification on sets of zero Lebesgue measure) with the norm:
$$\|\varphi\|_{L^p(O)} = \left(\int_O |\varphi(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}},$$
- $\mathscr{M}(O)$ is the space of positive finite measures on O, with the topology of weak convergence. That is $\lim_{\ell \to \infty} \mu^\ell = \mu$ in $\mathscr{M}(O)$ if for all f bounded and continuous:
$$\lim_{\ell \to \infty} \int_O f(x) \, \mathrm{d}\mu^\ell(x) = \int_O f(x) \, \mathrm{d}\mu(x),$$
- $\langle \cdot, \cdot \rangle_O$ indicates at the same time the dual pairing between $\mathscr{S}(O)$ and $\mathscr{S}'(O)$, the dual pairing between $L^p(O)$ and $L^{p'}(O)$ (with $p' = (1-1/p)^{-1}$) and the pairing between $\mathscr{M}(O)$ and $C_b(O)$, that is the integral of a continuous function against a measure, as in the point above. Whenever it is clear from the context, we will omit the subscript O,
- $\varphi * \psi$, for $\varphi \in \mathscr{S}'(O), \psi \in \mathscr{S}(O)$, is the convolution:
$$x \mapsto \varphi * \psi(x) = \langle \varphi(\cdot), \psi(x - \cdot) \rangle_O,$$
- $\mathbb{D}([0,\infty); Y)$, for a given metric space Y, is the space of cadlag functions $\varphi\colon [0,\infty) \to E$ endowed with the Skorohod topology. We may also consider the time interval $[0,T]$ instead of $[0,\infty)$, for some $T > 0$,

I.0.2 Scaling

We are interested in discrete approximations of continuous systems. To describe the scaling we will use a parameter $n \in \mathbf{N}$, and we will study systems in which particles evolve in space on distances of order $\Delta x = n^{-1}$. Hence define:

- $\mathbf{Z}_n^d = \frac{1}{n}\mathbf{Z}^d$,
- $\mathbf{T}_n^d = n\mathbf{T}^d = \mathbf{R}^d / n\mathbf{Z}^d$,
- $\mathscr{S}(\mathbf{Z}_n^d) = \Big\{\varphi\colon \mathbf{Z}_n^d \to \mathbf{R} \;\Big|\; \sup_{x\in\mathbf{Z}_n^d}(1+|x|)^a|\varphi(x)| < \infty, \;\; \forall a > 0\Big\}$,
- $\mathscr{S}'(\mathbf{Z}_n^d) = \Big\{\varphi\colon \mathbf{Z}_n^d \to \mathbf{R} \;\Big|\; \sup_{x\in\mathbf{Z}_n^d}(1+|x|)^a|\varphi(x)| < \infty, \quad \text{for some } a < 0\Big\}$,
- $\langle\cdot,\cdot\rangle_{\mathbf{Z}_n^d}\colon \mathscr{S}'(\mathbf{Z}_n^d)\times\mathscr{S}(\mathbf{Z}_n^d) \to \mathbf{R}$ is the pairing:
$$\langle\varphi,\psi\rangle_{\mathbf{Z}_n^d} = \frac{1}{n^d}\sum_{x\in\mathbf{Z}_n^d}\varphi(x)\psi(x),$$
- $\mathscr{S}(\mathbf{T}_n^d) = C^\infty(\mathbf{T}_n^d)$ is defined analogously to $\mathscr{S}(\mathbf{T}^d)$ and similarly also the pairing $\langle\cdot,\cdot\rangle_{\mathbf{T}_n^d}$ is an extension to distributions of:
$$\langle\varphi,\psi\rangle_{\mathbf{T}_n^d} = \int_{\mathbf{T}_n^d}\varphi(x)\psi(x)\,\mathrm{d}x,$$
- $\|\varphi\|_{L^p(\mathbf{Z}_n^d)} = \left(\frac{1}{n^d}\sum_{x\in\mathbf{Z}_n^d}|\varphi(x)|^p\right)^{\frac{1}{p}}$, for $p \in [1,\infty]$.

With this go on to the first chapter of the thesis.

I.1 Besov spaces & Co

In this section we are going to introduce the function spaces we will use throughout the thesis. In particular, we will make use of Besov spaces, which we will define via Fourier transformation. Let us start with the latter.

Fourier Transform

Consider $O = \mathbf{T}^d$ and $V = \mathbf{Z}^d$ or $O = \mathbf{R}^d, V = \mathbf{R}^d$ or $O = \mathbf{Z}_n^d, V = \mathbf{T}_n^d$ and $\varphi \in \mathscr{S}(O), \psi \in \mathscr{S}(V)$.

- Define the Fourier transform on $\mathbf{R}^d$ and its inverse by:
$$\mathscr{F}_{\mathbf{R}^d}(\varphi)(k) = \int_{\mathbf{R}^d}\varphi(x)e^{-2\pi\iota\langle x,k\rangle}\,\mathrm{d}x, \qquad \forall k \in \mathbf{R}^d$$
$$\mathscr{F}_{\mathbf{R}^d}^{-1}(\psi)(x) = \int_{\mathbf{R}^d}\psi(k)e^{2\pi\iota\langle x,k\rangle}\,\mathrm{d}k, \qquad \forall x \in \mathbf{R}^d.$$

- Analogously, on the torus:

$$\mathscr{F}_{\mathbf{T}^d}(\varphi)(k) = \int_{\mathbf{T}^d} \varphi(x) e^{-2\pi\iota\langle x,k\rangle}\,\mathrm{d}x, \qquad \forall k \in \mathbf{Z}^d$$
$$\mathscr{F}^{-1}_{\mathbf{T}^d}(\psi)(x) = \sum_{k\in\mathbf{Z}^d} \psi(k) e^{2\pi\iota\langle x,k\rangle}, \qquad \forall x \in \mathbf{T}^d.$$

- And in the discrete case:

$$\mathscr{F}_{\mathbf{Z}^d_n}(\varphi)(k) = \frac{1}{n^d} \sum_{x\in\mathbf{Z}^d_n} \varphi(x) e^{-2\pi\iota\langle x,k\rangle}, \qquad \forall k \in \mathbf{T}^d_n,$$
$$\mathscr{F}^{-1}_{\mathbf{Z}^d_n}(\psi)(x) = \int_{\mathbf{T}^d_n} \psi(k) e^{2\pi\iota\langle x,k\rangle}\,\mathrm{d}k, \qquad \forall x \in \mathbf{Z}^d_n.$$

The previous definitions are extended to any distribution $\varphi' \in \mathscr{S}'(O)$ via:

$$\langle \mathscr{F}_O\varphi', \varphi\rangle_O = \langle \varphi', \mathscr{F}_O\varphi\rangle_O, \qquad \forall \varphi \in \mathscr{S}(O),$$

and similarly for the inverse Fourier transforms. Eventually we consider

- $\psi(D)\colon \mathscr{S}'(O) \to \mathscr{S}'(O)$ the Fourier multiplier associated to $\psi \in \mathscr{S}(V)$, defined by:

$$\psi(D)\varphi = \mathscr{F}^{-1}_O\Big[\psi\mathscr{F}_O\varphi\Big], \qquad \forall \varphi \in \mathscr{S}'(O).$$

Finally, let us recall a connection between the Fourier transform on $\mathbf{T}^d$ and the one on $\mathbf{R}^d$.

Lemma I.1.1 (Poisson summation formula). *For $\varphi \in \mathscr{S}(\mathbf{R}^d)$ it holds that:*

$$\mathscr{F}^{-1}_{\mathbf{T}^d}\varphi(x) = \sum_{z\in\mathbf{Z}^d} \mathscr{F}^{-1}_{\mathbf{R}^d}\varphi(x+z).$$

In particular, this implies for $\varphi \in \mathscr{S}(\mathbf{R}^d)$ the bound:

$$\|\mathscr{F}^{-1}_{\mathbf{T}^d}\varphi\|_{L^1(\mathbf{T}^d)} \leq \|\mathscr{F}^{-1}_{\mathbf{R}^d}\varphi\|_{L^1(\mathbf{R}^d)}.$$

I.1.1 Besov spaces

Again, let $O = \mathbf{R}^d, V = \mathbf{R}^d$ or $O = \mathbf{T}^d, V = \mathbf{Z}^d$. Fix a dyadic partition of the unity

$$\{\varrho_j\}_{j\in\mathbf{Z}, j\geq -1}.$$

By this we mean that there exist two radial functions $\varrho_{-1}, \varrho_0 \in \mathscr{S}(\mathbf{R}^d)$ with supports in a ball $B = \{k \in \mathbf{R}^d \mid |k| \leq a_1\}$ about 0 and an annulus $\mathscr{A} = \{k \in \mathbf{R}^d \mid a_2 \leq |k| \leq a_3\}$ respectively, for some $a_1, a_2, a_3 \geq 0$ such that

$$\mathrm{supp}(\varrho_{-1}),\ \mathrm{supp}(\varrho_0) \subseteq (-1/2, 1/2)^d.$$

Then defining $\varrho_j(k) = \varrho_0(2^{-j}k)$ if $j \geq 0$, a dyadic partition of the unity must satisfy the following:

$$\sum_{j\geq -1} \varrho_j(k) = 1, \qquad \mathrm{supp}(\varrho_j) \cap \mathrm{supp}(\varrho_i) = \emptyset, \ \text{ if } |i-j| \geq 2.$$

Here the sum contains only finitely many non-zero terms. The existence of dyadic partitions of the unity is guaranteed for example by [BCD11, Proposition 2.10]. Having fixed such partition, define for $\varphi \in \mathscr{S}'(O)$:

- $\Delta_j \varphi = \varrho_j(D)\varphi$ is the j-th Paley block associated to φ, for $j \in \mathbf{Z}, j \geq -1$.
- $\|\varphi\|_{\mathscr{C}^\alpha_p(O)} = \sup_{j \geq -1} 2^{\alpha j} \|\Delta_j \varphi\|_{L^p(O)}$ for $\alpha \in \mathbf{R}, p \in [1, \infty]$, is the norm of the space:
$$\mathscr{C}^\alpha_p(O) = \Big\{ \varphi \in \mathscr{S}'(O) \mid \|\varphi\|_{\mathscr{C}^\alpha_p(O)} < \infty \Big\}.$$
- $\mathscr{C}^\alpha(O) = \mathscr{C}^\alpha_\infty(O)$.
- $B^\alpha_{p,q}(O)$, for $p, q \in [1, \infty]$ is similarly defined via the norm:
$$\|\varphi\|_{B^\alpha_{p,q}(O)} = \Bigg(\sum_{j \geq -1} 2^{\alpha j q} \|\Delta_j \varphi\|^q_{L^p(O)} \Bigg)^{\frac{1}{q}}$$

The following embedding holds true.

Proposition I.1.2 (Besov embedding). *Consider $O = \mathbf{T}^d$ or $O = \mathbf{R}^d$. For any $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq q_1 \leq q_2 \leq \infty$ the space $B^\alpha_{p_1,q_1}(O)$ is continuously embedded in $B^{\alpha - d(1/p_1 - 1/p_2)}_{p_2,q_2}(O)$. In other words, there exists a constant $C > 0$ such that:*

$$\|\varphi\|_{B^{\alpha - d(1/p_1 - 1/p_2)}_{p_2,q_2}(O)} \leq C \|\varphi\|_{B^\alpha_{p_1,q_1}(O)}$$

In addition, if $O = \mathbf{T}^d$, for $\alpha' < \alpha$ the embedding $B^\alpha_{p_2,q_2}(O) \subseteq B^{\alpha'}_{p_1,q_1}(O)$ is compact.

I.1.2 Products of distributions

In general, given two distributions $\varphi, \psi \in \mathscr{S}'(O)$, their product

$$\varphi \cdot \psi$$

is not well-defined. The Paley block decomposition we introduced singles out a part of the product that is in general not well-defined – the *resonant product*, indicated with $\odot$ – and a part that is always well-defined: the *paraproduct*, indicated with $\varolessthan$. We write formally (that is without considering whether the infinite sum is converging):

$$S_i \varphi := \sum_{j=-1}^{i-1} \Delta_j \varphi, \qquad \varphi \varolessthan \psi := \sum_{i \geq -1} S_{i-1} \varphi \Delta_i \psi, \qquad \varphi \varogreaterthan \psi = \psi \varolessthan \varphi, \qquad \varphi \odot \psi := \sum_{|i-j| \leq 1} \Delta_j \varphi \Delta_i \psi,$$

so that one can decompose a product:

$$\varphi \cdot \psi = \varphi \varolessthan \psi + \varphi \odot \psi + \varphi \varogreaterthan \psi.$$

The following result provides conditions under which the single terms in the product are well-defined.

Lemma I.1.3 (Theorems 2.82 and 2.85 [BCD11]). *Fix $\alpha,\beta \in \mathbf{R}$ and $p,q \in [1,\infty]$ such that $\frac{1}{r} = \frac{1}{p}+\frac{1}{q} \le 1$. For $\varphi,\psi \in \mathscr{S}'(O)$*

$$
\begin{aligned}
\|\varphi \varolessthan \psi\|_{\mathscr{C}^\alpha_r} &\lesssim \|\varphi\|_{L^p}\|\psi\|_{\mathscr{C}^\alpha_q},\\
\|\varphi \varolessthan \psi\|_{\mathscr{C}^{\alpha+\beta}_r} &\lesssim \|\varphi\|_{\mathscr{C}^\beta_p}\|\psi\|_{\mathscr{C}^\alpha_q}, \ \ \textit{if} \ \ \beta < 0,\\
\|\varphi \odot \psi\|_{\mathscr{C}^{\alpha+\beta}_r} &\lesssim \|\varphi\|_{\mathscr{C}^\beta_p}\|\psi\|_{\mathscr{C}^\alpha_q} \ \ \textit{if} \ \ \alpha+\beta > 0.
\end{aligned}
$$

I.1.3 Besov spaces, weighted and discretized

Many of the structures introduced in this section have been developed in [MP19]. We refer the reader to the latter work for a more complete description of these spaces.

The function $x \mapsto e^{|x|^\sigma}$, for some $\sigma \in (0,1)$, does not belong to the space of Schwartz distributions $\mathscr{S}'(\mathbf{R}^d)$ because it does not decay polynomially at infinity. Hence it falls outside the framework of the previous spaces, yet it will play a role in our study. To overcome this problem, we introduce *ultradistributions*. Consider

$$\varpi(x) = |x|^\sigma$$

for some $\sigma \in (0,1)$ which is henceforth fixed. Following [MP19, Definition 2.8] consider the set of admissible weights:

$$\rho(\varpi) = \left\{ z\colon \mathbf{R} \to \mathbf{R} \;\middle|\; \text{there exists a } \lambda > 0 \ \text{ such that } \ z(x) \lesssim z(y)e^{\lambda\varpi(x-y)}, \ \ \forall x,y \in \mathbf{R} \right\}.$$

For our purposes it suffices to know that for any $a \in \mathbf{R}_+, l \in \mathbf{R}$, the functions $p(a)$ and $e(l)$ belong to $\rho(\varpi)$, where

$$p(a)(x) = (1+|x|)^{-a}, \qquad e(l)(x) = e^{-l|x|^\sigma}.$$

Now consider the set of seminorms, for $k \in \mathbf{N}^d_0, \lambda \in \mathbf{R}$:

$$\|\varphi\|_{\varpi,k,\lambda} = \sup_{x\in\mathbf{R}^d} e^{\lambda\varpi(x)}|\partial^k\varphi(x)| < \infty.$$

The spaces $\mathscr{S}_\varpi(\mathbf{R}^d), \mathscr{S}_\varpi(\mathbf{Z}^d_n)$ of (discrete) test functions for ultradistributions are defined by

$$
\begin{aligned}
\mathscr{S}_\varpi(\mathbf{R}^d) &= \left\{\varphi\colon \mathbf{R}^d \to \mathbf{R} \;\middle|\; \|\varphi\|_{\varpi,k,\lambda} + \|\mathscr{F}_{\mathbf{R}^d}\varphi\|_{\varpi,k,\lambda} < \infty, \ \ \forall a,\lambda > 0, k \in \mathbf{N}^d_0\right\} \subseteq \mathscr{S}(\mathbf{R}^d),\\
\mathscr{S}_\varpi(\mathbf{Z}^d_n) &= \left\{\varphi\colon \mathbf{Z}^d_n \to \mathbf{R} \;\middle|\; \sup_{x\in\mathbf{Z}^d_n}|e^{\lambda\varpi(x)}\varphi(x)| < \infty, \ \ \forall\lambda > 0\right\}.
\end{aligned}
$$

The spaces $\mathscr{S}'_\varpi(\mathbf{R}^d), \mathscr{S}'_\varpi(\mathbf{Z}^d_n)$ are the topological dual spaces of the ones just defined. On these the Fourier transform is defined just as for classical distributions.
One can define weighted Besov spaces $\mathscr{C}^\alpha_p(\mathbf{R}^d;z)$, for a given $z \in \rho(\varpi)$, via the norm:

$$\|\varphi\|_{L^p(\mathbf{R}^d;z)} = \|\varphi\cdot z\|_{L^p(\mathbf{R}^d)}, \qquad \|\varphi\|_{\mathscr{C}^\alpha_p(\mathbf{R}^d;z)} = \sup_{j\ge -1} 2^{\alpha j}\|\Delta_j\varphi\|_{L^p(\mathbf{R}^d;z)}.$$

This definition can be extended to the discrete case, following the construction in [MP19]. Fix $n \in \mathbf{N}$ and write $j_n \in \mathbf{N}$ for the smallest index $j \geq -1$ such that $\mathrm{supp}(\varrho_j) \not\subseteq n[-1/2,1/2]^d$. By our assumptions on the support of ϱ_{-1}, ϱ_0 we have $j_n \geq 1$. For $j < j_n$ and $\varphi \in \mathscr{S}'_{\omega}(\mathbf{Z}^d_n)$ define the Littlewood-Paley blocks

$$\Delta^n_j \varphi = \mathscr{F}^{-1}_{\mathbf{Z}^d_n}\Big(\varrho_j \mathscr{F}_{\mathbf{Z}^d_n}(\varphi)\Big), \qquad \Delta^n_{j_n}\varphi = \mathscr{F}^{-1}_{\mathbf{Z}^d_n}\Big((1 - \sum_{-1\leq j<j_n} \varrho_j)\mathscr{F}_{\mathbf{Z}^d_n}(\varphi)\Big).$$

For $\alpha \in \mathbf{R}$, $p \in [1,\infty]$ and $z \in \varrho(\omega)$ the discrete weighted Besov spaces $\mathscr{C}^\alpha_p(\mathbf{Z}^d_n; z) \subseteq \mathscr{S}'_\omega(\mathbf{Z}^d_n)$ is defined via the norm:

$$\|\varphi\|_{L^p(\mathbf{Z}^d_n;z)} = \|\varphi \cdot z\|_{L^p(\mathbf{Z}^d_n)}, \qquad \|\varphi\|_{\mathscr{C}^\alpha_p(\mathbf{Z}^d_n;z)} = \sup_{-1\leq j\leq j_n} 2^{j\alpha}\|\Delta^n_j\varphi\|_{L^p(\mathbf{Z}^d_n;z)}.$$

The discretized and the continuous Besov spaces can be put in relationship to each other consistently via an extension operator as in [MP19, Lemma 2.24]

$$\mathscr{E}^n \colon \mathscr{C}^\alpha_p(\mathbf{Z}^d_n; z) \to \mathscr{C}^\alpha_p(\mathbf{R}^d; z), \qquad \mathscr{E}^n\varphi = \mathscr{F}^{-1}_{\mathbf{R}^d}\Big[\psi^{\mathscr{E}}(n\cdot)\mathscr{F}_{\mathbf{Z}^d_n}\varphi(\cdot)\Big],$$

where $\psi^{\mathscr{E}}$ is a smooth function with compact support in $(-1/2,1/2)^d$ (see the quoted article for the precise requirements). We said that the extension is *consistent*. By this we mean that uniformly over n:

$$\|\mathscr{E}^n\varphi\|_{\mathscr{C}^\alpha_p(\mathbf{R}^d;z)} \lesssim \|\varphi\|_{\mathscr{C}^\alpha_p(\mathbf{Z}^d_n;z)}$$

and for a smooth function φ, $\mathscr{E}^n(\varphi|_{\mathbf{Z}^d_n}) \to \varphi$ in the sense of distributions.

In this setting we can decompose the product of two discrete distributions as

$$\varphi \cdot \psi = \varphi \varolessthan \psi + \varphi \odot \psi + \psi \varogreaterthan \varphi,$$

with:

$$\varphi \varolessthan \psi = \sum_{1\leq i\leq j_n} S^n_{i-1}\varphi\Delta^n_i\psi, \qquad \varphi \odot \psi = \sum_{\substack{|i-j|\leq 1 \\ -1\leq i,j\leq j_n}} \Delta^n_i\varphi\Delta^n_j\psi,$$

where $S^n_i\varphi = \sum_{-1\leq j\leq i-1}\Delta^n_j\varphi$ (here we assume $i \leq j_n$). For simplicity, we do not include n in the notation for $\varolessthan$ and $\odot$. We can prove the discrete analogue of Lemma I.1.3.

Lemma I.1.4 (Lemma 4.2 [MP19])**.** *The estimates below hold uniformly over $n \in \mathbf{N}$ (as well as for $\mathbf{Z}^d$ replaced by $\mathbf{R}^d$). Consider $z_1, z_2 \in \varrho(\omega)$ and $\alpha, \beta \in \mathbb{R}$, $p, q \in [1,\infty]$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q} \leq 1$. We find that:*

$$\|\varphi \varolessthan \psi\|_{\mathscr{C}^\alpha_r(\mathbf{Z}^d_n;z_1z_2)} \lesssim \|\varphi\|_{L^p(\mathbf{Z}^d_n;z_1)}\|\psi\|_{\mathscr{C}^\alpha_q(\mathbf{Z}^d_n;z_2)},$$
$$\|\varphi \varolessthan \psi\|_{\mathscr{C}^{\alpha+\beta}_r(\mathbf{Z}^d_n;z_1z_2)} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p(\mathbf{Z}^d_n;z_1)}\|\psi\|_{\mathscr{C}^\alpha_q(\mathbf{Z}^d_n;z_2)}, \qquad \text{if } \beta < 0,$$
$$\|\varphi \odot \psi\|_{\mathscr{C}^{\alpha+\beta}_r(\mathbf{Z}^d_n;z_1z_2)} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p(\mathbf{Z}^d_n;z_1)}\|\psi\|_{\mathscr{C}^\alpha_q(\mathbf{Z}^d_n;z_2)} \qquad \text{if } \alpha+\beta > 0.$$

I.1.4 Time dependent functions

It will be also important to consider the temporal regularity of functions. Throughout the work $T > 0$ will always indicate an arbitrary time horizon. The spaces we introduce depend implicitly on the choice of T: we omit writing it explicitly to simplify the notation. In particular, since we work with the heat equation, we will consider spaces in which the regularity is parabolically scaled: namely to a spatial regularity of order α corresponds a temporal regularity of $\alpha/2$ (here $\alpha \in (0,2)$). We will also introduce a parameter $\gamma \in [0,1)$ which quantifies the blowup at time $t = 0$ of the regularity of a time-dependent function. This is used to deal with non-smooth initial conditions. For example, it $\varphi \in \mathscr{C}^\alpha(\mathbf{T}^d)$, $\alpha \in \mathbf{R}$, and P_t is the heat semigroup, we expect that $P_t\varphi$ is smooth for $t > 0$, and we can bound $\|P_t\varphi\|_{\mathscr{C}^{\alpha+2\gamma}(\mathbf{T}^d)} \lesssim t^{-\gamma}\|\varphi\|_{\mathscr{C}^\alpha(\mathbf{T}^d)}$.
In what follows, let X be an arbitrary Banach space and $O = \mathbf{T}^d, \mathbf{R}^d$ or $\mathbf{Z}^d_n$.

- CX is the space of continuous functions $\varphi\colon [0,T] \to X$, with the norm:

$$\|\varphi\|_{CX} = \sup_{0\leq t\leq T} \|\varphi(t)\|_X.$$

- $C^\alpha X$, for $\alpha \in (0,1)$, is the space of α–Hölder continuous functions $\varphi\colon [0,T] \to X$, with the norm

$$\|\varphi\|_{C^\alpha X} = \|\varphi\|_{CX} + [\varphi]_{C^\alpha X} = \|\varphi\|_{CX} + \sup_{0\leq s<t\leq T} \frac{\|\varphi(t)-\varphi(s)\|_X}{|t-s|^\alpha}.$$

- $\mathscr{M}^\gamma X$, for $\gamma \in (0,1)$, is the space of functions $\varphi\colon (0,T] \to X$ with a blow-up of order γ in $t = 0$:

$$\|\varphi\|_{\mathscr{M}^\gamma X} = \sup_{0<t\leq T} t^\gamma \|\varphi(t)\|_X.$$

- $\mathscr{L}^{\gamma,\alpha}_p(O)$, for $\alpha \in (0,2), \gamma \in [0,1), p \in [1,\infty]$, is the space of functions in $C([0,T];\mathscr{S}'(O))$ defined via the norm:

$$\|\varphi\|_{\mathscr{L}^{\gamma,\alpha}_p(O)} = \|t \mapsto t^\gamma\varphi(t)\|_{C^{\alpha/2}L^p(O)} + \|\varphi\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p(O)}.$$

- $\mathscr{L}^{\gamma,\alpha}_p(O;z)$, for $\alpha \in (0,2), \gamma \in [0,1), p \in [1,\infty], z \in \varrho(\varpi)$ and $O = \mathbf{R}^d$ or $O = \mathbf{Z}^d_n$ is defined analogously, via the norm:

$$\|\varphi\|_{\mathscr{L}^{\gamma,\alpha}_p(O,z)} = \|t \mapsto t^\gamma\varphi(t)\|_{C^{\alpha/2}L^p(O;z)} + \|\varphi\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p(O;z)}.$$

It will be useful to exchange regularity in time with a smaller blow-up at $t = 0$:

Lemma I.1.5. *The estimates below hold uniformly over $n \in \mathbf{N}$ (and also for $\mathbf{Z}^d_n$ replaced by $\mathbf{R}^d$ or $\mathbf{T}^d$). Consider $z \in \varrho(\varpi)$, $\alpha \in (0,2)$, $\gamma \in [0,1)$ and $\varepsilon \in [0,2\gamma] \cap [0,\alpha)$. We can bound:*

$$\|\varphi\|_{\mathscr{L}^{\gamma-\varepsilon/2,\alpha-\varepsilon}_p(\mathbf{Z}^d_n;z)} \lesssim \|\varphi\|_{\mathscr{L}^{\gamma,\alpha}_p(\mathbf{Z}^d_n;z)}. \tag{I.1}$$

Proof. The estimate is proven in [MP19, Lemma 3.11]. In that lemma the case $\varepsilon = 2\gamma < \alpha$ is not included, but it follows by the same arguments (since [GP17, Lemma A.1] still applies in that case). □

II

A Random walk in a random environment

II.1 Introduction

In this chapter we consider a branching random walk in a random environment (BRWRE). This is a process on the lattice $\mathbf{Z}^d_n$, for $n \in \mathbf{N}$ and $d = 1, 2$, and we are interested in the limit $n \to \infty$. The evolution of the BRWRE depends on the environment it lives in: every particle performs a simple random walk and it may give birth to a new particle or die according to how favorable the environment is.

In Assumption II.2.1 we state the probabilistic requirements on the random environment. These assumptions allow us to fix a null set outside of which certain analytical conditions are satisfied, see Lemma II.2.5 for details. We then introduce the model, (a rigorous construction of the random Markov process is postponed to Section A.1 of the Appendix). We also state the main results in Section II.2, namely the law of large numbers (Theorem II.2.10), the convergence to the rSBM (Theorem II.2.13), the representation as an SPDE in dimension $d = 1$ (Theorem II.2.19) and the persistence of the process (Theorem II.2.21). We then proceed to the proofs. In Section I.1 we introduce the function spaces we will need throughout this thesis. In Section II.3 we study the discrete and continuous PAM. We recall the results from [MP19] and adapt them to the current setting.

We prove the convergence in distribution of the BRWRE in Section II.4. First, we show tightness by using a mild martingale problem (see Remark II.4.1) which fits well with our analytical tools. We then show the duality between the rSBM and the solution to the SPDE (II.4) and use the duality to deduce the uniqueness in law of limit points of the BRWRE.

In Section II.6 we derive some properties of the rough super-Brownian motion: we show that in $d = 1$ it is the weak solution to an SPDE, where the key point is that the random measure admits a density w.r.t. the Lebesgue measure, as proven in Lemma II.6.1. We then apply the results of Section IV.3 to construct the rSBM with Dirichlet boundary conditions on large boxes. These constructions, along with the eigenvalue asymptotics for PAM derived in [Cv19, FN77] allow us to show that the process survives with positive probability.

II.2 The model & its scaling limits

Before we introduce the BRWRE, let us clarify our assumptions on the environment. A *deterministic environment* is a sequence $\{\xi^n\}_{n\in\mathbf{N}}$ of potentials on the lattice, i.e. functions $\xi^n\colon \mathbf{Z}^d_n \to \mathbf{R}$. A *random environment* is a sequence of probability spaces $(\Omega^{p,n}, \mathscr{F}^{p,n}, \mathbb{P}^{p,n})$ together with a sequence $\{\xi^n\}_{n\in\mathbf{N}}$ of measurable maps $\xi^n\colon \Omega^{p,n}\times\mathbf{Z}^d_n \to \mathbf{R}$.

Assumption II.2.1 (Random Environment)**.** *We assume that for every $n\in\mathbf{N}$, $\{\xi^n(x)\}_{x\in\mathbf{Z}^d_n}$ is a set of i.i.d random variables on a probability space $(\Omega^n, \mathscr{F}^n, \mathbb{P}^n)$ that satisfy:*

$$n^{-d/2}\xi^n(x) = \Phi \ \ \textit{in distribution}, \tag{II.1}$$

for a random variable Φ with finite moments of every order such that

$$\mathbb{E}[\Phi] = 0, \ \ \mathbb{E}[\Phi^2] = 1.$$

Next, let us recall the definition of space and space-time white noise.

Definition II.2.2 (White noise)**.** *White noise on $\mathbf{R}^d$ is a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ together with a measurable map:*

$$\xi\colon \Omega \to \mathscr{S}'(\mathbf{R}^d),$$

that is uniquely characterized by $\langle\xi,\varphi\rangle$ being a Gaussian random variable for all $\varphi\in\mathscr{S}(\mathbf{R}^d)$, with covariance:

$$\mathbb{E}\Big[\langle\xi,\varphi\rangle\langle\xi,\psi\rangle\Big] = \langle\varphi,\psi\rangle, \qquad \forall\varphi,\psi\in\mathscr{S}(\mathbf{R}^d).$$

By taking a limit in $L^2(\Omega)$ one can define $\langle\xi,\varphi\rangle$ also for $\varphi\in L^2(\mathbf{R}^d)$, as a Gaussian random variable with variance $\|\varphi\|_{L^2(\mathbf{R}^d)}$. We write:

$$\langle\xi,\varphi\rangle = \int_{\mathbf{R}^d}\varphi(x)\,\mathrm{d}\xi(x).$$

Since with d we indicate the spatial dimension of the processes we consider, we call a white noise on $\mathbf{R}^{d+1}$ a space-time *white noise.*

Remark II.2.3. *It follows that ξ^n converges in distribution to a white noise ξ on $\mathbf{R}^d$, in the sense that $\langle\xi^n,\varphi\rangle_{\mathbf{Z}^d_n} \to \xi(\varphi)$ in distribution for all $\varphi\in C_c(\mathbf{R}^d)$.*

To separate the randomness coming from the potential from that of the branching random walks it will be convenient to fix a realization $\xi^n(\omega)$ of ξ^n and consider it as a deterministic environment. But we cannot expect to obtain reasonable scaling limits for all deterministic environments. Therefore, we need to identify properties that hold for typical realizations of random potentials satisfying Assumption II.2.1. The reader only interested in random environments may skip the following assumption for deterministic environments and use it as a black box, since by Lemma II.2.5 below the assumption is satisfied a.s. by any random environment under II.2.1. In particular, we will make use of the weighted Hölder-Besov spaces and related constructions, whose definition can be found in Section I.1.1.

Below we indicate with Δ^n the discrete Laplacian (here for $x,y\in\mathbf{Z}^d_n$ we say $x\sim y$ if $|x-y| = n^{-1}$):

$$\Delta^n\varphi(x) = n^2\sum_{x\sim y=1}\Big(\varphi(y)-\varphi(x)\Big)$$

Assumption II.2.4 (Deterministic environment). *Let ξ^n be a deterministic environment and let X^n be the solution to the equation $-\Delta^n X^n = \chi(D)\xi^n = \mathscr{F}^{-1}_{\mathbf{Z}^d_n}(\chi \mathscr{F}_{\mathbf{Z}^d_n}\xi^n)$ in the sense explained in [MP19, Section 5.1], where χ is a smooth function equal to 1 outside of $(-1/4, 1/4)^d$ and equal to zero on $(-1/8, 1/8)^d$ (in this way the existence of X^n is guaranteed). Consider a regularity parameter*

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2.$$

We assume that the following holds:

(i) There exists $\xi \in \bigcap_{a>0} \mathscr{C}^{\alpha-2}(\mathbf{R}^d; p(a))$ such that for all $a > 0$:

$$\sup_n \|\xi^n\|_{\mathscr{C}^{\alpha-2}(\mathbf{Z}^d_n; p(a))} < +\infty \text{ and } \mathscr{E}^n \xi^n \to \xi \text{ in } \mathscr{C}^{\alpha-2}(\mathbf{R}^d; p(a)).$$

(ii) For any $a, \varepsilon > 0$ we can bound:

$$\sup_n \|n^{-d/2}\xi^n_+\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n; p(a))} + \sup_n \|n^{-d/2}|\xi^n|\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n; p(a))} < +\infty$$

as well as for any $b > d/2$:

$$\sup_n \|n^{-d/2}\xi^n_+\|_{L^2(\mathbf{Z}^d_n; p(b))} < +\infty.$$

Moreover, there exists $\nu \geq 0$ such that the following convergences hold:

$$\mathscr{E}^n n^{-d/2}\xi^n_+ \to \nu, \qquad \mathscr{E}^n n^{-d/2}|\xi^n| \to 2\nu$$

in $\mathscr{C}^{-\varepsilon}(\mathbf{R}^d; p(a))$.

(iii) If $d = 2$ there exists a sequence $c_n \in \mathbf{R}$ such that $n^{-d/2}c_n \to 0$ and distributions $X \in \bigcap_{a>0} \mathscr{C}^{\alpha}(\mathbf{R}^d, p(a))$ and $X \diamond \xi \in \bigcap_{a>0} \mathscr{C}^{2\alpha-2}(\mathbf{R}^d; p(a))$ which satisfy for all $a > 0$:

$$\sup_n \|X^n\|_{\mathscr{C}^{\alpha}(\mathbf{Z}^d_n; p(a))} + \sup_n \|(X^n \odot \xi^n) - c_n\|_{\mathscr{C}^{2\alpha-2}(\mathbf{Z}^d_n; p(a))} < +\infty$$

and $\mathscr{E}^n X^n \to X$ in $\mathscr{C}^{\alpha}(\mathbf{R}^d; p(a))$ and $\mathscr{E}^n\big((X^n \odot \xi^n) - c_n\big) \to X \diamond \xi$ in $\mathscr{C}^{2\alpha-2}(\mathbf{R}^d; p(a))$.

We say that $\xi \in \mathscr{S}'(\mathbf{R}^d)$ is a *deterministic environment satisfying Assumption II.2.4* if there exists a sequence $\{\xi^n\}_{n\in\mathbf{N}}$ such that the conditions of Assumption II.2.4 hold.

The next result establishes the connection between the previous probabilistic and analytical conditions. To formulate it we need the following sequence of diverging *renormalisation* constants:

$$\kappa_n = \int_{\mathbf{T}^2_n} \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k \simeq \log(n), \tag{II.2}$$

with χ as in the previous assumption and l^n being the Fourier multiplier associated to the negative discrete Laplacian $-\Delta^n$, i.e. $-\Delta^n \varphi = \mathscr{F}^{-1}_{\mathbf{Z}^d_n}[l^n(\cdot)\mathscr{F}_{\mathbf{Z}^d_n}\varphi(\cdot)]$. An explicit formula for l^n is simple to obtain, but is not important at the moment: it suffices to know that $l^n(k) \simeq |k|^2$, which explains the logarithmic divergence.

Lemma II.2.5. *Given a random environment $\{\bar{\xi}^n\}_{n\in\mathbf{N}}$ satisfying Assumption II.2.1, there exists a probability space $(\Omega,\mathscr{F},\mathbb{P})$ supporting random variables $\{\xi^n\}_{n\in\mathbf{N}}$ such that $\bar{\xi}^n = \xi^n$ in distribution and such that $\{\xi^n(\omega,\cdot)\}_{n\in\mathbf{N}}$ is a deterministic environment satisfying Assumption II.2.4 for all $\omega\in\Omega$. Moreover the sequence c_n in Assumption II.2.4 can be chosen equal to κ_n (see Equation* (II.2)*) outside of a null set. Similarly, ν is strictly positive and deterministic outside of a null set and equals the expectation $\mathbb{E}[\Phi_+]$.*

Proof. The existence of such a probability space is provided by the Skorohod representation theorem. Indeed it is a consequence of Assumption II.2.1 that all the convergences hold in the sense of distributions: The convergences in (i) and (iii) follow from Lemma II.7.2 if $d = 1$ and from [MP19, Lemmata 5.3 and 5.5] if $d = 2$ (where it is also shown that we can choose $c_n = \kappa_n$). The convergence in (ii) for $\nu = \mathbb{E}[\Phi_+]$ is shown in Lemma II.7.1. After changing the probability space the Skorohod representation theorem guarantees almost sure convergence, so setting $\xi^n,\xi,c^n,\nu = 0$ on a null set we find the result for every ω. (There is a small subtlety in the application of the Skorohod representation theorem because $\mathscr{C}^\gamma(\mathbf{R}^d;p(a))$ is not separable, but we can restrict our attention to the closure of smooth compactly supported functions in $\mathscr{C}^\gamma(\mathbf{R}^d;p(a))$, which is a closed separable subspace). □

Notation II.2.6. *A sequence of random variables $\{\xi^n\}_{n\in\mathbf{N}}$ defined on a common probability space $(\Omega,\mathscr{F},\mathbb{P})$ which almost surely satisfies Assumption II.2.4 is called a* controlled random environment. *By Lemma II.2.5, for any random environment satisfying Assumption II.2.1 we can find a controlled random environment with the same distribution. For a given controlled random environment we introduce the effective potential:*

$$\xi^n_e(\omega,x) = \xi^n(\omega,x) - c_n(\omega)1_{\{d=2\}}.$$

Given a controlled random environment we define $\mathscr{H}^\omega$ the random Anderson Hamiltonian and its domain $\mathscr{D}_{\mathscr{H}^\omega}$: see Lemma II.3.6. Roughly speaking:

$$\mathscr{H}^\omega = \Delta + \xi(\omega) = \lim_{n\to\infty} \Delta^n + \xi^n_e(\omega).$$

We pass to the description of the particle system. This will be a (random, i.e. dependent on the environment) Markov process on the space $E = \left(\mathbf{N}_0^{\mathbf{Z}^d_n}\right)_0$ of compactly supported functions $\eta\colon \mathbf{Z}^d_n \to \mathbf{N}_0$ endowed with the discrete topology. The rigorous construction of this process is discussed in Appendix A.1. We define

$$\eta^{x\mapsto y}(z) = \eta(z) + (1_{\{y\}}(z) - 1_{\{x\}}(z))1_{\{\eta(x)\geq 1\}}$$

and

$$\eta^{x\pm}(z) = (\eta(z) \pm 1_{\{x\}}(z))_+.$$

For $F\in C_b(E), x\in\mathbf{Z}^d_n$ we write:

$$\Delta^n_x F(\eta) = n^2 \sum_{y\sim x}(F(\eta^{x\mapsto y}) - F(\eta)), \qquad d^\pm_x F(\eta) = F(\eta^{x\pm}) - F(\eta).$$

Definition II.2.7. *Fix an "averaging parameter" $\varrho \geq 0$ and a controlled random environment ξ^n. Let $\mathbb{P}^n \ltimes \mathbb{P}^{\omega,n}$ be the measure on $\Omega \times \mathbb{D}([0,+\infty);E)$ defined as the "semidirect product measure" of $\mathbb{P}^n$ and $\mathbb{P}^{\omega,n}$ (see Equation* (A.1)*), where for $\omega \in \Omega$ the measure $\mathbb{P}^{\omega,n}$ on $\mathbb{D}([0,+\infty);E)$ is the law under which the canonical process $u^n(\omega,\cdot)$ started in $u^n(\omega,0) = \lfloor n^\varrho \rfloor 1_{\{0\}}(x)$ is the Markov process with generator*

$$\mathscr{L}^{n,\omega} \colon \mathscr{D}(\mathscr{L}^{n,\omega}) \to C_b(E),$$

where $\mathscr{L}^{n,\omega}(F)(\eta)$ is defined by:

$$\sum_{x \in \mathbf{Z}_n^d} \eta_x \cdot \Big[\Delta_x^n F(\eta) + (\xi_e^n)_+(\omega,x) d_x^+ F(\eta) + (\xi_e^n)_-(\omega,x) d_x^- F(\eta) \Big] \tag{II.3}$$

and the domain $\mathscr{D}(\mathscr{L}^{n,\omega})$ consists of all $F \in C_b(E)$ such that the right-hand side of (II.3) *lies in $C_b(E)$. To u^n we associate the process μ^n with the pairing*

$$\mu^n(\omega,t)(\varphi) := \sum_{x \in \mathbb{Z}_n^d} \lfloor n^\varrho \rfloor^{-1} u^n(\omega,t,x)\varphi(x)$$

for any function $\varphi \colon \mathbf{Z}_n^d \to \mathbf{R}$. Hence μ^n is a measure-valued stochastic process. We indicate with $\mathbb{P}^n \ltimes \mathbb{P}^{\omega,n}$ its law on the Skorohod space $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$.

Remark II.2.8. *Although not explicitly stated, it is part of the definition that $\omega \mapsto \mathbb{P}^{\omega,n}(A)$ is measurable for Borel sets $A \in \mathscr{B}(\mathbb{D}([0,+\infty);E))$.*

Since all particles evolve independently, we expect that for $\varrho \to \infty$ the law of large numbers applies. This is why we refer to ϱ as an averaging parameter.

Notation II.2.9. *In the terminology of stochastic processes in random media, $\mathbb{P}^{\omega,n}$ is the* quenched law *of the process u^n (or μ^n) given the noise ξ^n. We also call $\mathbb{P}^{\mathrm{tot},n} := \mathbb{P}^n \ltimes \mathbb{P}^{\omega,n}$ the* total law.

We can now state the main convergence results of this chapter. First, we prove quenched results: the corresponding total versions are then an easy corollary. We start with a law of large numbers.

Theorem II.2.10. *Let ξ be a deterministic environment satisfying Assumption II.2.4 and let $\varrho > d/2$. Let w be the solution of PAM* (II.7) *with initial condition $w(0,x) = \delta_0(x)$, as constructed in Proposition II.3.1 (see also Remark II.3.2). The measure-valued process μ^n from Definition II.2.7 converges to w in probability in the space $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$ as $n \to +\infty$.*

Proof. The proof can be found in Section II.4.1. □

If the averaging parameter takes the critical value $\varrho = d/2$, we see random fluctuations in the limit and we end up with the *rough super-Brownian motion* (rSBM). As in the case of the classical SBM, the limiting process can be characterized via duality with the following equation:

$$\partial_t \varphi = \mathscr{H}\varphi - \frac{\kappa}{2}\varphi^2, \qquad \varphi(0) = \varphi_0, \tag{II.4}$$

for $\varphi_0 \in C_c^\infty(\mathbf{R}^d)$, $\varphi_0 \geq 0$. With some abuse of notation (since the equation is not linear) we write $U_t\varphi_0 = \varphi(t)$. Here as before recall that $\mathscr{H}$ is the Anderson Hamiltonian as constructed in Lemma II.3.6.

Definition II.2.11. *Let ξ be a deterministic environment satisfying Assumption II.2.4, consider $\kappa > 0$ and let μ be a process with values in the space $C([0,+\infty);\mathscr{M}(\mathbf{R}^d))$, such that $\mu(0) = \delta_0$. Write $\mathscr{F} = \{\mathscr{F}_t\}_{t\in[0,+\infty)}$ for the completed and right-continuous filtration generated by μ. We call μ a* rough superBrownian motion (rSBM) with parameter κ *if it satisfies one of the three properties below:*

(i) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbf{R}^d), \varphi_0 \geq 0$ and for $U_\cdot\varphi_0$ the solution to Equation (II.4) *with initial condition φ_0, the process*

$$N_t^{\varphi_0}(s) = e^{-\langle \mu(s), U_{t-s}\varphi_0\rangle}, \qquad s \in [0,t],$$

is a bounded continuous $\mathscr{F}$–martingale.

(ii) For any $t \geq 0$ and $\varphi_0 \in C_c^\infty(\mathbf{R}^d)$ and $f \in C([0,t];\mathscr{C}^\zeta(\mathbf{R}^d;e(l)))$ for arbitrary $\zeta > 0$ and $l \in \mathbf{R}$, and for φ_t solving

$$\partial_s\varphi_t + \mathscr{H}\varphi_t = f, \qquad s \in [0,t], \qquad \varphi_t(t) = \varphi_0,$$

it holds that

$$s \mapsto M_t^{\varphi_0,f}(s) := \langle \mu(s), \varphi_t(s)\rangle - \langle \mu(0), \varphi_t(0)\rangle - \int_0^s \langle \mu(r), f(r)\rangle \mathrm{d}r,$$

defined for $s \in [0,t]$, is a continuous square integrable $\mathscr{F}$–martingale with quadratic variation

$$\langle M_t^{\varphi_0,f}\rangle_s = \kappa \int_0^s \langle \mu(r), (\varphi_t)^2(r)\rangle \mathrm{d}r.$$

(iii) For any $\varphi \in \mathscr{D}_{\mathscr{H}}$ the process:

$$L^\varphi(t) = \langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle - \int_0^t \langle \mu(r), \mathscr{H}\varphi\rangle \mathrm{d}r, \qquad t \in [0,+\infty),$$

is a continuous $\mathscr{F}$–martingale, square integrable on $[0,T]$ for all $T > 0$, with quadratic variation

$$\langle L^\varphi\rangle_t = \kappa \int_0^t \langle \mu(r), \varphi^2\rangle \mathrm{d}r.$$

Each of the three properties above characterizes the process uniquely:

Lemma II.2.12. *The three conditions of Definition II.2.11 are equivalent. Moreover, if μ is a rSBM with parameter κ, then its law is unique.*

Proof. The proof can be found at the end of Section II.4.1. □

Theorem II.2.13. *Let $\{\xi^n\}_{n\in\mathbf{N}}$ be a deterministic environment satisfying Assumption II.2.4 and let $\varrho = d/2$. Then the sequence $\{\mu^n\}_{n\in\mathbf{N}}$ converges to the rSBM μ with parameter $\kappa = 2\nu$ in distribution in $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$.*

Proof. The proof can be found at the end of Section II.4.1. □

Remark II.2.14. *Lemma II.2.12 gives the uniqueness of the rSBM for all parameters $\kappa > 0$, but Theorem II.2.13 only shows the existence conditionally on the existence of an environment which satisfies Assumption II.2.4. This leads to the constraint $\nu \in (0, \frac{1}{2}]$ because we should think of $\nu = \mathbb{E}[\Phi_+]$ for a centered random variable Φ with $\mathbb{E}[\Phi^2] = 1$. For general $\kappa > 0$ we establish the existence of the rSBM in Section II.5.1, as a perturbation of classical branching process.*

Remark II.2.15. *We restrict our attention to the Dirac delta initial condition for simplicity, but most of our arguments extend to initial conditions $\mu \in \mathscr{M}(\mathbf{R}^d)$ that satisfy $\langle \mu, e(l)\rangle < \infty$ for all $l < 0$. In this case only the construction of the initial value sequence $\{\mu^n(0)\}_{n\in\mathbf{N}}$ is more technical, because we need to come up with an approximation in terms of integer valued point measures (which we need as initial condition for the particle system). This can be achieved by discretizing the initial measure on a coarser grid.*

The previous results describe the scaling behavior of the BRWRE conditionally on the environment, and we now pass to the unconditional statements. To a given random environment ξ^n satisfying Assumption II.2.1 (not necessarily a *controlled* random environment) we associate a sequence of random variables in $\mathscr{S}'(\mathbf{R}^d)$ by defining $\xi^n(\varphi) = n^{-d}\sum_x \xi^n(x)\varphi(x)$ for $\varphi \in \mathscr{S}(\mathbf{R}^d)$. Recall that the sequence of measures $\mathbb{P}^{\mathrm{tot},n} = \mathbb{P}^n \ltimes \mathbb{P}^{\omega,n}$ on $\mathscr{S}'(\mathbf{R}^d) \times \mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$ is such that $\mathbb{P}^n$ is the law of ξ^n and $\mathbb{P}^{\omega,n}$ is the quenched law of the branching process μ^n given ξ^n.

Corollary II.2.16. *The sequence of measures $\mathbb{P}^{\mathrm{tot},n}$ converges weakly to $\mathbb{P}^{\mathrm{tot}} = \mathbb{P} \ltimes \mathbb{P}^{\omega}$ on $\mathscr{S}'(\mathbf{R}^d) \times \mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$, where $\mathbb{P}$ is the law of the space white noise ξ on $\mathscr{S}'(\mathbf{R}^d)$, and $\mathbb{P}^{\omega}$ is the quenched law of μ given ξ which is described by Theorem II.2.10 if $\varrho > d/2$ or by Theorem II.2.13 if $\varrho = d/2$.*

Proof. Consider a function F on $\mathscr{S}'(\mathbf{R}^d) \times \mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$ which is continuous and bounded. We need the convergence $\lim_n \mathbb{E}\Big[F(\xi^n,\mu^n)\Big] \to \mathbb{E}\Big[F(\xi,\mu)\Big]$. Up to changing the probability space (which does not affect the law) we may assume that ξ^n is a controlled random environment. We condition on the noise, rewriting the left-hand side as

$$\mathbb{E}\Big[F(\xi^n,\mu^n)\Big] = \int \mathbb{E}^{\omega,n}\Big[F(\xi^n(\omega),\mu^n)\Big]\mathrm{d}\mathbb{P}^n(\omega).$$

Under the additional property of being a controlled random environment and for fixed $\omega \in \Omega$, the conditional law $\mathbb{P}^{\omega,n}$ on the space $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$ converges weakly to the measure $\mathbb{P}^{\omega}$ given by Theorem II.2.10 or Theorem II.2.13, according to the value of ϱ. We can thus deduce the result by dominated convergence. □

For $\varrho > d/2$ the process of Corollary II.2.16 is simply the continuous parabolic Anderson model. For $\varrho = d/2$ it is a new process.

Definition II.2.17. *For $\varrho = d/2$ we call the process μ of Corollary II.2.16 an* SBM in static random environment *(of parameter $\kappa > 0$).*

In dimension $d = 1$ we characterize the process μ as the solution to the SPDE (II.5) below. First, we rigorously define solutions to such an equation.

Definition II.2.18. *Let $d = 1$, $\kappa > 0$, and $\mu_0 \in \mathscr{M}(\mathbf{R})$. A weak solution to*

$$\partial_t \mu(t,x) = \mathscr{H}^\omega \mu(t,x) + \sqrt{\kappa \mu(t,x)}\tilde{\xi}(t,x), \qquad \mu(0) = \mu_0, \tag{II.5}$$

is a couple formed by a probability space $(\Omega^{\mathrm{tot}}, \mathscr{F}^{\mathrm{tot}}, \mathbb{P}^{\mathrm{tot}})$ and a random process

$$\mu\colon \Omega^{\mathrm{tot}} \to C([0,+\infty); \mathscr{M}(\mathbf{R}))$$

such that $\Omega^{\mathrm{tot}} = \Omega \times \bar{\Omega}$ and $\mathbb{P}^{\mathrm{tot}}$ is of the form $\mathbb{P} \ltimes \mathbb{P}^\omega$ with $(\Omega, \mathbb{P})$ supporting a space white noise ξ and $(\Omega^{\mathrm{tot}}, \mathbb{P}^{\mathrm{tot}})$ supporting space-time white noise $\tilde{\xi}$ (see definition II.2.2) independent of ξ, such that the following properties are fulfilled for almost all $\omega \in \Omega$:

- *There exists a filtration $\{\mathscr{F}^\omega_t\}_{t \in [0,T]}$ on the space $(\bar{\Omega}, \mathbb{P}^\omega)$ which satisfies the usual conditions and such that $\mu(\omega, \cdot)$ is adapted and almost surely lies in $L^p([0,T]; L^2(\mathbf{R}; e(l)))$ for all $p < 2$ and $l \in \mathbf{R}$. Moreover, under $\mathbb{P}^\omega$ the process $\tilde{\xi}(\omega, \cdot)$ is a space-time white noise adapted to the same filtration.*

- *The random process μ satisfies for all $\varphi \in \mathscr{D}_{\mathscr{H}^\omega}$ and for all $t \geq 0$:*

$$\int_{\mathbf{R}} \mu(\omega,t,x)\varphi(x)\,\mathrm{d}x = \int_0^t \int_{\mathbf{R}} \mu(\omega,s,x)(\mathscr{H}^\omega \varphi)(x)\,\mathrm{d}s\,\mathrm{d}x$$
$$+ \int_0^t \int_{\mathbf{R}} \sqrt{\kappa \mu_p(\omega,s,x)}\varphi(x)\,\mathrm{d}\tilde{\xi}(\omega,s,x) + \int_{\mathbf{R}} \varphi(x)\,\mathrm{d}\mu_0(x),$$

  with the last integral understood in the sense of Walsh [Wal86].

Theorem II.2.19. *For $d = 1$ and $\mu_0 = \delta_0$ and any $\kappa > 0$ there exists a weak solution μ to the SPDE* (II.5) *in the sense of Definition II.2.18. The law of μ as a random process on $C([0,+\infty); \mathscr{M}(\mathbf{R}))$ is unique and corresponds to an SBM in static random environment of parameter κ.*

Proof. The proof can be found at the end of Section II.6.1. □

As a last result, we show that the rSBM is persistent in dimension $d = 1, 2$.

Definition II.2.20. *We say that a random process $\mu \in C([0,+\infty); \mathscr{M}(\mathbf{R}^d))$ with law P is* super-exponentially persistent *if for any nonzero positive function $\varphi \in C^\infty_c(\mathbf{R}^d)$ and for all $\lambda > 0$ it holds that:*

$$\mathbb{P}\Big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu(t), \varphi\rangle = \infty\Big) > 0.$$

Theorem II.2.21. *Let μ be an SBM in static random environment. Then for almost all $\omega \in \Omega$ the process $\mu(\omega, \cdot)$ is super-exponentially persistent.*

The result follows from Corollary II.6.6 and the preceding discussion.

II.3 Discrete and continuous Anderson model

Here we review the solution theory for the Parabolic Anderson Model (PAM) (3) on $\mathbf{R}^d$ in the discrete and continuous setting and the interplay between the two (we refer to [MP19] and [HL15, HL18], where the theory we present here was developed). The reader should be able to follow the arguments without understanding completely the machinery behind it: in any case, we will discuss further discretizations of the Anderson model in Chapter IV (without weights), where the problem is explained more thoroughly.

Recall that the regularity parameter α from Assumption II.2.4 satisfies:

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2. \tag{II.6}$$

We choose an initial condition $w_0 \in \mathscr{C}_p^{\zeta}(\mathbf{R}^d; e(l))$ and a forcing $f \in \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbf{R}^d; e(l))$, and we consider the equation

$$\partial_t w = \Delta w + \xi w + f, \qquad w(0) = w_0 \tag{II.7}$$

and its discrete counterpart

$$\partial_t w^n = (\Delta^n + \xi_e^n) w^n + f^n, \qquad w^n(0) = w_0^n. \tag{II.8}$$

To motivate the constraints on the parameters appearing in the proposition below, let us first formally discuss the solution theory in $d = 1$. Under Assumption II.2.4 it follows from the Schauder estimates in [MP19, Lemma 3.10] that the best regularity we can expect at a fixed time is $w(t) \in \mathscr{C}_p^{\alpha \wedge (\zeta+2) \wedge (\alpha_0+2)}(\mathbf{R}; e(k))$ for some $k \in \mathbf{R}$. In fact we lose a bit of regularity, so let $\vartheta < \alpha$ be "large enough" (we will see soon what we need from ϑ) and assume that $\zeta + 2 \geq \vartheta$ and $\alpha_0 + 2 \geq \vartheta$. Then we expect $w(t) \in \mathscr{C}_p^{\vartheta}(\mathbf{R}; e(k))$, and the Schauder estimates suggest the blow-up $\gamma = \max\{(\vartheta + \varepsilon - \zeta)_+/2, \gamma_0\}$ for some $\varepsilon > 0$, which has to be in $[0, 1)$ to be locally integrable, so in particular $\gamma_0 \in [0, 1)$. If $\vartheta + \alpha - 2 > 0$ (which is possible because in $d = 1$ we have $2\alpha - 2 > 0$), then the product $w(t)\xi$ is well defined and in $\mathscr{C}_p^{\alpha-2}(\mathbf{R}; e(k)p(a))$, so we can set up a Picard iteration. The loss of control in the weight (going from $e(k)$ to $e(k)p(a)$) is handled by introducing time-dependent weights so that $w(t) \in \mathscr{C}_p^{\vartheta}(\mathbf{R}^d; e(l + t))$. In the setting of singular SPDEs this idea was introduced by Hairer–Labbé [HL15], and it induces a small loss of regularity which explains why we only obtain regularity $\vartheta < \alpha$ for the solution and the additional $+\varepsilon/2$ in the blow-up γ.

In two dimensions the white noise is less regular, we no longer have $2\alpha - 2 > 0$, and we need paracontrolled analysis to solve the equation. The solution lives in a space of *paracontrolled distributions*, and now we take $\vartheta > 0$ such that $\vartheta + 2\alpha - 2 > 0$. We now need additional regularity requirements for the initial condition w_0 and for the forcing f. More precisely, we need to be able to multiply $(P_t w_0)\xi$ and $\left(\int_0^t P_{t-s} f(s)\, \mathrm{d}s\right)\xi$, and therefore we require now also $\zeta + 2 + (\alpha - 2) > 0$ and $\alpha_0 + 2 + (\alpha - 2) > 0$, i.e. $\zeta, \alpha_0 > -\alpha$.

We do not provide the details of the construction and refer to [MP19] instead, where the two-dimensional case is worked out (the one-dimensional case follows from similar, but much easier arguments).

Proposition II.3.1. *Consider α as in* (II.6)*, any $T > 0$, $p \in [1,+\infty]$, $l \in \mathbf{R}, \gamma_0 \in [0,1)$ and $\vartheta, \zeta, \alpha_0$ satisfying:*

$$\vartheta \in \begin{cases} (2-\alpha, \alpha), & d = 1, \\ (2-2\alpha, \alpha), & d = 2, \end{cases} \quad \zeta > (\vartheta-2) \vee (-\alpha), \quad \alpha_0 > (\vartheta-2) \vee (-\alpha), \tag{II.9}$$

and let $w_0^n \in \mathscr{C}_p^\zeta(\mathbf{Z}_n^d; e(l))$ and $f^n \in \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbf{Z}_n^d; e(l))$, such that

$$\mathscr{E}^n w_0^n \to w_0, \text{ in } \mathscr{C}_p^\zeta(\mathbf{R}^d; e(l)), \qquad \mathscr{E}^n f^n \to f \text{ in } \mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbf{R}^d; e(l)).$$

Then under Assumption II.2.4 there exist unique (paracontrolled) solutions w^n, w to Equation (II.8) *and* (II.7)*. Moreover, for all $\gamma > (\vartheta-\zeta)_+/2 \vee \gamma_0$ and for all $\hat{l} \geq l+T$, the sequence w^n is uniformly bounded in $\mathscr{L}_p^{\gamma,\vartheta}(\mathbf{Z}_n^d; e(\hat{l}))$:*

$$\sup_n \|w^n\|_{\mathscr{L}_p^{\gamma,\vartheta}(\mathbf{Z}_n^d; e(\hat{l}))} \lesssim \sup_n \|w_0^n\|_{\mathscr{C}_p^\zeta(\mathbf{Z}_n^d; e(l))} + \sup_n \|f^n\|_{\mathscr{M}^{\gamma_0}\mathscr{C}_p^{\alpha_0}(\mathbf{Z}_n^d; e(l))}, \tag{II.10}$$

where the proportionality constant depends on the time horizon T and the norms of the terms in Assumption II.2.4. Moreover

$$\mathscr{E}^n w^n \to w \text{ in } \mathscr{L}_p^{\gamma,\vartheta}(\mathbf{R}^d; e(\hat{l})).$$

Remark II.3.2. *We consider the case $p < \infty$ to start the equation in the Dirac measure δ_0. Indeed, δ_0 lies in $\mathscr{C}^{-d}(\mathbf{R}^d, e(l))$ for any $l \in \mathbf{R}$. This means that $\zeta = -d$, and in $d = 1$ we can choose ϑ small enough such that* (II.9) *holds. But in $d = 2$ this is not sufficient, so we use instead that $\delta_0 \in \mathscr{C}_p^{d(1-p)/p}(\mathbf{R}^d, e(l))$ for $p \in [1,\infty]$ and any $l \in \mathbf{R}$, so that for $p \in [1,2)$ the conditions in* (II.9) *are satisfied.*

It will be convenient to introduce, with a slight abuse of notation (as we will explain below) the following semigroup notation.

Notation II.3.3. *We write*

$$t \mapsto T_t^n w_0^n + \int_0^t T_{t-s}^n f_s^n \, \mathrm{d}s, \qquad t \mapsto T_t w_0 + \int_0^t T_{t-s} f_s \, \mathrm{d}s$$

for the solution to Equation (II.8) *and* (II.7)*, respectively.*

Proposition II.3.1 provides us with the tools to make sense of Property (ii) in the definition of the rSBM, Definition II.2.11. To make sense of the last Property (iii), we need to construct the Anderson Hamiltonian. In finite volume this was done in [FN77, AC15, GUZ20, Lab19], respectively, but the construction in infinite volume is more complicated, for example because the spectrum of $\mathscr{H}$ is unbounded from above and thus resolvent methods fail. Hairer-Labbé [HL18] suggest a construction based on spectral calculus, setting $\mathscr{H} = t^{-1} \log T_t$, but this gives insufficient information about the domain. Therefore, we use an ad-hoc approach which is sufficient for our purpose. We define the operator in terms of the solution map $(T_t)_{t\geq 0}$ to the parabolic equation. Strictly speaking, $(T_t)_{t\geq 0}$ does not define a semigroup, since due to the presence of the time-dependent weights it does not act on a fixed Banach space. But we simply ignore that and are still able to use standard arguments for semigroups on Banach spaces to identify a dense subset of the domain (compare the discussion below to [EK86, Proposition 1.1.5]). However, in that way we do not learn anything about the spectrum of $\mathscr{H}$. In finite volume, $(T_t)_{t\geq 0}$ is a strongly continuous semigroup of compact operators and we can simply define $\mathscr{H}$ as its infinitesimal generator. It seems that this would be equivalent to the construction of [AC15] through the resolvent equation.

We first discuss the case $d = 1$. Then $\xi \in \mathscr{C}^{\alpha-2}(\mathbf{R};p(a))$ for all $a > 0$ by assumption, where $\alpha \in (1, \frac{3}{2})$. In particular, $\mathscr{H}u = (\Delta+\xi)u$ is well defined for all $u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l))$ with $\vartheta > 2-\alpha$ and $l \in \mathbf{R}$, and $\mathscr{H}u \in \mathscr{C}^{\alpha-2}(\mathbf{R};e(l)p(a))$. Our aim is to identify a subset of $\mathscr{C}^{\vartheta}(\mathbf{R};e(l))$ on which $\mathscr{H}u$ is even a continuous function. We can do this by defining for $t > 0$

$$A_t u = \int_0^t T_s u \,\mathrm{d}s.$$

Then $A_t u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l+t))$, and by definition

$$\mathscr{H}A_t u = \int_0^t \mathscr{H}T_s u \,\mathrm{d}s = \int_0^t \partial_s T_s u \,\mathrm{d}s = T_t u - u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l+t)).$$

Moreover, the following convergence holds in $\mathscr{C}^{\vartheta}(\mathbf{R};e(l+t+\varepsilon))$ for all $\varepsilon > 0$:

$$\lim_{n\to\infty} n(T_{1/n}-\mathrm{id})A_t u = \lim_{n\to\infty} n\left(\int_t^{t+1/n} T_s u \,\mathrm{d}s - \int_0^{1/n} T_s u \,\mathrm{d}s\right) = \mathscr{H}A_t u.$$

Therefore, we define

$$\mathscr{D}_{\mathscr{H}} = \{A_t u : u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l)), l \in \mathbf{R}, t \in [0,T]\}.$$

Since for $u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l))$ the map $(t \mapsto T_t u)_{t\in[0,\varepsilon]}$ is continuous in the space $\mathscr{C}^{\vartheta}(\mathbf{R};e(l+\varepsilon))$ we can find for all $u \in \mathscr{C}^{\vartheta}(\mathbf{R};e(l))$ a sequence $\{u^m\}_{m\in\mathbf{N}} \subset \mathscr{D}_{\mathscr{H}}$ such that $\|u^m - u\|_{\mathscr{C}^{\vartheta}(\mathbf{R};e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. Indeed, it suffices to set $u^m = m^{-1}A_{m^{-1}}u$. The same construction also works for $\mathscr{H}^n$ instead of $\mathscr{H}$.

In the two-dimensional case $(\Delta+\xi)u$ would be well defined whenever $u \in \mathscr{C}^{\beta}(\mathbf{R}^2, e(l))$ with $\beta > 2-\alpha$ for $\alpha \in (\frac{2}{3}, 1)$. But in this space it seems impossible to find a domain that is mapped to continuous functions. And also $(\Delta+\xi)u$ is not the right object to look at: we have to take the renormalisation into account and should think of $\mathscr{H} = \Delta+\xi-\infty$. So we first need an appropriate notion of paracontrolled distributions u for which can define $\mathscr{H}u$ as a distribution. As in Proposition II.3.1 we let $\vartheta \in (2-2\alpha, \alpha)$.

The following lemma is crucial to understand the meaning of paracontrolled solution. It tells us that if $u = u' \varolessthan X + u^\sharp$, for sufficiently smooth $u', u^\sharp$, then the resonant product $u \odot \xi$ is well-defined, given the product $X \odot \xi$.

Lemma II.3.4 (Lemma 4.4 [MP19]). *The estimates below hold uniformly over $n \in \mathbf{N}$ (and also for $\mathbf{Z}_n^d$ replaced by $\mathbf{R}^d$). Consider $z_1, z_2, z_3 \in \rho(\varpi)$, $p \in [1,\infty]$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that $\beta+\gamma < 0$ and $\alpha+\beta+\gamma > 0$. Define the commutator*

$$C^{\odot}(\varphi,\psi,\zeta) = (\varphi \varolessthan \psi) \odot \zeta - \varphi(\psi \odot \zeta).$$

We have:

$$\|C^{\odot}(\varphi,\psi,\zeta)\|_{\mathscr{C}_p^{\beta+\gamma}(\mathbf{Z}_n^d;z_1z_2z_3)} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}(\mathbf{Z}_n^d;z_1)}\|\psi\|_{\mathscr{C}^{\beta}(\mathbf{Z}_n^d;z_2)}\|\zeta\|_{\mathscr{C}^{\gamma}(\mathbf{Z}_n^d;z_3)}.$$

We can now define paracontrolled distributions.

Definition II.3.5. *Consider $X = (-\Delta)^{-1}\chi(D)\xi$ and $X \diamond \xi$ defined as in Assumption II.2.4. We say that u (resp. u^n) is* paracontrolled *if $u \in \mathscr{C}^\vartheta(\mathbf{R}^2, e(l))$ for some $l \in \mathbf{R}$, and*

$$u^\sharp = u - u \otimes X \in \mathscr{C}^{\alpha+\vartheta}(\mathbf{R}^2, e(l)).$$

Then set

$$\mathscr{H}u = \Delta u + \xi \otimes u + u \otimes \xi + u^\sharp \odot \xi + C^\odot(u, X, \xi) + u(X \diamond \xi),$$

where $C^\odot$ is defined in Lemma II.3.4. The lemma, together with Lemma I.1.3 guarantees that $\mathscr{H}u$ is a well defined distribution in $\mathscr{C}^{\alpha-2}(\mathbf{R}^2, e(l)p(a))$. In the discrete setting the same holds, with $X, X \diamond \xi$ replaced by $X^n, X^n \odot \xi^n - c_n$ respectively.

The operator T_t leaves the space of paracontrolled distributions invariant, and therefore the same arguments as in $d = 1$ give us a domain $\mathscr{D}_{\mathscr{H}}$ such that for all paracontrolled u there exists a sequence $\{u^m\}_{m\in\mathbf{N}} \subset \mathscr{D}_{\mathscr{H}}$ with $\|u^m - u\|_{\mathscr{C}^\vartheta(\mathbf{R}^2, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. For general $u \in \mathscr{C}^\vartheta(\mathbf{R}^2, e(l))$ and $\varepsilon > 0$ we can find a paracontrolled $v \in \mathscr{C}^\vartheta(\mathbf{R}^2, e(l))$ with $\|u - v\|_{\mathscr{C}^\vartheta(\mathbf{R}^2, e(l+\varepsilon))} < \varepsilon$, because $T_t u$ is paracontrolled for all $t > 0$ and converges to u in $\mathscr{C}^\vartheta(\mathbf{R}^2, e(l+\varepsilon))$ as $t \to 0$. Thus, we have established the following result:

Lemma II.3.6. *Under Assumption II.2.4 let ϑ be as in Proposition II.3.1. There exists a domain $\mathscr{D}_{\mathscr{H}} \subset \bigcup_{l\in\mathbf{R}} \mathscr{C}^\vartheta(\mathbf{R}^d, e(l))$ such that $\mathscr{H}u = \lim_n n(T_{1/n} - \mathrm{id})u$ in $\mathscr{C}^\vartheta(\mathbf{R}^d, e(l+\varepsilon))$ for all $u \in \mathscr{D}_{\mathscr{H}} \cap \mathscr{C}^\vartheta(\mathbf{R}^d, e(l))$ and $\varepsilon > 0$ and such that for all $u \in \mathscr{C}^\vartheta(\mathbf{R}^d, e(l))$ there is a sequence $\{u^m\}_{m\in\mathbf{N}} \subset \mathscr{D}_{\mathscr{H}}$ with $\|u^m - u\|_{\mathscr{C}^\vartheta(\mathbf{R}^2, e(l+\varepsilon))} \to 0$ for all $\varepsilon > 0$. The same is true for the discrete operator $\mathscr{H}^n$ (with $\mathbf{R}^d$ replaced by $\mathbf{Z}^d_n$).*

II.4 The rough superBrownian motion

II.4.1 Scaling Limit of the BRWRE

In this section we consider a deterministic environment, that is a sequence $\{\xi^n\}_{n\in\mathbf{N}}$ of maps $\xi^n \colon \mathbf{Z}^d_n \to \mathbf{R}$, satisfying Assumption II.2.4, to which we associate the Markov process μ^n as in Definition II.2.7: our aim is to prove that the sequence μ^n converges weakly, with a limit depending on the value of ϱ. First, we prove tightness for the sequence μ^n in $\mathbb{D}([0, T]; \mathscr{M}(\mathbf{R}^d))$ for $\varrho \geq d/2$. Then, we prove uniqueness in law of the limit points and thus deduce the weak convergence of the sequence.

Remark II.4.1. *Fix $t > 0$. For any $\varphi \in L^\infty(\mathbf{Z}^d_n; e(l))$, for some $l \in \mathbf{R}$:*

$$[0, t] \ni s \mapsto M^{n,\varphi}_t(s) = \mu^n_s(T^n_{t-s}\varphi) - T^n_t\varphi(0) \tag{II.11}$$

is a centered martingale on $[0, t]$ with predictable quadratic variation

$$\langle M^{n,\varphi}_t \rangle_s = \int_0^s \mu^n_r\Big(n^{-\varrho}|\nabla^n T^n_{t-r}\varphi|^2 + n^{-\varrho}|\xi^n_e|(T^n_{t-r}\varphi)^2\Big)\mathrm{d}r.$$

Sketch of proof. Consider a differentiable function ψ: that is $\psi \in C([0,T];L^\infty(\mathbf{Z}_n^d;e(l)))$, $\partial_t\psi \in C([0,T];L^\infty(\mathbf{Z}_n^d;e(l)))$ for some $l \in \mathbf{R}, T > 0$. One can use Dynkin's formula and an approximation argument applied to the function $(s,\mu) \mapsto F_\psi^t(s,\mu^n) = \mu^n(\psi(s))$. By truncating F_ψ^t, discretizing time and then passing to the limit, we obtain that

$$\mu_s^n(\psi(s)) - \mu_0^n(\psi(0)) - \int_0^s \mu_r^n(\partial_r\psi(r) + \mathscr{H}^n\psi(r))\,\mathrm{d}r$$

is a martingale with the correct quadratic variation. Now it suffices to note that for $r \in [0,t]:\ \partial_r T_{t-r}^n\varphi = -\mathscr{H}^n T_{t-r}^n\varphi$. □

For the remainder of this section we assume that $\varrho \geq d/2$. To prove the tightness of the measure-valued process we use the upcoming auxiliary result, which provides tightness of the real-valued processes $\{t \mapsto \mu_t^n(\varphi)\}_{n\in\mathbf{N}}$ for smooth φ.

The main difficulty in the proof lies in handling the irregularity of the spatial environment. For this reason we consider first the martingale $[0,t] \ni s \mapsto \mu_s^n(T_{t-s}^n\varphi)$ (cf. (II.11)) instead of the more natural process $s \mapsto \mu_s^n(\varphi)$. We then exploit the martingale to prove tightness for $\mu^n(\varphi)$. Here we cannot apply the classical Kolmogorov continuity test, since we are considering a pure jump process. Instead we will use a slight variation, due to Chentsov [Che56] and conveniently exposed in [EK86, Theorem 3.8.8].

Lemma II.4.2. *For any $l \in \mathbf{R}$ and $\varphi \in C^\infty(\mathbf{R}^d;e(l))$ the processes $\{t \mapsto \mu_t^n(\varphi)\}_{n\in\mathbf{N}}$ form a tight sequence in $\mathbb{D}([0,+\infty);\mathbf{R})$.*

Proof. It is sufficient to prove that for arbitrary $T > 0$ the given sequence is tight in $\mathbb{D}([0,T];\mathbf{R})$. Hence fix $T > 0$ and consider $0 < \vartheta < 1$ as in Proposition II.3.1. In the following computation $k \in \mathbf{R}$ may change from line to line, but it is uniformly bounded for $l \in \mathbf{R}$ and $T > 0$ varying in a bounded set.

Step 1. Here the aim is to establish a second moment bound for the increment of the process. Let $(\mathscr{F}_t^n)_{t\geq 0}$ be the filtration generated by μ^n. We will prove that the following conditional expectation can be estimated uniformly over $0 \leq t \leq t+h \leq T$:

$$\mathbb{E}\Big[|\mu_{t+h}^n(\varphi) - \mu_t^n(\varphi)|^2 \big| \mathscr{F}_t^n\Big] \lesssim h^\vartheta\Big[\mu_t^n(e^{k|x|^\sigma}) + |\mu_t^n(e^{k|x|^\sigma})|^2\Big], \tag{II.12}$$

In fact, via the martingales defined in (II.11), one can start by observing that:

$$\begin{aligned}
&\mathbb{E}\Big[|\mu_{t+h}^n(\varphi) - \mu_t^n(\varphi)|^2 \big| \mathscr{F}_t^n\Big] \\
&\quad = \mathbb{E}\Big[|M_{t+h}^{n,\varphi}(t+h) - M_{t+h}^{n,\varphi}(t) + \mu_t^n(T_h^n\varphi - \varphi)|^2 \big| \mathscr{F}_t^n\Big] \\
&\quad \lesssim_{\varphi,T} \mathbb{E}\left[\int_t^{t+h} \mu_r^n\Big(n^{-\varrho}|\nabla^n T_{t+h-r}^n\varphi|^2 + n^{-\varrho}|\xi_e^n|(T_{t+h-r}^n\varphi)^2\Big)\mathrm{d}r\ \middle|\ \mathscr{F}_t^n\right] \\
&\qquad\qquad + h^\vartheta|\mu_t^n(e^{k|x|^\sigma})|^2,
\end{aligned}$$

where the last term appears since $h \mapsto T_h^n\varphi \in \mathscr{L}^\vartheta(\mathbf{Z}_n^d, e(k))$. The first term on the right-hand side can be bounded, for any $\varepsilon > 0$ by:

$$\begin{aligned}
&\int_t^{t+h} \mu_t^n\Big(T_{r-t}^n\Big(n^{-\varrho}|\nabla^n T_{t+h-r}^n\varphi|^2 + n^{-\varrho}|\xi_e^n|(T_{t+h-r}^n\varphi)^2\Big)\Big)\mathrm{d}r \\
&\qquad \lesssim \int_t^{t+h} \mu_t^n\Big(e^{k|x|^\sigma} + (r-t)^{-2\varepsilon}e^{k|x|^\sigma}\Big)\mathrm{d}r.
\end{aligned} \tag{II.13}$$

Here we have used Lemma II.9.1 to ensure that $\varphi|_{\mathbf{Z}^d_n}$ is smooth on the lattice together with the a-priori bound (II.10) of Proposition II.3.1 and with Lemmata II.9.2 and II.9.3, which show respectively a gain of regularity via the factor $n^{-\varrho}$ and a loss of regularity via the discrete derivative ∇^n, to obtain:

$$\lim_{n\to\infty} \sup_{r\in[0,T]} \| n^{-\varrho} |\nabla^n T^n_r \varphi|^2 \|_{\mathscr{C}^{\tilde\vartheta}(\mathbf{Z}^d_n; e(2(l+r)))} = 0,$$

for $0 < \tilde\vartheta < \vartheta - 1 + \varrho/2$ (we can choose ϑ sufficiently large so that the latter quantity is positive). Since $\vartheta > 0$ and the term is positive, one has by comparison:

$$T^n_{r-t}\Big(n^{-\varrho} |\nabla^n T^n_{t+h-r}\varphi|^2 \Big) \lesssim e^{k|x|^\sigma} \| n^{-\varrho} |\nabla^n T^n_{t+h-r}\varphi|^2 \|_{\mathscr{C}^{\tilde\vartheta}(\mathbf{Z}^d_n; e(2(l+T)))}.$$

Moreover, according to Assumption II.2.4 for $\varrho \geq d/2$ the term $n^{-\varrho}|\xi^n_e|$ is bounded in $\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n; p(a))$ whenever $\varepsilon > 0$. It then follows from the uniform bounds (II.10) from Proposition II.3.1 and by applying (I.1) from Lemma I.1.5, together with similar arguments to the ones just presented, that:

$$\begin{aligned}
&\sup_{n\in\mathbf{N}} \sup_{r\in[0,T]} \| s \mapsto T^n_s (n^{-\varrho} |\xi^n_e| (T^n_r \varphi)^2) \|_{\mathscr{M}^{2\varepsilon}\mathscr{C}^{\varepsilon}(\mathbf{Z}^d_n; e(k))} \\
&\qquad \lesssim \sup_{n\in\mathbf{N}} \sup_{r\in[0,T]} \| s \mapsto T^n_s (n^{-\varrho} |\xi^n_e| (T^n_r \varphi)^2) \|_{\mathscr{L}^{\frac{\vartheta+\varepsilon}{2}+\varepsilon, \vartheta}(\mathbf{Z}^d_n; e(k))} \\
&\qquad \lesssim \sup_{n\in\mathbf{N}} \sup_{r\in[0,T]} \| n^{-\varrho} |\xi^n_e| (T^n_r \varphi)^2 \|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n; e(k))} < \infty.
\end{aligned}$$

This completes the explanation of (II.13). So overall, integrating over r we can bound the conditional expectation by:

$$h^{1-2\varepsilon} \mu^n_t(e^{k|x|^\sigma}) + h^{\vartheta} |\mu^n_t(e^{k|x|^\sigma})|^2 \leq h^{\vartheta}\Big[\mu^n_t(e^{k|x|^\sigma}) + |\mu^n_t(e^{k|x|^\sigma})|^2 \Big],$$

assuming $1 - 2\varepsilon \geq \vartheta$. This completes the proof of (II.12).

Step 2. Now we are ready to apply Chentsov's criterion [EK86, Theorem 3.8.8]. We have to multiply two increments of $\mu^n(\varphi)$ on $[t-h, h]$ and on $[t, t+h]$ and show that for some $\kappa > 0$:

$$\mathbb{E}\Big[(|\mu^n_{t+h}(\varphi) - \mu^n_t(\varphi)| \wedge 1)^2 (|\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \wedge 1)^2 \Big] \lesssim h^{1+\kappa} \tag{II.14}$$

We use (II.12) to bound:

$$\begin{aligned}
&\mathbb{E}\Big[(|\mu^n_{t+h}(\varphi) - \mu^n_t(\varphi)| \wedge 1)^2 (|\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \wedge 1)^2 \Big] \\
&\quad \leq \mathbb{E}\Big[|\mu^n_{t+h}(\varphi) - \mu^n_t(\varphi)|^2 |\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \Big] \\
&\quad \lesssim h^{\vartheta} \mathbb{E}\Big[\Big(\mu^n_t(e^{k|x|^\sigma}) + |\mu^n_t(e^{k|x|^\sigma})|^2 \Big) |\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \Big] \\
&\quad \lesssim h^{\vartheta} \mathbb{E}\Big[\Big(1 + |\mu^n_t(e^{k|x|^\sigma})|^2 \Big) |\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \Big].
\end{aligned}$$

By the Cauchy-Schwarz inequality together with (II.12) and the moment bound for $|\mu^n_t(e^{k|x|^\sigma})|^4$ from Lemma II.8.1 one obtains:

$$\begin{aligned}
&\mathbb{E}\Bigg[(1 + |\mu^n_t(e^{k|x|^\sigma})|^2) |\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)| \Bigg] \\
&\quad \lesssim \Big(1 + \mathbb{E}\Big[|\mu^n_t(e^{k|x|^\sigma})|^4 \Big]^{1/2} \Big) \mathbb{E}\Big[|\mu^n_t(\varphi) - \mu^n_{t-h}(\varphi)|^2 \Big]^{1/2} \lesssim h^{\vartheta/2}.
\end{aligned}$$

Combining all the estimates one finds:

$$\mathbb{E}\Big[(|\mu^n_{t+h}(\varphi)-\mu^n_t(\varphi)|\wedge 1)^2(|\mu^n_t(\varphi)-\mu^n_{t-h}(\varphi)|\wedge 1)^2\Big] \lesssim h^{\frac{3}{2}\vartheta}.$$

Since $\vartheta > \frac{2}{3}$, this proves Equation (II.14) for some $\kappa > 0$. In particular, we can apply [EK86, Theorem 3.8.8] with $\beta = 4$, which in turn implies that the tightness criterion of Theorem 3.8.6 (b) of the same book is satisfied. This concludes the proof of tightness for $\{t \mapsto \mu^n_t(\varphi)\}_{n\in\mathbf{N}}$. □

Consequently, we find tightness of the process μ^n in the space of measures.

Corollary II.4.3. *The processes* $\{t \mapsto \mu^n(t)\}_{n\in\mathbf{N}}$ *form a tight sequence in* $\mathbb{D}([0,\infty);\mathscr{M}(\mathbf{R}^d))$.

Proof. We apply Jakubowski's criterion [DMS93, Theorem 3.6.4], which we recall in the appendix, Proposition A.2.1. We first need to verify the compact containment condition. For that purpose note that for all $R > 0$ the set $K_R = \{\mu \in \mathscr{M}(\mathbf{R}^d) \mid \mu(|\cdot|^2) \le R\}$ is compact in $\mathscr{M}(\mathbf{R}^d)$. Here $\mu(|\cdot|^2) = \int_{\mathbf{R}^d} |x|^2 \,\mathrm{d}\mu(x)$. Since the sequence of processes $\{\mu^n(|\cdot|^2)\}_{n\in\mathbf{N}}$ are tight by Lemma II.4.2, we find for all $T, \varepsilon > 0$ an $R(\varepsilon)$ such that

$$\sup_n \mathbb{P}\Bigg(\sup_{t\in[0,T]} \mu^n(t)(|\cdot|^2) \ge R(\varepsilon)\Bigg) \le \varepsilon,$$

as required. Second we note that $C^\infty_c(\mathbf{R}^d)$ is closed under addition and the maps $\mu \mapsto \{\mu(\varphi)\}_{\varphi\in C^\infty_c(\mathbf{R}^d)}$ separate points in $\mathscr{M}(\mathbf{R}^d)$. Since Lemma II.4.2 shows that $t \mapsto \mu^n(t)(\varphi)$ is tight for any $\varphi \in C^\infty_c(\mathbf{R}^d)$, we can conclude. □

Next we show that any limit point is a solution to a martingale problem.

Lemma II.4.4. *Any limit point of the sequence* $\{t \mapsto \mu^n(t)\}_{n\in\mathbf{N}}$ *is supported in the space of continuous function* $C([0,+\infty);\mathscr{M}(\mathbf{R}^d))$*, and it satisfies Property* (ii) *of Definition II.2.11 with* $\kappa = 0$ *if* $\varrho > d/2$*, and* $\kappa = 2\nu$ *if* $\varrho = d/2$.

Proof. First, we address the continuity of an arbitrary limit point μ. Since $\mathscr{M}(\mathbf{T}^d)$ is endowed with the weak topology, it is sufficient to prove the continuity of $t \mapsto \langle \mu(t), \varphi\rangle$ for all $\varphi \in C_b(\mathbf{R}^d)$. In view of Corollary II.4.3, up to a subsequence:

$$\langle \mu^n, \varphi\rangle \to \langle \mu, \varphi\rangle \quad \text{in} \quad \mathbb{D}([0,\infty);\mathbf{R}).$$

Then by [EK86, Theorem 3.10.2] in order to obtain the continuity of the limit point it is sufficient to observe that the maximal jump size is vanishing in n:

$$\sup_{t\ge 0} |\langle \mu^n_t, \varphi\rangle - \langle \mu^n_{t-}, \varphi\rangle| \lesssim n^{-\varrho}\|\varphi\|_{L^\infty}.$$

Next, we study the limiting martingale problem. First we will prove that the process $M^{\varphi_0,f}_t$ from Definition II.2.11 is a martingale. Then we will compute its quadratic variation.

Step 1. We fix a limit point μ and study the required martingale property. For f, φ_0 as required, observe that $\varphi^n_0 = \varphi_0|_{\mathbf{Z}^d_n}$ is uniformly bounded in $\mathscr{C}^{\zeta_0}(\mathbf{Z}^d_n; e(l))$ for any $\zeta_0 > 0$

and $l \in \mathbf{R}$, and similarly $f^n = f|_{\mathbf{Z}^d_n}$ is uniformly bounded in $C([0,t];\mathscr{C}^\zeta(\mathbf{Z}^d_n))$, with an application of Lemma II.9.1. Hence by Proposition II.3.1 the solutions φ^n_t to the discrete equations

$$\partial_s \varphi^n_t + \mathscr{H}^n \varphi^n_t = f^n, \qquad \varphi^n_t(t) = \varphi^n_0$$

converge (when extended via $\mathscr{E}^n$) in $\mathscr{L}^\vartheta(\mathbf{R}^d; e(l))$, for $0 < \vartheta < 1$, to φ_t. To be accurate, we might need to increase the value of l: so let us assume that $l > 0$ is large enough so that the convergence holds. At the discrete level we find, analogously to (II.11), that

$$M^{\varphi_0,f,n}_t(s) := \langle \mu^n(s), \varphi^n_t(s)\rangle - \langle \mu(0), \varphi^n_t(0)\rangle - \int_0^s \langle \mu^n(r), f^n(r)\rangle \,\mathrm{d}r, \qquad s \in [0,t]$$

is a square integrable martingale. Moreover this martingale is bounded in L^2 uniformly over n, since the second moment can be bounded via the initial value and the predictable quadratic variation by

$$\mathbb{E}\Big[|M^{\varphi_0,f,n}_t|^2\Big] \lesssim \int_0^t T^n_r\Big(n^{-\varrho}|\nabla^n \varphi^n_t(r)|^2 + n^{-\varrho}|\xi^n|(\varphi^n_t(r))^2\Big)\mathrm{d}r$$

and the latter quantity is uniformly bounded in n. To conclude that $M^{\varphi_0,f}_t$ is an $\mathscr{F}$–martingale note that by assumption $M^{\varphi_0,f,n}_t$ converges to the continuous process $M^{\varphi_0,f}_t$. Then by [EK86, Theorem 3.7.8], for $0 \le s \le r \le t$ and for bounded and continuous $\Phi \in C_b(\mathbb{D}([0,s];\mathscr{M}))$

$$\begin{aligned}\mathbb{E}[\Phi(\mu|_{[0,s]})(M^{\varphi_0,f}_t(r) - M^{\varphi^0,f}_t(s))]& \\ &= \lim_n \mathbb{E}[\Phi(\mu^n|_{[0,s]})(M^{\varphi_0,f,n}_t(r) - M^{\varphi^0,f,n}_t(s))] = 0\end{aligned}$$

by the martingale property of $M^{\varphi_0,f,n}$. From here we easily deduce the martingale property of $M^{\varphi_0,f}_t$.

Step 2. We show that $M^{\varphi_0,f}_t$ has the correct quadratic variation, which should be given as the limit of

$$\langle M^{\varphi_0,f,n}_t\rangle_s = \int_0^s \mu^n(r)\Big(n^{-\varrho}|\nabla^n \varphi^n_t(r)|^2 + n^{-\varrho}|\xi^n|(\varphi^n_t(r))^2\Big)\mathrm{d}r.$$

We only treat the case $\varrho = d/2$, the case $\varrho > d/2$ is similar but easier because then we can use Lemma II.9.2 to gain some regularity from the factor $n^{d/2-\varrho}$, so that

$$\|n^{-\varrho}|\xi^n|\|_{\mathscr{C}^\varepsilon(\mathbf{Z}^d_n;p(a))} \to 0$$

for some $\varepsilon > 0$ and for all $a > 0$.

First we assume, leaving the proof for later, that for any sequence $\{\psi^n\}_{n\in\mathbf{N}}$ with

$$\lim_n \|\psi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbf{R}^d;p(a))} = 0$$

for some $a > 0$ and all $\varepsilon > 0$:

$$\mathbb{E}\bigg[\sup_{s\le t}\bigg|\int_0^s \mu^n(r)\Big(\psi^n\cdot(\varphi^n_t(r))^2\Big)\,\mathrm{d}r\,\bigg|^2\bigg] \longrightarrow 0. \tag{II.15}$$

By Assumption II.2.4 we can apply this to $\psi^n = n^{-\varrho}|\xi^n| - 2\nu$, and deduce that along a subsequence we have the following weak convergence in $\mathbb{D}([0,t];\mathbf{R})$:

$$\Big(M_t^{\varphi_0,f,n}\Big)^2_\cdot - \langle M_t^{\varphi_0,f,n}\rangle_\cdot \longrightarrow \Big(M_t^{\varphi_0,f}\Big)^2_\cdot - \int_0^\cdot \mu(r)\Big(2\nu(\varphi_t)^2(r)\Big)\,\mathrm{d}r.$$

Note also that the limit lies in $C([0,t];\mathbf{R})$. If the martingales on the left-hand side are uniformly bounded in L^2 we can deduce as before that the limit is a continuous L^2–martingale, and conclude that

$$\langle M_t^{\varphi_0,f}\rangle_s = \int_0^s \mu(r)\Big(2\nu(\varphi_t)^2(r)\Big)\,\mathrm{d}r.$$

As for the uniform bound in L^2, note that it follows from Lemma II.8.1 that

$$\sup_n \sup_{0\le s\le t} \mathbb{E}\Big[|M_t^{\varphi_0,f,n}(s)|^4\Big] < +\infty.$$

For the quadratic variation term we estimate:

$$\mathbb{E}\Big[|\langle M_t^{\varphi_0,f,n}\rangle_s|^2\Big] \le s\int_0^s \mathbb{E}\Big[\Big|\mu^n(r)\Big(n^{-\varrho}|\nabla^n\varphi_t^n(r)|^2 + n^{-\varrho}|\xi^n|(\varphi_t^n(r))^2\Big)\Big|^2\Big]\mathrm{d}r,$$

which can be bounded via the second estimate of Lemma II.8.1.

Step 3. Thus, we are left with the convergence in (II.15). By introducing the martingale from Equation (II.11) we find that

$$\begin{aligned}&\mathbb{E}\Big[|\mu^n(r)\Big(\psi^n(\varphi_t^n(r))^2\Big)|^2\Big]\\ &\quad\lesssim |T_r^n\Big[\psi^n(\varphi_t^n(r))^2\Big]|^2(0) + \int_0^r T_q^n\Big[n^{-\varrho}\Big|\nabla^n\Big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\Big]\Big|^2 \\ &\qquad\qquad + n^{-\varrho}|\xi^n|(T_{r-q}^n[\psi^n(\varphi_t^n(r))^2])^2\Big](0)\,\mathrm{d}q.\end{aligned} \tag{II.16}$$

We start with the first term. By Proposition II.3.1 we know that for all $\varepsilon > 0$ and $0 < \vartheta < 1$ satisfying $\vartheta + 3\varepsilon < 1$ and for $l > 0$ sufficiently large:

$$\begin{aligned}&\|r \mapsto T_r^n[\psi^n(\varphi_t^n(r))^2]\|_{\mathscr{L}^{\frac{\vartheta+\varepsilon}{2}+\varepsilon,\vartheta}(\mathbf{Z}_n^d;e(3l))}\\ &\qquad\lesssim \|\psi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}\|\varphi^n\|^2_{\mathscr{L}^{\vartheta}(\mathbf{Z}_n^d;e(l))}\\ &\qquad\lesssim \|\psi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}.\end{aligned} \tag{II.17}$$

Together with Equation (I.1) from Lemma I.1.5 and (II.17), we thus bound:

$$\begin{aligned}|T_r^n\Big[\psi^n(\varphi_t^n(r))^2\Big]|^2(0) &\lesssim r^{-4\varepsilon}\|r \mapsto |T_r^n\Big[\psi^n(\varphi_t^n(r))^2\Big]\|^2_{\mathscr{L}^{2\varepsilon,\varepsilon}(\mathbf{Z}_n^d;e(l))}\\ &\lesssim r^{-4\varepsilon}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}.\end{aligned}$$

Now we can treat the first term in the integral in (II.16). We can choose $0 < \vartheta < 1$ and $\varepsilon > 0$ with $\vartheta + 3\varepsilon < 1$ such that $0 < \tilde{\vartheta} = \vartheta - 1 + d/4$. We then apply Lemmata II.9.2 and II.9.3, which guarantee us respectively a regularity gain from the factor $n^{-\frac{d}{4}}$ and a regularity loss from the derivative ∇^n, to obtain:

$$\begin{aligned}\||n^{-d/4}\nabla^n\Big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\Big]\Big|^2\|_{\mathscr{C}^{\tilde{\vartheta}}(\mathbf{Z}_n^d;e(6l))} &\lesssim \|T_{r-q}^n[\psi^n(\varphi_t^n(r))^2\|^2_{\mathscr{C}^{\vartheta}(\mathbf{Z}_n^d;e(3l))}\\ &\lesssim (r-q)^{-(\vartheta+3\varepsilon)}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))},\end{aligned}$$

where the last step follows similarly to (II.17). Overall we thus obtain the estimate:

$$\begin{aligned}\int_0^r T_q^n\Big(n^{-\varrho}\Big|\nabla^n\Big[T_{r-q}^n[\psi^n(\varphi_t^n(r))^2]\Big]\Big|^2\Big)(0)\,\mathrm{d}q \\ \lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}\int_0^r (r-q)^{-(\vartheta+3\varepsilon)} \lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}\,\mathrm{d}q.\end{aligned}$$

Following the same steps, one can treat the second term in the integral in (II.16). We now use the same parameter ε both for the regularity of $n^{-\varrho}|\xi^n|$ and of ψ^n, in view of Assumption II.2.4, and choose ϑ,ε as above with the additional constraint $\vartheta+5\varepsilon<1$. Then we can argue as follows:

$$\|n^{-\varrho}|\xi^n|(T_q^n[\psi^n(\varphi_t^n(r))^2])^2\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;e(2l)p(a))} \lesssim q^{-(\vartheta+3\varepsilon)}\|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}$$

and hence:

$$\begin{aligned}\int_0^r T_q^n(n^{-\varrho}|\xi^n|(T_q^n[\psi^n(\varphi_t^n(r))^2])^2)(0)\,\mathrm{d}q \\ \lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))}\int_0^r (r-q)^{-(\vartheta+3\varepsilon)}q^{-2\varepsilon}\,\mathrm{d}q \\ \lesssim \|\psi^n\|^2_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}_n^d;p(a))},\end{aligned}$$

where in the last step we used that $\vartheta+5\varepsilon<1$. This concludes the proof. □

Our first main result, the law of large numbers, is now an easy consequence.

Proof of Theorem II.2.10. Recall that now we assume $\varrho>d/2$. In view of Corollary II.4.3 we can assume that along a subsequence $\mu^{n_k}\Rightarrow\mu$ in distribution in $\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d))$. It thus suffices to prove that $\mu=w$. The previous lemma shows that for $\varphi\in C_c^\infty(\mathbf{R}^d)$ the process $s\mapsto\mu(s)(T_{t-s}\varphi)-T_t\varphi(0)$ is a continuous centered square integrable martingale with vanishing quadratic variation. Hence, it is constantly zero and $\mu(t)(\varphi)=T_t\varphi(0)=(T_t\delta_0)(\varphi)$ almost surely for each fixed $t\geq 0$. Note that $T_\cdot\delta_0$ is well-defined, as explained in Remark II.3.2. Since μ is continuous, the identity holds almost surely for all $t>0$. The identity $\mu(t)=T_t\delta_0$ then follows by choosing a countable separating set of smooth functions in $C_c^\infty(\mathbf{R}^d)$. □

Now we pass to the case $\varrho=d/2$. To deduce the weak convergence of the sequence μ^n we have to prove that the distribution of the limit points is unique. For that purpose we first introduce a duality principle for the Laplace transform of our measure-valued process, for which we have to study Equation (II.4). We will consider mild solutions, i.e. φ solves (II.4) if and only if

$$\varphi(t)=T_t\varphi_0-\frac{\kappa}{2}\int_0^t \mathrm{d}s\; T_{t-s}(\varphi(s)^2).$$

We shall denote the solution by $\varphi(t)=U_t\varphi_0$, which is justified by the following existence and uniqueness result:

Proposition II.4.5. *Consider $T, \kappa > 0$, $l_0 < -T$ and $\varphi_0 \in C^\infty(\mathbf{R}^d, e(l_0))$ with $\varphi_0 \geq 0$. For $l = l_0 + T$ and ϑ as in Proposition II.3.1 there is a unique mild solution $\varphi \in \mathscr{L}^\vartheta(\mathbf{R}^d; e(l))$ to Equation* (II.4)*:*

$$\partial_t \varphi = \mathscr{H}\varphi - \frac{\kappa}{2}\varphi^2, \qquad \varphi(0) = \varphi_0.$$

We write $U_t\varphi_0 := \varphi(t)$ and we have the following bounds:

$$0 \leq U_t\varphi_0 \leq T_t\varphi_0, \quad \|\{U_t\varphi_0\}_{t\in[0,T]}\|_{\mathscr{L}^\vartheta(\mathbf{R}^d;e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbf{R}^d;e(l))}}.$$

Proof. We define the map $\mathscr{I}(\psi) = \varphi$, where φ is the solution to

$$\partial_t \varphi = \Big(\mathscr{H} - \frac{\kappa}{2}\psi\Big)\varphi, \qquad \varphi(0) = \varphi_0.$$

If $l_0 < -T$, then $(T_t\varphi_0)_{t\in[0,T]} \in \mathscr{L}^\vartheta(\mathbf{R}^d; e(l))$ for $l = l_0 + T$, and thus a slight adaptation of the arguments for Proposition II.3.1 shows that $\mathscr{I}$ satisfies

$$\mathscr{I} : \mathscr{L}^\vartheta(\mathbf{R}^d; e(l)) \to \mathscr{L}^\vartheta(\mathbf{R}^d; e(l)), \qquad \|\mathscr{I}(\psi)\|_{\mathscr{L}^\vartheta(\mathbf{R}^d;e(l))} \lesssim e^{C\|\psi\|_{CL^\infty(\mathbf{R}^d;e(l))}}$$

for some $C > 0$. Moreover, for positive ψ this map satisfies the bound $0 \leq \mathscr{I}(\psi)(t) \leq T_t\varphi_0$, so in particular we can bound $\|\mathscr{I}(\psi)\|_{CL^\infty(\mathbf{R}^d;e(l))} \leq \|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbf{R}^d;e(l))}$. Now, define $\varphi^0(t,x) = T_t\varphi_0(x)$ and then iteratively $\varphi^m = \mathscr{I}(\varphi^{m-1})$ for $m \geq 1$. This means that φ^m solves the equation:

$$\partial_t \varphi^m = \mathscr{H}\varphi - \frac{\kappa}{2}\varphi^{m-1}\varphi^m.$$

Hence our a-priori bounds guarantee that

$$\sup_m \|\varphi^m\|_{\mathscr{L}^\vartheta(\mathbf{R}^d;e(l))} \lesssim e^{C\|\{T_t\varphi_0\}_{t\in[0,T]}\|_{CL^\infty(\mathbf{R}^d;e(l))}}.$$

By compact embedding of $\mathscr{L}^\vartheta(\mathbf{R}^d; e(l)) \subset \mathscr{L}^\zeta(\mathbf{R}^d; e(l'))$ for $\zeta < \vartheta$, $l' < l$ we obtain convergence of a subsequence in the latter space. The regularity ensures that the limit point is indeed a solution to Equation (II.4). The uniqueness of such a fixed-point follows from the fact that the difference $z = \varphi - \psi$ of two solutions φ and ψ solves the well posed linear equation: $\partial_t z = \big(\mathscr{H} + \frac{\kappa}{2}(\varphi + \psi)\big)z$ with $z(0) = 0$, and thus $z(t) = 0$, $\forall t \geq 0$. □

We proceed by proving some implications between Properties $(i)-(iii)$ of Definition II.2.11.

Lemma II.4.6. *In Definition II.2.11 the following implications hold between the three properties:*

$$(ii) \Rightarrow (i), \qquad (ii) \Leftrightarrow (iii).$$

Proof. $(ii) \Rightarrow (i)$: Consider $U_\cdot\varphi_0$ as in point (i) of Definition II.2.11, which is well defined in view of Proposition II.4.5. An application of Itô's formula and Property (ii) of Definition II.2.11 with $\varphi_t(s) = U_{t-s}\varphi_0$, guarantee that for any $F \in C^2(\mathbf{R})$, and for $f(r) = \frac{\kappa}{2}(U_{t-r}\varphi_0)^2$:

$$\begin{aligned}F(\langle\mu(t),\varphi_0\rangle) = F(\langle\mu(s),U_{t-s}\varphi_0\rangle) + \int_s^t F'(\langle\mu(r),U_{t-r}\varphi_0\rangle)\langle\mu(r),f(r)\rangle\,\mathrm{d}r\\ +\frac{1}{2}\int_s^t F''(\langle\mu(r),U_{t-r}\varphi_0\rangle)\,\mathrm{d}\langle M_t^{\varphi_0,f}\rangle_r + \int_s^t F'(\langle\mu(r),U_{t-r}\varphi_0\rangle)\,\mathrm{d}M_t^{\varphi_0,f}(r),\end{aligned}$$

where $\mathrm{d}\langle M_t^{\varphi_0,f}\rangle_r = \langle\mu(r),\kappa(U_{t-r}\varphi_0)^2\rangle\,\mathrm{d}r = \langle\mu(r),2f(r)\rangle\,\mathrm{d}r$. We apply this for $F(x) = e^{-x}$, so that $F'' = -F'$ and the two Lebesgue integrals cancel. Since F' is bounded for positive x the stochastic integral is a true martingale and we deduce property (i).

$(ii) \Rightarrow (iii)$: Let $\varphi \in \mathscr{D}_{\mathscr{H}}$ and $t > 0$ and let $0 = t_0^n \le t_1^n \le \ldots \le t_n^n = t$, $n \in \mathbf{N}$, be a sequence of partitions of $[0,t]$ with $\max_{k\le n-1}\Delta_k^n := \max_{k\le n-1}(t_{k+1}^n - t_k^n) \to 0$. Then

$$\begin{aligned}&\langle\mu(t),\varphi\rangle - \langle\mu(0),\varphi\rangle\\ &\quad= \sum_{k=0}^{n-1}\Big[\Big(\langle\mu(t_{k+1}^n),\varphi\rangle - \langle\mu(t_k^n),T_{\Delta_k^n}\varphi\rangle\Big) + \langle\mu(t_k^n),T_{\Delta_k^n}\varphi-\varphi\rangle\Big]\\ &\quad= \sum_{k=0}^{n-1}\Big[\Big(M_{t_{k+1}^n}^{\varphi,0}(t_{k+1}^n) - M_{t_{k+1}^n}^{\varphi,0}(t_k^n)\Big) + \Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n}\rangle\Big].\end{aligned}$$

We start by studying the second term on the right-hand side:

$$\begin{aligned}&\sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n}\rangle\\ &\qquad= \sum_{k=0}^{n-1}\Big[\Delta_k^n\langle\mu(t_k^n),\frac{T_{\Delta_k^n}\varphi-\varphi}{\Delta_k^n} - \mathscr{H}\varphi\rangle + \Delta_k^n\langle\mu(t_k^n),\mathscr{H}\varphi\rangle\Big]\\ &\qquad=: R_n + \sum_{k=0}^{n-1}\Delta_k^n\langle\mu(t_k^n),\mathscr{H}\varphi\rangle.\end{aligned}$$

By continuity of μ the second term on the right-hand side of the latter equation converges almost surely to the Riemann integral $\int_0^t\langle\mu(r),\mathscr{H}\varphi\rangle\,\mathrm{d}r$. Moreover, from the characterization (ii) we get $\mathbb{E}[\mu(s)(\psi)] = \langle\mu(0),T_s\psi\rangle$ and

$$\mathbb{E}[\mu(s)(\mathscr{H}\varphi)^2] \lesssim \langle\mu(0),(T_s(\mathscr{H}\varphi))^2\rangle + \int_0^s\langle\mu(0),T_r\Big[(T_{s-r}\mathscr{H}\varphi)^2\Big]\rangle\,\mathrm{d}r,$$

which is uniformly bounded in $s \in [0,t]$. So the sequence is uniformly integrable and converges also in L^1 and not just almost surely. Moreover,

$$\mathbb{E}[|R_n|] \lesssim \sum_{k=0}^{n-1}\Delta_k^n\Big\langle\mu_0,T_{t_k^n}(|(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi-\varphi)-\mathscr{H}\varphi|)\Big\rangle,$$

and since Lemma II.3.6 implies that $\max_{k\le n-1}(\Delta_k^n)^{-1}(T_{\Delta_k^n}\varphi-\varphi)$ converges to $\mathscr{H}\varphi$ in $\mathscr{C}^\vartheta(\mathbf{R}^d;e(l))$ for some $l \in \mathbf{R}$ and $\vartheta > 0$ (so in particular uniformly), it follows from Proposition II.3.1

and the assumption $\langle \mu_0, e(l)\rangle < \infty$ for all $l \in \mathbf{R}$ that $\mathbb{E}[|R_n|] \to 0$. Thus, we showed that

$$\begin{aligned} L_t^{\varphi} &= \langle \mu(t), \varphi\rangle - \langle \mu(0), \varphi\rangle - \int_0^t \langle \mu(r), \mathscr{H}\varphi\rangle \,\mathrm{d}r \\ &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \Big(M^{\varphi,0}_{t^n_{k+1}}(t^n_{k+1}) - M^{\varphi,0}_{t^n_{k+1}}(t^n_k) \Big), \end{aligned}$$

and the convergence is in L^1. By taking partitions that contain $s \in [0,t)$ and using the martingale property of $M_r^{\varphi,0}$ we get $\mathbb{E}[L^{\varphi}(t)|\mathscr{F}_s] = L^{\varphi}(s)$, i.e. L^{φ} is a martingale. By the same arguments that we used to show the uniform integrability above, $L^{\varphi}(t)$ is square integrable for all $t > 0$. To derive the quadratic variation we use again a sequence of partitions containing $s \in [0,t)$ and obtain

$$\begin{aligned} \mathbb{E}\Big[L^{\varphi}(t)^2 - L^{\varphi}(s)^2 \Big| \mathscr{F}_s\Big] &= \mathbb{E}\Big[(L^{\varphi}(t) - L^{\varphi}(s))^2 \Big| \mathscr{F}_s\Big] \\ &= \lim_{n\to\infty} \sum_{k: t^n_{k+1} > s} \mathbb{E}\Big[\Big(M^{\varphi,0}_{t^n_{k+1}}(t^n_{k+1}) - M^{\varphi,0}_{t^n_{k+1}}(t^n_k)\Big)^2 \Big| \mathscr{F}_s\Big] \\ &= \lim_{n\to\infty} \sum_{k: t^n_{k+1} > s} \mathbb{E}\Big[\kappa \int_{t^n_k}^{t^n_{k+1}} \langle \mu(r), (T_{t^n_{k+1}-r}\varphi)^2\rangle \,\mathrm{d}r \Big| \mathscr{F}_s\Big] \\ &= \mathbb{E}\Big[\kappa \int_s^t \langle \mu(r), \varphi^2\rangle \,\mathrm{d}r \Big| \mathscr{F}_s\Big]. \end{aligned}$$

Since the process $\kappa \int_0^{\cdot} \langle \mu(r), \varphi^2\rangle \mathrm{d}r$ is increasing and predictable, it must be equal to $\langle L^{\varphi}\rangle$.

$(iii) \Rightarrow (ii)$: Let $t \geq 0$, $\varphi_0 \in \mathscr{D}_{\mathscr{H}}$, and let $f\colon [0,t] \to \mathscr{D}_{\mathscr{H}}$ be a piecewise constant function in time (it might seem more natural to take f continuous, but since we did not equip $\mathscr{D}_{\mathscr{H}}$ with a topology this has no clear meaning). We write φ for the solution to the backward equation

$$(\partial_s + \mathscr{H})\varphi = f, \qquad \varphi(t) = \varphi_0,$$

which is given by $\varphi(s) = T_{t-s}\varphi_0 + \int_s^t T_{r-s} f(r)\mathrm{d}r$. Note that by assumption $\varphi(r) \in \mathscr{D}_{\mathscr{H}}$ for all $r \leq t$. For $0 \leq s \leq t$, let $0 = t_0^n \leq t_1^n \leq \ldots \leq t_n^n = s$, $n \in \mathbf{N}$, be a sequence of partitions of $[0,s]$ with $\max_{k\leq n-1} \Delta_k^n := \max_{k\leq n-1}(t^n_{k+1} - t^n_k) \to 0$. Similarly to the computation in the step "$(ii) \Rightarrow (iii)$" we can decompose:

$$\begin{aligned} &\langle \mu(s), \varphi(s)\rangle - \langle \mu(0), \varphi(0)\rangle = \\ &\quad = \sum_{k=0}^{n-1} \Big[L^{\varphi(t^n_{k+1})}(t^n_{k+1}) - L^{\varphi(t^n_{k+1})}(t^n_k) + \int_{t^n_k}^{t^n_{k+1}} \langle \mu(r), f(r)\rangle \,\mathrm{d}r \Big] + R_n, \end{aligned}$$

with

$$\begin{aligned} R_n = \sum_{k=0}^{n-1} \int_{t^n_k}^{t^n_{k+1}} \Big[&\langle \mu(r), \mathscr{H}\varphi(t^n_{k+1})\rangle - \langle \mu(t^n_k), (\Delta^n_k)^{-1}(T_{\Delta^n_k} - \mathrm{id})\varphi(t^n_{k+1})\rangle \\ &+ \langle \mu(t^n_k), T_{r-t^n_k} f(r)\rangle - \langle \mu(r), f(r)\rangle \Big] \mathrm{d}r. \end{aligned}$$

As before we see that R_n converges to zero in L^1, and therefore

$$s \mapsto \langle \mu(s), \varphi(s)\rangle - \langle \mu(0), \varphi(0)\rangle - \int_0^s \langle \mu(r), f(r)\rangle \, \mathrm{d}r$$

is a martingale. Square integrability and the right form of the quadratic variation are shown again by similar arguments as in the previous step.

By density of $\mathscr{D}_{\mathscr{H}}$ it follows that $M_t^{\varphi_0, f}$ is a martingale on $[0,t]$ with the required quadratic variation for any $\varphi_0 \in C_c^\infty(\mathbf{R}^d)$ and $f \in C([0,t]; \mathscr{C}^\zeta(\mathbf{R}^d))$ for $\zeta > 0$. This concludes the proof. □

Characterization (i) of Definition II.2.11 enables us to deduce the uniqueness in law and then to conclude the proof of the equivalence of the different characterizations in Definition II.2.11.

Proof of Lemma II.2.12. First, we claim that uniqueness in law follows from Property (i) of Definition II.2.11. Indeed, we have for $0 \le s \le t$ and $\varphi \in C_c^\infty(\mathbf{R}^d)$, $\varphi \ge 0$ that $\mathbb{E}\Big[e^{-\langle \mu(t), \varphi\rangle} \Big| \mathscr{F}_s\Big] = e^{-\langle \mu(s), U_{t-s}\varphi\rangle}$. For $s = 0$ we can use the Laplace transform and the linearity of $\varphi \mapsto \langle \mu(t), \varphi\rangle$ to deduce that the law of $(\langle \mu(t), \varphi_1\rangle, \dots, \langle \mu(t), \varphi_n\rangle)$ is uniquely determined by (i) whenever $\varphi_1, \dots, \varphi_n$ are positive functions in $C_c^\infty(\mathbf{R}^d)$. By a monotone class argument (cf. [DMS93, Lemma 3.2.5]) the law of $\mu(t)$ is unique. We then see inductively that the finite-dimensional distributions of $\mu = \{\mu(t)\}_{t \ge 0}$ are unique, and thus that the law of μ is unique.

It remains to show the implication $(i) \Rightarrow (ii)$ to conclude the proof of the equivalence of the characterizations in Definition II.2.11. But we showed in Lemma II.4.4 that there exists a process satisfying (ii), and in Lemma II.4.6 we showed that then it must also satisfy (i). And since we just saw that there is uniqueness in law for processes satisfying (i) and since Property (ii) only depends on the law and it holds for one process satisfying (i), it must hold for all processes satisfying (i) (strictly speaking Lemma II.4.4 only gives the existence for $\kappa = 2\nu \in (0,1]$, but see Section II.5.1 below for general κ). □

Now the convergence of the sequence $\{\mu^n\}_{n \in \mathbf{N}}$ is an easy consequence:

Proof of Theorem II.2.13. This follows from the characterization of the limit points from Lemma II.4.4 together with the uniqueness result from Lemma II.2.12. □

II.5 Variations on the theme

II.5.1 Mixing with a classical superprocess

In Section II.4.1 we constructed the rSBM of parameter $\kappa = 2\nu$, for ν defined via Assumption II.2.1 which leads to the restriction $\nu \in (0, \frac{1}{2}]$. This section is devoted to constructing the rSBM for arbitrary $\kappa > 0$. We do so by means of an interpolation between the rSBM and a Dawson-Watanabe superprocess (cf. [Eth00, Chapter 1]). Let Ψ be the generating function of a discrete finite positive measure $\Psi(s) = \sum_{k \ge 0} p_k s^k$ (so $p_k \ge 0$ and $\sum_k p_k < \infty$)

and let ξ^n be a controlled random environment associated to a parameter $\nu = \mathbb{E}[\Phi_+]$ (recall Notation II.2.6). We consider the quenched generator:

$$\mathscr{L}_{\psi}^{n,\omega}(F)(\eta) = \sum_{x\in\mathbf{Z}_n^d} \eta_x \cdot \Big[\Delta^n F(\eta) + (\xi_e^n)_+(\omega,x) d_x^1 F(\eta) + (\xi_e^n)_-(\omega,x) d_x^{-1} F(\eta) + n^{\varrho} \sum_{k\geq 0} p_k d_x^{(k-1)} F(\eta)\Big]$$

with the notation $d_x^k F(\eta) = F(\eta^{x;k}) - F(\eta)$, where for $k \geq -1$ we write $\eta^{x;k}(y) = (\eta(y) + k 1_{\{x\}}(y))_+$.

Assumption II.5.1 (On the Moment generating function). *We assume that $\Psi'(1) = 1$ (critical branching, i.e. the expected number of offsprings in one branching/killing event is 1) and we write $\sigma^2 = \Psi''(1)$ for the variance of the offspring distribution.*

Now we introduce the associated process. The construction of the process $\bar{u}^n$ is analogous to the case without Ψ, which is treated in Appendix A.1.

Definition II.5.2. *Let $\varrho \geq d/2$ and let Ψ be a moment generating function satisfying the previous assumptions. Consider a controlled random environment ξ^n associated to a parameter $\nu \in (0, \frac{1}{2}]$. Let $\mathbb{P}_{\Psi}^{\mathrm{tot},n} = \mathbb{P} \ltimes \mathbb{P}_{\Psi}^{\omega,n}$ be the measure on $\Omega \times \mathbb{D}([0,+\infty); E)$ such that for fixed $\omega \in \Omega$, under the measure $\mathbb{P}_{\Psi}^{\omega,n}$ the canonical process on $\mathbb{D}([0,+\infty); E)$ is the Markov process $\bar{u}^n(\omega,\cdot)$ started in $\bar{u}^n(0) = \lfloor n^{\varrho} \rfloor 1_{\{0\}}(x)$ associated to the generator $\mathscr{L}_{\Psi}^{\omega,n}$ defined as above. To $\bar{u}^n$ we associate the measure valued process*

$$\langle \bar{\mu}^n(\omega,t), \varphi\rangle = \sum_{x\in\mathbb{Z}_n^d} \bar{u}^n(\omega,t,x)\varphi(x)\lfloor n^{\varrho}\rfloor^{-1}$$

for any bounded $\varphi\colon \mathbf{Z}_n^d \to \mathbf{R}$. With this definition $\bar{\mu}^n$ is defined on $\Omega \times \mathbb{D}([0,T]; \mathscr{M}(\mathbf{R}^d))$ with the law induced by $\mathbb{P}_{\Psi}^{\mathrm{tot},n}$.

Remark II.5.3. *As in Remark II.4.1 we see that for fixed $\omega \in \Omega$ and any $\varphi \in L^{\infty}(\mathbf{Z}_n^d; e(l))$ with $l \in \mathbf{R}$ the process $\bar{M}_t^{n,\varphi}(s) := \bar{\mu}_s^n(\omega)(T_{t-s}^n\varphi) - T_t^n\varphi(0)$ is a martingale (under $\mathbb{P}_{\Psi}^{\omega,n}$ and with respect to the filtration $\mathscr{F}^{\omega,n}$ generated by $\overline{\mu}^n(\omega,\cdot)$), with predictable quadratic variation:*

$$\langle \bar{M}_t^{n,\varphi}\rangle_s = \int_0^s \bar{\mu}_r^n(\omega)\Big(n^{-\varrho}|\nabla^n T_{t-r}^n\varphi|^2 + (n^{-\varrho}|\xi_e^n| + \sigma^2)(T_{t-r}^n\varphi)^2\Big)\mathrm{d}r.$$

In view of this Remark, we can follow the discussion of Section II.4.1 to deduce the following result (cf. Corollary II.2.16).

Proposition II.5.4. *The sequence of measures $\mathbb{P}_{\Psi}^{\mathrm{tot},n}$ as in Definition II.5.2 converge weakly as measures on $\Omega \times \mathbb{D}([0,T]; \mathscr{M}(\mathbf{R}^d))$ to the measure $\mathbb{P} \times \mathbb{P}_{\kappa}^{\omega}$ associated to a rSBM of parameter $\kappa = 1_{\{\varrho = \frac{d}{2}\}} 2\nu + \sigma^2$, in the sense of Theorem II.2.13 and Corollary II.2.16.*

In particular the rSBM is also the scaling limit of critical branching random walks whose branching rates are perturbed by small random potentials.

II.5.2 Killed rSBM

In this section we introduce a rSBM with Dirichlet boundary conditions. For arbitrary $L \in 2\mathbf{N}$, we want to consider particles that are spatially distributed on the lattice

$$\Lambda_n^L = \{x \in \mathbf{Z}_n^d \ : \ x \in [-L/2, L/2]^d\}.$$

We will make use of the tools for discrete paracontrolled calculus with Dirichlet boundary conditions from Section IV.3, where for convenience we used the lattice $\widetilde{\Lambda}_n^L = \{x \in \mathbf{Z}_n^d \ : \ x \in [0,L]^d\}$. Of course, since $L \in 2\mathbf{N}$, up to translation the two lattices are the same and all the results of that section remain valid. Define in addition the space of functions $E^L = \left\{\eta \in \mathbf{N}_0^{\Lambda_n^L} \ : \ \eta(x) = 0, \forall x \in \partial\Lambda_n^L\right\}$, while previously we worked only with $E = \left(\mathbf{N}_0^{\mathbf{Z}_n^d}\right)_0$. We work in the following framework.

Assumption II.5.5. *Let $\{\xi^n\}$ be a random environment on a probability $(\Omega, \mathscr{F}, \mathbb{P})$ satisfying Assumption II.2.1. Assume in addition (up to changing probability space: see Lemmata II.2.5 and IV.3.3) that both Assumption II.2.4 and the results of Lemma IV.3.3 are satisfied. Then define:*

$$\xi_e^n(\omega, x) = \xi^n(\omega, x) - c_n(\omega) 1_{\{d=2\}}.$$

Let $u^n(\omega, t, x)$ be the (random) stochastic process constructed in II.2.7, in the case

$$\varrho = \frac{d}{2},$$

and let $\mu^n(\omega, t)$ be the measure associated to it. Such process lives on:

$$\left(\Omega \times \mathbb{D}([0,\infty); E), \mathscr{F}, \mathbb{P} \ltimes \mathbb{P}^{\omega,n}\right),$$

where $\mathbb{P}^{\omega,n}$ is the quenched law of u^n, conditional on the environment $\xi^n(\omega)$, for $\omega \in \Omega$.

Observe that previously any $\varrho \geq \frac{d}{2}$ was allowed. We restrict to $\varrho = \frac{d}{2}$, since we are only interested in the fluctuations.

The process u^n does not keep track of the individual particles (all particles are identical, only their position matters). Instead, consider a labeled process that distinguishes individual particles in which all particles that leave the box $(-L/2, L/2)^d$ are killed. Hence introduce the space $E_{\text{lab}}^n = \bigsqcup_{m \in \mathbf{N}} (\frac{1}{n}\mathbf{Z}^d \cup \{\Delta\})^m$, where $\bigsqcup$ denotes the disjoint union, endowed with the discrete topology. Here Δ is a cemetery state. For $\eta \in E_{\text{lab}}^n$ we write $\dim(\eta) = m$ if $\eta \in (\frac{1}{n}\mathbf{Z}^d \cup \{\Delta\})^m$. A rigorous construction of the process below follows as in Appendix A.1.

Definition II.5.6. *Fix $\omega \in \Omega$ and $X_0^n \in E_{\text{lab}}^n$ with $\dim(X_0^n) = \lfloor n^\varrho \rfloor, (X_0^n)_i = 0, i = 1 \dots \lfloor n^\varrho \rfloor$. Let $X^n(\omega)$ be the Markov jump process on E_{lab}^n with initial condition $X^n(0) = X_0^n$ and with generator:*

$$\begin{aligned}\mathscr{L}_{\text{lab}}^\omega(F)(\eta) = \sum_{i=1}^{\dim(\eta)} 1_{\{\frac{1}{n}\mathbf{Z}^d\}}(\eta_i) \Big[\sum_{|y-\eta_i|=n^{-1}} \left(F(\eta^{i\mapsto y}) - F(\eta)\right) \\ + (\xi^n)_+(\omega, \eta_i)\left(F(\eta^{i,+}) - F(\eta)\right) + (\xi^n)_-(\omega, \eta_i)\left(F(\eta^{i,-}) - F(\eta)\right)\Big],\end{aligned}$$

where

$$\eta_j^{i\mapsto y} = \eta_j(1-1_{\{i\}}(j))+y1_{\{i\}}(j), \qquad \eta_j^{i,+} = \eta_j 1_{[0,\dim(\eta)]}(j)+\eta_i 1_{\{\dim(\eta)+1\}}(j)$$

as well as $\eta_j^{i,-} = \eta_j(1-1_{\{i\}}(j))+\Delta 1_{\{i\}}(j)$, *on the domain* $\mathscr{D}(\mathscr{L}^{\omega}_{\mathrm{lab}})$ *of functions* $F \in C_b(E^n_{\mathrm{lab}})$ *is such that the right hand-side is lies in* $C_b(E^n_{\mathrm{lab}})$. *We can then redefine the process*

$$u^n(\omega,t,x) = \#\{i \in \{1,\dots,\dim(X^n(\omega,t))\} \,:\, X^n_i(\omega,t) = x\}$$

which has the same quenched law $\mathbb{P}^{\omega,n}$ *as the process above.*

Similarly, for $i \in \mathbf{N}$ *consider stopping times*

$$\tau_i^{n,L}(\omega) = \inf\{t \geq 0 \colon \dim(X^n(\omega,t)) \geq i \text{ and } X^n_i(t) \in \partial\Lambda^L_n\},$$

so that we can define $X^{n,L}(\omega,t) \in E^n_{\mathrm{lab}}$ *by* $\dim(X^{n,L}(\omega,t)) = \dim(X^n(\omega,t))$ *and*

$$X_i^{n,L}(\omega,t) = X^n_i(\omega,t)1_{\{t<\tau_i^{n,L}(\omega)\}} + \Delta 1_{\{\tau_i^{n,L}(\omega)\leq t\}}.$$

Then as before construct $u^{n,L}$ *taking values in* E^L *by*

$$u^{n,L}(\omega,t,x) = \#\{i \in \{1,\dots,\dim(X^{n,L}(\omega,t))\} \,:\, X^{n,L}_i(\omega,t) = x\}, \qquad \forall t \geq 0, x \in \mathbf{Z}^d_n.$$

Write $\mathscr{M}((-L/2,L/2)^d)$ for the set of all finite positive measures on $(-L/2,L/2)^d$ and for μ,ν in this space we say $\mu \geq \nu$ if also $\mu-\nu$ is a *positive* measure. The following result is now easy to verify (cf. Appendix A.1).

Lemma II.5.7. *For any* $\omega \in \Omega$ *and* $L \in 2\mathbf{N}$ *the process* $t \mapsto u^{n,L}(\omega,t,\cdot)$ *is a Markov process with paths in* $\mathbb{D}([0,+\infty);E^L)$, *associated to the generator* $\mathscr{L}^{n,\omega}_L \colon C_b(E^L) \to C_b(E^L)$ *defined via:*

$$\begin{aligned}\mathscr{L}^{n,\omega}_L(F)(\eta) = \sum_{x\in\Lambda^L_n\setminus\partial\Lambda^L_n} \eta_x \cdot \Bigg[&\sum_{x\sim y} n^2(F(\eta^{x\mapsto y})-F(\eta)) \\ &+(\xi^n_e)_+(\omega,x)[F(\eta^{x+})-F(\eta)] + (\xi^n_e)_-(\omega,x)[F(\eta^{x-})-F(\eta)]\Bigg],\end{aligned}$$

where for $\eta \in E^L$ *we define* $\eta^{x\mapsto y}(z) = (\eta(z)-1_{\{z=x\}}+1_{\{z=y,\ y\notin\partial\Lambda^L_n\}})_+$ *and* $\eta^{x\pm}(z) = (\eta(z)\pm 1_{\{z=x\}})_+$. *We associate to* $u^{n,L}(\omega,t)$ *a measure:*

$$\mu^{n,L}(\omega,t)(\varphi) = \sum_{x\in\Lambda^L_n} \lfloor n^{-\varrho}\rfloor u^{n,L}(\omega,t,x)\varphi(x), \qquad \forall \varphi \in C((-L/2,L/2)^d). \tag{II.18}$$

Finally:

$$\mu^{n,L}(\omega,t) \leq \mu^{n,L+2}(\omega,t) \leq \cdots \leq \mu^n(\omega,t) \qquad \forall \omega\in\Omega, t\geq 0. \tag{II.19}$$

When studying the convergence of the process $\mu^{n,L}$, special care has to be taken with regard to what happens on the boundary of the box. Indeed a function $\varphi \in C^\infty([-L/2,L/2]^d)$ (i.e. smooth in the interior with all derivatives continuous on the entire box) is not smooth in the scale of spaces $B^{\mathfrak{l},\alpha}_{p,q}$ for $\mathfrak{l} \in \{\mathfrak{d},\mathfrak{n}\}$ that are introduced in Section IV.3 and encode, respectively, Dirichlet and Neumann boundary conditions. In fact such φ does not satisfy the required boundary conditions. For this reason we consider only vague convergence for the processes $\mu^{n,L}$. We write

$$\mathscr{M}_0^L = \Big(\mathscr{M}((-L/2,L/2)^d),\tau_v\Big)$$

for the set of finite positive measures on $(-L/2,L/2)^d$ endowed with the vague topology τ_v (cf. [DMS93, Section 3]), i.e. $\mu^n \to \mu$ in $\mathscr{M}_0^L$ if $\mu^n(\varphi)\to\mu(\varphi)$, for all $\varphi \in C_0((-L/2,L/2)^d)$, the space of continuous functions that vanish on the boundary of the box (the latter is a Banach space, when endowed with the uniform norm). This topology is convenient because sets of the form $K_R \subset \mathscr{M}_0^L$, with $K_R = \{\mu \in \mathscr{M}_0^L \ : \ \mu(1) \le R\}$ are compact. The observation below now follows from a short calculation.

Remark II.5.8. *For $\alpha > 0$ there is a continuous embedding of Banach spaces*

$$\mathscr{C}^\alpha_{\mathfrak{d}}([-L/2,L/2]^d) \hookrightarrow C_0((-L/2,L/2)^d).$$

Moreover, if $\{\mu^n\}_{n\in\mathbf{N}} \subseteq \mathscr{M}_0^L$ satisfies that for some $R>0$, $\{\mu^n\}_{n\in\mathbf{N}} \subseteq R$, then $\mu^n \to \mu$ in $\mathscr{M}_0^L$ is equivalent to:

$$\mu^n(\varphi)\to\mu(\varphi), \qquad \forall \varphi \in C_c^\infty((-L/2,L/2)^d).$$

We study the convergence of the killed process. First observe that one can bound its total mass locally uniformly in time.

Lemma II.5.9. *For all $\omega\in\Omega$ it holds that:*

$$\lim_{R\to\infty}\sup_n \mathbb{P}^{\omega,n}\Big(\sup_{t\in[0,T]}\mu^{n,L}(\omega,t)(1)\ge R\Big)=0, \qquad \sup_n\sup_{t\in[0,T]}\|T_t^{n,\mathfrak{d},L,\omega}1\|_\infty<+\infty.$$

Proof. The first bound follows from comparison with the process on the whole real line (i.e. Equation (II.19)), see Corollary II.4.3. The second bound follows from Theorem IV.3.4 because the antisymmetric extension of 1 is bounded: $|\Pi_o 1(\cdot)|\equiv 1$. Hence by comparison and the discussion preceding Equation (IV.30): $\|T_t^{n,\mathfrak{d},L,\omega}1\|_\infty \le \|\widetilde{w}(t)\|_\infty$, with $\widetilde{w}$ solving:

$$\partial_t\widetilde{w} = \Delta^n\widetilde{w} + \Pi_e(\xi^n(\omega) - c_n(\omega)1_{\{d=2\}})\widetilde{w}, \qquad \widetilde{w}(0)\equiv 1.$$

□

Lemma II.5.10. *For every $\omega\in\Omega$ the sequence $\{t\mapsto\mu^{n,L}(\omega,t)\}_{n\in\mathbf{N}}$ is tight in the space $\mathbb{D}([0,\infty);\mathscr{M}_0^L)$. Any limit point $\mu^L(\omega)$ lies in $C([0,\infty);\mathscr{M}_0^L)$.*

Proof. We want to apply Jakubowski's tightness criterion [DMS93, Theorem 3.6.4]. The sequence $\mu^{n,L}$ satisfies the compact containment condition in view of Lemma II.5.9. The tightness of the entire process is guaranteed if we prove that the sequence $\{t\mapsto\mu^{n,L}(t)(\varphi)\}_{n\in\mathbf{N}}$ is tight in $\mathbb{D}([0,T];\mathbf{R})$ for any $\varphi\in C_c^\infty((-L/2,L/2)^d)$. Here we can follow the calculation of Lemma II.4.2 (only simpler, since we do not need weights), using the results from Theorem IV.3.4. The continuity of the limit points is shown as in II.4.4.

□

One can characterize the limit points of $\{\mu^{n,L}\}_{n\in\mathbf{N}}$ in a similar way as the rough super-Brownian motion, and for that purpose we need to solve the following equation (for any $\omega \in \Omega, L \in 2\mathbf{N}$):

$$\partial_t \varphi = \mathscr{H}^{\omega}_{\mathfrak{d},L}\varphi - \nu\varphi^2, \ \ \varphi(0) = \varphi_0, \ \ \varphi(t,x) = 0, \ \forall (t,x) \in (0,T] \times \partial[-L/2, L/2]^d. \tag{II.20}$$

We say that φ is a solution to (II.20) if

$$\varphi(t) = T^{\mathfrak{d},L,\omega}_t \varphi_0 - \nu \int_0^t T^{\mathfrak{d},L,\omega}_{t-s}[\varphi^2(s)]\,\mathrm{d}s.$$

Lemma II.5.11. *Fix $\omega \in \Omega, L \in 2\mathbf{N}$. For $T > 0$ and $\varphi_0 \in C^\infty_c((-L/2, L/2)^d)$ with $\varphi_0 \geq 0$ and ϑ as in Theorem IV.3.4, there exists a unique solution $\varphi \in \mathscr{L}^{\vartheta}_{\mathfrak{d}}([-L/2, L/2]^d)$ to* (II.20) *and the following bounds hold:*

$$0 \leq \varphi(t) \leq T^{\mathfrak{d},L,\omega}_t \varphi_0, \qquad \|\varphi\|_{\mathscr{L}^{\vartheta}_{\mathfrak{d}}([-L/2,L/2]^d)} \lesssim e^{C\|\{T^{\mathfrak{d},L,\omega}_t \varphi_0\}_{t\in[0,T]}\|_{CL^\infty([-L/2,L/2]^d)}}.$$

The proof is analogous to the one of Proposition II.4.5. As before we write $U^{\mathfrak{d},L,\omega}_t \varphi_0$ for the solution $\varphi(t)$ to Equation (II.20) started in φ_0. We thus arrive at the following description of the limit points of $\{\mu^{n,L}\}_{n\in\mathbf{N}}$.

Theorem II.5.12. *For any $\omega \in \Omega$ and $L \in 2\mathbf{N}$, under Assumption II.5.5, there exists $\mu^L(\omega) \in C(\mathbf{R}_{\geq 0}; \mathscr{M}^L_0)$ such that $\mu^{n,L}(\omega) \to \mu^L(\omega)$ in distribution in $\mathbb{D}(\mathbf{R}_{\geq 0}; \mathscr{M}^L_0)$. The process $\mu^L(\omega)$ is the unique (in law) process in $C(\mathbf{R}_{\geq 0}; \mathscr{M}^L_0)$ which satisfies one (and then all) of the following equivalent properties with $\mathscr{F}^\omega = \{\mathscr{F}^\omega_t\}_{t\geq 0}$ being the usual augmentation of the filtration generated by $\mu^L(\omega)$.*

(i) For any $t \geq 0$ and $\varphi_0 \in C^\infty_c((-L/2, L/2)^d), \varphi_0 \geq 0$ and for $U^{\mathfrak{d},L,\omega}_t \varphi_0$ the solution to Equation (II.20) *with initial condition φ_0 the process*

$$N^{\varphi_0}_t(s) = e^{-\langle \mu^L(\omega,s), U^{\mathfrak{d},L,\omega}_{t-s}\varphi_0\rangle}, \qquad s \in [0,t]$$

is a bounded continuous $\mathscr{F}^\omega$–martingale.

(ii) For any $\varphi \in \mathscr{D}_{\mathscr{H}^\omega_{\mathfrak{d},L}}$ the process:

$$K^\varphi(t) = \langle \mu^L(\omega,t), \varphi\rangle - \langle \delta_0, \varphi\rangle - \int_0^t \langle \mu^L(\omega,r), \mathscr{H}^\omega_{\mathfrak{d},L}\varphi\rangle\,\mathrm{d}r, \qquad t \in [0,T]$$

is a continuous $\mathscr{F}^\omega$–martingale, square integrable on $[0,T]$ for all $T > 0$, with quadratic variation

$$\langle K^\varphi\rangle_t = 2\nu \int_0^t \langle \mu^L(\omega,r), \varphi^2\rangle\,\mathrm{d}r.$$

Proof. The proof is almost identical to the one of Theorem II.2.13. The main difference is that here we only test against functions with zero boundary conditions and thus use the results from Section IV.3. □

We call the above process the killed rSBM on $(-\frac{L}{2},\frac{L}{2})^d$. Note that one can interpret the killed rSBM as an element of $C([0,\infty);\mathcal{M}(\mathbf{R}^d))$ extending it by zero, i.e. $\mu^L(\omega,t,A) = \mu^L(\omega,t,A\cap(-L/2,L/2)^d)$ for any measurable $A\subset\mathbf{R}^d$. This allows us to couple infinitely many killed rSBMs with a rSBM on $\mathbf{R}^d$ so that they are ordered in the natural way.

Corollary II.5.13. *For any $\omega\in\Omega$, under Assumption II.5.5, there exists a process*

$$(\mu(\omega,\cdot),\mu^2(\omega,\cdot),\mu^4(\omega,\cdot),\dots)$$

taking values in $C([0,\infty);\mathcal{M}(\mathbf{R}^d))^{\mathbf{N}}$ (equipped with the product topology) such that μ is an rSBM and μ^L is a killed rSBM for all $L\in 2\mathbf{N}$ (all associated to the environment $\{\xi^n\}_{n\in\mathbf{N}}$), and such that:

$$\mu^2(\omega,t,A)\le\mu^4(\omega,t,A)\le\cdots\le\mu(\omega,t,A) \tag{II.21}$$

for all $t\ge0$ and all Borel sets $A\subset\mathbf{R}^d$.

Proof. The construction (II.18) of μ^n and $\mu^{n,L}$ based on the labeled particle system gives us a coupling $(\mu^n,\mu^{n,2},\mu^{n,4},\dots)$ such that for all $\omega\in\Omega$

$$\mu^{n,2}(\omega,t,A)\le\mu^{n,4}(\omega,t,A)\le\cdots\le\mu^n(\omega,t,A)$$

for all $t\ge0$ and all Borel sets $A\subset\mathbf{R}^d$, where as above we extend $\mu^{n,L}$ to $\mathbf{R}^d$ by setting it to zero outside of $(-\frac{L}{2},\frac{L}{2})^d$ (cf. Equation (II.19)). By Theorem II.2.13 one obtains tightness of the finite-dimensional projections $(\mu^n,\mu^{n,2},\dots,\mu^{n,L})$ for $L\in2\mathbf{N}$, and this gives tightness of the whole sequence in the product topology. Moreover, for any subsequential limit $(\mu,\mu^2,\mu^4,\dots)$, μ is an rSBM and μ^L is a killed rSBM on $(-\frac{L}{2},\frac{L}{2})^d$. It is however a little subtle to obtain the ordering (II.21), because we only showed tightness in the vague topology on $\mathcal{M}_0^L$ for the $\mu^{n,L}$ component. So we introduce suitable cut-off functions to show that the ordering is preserved along any (subsequential) limit: let $\chi^m\in C_c^\infty((-L/2,L/2)^d)$, $\chi^m\ge0$ such that $\chi^m=1$ on a sequence of compact sets K^m which increase to $(-L/2,L/2)^d$ as $m\to\infty$. Note that on compact sets the sequence $\mu^{n,L}$ converges weakly (and not just vaguely). We then estimate (in view of Equation (II.19)) for $\varphi\in C_b(\mathbf{R}^d)$ with $\varphi\ge0$:

$$\begin{aligned}\langle\mu^L(t),\varphi\rangle=\lim_{m\to\infty}\langle\mu^L(t),\varphi\cdot\chi^m\rangle&=\lim_{m\to\infty}\lim_{n\to\infty}\langle\mu^{n,L}(t),\varphi\cdot\chi^m\rangle\\&\le\lim_{m\to\infty}\langle\mu(t),\varphi\cdot\chi^m\rangle=\langle\mu(t),\varphi\rangle,\end{aligned}$$

and similarly one obtains $\langle\mu^L(t),\varphi\rangle\le\langle\mu^{L'}(t),\varphi\rangle$ for $L\le L'$. Since a signed measure that has a positive integral against every positive continuous function must be positive, our claim follows. □

II.6 Properties of the rough superBrownian motion

II.6.1 Scaling Limit as SPDE in d=1

In this section we characterize the rSBM in dimension $d=1$ as the solution to the SPDE (II.5) in the sense of Definition II.2.18. For that purpose we first show that the random measure μ_p admits a density with respect to the Lebesgue measure.

Lemma II.6.1. *Let μ be a one-dimensional rSBM of parameter ν. For any $\beta < 1/2$, $p \in [1, 2/(\beta+1))$ and $l \in \mathbf{R}$, we have:*

$$\mathbb{E}\Big[\|\mu\|^p_{L^p([0,T];B^{\beta}_{2,2}(\mathbf{R};e(l)))}\Big] < \infty.$$

Proof. Consider $t > 0$ and $\varphi \in C^\infty_c(\mathbf{R})$. By Point (ii) of Definition II.2.11 the process $M^{\varphi}_t(s) = \langle \mu(s), T_{t-s}\varphi\rangle - \langle \mu(0), T_t\varphi\rangle$, $s \in [0,t]$, is a continuous square integrable martingale with quadratic variation $\langle M^{\varphi}_t\rangle_s = \int_0^s \langle \mu(r), (T_{t-r}\varphi)^2\rangle \mathrm{d}r$. Using the moment estimates of Lemma II.8.1, which by Fatou's lemma hold also for the limit μ of the sequence $\{\mu^n\}_{n\in\mathbf{N}}$, this martingale property extends to $\varphi \in \mathscr{C}^{\vartheta}(\mathbf{R}; e(k))$ for arbitrary $k \in \mathbf{R}$ and $\vartheta > 0$. In particular, for such φ we get

$$\mathbb{E}[\langle \mu(t), \varphi\rangle^2] \lesssim \int_0^t T_r((T_{t-r}\varphi)^2)(0)\,\mathrm{d}r + (T_t\varphi)^2(0).$$

Now note that

$$\mathbb{E}\Big[\|\mu(t)\|^2_{B^{\beta}_{2,2}(\mathbf{R};e(l))}\Big] = \sum_{j\geq -1} 2^{2j\beta}\int \mathbb{E}[\langle \mu(t), K_j(x-\cdot)\rangle^2] e^{-2l|x|^\sigma}\,\mathrm{d}x,$$

where $K_j(x) = \mathscr{F}^{-1}_{\mathbf{R}}\varrho_j(k)$ (so that for $j \geq 0:\ K_j(x) = 2^{jd}K_0(2^j x)$) so we apply the estimate above to $\varphi = K_j(\cdot - x)$:

$$\mathbb{E}[\langle \mu(t), K_j(x-\cdot)\rangle^2] \lesssim \int_0^t T_r((T_{t-r}K_j(x-\cdot))^2)(0)\mathrm{d}r + (T_tK_j(x-\cdot))^2(0). \tag{II.22}$$

We start by proving that $\|K_j(x-\cdot)\|_{\mathscr{C}^{\alpha}_1(\mathbf{R},e(k))} \lesssim 2^{j\alpha}e^{-k|x|^\sigma}$ for any $k > 0$. Indeed, using that K_i is an even function and writing $\tilde{K}_{i-j} = 2^{(i-j)d}K_0(2^{i-j}\cdot) * K_0$ if $i, j \geq 0$ (appropriately adapting the definition via K_{-1} if $i = -1$ or $j = -1$), we have:

$$\begin{aligned}
\|\Delta_i(K_j(x-\cdot))e(k)\|_{L^1(\mathbf{R})} &= 1_{\{|i-j|\leq 1\}}\int_{\mathbf{R}} |K_i * K_j(x-y)|e^{-k|y|^\sigma}\,\mathrm{d}y \\
&= 1_{\{|i-j|\leq 1\}}\int_{\mathbf{R}} |\tilde{K}_{i-j}(y)|e^{-k|x-2^{-j}y|^\sigma}\,\mathrm{d}y \\
&\lesssim 1_{\{|i-j|\leq 1\}}\int_{\mathbf{R}} |\tilde{K}_{i-j}(y)|e^{k|2^{-j}y|^\sigma - k|x|^\sigma}\,\mathrm{d}y \lesssim 1_{\{|i-j|\leq 1\}}e^{-k|x|^\sigma},
\end{aligned}$$

where in the last step we used that $|\tilde{K}_{i-j}(y)| \lesssim e^{-2k|y|^\sigma}$ and $2^{-j\sigma} \leq 2^\sigma < 2$.

Now, for $\zeta < 0$ satisfying the assumptions of Proposition II.3.1 and for $p \in [1, \infty]$ and sufficiently small $\varepsilon > 0$:

$$\begin{aligned}
\|T_sK_j(x-\cdot)\|_{\mathscr{C}^{\varepsilon}_p(\mathbf{R};e(k+s))} &\lesssim \|T_sK_j(x-\cdot)\|_{\mathscr{C}_1^{1-\frac{1}{p}+\varepsilon}(\mathbf{R};e(k+s))} \\
&\lesssim 2^{j\zeta}s^{(\zeta-1+\frac{1}{p}-2\varepsilon)/2}e^{-k|x|^\sigma}.
\end{aligned}$$

To control the first term on the right hand side of (II.22), we apply the last estimate with

$p = 2$ and obtain for $t \in [0,T]$ and $\zeta > -1/2$

$$\begin{aligned}\int_0^t T_r((T_{t-r}K_j(x-\cdot))^2)(0)\,\mathrm{d}r \\ &\lesssim \int_0^t \|T_r((T_{t-r}K_j(x-\cdot))^2)\|_{\mathscr{C}^{\varepsilon}_{\infty}(\mathbf{R};e(2k+T))}\,\mathrm{d}r \\ &\lesssim \int_0^t \|T_r((T_{t-r}K_j(x-\cdot))^2)\|_{\mathscr{C}^{1+\varepsilon}_{1}(\mathbf{R};e(2k+T))}\mathrm{d}r \\ &\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\|(T_{t-r}K_j(x-\cdot))^2\|_{\mathscr{C}^{\varepsilon}_{1}(\mathbf{R};e(2k))}\,\mathrm{d}r \\ &\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}\|T_{t-r}K_j(x-\cdot)\|^2_{\mathscr{C}^{\varepsilon}_{2}(\mathbf{R};e(k))}\mathrm{d}r \\ &\lesssim \int_0^t r^{-\frac{1+2\varepsilon}{2}}(2^{j\zeta}(t-r)^{\frac{\zeta-\frac{1}{2}-2\varepsilon}{2}}e^{-k|x|^{\sigma}})^2\mathrm{d}r \\ &\simeq 2^{2j\zeta}e^{-2k|x|^{\sigma}}t^{1-\frac{1+2\varepsilon}{2}+\zeta-\frac{1}{2}-2\varepsilon} = 2^{2j\zeta}e^{-2k|x|^{\sigma}}t^{\zeta-3\varepsilon},\end{aligned}$$

where we used that $\int_0^t r^{-\alpha}(t-r)^{-\beta}\,\mathrm{d}r \simeq t^{1-\alpha-\beta}$ for $\alpha,\beta < 1$. The second term on the right hand side of (II.22) is bounded by

$$\begin{aligned}(T_tK_j(x-\cdot))^2(0) &\lesssim \|(T_tK_j(x-\cdot))^2\|_{\mathscr{C}^{\varepsilon}_{\infty}(\mathbf{R};e(2k+2T))} \\ &\lesssim \|T_tK_j(x-\cdot)\|^2_{\mathscr{C}^{\varepsilon}_{\infty}(\mathbf{R};e(k+T))} \lesssim 2^{2j\zeta}t^{\zeta-1-2\varepsilon}e^{-2k|x|^{\sigma}}.\end{aligned}$$

Note that this estimate is much worse than the first one (because $t \in [0,T]$ is bounded above). We plug both those estimates into (II.22) and set $\zeta = -\beta-\varepsilon$ and $k > -l$ to obtain $\mathbb{E}\Big[\|\mu(t)\|^2_{B^{\beta}_{2,2}(\mathbf{R};e(l))}\Big] \lesssim t^{-\beta-1-3\varepsilon}$ for $\beta < 1/2$ and for $l \in \mathbf{R}$. So finally for $p \in [1,2)$

$$\mathbb{E}\Big[\|\mu\|^p_{L^p([0,T];B^{\beta}_{2,2}(\mathbf{R};e(l)))}\Big] = \int_0^T \mathbb{E}\Big[\|\mu(t)\|^p_{B^{\beta}_{2,2}(\mathbf{R};e(l))}\Big]\mathrm{d}t \lesssim \int_0^T t^{(-\beta-1-3\varepsilon)\frac{p}{2}}\,\mathrm{d}t,$$

and now it suffices to note that there exists $\varepsilon > 0$ with $(-\beta-1-3\varepsilon)\frac{p}{2} > -1$ if and only if $p < 2/(\beta+1)$. □

Corollary II.6.2. *In the setting of Proposition II.6.1 we have almost surely*

$$\sqrt{\mu} \in L^2([0,T];L^2(\mathbf{R};e(l)))$$

for all $T > 0$ and $l \in \mathbf{R}$.

Proof of Theorem II.2.19. We follow the approach of Konno and Shiga [KS88]. Applying Corollary II.2.16 for $\kappa \in (0,1/2]$ or Proposition II.5.4 for $\kappa > 1/2$, we obtain an SBM in static random environment μ, which is a process on $(\Omega\times\mathbb{D}([0,T];\mathscr{M}(\mathbf{R})),\mathscr{F},\mathbb{P}\ltimes\mathbb{P}^{\omega})$, with $\mathscr{F}$ being the product sigma algebra. Enlarging the probability space, we can moreover assume that the process is defined on $(\Omega\times\bar{\Omega},\mathscr{F}\otimes\bar{\mathscr{F}},\mathbb{P}\ltimes\bar{\mathbb{P}}^{\omega})$ such that the probability space $(\bar{\Omega},\bar{\mathscr{F}},\bar{\mathbb{P}})$ supports a space-time white noise $\bar{\xi}$ which is independent of ξ. More precisely, we are given a map $\overline{\xi}:\Omega\times\overline{\Omega}\to\mathscr{S}'_{\omega}(\mathbf{R}^d\times[0,T])$ which has the law of space-time white noise and does not depend on Ω, i.e. $\overline{\xi}(\omega,\overline{\omega}) = \overline{\xi}(\overline{\omega})$.

For $\omega \in \Omega$ let $\{\mathscr{F}^{\omega}_t\}_{t\in[0,T]}$ be the usual augmentation of the (random) filtration generated by $\mu(\omega,\cdot)$ and $\bar{\xi}$. For almost all $\omega \in \Omega$ the collection of martingales $t \mapsto L^{\varphi}(\omega,t)$ for $t \in [0,T]$, $\varphi \in \mathscr{D}_{\mathscr{H}^{\omega}}$ defines a (random) worthy orthogonal martingale measure $M(\omega, \mathrm{d}t, \mathrm{d}x)$ in the sense of [Wal86], with quadratic variation $Q(A\times B\times[s,t]) = \int_s^t \mu(r)(A\cap B)\,\mathrm{d}r$ for all Borel sets $A, B \subset \mathbf{R}$ (first we define $Q(\varphi\times\psi\times[s,t]) = \int_s^t \langle \mu(r), \varphi\psi\rangle\,\mathrm{d}r$ for $\varphi,\psi \in \mathscr{D}_{\mathscr{H}^{\omega}}$, then we use Lemma II.6.1 with $p = 1$ and $\beta \in (0,1/2)$ to extend the quadratic variation and the martingales to indicator functions of Borel sets). We can thus build a space-time white noise $\tilde{\xi}$ by defining for $\varphi \in L^2([0,T]\times\mathbf{R})$:

$$\int_{[0,T]\times\mathbf{R}} \varphi(s,x)\tilde{\xi}(\omega,\mathrm{d}s,\mathrm{d}x) := \int_{[0,T]\times\mathbf{R}} \frac{\varphi(s,x)}{\sqrt{\mu(\omega,s,x)}} 1_{\{\mu(\omega,s,x)>0\}} M(\omega,\mathrm{d}s,\mathrm{d}x) + \int_{[0,T]\times\mathbf{R}} \varphi(s,x) 1_{\{\mu(\omega,s,x)=0\}} \bar{\xi}(\mathrm{d}s,\mathrm{d}x).$$

By taking conditional expectations with respect to ξ we see that $\tilde{\xi}$ and ξ are independent, and by definition the SBM in static random environment solves the SPDE (II.5).

Conversely, it is straightforward to see that any solution to the SPDE is a SBM in static random environment of parameter $\nu = \kappa/2$. Uniqueness in law of the latter then implies uniqueness in law of the solution to the SPDE. □

II.6.2 Persistence

In this section we study the persistence of the SBM in static random environment μ and we prove Theorem II.2.21, namely that μ is super-exponentially persistent. For the proof we rely on the results of Section II.5.2, where we constructed the rSBM μ^L with Dirichlet boundary conditions on the box $(-L/2, L/2)^d$, for integer $L \in 2\mathbf{N}$. This process is obtained from the one considered so far by killing particles that reach the boundary of the box. In this way, the processes $\{\mu^L\}_{L\in\mathbf{N}}$ are naturally coupled and satisfy:

$$\mu^{L_1} \le \mu^{L_2} \le \mu,$$

for $L_1 \le L_2$. In particular, the following result holds.

Lemma II.6.3. *Let $\bar{\mu}$ be a rSBM associated to a random environment $\{\xi^n\}_{n\in\mathbf{N}}$ satisfying Assumption II.2.1. There exists a probability space of the form $(\Omega\times\mathbb{D}([0,+\infty);\mathscr{M}(\mathbf{R}^d)), \mathscr{F}, \mathbb{P}\ltimes\mathbb{P}^{\omega})$ supporting a rSBM μ such that $\mu = \overline{\mu}$ in distribution. Moreover Ω supports a spatial white noise ξ and there exists a null set $N_0 \subseteq \Omega$ such that:*

- *i For all $\omega \in N_0^c$ and $L \in \mathbf{N}$ the random Anderson Hamiltonian associated to ξ with Dirichlet boundary conditions on $(-L/2,L/2)^d$, $\mathscr{H}^{\omega}_{\mathfrak{d},L}$, on the domain $\mathscr{D}_{\mathscr{H}^{\omega}_{\mathfrak{d},L}}$ is well defined (cf. [Cv19] and Section IV.3). Moreover, $\mathscr{D}_{\mathscr{H}^{\omega}_{\mathfrak{d},L}} \subseteq C^{\vartheta}((-L/2,L/2)^d)$ for any $\vartheta < 2-d/2$. Finally the operator has discrete spectrum. If $\lambda_1(\omega,L)$ is the largest eigenvalue of $\mathscr{H}^{\omega}_{\mathfrak{d},L}$, then the associated eigenfunction $e_{\lambda_1(\omega,L)}$ satisfies $e_{\lambda_1(\omega,L)}(x) > 0$ for all $x \in (-\frac{L}{2},\frac{L}{2})^d$.*
- *ii There exist random variables $\{\mu^L\}_{L\in\mathbf{N}}$ with values in $\mathbb{D}([0,\infty);\mathscr{M}(\mathbf{R}^d))$ satisfying $\mu^L(\omega,t) \le \mu^{L+1}(\omega,t) \le \cdots \le \mu(\omega,t)$ and $\mu^L(0) = \delta_0$. Moreover, for all $\omega \in N_0^c, L \in 2\mathbf{N}$ and $\varphi \in \mathscr{D}_{\mathscr{H}^{\omega}_{\mathfrak{d},L}}$:*

$$K^{\varphi}_L(\omega,t) = \langle \mu^L(\omega,t),\varphi\rangle - \langle \mu^L(\omega,0),\varphi\rangle - \int_0^t \langle \mu^L(\omega,r), \mathscr{H}^{\omega}_{\mathfrak{d},L}\varphi\rangle\,\mathrm{d}r, \quad t \ge 0$$

is a continuous centered martingale (w.r.t. the filtration generated by $\mu^L(\omega,\cdot)$) with quadratic variation $\langle K_L^{\varphi}(\omega)\rangle_t = 2\nu\int_0^t \langle \mu(\omega,r),\varphi^2\rangle \mathrm{d}r$.

Proof. For the first point see [Cv19] and Lemma IV.3.5. The second statement is proved in Theorem II.5.12 and Corollary II.5.13. □

Analogously to the previous section and following the notation of Section IV.3, we denote with $t \mapsto T_t^{\mathfrak{d},\omega}$ the semigroup associated to $\mathscr{H}^{\omega}_{\mathfrak{d},L}$ for some fixed L,ω. Now we prove that given a nonzero positive $\varphi \in C_c^\infty(\mathbf{R}^d)$ and $\lambda > 0$, for almost all ω there exists an $L = L(\omega)$ with

$$\mathbb{P}^{\omega}\Big(\lim_{t\to\infty} e^{-t\lambda}\langle \mu^L(\omega,t,\cdot),\varphi\rangle = \infty\Big) > 0. \tag{II.23}$$

This implies Theorem II.2.21.

The reason for working with μ^L is that the spectrum of the Anderson Hamiltonian on $(-L/2,L/2)^d$ is discrete, and its largest eigenvalue almost surely becomes bigger than λ for $L \to \infty$. Given this information, (II.23) follows from a simple martingale convergence argument, see Corollary II.6.6 below.

Remark II.6.4. *For simplicity we only treat the case of a (killed) rSBM with parameter $\nu \in (0,1/2]$. For $\nu > 1/2$ we need to use the constructions of Section II.5.1, after which we can follow the same arguments to show persistence.*

Let us write $\lambda_1(\omega,L)$ for the largest eigenvalue of the Anderson Hamiltonian $\mathscr{H}^{\omega}_{\mathfrak{d},L}$ with Dirichlet boundary conditions on $(-L/2,L/2)^d$.

Lemma II.6.5. *There exist $c_1,c_2 > 0$ such that for almost all $\omega \in \Omega$:*

(i) In $d = 1$ (by [Che14, Lemmata 2.3 and 4.1]):

$$\lim_{L\to+\infty} \frac{\lambda_1(\omega,L)}{\log(L)^{2/3}} = c_1.$$

(ii) In $d = 2$ (by [Cv19, Theorem 10.1]):

$$\lim_{L\to+\infty} \frac{\lambda_1(\omega,L)}{\log(L)} = c_2.$$

Corollary II.6.6. *Consider $d \le 2$ and $\lambda > 0$ and let μ be an SBM in static random environment, coupled for all $L \in \mathbf{N}$ to a killed SBM in static random environment μ^L on $(-\frac{L}{2},\frac{L}{2})^d$ with $\mu^L \le \mu$ (as described in Lemma II.6.3). For almost all $\omega \in \Omega$ there exists an $L_0(\omega) > 0$ such that for all $L \ge L_0(\omega)$ the killed rSBM $\mu^L(\omega,\cdot)$ satisfies* (II.23). *In particular, for almost all $\omega \in \Omega$ the process $\mu(\omega,\cdot)$ is super-exponentially persistent.*

Proof. In view of Lemma II.6.5, for almost all $\omega \in \Omega$ we can choose $L_0(\omega)$ such that the largest eigenvalue of the Anderson Hamiltonian $\lambda_1(\omega,L)$ is bigger than λ for all $L \ge L_0(\omega)$. Now we fix ω such that the above holds true and work conditional on the environment (we omit ω from the notation to improve the readability). We also fix some $L \ge L_0(\omega)$ and write λ_1 instead of $\lambda_1(\omega,L)$ for the largest eigenvalue. Finally, let e_1 be the strictly positive eigenfunction with $\|e_1\|_{L^2((-\frac{L}{2},\frac{L}{2})^d)} = 1$ associated to λ_1. By Lemma II.6.3 we find for $0 \le s < t$:

$$\mathbb{E}[\langle \mu^L(t), e_1\rangle | \mathscr{F}_s] = \langle \mu^L(s), T^{\mathfrak{b}}_{t-s} e_1\rangle = \langle \mu^L(s), e^{(t-s)\lambda_1} e_1\rangle,$$

and thus the process $E(t) = \langle \mu^L(t), e^{-\lambda_1 t} e_1\rangle$, $t \ge 0$, is a martingale. Moreover, the variance of this martingale is bounded uniformly in t. Indeed:

$$\mathbb{E}\Big[|E(t)-E(0)|^2\Big] \simeq \int_0^t T^{\mathfrak{b}}_r((e^{-\lambda_1 r} e_1)^2)(0)\,\mathrm{d}r \lesssim \int_0^t e^{-\lambda_1 r}\,\mathrm{d}r \lesssim 1,$$

where we used that by Lemma II.6.3 we have $e_1 \in \mathscr{C}^{\vartheta}((-\frac{L}{2},\frac{L}{2})^d)$ for some admissible $\vartheta > 0$, and therefore

$$\begin{aligned} T^{\mathfrak{b}}_r((e^{-\lambda_1 r} e_1)^2)(0) &\le \|e_1\|_\infty e^{-\lambda_1 r} T^{\mathfrak{b}}_r(e^{-\lambda_1 r} e_1)(0) \\ &= \|e_1\|_\infty e^{-\lambda_1 r} e_1(0) \lesssim e^{-\lambda_1 r}. \end{aligned}$$

It follows that $E(t)$ converges almost surely and in L^2 to a random variable $E(\infty) \ge 0$ as $t \to \infty$, and since $\mathbb{E}[E(\infty)] = E(0) = e_1(0) > 0$ we know that $E(\infty)$ is strictly positive with positive probability. For $\varphi \ge 0$ nonzero with support in $(-L/2, L/2)^d$ we show in Lemma II.6.7 that:

$$e^{-\lambda_1 t}\langle \mu^L(t), \varphi\rangle \to \langle e_1, \varphi\rangle E(\infty), \quad \text{as } t \to \infty, \quad \text{in } L^2(\mathbb{P}^\omega) \tag{II.24}$$

so that we get from the strict positivity of e_1 and from the fact that $\lambda_1 > \lambda$

$$\mathbb{P}\Big(\lim_{t\to\infty} e^{-\lambda t}\langle \mu^L(t), \varphi\rangle = \infty\Big) \ge \mathbb{P}(E(\infty) > 0) > 0.$$

This completes the proof.

□

Lemma II.6.7. *In the setting of Corollary II.6.6, let $\varphi \in \mathscr{C}^{\vartheta}_{\mathfrak{b}}$ and let $\psi = \varphi - \langle e_1, \varphi\rangle e_1$. Then*

$$\lim_{t\to\infty} \mathbb{E}^\omega\Big[|e^{-\lambda_1 t}\langle \mu^L(\omega, t), \psi\rangle|^2\Big] = 0. \tag{II.25}$$

Proof. As before we omit the dependence on the realization ω of the noise. Using the martingale $\langle \mu^L(s), T^{\mathfrak{b}}_{t-s}\psi\rangle$, we get

$$\mathbb{E}\Big[|\langle \mu^L(t), \psi\rangle|^2\Big] \lesssim |T^{\mathfrak{b}}_t(\psi)|^2(0) + \int_0^t T^{\mathfrak{b}}_r\Big[(T^{\mathfrak{b}}_{t-r}\psi)^2\Big](0)\,\mathrm{d}r. \tag{II.26}$$

Let $\lambda_2 < \lambda_1$ be the second eigenvalue of the Anderson Hamiltonian (the strict inequality is a consequence of the Krein-Rutman theorem, see also Lemma IV.3.5). The main idea is to leverage that:

$$\|T^{\mathfrak{b}}_t\psi\|_{L^2} \le e^{\lambda_2 t}\|\psi\|_{L^2},$$

since ψ is orthogonal to the first eigenfunction. The only subtlety is that of course the value of a function in 0 is not controlled by its L^2 norm. To go from L^2 to a space of continuous functions, we use that for all ϑ as in Equation (II.9) and sufficiently close to 1:

$$\begin{aligned}\|T^{\mathfrak{d}}_1 f\|_{\mathscr{C}^{\vartheta}_{\mathfrak{d}}} &\lesssim \|T^{\mathfrak{d}}_{2/3} f\|_{\mathscr{C}^{\vartheta-\frac{d}{2}}_{\mathfrak{d}}} \lesssim \|T^{\mathfrak{d}}_{2/3} f\|_{\mathscr{C}^{\vartheta}_{\mathfrak{d},2}} \\ &\lesssim \|T^{\mathfrak{d}}_{1/3} f\|_{\mathscr{C}^{\vartheta-\frac{d}{2}}_{\mathfrak{d},2}} \lesssim \|T^{\mathfrak{d}}_{1/3} f\|_{\mathscr{C}^{\vartheta}_{2}} \lesssim \|f\|_{L^2},\end{aligned}$$

in view of the regularizing properties of the semigroup $T^{\mathfrak{d}}$ (which hold with the same parameters as in Proposition II.3.1, see Theorem IV.3.4) and by Besov embedding. We refer the reader to Section IV.3 for the definition of Besov spaces with Dirichlet boundary conditions. For all present purposes these spaces behave identically to their counterpart on $\mathbf{R}^d$.

Let us consider the second term in (II.26), for $t \geq 2$. With the previous estimates, we bound it as follows:

$$\begin{aligned}&\int_0^1 T^{\mathfrak{d}}_r\Big[(T^{\mathfrak{d}}_{t-r}\psi)^2\Big](0)\,\mathrm{d}r + \int_1^t T^{\mathfrak{d}}_r\Big[(T^{\mathfrak{d}}_{t-r}\psi)^2\Big](0)\,\mathrm{d}r \\ &\quad\lesssim \int_0^1 \|T^{\mathfrak{d}}_{t-r}\psi\|^2_{\mathscr{C}^{\vartheta}_{\mathfrak{d}}}\,\mathrm{d}r + \int_1^t \|T^{\mathfrak{d}}_{r-1}(T^{\mathfrak{d}}_{t-r}\psi)^2\|_{L^2}\,\mathrm{d}r \\ &\quad\lesssim \int_0^1 \|T^{\mathfrak{d}}_{t-r-1}\psi\|^2_{L^2}\,\mathrm{d}r + \int_1^t e^{\lambda_1(r-1)}\|(T^{\mathfrak{d}}_{t-r}\psi)^2\|_{L^2}\,\mathrm{d}r \\ &\quad\lesssim \int_0^1 \|T^{\mathfrak{d}}_{t-r-1}\psi\|^2_{L^2}\,\mathrm{d}r + \int_1^{t-1} e^{\lambda_1(r-1)}\|T^{\mathfrak{d}}_{t-r-1}\psi\|^2_{L^2}\,\mathrm{d}r + \int_{t-1}^t e^{\lambda_1(r-1)}\|\psi\|^2_{\mathscr{C}^{\vartheta}_{\mathfrak{d}}}\,\mathrm{d}r \\ &\quad\lesssim \int_0^t e^{2\lambda_2(t-r)+\lambda_1 r}\,\mathrm{d}r \lesssim e^{2\lambda_2 t}(1+e^{(\lambda_1-2\lambda_2)t}+t) \lesssim (e^{2\lambda_2 t}+e^{\lambda_1 t})(1+t),\end{aligned}$$

where we used that for any $\lambda \in \mathbf{R}$ one can bound $\int_0^t e^{\lambda s}\,\mathrm{d}s \leq \frac{1}{|\lambda|}(1+e^{\lambda t}+t)$. Plugging this estimate into (II.26), we obtain

$$\begin{aligned}\mathbb{E}\Big[|e^{-\lambda_1 t}\langle\mu^L(t),\psi\rangle|^2\Big] &\lesssim e^{-2\lambda_1 t}e^{2\lambda_2(t-1)} + e^{-2\lambda_1 t}\Big(e^{2\lambda_2 t}+e^{\lambda_1 t}\Big)(1+t) \\ &\lesssim e^{-\lambda_1 t} + e^{-2(\lambda_1-\lambda_2)t}(1+t).\end{aligned}$$

This proves (II.25). □

Remark II.6.8. *The connection of extinction or persistence of a branching particle system to the largest eigenvalue of the associated Hamiltonian is reminiscent of conditions appearing in the theory of multi-type Galton-Watson processes: see for example [Har02, Section 2.7]. The martingale argument in our proof can be traced back at least to Everett and Ulam, as explained in [Har51, Theorem 7b].*

II.7 Stochastic estimates

In this section we prove parts of Lemma II.2.5, i.e. that a random environment satisfying Assumption II.2.1 gives rise to a deterministic environment satisfying Assumption II.2.4.

Lemma II.7.1. *Consider $a, \varepsilon, q > 0$ and $b > d/2$. Under Assumption II.2.1 we have*

$$\sup_n \left[\mathbb{E}\|n^{-d/2}(\xi^n)_+\|^q_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n;p(a))} + \mathbb{E}\|n^{-d/2}(\xi^n)_+\|^2_{L^2(\mathbf{Z}^d_n;p(b))} \right] < +\infty,$$

and the same holds if we replace $(\xi^n)_+$ with $|\xi^n|$. Furthermore, for $\nu = \mathbb{E}[\Phi_+]$, the following convergences hold true in distribution in $\mathscr{C}^{-\varepsilon}(\mathbf{R}^d;p(a))$:

$$\mathscr{E}^n n^{-d/2}(\xi^n)_+ \longrightarrow \nu, \qquad \mathscr{E}^n n^{-d/2}|\xi^n| \longrightarrow 2\nu.$$

Proof. We prove the result only for $(\xi^n)_+$, since then we can treat $(\xi^n)_-$ by considering $-\xi^n$ ($-\Phi$ is still a centered random variable). Now note that

$$\begin{aligned}\mathbb{E}[\|n^{-d/2}(\xi^n)_+\|^q_{L^q(\mathbf{Z}^d_n;p(a))}] &= \sum_{x\in\mathbf{Z}^d_n} n^{-d}\mathbb{E}[|n^{-d/2}(\xi^n)_+|^q(x)]|p(a)(x)|^q \\ &\lesssim \mathbb{E}[|\Phi|^q]\int_{\mathbf{R}^d}(1+|y|)^{-aq}\,\mathrm{d}y,\end{aligned}$$

which is finite whenever $aq > d$. From here the uniform bound on the expectations follows by Besov embedding.

Convergence to ν is then a consequence of the spatial independence of the noise ξ^n, since it is easy to see that $\mathbb{E}\Big[\langle \mathscr{E}^n(\xi^n)_+ - \nu, \varphi\rangle\Big] = O(n^{-d})$ for all φ with compactly supported Fourier transform. □

The following result is a simpler variant of [MP19, Lemma 5.5] for the case $d = 1$, hence we omit the proof.

Lemma II.7.2. *Fix ξ^n satisfying Assumption II.2.1, $d = 1$, $a, q > 0$ and $\alpha < 2 - d/2$. We have:*

$$\sup_n \mathbb{E}\Big[\|\xi^n\|^q_{\mathscr{C}^{\alpha-2}(\mathbf{Z}_n;p(a))}\Big] < +\infty, \qquad \mathscr{E}^n\xi^n \to \xi,$$

where ξ is a white noise on $\mathbf{R}$ and the convergence holds in distribution in $\mathscr{C}^{\alpha-2}(\mathbf{R};p(a))$.

II.8 Moment estimates

Here we derive uniform bounds for the moments of the processes $\{\mu^n\}_{n\in\mathbf{N}}$. As a convention, in the following we will write $\mathbb{E}$ and $\mathbb{P}$ for the expectation and the probability under the distribution of u^n conditional on the realization of the environment.

Lemma II.8.1. *Fix $q, T > 0$. For all $n \in \mathbf{N}$, consider the process $\{\mu^n(t)\}_{t\geq 0}$ as in Definition II.2.7. Consider then $\varphi^n\colon \mathbf{Z}^d_n \to \mathbf{R}$ with $\varphi^n \geq 0$, $\varphi^n = \varphi|_{\mathbf{Z}^d_n}$ with $\varphi \in \mathscr{C}^2(\mathbf{R}^d;e(l))$ for some $l \in \mathbf{R}$. Then*

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}\Big[|\mu^n(t)(\varphi^n)|^q\Big] < +\infty.$$

If for all $\varepsilon > 0$ *there exists an* $l \in \mathbf{R}$ *such that* $\sup_n \|\varphi^n\|_{\mathscr{C}^{-\varepsilon}(\mathbf{R}^d;e(l))} < +\infty$, *we can bound for all* $\gamma \in (0,1)$*:*

$$\sup_n \sup_{t\in[0,T]} t^\gamma \mathbb{E}\Big[|\mu^n(t)(\varphi^n)|^q\Big] < +\infty.$$

Proof. We prove the second estimate, since the first estimate is similar but easier (Lemma II.9.1 below controls $\|\varphi^n\|_{\mathscr{C}^\vartheta(\mathbf{Z}^d_n;e(l))}$ for all $\vartheta < 2$ in that case). Also, we assume without loss of generality that $q \geq 2$. As usual, we use the convention of freely increasing the value of l in the exponential weight. Let us start by recalling that $\mathbb{E}\Big[\mu^n(t)(\varphi^n)\Big] = T^n_t\varphi^n(0)$. Moreover, via the assumption on the regularity, Proposition II.3.1 and Equation (I.1) from Lemma I.1.5 guarantees that for any $\gamma \in (0,1)$ there exists a $\delta = \delta(\gamma,q) > 0$ such that

$$\sup_n \|t \mapsto T^n_t\varphi^n\|_{\mathscr{L}^{\gamma/q,\delta}(\mathbf{Z}^d_n;e(l))} < +\infty.$$

By the triangle inequality it thus suffices to prove that for any $\gamma > 0$:

$$\sup_n \sup_{t\in[0,T]} t^\gamma \mathbb{E}\Big[|\mu^n(t)(\varphi^n) - T^n_t\varphi^n(0)|^q\Big] < +\infty.$$

We can interpret the particle system u^n as the superposition of $\lfloor n^\varrho \rfloor$ independent particle systems, each started with one particle in zero; we write $u^n = u^n_1 + \dots + u^n_{\lfloor n^\varrho\rfloor}$. To lighten the notation we assume that $n^\varrho \in \mathbf{N}$. We then apply Rosenthal's inequality, [Pet95, Theorem 2.9] (recall that $q \geq 2$) and obtain (with $\langle f,g\rangle = \sum_{x\in\mathbf{Z}^d_n} f(x)g(x)$):

$$\begin{aligned}
\mathbb{E}\Big[|\mu^n(t)(\varphi^n) - T^n_t\varphi^n(0)|^q\Big] &= \mathbb{E}\Bigg[\Bigg|\sum_{k=1}^{n^\varrho}\Big[n^{-\varrho}\langle u^n_k(t),\varphi^n\rangle - n^{-\varrho}T^n_t\varphi^n(0)\Big]\Bigg|^q\Bigg] \\
&\lesssim n^{-\varrho q}\sum_{k=1}^{n^\varrho}\mathbb{E}\Big[|\langle u^n_k(t),\varphi^n\rangle - T^n_t\varphi^n(0)|^q\Big] \\
&\quad + n^{-\varrho q}\Bigg(\sum_{k=1}^{n^\varrho}\mathbb{E}\Big[|\langle u^n_k(t),\varphi^n\rangle - T^n_t\varphi^n(0)|^2\Big]\Bigg)^{\frac{q}{2}} \\
&\lesssim n^{-\varrho(q-1)}\mathbb{E}\Big[|\langle u^n_1(t),\varphi^n\rangle|^q\Big] + \Big(n^{-\varrho}\mathbb{E}\Big[|\langle u^n_1(t),\varphi^n\rangle|^2\Big]\Big)^{q/2} \\
&\quad + n^{-\frac{\varrho q}{2}}t^{-\gamma}\|t\mapsto T^n_t\varphi^n\|^q_{\mathscr{L}^{\gamma/q,\delta}(\mathbf{Z}^d_n;e(l))}
\end{aligned}$$

for the same $\delta > 0$ and $l \in \mathbf{R}$ as above. The two scaled expectations are of the same form $\mathbb{E}\Big[|\langle u^n_1(t),\varphi^n\rangle|^p\Big]$, for some $p \in [1,\infty]$. To control them, we define for $p \in \mathbf{N}$ the map

$$m^{p,n}_{\varphi^n}(t,x) = n^{\varrho(1-p)}\mathbb{E}\Big[|(u^n_1(t),\varphi^n)|^p\Big].$$

As a consequence of Kolmogorov's backward equation each $m^{p,n}_{\varphi^n}$ solves the discrete PDE (see also Equation (2.4) in [ABMY00]):

$$\partial_t m^{p,n}_{\varphi^n}(t,x) = \mathscr{H}^n m^{p,n}_{\varphi^n}(t,x) + n^{-\varrho}(\xi^n_e)_+(x)\sum_{i=1}^{p-1}\binom{p}{i}m^{i,n}_{\varphi^n}(t,x)m^{p-i,n}_{\varphi^n}(t,x),$$

with initial condition $m^{p,n}_{\varphi^n}(0,x) = n^{\varrho(1-p)}|\varphi^n(x)|^p$. We claim that this equation has a unique (paracontrolled in $d=2$) solution $m^{p,n}_{\varphi^n}$, such that for all $\gamma > 0$ there exists $\delta = \delta(\gamma,p) > 0$

with $\sup_n \|m^{n,p}_{\varphi^n}\|_{\mathscr{L}^{\gamma,\delta}(\mathbf{Z}^d_n;e(l))} < \infty$. Once this is shown, the proof is complete. We proceed by induction over p. For $p = 1$ we simply have $m^{n,1}_{\varphi^n}(t,x) = T^n_t \varphi^n(x)$. For $p \geq 2$ we use that by Lemma II.9.2 we have $\|n^{\varrho(1-p)}|\varphi^n(x)|^p\|_{\mathscr{C}^\kappa(\mathbf{Z}^d_n;e(l))} \to 0$ for some $\kappa > 0$ and we assume that the induction hypothesis holds for all $p' < p$. Since it suffices to prove the bound for small $\gamma > 0$, we may assume also that $\kappa > \gamma$. We choose then $\gamma' < \gamma$ such that for some $\delta(\gamma', p) > 0$:

$$\sup_n \Big\| \sum_{i=1}^{p-1} m^{i,n}_{\varphi^n} m^{p-i,n}_{\varphi^n} \Big\|_{\mathscr{M}^{\gamma'}\mathscr{C}^{\delta(\gamma',p)}(\mathbf{Z}^d_n;e(l))} < +\infty.$$

Since by Assumption II.2.4 $\|n^{-\varrho}(\xi^n_e)_+\|_{\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n;p(a))}$ is uniformly bounded in n for all $\varepsilon, a > 0$, the above bound is sufficient to control the product:

$$\sup_n \Big\| n^{-\varrho}(\xi^n_e)_+ \sum_{i=1}^{p-1} m^{i,n}_{\varphi^n} m^{p-i,n}_{\varphi^n} \Big\|_{\mathscr{M}^{\gamma'}\mathscr{C}^{-\varepsilon}(\mathbf{Z}^d_n;e(l))} < +\infty.$$

Now the claimed bound for $m^{n,p}_{\varphi^n}$ follows from an application of Proposition II.3.1. For non-integer q we simply use interpolation between the bounds for $p < q < p'$ with $p, p' \in \mathbf{N}$. □

II.9 Some estimates in Besov spaces

Here we prove some results concerning discrete and continuous Besov spaces. First, we show that restricting a function to the lattice preserves its regularity.

Lemma II.9.1. *Consider $\varphi \in \mathscr{C}^\alpha(\mathbf{R}^d)$ for $\alpha \in (0,\infty) \setminus \mathbf{N}$. Then the restriction $\varphi|_{\mathbf{Z}^d_n}$ belongs to $\mathscr{C}^\alpha(\mathbf{Z}^d_n)$ and*

$$\sup_{n\in\mathbf{N}} \|\varphi|_{\mathbf{Z}^d_n}\|_{\mathscr{C}^\alpha(\mathbf{Z}^d_n)} \lesssim \|\varphi\|_{\mathscr{C}^\alpha(\mathbf{R}^d)}.$$

For the extension of $\varphi|_{\mathbf{Z}^d_n}$ we have $\mathscr{E}^n(\varphi|_{\mathbf{Z}^d_n}) \to \varphi$ in $\mathscr{C}^\beta(\mathbf{R}^d)$ for all $\beta < \alpha$.

Proof. Let us write $\varphi^n = \varphi|_{\mathbf{Z}^d_n}$. We have to estimate $\|\Delta^n_j \varphi^n\|_{L^\infty(\mathbf{Z}^d_n)}$, and for that purpose we consider the cases $j < j_n$ and $j = j_n$ separately. In the first case we have $\Delta^n_j \varphi^n(x) = K_j * \varphi(x) = \Delta_j \varphi(x)$ for $x \in \mathbf{Z}^d_n$ because $\mathrm{supp}(\varrho_j) \subset n(-1/2,1/2)^d$. Therefore:

$$\|\Delta^n_j \varphi\|_{L^\infty(\mathbf{Z}^d_n)} \leq \|\Delta_j \varphi\|_{L^\infty(\mathbf{R}^d)} \leq 2^{j\alpha}\|\varphi\|_{\mathscr{C}^\alpha}.$$

For $j = j_n$ we have $\varrho^n_{j_n}(\cdot) = 1 - \varrho_{-1}(2^{-j_n}\cdot)$, where $\varrho_{-1} \in \mathscr{S}_\varpi$ is one of the two functions generating the dyadic partition of unity. By construction we have $\varrho^n_{j_n}(x) \equiv 1$ for x near the boundary of $n(-1/2,1/2)^d$, since $\mathrm{supp}(\varrho_{-1}(2^{-j_n}\cdot)) \subset n(-1/2,1/2)^d$. Let us define $\psi_n = \mathscr{F}^{-1}_{\mathbf{Z}^d_n}\varrho_{-1}(2^{-j_n}\cdot) = \mathscr{F}^{-1}_{\mathbf{R}^d}\varrho_{-1}(2^{-j_n}\cdot)$, so that

$$\sum_{x\in\mathbf{Z}^d_n} n^{-d}\psi_n(x) = \mathscr{F}_n\psi_n(0) = \varrho_{-1}(2^{-j_n}\cdot 0) = 1.$$

To avoid confusion, write:

$$\varphi *_n \psi(x) = \frac{1}{n^d}\sum_{y\in\mathbf{Z}^d_n} \varphi(x-y)\psi(y).$$

Then, for every monomial M of strictly positive degree we have, since ψ_n is an even function,

$$\sum_{x\in\mathbf{Z}_n^d} n^{-d}\psi_n(x)M(x) = (\psi_n *_n M)(0) = \mathscr{F}^{-1}_{\mathbf{R}^d}(\varrho_{-1}(2^{-j_n}\cdot)\mathscr{F}_{\mathbf{R}^d}M)(0) = M(0) = 0,$$

where we used that the Fourier transform of a polynomial is supported in 0. Thus for $x\in\mathbf{Z}_n^d$ we get $\Delta^n_{j_n}\varphi^n(x) = \varphi(x) - (\psi_n *_n \varphi)(x)$, that is:

$$\varphi(x) - (\psi_n *_n \varphi)(x) = -\psi_n *_n \left(\varphi(\cdot) - \varphi(x) - \sum_{1\leq|k|\leq\lfloor\alpha\rfloor}\frac{1}{k!}\partial^k\varphi(x)(\cdot - x)^k\right)(x),$$

with the usual multi-index notation and where as above we could replace the discrete convolution $*_n$ with the continuous convolution on $\mathbf{R}^d$. Moreover, since $\varphi\in\mathscr{C}^\alpha(\mathbf{R}^d)$ and $\alpha > 0$ is not an integer, we can estimate

$$\left\|\varphi(\cdot) - \sum_{0\leq|k|\leq\lfloor\alpha\rfloor}\frac{1}{k!}\partial^k\varphi(x)(\cdot - x)^{\otimes k}\right\|_{L^\infty(\mathbf{R}^d)} \lesssim |y|^\alpha\|\varphi\|_{\mathscr{C}^\alpha(\mathbf{R}^d)},$$

and from here the estimate for the convolution holds by a scaling argument. The convergence then follows by interpolation. □

The following result shows that multiplying a function on $\mathbf{Z}_n^d$ by $n^{-\kappa}$ for some $\kappa > 0$ gains regularity and gives convergence to zero under a uniform bound for the norm.

Lemma II.9.2. *Consider $z\in\varpi$ and $p\in[1,\infty], \alpha\in\mathbf{R}$ and a sequence of functions $\varphi^n\in\mathscr{C}^\alpha_p(\mathbf{Z}_n^d;z)$ with uniformly bounded norm:*

$$\sup_n\|\varphi^n\|_{\mathscr{C}^\alpha_p(\mathbf{Z}_n^d;z)} < +\infty.$$

Then for any $\kappa > 0$ the sequence $n^{-\kappa}f^n$ is bounded in $\mathscr{C}^{\alpha+\kappa}_p(\mathbf{Z}_n^d;z)$:

$$\sup_n\|n^{-\kappa}\varphi^n\|_{\mathscr{C}^{\alpha+\kappa}_p(\mathbf{Z}_n^d;z)} \lesssim \sup_n\|f^n\|_{\mathscr{C}^\alpha_p(\mathbf{Z}_n^d;z)}$$

and $n^{-\kappa}\mathscr{E}^n f^n$ converges to zero in $\mathscr{C}^\beta_p(\mathbf{R}^d;z)$ for any $\beta < \alpha+\kappa$.

Proof. By definition, we only encounter Littlewood-Paley blocks up to an order $j_n \simeq \log_2(n)$. Hence $2^{j(\alpha+\kappa-\varepsilon)}n^{-\kappa} \lesssim 2^{j\alpha}n^{-\varepsilon}$ for $j\leq j_n$ and $\varepsilon\geq 0$, from where the claim follows. □

Now we study the action of discrete gradients. We write $\mathscr{C}^\alpha_p(\mathbf{Z}_n^d;z;\mathbf{R}^d)$ for the space of maps $\varphi\colon\mathbf{Z}_n^d\to\mathbf{R}^d$ such that each component lies in $\mathscr{C}^\alpha_p(\mathbf{Z}_n^d;z)$ with the naturally induced norm. The following result is analogous to [MP19, Lemma 3.4], hence we omit the proof.

Lemma II.9.3 (Lemma 3.4, [MP19])**.** *The discrete gradient*

$$(\nabla^n\varphi)_i(x) = n(\varphi(x+\frac{e_i}{n}) - \varphi(x))$$

for $i = 1, \dots, d$ (with $\{e_i\}_i$ the standard basis in $\mathbf{R}^d$) and the discrete Laplacian

$$\Delta^n \varphi(x) = n^2 \sum_{i=1}^{d} (\varphi(x+\frac{e_i}{n}) - 2\varphi(x) + \varphi(x-\frac{e_i}{n}))$$

satisfy:

$$\|\nabla^n \varphi\|_{\mathscr{C}_p^{\alpha-1}(\mathbf{Z}_n^d; z; \mathbf{R}^d)} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}(\mathbf{Z}_n^d; z)}, \qquad \|\Delta^n \varphi\|_{\mathscr{C}_p^{\alpha-2}(\mathbf{Z}_n^d; z)} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}(\mathbf{Z}_n^d; z)},$$

for all $\alpha \in \mathbf{R}$ and $p \in [1, \infty]$, where both estimates hold uniformly in $n \in \mathbf{N}$.

III

The spatial Λ-Fleming-Viot model

III.1 Introduction

In this chapter we are going to work on the torus $\mathbf{T}^d$ in dimensions $d = 1, 2$. The model we will consider, called spatial Λ–Fleming-Viot process, describes a population of two types distributed in the spatial continuum. Reproductive events will not affect only a single particle, as is the case in the branching random walk of the previous chapter, but have an impact on a macroscopic area of diameter approximately $\frac{1}{n}$, for $n \in \mathbf{N}$. We study the behaviour of this process in different regimes for $n \to \infty$.

In Section III.2 we describe the SLFV with random selection, the related martingale problem and a dual process. Rigorous constructions and proofs of all results are deferred to Section III.8. Then, in Section III.3 we state the main results of this chapter, namely the convergence to the rSBM in the sparse regime (see Theorem III.3.5, under Assumptions III.3.2 and III.3.1), the convergence to the Fisher-KPP equation (see Theorem III.3.11, under Assumptions III.3.7 and III.3.8). We also state two technical results that are at the heart of our proof methods. On the one hand a two-scale regularization results for the semidiscrete Laplacian (see Theorem III.3.13). On the other hand an approximation result for the Anderson Hamiltonian (see Theorem III.3.15), which we will prove in the upcoming chapter.

Section III.4 is devoted to the proof of the convergence of the Spatial Lambda-Fleming-Viot process with selection in rough potential to the rSBM in the sparse regime, whereas in Section III.5 we establish the convergence to the Fisher-KPP equation. Section III.6 covers the Schauder estimates.

Notations

Indicate with $|A|$ the Lebesgue measure of a Borel set $A \subseteq \mathbf{T}^d$. Let $\overline{B}_n(x) \subseteq \mathbf{R}^d$ be the ball (with respect to the Euclidean norm) of volume n^{-d} about x. Similarly, let $\overline{Q}_n(x) \subset \mathbf{R}^d$ be the d-dimensional cube

$$y \in \overline{Q}_n(x) \iff (y-x)_i \in \Big[-\frac{1}{2n}, \frac{1}{2n}\Big), \qquad \forall i \in \{1, \dots, d\}.$$

As we work on the $d-$dimensional torus we denote with we define

$$B_n(x) = \overline{B}_n(x)/\mathbf{Z}^d \subseteq \mathbf{T}^d, \quad Q_n(x) = \overline{Q}_n(x)/\mathbf{Z}^d \subseteq \mathbf{T}^d$$

for the projections of $\overline{Q}_n, \overline{B}_n$ on the torus. To make sure that these sets still satisfy the normalization

$$|B_n(x)| = |Q_n(x)| = n^{-d},$$

observe that for every $d \in \mathbf{N}$ there exists a $c(d) \in \mathbf{N}$ such that

$$\overline{B}_n(0), \overline{Q}_n(0) \subseteq \Big(-\frac{1}{2}, \frac{1}{2}\Big)^d, \qquad \forall n \geqslant c(d).$$

For this reason, throughout this chapter we consider only

$$n \geqslant c(d).$$

We will not repeat this assumption to avoid an additional burden on the notation. Now, since $Q_n(x)$ for $x \in \mathbf{Z}_n^d \cap \mathbf{T}^d$ are disjoint, we can decompose the torus in the disjoint union

$$\mathbf{T}^d = \bigcup_{x \in \mathbf{Z}_n^d \cap \mathbf{T}^d} Q_n(x).$$

For integrable $w \colon \mathbf{T}^d \to \mathbf{R}$ define $\Pi_n w(x)$ as an average integral of w over $B_n(x)$, that is

$$\Pi_n w(x) := \fint_{B_n(x)} w(y)\,\mathrm{d}y := \frac{1}{|B_n(x)|}\int_{B_n(x)} w(y)\,\mathrm{d}y.$$

Since characteristic functions normalized to integrate to 1 enter the calculations repeatedly, for a set A we write:

$$\chi_A(x) = \frac{1}{|A|} 1_A(x).$$

In the special case of balls and cubes we additionally define

$$\chi_n(x) := n^d 1_{B_n(0)}(x), \qquad \widehat{\chi}_n(k) = \widehat{\chi}(n^{-1}k) := \mathscr{F}_{\mathbf{T}^d}\chi_n(k) = \mathscr{F}_{\mathbf{R}^d}\chi_n(k), \qquad \forall x \in \mathbf{T}^d,\ k \in \mathbf{Z}^d,$$
$$\chi_{Q_n}(x) := n^d 1_{Q_n(0)}(x), \qquad \widehat{\chi}_{Q_n}(k) = \widehat{\chi}_Q(n^{-1}k) := \mathscr{F}_{\mathbf{T}^d}\chi_{Q_n}(k) = \mathscr{F}_{\mathbf{R}^d}\chi_{Q_n}(k), \quad \forall x \in \mathbf{T}^d,\ k \in \mathbf{Z}^d.$$

Observe that in order to obtain the identity between the Fourier transform on the torus and in the full space, we have used that $n \geqslant c(d)$.
A special role will be played by the *semidiscrete Laplace* operator $\mathscr{A}_n$:

$$\mathscr{A}_n(\varphi)(x) = n^2 \fint_{B_n(x)} \fint_{B_n(y)} \fint_{B_n(z)} \fint_{B_n(r)} \varphi(s) - \varphi(x)\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}y = n^2\Big(\Pi_n^4\varphi - \varphi\Big)(x). \tag{III.1}$$

Such an operator is a Fourier multiplier with

$$\mathscr{A}_n = \vartheta_n(D), \qquad \vartheta_n(k) = n^2\Big(\widehat{\chi}^4(n^{-1}k) - 1\Big).$$

III.2 The spatial Λ-Fleming-Viot process in a random environment

We now describe the Spatial Lambda-Fleming-Viot process. In addition to the original neutral process we consider the effect of a randomly chosen spatially inhomogeneous selection. We consider a population that presents two genetic types, $\mathfrak{a}$ and $\mathfrak{A}$. At each time $t \geq 0$, X^n_t is a random function such that

$$X^n_t(x) = \text{ proportion of individuals of type } \mathfrak{a} \text{ at time } t \text{ and at position } x.$$

The dynamics of the Spatial Lambda-Fleming-Viot model are determined by reproduction events. In order to incorporate selection, we distinguish two types of reproduction events, neutral and selective. These events are driven by independent Poisson point processes. In simple terms

Neutral event: Both types have the same chance of reproducing,

Selective event: One of the two types is more likely to reproduce than the other.

The strength, and the direction of the selection are encoded respectively by the magnitude and sign of a random function $s_n(\omega)$. The function s_n should satisfy the following requirements.

Assumption III.2.1. *Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ and fix $n \in \mathbf{N}$. We assume that s_n is a measurable map*

$$s_n \colon \Omega \to L^\infty(\mathbf{T}^d; \mathbf{R}),$$

such that:

$$|s_n(\omega, x)| < 1, \qquad \forall \omega \in \Omega, x \in \mathbf{T}^d.$$

Conditional on the realization $s_n(\omega)$ of the environment, the process $X^n(\omega)$ will be a Markov process. Its dynamics are defined below, deferring some technical steps regarding the probability space on which the process is defined until Section III.8.1. We write:

$$M = \Big\{ w \colon \mathbf{T}^d \to [0,1], \ w \ \text{ measurable} \Big\}.$$

Definition III.2.2 (Spatial Λ-Fleming-Viot process with random selection)**.** *Fix $n \in \mathbf{N}, \mathfrak{u} \in (0,1)$ and consider s_n and Ω satisfying Assumption III.2.1 and $X^{n,0} \in M$. Define the process X^n on the probability space*

$$(\Omega \times \mathbb{D}([0,\infty); M), \mathscr{F} \otimes \mathscr{B}(\mathbb{D}([0,\infty); M)), \mathbb{P} \ltimes \mathbb{P}^{\omega,n}),$$

so that for every $\omega \in \Omega$ it holds that

i) The space $(\mathbb{D}([0,\infty); M), \mathbb{P}^{\omega,n})$ supports a pair of independent Poisson point processes $\Pi^{\mathrm{neu}}_\omega$ and $\Pi^{\mathrm{sel}}_\omega$ on $\mathbf{R}_+ \times \mathbf{T}^d$ with intensity measures $\mathrm{d}t \otimes (1 - |s_n(\omega, x)|)\mathrm{d}x$ and $\mathrm{d}t \otimes |s_n(\omega, x)|\mathrm{d}x$ respectively.

ii) The random process $\mathbf{R}_+ \ni t \mapsto X^n_t(\omega)$ *is the canonical process on* $\mathbb{D}([0,\infty);M)$. *It is the Markov process with law* $\mathbb{P}^{\omega,n}$ *started in* $X^{n,0}$ *with values in* M *associated to the generator*

$$\mathscr{L}(n, s_n(\omega), \mathfrak{u})\colon C_b(M;\mathbf{R}) \to C_b(M;\mathbf{R})$$

(see Lemma III.8.2 for its construction), that can be described by the following dynamics.

1. *If* $(t,x) \in \Pi^{\mathrm{neu}}_\omega$, *a neutral event occurs at time* t *in the ball* $B_n(x)$, *namely:*
   - *(a) Choose a parental location* y *uniformly in* $B_n(x)$.
   - *(b) Choose the parental type* $\mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$ *according to the distribution*

   $$\mathbb{P}[\mathfrak{p} = \mathfrak{a}] = \Pi^2_n X^n_{t-}(\omega, y), \quad \mathbb{P}[\mathfrak{p} = \mathfrak{A}] = 1 - \Pi^2_n X^n_{t-}(\omega, y).$$

   - *(c) A proportion* $\mathfrak{u}$ *of the population within* $B_n(x)$ *dies and is replaced by offspring with type* $\mathfrak{p}$. *Therefore, for each point* $z \in B_n(x)$,

   $$X^n_t(\omega, z) = X^n_{t-}(\omega, z)(1 - \mathfrak{u}) + \mathfrak{u}\chi_{\{\mathfrak{p}=\mathfrak{a}\}}.$$

2. *If* $(t,x) \in \Pi^{\mathrm{sel}}_\omega$, *a selective event occurs in the ball* $B_n(x)$, *namely:*
   - *(a) Choose two parental locations* y_0, y_1 *independently, uniformly in* $B_n(x)$.
   - *(b) Choose the two parental types,* $\mathfrak{p}_0, \mathfrak{p}_1$, *independently, according to*

   $$\mathbb{P}[\mathfrak{p}_i = \mathfrak{a}] = \Pi^2_n X^n_{t-}(\omega, y_i), \quad \mathbb{P}[\mathfrak{p}_i = \mathfrak{A}] = 1 - \Pi^2_n X^n_{t-}(\omega, y_i).$$

   - *(c) A proportion* $\mathfrak{u}$ *of the population within* $B_n(x)$ *dies and is replaced by an offspring with type chosen as follows:*
     - *i. If* $s_n(\omega, x) < 0$, *their type is set to be* $\mathfrak{a}$ *if* $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$, *and* $\mathfrak{A}$ *otherwise. Thus for each* $z \in B_n(x)$

     $$X^n_t(\omega, x) = (1 - \mathfrak{u}) X^n_{t-}(\omega, z) + \mathfrak{u}\chi_{\{\mathfrak{p}_0=\mathfrak{p}_1=\mathfrak{a}\}}.$$

     - *ii. If* $s_n(\omega, x) > 0$, *their type is set to be* $\mathfrak{a}$ *if* $\mathfrak{p}_0 = \mathfrak{p}_1 = \mathfrak{a}$ *or* $\mathfrak{p}_0 \neq \mathfrak{p}_1$ *and* $\mathfrak{A}$ *otherwise, so that for each* $z \in B_n(x)$,

     $$X^n_t(\omega, z) = (1 - \mathfrak{u}) X^n_{t-}(\omega, z) + \mathfrak{u}(1 - \chi_{\{\mathfrak{p}_1=\mathfrak{p}_2=\mathfrak{A}\}}).$$

Remark III.2.3. *In the original SLFV process the probabilities at points* 1.*b*, 2.*b of the definition do not depend on the local average* $\Pi^2_n X_{t-}(y)$. *Instead they depended only on the evaluation at the exact point* $X_{t-}(y)$. *Introducing the local average is a mathematical simplification of the model: the main implication is that the operator* $\mathscr{H}^\omega_n$ *considered in Theorem III.3.15 will be selfadjoint.*

Remark III.2.4. *In Section III.8.1 we construct only the Markov jump process* $X^n(\omega)$. *The Poisson point processes mentioned in Definition III.2.2 are not described explicitly, but can be reconstructed from the jump times and jump locations.*

Most of the arguments we use take advantage of the martingale representation of the process. We record this representation as a lemma. The proof can be found in Section III.8.1. For a function $\varphi\colon [0,\infty) \to \mathbf{R}$ we write

$$\varphi_{t,s} = \varphi_t - \varphi_s.$$

Lemma III.2.5. *Fix $\omega \in \Omega$ and X^n the SLFV as in the previous definition. For every $\varphi \in L^\infty(\mathbf{T}^d)$ the process $t \mapsto \langle X^n_t(\omega), \varphi\rangle$ satisfies the following martingale problem, for $t \geq s \geq 0$:*

$$\begin{aligned}\langle X^n_{t,s}(\omega), \varphi\rangle = \mathfrak{u} n^{-d} \int_s^t & \langle \big(\Pi^4_n - \mathrm{Id}\big)(X^n_r(\omega)), \varphi\rangle \\ & + \langle \Pi_n \Big[s_n(\omega)\Big(\Pi^3_n X^n_r(\omega) - (\Pi^3_n X^n_r(\omega))^2\Big)\Big], \varphi\rangle \,\mathrm{d}r + M^n_{t,s}(\varphi)\end{aligned}$$

where $M^n_{t,s}(\varphi)$ is the increment of a square integrable martingale with predictable quadratic variation given by

$$\begin{aligned}\langle M^n(\varphi)\rangle_t = \mathfrak{u}^2 n^{-2d} \int_0^t & \langle (1+s_n(\omega))\Pi^3_n X^n_r(\omega), (\Pi_n\varphi)^2 - 2(\Pi_n\varphi)\Big(\Pi_n(X^n_r(\omega)\varphi)\Big)\rangle \\ & + \langle \Big(\Pi_n(X^n_r(\omega)\varphi)\Big)^2, 1\rangle \\ & - \langle s_n(\omega)(\Pi^3_n X^n_r(\omega))^2, (\Pi_n\varphi)^2 - 2(\Pi_n\varphi)\Big(\Pi_n(X^n_r(\omega)\varphi)\Big)\rangle \,\mathrm{d}r.\end{aligned}$$

III.3 Scaling limits and main results

III.3.1 Sparse regime

First, we consider a scaling regime in which the part of the population of type $\mathfrak{a}$ is rare, which means that X^n_t is very close to 0. To quantify what we mean with "close to zero", we introduce a smallness parameter $\varrho > 0$. We assume that the initial condition $X^{n,0}$ is of order $n^{-\varrho}$ and we will work under the following, mostly technical, assumptions on the parameter ϱ.

Assumption III.3.1 (Sparsity)**.** *Fix any*

$$\varrho > \frac{3d}{2}$$

and a sequence $X^{n,0} \in M$ such that for some $Y^0 \in \mathscr{M}(\mathbf{T}^d)$

$$\lim_{n\to\infty} n^{\varrho} X^{n,0} = Y^0 \text{ in } \mathscr{M}(\mathbf{T}^d).$$

Our selection coefficient will converge to space white noise, similarly to what we have done in the previous chapter. To obtain a non-trivial scaling limit in dimension $d = 2$, renormalisation has to be taken into account. Hence we define c_n as

$$c_n = \sum_{k\in\mathbf{Z}^2} \frac{\hat{\chi}^2(n^{-1}k)\hat{\chi}_Q(n^{-1}k)}{-\vartheta_n(k)+1}. \tag{III.2}$$

The assumptions on the noise are summarized in what follows.

Assumption III.3.2 (White noise scaling)**.** *Fix $d = 1$ or 2 and consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ supporting for any $n \in \mathbf{N}$ a sequence of i.i.d. random variables $\{Z_n(x)\}_{x\in\mathbf{Z}_n^d}$ satisfying:*

$$\mathbb{E}\Big[Z_n^2(x)\Big] = 1, \qquad Z_n(\omega, x) \in (-2,2), \qquad \text{for all } \ x \in \mathbf{Z}_n^d, \ \ \omega \in \Omega.$$

Then define

$$s_n(\omega, y) = Z_n(\omega, x) - n^{-\frac{d}{2}} c_n 1_{\{d=2\}}, \ \ \text{if } y \in Q_n(x), \qquad \forall \omega \in \Omega, x \in \mathbf{T}^d$$

and write:

$$\xi_e^n(\omega, x) = n^{\frac{d}{2}} s_n(\omega, x), \qquad \xi^n(\omega, x) = \xi_e^n(\omega, x) + c_n 1_{\{d=2\}}.$$

Under appropriate scaling, we will prove that the process X^n converges to a rough superBrownian motion. First, we recall the Anderson Hamiltonian on the torus, and its relationship to our setting.

Lemma III.3.3. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a white noise $\xi : \Omega \to \mathscr{S}'(\mathbf{T}^d)$. For almost all $\omega \in \Omega$ there exists an operator*

$$\mathscr{H}^\omega : \mathscr{D}_\omega \subseteq C(\mathbf{T}^d) \to C(\mathbf{T}^d),$$

with a dense domain $\mathscr{D}_\omega \subseteq C(\mathbf{T}^d)$, such that

$$\mathscr{H}^\omega = \lim_{n\to\infty} \Big[\mathscr{A}_n + \Pi_n^2(\xi^n - c_n 1_{\{d=2\}})\Pi_n^2\Big] =: \nu_0 \Delta + \xi,$$

with ν_0 defined by:

$$\nu_0 = \frac{1}{3} \ \ in \ \ d = 1, \qquad \nu_0 = \frac{1}{\pi} \ \ in \ \ d = 2.$$

The limit is taken in distribution, with ξ^n as in Assumption III.3.2. The precise meaning of the limit is provided in Theorem III.3.15.

This lemma is a consequence of Proposition III.3.14 and Theorem III.3.15 below. Now let us recall the definition of the rSBM in this setting. We provide only one of the equivalent characterizations of Definition II.2.11. One can follow the same calculations as in the previous chapter and show all other properties as well. In fact, we will silently use duality to obtain uniqueness in law of the process. In contrast to the previous chapter, though, we do not rely on the mild martingale problem formulation. Instead our tightness proof will rely on the convergence, in an appropriate sense, of the eigenfunctions and eigenvalues of the semidiscrete Anderson Hamiltonian.

Definition III.3.4. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a white noise ξ and consider $Y^0 \in \mathscr{M}(\mathbf{T}^d)$. Consider an enlarged probability space $(\Omega \times \overline{\Omega}, \mathscr{F} \otimes \overline{\mathscr{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^\omega)$, where $\mathscr{F} \otimes \overline{\mathscr{F}}$ is the product sigma-field and $\mathbb{P}^\omega$ is the conditional (given the realization ω of the environment) law of a process*

$$Y : \Omega \times \overline{\Omega} \to C([0,\infty); \mathscr{M}(\mathbf{T}^d)).$$

For any $\omega \in \Omega$ let $\{\mathscr{F}_t^\omega\}_{t\geq 0}$ be the filtration generated by $t \mapsto Y_t(\omega)$, right-continuous and enlarged with all null sets. And let $\mathscr{H}^\omega$ be the operator in the definition above. Y is a rough superBrownian motion, if for all $\varphi \in \mathscr{D}_\omega$ and $T > 0$, the process

$$M_t^\varphi := \langle Y_t(\omega), \varphi\rangle - \langle Y^0, \varphi\rangle - \int_0^t \langle Y_s(\omega), \mathscr{H}^\omega \varphi\rangle \mathrm{d}s$$

is a centered continuous, square integrable $\mathscr{F}_t^\omega$-martingale on $[0, T]$ with quadratic variation

$$\langle M^\varphi\rangle_t = \int_0^t \langle Y_s(\omega), \varphi^2\rangle \,\mathrm{d}s.$$

We are now in position to state the first main result of this chapter.

Theorem III.3.5. *For any $\varrho > \frac{5}{2}d$ consider a random environment s_n as in Assumption III.3.2, and initial conditions $X^{n,0}$ as in Assumption III.3.1. Consider the process X^n as in Definition III.2.2, associated, for each $\omega \in \Omega$, to the generator*

$$n^{d+2+\eta}\mathscr{L}(n, n^{\frac{d}{2}-2}s_n(\omega), n^{-\eta}),$$

with η defined by

$$\eta := \varrho + 2 - d. \tag{III.3}$$

Then the process $t \mapsto Y_t^n = n^\varrho X_t^n$ converges in distribution:

$$\lim_{n\to\infty} Y^n = Y \quad in \quad \mathbb{D}([0,\infty); \mathscr{M}(\mathbf{T}^d)),$$

where Y is the unique in distribution rough superBrownian motion as in Definition III.3.4, started in Y^0.

Remark III.3.6. *Let us comment on the scaling in the previous theorem. The temporal speed of order $n^{d+2+\eta}$ corresponds to parabolic scaling. The factor n^d is payed to cancel the corresponding factor appearing in Lemma III.2.5. The factor n^η instead cancels with the size of the impact. So we are left with a factor n^2, which corresponds to parabolic scaling, since spatial distances are of order $1/n$.*

As for the selection, we necessarily consider a weak *regime, that is $|s_n| \simeq n^{-2}$, which cancels with the temporal speed up, providing a term of macroscopic order.*

Finally, the smallness of the impact enters only to see fluctuations of the correct order. It is clear that impacts should be small, since we expect at least jumps to become small. For a given positive smooth test function φ a jump is of magnitude:

$$|\langle Y_t^n, \varphi\rangle - \langle Y_{t-}^n, \varphi\rangle| \lesssim n^\varrho n^{-\eta}\int_{B_n(x)} |1_{\mathfrak{a}} - X_{t-}^n(y)|\varphi(y)\,\mathrm{d}y \lesssim n^\varrho n^{-\eta}\int_{B_n(x)} \varphi(y)\,\mathrm{d}y.$$

So we expect at least $\eta > \varrho - d$ (here with $1_{\mathfrak{a}}$ we indicate events in which particles of type $\mathfrak{a}$ are produced). This is not enough, since we would like the quadratic variation to converge, so we should impose that the sum of the squared jumps is finite, which brings to the pessimistic bound:

$$\sum_{t\leq n^{2+d+\eta}} |\langle Y_t^n, \varphi\rangle - \langle Y_{t-}^n, \varphi\rangle|^2 \lesssim n^{2+d+\eta} n^{2(\varrho-d-\eta)}.$$

This would lead us to require $\eta \geq 2\varrho + 2 - d$. In doing so, though, we did not consider the sparsity assumption. In fact by sparsity events that produce particles of type $\mathfrak{a}$ will be much less common (in fact they happen with probability $\simeq n^{-\varrho}$), so the typical jump will be of order

$$|\langle Y_t^n, \varphi\rangle - \langle Y_{t-}^n, \varphi\rangle| \lesssim n^{\varrho} n^{-\eta} \int_{B_n(x)} X_{t-}^n(y)\varphi(y)\,\mathrm{d}y \lesssim n^{-\eta-d}\|\varphi\|_\infty.$$

In this way we obtain the sharper bound:

$$\mathbb{E}\Bigg(\sum_{t \leq n^{2+d+\eta}} |\langle Y_t^n, \varphi\rangle - \langle Y_{t-}^n, \varphi\rangle|^2\Bigg) \lesssim n^{2+d+\eta}(n^{2(\varrho-d-\eta)}n^{-\varrho} + n^{-2(d+\eta)}).$$

So we finally get the correct scaling $\eta \geq \varrho + 2 - d$. Of course, a more efficient derivation of the required scaling follows from the predictable quadratic variation in Lemma III.2.5.

III.3.2 Diffusive regime

The second scaling regime we consider is a purely diffusive one. As before, the impact parameter $\mathfrak{u}$ is scaled as $n^{-\eta}$. The restrictions on the value of η follows

Assumption III.3.7. *Choose η such that*

$$\eta = 1 \;\; if \;\; d = 1, \qquad \eta > 0 \;\; if \;\; d = 2.$$

In this diffusive regime we still assume that the selection coefficient may be random, but we restrict to smooth selection.

Assumption III.3.8. *Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\overline{\xi}$ be a measurable map:*

$$\overline{\xi} \colon \Omega \to \mathcal{S}(\mathbf{T}^d).$$

Then define:

$$s_n(\omega, x) = (n^{-2}\bar{\xi}(\omega, x)) \vee 1 \wedge (-1).$$

The limiting process in this setting will be the (stochastic if $d = 1$) Fisher–KPP equation in a random potential, defined as follows.

Definition III.3.9. *Consider Ω and $\overline{\xi}$ as in Assumption III.3.8. Fix any $\alpha > 0$ and $X^0 \in B^{\alpha}_{2,2}$. A (stochastic if $d = 1$) Fisher-KPP process in random potential is a couple given by a probability space $(\Omega \times \overline{\Omega}, \mathcal{F} \otimes \overline{\mathcal{F}}, \mathbb{P} \ltimes \overline{\mathbb{P}}^{\omega})$ (cf. Definition III.3.4) and a map*

$$X \colon \Omega \times \overline{\Omega} \to L^2_{\mathrm{loc}}([0,\infty); B^{\alpha}_{2,2}).$$

For $\omega \in \Omega$ let $\{\mathcal{F}^{\omega}_t\}_{t\geq 0}$ be the filtration generated by $t \mapsto X_t(\omega)$, right-continuous and enlarged with all null sets. Then for all $\omega \in \Omega$ it is required that, depending on the dimension:

i In dimension $d = 1$ for all $\varphi \in C^\infty(\mathbf{T})$:

$$N_t^{\varphi} := \langle X_t(\omega), \varphi\rangle - \langle X^0, \varphi\rangle - \int_0^t \langle X_s(\omega), \nu_0 \Delta\varphi\rangle - \langle \overline{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi\rangle \mathrm{d}s$$

is a continuous in time, square integrable martingale with quadratic variation

$$\langle N^{\varphi}\rangle_t = \int_0^t \langle X_s(\omega)(1 - X_s(\omega)), \varphi^2\rangle \mathrm{d}s.$$

ii In dimension $d = 2$, X is a solution to

$$\begin{aligned} \partial_t X_t(\omega) &= \nu_0 \Delta X_t(\omega) + \overline{\xi}(\omega) X_t(\omega)(1 - X_t(\omega)), \\ X_0(\omega, x) &= X^0(\omega, x), \qquad \forall x \in \mathbf{T}^d. \end{aligned}$$

The solution is interpreted in the sense that for all $\varphi \in C^\infty(\mathbf{T}^2)$

$$\langle X_t(\omega), \varphi \rangle = \langle X^0, \varphi \rangle + \int_0^t \langle X_s(\omega), \nu_0 \Delta \varphi \rangle + \langle \overline{\xi}(\omega) X_s(\omega)(1 - X_s(\omega)), \varphi \rangle \, \mathrm{d}s.$$

Remark III.3.10. *Note that in the previous definition, since $X \in L^2_{\mathrm{loc}}([0,\infty); B^\alpha_{2,2})$, the quadratic non-linearity:*

$$\int_0^t \langle X_s^2, \varphi \rangle \, \mathrm{d}s$$

is well-defined. Moreover, up to enlarging the probability space, the process can be represented in $d = 1$ as a solution to an SPDE of the form

$$\partial_t X = \nu_0 \Delta X + \overline{\xi} X(1 - X) + \sqrt{X(1 - X)} \widetilde{\xi},$$

where the spatial noise $\overline{\xi}$ is independent of the space-time white noise $\widetilde{\xi}$, following the classical construction by Konno and Shiga [KS88] (see also Theorem II.2.19).

In this setting, we can prove the following scaling limit.

Theorem III.3.11. *Let η satisfy Assumption III.3.7 and s_n be as in Assumption III.3.8. Consider $X_0 \in \mathscr{S}(\mathbf{T}^d)$ with $0 \leqslant X_0(x) \leqslant 1$, $\forall x \in \mathbf{T}^d$, and let $X^n(\omega)$ be the Markov process associated to the generator*

$$n^{\eta + d + 2} \cdot \mathscr{L}(n, s_n(\omega), n^{-\eta})$$

and started in X_0, as Definition III.2.2. There exists an $\alpha > 0$ such that for every $\omega \in \Omega$

$$\{t \mapsto \Pi_n X_t^n(\omega)\}_{n \in \mathbf{N}}$$

is tight in the space $L^2_{\mathrm{loc}}([0,\infty); B^\alpha_{2,2}(\mathbf{T}^d))$. Similarly, the sequence

$$\{t \mapsto X_t^n(\omega)\}_{n \in \mathbf{N}}$$

is tight in $\mathbb{D}([0,\infty); \mathscr{M}(\mathbf{T}^d))$. In particular:

i In dimension $d = 1$ both sequences converge in distribution to the unique in law solution to the martingale problem of the stochastic Fisher-KPP process in a random potential, as in Definition III.3.9.

ii In dimension $d = 2$ both sequences converge in distribution to the unique solution to the Fisher-KPP equation in a random potential as in Definition III.3.9.

Remark III.3.12. *The scaling in Theorem III.3.11 is similar to that of Theorem III.3.5 (in the case $\varrho = 0$). The only difference is the assumption $\eta > 0$ in $d = 2$. As we already commented, the parameter η tunes the strength of the noise, and we expect that at $\eta = 2 - d$ we see fluctuations.*

One would thus naturally also expect that by choosing $\eta = 0$ in $d = 2$ one obtains as a scaling limit the stochastic *Fisher-KPP equation. But, even assuming $\overline{\xi} = 0$ we cannot hope to make sense of the product X_t^2 in the quadratic variation term, since a solution X_t to a stochastic Fisher-KPP equation should not live in a space of positive regularity. This point is not just technical: instead the limit is expected to be deterministic also if $\eta = 0$. If $\overline{\xi} = 0$ one can show that the dual converges to a system of coalescing Brownian motions: in dimension $d = 2$ Brownian motions can get arbitrarily close, but cannot meet. Hence the dual is a system of independent Brownian motions: so the correct scaling limit is the heat equation. In our setting we expect that the same argument holds and the correct scaling limit for $\eta = 0$ should still be the deterministic Fisher-KPP equation. Since we do not have a complete understanding of the dual, this study is left for a future work.*

III.3.3 Proof methods

The main ingredient of the proofs of the scaling limits in the previous sections is a careful study of the semidiscrete Laplace operator $\mathscr{A}_n$. Intuitively, one expects that this operator approximates the Laplacian with periodic boundary conditions and therefore has similar regularizing properties. To quantify this intuition we introduce a division of scales. On large scales, namely for Fourier modes k of order $|k| \lesssim n$ we show that $\mathscr{A}_n$ has the required regularizing properties. On small scales, that is for modes of order $|k| \gtrsim n$ we do not expect any regularization. Instead we prove that small scales are negligible in terms of powers of n. Below we state a slimmed version of the results we require. The proof of the following theorem, as well as additional side results, is the content of Section III.6.

Theorem III.3.13. *Fix any smooth radial function with compact support $\daleth \colon \mathbf{R}^d \to \mathbf{R}$ such that for some $0 < r < R$*

$$\daleth(k) = 1,\ \forall |k| \leq r, \qquad \daleth(k) = 0,\ \forall |k| \geq R.$$

Define

$$\mathscr{P}_n = \daleth(n^{-1}D), \qquad \mathscr{Q}_n = (1-\daleth)(n^{-1}D).$$

For any $\alpha \in \mathbf{R}, p \in [1,\infty]$ the following holds:

i) For any $\zeta > 0$ and $\varphi \in \mathscr{C}^\alpha_p$

$$\mathscr{A}_n\varphi \to \nu_0 \Delta\varphi \ \text{ in } \ \mathscr{C}^{\alpha-2-\zeta}_p, \qquad \text{as } \ n \in \mathbf{N},$$

where

$$\nu_0 = \frac{1}{3} \quad \text{in} \quad d = 1, \qquad \nu_0 = \frac{1}{\pi} \quad \text{in} \quad d = 2. \tag{III.4}$$

ii) Uniformly over $\lambda > 1, n \in \mathbf{N}$ and $\varphi \in \mathscr{C}^\alpha_p$ the following estimates hold:

$$\|\mathscr{P}_n(-\mathscr{A}_n + \lambda)^{-1}\varphi\|_{\mathscr{C}^{\alpha+2}_p} + n^2 \|\mathscr{Q}_n(-\mathscr{A}_n + \lambda)^{-1}\varphi\|_{\mathscr{C}^\alpha_p} \lesssim \|\varphi\|_{\mathscr{C}^\alpha_p}.$$

A precise control of the regularization effects of the semidiscrete Laplacian $\mathscr{A}_n$ allows us to treat semidiscrete approximations of the Anderson model that appear in the study of the rough superBrownian motion. In the next proposition we recall some salient features of the continuous Anderson Hamiltonian.

Proposition III.3.14. *Fix $d = 1$ or 2, $\kappa > 0$ and $(\Omega, \mathscr{F}, \mathbb{P})$ a probability space supporting a space white noise $\xi : \Omega \to \mathscr{S}'(\mathbf{T}^d)$. Then the following hold true for all $\omega \in \Omega$. The Anderson Hamiltonian*

$$\mathscr{H}^\omega = \nu_0 \Delta + \xi(\omega)$$

associated to $\xi(\omega)$ is defined, as constructed[1] in [FN77] in $d = 1$ and [AC15] in $d = 2$. The Hamiltonian, as an unbounded selfadjoint operator on $L^2(\mathbf{T}^d)$, has a discrete spectrum given by pairs of eigenvalues and eigenfunctions $\{(\lambda_k(\omega), e_k(\omega))\}_{k \in \mathbf{N}}$ such that:

$$\lambda_1(\omega) > \lambda_2(\omega) \geq \lambda_3(\omega) \geq \dots, \qquad \lim_{k \to \infty} \lambda_k(\omega) = -\infty, \qquad e_1(\omega, x) > 0, \forall x \in \mathbf{T}^d.$$

In addition, for every $k \in \mathbf{N}$, $e_k(\omega) \in \mathscr{C}^{2 - \frac{d}{2} - \kappa}(\mathbf{T}^d)$, and the set

$$\mathscr{D}_\omega = \{\textit{Finite linear combination of } \{e_k(\omega)\}_{k \in \mathbf{N}}\}$$

is dense in $C(\mathbf{T}^d)$.

The proof of this proposition is postponed to Chapter IV, in Lemmata IV.2.1 and IV.2.2. For the semidiscrete Laplace operator $\mathscr{A}_n$ the following holds.

Theorem III.3.15. *Fix $d = 1$ or 2, $\kappa > 0$ and ξ^n satisfying Assumption III.3.2. Up to changing probability space Ω, the following hold true for almost all ω in Ω.*

For every $k \in \mathbf{N}$ let $m(\lambda_k)$ be the multiplicity of the eigenvalue λ_k of $\mathscr{H}^\omega$ (as in Proposition III.3.14) and let $\{e_k^i(\omega)\}_{i=1}^{m(\lambda_k)}$ be an associated set of orthonormal eigenfunctions. Then $m(\lambda_1) > 0$.

For every $k \in \mathbf{N}$ there exists an $n_0(\omega, k) \in \mathbf{N}$ such that for every $n \geq n_0(\omega, k)$ there exist orthonormal functions $\{e_k^{i,n}(\omega)\}_{i=1}^{m(\lambda_k)} \subseteq L^2(\mathbf{T}^d)$ such that, considering the operator

$$\mathscr{H}_n^\omega := \mathscr{A}_n + \Pi_n^2(\xi^n(\omega) - c_n)\Pi_n^2, \qquad \mathscr{H}_n^\omega : L^2(\mathbf{T}^d) \to L^2(\mathbf{T}^d),$$

with c_n as in (III.2)*, one has for some $\varepsilon > 0$:*

$$\lim_{n \to \infty} e_k^{i,n}(\omega) = e_k^i(\omega), \textit{ in } L^2(\mathbf{T}^d) \qquad \lim_{n \to \infty} \Pi_n e_k^{i,n}(\omega) = e_k^i(\omega) \textit{ in } \mathscr{C}^\varepsilon(\mathbf{T}^d),$$

and

$$\lim_{n \to \infty} \mathscr{H}_n^\omega e_k^{i,n}(\omega) = \lambda_k e_k^i(\omega), \textit{ in } L^2(\mathbf{T}^d), \qquad \lim_{n \to \infty} \Pi_n \mathscr{H}_n^\omega e_k^i(\omega) = \lambda_k e_k^i(\omega) \textit{ in } \mathscr{C}^\varepsilon(\mathbf{T}^d).$$

If the eigenvalue is simple, i.e. $m(\lambda_k) = 1$, then in addition $e_k^n(\omega)$ is an eigenfunction for $\mathscr{H}_n^\omega$:

$$\mathscr{H}_n^\omega e_k^n(\omega) = \lambda_k^n e_k^n(\omega),$$

with $\lim_{n \to \infty} \lambda_k^n = \lambda_k$.

The proof of this result is the content of Section IV.2.4.

[1] To be precise, [FN77] constructs the operator in dimension $d = 1$ with Dirichlet boundary conditions, but their construction can be extended to periodic boundary conditions. Alternatively, the operator can be constructed with arguments similar to the ones presented in Section IV.2.

III.4 Scaling to the rough superBrownian motion

This section is devoted to the proof of Theorem III.3.5. Since we want to prove convergence in distribution for the sequence Y^n, the exact choice of the probability space Ω of Definition III.2.2 is not important. For this reason we adopt the following standing assumption that allows us to work with a suitably chosen probability space.

Assumption III.4.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$, the probability space appearing in Definition III.2.2 and Assumption III.3.2 be such that the results of Proposition III.3.14 and Theorem III.3.15 hold true for almost all $\omega \in \Omega$.*

The first step towards establishing tightness is to restate the martingale problem of Lemma III.2.5 to take into account the scaling assumed in Theorem III.3.5.

Lemma III.4.2. *In the setting of Theorem III.3.5 and under Assumption III.4.1, for every $\omega \in \Omega$ and $n \in \mathbf{N}$, under the law $\mathbb{P}^\omega$, and for every $\varphi \in L^\infty(\mathbf{T}^d)$ the process $t \mapsto \langle Y^n_t(\omega), \varphi\rangle$ satisfies the following martingale problem:*

$$\begin{aligned}\langle Y^n_{t,s}(\omega), \varphi\rangle = \int_s^t & \langle \mathscr{A}_n(Y^n_r(\omega)) + \Pi_n[\xi^n_e(\omega)\Pi^3_n Y^n_r(\omega)], \varphi\rangle \\ & - n^{-\varrho}\langle \Big(\Pi^3_n Y^n_r(\omega)\Big)^2, \xi^n_e(\omega)\Pi_n(\varphi)\rangle \,\mathrm{d}r + M^n_{t,s}(\varphi),\end{aligned} \tag{III.5}$$

where $M^n_\cdot(\varphi)$ is a square integrable martingale with predictable quadratic variation given by:

$$\begin{aligned}\langle M^n(\varphi)\rangle_t = \int_0^t & \langle (1+n^{-2+\frac{d}{2}}s_n(\omega))\Pi^3_n Y^n_r(\omega), (\Pi_n\varphi)^2 - 2n^{-\varrho}\Pi_n(\varphi)\Pi_n(Y^n_r(\omega)\varphi)\rangle \\ & + n^{-\varrho}\langle \Big(\Pi_n(Y^n_r(\omega)\varphi)\Big)^2, 1\rangle \\ & - n^{-\varrho}\langle n^{-2+\frac{d}{2}}s_n(\omega)(\Pi^3_n Y^n_r(\omega))^2, (\Pi_n\varphi)^2 - 2n^{-\varrho}\Pi_n(\varphi)\Pi_n(Y^n_r(\omega)\varphi)\rangle \,\mathrm{d}r.\end{aligned} \tag{III.6}$$

Remark III.4.3. *The only term that is not of lower order in the quadratic variation is*

$$\langle \Pi^3_n Y^n_r, (\Pi_n\varphi)^2\rangle,$$

which explains the superBrownian noise in the limit.

Remark III.4.4. *At first sight this martingale problem has no relationship with the operator*

$$\mathscr{H}^\omega_n = \mathscr{A}_n + \Pi^2_n\xi^n_e(\omega)\Pi^2_n$$

we introduced earlier. The reason for our choice of the approximating operator is that if we test the martingale problem on $\varphi = \Pi_n e^n$, with e^n in the domain of $\mathscr{H}^\omega_n$ (say an eigenfunction), then the first line of the drift becomes

$$\langle Y_r(\omega), \Pi_n \mathscr{H}^\omega_n e^n\rangle,$$

which is exactly the kind of term that Theorem III.3.15 aims at controlling.

In order to obtain the convergence, the first step is to prove a tightness result.

Proposition III.4.5. *In the setting of Theorem III.3.5 and under Assumption III.4.1 fix any $\omega \in \Omega$. For any $T > 0$ the sequence $\{Y^n(\omega)\}_{n\in\mathbf{N}}$ is tight in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$. Moreover any limit point is continuous, i.e. lies in $C([0,T];\mathscr{M}(\mathbf{T}^d))$.*

Proof of Proposition III.4.5. Since $\omega \in \Omega$ is fixed, we omit the dependence on it. The proof relies on Jakubowski's tightness criterion, which we recall in Proposition A.2.1. The criterion consists of a compact containment condition and the tightness of one-dimensional projections of the process.

In a first step of the proof, we establish the compact containment condition. Since for $R > 0$ sets of the form $K_R = \{\mu \colon \langle \mu, 1\rangle \leq R\} \subseteq \mathscr{M}(\mathbf{T}^d)$ are compact in the weak topology, it is sufficient to show that

$$\forall \delta > 0, \quad \exists R(\delta) > 0,\ n(\delta) \in \mathbf{N} \text{ such that } \inf_{n \geq n(\delta)} \mathbb{P}\Big(\sup_{t\in[0,T]} \langle Y^n_t, 1\rangle \leq R(\delta)\Big) \geq 1 - \delta. \tag{III.7}$$

In a second step, we establish the one-dimensional tightness. By Theorem III.3.15 (since the domain $\mathscr{D}_\omega$ is dense in $C(\mathbf{T}^d)$), it is sufficient to show that for every $k \in \mathbf{N}$ the process $\langle Y^n_t, e_k\rangle$ is tight in $\mathbb{D}([0,T];\mathbf{R})$, where the sequence $\{e_k\}_{k\in\mathbf{N}}$ is an orthonormal basis of $L^2(\mathbf{T}^d)$ consisting of eigenfunctions of $\mathscr{H}$, as in Proposition III.3.14. By Aldous' tightness criterion [Ald78, Theorem 1], this reduces to proving that for any sequence of stopping times τ_n, taking finitely many values and adapted to the filtration of Y^n, and any sequence δ_n of constants such that $\delta_n \to 0$ as $n \to \infty$

$$\forall \delta > 0, \qquad \lim_{n\to\infty} \mathbb{P}\Big(|\langle Y^n_{\tau_n+\delta_n}, e_k\rangle - \langle Y^n_{\tau_n}, e_k\rangle| \geq \delta\Big) = 0. \tag{III.8}$$

In the third step we address the continuity of the limiting process.

Step 1. By Theorem III.3.15, for any $k \in \mathbf{N}$ and $n \geq n_0(k)$ there exists a function $e^n_k \in L^2(\mathbf{T}^d)$ such that $\Pi_n e^n_k \to e_k$ in $\mathscr{C}^\varepsilon(\mathbf{T}^d)$, and $\Pi_n \mathscr{H}_n e^n_k \to \lambda_k e_k$ in $\mathscr{C}^\varepsilon(\mathbf{T}^d)$ for some $\varepsilon > 0$. In particular, choose $k = 1$. Then λ_1 is simple and we can choose e^n_1 to be an eigenfunction of $\mathscr{H}_n$ of eigenvalue $\lambda^n_1 \to \lambda_1$. Since $e_1 > 0$, we may assume that $\Pi_n e^n_1 > 0, \forall n \geq n_0(1)$ and hence for any positive measure μ there exists a $C > 0$ such that

$$\langle \mu, 1\rangle \leq C\langle \mu, \Pi_n e^n_1\rangle, \qquad \forall n \geq n_0(1).$$

Therefore (III.7) follows if one can show that

$$\forall \delta > 0, \quad \exists R(\delta) > 0,\ n(\delta) \geq n_0(1) \text{ such that } \inf_{n\geq n(\delta)} \mathbb{P}\Big(\sup_{t\in[0,T]} \langle Y^n_t, \Pi_n e^n_1\rangle \leq R(\delta)\Big) \geq 1 - \delta.$$

Let us focus our attention on $\langle Y^n_t, \Pi_n e^n_1\rangle$. By the martingale representation (III.5) one obtains

$$\langle Y^n_t, \Pi_n e^n_1\rangle = \langle Y^n_0, \Pi_n e^n_1\rangle + \int_0^t \lambda^n_0 \langle Y^n_r, \Pi_n e^n_1\rangle - n^{-\varrho}\langle \big(\Pi^3_n Y^n_r\big)^2, \xi^n_\varepsilon \Pi^2_n e^n_1\rangle \mathrm{d}r + M^n_t(\Pi_n e^n_1).$$

To treat the nonlinear quadratic term, we shall consider a stopped process. Let us fix $R > 0$ and consider the stopping time τ_R and a parameter ϱ_0, defined as

$$\tau_R := \inf\{t \geq 0 \,:\, \langle Y^n_t, \Pi_n e^n_1\rangle \geq R\}, \qquad \varrho_0 = \varrho - \frac{d}{2} - d.$$

Since $|\xi^n(x)| \lesssim n^{\frac{d}{2}}$ and since

$$\|\Pi_n^3 Y_r^n\|_\infty \leqslant \|\Pi_n Y_r^n\|_\infty \leqslant n^d \langle Y_r^n, 1\rangle \lesssim n^d \langle Y_r^n, \Pi_n e_1^n\rangle,$$

one can bound

$$n^{-\rho}|\langle (\Pi_n^3 Y_{r\wedge\tau_R}^n)^2, \xi^n \Pi_n^2 e_1^n\rangle| \lesssim n^{-\rho+\frac{d}{2}+d}\langle Y_{r\wedge\tau_R}^n, \Pi_n e_1^n\rangle^2 \lesssim Rn^{-\rho_0}\langle Y_{r\wedge\tau_R}^n, \Pi_n e_1^n\rangle,$$

and therefore

$$\mathbb{E}|\langle Y_{t\wedge\tau_R}^n, \Pi_n e_1^n\rangle|^2 \lesssim \langle Y_0^n, 1\rangle + (1+Rn^{-\rho_0})\int_0^t \mathbb{E}|\langle Y_{r\wedge\tau_R}^n, \Pi_n e_1^n\rangle|^2 \,\mathrm{d}r + \mathbb{E}\langle M^n(\Pi_n e_1^n)\rangle_{t\wedge\tau_R}.$$

Furthermore, using the formula for the predictable quadratic variation from Lemma III.4.2 one obtains

$$\begin{aligned}\mathbb{E}\langle M^n(\Pi_n e_1^n)\rangle_{t\wedge\tau_R} \leqslant \mathbb{E}\int_0^t &\langle (1+n^{-2+\frac{d}{2}}s_n)\Pi_n^3 Y_{r\wedge\tau_R}^n, (\Pi_n^2 e_1^n)^2\rangle + n^{-\rho}\langle \big(\Pi_n(Y_{r\wedge\tau_R}^n \Pi_n e_1^n)\big)^2, 1\rangle \\ &+ n^{-\rho}\langle n^{-2+\frac{d}{2}}|s_n|(\Pi_n^3 Y_{r\wedge\tau_R}^n)^2, (\Pi_n^2 e_1^n)^2 + 2n^{-\rho}(\Pi_n^2 e_1^n)\Pi_n(Y_{r\wedge\tau_R}^n \Pi_n e_1^n)\,\mathrm{d}r.\end{aligned}$$

Since by Assumption III.3.2 $n^{-2+\frac{d}{2}}|s_n| \leqslant 2n^{-2+\frac{d}{2}}$, and since $\sup_{n\geqslant n_0(1)}\|\Pi_n e_1^n\|_\infty < \infty$ as well as $0 \leqslant Y_r \leqslant n^\rho$, we can rewrite the bound as:

$$\begin{aligned}\mathbb{E}\langle M^n(\Pi_n e_1^n)\rangle_{t\wedge\tau_R} &\lesssim \mathbb{E}\int_0^t \langle \Pi_n^3 Y_{r\wedge\tau_R}^n, \Pi_n^2 e_1^n\rangle + \langle \Pi_n(Y_{r\wedge\tau_R}^n \Pi_n e_1^n), 1\rangle \\ &\qquad + \langle \Pi_n^3 Y_{r\wedge\tau_R}^n, \Pi_n^2 e_1^n\rangle \,\mathrm{d}r \\ &\lesssim \mathbb{E}\int_0^t \langle Y_{r\wedge\tau_R}, \Pi_n e_1^n\rangle \,\mathrm{d}r.\end{aligned}$$

Therefore, by Gronwall's inequality, there exists a $C > 0$ such that

$$\sup_{0\leq t\leq T} \mathbb{E}|\langle Y_{t\wedge\tau_R}^n, \Pi_n e_1^n\rangle|^2 \lesssim e^{C(1+Rn^{-\rho_0})}. \tag{III.9}$$

It follows that if $n \geq R^{\frac{1}{\rho_0}}$

$$\mathbb{P}\Big(\sup_{0\leq t\leq T} |\langle Y_t^n, \Pi_n e_1^n\rangle| \geq R\Big) = \mathbb{P}\Big(|\langle Y_{\tau_R\wedge T}^n, \Pi_n e_1^n\rangle| = R\Big) \lesssim R^{-2}.$$

This concludes the proof of the compact containment condition (III.7).

Step 2. Next we want to prove (III.8), so let us fix $k \in \mathbf{N}, \gamma > 0$ and $\delta > 0$. In view of calculations from Step 1 there exist $R(\gamma), n(\gamma)$ for which (III.7) holds (with δ replaced by γ). In addition, for some $n(\gamma,\delta) \geqslant n(\gamma)$ we may also assume that

$$\forall n \geq n(\gamma,\delta) \qquad \|e_k - \Pi_n e_k^n\|_{L^\infty} \leq \frac{\delta}{2R(\gamma)}.$$

Hence, for every $n \geq n(\gamma,\delta)$

$$\mathbb{P}\Big(|\langle Y_{\tau_n+\delta_n}^n, e_k\rangle - \langle Y_{\tau_n}^n, e_k\rangle| \geq \delta\Big) \leq \gamma + \mathbb{P}\Big(|\langle Y_{\tau_n+\delta_n}^n, \Pi_n e_k^n\rangle - \langle Y_{\tau_n}^n, \Pi_n e_k^n\rangle| \geq \delta\Big).$$

Now, using the definition of $R(\gamma)$ (and writing for simplicity R instead of $R(\gamma)$):

$$\mathbb{P}\Big(|\langle Y^n_{\tau_n+\delta_n}, \Pi_n e^n_k\rangle - \langle Y^n_{\tau_n}, \Pi_n e^n_k\rangle| \geq \delta\Big) \leq \gamma + \mathbb{P}\Big(|\langle Y^n_{(\tau_n+\delta_n)\wedge\tau_R}, \Pi_n e^n_k\rangle - \langle Y^n_{\tau_n\wedge\tau_R}, \Pi_n e^n_k\rangle| \geq \delta\Big).$$

At this point, via the representation of Lemma III.4.2 we have that

$$\langle Y^n_{(\tau_n+\delta_n)\wedge\tau_R} - Y^n_{\tau_n\wedge\tau_R}, \Pi_n e^n_k\rangle = \int_{\tau_n\wedge\tau_R}^{(\tau_n+\delta_n)\wedge\tau_R} \langle Y^n_r, \Pi_n \mathscr{H}_n e^n_k\rangle - n^{-\varrho}\langle\big(\Pi_n Y^n_r\big)^2, \xi^n_e \Pi^2_n e^n_k\rangle \,\mathrm{d}r$$
$$+ M^n_{\tau_n+\delta_n}(\Pi_n e^n_k) - M^n_{\tau_n}(\Pi_n e^n_k).$$

Hence we obtain

$$\begin{aligned}
&\mathbb{P}\Big(|\langle Y^n_{(\tau_n+\delta_n)\wedge\tau_R}, \Pi_n e^n_k\rangle - \langle Y^n_{\tau_n\wedge\tau_R}, \Pi_n e^n_k\rangle| \geq \delta\Big) \\
&\qquad\leqslant \mathbb{P}\Big(\Big|\int_{\tau_n\wedge\tau_R}^{(\tau_n+\delta_n)\wedge\tau_R} \langle Y^n_r, \Pi_n \mathscr{H}_n e^n_k\rangle - n^{-\varrho}\langle\big(\Pi_n Y^n_r\big)^2, \xi^n_e \Pi^2_n e^n_k\rangle \,\mathrm{d}r\Big| \geq \frac{\delta}{2}\Big) \\
&\qquad\quad + \frac{4}{\delta^2}\mathbb{E}|M^n_{(\tau_n+\delta_n)\wedge\tau_R}(\Pi_n e^n_k) - M^n_{\tau_n\wedge\tau_R}(\Pi_n e^n_k)|^2,
\end{aligned}$$

where we used Markov's inequality in the last line. Following the calculations of Step 1 and using that both $\Pi_n\mathscr{H}_n e^n_k$ and $\Pi_n e^n_k$ are uniformly bounded in $\mathscr{C}^\varepsilon(\mathbf{T}^d)$ for some $\varepsilon > 0$, we now find

$$\begin{aligned}
&\Big|\int_{\tau_n\wedge\tau_R}^{(\tau_n+\delta_n)\wedge\tau_R} \langle Y^n_r, \Pi_n \mathscr{H}_n e^n_k\rangle - n^{-\varrho}\langle\big(\Pi_n Y^n_r\big)^2, \xi^n_e \Pi^2_n e^n_k\rangle \,\mathrm{d}r\Big| \\
&\qquad\qquad \lesssim \int_{\tau_n}^{\tau_n+\delta_n} \langle Y^n_{r\wedge\tau_R}, 1\rangle \,\mathrm{d}r \lesssim \delta_n R(\gamma).
\end{aligned}$$

Similar calculations for the quadratic variation show that

$$\mathbb{E}|M^n_{(\tau_n+\delta_n)\wedge\tau_R}(\Pi_n e^n_k) - M^n_{\tau_n\wedge\tau_R}(\Pi_n e^n_k)|^2 \lesssim \delta_n R(\gamma)$$

Collecting all the bounds we proves so far and passing to the limit $n \to \infty$ we obtain that

$$\limsup_{n\to\infty} \mathbb{P}\Big(|\langle Y^n_{\tau_n+\delta_n}, e_k\rangle - \langle Y^n_{\tau_n}, e_k\rangle| \geq \delta\Big) \leq 2\gamma$$

Since γ is arbitrary, this proves (III.8).

Step 3. So far any limit point Y of the sequence Y^n lies in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$. Since $\mathscr{M}(\mathbf{T}^d)$ is endowed with the weak topology, to prove that actually $Y \in C([0,T];\mathscr{M}(\mathbf{T}^d))$, it is sufficient to show that for any continuous function φ, $\langle Y_t, \varphi\rangle$ is continuous in time. Here one can apply a criterion [EK86, Theorem 3.10.2] according to which it is sufficient to prove that the maximum size of a jump converges weakly to zero. In our case such convergence is even almost sure, since:

$$|\langle Y^n_t, \varphi\rangle - \langle Y^n_{t-}, \varphi\rangle| \lesssim n^{\varrho-d-\eta}\|\varphi\|_{C(\mathbf{T}^d)} = n^{-2}\|\varphi\|_{C(\mathbf{T}^d)}.$$

This follows from the definition of the generator, as well as the exact definition of η (cf. Equation (III.3)), which imply that jumps are bounded as follows:

$$\|Y_t^n - Y_{t-}^n\|_{L^\infty} \lesssim n^{\varrho-\eta} \lesssim 1.$$

Since a jump has an impact only in a ball $B_n(x)$ for some $x \in \mathbf{T}^d$, integrating φ over such ball guarantees the previous bound. □

Finally we are in position to deduce Theorem III.3.5.

Proof of Theorem III.3.5. By Proposition III.4.5 the sequence $Y^n(\omega)$ is tight, for every $\omega \in \Omega$, under Assumption III.4.1 (recall that we can always put ourselves in the setting of this assumption by changing probability space, which does not affect the convergence in distribution). It remains to show that, for a fixed realization $\omega \in \Omega$, every limit point satisfies the martingale problem for the rough superBrownian motion as in Definition III.3.4, which is covered by Steps 1 and 2, and that solutions to such martingale problems are unique, which is covered by Step 3.

Step 1. As in the proof of Proposition III.4.5, since $\omega \in \Omega$ is fixed we omit writing it. Moreover it is sufficient to fix a finite but arbitrary time horizon $T > 0$ and check the martingale property until that time. Assume that (up to taking a subsequence and applying the Skorohod representation theorem) $Y^n \to Y$ almost surely in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$. Recall that the domain $\mathscr{D}$ of the Anderson Hamiltonian is composed of finite linear combinations of eigenfunctions, hence we have to check the martingale property for φ of the form

$$\varphi = \sum_{i=1}^m \alpha_{k_i} e_{k_i},$$

for some $m \in \mathbf{N}$, $k_1, \dots k_m \in \mathbf{N}$, $\alpha_{k_i} \in \mathbf{R}$, and where $\{e_k\}_{k\in\mathbf{N}}$ is the set of eigenfunctions of $\mathscr{H}$. Now consider the approximate eigenfunctions e_k^n from Theorem III.3.15 and define φ^n as

$$\varphi^n = \sum_{i=1}^m \alpha_{k_i} e_{k_i}^n.$$

Then Theorem III.3.15 implies that for some $\varepsilon > 0$

$$\lim_{n\to\infty} \Pi_n \varphi^n = \varphi, \qquad \lim_{n\to\infty} \Pi_n \mathscr{H}_n \varphi^n = \mathscr{H}\varphi = \sum_{i=1}^m \alpha_{k_i}\lambda_{k_i} e_{k_i}, \quad \text{in} \qquad \mathscr{C}^\varepsilon.$$

In this setting, recalling the definition of the martingales M^n from Lemma III.4.2, one has that almost surely

$$\begin{aligned}
M_t^\varphi &= \langle Y_{t,0}, \varphi\rangle - \int_0^t \langle Y_s, \mathscr{H}\varphi\rangle \mathrm{d}s \\
&= \lim_{n\to\infty}\left[\langle Y_{t,0}^n, \Pi_n\varphi^n\rangle - \int_0^t \langle Y_r^n, \Pi_n \mathscr{H}_n\varphi^n\rangle - n^{-\varrho}\langle\left(\Pi_n^3 Y_r^n\right)^2, \xi^n \Pi_n^2\varphi^n\rangle \mathrm{d}r\right] \\
&= \lim_{n\to\infty}\left[\langle Y_{t,0}^n, \Pi_n\varphi^n\rangle - \int_0^t \langle \mathscr{A}_n(Y_r^n) + \Pi_n[\xi_e^n \Pi_n^3 Y_r^n], \Pi_n\varphi^n\rangle - n^{-\varrho}\langle\left(\Pi_n^3 Y_r^n\right)^2, \xi_e^n \Pi_n^2\varphi^n\rangle \mathrm{d}r\right] \\
&= \lim_{n\to\infty} M_t^n(\Pi_n\varphi^n).
\end{aligned}$$

Here the convergence to zero of the non-linear term follows as in the proof of Proposition III.4.5:

$$\langle \big(\Pi_n Y^n_r\big)^2, \xi^n_e \Pi^2_n \varphi^n\rangle \lesssim n^{-\varrho+d+\frac{d}{2}} \|\Pi_n \varphi^n\|_{\mathscr{C}^\varepsilon} \langle Y^n_r, 1\rangle^2 \to 0,$$

by the assumption on ϱ. Our aim is to establish the martingale property for M^φ_t with respect to the filtration $\mathscr{F}_t$ generated by Y_t. The almost sure convergence $M^n_t(\Pi_n\varphi^n) \to M^\varphi_t$ is not sufficient. Instead, we will pick a sequence of stopped martingales $\widetilde{M}^n_t(\Pi_n\varphi^n)$, such that $\widetilde{M}^n_t(\Pi_n\varphi^n) \to M^\varphi_t$ almost surely and in L^1, for all $t \in [0,T]$. As we will see, the additional convergence in L^1 will guarantee that the limit M^φ is a martingale. Hence, let us define the following stopping time, for any path $z \in \mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$:

$$\tau_R(z) \colon = \inf\{t \in [0,T] : |\langle z_t, 1\rangle| \geqslant R\}.$$

Sine Y takes values in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$ we have that

$$\lim_{R\to\infty} \tau_R(Y) = \infty$$

Now, Lemma III.4.7 guarantees that almost surely (that is, on the events in which $Y^n \to Y$ in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$) for any $0 < \varepsilon < R$:

$$\tau_{R-\varepsilon}(Y) \leqslant \liminf_{n\to\infty} \tau_R(Y^n).$$

We deduce, using the monotonicity $\tau_R(z) \leqslant \tau_{R'}(z)$ if $R \leqslant R'$, that for $\varrho_0 = \varrho - \frac{d}{2} - d > 0$ (by Assumption III.3.1) almost surely:

$$\lim_{n\to\infty} \tau_{n^{\varrho_0}}(Y^n) = \infty.$$

Now, Equation (III.9) implies that

$$\sup_{n\in\mathbf{N}} \sup_{0\leqslant t\leqslant T} \mathbb{E}\Big[\Big|\Big\langle Y^n_{t\wedge\tau_{n^{\varrho_0}}(Y^n)}, 1\Big\rangle\Big|^2\Big] < \infty.$$

In particular, following the calculations of Proposition III.4.5 the sequence of stopped martingales

$$\Big\{M^n_{t\wedge\tau_{n^{\varrho_0}}(Y^n)}(\Pi_n\varphi^n)\Big\}_{n\in\mathbf{N}}$$

is uniformly integrable:

$$\sup_{n\in\mathbf{N}} \sup_{t\in[0,T]} \mathbb{E}|M^n_{t\wedge\tau_{n^{\varrho_0}}(Y^n)}(\Pi_n\varphi^n)|^2 < \infty.$$

Moreover, following from the previous observations $\widetilde{M}^n_t(\Pi_n\varphi^n) \colon = M^n_{t\wedge\tau_{n^{\varrho_0}}(Y^n)}(\Pi_n\varphi^n)$ converges almost surely to M^φ_t. The uniform integrability implies that the convergence holds also in L^1. In order to conclude that M^φ is a martingale with respect to $\mathscr{F}$ it suffices to show that for every $s < t$, $m \in \mathbf{N}$, $0 \leqslant s_1 \leqslant \cdots \leqslant s_m \leqslant s$ and every bounded measurable function $h\colon \mathbf{R}^m \to \mathbf{R}$, that

$$\mathbb{E}\Big[M^\varphi_t h(Y_{s_1},\dots,Y_{s_m})\Big] = \mathbb{E}\Big[M^\varphi_s h(Y_{s_1},\dots,Y_{s_m})\Big].$$

From the convergence in L^1 and almost surely that we just proved we obtain that

$$\begin{aligned}\mathbb{E}\Big[M_t^{\varphi} h(Y_{s_1},\dots,Y_{s_m})\Big] &= \lim_{n\to\infty}\mathbb{E}\Big[\widetilde{M}_t^n(\Pi_n\varphi^n)h(Y^n_{s_1\wedge\tau_{n^{\varrho_0}}(Y^n)},\dots,Y^n_{s_m\wedge\tau_{n^{\varrho_0}}(Y^n)})\Big]\\ &= \lim_{n\to\infty}\mathbb{E}\Big[\widetilde{M}_s^n(\Pi_n\varphi^n)h(Y^n_{s_1\wedge\tau_{n^{\varrho_0}}(Y^n)},\dots,Y^n_{s_m\wedge\tau_{n^{\varrho_0}}(Y^n)})\Big] \qquad \text{(III.10)}\\ &= \mathbb{E}\Big[M_s^{\varphi} h(Y_{s_1},\dots,Y_{s_m})\Big],\end{aligned}$$

where in the second line we used the martingale property for $\widetilde{M}^n(\Pi_n\varphi^n)$.

Step 2. Now we have to show that the martingale has the correct quadratic variation, namely that

$$\langle M^{\varphi}\rangle_t = \int_0^t \langle Y_s,\varphi^2\rangle\,\mathrm{d}s.$$

Here the problem is that we do not control moments of $\widetilde{M}_t^n(\Pi_n\varphi^n)$ higher than the second one. So proving that the martingale property of $(\widetilde{M}_t^n)^2-\langle\widetilde{M}^n\rangle_t$ is preserved in the limit does not follow from the same arguments we just used. Instead we stop the martingales in a different way. Consider the following stopping times as a sequence indexed by $R\in\mathbf{N}$:

$$\{\tau_R(Y^n)\wedge T\}_{R\in\mathbf{N}}\in[0,T]^{\mathbf{N}}.$$

Here the space $[0,T]^{\mathbf{N}}$ is endowed with the product topology and under this topology it is both compact and separable. In particular, since we are assuming that $Y^n\to Y$ in distribution in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$, the sequence

$$(\{\tau_R(Y^n)\wedge T\}_{R\in\mathbf{N}},Y^n)_{n\in\mathbf{N}}$$

is tight in the space

$$[0,T]^{\mathbf{N}}\times\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d)).$$

Hence let $(\{\bar\tau_R\}_{R\in\mathbf{N}},Y)$ be any limit point of the joint distribution. Since the space $[0,T]^{\mathbf{N}}\times\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$ is separable, by the Skorohod representation theorem, up to changing probability space, we can pick a subsequence n_k, for $k\in\mathbf{N}$ such that almost surely

$$\lim_{k\to\infty}(\{\tau_R(Y^{n_k})\wedge T\}_{R\in\mathbf{N}},Y^{n_k})=(\{\bar\tau_R\}_{R\in\mathbf{N}},Y),\quad \text{in}\quad [0,T]^{\mathbf{N}}\times\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d)).$$

The limiting random variables still satisfy the ordering:

$$\bar\tau_R\leqslant\bar\tau_{R+m},\qquad\forall m\in\mathbf{N},$$

as well as, by Lemma III.4.7:

$$\tau_{R-\varepsilon}(Y)\wedge T\leq\bar\tau_R\leq\tau_{R+\varepsilon}(Y)\wedge T,\qquad\forall\varepsilon>0. \qquad \text{(III.11)}$$

Now, the same calculations leading to Equation (III.10) show that for any $R\in\mathbf{N}$ the stopped martingales $M^{n_k}_{t\wedge\tau_R(Y^{n_k})}(\Pi_{n_k}\varphi^{n_k})$ converge to $M^{\varphi}_{t\wedge\bar\tau_R}$ almost surely and in L^1 (note that now the martingales $M^{n_k}_{t\wedge\tau_R(Y^{n_k})}(\Pi_{n_k}\varphi^{n_k})$ are even bounded). Similarly we obtain that

$M^{\varphi}_{t\wedge\bar{\tau}_R}$ is a martingale with respect to the filtration $\overline{\mathscr{F}}^R_t$ generated by $Y_{t\wedge\bar{\tau}_R}$. Following the calculations of Proposition III.4.5 we observe that

$$\langle M^{n_k}(\Pi_{n_k}\varphi^{n_k})\rangle_{t\wedge\tau_R(Y^{n_k})} \leqslant C\int_0^{t\wedge\tau_R(Y^{n_k})}\langle Y^{n_k}_s, 1\rangle \mathrm{d}s,$$

for some deterministic $C > 0$. In particular, following once more the calculations of Proposition III.4.5, we deduce that the martingale

$$\Big(M^{n_k}_{t\wedge\tau_R(Y^{n_k})}(\Pi_{n_k}\varphi^{n_k})\Big)^2 - \langle M^{n_k}(\Pi_{n_k}\varphi^{n_k})\rangle_{t\wedge\tau_R(Y^{n_k})}$$

is bounded and converges almost surely to:

$$\Big(M^{\varphi}_{t\wedge\overline{\tau}_R}\Big)^2 - \int_0^{t\wedge\overline{\tau}_R}\langle Y_s, \varphi^2\rangle \mathrm{d}s.$$

We then conclude that

$$\langle M^{\varphi}_{\cdot\wedge\bar{\tau}_R}\rangle_t = \int_0^{t\wedge\bar{\tau}_R}\langle Y_s, \varphi^2\rangle \mathrm{d}s.$$

Now, defining $t^n_k = \frac{kT}{n}$, for $k \leqslant n \in \mathbf{N}$, we can view the quadratic variation as the limit in probability:

$$\langle M^{\varphi}_{\cdot\wedge\overline{\tau}_R}\rangle_t = \mathbb{P} - \lim_{n\to\infty}\sum_{k=0}^{n}(M_{t\wedge\bar{\tau}_R\wedge t^n_{k+1}} - M_{t\wedge\bar{\tau}_R\wedge t^n_k})^2$$

Similarly also for the martingale whose quadratic variation we would actually like to compute:

$$\langle M^{\varphi}\rangle_t = \mathbb{P} - \lim_{n\to\infty}\sum_{k=0}^{n}(M_{t\wedge t^n_{k+1}} - M_{t\wedge t^n_k})^2.$$

Now, for any $\delta > 0$ and $t \in [0, T)$ we can choose an $R \in \mathbf{N}$ such that

$$\mathbb{P}(\bar{\tau}_R > t) \geq 1 - \delta,$$

by comparison with the stopping time $\tau_{R-\varepsilon}(Y)$ for any $\varepsilon > 0$ (see Equation (III.11)) and since $\lim_{R\to\infty}\tau_R(Y) = \infty$. So we finally conclude that for any $t < T$:

$$\mathbb{P}\bigg(\langle M\rangle_t = \int_0^t \langle Y_s, \varphi^2\rangle \mathrm{d}s\bigg) \geq 1 - \delta.$$

Since $\delta, T > 0$ are arbitrary we obtain the correct quadratic variation for all times.

Step 3. We conclude by explaining the uniqueness in law of any process Y satisfying the martingale problem of the rough superBrownian motion (in the following as always $\omega \in \Omega$ is fixed, and we omit from writing it. In particular, all averages are still conditional on the realization of the environment). The uniqueness is the consequence of a duality argument. For any $\varphi \geq 0, \varphi \in C^\infty$ we find a process $t \mapsto U_t\varphi$ such that

$$\mathbb{E}\Big[e^{-\langle Y_t, \varphi\rangle}\Big] = e^{-\langle Y^0, U_t\varphi\rangle}. \tag{III.12}$$

Hence the distribution of $\langle Y_t, \varphi\rangle$ is uniquely characterized by its Laplace transform. This also characterizes the law of the entire process $\langle Y_t, \varphi\rangle$ through a monotone class argument (see [DMS93, Lemma 3.2.5]), proving the required result.

We are left with the task of describing the process $U_t\varphi$. This is the solution, evaluated at time $t \geq 0$, of the nonlinearly damped parabolic equation

$$\partial_t(U_\cdot\varphi) = \mathscr{H}(U_\cdot\varphi) - \frac{1}{2}(U_\cdot\varphi)^2, \qquad U_0\varphi = \varphi,$$

where we consider the solutions in the mild sense, namely

$$U_t\varphi = e^{t\mathscr{H}}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathscr{H}}(U_s\varphi)^2\,\mathrm{d}s,$$

as constructed in Lemma III.4.6. To obtain Equation (III.12) consider some $\zeta > 0$ and a process $\psi \in C([0,T];\mathscr{C}^\zeta)$ of the form

$$\psi_t = e^{t\mathscr{H}}\psi_0 + \int_0^t e^{(t-s)\mathscr{H}} f_s\,\mathrm{d}s,$$

with $f \in C([0,T];\mathscr{C}^\zeta), \psi_0 \in \mathscr{C}^\zeta$. Now approximate f through a piecewise constant function in time $\widetilde{f}$ and in turn approximate both $\widetilde{f}$ and ψ_0 via a finite number of eigenfunctions (here we use the density of the domain proved in Lemma IV.2.2). Using the continuity of the semigroup as in Equation (IV.2), it follows from the definition of the rough superBrownian motion that for $0 \leq s \leq t$:

$$\langle Y_s, \psi_{t-s}\rangle - \langle Y_0, \psi_t\rangle - \int_0^s \langle Y_r, f_r\rangle\,\mathrm{d}r =: \widetilde{M}_s(\psi)$$

is a continuous martingale with quadratic variation

$$\langle \widetilde{M}(\psi)\rangle_s = \int_0^s \langle Y_r, \psi_{t-r}^2\rangle\,\mathrm{d}r.$$

Now we apply this observation together with Itô's formula to deduce that

$$[0,t] \ni s \mapsto e^{-\langle Y_s, U_{t-s}\varphi\rangle}$$

is a martingale on $[0,t]$. In particular, this implies Equation (III.12) and concludes the proof. □

The following result states the wellposedness of the dual PDE to the rough superBrownian motion. The proof is identical to that of Proposition II.4.5, only here there is no necessity to consider weights.

Lemma III.4.6. *Under Assumption III.4.1, fix $\omega \in \Omega$. For any $\varphi \geq 0, \varphi \in C^\infty$, time horizon $T > 0$ and $\zeta < 2 - \frac{d}{2}$, there exists a unique function $(t,x) \mapsto (U_t^\omega\varphi)(x)$ such that $U^\omega\varphi \in C([0,T];\mathscr{C}^\zeta)$, where*

$$U_t^\omega\varphi = e^{t\mathscr{H}^\omega}\varphi - \frac{1}{2}\int_0^t e^{(t-s)\mathscr{H}^\omega}(U_s^\omega\varphi)^2\,\mathrm{d}s.$$

We conclude the section with a consideration on stopping times and convergence in the Skorohod topology, which is used in the proofs above.

Lemma III.4.7. *Consider* $T > 0$ *and* $\{z^n\}_{n\in\mathbf{N}}, z \in \mathbb{D}([0,T];\mathbf{R})$ *such that* $z^n \to z$ *in* $\mathbb{D}([0,T],\mathbf{R})$. *Define, for* $R > 0$*:*

$$\tau_R(z) = \inf\{t \in [0,T] \ : \ |z_t| \geqslant R\},$$

and identically also $\tau_R(z^n)$*, with the convention that* $\inf \emptyset = \infty$. *Then, for any* $\varepsilon > 0$

$$\tau_{R-\varepsilon}(z) \leqslant \liminf_{n\to\infty} \tau_R(z^n) \leqslant \limsup_{n\to\infty} \tau_R(z^n) \leqslant \tau_{R+\varepsilon}(z).$$

Proof. Let us distinguish the cases $\tau_{R-\varepsilon}(z) = \infty$ and $\tau_{R-\varepsilon}(z) < \infty$.

Step 1. Assume that $\tau_{R-\varepsilon}(z) = \infty$. Then also $\tau_{R+\varepsilon}(z) = \infty$ an we only have to prove that

$$\liminf_{n\to\infty} \tau_R(z^n) = \infty.$$

Suppose on the contrary that for some $\alpha > 0$ the following holds: for every $m \in \mathbf{N}$ there exists an $n_m \geqslant m$ such that $\tau_R(z^{n_m}) \leqslant \alpha$. Then there exists a sequence of times $t^{n_m} \leqslant \alpha$ such that $|z^{n_m}_{t^{n_m}}| \geqslant R$. Then recall the Skorohod distance for $z, z' \in \mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$:

$$d_{\mathbb{D}}(z,z') = \inf_{\lambda\in\Lambda} d(z,z',\lambda) := \inf_{\lambda\in\Lambda}\left\{\gamma(\lambda) \vee \sup_{0\leqslant t\leqslant T} |z_t - z'_{\lambda(t)}|\right\}.$$

Here Λ is the set of time changes

$$\Lambda = \left\{ \text{Strictly increasing bijections } \lambda : [0,T] \to [0,T]\right\},$$

and

$$\gamma(\lambda) = \sup_{0\leqslant s<t\leqslant T} \left|\log\left(\frac{\lambda(t)-\lambda(s)}{t-s}\right)\right|.$$

Now, from the convergence $z^n \to z$, choose an $m \in \mathbf{N}$ such that $d_{\mathbb{D}}(z^n, z) \leqslant \frac{\varepsilon}{4}, \forall n \geqslant m$, for such m we can estimate

$$\begin{aligned}\sup_{t\in[0,T]} |z^{n_m}_t| &\leqslant \sup_{t\in[0,T]} |z_{\lambda(t)}| + \sup_{t\in[0,T]} |z^{n_m}_t - z_{\lambda(t)}| \\ &\leqslant (R-\varepsilon) + \frac{\varepsilon}{2} \\ &\leqslant R - \frac{\varepsilon}{2},\end{aligned}$$

where we chose λ such that $d(z^{n_m}, z, \lambda) \leqslant \frac{\varepsilon}{2}$. This is in contradiction with $|z^{n_m}_{t^{n_m}}| \geqslant R$.

Step 2. Now we assume that $\tau_{R-\varepsilon}(z) < \infty$, as well as $\tau_{R+\varepsilon}(z) < \infty$ (if the latter does not hold the second of the claimed inequalities is trivial). Suppose again that there exists a $0 < \alpha < \tau_{R-\varepsilon}(z)$ such that for every $m \in \mathbf{N}$ we can find an $n_m \geqslant m$ such that

$$\tau_R(z^{n_m}) \leqslant \alpha < \tau_{R-\varepsilon}(z).$$

In particular, we have a sequence of times $t^{n_m} \leqslant \alpha < \tau_{R-\varepsilon}(z)$ such that

$$z^{n_m}_{t^{n_m}} \geqslant R.$$

Now for any given $\delta > 0$, if m is sufficiently large, we have that for some $\lambda \in \Lambda$:

$$d(z^{n_m}, z, \lambda) \leqslant \delta.$$

In particular we can choose δ small enough, so that

$$\lambda(\alpha) < \tau_{R-\varepsilon}(z).$$

Under this assumption we have that

$$\begin{aligned}\delta \geqslant d(z^{n_m}, z, \lambda) &\geqslant \sup_{0\leqslant t\leqslant\alpha} |z^{n_m}_t - z_{\lambda(t)}| \geqslant |z^{n_m}_{t_m} - z_{\lambda(t_m)}| \\ &\geqslant |z^{n_m}_{t_m}| - |z_{\lambda(t)}| \\ &\geqslant R - (R-\varepsilon) = \varepsilon.\end{aligned}$$

Since this should hold for any δ we can choose $\delta < \varepsilon$, obtaining a contradiction.

As for the upper bound, assume that there exists an $\alpha > \tau_{R+\varepsilon}(z)$ such that for every $m \in \mathbf{N}$ we can find an $n_m \geqslant m$ satisfying

$$\tau_{R+\varepsilon}(z) < \alpha \leqslant \tau_R(z^{n_m}).$$

As before, for any $\delta > 0$, if m is sufficiently large, there exists a λ such that:

$$d(z, z^{n_m}, \lambda) \leqslant \delta,$$

and if δ is sufficiently small this implies that

$$\lambda(\tau_{R+\varepsilon}(z)) < \alpha.$$

Then we find that

$$\begin{aligned}\delta \geqslant d(z, z^{n_m}, \lambda) &\geqslant \sup_{0\leqslant t\leqslant T} |z_t - z^{n_m}_{\lambda(t)}| \\ &\geqslant |z_{\tau_{R+\varepsilon}(z)}| - \sup_{t\in[0,\lambda(\tau_{R+\varepsilon}(z))]} |z^{n_m}_t| \\ &\geqslant R + \varepsilon - R = \varepsilon,\end{aligned}$$

which is a contradiction as soon as $\delta < \varepsilon$. □

III.5 Scaling to Fisher-KPP

As in Section III.4, we will fix one realization $\omega \in \Omega$ of the environment and work conditional on that realization.

The first step towards the scaling limit is to restate the martingale problem of Lemma III.2.5 in the current setting. The proof is an immediate consequence of the aforementioned lemma.

Lemma III.5.1. *Under the assumptions of Theorem III.3.11 fix any $\omega \in \Omega$. For all $\varphi \in L^\infty(\mathbf{T}^d)$, the process $t \mapsto \langle X^n_t(\omega), \varphi\rangle$ satisfies*

$$\langle X^n_{t,s}(\omega), \varphi\rangle = \int_s^t \langle \mathscr{A}_n(X^n_r(\omega)), \varphi\rangle + \langle \Pi_n\Big[\bar{\xi}(\omega)(\Pi^3_n X^n_r(\omega) - (\Pi^3_n X^n_r(\omega))^2)\Big], \varphi\rangle \,\mathrm{d}r + M^n_{t,s}(\varphi), \tag{III.13}$$

where $M^n_\cdot(\varphi)$ is a centered square integrable martingale with predictable quadratic variation

$$\begin{aligned}\langle M^n(\varphi)\rangle_t = n^{-\eta-d+2}\int_0^t &\langle (1+s_n(\omega))\Pi^3_n X^n_r(\omega), (\Pi_n\varphi)^2 - 2\Pi_n(\varphi)\Pi_n(X^n_r(\omega)\varphi)\rangle \\ &+ \langle \Big(\Pi_n(X^n_r(\omega)\varphi)\Big)^2, 1\rangle \\ &- \langle s_n(\omega)(\Pi^3_n X^n_r(\omega))^2, (\Pi_n\varphi)^2 - 2\Pi_n(\varphi)\Pi_n(X^n_r(\omega)\varphi)\rangle \,\mathrm{d}r.\end{aligned} \tag{III.14}$$

Now we are able to show tightness for the process.

Proposition III.5.2. *Under the assumptions of Theorem III.3.11 fix any $\omega \in \Omega$. Fix $T > 0$ and α such that*

$$\begin{cases} \alpha \in (0, 1/2) & \text{if } d = 1, \\ \alpha \in (0, \min\{\eta, 1\}) & \text{if } d = 2. \end{cases}$$

The sequence $\{s \mapsto \Pi_n X^n_s(\omega)\}_{n\in\mathbf{N}}$ is tight in the space

$$L^2([0,T]; B^\alpha_{2,2}).$$

In addition, the sequence $\{s \mapsto X^n_s(\omega)\}_{n\in\mathbf{N}}$ is tight in $\mathbb{D}([0,T]; \mathscr{M}(\mathbf{T}^d))$, and any limit point lies in $C([0,T]; \mathscr{M}(\mathbf{T}^d))$.

To prove the proposition we will make use of the regularizing properties of the semigroup $e^{t\mathscr{A}_n}$ as described in the following lemma.

Lemma III.5.3. *For any $\gamma \in [0,1), p \in [1,\infty], T > 0$ and $\alpha \in \mathbf{R}$ one can bound, uniformly over $n \in \mathbf{N}$, $\varphi \in \mathscr{C}^\alpha_p, t \in [0,T]$:*

$$\|\Pi_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{C}^{\alpha+\gamma}_p} \lesssim t^{-\frac{\gamma}{2}}\|\varphi\|_{\mathscr{C}^\alpha_p}.$$

Proof. We can bound

$$\begin{aligned}\|\mathscr{P}_n e^{t\mathscr{A}_n}\Pi_n\varphi\|_{\mathscr{C}^{\alpha+\gamma}_p} &\lesssim \|\mathscr{P}_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{C}^{\alpha+\gamma}_p} \\ &\lesssim t^{-\frac{\gamma}{2}}\|\varphi\|_{\mathscr{C}^\alpha_p},\end{aligned}$$

where in the first step we applied Corollary III.7.4 and in the last step the large scale estimate of Proposition III.6.7. Instead, on small scales we find:

$$\begin{aligned}\|\mathscr{Q}_n e^{t\mathscr{A}_n}\Pi_n\varphi\|_{\mathscr{C}^{\alpha+\gamma}_p} &\lesssim n^\gamma\|\mathscr{Q}_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{C}^\alpha_p} \\ &\lesssim t^{-\frac{\gamma}{2}}\|\varphi\|_{\mathscr{C}^\alpha_p},\end{aligned}$$

where we again applied Corollary III.7.4 in the first step and Proposition III.6.7 in the second step. □

Proof of Proposition III.5.2. Since $\omega \in \Omega$ is fixed throughout the proof, we omit writing it, to lighten the notation.

*Step 1.*Tightness of the sequence X^n in $\mathbb{D}([0,T];\mathscr{M}(\mathbf{T}^d))$ is a consequence of the bound $0 \le X^n_t \le 1$. In fact, we can apply Jakubowski's tightness criterion, which we recall in Proposition A.2.1. The criterion consists in proving first a compact containment condition. This is immediately satisfied since

$$\mathbb{P}(\sup_{0 \leqslant t \leqslant T} |\langle X^n_t, 1\rangle| > 1) = 0$$

from the boundedness of X^n. The second and last requirement for Jakubowski's tightness criterion is the tightness of one dimensional distributions. Namely it suffices to prove that for any $\varphi \in C^\infty(\mathbf{T}^d)$ the sequences of process

$$\{t \mapsto \langle X^n_t, \varphi\rangle\}_{n \in \mathbf{N}}$$

is tight in $\mathbb{D}([0.T];\mathbf{R})$. For this purpose we use Aldous' tightness criterion (note that this is the same approach as in the proof of Proposition III.4.5). Let us define

$$D^n_{t,s}(\varphi) = \langle X^n_{t,s}, \varphi\rangle - M^n_{t,s}(\varphi),$$

where we used the notations of Lemma III.5.1. Now to prove tightness of the one-dimensional distributions Aldous' criterion guarantees that it suffices to show that for any sequence of stopping times τ^n and any deterministic sequence δ_n with $\delta_n \to 0$ one has

$$\forall \delta > 0 \qquad \lim_{n\to\infty} \mathbb{P}\Big(|\langle X^n_{\tau_n+\delta_n, \tau_n}, \varphi\rangle| \geqslant \delta\Big) = 0.$$

In particular it suffices to show that for any $\delta > 0$

$$\lim_{n\to\infty} \mathbb{P}\Big(|D^n_{\tau_n+\delta_n,\tau_n}(\varphi)| \geqslant \delta\Big) = \lim_{n\to\infty} \mathbb{P}\Big(|M^n_{\tau_n+\delta_n,\tau_n}(\varphi)| \geqslant \delta\Big) = 0.$$

Now by Proposition III.6.6 we find that (since φ is smooth)

$$\sup_{n\in\mathbf{N}} \|\mathscr{A}_n \varphi\|_{L^\infty} < \infty.$$

Hence the following deterministic bound holds (since $0 \leqslant X^n_t \leqslant 1$):

$$|D^n_{\tau_n+\delta_n,\tau_n}(\varphi)| \lesssim_\varphi \delta_n,$$

which proves the first limit. As for the second one, we observe that

$$\mathbb{P}\Big(|M^n_{\tau_n+\delta_n,\tau_n}(\varphi)| \geqslant \delta\Big) \leqslant \frac{1}{\delta^2}\mathbb{E}\Big[\langle M^n(\varphi)\rangle_{\tau_n+\delta_n,\tau_n}\Big] \lesssim \frac{\delta_n}{\delta^2} \to 0,$$

where for the quadratic variation we used similar bounds as for the drift. Finally, to show that any limit point lies in $C([0,T];\mathscr{M}(\mathbf{T}^d))$ note that for any $\varphi \in C(\mathbf{T}^d)$

$$|\langle X^n_t, \varphi\rangle - \langle X^n_{t-}, \varphi\rangle| \lesssim n^{-\eta-d}\|\varphi\|_{L^\infty} \le n^{-2}\|\varphi\|_{L^\infty},$$

so that the maximal jump size is vanishing as $n \to \infty$. The continuity of the limit points follows then through [EK86, Theorem 3.10.2].

Tightness in the space of measures is not sufficient to make sense of the nonlinearity in the limit. Hence from now on we now concentrate on proving the tightness of the sequence $\Pi_n X^n_s$ in $L^2([0,T]; B^{\alpha}_{2,2})$ for some $\alpha > 0$. Our aim is to apply Simon's tightness criterion, which we recall in Proposition A.2.2, with

$$X = B^{\alpha'}_{2,2}, \quad Y = B^{\alpha}_{2,2}, \quad Z = B^{\alpha''}_{2,2},$$

for appropriate $\alpha' > \alpha > \alpha''$.

Step 2. First, we derive a uniform bound for the second moment of the $B^{\alpha}_{2,2}$ norm (this in particular implies boundedness of the sequence $\Pi_n X^n$ in $L^2([0,T]; B^{\alpha}_{2,2})$):

$$\sup_{n \in \mathbf{N}} \sup_{0 \le t \le T} \mathbb{E} \|\Pi_n X^n_t\|^2_{B^{\alpha}_{2,2}} < \infty. \tag{III.15}$$

To obtain this bound it is convenient to prove the following stronger estimate. Uniformly over $s \in [0,T]$

$$\sup_{s \le t \le T} \mathbb{E}\Big[\|\Pi_n X^n_t\|^2_{B^{\alpha}_{2,2}} \Big| \mathscr{F}_s \Big] \lesssim_T 1 + \|\Pi_n X^n_s\|^2_{B^{\alpha}_{2,2}}, \tag{III.16}$$

where $\{\mathscr{F}_t\}_{t \ge 0}$ is the natural filtration generated by X^n (we omit the dependence of the filtration on n). We state the bound with the conditional expectation, since in this form it is simpler to derive, via a Gronwall-type argument. For brevity, fix the notation

$$\overline{X}^n = \Pi_n X^n.$$

By the martingale representation of Lemma III.5.1 and a change of variables formula

$$\overline{X}^n_t = e^{(t-s)\mathscr{A}_n} \overline{X}^n_s + \int_s^t e^{(t-r)\mathscr{A}_n} \Pi^2_n \Big[\bar{\xi} \Pi^2_n \Big(\overline{X}^n_r - (\overline{X}^n_r)^2 \Big) \Big] \mathrm{d}r + \int_{s+}^t \Pi_n e^{(t-r)\mathscr{A}_n} \, \mathrm{d}M^n_r,$$

where the last integral is understood as a stochastic integral against a martingale measure (cf. [Wal86]). For the purpose of the proof it is sufficient to consider its one dimensional projections, that is for $\varphi \in C(\mathbf{T}^d)$

$$\langle \overline{X}^n_t, \varphi \rangle = \langle \overline{X}^n_s, e^{(t-s)\mathscr{A}_n} \varphi \rangle + \int_s^t \langle \Pi^2_n \Big[\bar{\xi} \Pi^2_n \Big(\overline{X}^n_r - (\overline{X}^n_r)^2 \Big) \Big], e^{(t-r)\mathscr{A}_n} \varphi \rangle \, \mathrm{d}r + \int_{s+}^t \mathrm{d}M^n_r (\Pi_n e^{(t-r)\mathscr{A}_n} \varphi).$$

The $B^{\alpha}_{2,2}$ norm is estimated by

$$\begin{aligned} \mathbb{E}\Big[\|\overline{X}^n_t\|^2_{B^{\alpha}_{2,2}} \Big| \mathscr{F}_s \Big] \lesssim \|\overline{X}^n_s\|^2_{B^{\alpha}_{2,2}} &+ \mathbb{E}\bigg[\bigg\| \int_s^t e^{(t-r)\mathscr{A}_n} \Pi^2_n \Big[\bar{\xi} \Pi^2_n (\overline{X}^n_r - (\overline{X}^n_r)^2) \Big] \mathrm{d}r \bigg\|^2_{B^{\alpha}_{2,2}} \bigg| \mathscr{F}_s \bigg] \\ &+ \mathbb{E}\bigg[\bigg\| \int_{s+}^t \Pi_n e^{(t-r)\mathscr{A}_n} \, \mathrm{d}M^n_r \bigg\|^2_{B^{\alpha}_{2,2}} \bigg| \mathscr{F}_s \bigg]. \end{aligned}$$

An extension of the paraproduct estimates of Lemma I.1.3 to the $B^{\alpha}_{p,q}$ scale (see [BCD11, Theorems 2.82, 2.85]) guarantees that

$$\|f^2\|_{B^{\alpha}_{2,2}} \le 2\|f \varolessthan f\|_{B^{\alpha}_{2,2}} + \|f \odot f\|_{B^{\alpha}_{2,2}} \lesssim \|f\|_{L^\infty} \|f\|_{B^{\alpha}_{2,2}}.$$

Now we apply the Schauder estimates of Proposition III.6.7. Note that here we do not need the real strength of the estimates, as we do not need to gain any regularity. Note also that the estimates are proven on the scale of $B^{\alpha}_{p,\infty}$ spaces but extend verbatim to $B^{\alpha}_{p,q}$ spaces for $q \in [1,\infty)$. Hence, using the L^{∞} bound on $\overline{X}^n$ and the fact that $\overline{\xi}$ is smooth one obtains

$$\begin{aligned}\Big\| e^{(t-r)\mathscr{A}_n}\Pi_n^2\Big[\bar{\xi}\Pi_n^2(\overline{X}_r^n-(\overline{X}_r^n)^2)\Big]\Big\|_{B^{\alpha}_{2,2}} &\lesssim \Big\|\Pi_n^2\Big[\bar{\xi}\Pi_n^2(\overline{X}_r^n-(\overline{X}_r^n)^2)\Big]\Big\|_{B^{\alpha}_{2,2}}\\ &\lesssim \|\overline{X}_r^n\|_{B^{\alpha}_{2,2}},\end{aligned}$$

so that:

$$\mathbb{E}\Big[\Big\|\int_s^t e^{(t-r)\mathscr{A}_n}\Pi_n^2\Big[\bar{\xi}\Pi_n^2(\overline{X}_r^n-(\overline{X}_r^n)^2)\Big]\mathrm{d}r\Big\|^2_{B^{\alpha}_{2,2}}\Big|\mathscr{F}_s\Big]\lesssim |t-s|^2\sup_{s\le t\le T}\mathbb{E}\Big[\|\overline{X}_t^n\|^2_{B^{\alpha}_{2,2}}\Big|\mathscr{F}_s\Big].$$

As for the martingale term, let us introduce a parameter λ according to the following definition:

$$\begin{cases}\text{If } d=1, & \eta=1 \;\Rightarrow\; \text{set } \lambda=0,\\ \text{If } d=2, & \eta>0 \;\Rightarrow\; \text{set } \lambda=\min\{\eta,1\}.\end{cases}$$

Then, from the definition of the space $B^{\alpha}_{2,2}$ one has

$$\begin{aligned}&\mathbb{E}\Big[\Big\|\int_{s+}^t \Pi_n e^{(t-r)\mathscr{A}_n}\,\mathrm{d}M_r^n\Big\|^2_{B^{\alpha}_{2,2}}\Big|\mathscr{F}_s\Big]\\ &\quad=\sum_{j\ge -1}2^{2\alpha j}\int_{\mathbf{T}^d}\mathbb{E}\Big[\Big|\int_{s+}^t \mathrm{d}M_r^n(e^{(t-r)\mathscr{A}_n}\Pi_n K_j^x)\Big|^2\;\Big|\;\mathscr{F}_s\Big]\mathrm{d}x\end{aligned}$$

where with K_j^x we indicate the function:

$$K_j^x(y)=\mathscr{F}^{-1}_{\mathbf{T}^d}\varrho_j(x-y),$$

with ϱ_j the elements of the dyadic partition of the unity that define the Besov spaces. Using the predictable quadratic variation computed in Lemma III.5.1 one obtains, uniformly over x

$$\begin{aligned}&\mathbb{E}\Big[\Big|\int_{s+}^t \mathrm{d}M_r^n(e^{(t-r)\mathscr{A}_n}\Pi_n K_j^x)\Big|^2\;\Big|\;\mathscr{F}_s\Big]\\ &\quad\leqslant n^{-\lambda}\mathbb{E}\Big[\int_s^t\langle\Pi_n^2\overline{X}_r^n,(1+s_n)\Big[(\Pi_n^2e^{(t-r)\mathscr{A}_n}K_j^x)^2-2\Pi_n^2(e^{(t-r)\mathscr{A}_n}K_j^x)\Pi_n(X_r^n\Pi_ne^{(t-r)\mathscr{A}_n}K_j^x)\Big]\rangle\\ &\qquad+\langle\Big(\Pi_n(X_r^n\Pi_ne^{(t-r)\mathscr{A}_n}K_j^x)\Big)^2,1\rangle\\ &\qquad-\langle(\Pi_n^2\overline{X}_r^n)^2,s_n\Big[(\Pi_n^2e^{(t-r)\mathscr{A}_n}K_j^x)^2-2\Pi_n^2(e^{(t-r)\mathscr{A}_n}K_j^x)\Pi_n(X_r^n\Pi_ne^{(t-r)\mathscr{A}_n}K_j^x)\Big]\rangle\,\mathrm{d}r\;\Big|\;\mathscr{F}_s\Big]\\ &\quad\lesssim n^{-\lambda}\int_s^t\Big\|\Pi_n\Big|\Pi_ne^{(t-r)\mathscr{A}_n}K_j^x\Big|\Big\|^2_{L^2}\,\mathrm{d}r,\end{aligned}\tag{III.17}$$

since $|s_n|, |X^n|, |\overline{X}^n| \leq 1$. Now, for $\zeta \in \mathbf{R}$, for example via the Poisson summation formula in Lemma I.1.1 and a scaling argument on $\mathbf{R}^d$

$$\|K_j^x\|_{\mathscr{C}_1^\zeta} \lesssim 2^{j\zeta}$$

and therefore by the Schauder estimates that we recalled in Lemma III.5.3, for $\gamma \in [0,1)$

$$\|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\|_{\mathscr{C}_1^{\zeta+\gamma}} \lesssim (t-r)^{-\frac{\gamma}{2}} 2^{j\zeta}.$$

Now, for clarity, dimension $d = 1$ and dimension $d = 2$ will be treated separately. In dimension $d = 1$ choose $-\frac{1}{2} < \zeta < -\alpha$ and fix $\gamma \in (0,1)$ such that $\zeta + \gamma > \frac{1}{2}$. Then, by Besov embedding, one has

$$\|\Pi_n \big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{L^2}^2 \leq \|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\|_{L^2}^2 \lesssim \|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\|_{\mathscr{C}_1^{\zeta+\gamma}}^2 \lesssim (t-r)^{-\gamma} 2^{2j\zeta}.$$

In dimension $d = 2$, we make additional use of the regularizing properties of Π_n together with the factor $n^{-\lambda}$ appearing in front of the quadratic variation. Note that Corollary III.7.4 allows only to gain one degree of regularity, which is why we have defined $\lambda = \min\{1, \eta\}$ (we have no use for additional powers of n). Now, choose $\kappa > 0$ such that $\alpha < \lambda - 5\kappa$ and set $\gamma = 1 - \kappa$. Then Corollary III.7.4 implies that

$$\|\Pi_n \big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{L^2} \lesssim n^{\lambda-\kappa} \|\big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{\mathscr{C}_2^{-\lambda+2\kappa}},$$

and Besov embeddings I.1.2 additionally guarantee the following chain of inequalities (here the main aim is to get rid of the absolute value):

$$\begin{aligned}
\|\big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{\mathscr{C}_2^{-\lambda+2\kappa}} &\lesssim \|\big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{\mathscr{C}^{-\kappa}_{\frac{2}{1+\lambda-3\kappa}}} \\
&\lesssim \|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\|_{L^{\frac{2}{1+\lambda-3\kappa}}} \\
&\lesssim \|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\|_{\mathscr{C}_1^{1-\lambda+4\kappa}} \\
&\lesssim (t-r)^{\frac{1-\kappa}{2}} \|K_j^x\|_{\mathscr{C}_1^{-\lambda+5\kappa}} \\
&\lesssim (t-r)^{\frac{\gamma}{2}} 2^{-j(\lambda-5\kappa)}.
\end{aligned}$$

Overall, we have obtained that

$$\|\Pi_n \big|\Pi_n e^{(t-r)\mathscr{A}_n} K_j^x\big|\|_{L^2} \lesssim n^{\lambda-\kappa} (t-r)^{\frac{\gamma}{2}} 2^{-j(\lambda-5\kappa)}.$$

In this way, in both dimensions, substituting the estimate into (III.17) one obtains

$$\mathbb{E}\left[\left\| \int_s^t \Pi_n e^{(t-r)\mathscr{A}_n} \, \mathrm{d}M_r^n \right\|_{B_{2,2}^\alpha}^2 \middle| \mathscr{F}_s \right] \lesssim |t-s|^{1-\gamma}.$$

For sufficiently small, deterministic T^*, chosen uniform over all parameters, inequality (III.16) is shown for all $(t-s) \leq T^*$. Due to the presence of the conditional expectation, one can exploit this argument for general t, s via a Gronwall-type argument. Indeed, to extend the estimate to $2T^*$, observe there exists a $C(T^*)$ such that

$$\sup_{t\in[s,s+2T^*]} \mathbb{E}\Big[\|\Pi_n X^n_t\|^2_{B^\alpha_{2,2}}\big|\mathscr{F}_s\Big] \leq C(T^*)\bigg(1+\sup_{t\in[s,s+T^*]}\mathbb{E}\Big[\|\Pi_n X^n_t\|^2_{B^\alpha_{2,2}}|\mathscr{F}_s\Big]\bigg)$$
$$\leq C(T^*)\bigg(1+C(T^*)\Big(1+\mathbb{E}\Big[\|\Pi_n X^n_s\|^2_{B^\alpha_{2,2}}\Big]\Big)\bigg).$$

Iterating this argument yields the bound for arbitrary T.

Step 3. The next goal is a bound for the expectation of an increment. For this reason fix

$$0 < \beta < \alpha,$$

with α as in Step 1. We shall prove that there exists a $\zeta > 0$ satisfying:

$$\mathbb{E}\Big[\|\overline{X}^n_t - \overline{X}^n_s\|^2_{B^\beta_{2,2}}\Big] \lesssim |t-s|^{4\zeta}. \tag{III.18}$$

Indeed, arguments similar to those in Step 1 show that

$$\mathbb{E}\Big[\|\overline{X}^n_t - \overline{X}^n_s\|^2_{B^\beta_{2,2}}\Big] \lesssim \mathbb{E}\Big[\|\overline{X}^n_t - e^{(t-s)\mathscr{A}_n}\overline{X}^n_s\|^2_{B^\beta_{2,2}}\Big] + \mathbb{E}\Big[\|e^{(t-s)\mathscr{A}_n}\overline{X}^n_s - \overline{X}^n_s\|^2_{B^\beta_{2,2}}\Big]$$
$$\lesssim \mathbb{E}\Big[\|\overline{X}^n_t - e^{(t-s)\mathscr{A}_n}\overline{X}^n_s\|^2_{B^\beta_{2,2}}\Big] + |t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}^n_s\|^2_{B^\alpha_{2,2}}$$
$$\lesssim |t-s|^{1-\gamma}(1+\mathbb{E}\|\overline{X}^n_s\|^2_{B^\beta_{2,2}}) + |t-s|^{\alpha-\beta}\mathbb{E}\|\overline{X}^n_s\|^2_{B^\alpha_{2,2}},$$

where the penultimate step follows from Lemma III.6.8. This is enough to establish (III.18).

Step 4. Notice that (III.15) and (III.18) together guarantee that

$$\sup_{n\in\mathbf{N}} \mathbb{E}\Big[\|\overline{X}^n\|^2_{L^2([0,T];B^\alpha_{2,2})} + \|\overline{X}^n\|_{W^{2,\zeta}([0,T];B^\beta_{2,2})}\Big] < \infty,$$

with ζ as in (III.18). Note that this implies tightness in $L^2([0,T];B^{\alpha'}_{2,2})$ for any $\alpha' < \alpha$, which is still sufficient for the result, since α varies in an open set.

□

At this point, the last step is to prove that any limit point satisfies the required martingale problem (in $d = 1$) or solves the required PDE (in $d = 2$).

Proof of Theorem III.3.11. As in all previous cases, we fix $\omega \in \Omega$ and do not state explicitly the dependence on it. We treat the drift and the martingale part differently.

Step 1. We start with the drift, which is the same in both dimensions. Since Let X be any limit point of X^n in $C([0,T];\mathscr{M}(\mathbf{T}^d))$. The previous proposition guarantees that any such X lies almost surely in $L^2([0,T];B^\alpha_{2,2})$ for some $\alpha > 0$. In addition, through Skorohod representation, we can assume that $\Pi_n X^n \to X$ in $L^2([0,T];B^\alpha_{2,2})$ almost surely. In particular, for $\varphi \in C^\infty(\mathbf{T}^d)$, defining

$$N^\varphi_t = \langle X_{t,0}, \varphi\rangle - \int_0^t \langle X_s, \nu_0\Delta\varphi\rangle + \langle\bar{\xi}(X_s - X_s^2), \varphi\rangle \,\mathrm{d}s,$$

and since regarding the nonlinear term one can estimate:

$$\int_0^t \int_{\mathbf{T}^d} |X_s^2 - (\Pi_n X^n)^2| \mathrm{d}x \, \mathrm{d}s \leq \int_0^t \int_{\mathbf{T}^d} 2|X_s - \Pi_n X^n| \mathrm{d}x \, \mathrm{d}s \lesssim \|X_s - \Pi_n X^n\|_{L^2([0,T];B^\alpha_{2,2})}$$

and applying Lemma III.6.5, one has almost surely:

$$\begin{aligned} N_t^\varphi &= \lim_{n\to\infty} \Big[\langle \Pi_n X^n_{t,0}, \varphi\rangle - \int_0^t \langle \mathscr{A}_n X^n_s, \varphi\rangle + \langle \xi \Pi_n^2 \big[\Pi_n X_s^n - (\Pi_n X_s^n)^2 \big], \Pi_n^2 \varphi \rangle \mathrm{d}s \Big] \\ &=: \lim_{n\to\infty} N_t^{n,\varphi}. \end{aligned}$$

Step 2. Now we prove that N_t^φ is a centered continuous martingale, with quadratic variation depending on the dimension. In $d=2$ the quadratic variation will be zero and hence $N^\varphi \equiv 0$, proving that the limit is deterministic (conditional on the environment). Since $N_t^{n,\varphi}$ is a sequence of martingales, by Lemma III.5.1, the fact that also N_t^φ is a martingale follows from the uniform bound of Equation (III.15) (the continuity of N^φ is as well a consequence of that proposition). The quadratic variation of $N^{n,\varphi}$ is given by:

$$\begin{aligned} \langle N^{n,\varphi}\rangle_t = n^{-\lambda} \int_0^t & \langle (1+s_n)\Pi_n^3 X_r^n, (\Pi_n^2\varphi)^2 - 2\Pi_n^2(\varphi)\Pi_n(X_r^n\varphi)\rangle \\ & + \langle \big(\Pi_n(X_r^n \Pi_n\varphi)\big)^2, 1\rangle - \langle s_n(\Pi_n^3 X_r^n)^2, (\Pi_n^2\varphi)^2 - 2\Pi_n^2(\varphi)\Pi_n(X_r^n\Pi_n\varphi)\rangle \mathrm{d}r, \end{aligned}$$

with $\lambda = 0$ in $d=1$ and $\lambda = \eta > 0$ in $d = 2$. In the latter case ($d=2, \lambda > 0$) the bounds $0 \leqslant X^n \leqslant 1, |s_n| \lesssim n^{-2}$ guarantee that

$$\lim_{n\to\infty} \langle N^{n,\varphi}\rangle_t = 0.$$

Instead if $d = 1, \lambda = 0$ we have to take more care. As before, the bound $|s_n| \lesssim n^{-2}$ guarantees that all terms multiplied by s_n vanish in the limit, so we are left with considering

$$\lim_{n\to\infty} \int_0^t \langle \Pi_n^3 X_r^n, (\Pi_n^2\varphi)^2 - 2\Pi_n^2(\varphi)\Pi_n(X_r^n\varphi)\rangle + \langle \big(\Pi_n(X_r^n\Pi_n\varphi)\big)^2, 1\rangle \mathrm{d}r.$$

We can rewrite the quantity in the limit as:

$$\begin{aligned} &\int_0^t \langle \Pi_n^3 X_r^n, (\Pi_n^2\varphi)^2 - 2(\Pi_n^2\varphi)[(\Pi_n X_r^n)\varphi]\rangle + \langle \big((\Pi_n X_r^n)\Pi_n\varphi\big)^2, 1\rangle \mathrm{d}r \\ &\quad + \int_0^t -2\langle \Pi_n^3 X_r^n, (\Pi_n^2\varphi)[D^{\Pi,n}(X_r^n, \varphi)]\rangle + \langle (D^{\Pi,n}(X_r^n, \Pi_n\varphi))^2, 1\rangle \\ &\qquad + 2\langle D^{\Pi,n}(X_r^n, \Pi_n\varphi), (\Pi_n X_r^n)\Pi_n\varphi\rangle \mathrm{d}r, \end{aligned}$$

where we have defined the commutator (cf. Lemma IV.2.16 for a similar construction)

$$D^{\Pi,n}(\psi,\psi') = \Pi_n(\psi\cdot\psi') - (\Pi_n\psi)\cdot\psi'.$$

Now we observe that for $\delta \in [0,1]$:

$$\begin{aligned} \sup_{x\in\mathbf{T}^d} |D^{\Pi,n}(\psi,\psi')|(x) &= \sup_{x\in\mathbf{T}^d} \Big| \fint_{B_n(x)} \psi(y)(\psi'(y) - \psi'(x))\mathrm{d}y \Big| \\ &\lesssim n^{-\delta} \|\psi\|_{L^\infty} \|\psi'\|_{\mathscr{C}^\delta}. \end{aligned}$$

We can apply this bound to our quadratic variation, observing that $\varphi \in C^{\infty}(\mathbf{T}^d)$ and $\|X^n\|_{\infty} \leqslant 1$, so that:

$$\begin{aligned}\lim_{n\to\infty}\int_0^t & -2\langle \Pi_n^3 X_r^n, (\Pi_n^2\varphi)[D^{\Pi,n}(X_r^n,\varphi)]\rangle + \langle (D^{\Pi,n}(X_r^n,\Pi_n\varphi))^2, 1\rangle \\ & \qquad + 2\langle D^{\Pi,n}(X_r^n,\Pi_n\varphi),(\Pi_n X_r^n)\Pi_n\varphi\rangle \mathrm{d}r \\ &= 0.\end{aligned}$$

Finally we are left with computing the limit

$$\begin{aligned}\lim_{n\to\infty}\langle N^{n,\varphi}\rangle_t &= \lim_{n\to\infty}\int_0^t \langle \Pi_n^3 X_r^n, (\Pi_n^2\varphi)^2 - 2(\Pi_n^2\varphi)[(\Pi_n X_r^n)\varphi]\rangle + \langle \Big((\Pi_n X_r^n)\Pi_n\varphi\Big)^2, 1\rangle \mathrm{d}r \\ &= \int_0^t \langle X_r, \varphi^2 - 2X_r\varphi^2\rangle + \langle X_r^2,\varphi^2\rangle \mathrm{d}r \\ &= \int_0^t \langle X_r(1-X_r),\varphi^2\rangle \mathrm{d}r.\end{aligned}$$

Here the second equality follows by calculations analogous to those in Step 1, since now the quadratic nonlinearity is a function of $\Pi_n X^n$ and the latter is converging in $L^2([0,T];B^{\alpha}_{2,2,})$.

Finally, since the martingale $(N_t^{n,\varphi})^2 - \langle N^{n,\varphi}\rangle_t$ is bounded (using that $0 \leqslant X^n \leqslant 1$), also the limiting process $(N_t^{\varphi})^2 - \lim_{n\to\infty}\langle N^{n,\varphi}\rangle_t$ is a martingale, implying that $\langle N^{\varphi}\rangle_t = \lim_{n\to\infty}\langle N^{n,\varphi}\rangle_t$. Hence the quadratic variation is of the required form for Theorem III.3.11.

So far we have proven that any limit point solves the required equation. To conclude the convergence, we have to prove that such solutions are unique. In $d = 2$, that for every $\omega \in \Omega$ there exists a unique solution to the equation

$$\partial_t X = \nu_0 \Delta X + \overline{\xi}(\omega)X(1-X), \qquad X(0) = X_0$$

follows from classical solution theory. Instead in $d = 2$ uniqueness in law can be established via a Girsanov transform, as we show in Lemma III.5.4 below.

□

Lemma III.5.4. *In $d = 1$ and under Assumption III.3.8, solutions to the stochastic Fisher-KPP equation as in Definition III.3.9 are unique in distribution.*

Proof. As usual, the argument works for fixed $\omega \in \Omega$, so we omit writing the dependence on it. First, the same calculations as in Proposition III.5.2 prove that any solution X to the martingale problem of the stochastic Fisher-KPP equation lives in $L^2([0,T];B^{\alpha}_{2,2})$, for some $\alpha > 0$ and arbitrary $T > 0$. Then, following the same arguments as in the proof of Theorem II.2.19, we see that (up to enlarging the probability space) X is a solution to the SPDE:

$$\partial_t X = \nu_0\Delta X + \overline{\xi} X(1-X) + \sqrt{X(1-X)}\tilde{\xi}, \qquad X(0) = X_0,$$

where $\tilde{\xi}$ is a space time white noise. Here we mean solutions in the sense that for any $\varphi \in C^\infty(\mathbf{T}^d)$ and $t \in [0,T]$:

$$\langle X_t, \varphi\rangle = \langle X_0, \varphi\rangle = \int_0^t \langle X_s, \nu_0 \Delta\varphi\rangle + \langle \overline{\xi} X_s(1-X_s), \varphi\rangle \,\mathrm{d}s \\ + \int_0^t \int_{\mathbf{T}^d} \sqrt{X_s(x)(1-X_s(x))}\varphi(x)\,\mathrm{d}\tilde{\xi}(s,x),$$

where the latter is understood as an integral against a martingale measure, in the sense of Walsh [Wal86]. Now we can use a Girsanov transform [Daw78, Theorem 5.1] (see also [Per02, Theorem IV.1.6] and [MMR19, Section 2.2] for more recent accounts). Let us denote with $\mathbb{P}$ the law of X on $L^2([0,T]; B^\alpha_{2,2})$ and define the measure $\mathbb{Q}$ by:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}} = \exp\left(-\int_0^T \int_{\mathbf{T}^d} \frac{\overline{\xi}(x)X_s(x)(1-X_s(x))}{\sqrt{X_s(x)(1-X_s(x))}}\,\mathrm{d}\tilde{\xi}(s,x) - \frac{1}{2}\int_0^T \int_{\mathbf{T}^d} \frac{\left(\overline{\xi}(x)X_s(x)(1-X_s(x))\right)^2}{X_s(x)(1-X_s(x))}\,\mathrm{d}s\,\mathrm{d}x\right).$$

Clearly, this transformation defines a change of measure, since

$$\int_0^T \int_{\mathbf{T}^d} \frac{\left(\overline{\xi}(x)X_s(x)(1-X_s(x))\right)^2}{X_s(x)(1-X_s(x))}\,\mathrm{d}s\,\mathrm{d}x \le T\|\overline{\xi}\|^2_\infty.$$

Under this change of measure, for every $\varphi \in C^\infty(\mathbf{T}^d)$, the process:

$$\langle X_t, \varphi\rangle - \langle X_0, \varphi\rangle - \int_0^t \langle X_s, \nu_0\Delta\varphi\rangle\,\mathrm{d}s =: L^\varphi_t$$

is a continuous $\mathbb{Q}$–martingale with quadratic variation:

$$\langle L^\varphi\rangle_t = \int_0^t \langle X_s(1-X_s), \varphi^2\rangle\,\mathrm{d}s.$$

This means that under $\mathbb{Q}$, the process X_t is the unique (in law) solution to the SPDE:

$$\partial_t X = \nu_0 \Delta X + \sqrt{X(1-X)}\tilde{\xi}, \qquad X(0) = X_0.$$

The uniqueness in law of solutions to the latter equation follows by duality, see for example [Shi88].

□

III.6 Schauder estimates

This section is devoted to the proof of Theorem III.3.13 and other similar results. Since the central object in this section, the semidiscrete Laplace operator $\mathscr{A}_n$, is defined through convolutions with characteristic functions, the following result collects some information that will be useful in the upcoming discussion.

Lemma III.6.1. *Let $(D\varphi)_i = \frac{d\varphi}{dx_i}$ and $(D^2\varphi)_{i,j} = \frac{d^2\varphi}{dx_i dx_j}$ indicate the gradient and the Hessian matrix of a smooth function $\varphi\colon \mathbf{R}^d \to \mathbf{R}$ respectively. Recall that $\hat{\chi}_n(k) = \hat{\chi}(n^{-1}k) = \mathscr{F}_{\mathbf{R}^d}(n^d 1_{\{B_n(0)\}})(k)$. Then:*

$$D\hat{\chi}(0) = 0, \qquad D^2\hat{\chi}(0) = -\frac{(2\pi)^2}{4}\nu_0 \mathrm{Id},$$

with

$$\nu_0 = \frac{1}{3} \;\; in \;\; d = 1, \qquad \nu_0 = \frac{1}{\pi} \;\; in \;\; d = 2.$$

Next, recall that

$$\vartheta_n(k) = n^2\Big(\hat{\chi}_n^4(k) - 1\Big).$$

Then for any choice of constants $c < 1 < C$, there exists a $\kappa(c,C) > 0$ such that

$$c \le \frac{\vartheta_n(k)}{-(2\pi)^2\nu_0|k|^2} \le C, \qquad \forall k\colon \; |k|n^{-1} \le \kappa(c,C).$$

Finally, the decay of $\hat{\chi}$ can be controlled as follows for any $n \in \mathbf{N} \cup \{0\}$ and $i_1, \dots, i_n \in \{1, \dots, d\}$:

$$\Big|\frac{d^n \hat{\chi}(k)}{dx_{i_1} \cdots dx_{i_n}}\Big| \lesssim_n (1+|k|)^{-\frac{d+1}{2}}.$$

The proof of this result is deferred to Section III.7.1. Instead, we pass to the central result of this section, from which all other will follow. Recall that $\mathscr{A}_n$ is a Fourier multiplier, therefore also the exponential $e^{t\mathscr{A}_n}$ and the resolvent $(-\mathscr{A}_n + \lambda)^{-1}$ (for $\lambda > 1$) are naturally defined as Fourier multipliers. As explained already in other points, the action of $\mathscr{A}_n$ is different on large and small Fourier modes. The next result provides the correct choice for this division of scales.

Proposition III.6.2. *For some, and hence for all, $\kappa_0 > 0$ the following holds. For any $p \in [1,\infty], \alpha \in \mathbf{R}$ and $j \ge -1$ there exists a $c > 0$ such that uniformly over $n \in \mathbf{N}, t \ge 0, j \ge -1$ and $\varphi \in \mathscr{C}^\alpha_p$ one can bound:*

$$\begin{aligned} \|\Delta_j \mathscr{A}_n \varphi\|_{L^p(\mathbf{T}^d)} &\lesssim 2^{-(\alpha-2)j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; if \;\; 2^j \le \kappa_0 n, \\ \|\Delta_j \mathscr{A}_n \varphi\|_{L^p(\mathbf{T}^d)} &\lesssim n^2 2^{-\alpha j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; if \; 2^j > \kappa_0 n. \end{aligned} \tag{III.19}$$

And similarly for the exponential:

$$\begin{aligned} \|\Delta_j e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{T}^d)} &\lesssim e^{-ct2^{2j}} 2^{-\alpha j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; for \;\; 2^j \le \kappa_0 n, \\ \|\Delta_j e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{T}^d)} &\lesssim e^{-ctn^2} 2^{-\alpha j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; for \;\; 2^j > \kappa_0 n, \end{aligned} \tag{III.20}$$

and for the resolvent (uniformly over $\lambda > 1$):

$$\begin{aligned} \|\Delta_j(-\mathscr{A}_n + \lambda)^{-1}\varphi\|_{L^p(\mathbf{T}^d)} &\lesssim \frac{1}{2^{2j} + \lambda} 2^{-\alpha j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; for \;\; 2^j \le \kappa_0 n, \\ \|\Delta_j(-\mathscr{A}_n + \lambda)^{-1}\varphi\|_{L^p(\mathbf{T}^d)} &\lesssim \frac{1}{n^2 + \lambda} 2^{-\alpha j}\|\varphi\|_{\mathscr{C}^\alpha_p}, \;\; for \;\; 2^j > \kappa_0 n, \end{aligned} \tag{III.21}$$

Proof. If the estimates hold for a certain $\kappa_0 > 0$, it is evident that they hold for all $\kappa_0 > 0$ (up to changing proportionality constants). In fact, for $2^j \simeq n$ the first and second estimate in every pair are equivalent.

Since all of the estimates follow the same pattern and the first one is particularly simple, we will mainly discuss the proof of the inequalities in (III.20), pointing out how to adapt the calculations to the other cases. We also restrict to the case

$$j \geq 0,$$

since the case $j = -1$ is immediate. We begin by restating the inequalities for distributions on $\mathbf{R}^d$. This is useful because on the entire space we can use scaling arguments. Then we examine the behaviour on large and small scales separately. The precise separation of modes is chosen based on Lemma III.6.1.

Step 1. To restate the problem on $\mathbf{R}^d$ we extend distributions on the torus periodically. Let $\pi\colon \mathscr{S}'(\mathbf{T}^d) \to \mathscr{S}'(\mathbf{R}^d)$ denote the such periodic extension operator of distribution on $\mathbf{T}^d$ to the full space. Its adjoint is the operator $\pi^*\colon \mathscr{S}(\mathbf{R}^d) \to \mathscr{S}(\mathbf{T}^d)$, given by

$$\pi^*\varphi(\cdot) = \sum_{k \in \mathbf{Z}^d} \varphi(\cdot + k).$$

We observe that $\pi(\mathscr{A}_n\varphi) = \mathscr{A}_n\pi(\varphi)$, where with a slight abuse of notation we have extended $\mathscr{A}_n$ to act on distributions on the whole space (simply through Equation (III.1) – and note that it is still a Fourier multiplier, since for $\varphi\colon \mathbf{R}^d \to \mathbf{R}$, $\mathscr{A}_n\varphi = \mathscr{F}^{-1}_{\mathbf{R}^d}\vartheta_n\mathscr{F}_{\mathbf{R}^d}\varphi$). Similarly, by the Poisson summation formula (Lemma I.1.1), $\pi(\Delta_j\varphi) = \Delta_j\pi(\varphi)$. As a consequence of this last observation, and since $\pi(\Delta_j\varphi)$ is periodic, for any $a > \frac{d}{p}$ (or $a \geq 0$ if $p = \infty$):

$$\|\Delta_j\pi(\varphi)\|_{L^p(\mathbf{R}^d;p(a))} \simeq_{a,p} \|\Delta_j\varphi\|_{L^p(\mathbf{T}^d)},$$

Here we have used the weighted spaces introduced in Section I.1. Therefore in order to show (III.20) it is sufficient to show that for all $\varphi \in \mathscr{S}'(\mathbf{R}^d)$ and setting $a = d + 1$:

$$\|\Delta_j e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{R}^d;p(d+1))} \lesssim e^{-ct2^{2j}}\|\Delta_j\varphi\|_{L^p(\mathbf{R}^d;p(d+1))}, \quad \text{for} \quad 2^j \leq \kappa_0 n$$
$$\|\Delta_j e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{R}^d;p(d+1))} \lesssim e^{-ctn^2}\|\Delta_j\varphi\|_{L^p(\mathbf{R}^d;p(d+1))}, \quad \text{for} \quad 2^j > \kappa_0 n.$$

The same holds for (III.19) and (III.21), with the natural changes. Hence, from now on let us consider all functions and operators defined on $\mathbf{R}^d$. Let ψ be a smooth radial function with compact support in an annulus (i.e. $\psi(k) = 0$ if $|k| \leq c_1$ or $|k| \geq c_2$ for some $0 < c_1 < c_2$) such that $\rho\psi = \rho$ (here ρ is associated to the dyadic partition of the unity through which we define Besov spaces: see Section I.1). By Young's inequality for convolutions and by estimating uniformly over $x, y \in \mathbf{R}^d$

$$(1 + |x|^2)^{-\frac{(d+1)}{2}} \lesssim (1 + |y|^2)^{-\frac{(d+1)}{2}}(1 + |x - y|^2)^{\frac{d+1}{2}},$$

one obtains:

$$\|\Delta_j e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{R}^d;p(d+1))} \lesssim \|\mathscr{F}^{-1}_{\mathbf{R}^d}(e^{t\vartheta_n(\cdot)}\psi(2^{-j}\cdot))\|_{L^1(\mathbf{R}^d;p(-d-1))}\|\Delta_j\varphi\|_{L^p(\mathbf{R}^d;p(d+1))}.$$

In this way, through a change of variables, we reduced the problem to a bound for

$$\int_{\mathbf{R}^d} (1+2^{-2j}|x|^2)^{\frac{d+1}{2}} \Big| \mathscr{F}^{-1}_{\mathbf{R}^d}\Big[e^{t\vartheta_n(2^j\cdot)}\psi(\cdot)\Big](x)\Big| \mathrm{d}x \tag{III.22}$$

(and similarly for (III.19) and (III.21), with $e^{t\vartheta_n}$ replaced by ϑ_n and $(-\vartheta_n+\lambda)^{-1}$ respectively). Before we move on, we finally observe that by Lemma III.6.1, there exists a $\kappa_0 > 0$ such that for $2^j n^{-1} \le \kappa_0$:

$$\frac{1}{2} \le \frac{\vartheta_n(2^j k)}{-(2\pi)^2 \nu_0 2^{2j}|k|^2} \le \frac{3}{2}, \qquad \forall k \in \operatorname{supp}(\psi).$$

Step 2. We now estimate (III.22) on large scales, i.e. $2^j n^{-1} \le \kappa_0$. In this case the term can be bounded by:

$$\begin{aligned}\Big\| \Big|\mathscr{F}^{-1}_{\mathbf{R}^d}[e^{t\vartheta_n(2^j\cdot)}\psi(\cdot)] + \sum_{i=1}^d \Big|\mathscr{F}^{-1}_{\mathbf{R}^d}[\partial_{k_i}^{2(d+1)} e^{t\vartheta_n(2^j\cdot)}\psi(\cdot)]\Big|\Big\|_{L^\infty(\mathbf{R}^d)} \\ \lesssim \sup_{k\in\operatorname{supp}(\psi)} \Big[\Big|e^{t\vartheta_n(2^jk)}\psi(k)\Big| + \sum_{i=1}^d \Big|\partial_{k_i}^{2(d+1)} e^{t\vartheta_\vartheta(2^jk)}\psi(k)\Big|\Big].\end{aligned}$$

To bound the term involving derivatives we observe that:

$$D[t\vartheta_n(2^j\cdot)](k) = f(2^j n^{-1}k)t2^{2j}|k|, \qquad f(k) = 4\hat{\chi}^3(k)\frac{D\hat{\chi}(k)}{|k|}.$$

where f is smooth on $\mathbf{R}^d$, again by Lemma III.6.1. In particular, since $2^j \lesssim n$, taking higher order derivatives one has for any $\ell \in \mathbf{N}$: $\Big|\partial_{k_i}^\ell[t\vartheta_n(2^j\cdot)]\Big|(k) \lesssim t2^{2j}$ for $k \in \operatorname{supp}(\psi)$. Now recall Faá di Bruno's formula for $\ell \in \mathbf{N}$:

$$\partial_x^\ell f(g(x)) = \sum_{\{m\}} C(\{m\},\ell)(\partial_x^{m_1+\dots+m_\ell} f)(g(x)) \prod_{r=1}^\ell \Big(\partial_x^r g(x)\Big)^{m_j},$$

where the sum runs over all $\{m\} := (m_1,\dots,m_\ell)$ such that $m_1+2m_2+\dots+\ell m_\ell = \ell$. Applying this formula and by our choice of κ_0, there exists a constant $c>0$ such that:

$$\sup_{k\in\operatorname{supp}(\psi)} \Big[|e^{t\vartheta_n(2^jk)\psi(k)}| + \sum_{i=1}^d \Big|\partial_{k_i}^{2(d+1)} e^{t\vartheta_n(2^jk)}\psi(k)\Big|\Big] \lesssim e^{-\frac{1}{2}(2\pi)^2\nu_0 t2^{2j}}(1+t2^{2j})^{2(d+1)} \lesssim e^{-c(t2^{2j})}.$$

This concludes the proof of the large-scale bound in (III.20). For the resolvent equation one similarly has to bound:

$$\sup_{k\in\operatorname{supp}(\psi)} \Big[\Big|\frac{\psi(k)}{-\vartheta_n(2^jk)+\lambda}\Big| + \sum_{i=1}^d \Big|\partial_{k_i}^{2(d+1)} \frac{\psi(k)}{-\vartheta_n(k)+\lambda}\Big|\Big].$$

Here as before, for the derivative term one has, through the choice of κ_0:

$$\begin{aligned}\Big|\partial_{k_i}^\ell \frac{1}{-\vartheta_n(k)+\lambda}\Big| &\lesssim \sum_{\{m\}} \Big|\frac{1}{-\vartheta_n(k)+\lambda}\Big|^{1+m_1+\dots+m_\ell} \prod_{r=1}^\ell \Big(2^{2j}\Big)^{m_r} \\ &\lesssim \sum_{\{m\}} \Big|\frac{1}{\frac{3}{2}(2\pi)^2\nu_0 2^{2j}+\lambda}\Big|^{1+m_1+\dots+m_\ell} (2^{2j})^{m_1+\dots+m_\ell} \\ &\lesssim \frac{1}{\frac{1}{2}(2\pi)^2\nu_0 2^{2j}+\lambda} \lesssim \frac{1}{2^{2j}+\lambda},\end{aligned}$$

as requested for (III.21). The estimate (III.19) follows similarly.

Step 3. We pass to the small-scale estimates, namely for j such that $2^j n^{-1} > \kappa_0$. Here we will need tighter control on the decay of $\hat{\chi}(k)$: since χ is not smooth, the decay at infinity is not faster than any polynomial and is quantified in Lemma III.6.1. We now estimate (III.22) by:

$$\left(\int_{\mathbf{R}^d} \frac{1}{(1+|x|)^{d+1}} \, \mathrm{d}x\right) \sup_{x \in \mathbf{R}^d} \left[(1+|x|^{d+1} + 2^{-j(d+1)}|x|^{2(d+1)}) \left| \mathscr{F}^{-1}_{\mathbf{R}^d} \left[e^{t\vartheta_n(2^j \cdot)} \psi(\cdot) \right] \right| (x) \right]$$
$$\lesssim \|e^{\vartheta_n(2^j \cdot)} \psi(\cdot)\|_{L^\infty} + \|(1-\Delta)^{\frac{d+1}{2}} e^{t\vartheta_n(2^j \cdot)} \psi(\cdot)\|_{L^p(\mathbf{R}^d)} + \sum_{i=1}^d 2^{-j(d+1)} \|\partial_{k_i}^{2(d+1)} e^{t\vartheta_n(2^j \cdot)} \psi(\cdot)\|_{L^\infty},$$

for any $p \in (1, \infty)$. As for the first term, since $|\hat{\chi}(k)| < 1$ for $k \neq 0$ and it decays to zero at infinity, up to reducing the value of $c > 0$ we can assume that:

$$\vartheta_n(2^j k) \leq -cn^2.$$

This is sufficient to show:

$$\|e^{t\vartheta_n(2^j \cdot)} \psi(\cdot)\|_{L^\infty} \lesssim e^{-ctn^2},$$

which is a bound of the required order.

Now bounding these derivatives is similar to bounding the last term:

$$\sum_{i=1}^d 2^{-j(d+1)} \|\partial_{k_i}^{2(d+1)} e^{t\vartheta_n(2^j \cdot)} \psi(\cdot)\|_{L^\infty},$$

so we concentrate on the latter, which has the added difficulty of containing derivatives of higher order, counterbalanced by the factor $2^{-j(d+1)}$. Here observe that for $1 \leq \ell \leq 2(d+1)$:

$$\partial_{k_i}^\ell e^{t\vartheta_n(2^j k)} = \partial_{k_i}^{\ell-1} \left[e^{t\vartheta_n(2^j k)} 4\hat{\chi}^3(2^j n^{-1} k)[\partial_{k_i} \hat{\chi}](2^j n^{-1} k) \right] \cdot (2^j n^{-1}) \cdot (tn^2).$$

Iterating the above procedure, we apply Faá Di Bruno's formula again to obtain

$$\left| 2^{-j(d+1)} \partial_{k_i}^\ell e^{t\vartheta_n(2^j k)} \right| \lesssim 2^{-j(d+1)} e^{t\vartheta_n(2^j k)} \sum_{\{m\}} \prod_{r=1}^\ell \left(\partial_{k_i}^{r-1} \left[4\hat{\chi}^3(\cdot)[\partial_{k_i} \hat{\chi}(\cdot)] \right] \Big|_{2^j n^{-1} k} \cdot (2^j n^{-1})^r \right)^{m_r} \cdot (tn^2)^{m_r}.$$

In view of Lemma III.6.1, for any $r \in \mathbf{N}$:

$$\sup_{k \in \mathrm{supp}(\psi)} \left| \partial_{k_i}^{r-1} \left[4\hat{\chi}^3(\cdot)[\partial_{k_i} \hat{\chi}(\cdot)] \right] \Big|_{2^j n^{-1} k} \right| \lesssim \frac{1}{1+|2^j n^{-1}|^{2(d+1)}}.$$

Hence, as before up to further reducing the value of $c > 0$:

$$\|\partial_{k_i}^\ell e^{t\vartheta_n(2^j \cdot)}\|_{L^\infty} \lesssim e^{-ctn^2} 2^{-j(d+1)} (2^j n^{-1})^\ell \sum_{\{m\}} \prod_{r=1}^\ell (1+|2^j n^{-1}|)^{-2m_r(d+1)}$$
$$\lesssim e^{-ctn^2} 2^{-j(d+1)} (2^j n^{-1})^{\ell - 2(d+1)} \lesssim e^{-ctn^2},$$

since at least one of the elements of the sequence m_r is strictly positive and since $\ell \leq 2(d+1)$. This concludes the proof of (III.20). Regarding the resolvent, one can follow mutatis mutandis the previous discussion until one has, as before, to bound:

$$\sum_{i=1}^{d} 2^{-j(d+1)} \Big\| \partial_{k_i}^{2(d+1)} \frac{\psi(\cdot)}{-\vartheta_n(2^j \cdot) + \lambda} \Big\|_\infty \lesssim \sum_{i=1}^{d} \sum_{\ell=0}^{2(d+1)} 2^{-j(d+1)} \Big\| \partial_{k_i}^{\ell} \frac{1}{-\vartheta_n(2^j \cdot) - \lambda} \Big\|_{L^\infty}.$$

Then again, with Faá di Bruno's formula:

$$\begin{aligned} \Big| \partial_{k_i}^{\ell} \frac{1}{-\vartheta_n(2^j k) + \lambda} \Big| &\lesssim \sum_{\{m\}} \Big| \frac{1}{-\vartheta_n(2^j k) + \lambda} \Big|^{1+m_1+\dots+m_\ell} \prod_{r=1}^{\ell} \big| \partial_{k_i}^{r-1} (\hat{\chi}^3(\cdot) \partial_{k_i} \hat{\chi}(\cdot)) |_{2^j n^{-1} k} \big|^{m_r} \cdot (2^j n^{-1})^{r m_r} \\ &\lesssim \frac{1}{n^2 + \lambda} \sum_{\{m\}} \Big| \frac{1}{n^2+\lambda} \Big|^{m_1+\dots+m_\ell} \prod_{r=1}^{\ell} \Big(\frac{1}{1 + |2^j n^{-1}|} \Big)^{2 m_r (d+1)} (2^j n^{-1})^{r m_r} \\ &\lesssim \frac{1}{n^2+\lambda} 2^{j(d+1)}. \end{aligned}$$

Plugging this into the previous formula provides us the correct bound. Similarly one can also treat the small-scale estimate for (III.19). □

The previous proposition motivates the introduction of cut-off operators as follows.

Definition III.6.3. *Let* $\urcorner \colon \mathbf{R}^d \to \mathbf{R}$ *be a smooth radial function with compact support. Let us define the annulus* $A_r^R = \{x \in \mathbf{R}^d \colon r \leq |x| \leq R\}$ *for* $0 < r < R$*. Then we additionally assume that:*

$$\urcorner(x) = 1, \ \forall x \in A_0^r, \qquad \urcorner(x) = 0, \ \forall x \in A_R^\infty,$$

for some $0 < r < R < \infty$*. Define*

$$\mathscr{P}_n = \urcorner(n^{-1} D), \qquad \mathscr{Q}_n = (1 - \urcorner)(n^{-1} D).$$

We say that $\mathscr{P}_n$ *is a projection on* ***large scales****, since those Fourier modes describe a function macroscopically, whereas* $\mathscr{Q}_n$ *is a projection on* ***small scales****.*

The next lemma states that the cut-off operators are bounded.

Lemma III.6.4. *Consider* $\alpha \in \mathbf{R}$ *and* $p \in [1, \infty]$*. For* $\urcorner$ *as in Definition III.6.3 one can bound uniformly over* $n \in \mathbf{N}$*:*

$$\|\mathscr{P}_n \varphi\|_{\mathscr{C}_p^\alpha} \lesssim \|\varphi\|_{\mathscr{C}_p^\alpha}, \qquad \|\mathscr{Q}_n \varphi\|_{\mathscr{C}_p^\alpha} \lesssim \|\varphi\|_{\mathscr{C}_p^\alpha}.$$

Proof. Define the inverse Fourier transform $\widehat{\urcorner}(x) = \mathscr{F}_{\mathbf{R}^d}^{-1} \urcorner(x)$. By an application of the Poisson summation formula (Lemma I.1.1) and a scaling argument:

$$\begin{aligned} \|\urcorner(n^{-1} D) \varphi\|_{\mathscr{C}_p^\alpha} &= \sup_{j \geq -1} 2^{j\alpha} \| (\mathscr{F}_{\mathbf{T}^d}^{-1} [\urcorner(n^{-1} \cdot)]) * \Delta_j \varphi \|_{L^p} \lesssim \| \mathscr{F}_{\mathbf{T}^d}^{-1} [\urcorner(n^{-1} \cdot)] \|_{L^1(\mathbf{T}^d)} \|\varphi\|_{\mathscr{C}_p^\alpha} \\ &\lesssim \| n^d \widehat{\urcorner}(n \cdot) \|_{L^1(\mathbf{R}^d)} \|\varphi\|_{\mathscr{C}_p^\alpha} \lesssim \|\varphi\|_{\mathscr{C}_p^\alpha}. \end{aligned}$$

The same argument shows that $(1 - \urcorner(a \cdot))$ is bounded. □

III.6.1 Elliptic regularity

In this subsection we prove Theorem III.3.13. The theorem is a direct consequence of the lemma and the proposition that follows.

Lemma III.6.5. *Fix any $\alpha \in \mathbf{R}, \zeta > 0, p \in [1,\infty]$. Uniformly over $\varphi \in \mathscr{C}^{\alpha}_p$ and $n \in \mathbf{N}$:*

$$\|\mathscr{A}_n \mathscr{P}_n \varphi\|_{\mathscr{C}^{\alpha-2}_p} \lesssim \|\varphi\|_{\mathscr{C}^{\alpha}_p}.$$

Moreover, as $n \to \infty$

$$\mathscr{A}_n \varphi \to \nu_0 \Delta \varphi \quad in \quad \mathscr{C}^{\alpha-2-\zeta}_p,$$

where

$$\nu_0 = \frac{1}{3} \quad for \quad d = 1, \qquad \nu_0 = \frac{1}{\pi} \quad for \quad d = 2.$$

Proof. On large scales, Proposition III.6.2 and Lemma III.6.4 imply that

$$\|\mathscr{A}_n \mathscr{P}_n \varphi\|_{\mathscr{C}^{\alpha-2}_p} \lesssim \|\mathscr{P}_n \varphi\|_{\mathscr{C}^{\alpha}_p} \lesssim \|\varphi\|_{\mathscr{C}^{\alpha}_p}.$$

Moreover on small scales the same results guarantee that for any $\zeta \geq 0$:

$$\|\mathscr{Q}_n \mathscr{A}_n \varphi\|_{\mathscr{C}^{\alpha-2-\zeta}_p} \lesssim n^2 \sup_{2^j \gtrsim n} 2^{j(\alpha-2-\zeta)} \|\Delta_j \mathscr{Q}_n \varphi\|_{L^p} \lesssim n^{-\zeta} \|\varphi\|_{\mathscr{C}^{\alpha}_p},$$

which tends to 0 as n tends to ∞ if $\zeta > 0$. Combining these two observations provides the first bound and guarantees compactness in $\mathscr{C}^{\alpha-2-\zeta}_p$. Convergence follows since, by Lemma III.6.1, for any $k \in \mathbf{Z}^d$:

$$\mathscr{F}_{\mathbf{T}^d}[\mathscr{A}_n \mathscr{P}_n \varphi](k) = \urcorner(n^{-1}k) n^2 \Big(\hat{\chi}^2(n^{-1}k) - 1\Big) \hat{\varphi}(k) \to -(2\pi)^2 \nu_0 |k|^2 \hat{\varphi}(k) = \mathscr{F}_{\mathbf{T}^d}[\nu_0 \Delta \varphi](k).$$

□

The regularity gain provided by the operator $\mathscr{A}_n$ can be described as follows (for the proof of Theorem III.3.13 we require the result only for $\delta = 0$).

Proposition III.6.6. *Fix any $\alpha \in \mathbf{R}$, $\delta \in [0,1]$ and $p \in [1,\infty]$. Uniformly over $\lambda > 1, n \in \mathbf{N}$ and $\varphi \in \mathscr{C}^{\alpha}_p$ the following estimates hold:*

$$\lambda^{\delta} \|\mathscr{P}_n(-\mathscr{A}_n + \lambda)^{-1} \varphi\|_{\mathscr{C}^{\alpha+2(1-\delta)}_p} + \lambda^{\delta} n^{2(1-\delta)} \|\mathscr{Q}_n(-\mathscr{A}_n + \lambda)^{-1} \varphi\|_{\mathscr{C}^{\alpha}_p} \lesssim \|\varphi\|_{\mathscr{C}^{\alpha}_p}.$$

Moreover, as $n \to \infty$,

$$\mathscr{P}_n(-\mathscr{A}_n + \lambda)^{-1} \varphi \to (-\nu_0 \Delta + \lambda)^{-1} \varphi$$

where the convergence is in $\mathscr{C}^{\alpha+2-\zeta}_p$ for any $\zeta > 0$ and ν_0 is as in Lemma III.6.5.

Proof. Consider the large-scale estimate. Proposition III.6.2 and Lemma III.6.4 guarantee that for $2^j \lesssim n$:

$$\|\Delta_j \mathscr{P}_n(-\mathscr{A}_n + \lambda)^{-1} \varphi\|_{L^p} \lesssim \frac{1}{2^{2j} + \lambda} 2^{-\alpha j} \|\mathscr{P}_n \varphi\|_{\mathscr{C}^{\alpha}_p} \lesssim 2^{-2j(1-\delta)-\alpha j} \lambda^{-\delta} \|\varphi\|_{\mathscr{C}^{\alpha}_p},$$

which is a bound of the correct order. All other bounds follow similarly, and the proof of the convergence is analogous to the one in Lemma III.6.5. □

III.6.2 Parabolic regularity

In this subsection we study the regularization effect of the semigroup $e^{t\mathscr{A}_n}$. This discussion requires the spaces of time-dependent functions introduced in Section I.1.4. Throughout this section we fix an arbitrary time horizon $T>0$. All function spaces will depend implicitly on this choice. All estimates hold locally uniformly over the choice of T, unless stated otherwise.

Now we state the main result of this section, the parabolic Schauder estimates.

Proposition III.6.7. *Fix $p\in[1,\infty], T>0, \gamma\in[0,1)$ and $\alpha\in(-2,0), \beta\in[\alpha,\alpha+2)\cap(0,2)$. Uniformly over $\varphi\in\mathscr{C}^\alpha_p$ and $f\in\mathscr{M}^\gamma\mathscr{C}^\alpha_p$ and locally uniformly over $T>0$:*

$$\|t\mapsto \mathscr{P}_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{L}_p^{(\beta-\alpha)/2,\beta}}\lesssim\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p},\tag{III.23}$$

$$\Big\|t\mapsto\int_0^t\mathscr{P}_n e^{(t-s)\mathscr{A}_n}f(s)\,\mathrm{d}s\Big\|_{\mathscr{L}_p^{\gamma,\alpha+2}}\lesssim\|\mathscr{P}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}.\tag{III.24}$$

In addition, let $\zeta_1,\zeta_2\in[0,1)$ such that $\zeta_1+\zeta_2<1$ and $\delta_1,\delta_2,\delta_3\in[0,1]$ such that $\delta_1+\delta_2+\delta_3=1$. Then:

$$\|t\mapsto t^{\zeta_1+\zeta_2}\mathscr{Q}_n e^{t\mathscr{A}_n}\varphi\|_{C^{\zeta_1}\mathscr{C}^\alpha_p}\lesssim n^{-2\zeta_2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p},\tag{III.25}$$

$$\Big\|t\mapsto t^\gamma\int_0^t e^{(t-s)\mathscr{A}_n}\mathscr{Q}_n f(s)\,\mathrm{d}s\Big\|_{C^{\delta_1}\mathscr{C}^\alpha_p}\lesssim n^{-2\delta_2}T^{\delta_3}\|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}.\tag{III.26}$$

with constants independent of f,φ,T.

In many steps the proof mimics proofs in [GIP15] and [GP17], to which we refer the reader for simple proofs of classical Schauder estimates in the setting of stochastic PDEs.

Proof. Step 1. We begin with large scales, namely (III.23). By Proposition III.6.2:

$$\begin{aligned}\sup_{j\geq-1}2^{\beta j}\|\Delta_j\mathscr{P}_n e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{T}^d)}&\lesssim\sup_{j\geq-1}e^{-ct2^{2j}}2^{(\beta-\alpha)j}\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}\\&=t^{-\frac{\beta-\alpha}{2}}\sup_{j\geq-1}e^{-ct2^{2j}}(t2^{2j})^{\frac{\beta-\alpha}{2}}\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}\lesssim t^{-\frac{\beta-\alpha}{2}}\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}.\end{aligned}$$

Therefore

$$\|t\mapsto\mathscr{P}_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{M}^{(\beta-\alpha)/2}\mathscr{C}^\beta_p}\lesssim\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}.$$

Similarly, for (III.24)

$$\sup_{j\geq-1}2^{j(\alpha+2)}\Big\|\int_0^t\Delta_j e^{(t-s)\mathscr{A}_n}\mathscr{P}_n f(s)\,\mathrm{d}s\Big\|_{L^p(\mathbf{T}^d)}\lesssim\|\mathscr{P}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}\sup_{j\geq-1}2^{j2}\int_0^t e^{-cs2^{2j}}(t-s)^{-\gamma}\,\mathrm{d}s.$$

which can be bounded by $\|\mathscr{P}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}$ by the same arguments as in the proof of [GIP15, Lemma A.9]. We still need to address the temporal regularity for both terms. Again, Proposition III.6.2 leads to:

$$\begin{aligned}\|(e^{t\mathscr{A}_n}-\mathrm{Id})\mathscr{P}_n\varphi\|_{L^p(\mathbf{T}^d)}&=\Big\|\int_0^t e^{s\mathscr{A}_n}\mathscr{A}_n\mathscr{P}_n\varphi\,\mathrm{d}s\Big\|_{L^p(\mathbf{T}^d)}\\&\lesssim\int_0^t s^{-1+\frac{\alpha}{2}}\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}\,\mathrm{d}s\simeq t^{\frac{\alpha}{2}}\|\mathscr{P}_n\varphi\|_{\mathscr{C}^\alpha_p}.\end{aligned}\tag{III.27}$$

To conclude the proof of both (III.23) and (III.24) it is now sufficient to follow the same steps as in [GP17, Lemma 6.6].

Step 2. We turn our attention to the small scale bounds (III.25) and (III.26). Fix $\zeta_1 = \delta_1 = 0$ first. With calculations in the same spirit as in the Step 1, we arrive at:

$$\|\mathscr{Q}_n e^{t\mathscr{A}_n}\varphi\|_{\mathscr{C}^\alpha_p} = \sup_{j\geq -1} 2^{\alpha j}\|\Delta_j \mathscr{Q}_n e^{t\mathscr{A}_n}\varphi\|_{L^p(\mathbf{R}^d)} \lesssim e^{-ctn^2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p} \lesssim (tn^2)^{-\zeta_2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p}.$$

For the inequality (III.26), if $\delta_3 > 0$ the spatial bound follows from the previous result. If $\delta_3 = 0$, we observe that

$$\Big\|\int_0^t \mathscr{Q}_n e^{(t-s)\mathscr{A}_n} f(s)\,\mathrm{d}s\Big\|_{\mathscr{C}^\alpha_p} \lesssim \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}\int_0^t e^{-csn^2}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim n^{-2}t^{-\gamma}\|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma\mathscr{C}^\alpha_p}.$$

The last bound in the above inequality is obtained in the same spirit as [GIP15, Lemma A.9]. Namely, choose $\lambda \in (0, t/2)$ and split the integral at time λ. We note that

$$\int_0^\lambda e^{-csn^2}(t-s)^{-\gamma}\,\mathrm{d}s \leq \int_0^\lambda (t-s)^{-\gamma}\,\mathrm{d}s = t^{-\gamma+1}\int_0^{\lambda/t}(1-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{-\gamma}\lambda,$$

since, as $\lambda/t \leq 1/2$, $1-(1-\lambda/t)^{(1-\gamma)} \lesssim \lambda/t$. We then observe that for any $\rho \in (0,1)$,

$$\begin{aligned}\int_\lambda^t e^{-csn^2}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim \int_\lambda^t (sn^2)^{-(1+\rho)}(t-s)^{-\gamma}\,\mathrm{d}s &\lesssim t^{-\gamma-\rho}n^{-2(1+\rho)}\int_{\lambda/t}^1 s^{-(1+\rho)}(1-s)^{-\gamma}\,\mathrm{d}s\\ &\lesssim t^{-\gamma}n^{-2(1+\rho)}\lambda^{-\rho}.\end{aligned}$$

If $n^{-2} \leq t/2$, choosing $\lambda = n^{-2}$ provides the result. Otherwise, one simply notes that

$$\int_0^t e^{-csn^2}(t-s)^{-\gamma}\,\mathrm{d}s \lesssim t^{1-\gamma} \lesssim t^{-\gamma}n^{-2}.$$

Step 3. We now investigate the full temporal regularity for (III.25) and (III.26), that is, we allow for $\zeta_1, \delta_1 > 0$. We first observe that for $\delta \in [0,1)$

$$\begin{aligned}\|(e^{t\mathscr{A}_n}-\mathrm{Id})\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p} &= \Big\|\int_0^t e^{s\mathscr{A}_n}\mathscr{A}_n\mathscr{Q}_n\varphi\,\mathrm{d}s\Big\|_{\mathscr{C}^\alpha_p}\\ &\lesssim \|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p}\int_0^t (sn^2)^{-\delta}n^{-2}\,\mathrm{d}s = \|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p}n^{-2(\delta-1)}t^{1-\delta}.\end{aligned} \tag{III.28}$$

Hence for $\zeta = \zeta_1+\zeta_2 \in [0,1)$, the temporal regularity of the first terms can be established via

$$\begin{aligned}\|t^\zeta e^{t\mathscr{A}_n}\mathscr{Q}_n\varphi - s^\zeta e^{s\mathscr{A}_n}\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p} &\lesssim (t^\zeta - s^\zeta)t^{-\zeta_2}n^{-2\zeta_2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p} + s^\zeta\|(e^{(t-s)\mathscr{A}_n}-\mathrm{Id})e^{s\mathscr{A}_n}\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p}\\ &\lesssim (t^\zeta - s^\zeta)t^{-\zeta_2}n^{-2\zeta_2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p} + s^\zeta(t-s)^{1-\delta}n^{-2(\delta-1)}\|e^{s\mathscr{A}_n}\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p}\\ &\lesssim [(t^\zeta - s^\zeta)t^{-\zeta_2}n^{-2\zeta_2} + (t-s)^{1-\delta}n^{-2(\delta-1)}n^{-2\zeta}]\|\mathscr{Q}_\varepsilon\varphi\|_{\mathscr{C}^\alpha_p}\\ &\lesssim (t-s)^{\zeta_1}n^{-2\zeta_2}\|\mathscr{Q}_n\varphi\|_{\mathscr{C}^\alpha_p},\end{aligned}$$

where in the last step we set $\delta = 1-\zeta_1$ and notice that $(t^\zeta - s^\zeta)t^{-\zeta_2} \lesssim (t-s)^{\zeta_1}$.

The bound for (III.26) follows similar pattern. For simplicity write $V(t) = \int_0^t e^{(t-s)\mathscr{A}_n} \mathscr{Q}_n f(s)\,\mathrm{d}s$. Then

$$\|t^\gamma V(t) - s^\gamma V(s)\|_{\mathscr{C}^\alpha_p} \le (t^\gamma - s^\gamma)\|V(t)\|_{\mathscr{C}^\alpha_p} + s^\gamma \left\| \int_s^t e^{(t-r)\mathscr{A}_n} \mathscr{Q}_n f(r)\,\mathrm{d}r \right\|_{\mathscr{C}^\alpha_p} + s^\gamma \|(e^{(t-s)\mathscr{A}_n} - \mathrm{Id})V(s)\|_{\mathscr{C}^\alpha_p}.$$

The only term for which the estimation does not follow the already established pattern is the one in the middle, for which we observe that

$$\begin{aligned}
s^\gamma \left\| \int_s^t e^{(t-r)\mathscr{A}_n} \mathscr{Q}_n f(r)\,\mathrm{d}r \right\|_{\mathscr{C}^\alpha_p} &\lesssim s^\gamma \int_s^t ((t-r)n^2)^{-\delta_2} r^{-\gamma}\,\mathrm{d}r \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} \\
&\lesssim \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} n^{-2\delta_2} s^\gamma t^{-\delta_2 - \gamma + 1} \int_{s/t}^1 (1-r)^{-\delta_2} r^{-\gamma}\,\mathrm{d}r \\
&\lesssim \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} n^{-2\delta_2} t^{1-\delta_2} \int_{s/t}^1 (1-r)^{-\delta_2}\,\mathrm{d}r \\
&\lesssim \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} n^{-2\delta_2} t^{1-\delta_2} (1-s/t)^{1-\delta_2} \\
&\le \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} n^{-2\delta_2} (t-s)^{1-\delta_2} \\
&\le \|\mathscr{Q}_n f\|_{\mathscr{M}^\gamma \mathscr{C}^\alpha_p} n^{-2\delta_2} T^{\delta_3} (t-s)^{\delta_1},
\end{aligned}$$

which completes the proof of the proposition. □

The following result is essentially a by-product of the previous proof, but deserves a separate statement, for later use.

Lemma III.6.8. *Consider $\alpha, \beta \in \mathbf{R}$ and $p \in [1, \infty]$ with $\gamma := \alpha - \beta \in [0, 2]$. Then uniformly over $\varphi \in \mathscr{C}^\alpha_p$:*

$$\|(e^{t\mathscr{A}_n} - \mathrm{Id})\varphi\|_{\mathscr{C}^\beta_p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathscr{C}^\alpha_p}.$$

Proof. The proof follows from Proposition III.6.7. Indeed, Equation (III.27) implies that for $2^j \lesssim n$ one has:

$$2^{j\beta} \|(e^{t\mathscr{A}_n} - \mathrm{Id})\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{j\beta} \|\Delta_j \varphi\|_{\mathscr{C}^\gamma_p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathscr{C}^\alpha_p}.$$

While a slight modification (to L^p spaces) of (III.28) guarantees that for $2^j \gtrsim n$:

$$2^{j\beta} \|(e^{t\mathscr{A}_n} - \mathrm{Id})\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{j\beta} n^\gamma \|\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} 2^{j\alpha} \|\Delta_j \varphi\|_{L^p} \lesssim t^{\frac{\gamma}{2}} \|\varphi\|_{\mathscr{C}^\alpha_p}.$$

This concludes the proof. □

III.7 Some analytic results

In this appendix we recall some of the analytic theory we require. First we concentrate on special properties of Besov spaces and the regularity of characteristic functions. Later we will address some relevant points in paracontrolled calculus.

III.7.1 Besov spaces & characteristic functions

In certain cases, it will be convenient to use the following alternative characterization of certain Besov spaces.

Proposition III.7.1 (Sobolev-Slobodeckij norm)**.** *For every $\alpha \in \mathbf{R}_+ \setminus \mathbf{N}$ and for every $p \in [1,\infty)$ define the Sobolev-Slobodeckij norm for $\varphi \in \mathscr{S}'(\mathbf{T}^d)$ as:*

$$\|\varphi\|_{W^\alpha_p} := \|\varphi\|_{L^p} + \sum_{|m|=\lfloor\alpha\rfloor}\left(\int_{\mathbf{T}^d\times\mathbf{T}^d}\frac{|D^m\varphi(x)-D^m\varphi(y)|^p}{|x-y|^{d+(\alpha-\lfloor\alpha\rfloor)p}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/p}\in[0,\infty].$$

There exist constants a pair of constants $c(p), C(p) > 0$ such that for $\varphi \in \mathscr{S}'(\mathbf{T}^d)$

$$c\|\varphi\|_{B^\alpha_{p,p}} \le \|\varphi\|_{W^\alpha_p} \le C\|\varphi\|_{B^\alpha_{p,p}}.$$

For a proof consult e.g. [Tri10] Theorem 2.5.7 and the discussion in Section 2.2.2. The next result states the regularizing properties of convolutions.

Lemma III.7.2. *For $p,q,r \in [1,\infty]$ satisfying $\frac{1}{r} = \frac{1}{p}+\frac{1}{q}-1$ and for any $\varphi,\psi \in \mathscr{S}'(\mathbf{T}^d)$:.*

$$\|\varphi * \psi\|_{C^{\alpha+\beta}_r} \lesssim \|f\|_{C^\alpha_p}\|g\|_{\mathscr{C}^\beta_q}.$$

Proof. By Young convolution inequality

$$\|\Delta_i(f*g)\|_{L^r} = \|\Delta_i f * \overline{\Delta}_i g\|_{L^r} \lesssim \|\Delta_i f\|_{L^p}\|\overline{\Delta}_i g\|_{L^q}, \tag{III.29}$$

where $\overline{\Delta}_i$ is associated with a dyadic partition of the unity different from the one we use for most of the proofs. Namely we require that it satisfies $\{\overline{\varrho_j}\}_{j\ge-1}$ such that $\varrho_j\overline{\varrho_j} = \varrho_j$. Then the bound follows immediately, since the Besov norms associated to different dyadic partitions are equivalent (cf. [BCD11, Remark 2.17]). □

The following lemma is a special case of results obtained by [Sic99]. The proof is included for completeness.

Lemma III.7.3. *Fix $p \in [1,\infty)$, $\zeta \in [0,\frac{1}{p})$. Then:*

$$\sup_{n\in\mathbf{N}} n^{-\zeta-d+\frac{d}{p}}\|\chi_n\|_{W^\zeta_p} < \infty.$$

Proof. We shall make use of the characterization of fractional Sobolev spaces in terms of Sobolev-Slobodeckij norms. A direct computation shows that

$$\begin{aligned}\|\chi_n\|_{W^\zeta_p} &= \|\chi_n\|_{L^p} + \left(\int_{\mathbf{T}^d\times\mathbf{T}^d} n^{dp}\frac{|1_{B_n}(x)-1_{B_n}(y)|^p}{|x-y|^{d+\zeta p}}\mathrm{d}x\mathrm{d}y\right)^{1/p}\\ &\le n^{d-\frac{d}{p}} + \left(2\int_{B_n}\int_{\mathbf{T}^d\setminus B_n} n^{dp}\frac{|1_{B_n}(x)-1_{B_n}(y)|^p}{|x-y|^{d+\zeta p}}\mathrm{d}x\mathrm{d}y\right)^{1/p}.\end{aligned}$$

Now let $d_n(z)$ be the Euclidean distance of a point z from the boundary ∂B_n and let $\overline{B}_{d_n(z)}(y)$ be the ball of radius $d_n(z)$ about y. Then the previous integral can be estimated by:

$$\left(\int_{B_n}\int_{\mathbf{T}^d\setminus B_n} n^{dp}\frac{|1_{B_n}(x)-1_{B_n}(y)|^p}{|x-y|^{d+\zeta p}}\mathrm{d}x\mathrm{d}y\right)^{1/p} \le \left(\int_{B_n}\int_{\mathbf{T}^d\setminus \overline{B}_{d_n(y)}(y)} n^{dp}\frac{1}{|x-y|^{d+\zeta p}}\mathrm{d}x\mathrm{d}y\right)^{1/p}$$
$$= \left(\int_{B_n}\int_{\mathbf{T}^d\setminus \overline{B}_{d_n(y)}(0)} n^{dp}\frac{1}{|x|^{d+\zeta p}}\mathrm{d}x\mathrm{d}y\right)^{1/p} \lesssim \left(\int_{B_n} n^{dp} d_n(y)^{-\zeta p}\mathrm{d}y\right)^{1/p}$$
$$\lesssim \left(\int_0^{\frac{c}{n}} n^{dp}(c/n-r)^{-\zeta p} r^{d-1}\mathrm{d}r\right)^{\frac{1}{p}} \lesssim n^d\left(n^{\zeta p-d}\right)^{1/p} \le n^{d+\zeta-d/p}.$$

□

Corollary III.7.4. *Recall that we define the operator Π_n by*

$$\Pi_n\varphi(x) = \chi_n * \varphi(x).$$

Then, for $\zeta\in[0,1), p\in[1,\infty]$ and $\alpha\in\mathbf{R}$

$$\sup_{n\in\mathbf{N}} n^{-\zeta}\|\Pi_n\varphi\|_{\mathscr{C}_p^{\alpha+\zeta}} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}}.$$

Proof. This is now a direct consequence of Lemma III.7.2 and III.7.3 (the latter with $p = 1$).

□

The rest of this subsection is devoted to the proof of Lemma III.6.1.

Proof of Lemma III.6.1. Let us start with the term involving the gradient. We have that for $i = 1,\dots,d$:

$$(D\widehat{\chi})_i(0) = -2\pi\iota \fint_{B_1(0)} x_i e^{-2\pi\iota\langle k,x\rangle}\mathrm{d}x\Big|_{k=0} = 0.$$

For the term involving the Hessian, we observe that an analogous computation for $i\neq j$ shows that $(D^2\widehat{\chi})_{i,j}(0) = 0$. If $i = j$ we find that

$$(D^2\widehat{\chi})_{i,i}(0) = -(2\pi)^2 \fint_{B_1(0)} x_i^2 e^{-2\pi\iota\langle k,x\rangle}\mathrm{d}x\Big|_{k=0} =: \frac{-(2\pi)^2}{4}\nu_0,$$

with the value of ν_0 as in the statement. The two-sided inequality follows by a Taylor approximation.

We are left with a bound on the decay of $\hat{\chi}$:

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}x_{i_1}\dots\mathrm{d}x_{i_n}}\hat{\chi}_B(k)\right| \lesssim (1+|k|)^{-\frac{d+1}{2}}.$$

For this purpose let $J_\nu(\cdot)$ be the Bessel function of the first kind with parameter ν, that is

$$J_\nu(k) = \sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(m+\nu+1)}\left(\frac{k}{2}\right)^{2m+\nu}.$$

The Fourier transform of χ_B can be written, for some $c, C > 0$, as

$$\hat{\chi}_B(k) = c(d)\int_0^{\pi} \sin^d(t) e^{-2\pi\imath|k|\cos(t)/4}\,\mathrm{d}t = C|k|^{-d/2} J_{d/2}(\pi|k|/2). \tag{III.30}$$

In the last step we used one of the alternative representations of Bessel functions, see e.g. [Wat95, Section 6.15, Equation (5)] (the author uses the notation K_n for the real part of J_ν, but in our case the Bessel function is real valued). Since $J_{\frac{1}{2}}(k) = \sqrt{\frac{2}{\pi k}}\sin k$, the bound for $d = 1$ is immediate. For $d = 2$, we make use of an asymptotic bound for Bessel functions:

$$\sup_{\rho\geq 1} \rho^{1/2}|J_\nu(\rho)| < +\infty.$$

We provide a proof of this bound in the next Lemma. The bound for the derivatives then follows from (III.30), the asymptotic result for Bessel functions, and the following pair of identities

$$\partial_x J_n(x) = \frac{1}{2}(J_{n-1}(x)+J_{n+1}(x)), \qquad \forall n \in \mathbf{Z},$$
$$J_{-n}(\cdot) = (-1)^n J_n(\cdot) \qquad \forall n \in \mathbf{N}_0.$$

□

The following result is well-known (see e.g. [Wat95], where many deeper results are presented). For completeness we provide a proof that satisfies all our purposes.

Lemma III.7.5. *Fix $\nu \in \mathbf{R}$. Then*

$$\sup_{\rho\geq 1} \rho^{1/2}|J_\nu(\rho)| < +\infty,$$

Proof. Through (III.30) and by changing variables $x = cos(t)$ we rewrite the Bessel function as

$$\int_{-1}^{1} (1-x^2)^{\frac{d-1}{2}} e^{\imath\rho x}\,\mathrm{d}x = 2\mathrm{Re}\bigg(\int_0^1 (1-x^2)^{\frac{d-1}{2}} e^{\imath\rho x}\bigg)\mathrm{d}x.$$

A change variables $x = 1-u^2$. yields

$$e^{i\rho}\int_0^1 \Big(u^2(2-u^2)\Big)^{\frac{d-1}{2}} e^{-\imath\rho u^2} u\,\mathrm{d}u = \frac{e^{i\rho}}{\rho^{\frac{d+1}{2}}}\int_0^{\sqrt{\rho}} \Big(w^2(2-\frac{w^2}{\rho})\Big)^{\frac{d-1}{2}} e^{-\imath w^2} w\,\mathrm{d}w.$$

Observe that in order to obtain the desired bound it is now sufficient to show that the integral terms is bounded uniformly in ρ. After another change of variable $w = e^{-\imath\frac{\pi}{4}} z$ we obtain

$$\int_0^{e^{\frac{\imath\pi}{4}}\sqrt{\rho}} \Big(-\imath z^2(2+\imath z^2/\rho)\Big)^{\frac{d-1}{2}} e^{-z^2} z\,\mathrm{d}z$$
$$= \int_0^{\sqrt{\rho}} \Big(-\imath z^2(2+\imath z^2/\rho)\Big)^{\frac{d-1}{2}} e^{-z^2} z\,\mathrm{d}z + \int_0^{\pi/4} \Big(-\imath\rho e^{2\imath\varphi}(2+\imath e^{2\imath\varphi})\Big)^{\frac{d-1}{2}} e^{-\rho e^{2\imath\varphi}} \rho e^{2\imath\varphi}\,\mathrm{d}\varphi.$$

The first integral can be trivially bounded uniformly over ρ while the second one is tends to 0 as ρ tends to infinity since the exponential term dominates all the others. □

III.8 Discrete results

III.8.1 The SLFV in a random environment

In this section we provide a rigorous construction of the spatial Λ-Fleming-Viot process (SLFV) in a random environment. We work under the following assumptions.

Assumption III.8.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. Fix $n \in \mathbf{N}$ and $\mathfrak{u} \in (0,1), d = 1,2$ and let $w^0 \colon \mathbf{T}^d \to [0,1]$ and $s_n \colon \Omega \times \mathbf{T}^d \to (-1,1)$ be two measurable functions.*

The natural state space of the spatial SLFV process is:

$$M = \{w \colon \mathbf{T}^d \to [0,1], \ \ w \text{ measurable}\},$$

which is a metric space when endowed with the distance $d_M(u,w) = \sup_{x \in \mathbf{T}^d} |u(x) - w(x)|$. Then under the assumption above, for $x \in \mathbf{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$ and any function $w : \mathbf{T}^d \to [0,1]$ define the operator $\Theta^{\mathfrak{p}}_x \colon M \to M$ by

$$\begin{aligned} \Theta^{\mathfrak{p}}_x w(y) &= w(y) 1_{\{B^c_n(x)\}}(y) + (\mathfrak{u} 1_{\{\mathfrak{p}=\mathfrak{a}\}} + (1-\mathfrak{u}) w(y)) 1_{\{B_n(x)\}}(y) \\ &= w(y) + \mathfrak{u}(1_{\{\mathfrak{p}=\mathfrak{a}\}} - w(y)) 1_{\{B_n(x)\}}(y). \end{aligned}$$

In the discussion below, let $\mathscr{B}(E)$ be the Borel sigma-algebra associated to some metric space E. We say that a probability measure $\mathbb{P}^\omega$ on $(E, \mathscr{B}(E))$ indexed by $\omega \in \Omega$ is a Markov kernel, if for any $A \in \mathscr{B}(E)$ the map $\omega \mapsto \mathbb{P}^\omega(A)$ is measurable. Then one can build the semidirect product measure $\mathbb{P} \ltimes \mathbb{P}^\omega$ on $\Omega \times E$ (with the product sigma-algebra), characterized, for $A \in \mathscr{F}, B \in \mathscr{B}(E)$, by:

$$\mathbb{P} \ltimes \mathbb{P}^\omega (A \times B) = \int_A \mathbb{P}^\omega(B) \mathbb{P}(\mathrm{d}\omega).$$

In the definition below we write:

$$s_+(x) = \max\{s(x), 0\}, \qquad s_-(x) = \max\{-s(x), 0\}.$$

Lemma III.8.2. *Under Assumption III.8.1, fix $\omega \in \Omega$. There exists a unique Markov jump process $t \mapsto w(t)$ in $\mathbb{D}([0,\infty); M)$ started in $w(0) = w^0$, associated to the generator*

$$\mathscr{L}(n, s_n(\omega), \mathfrak{u}) \colon C_b(M; \mathbf{R}) \to C_b(M; \mathbf{R}),$$

defined by

$$\mathscr{L}(f)(w) = \int_M (f(w') - f(w)) \mu(w, \mathrm{d}w'), \qquad f \in C_b(M; \mathbf{R}),$$

where the transition function $\mu \colon M \times \mathscr{B}(M) \to \mathbf{R}$ (depending on $s_n(\omega), \mathfrak{u}, n$) is defined by:

$$\mu(w, \mathrm{d}w') = 0 \ \ \text{unless there exist} \ \ x \in \mathbf{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\} \ \ \text{such that} \ \ w' = \Theta^{\mathfrak{p}}_x w.$$

And if $w' = \Theta^{\mathfrak{p}}_x w$ for some $x \in \mathbf{T}^d, \mathfrak{p} \in \{\mathfrak{a}, \mathfrak{A}\}$:

$$\begin{aligned} \mu(w, \mathrm{d}w') = \Big\{ & (1 - |s_n(\omega, x)|) \Big[\Pi^3_n w 1_{\{\mathfrak{p}=\mathfrak{a}\}} + (1 - \Pi^3_n w) 1_{\{\mathfrak{p}=\mathfrak{A}\}} \Big](x) \\ & + (s_n)_-(\omega, x) \Big[\big(\Pi^3_n w\big)^2 1_{\{\mathfrak{p}=\mathfrak{a}\}} + \big(1 - \big(\Pi^3_n w\big)^2\big) 1_{\{\mathfrak{p}=\mathfrak{A}\}} \Big](x) \\ & + (s_n)_+(\omega, x) \Big[\Pi^3_n w (2 - \Pi^3_n w) 1_{\{\mathfrak{p}=\mathfrak{a}\}} + (1 - \Pi^3_n w)^2 1_{\{\mathfrak{p}=\mathfrak{A}\}} \Big](x) \Big\} \mathrm{d}x. \end{aligned}$$

The law $\mathbb{P}^\omega$ of w in $\mathbb{D}([0,\infty);M)$ is a Markov kernel and induces the semidirect product measure $\mathbb{P}\ltimes\mathbb{P}^\omega$ on $\Omega\times\mathbb{D}([0,\infty);M)$.

Proof. Note that μ defined as above is a Markov kernel on $M\times\mathscr{B}(M)$ (to be precise, here we have to observe that for fixed w the set $\{\Theta^{\mathfrak{p}}_x w,\ \ x\in\mathbf{T}^d,\mathfrak{p}\in\{\mathfrak{a},\mathfrak{A}\}\}$ is closed and hence measurable in M). Hence, the Markov process is constructed following [EK86, Section 4.2]. In addition, for $f\in C_b(M;\mathbf{R})$ measurable and bounded the map $\omega\mapsto\int_M f(w')\mu_\omega(w,\mathrm{d}w')$ is measurable (we made explicit the dependence of μ on ω). This implies, e.g. by [EK86, Equation 4.2.8], that the map $\omega\mapsto\mathbb{P}^\omega(A)$ is measurable, for $A\in\mathscr{B}(\mathbb{D}([0,\infty);M))$. So the proof is complete. □

Lemma III.8.3. *Under Assumption III.8.1 fix $\omega\in\Omega$ and let w be the Markov process as in the previous result. For any $\varphi\in L^\infty(\mathbf{T}^d)$ the process $t\mapsto\langle w(t),\varphi\rangle$ satisfies the martingale problem of Lemma III.2.5.*

Proof. In the discussion below we omit the dependence of $s_n(\omega)$ on n and ω, since such dependence is not relevant here. We will apply the generator to functions of the form $F_\varphi(w)=F(\langle w,\varphi\rangle)$, with $F\in C(\mathbf{R};\mathbf{R}),\varphi\in L^\infty(\mathbf{T}^d)$. For simplicity we divide the operator $\mathscr{L}=\mathscr{L}(n,s,\mathfrak{u})$ in three parts:

$$\begin{aligned}\mathscr{L}(F_\varphi)(w)&:=\mathscr{L}^{\mathrm{neu}}(F_\varphi)(w)+\mathscr{L}^{\mathrm{sel}}(F_\varphi)(w)\\&:=\mathscr{L}^{\mathrm{neu}}(F_\varphi)(w)+\mathscr{L}^{\mathrm{sel}}_<(F_\varphi)(w)+\mathscr{L}^{\mathrm{sel}}_>(F_\varphi)(w)\end{aligned}$$

(the first is the neutral part, the second two are the selective parts of the operator), where

$$\mathscr{L}^{\mathrm{neu}}(F_\varphi)(w)=\int_{\mathbf{T}^d}(1-|s(x)|)\Big[\Pi^3_n w[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+(1-\Pi^3_n w)[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathscr{L}^{\mathrm{sel}}_<(F_\varphi)(w)=\int_{\mathbf{T}^d}s_-(x)\Big[\big(\Pi^3_n w\big)^2[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+\Big(1-\big(\Pi^3_n w\big)^2\Big)[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

$$\mathscr{L}^{\mathrm{sel}}_>(F_\varphi)(w)=\int_{\mathbf{T}^d}s_+(x)\Big[\Pi^3_n w(2-\Pi^3_n w)[F_\varphi(\Theta^{\mathfrak{a}}_x w)-F_\varphi(w)]+(1-\Pi^3_n w)^2[F_\varphi(\Theta^{\mathfrak{A}}_x w)-F_\varphi(w)]\Big](x)\,\mathrm{d}x$$

Now, in the special case of $F=\mathrm{Id}_\varphi$, the neutral part of the generator takes the form

$$\mathscr{L}^{\mathrm{neu}}(\mathrm{Id}_\varphi)(w)=\mathfrak{u}n^{-d}\int_{\mathbf{T}^d}(1-|s(x)|)[(\Pi^3_n w)(\Pi_n\varphi)-\Pi_n(w\varphi)](x)\,\mathrm{d}x,$$

Analogously, the selective part can be written as

$$\mathscr{L}^{\mathrm{sel}}(\mathrm{Id}_\varphi)(w)=\mathfrak{u}n^{-d}\int_{\mathbf{T}^d}s(x)[\Pi_n(w\varphi)-\big(\Pi^3_n w\big)^2\Pi_n\varphi](x)+2s_+(x)[\Pi^3_n w\Pi_n\varphi-\Pi_n(w\varphi)](x)\,\mathrm{d}x.$$

Adding those two we conclude that

$$\mathscr{L}(\mathrm{Id}_\varphi)(w)=\mathfrak{u}n^{-d}\int_{\mathbf{T}^d}[(\Pi^3_n w)(\Pi_n\varphi)-\Pi_n(w\varphi)](x)+s(x)[(\Pi^3_n w)(\Pi_n\varphi)-(\Pi^3_n w)^2\Pi_n\varphi](x)\,\mathrm{d}x.$$

This justifies the drift in the required decomposition. To obtain the predictable quadratic variation of the martingale make use of Dynkin's formula, that is

$$\langle M^n(\varphi)\rangle_t = \int_0^t \mathscr{L}(\mathrm{Id}_\varphi^2) - 2\Big(\mathrm{Id}_\varphi \mathscr{L}(\mathrm{Id}_\varphi)\Big)(w_r)\,\mathrm{d}r.$$

Once again, it is natural to treat the terms involving $\mathscr{L}^{\mathrm{neu}}$ and $\mathscr{L}^{\mathrm{sel}}$ separately. For the neutral term:

$$\begin{aligned}&\Big(\mathscr{L}^{\mathrm{neu}}(\mathrm{Id}_\varphi^2) - 2F_\varphi \mathscr{L}^{\mathrm{neu}}(\mathrm{Id}_\varphi)\Big)(w)\\ &\quad = \mathfrak{u}^2 n^{-2d}\int_{\mathbf{T}^d}(1-|s(x)|)\Big[\Pi_n^3 w\Big(\Pi_n\varphi - \Pi_n(w\varphi)\Big)^2 + \Big(1-\Pi_n^3 w\Big)\Big(\Pi_n(w\varphi)\Big)^2\Big](x)\,\mathrm{d}x,\end{aligned}$$

which can be written as

$$\mathfrak{u}^2 n^{-2d}\int_{\mathbf{T}^d}(1-|s(x)|)\Big[\Pi_n^3 w\Big[\Big(\Pi_n\varphi\Big)^2 - 2\Pi_n\varphi(x)\Pi_n(w\varphi)\Big] + \Big[\Pi_n(w\varphi)\Big]^2\Big](x)\,\mathrm{d}x.$$

Analogous calculations for $\mathscr{L}^{\mathrm{sel}}_<$ lead to

$$\begin{aligned}&\Big(\mathscr{L}^{\mathrm{sel}}_<(\mathrm{Id}_\varphi^2) - 2\mathrm{Id}_\varphi \mathscr{L}^{\mathrm{sel}}_<\mathrm{Id}_\varphi\Big)(w) =\\ &\quad = \mathfrak{u}^2 n^{-2d}\int_{\mathbf{T}^d} s_-(x)\Big[(\Pi_n^3 w)^2\Big[(\Pi_n\varphi)^2 - 2\Pi_n\varphi\Pi_n(w\varphi)\Big] + \Big[\Pi_n(w\varphi)\Big]^2\Big](x)\,\mathrm{d}x.\end{aligned}$$

Whereas for $\mathscr{L}^{\mathrm{sel}}_>$ they lead to

$$\begin{aligned}&\Big(\mathscr{L}^{\mathrm{sel}}_>(\mathrm{Id}_\varphi^2) - 2\mathrm{Id}_\varphi \mathscr{L}^{\mathrm{sel}}_>\mathrm{Id}_\varphi\Big)(w)\\ &\quad = \mathfrak{u}^2 n^{-2d}\int_{\mathbf{T}^d} s_+(x)\Big[(\Pi_n^3 w)(2-\Pi_n^3 w)\Big(\Pi_n\varphi - \Big(\Pi_n(w\varphi)\Big)^2 + \Big(1-\Pi_n^3 w\Big)^2\Big(\Pi_n w\Big)^2\Big](x)\,\mathrm{d}x\\ &\quad = \mathfrak{u}^2 n^{-2d}\int_{\mathbf{T}^d} s_+(x)\Big[(\Pi_n^3 w)(2-\Pi_n^3 w)\Big[\Big(\Pi_n\varphi\Big)^2 - 2\Pi_n\varphi\Pi_n(w\varphi)\Big] + \Big[\Pi_n(w\varphi)\Big]^2(x)\Big]\,\mathrm{d}x.\end{aligned}$$

Summing neutral and selective terms one obtains

$$\begin{aligned}&\mathfrak{u}^2 n^{-2d}\langle \Pi_n^3 w, (1-|s|)\Big[\Big(\Pi_n\varphi\Big)^2 - 2\Pi_n\varphi\Pi_n(w\varphi)\Big]\rangle + \langle\Big(\Pi_n(w\varphi)\Big)^2, (1-|s|)\rangle\\ &\quad + \mathfrak{u} n^{-2d}\langle(\Pi_n^3 w)^2, s_-\Big[(\Pi_n\varphi)^2 - 2(\Pi_n\varphi)\Big(\Pi_n(w\varphi)\Big)\Big] + \langle\Big(\Pi_n(w\varphi)\Big)^2, s_-\rangle\\ &\quad + \mathfrak{u}^2 n^{-2d}\langle \Pi_n^3 w, s_+\Big[(2-\Pi_n^3 w)\Big((\Pi_n\varphi)^2 - 2(\Pi_n\varphi)\Big(\Pi_n(w\varphi)\Big)\Big)\Big]\rangle + \langle\Big(\Pi_n(w\varphi)\Big)^2, s_+\rangle,\end{aligned}$$

which can be written in the form from the statement of the Lemma.

□

IV

Discretizations of the Anderson model

IV.1 Introduction

In this chapter we study discretizations of the Anderson Hamiltonian (and the associated semigroup)

$$\mathscr{H} = \Delta + \xi$$

with ξ space white noise on a box in dimension $d = 1, 2$.

In Section IV.2 we consider semidiscrete approximations with periodic boundary conditions. The main result is the proof of Theorem III.3.15 stated in the previous chapter. The proof relies on some stochastic estimates, which we provide in Section IV.2.5 and some commutator estimates we prove in Section IV.2.6.

In Section IV.3 we consider instead lattice approximations of the Anderson model with Dirichlet boundary conditions. We recall the approach of [Cv19] for paracontrolled distributions with Dirichlet boundary conditions and show some required stochastic estimates in Section IV.3.3.

The Anderson Hamiltonian was introduced in $d = 1$ by [FN77], in $d = 2$ by [AC15] and $d = 3$ by [Lab19]. In the last two cases the construction relies on theories in singular stochastic PDEs [Hai14, GIP15]. In higher dimensions the $d = 4$ or higher these solution theories do not work, because the noise becomes too rough (a problem known as *supercriticality* [Hai14]). To see why the problem becomes more complicated as the dimension increases, consider the resolvent equation for $\lambda > 0$:

$$(\Delta + \xi - \lambda)\psi = \varphi.$$

If we can solve this equation, for example for any $\varphi \in L^2$ and some $\lambda > 0$ large enough, we would have:

$$\psi = (-\Delta + \lambda)^{-1}(\xi\psi - \varphi).$$

Now, assume that such ψ has a regularity $\psi \in \mathscr{C}_2^{\alpha}$ for some $\alpha > 0$ (if $\xi = 0$ Schauder estimates would guarantee this regularity for any $\alpha < 2$). Since space white noise has regularity $\xi \in \mathscr{C}^{-\frac{d}{2}-\kappa}$ for any $\kappa > 0$ (see for example the discussion in the upcoming section), the product $\xi\psi$ lies in $\mathscr{C}_2^{-\frac{d}{2}-\kappa}$ and hence we cannot expect any better than $\alpha <$

$2-\frac{d}{2}$. Yet to define the product $\xi\psi$ (see the paraproduct estimates Lemma I.1.3) we would need $\alpha-\frac{d}{2}-\kappa > 0$, which can be the case only in $d = 1$. In this sense, in $d = 2, 3$ the equation is *singular*. The solution theory involves a Taylor expansion in terms of functionals of the noise, which is encoded in the theories of regularity structures or paracontrolled distributions.

In the next section we will construct semidiscrete approximations of the Anderson Hamiltonian. With "semidiscrete" we refer to the setting of Chapter III.

In the second half of this chapter we will consider a lattice approximations with Dirichlet boundary conditions of the Parabolic Anderson Model (PAM) – that is, we construct the semigroup $e^{t\mathscr{H}}$ – in the framework of [Cv19].

IV.2 Semidiscrete Anderson Hamiltonian

This section is devoted to the proof of Theorem III.3.15. This theorem is an approximation result for the continuous Anderson Hamiltonian in dimension $d = 1$ and $d = 2$. Before we proceed, let us collect some basic ideas of the proof that will follow.

The proof of the theorem concentrates on the two-dimensional case, since here the resolvent equation is a singular stochastic PDE. In the construction of the Hamiltonian in $d = 2$ we follow the results in [AC15] that rely on paracontrolled calculus (we refer the reader to [GIP15] and [GP17] for a more in-depth discussion).

IV.2.1 Density of the domain

We start with some results regarding the continuous Anderson Hamiltonian, which imply Proposition III.3.14.

Lemma IV.2.1. *Consider a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ supporting a space white noise $\xi : \Omega \to \mathscr{S}'(\mathbf{T}^d)$. Fix any $\kappa > 0$. The following hold true for almost all $\omega \in \Omega$. The Anderson Hamiltonian*

$$\mathscr{H}^\omega = \nu_0 \Delta + \xi(\omega)$$

associated to $\xi(\omega)$ is defined, as constructed in [FN77] in $d = 1$ and [AC15] in $d = 2$. The Hamiltonian, as an unbounded selfadjoint operator on $L^2(\mathbf{T}^d)$, has a discrete spectrum given by pairs of eigenvalues and eigenfunctions $\{(\lambda_k(\omega), e_k(\omega))\}_{k\in\mathbf{N}}$ such that:

$$\lambda_1(\omega) > \lambda_2(\omega) \geq \lambda_3(\omega) \geq \dots, \qquad \lim_{k\to\infty} \lambda_k(\omega) = -\infty, \qquad e_1(\omega, x) > 0, \forall x \in \mathbf{T}^d.$$

Proof. The Hamiltonian $\mathscr{H}^\omega$ has been constructed in dimension $d = 1$ in [FN77] (albeit with Dirichlet boundary conditions, but the construction for periodic boundary conditions is identical) and in dimension $d = 2$ in [AC15], for almost all $\omega \in \Omega$. In both cases $\mathscr{H}^\omega$ is an unbounded, selfadjoint operator on L^2, that is:

$$\mathscr{H}^\omega : \mathscr{D}(\mathscr{H}^\omega) \subset L^2 \to L^2.$$

In particular, in $d = 2$ [AC15, Proposition 4.13] implies that the operator $\mathscr{H}^\omega$ admits compact resolvents (cf. [FN77, Section 2] for the analogous discussion in $d = 1$). This means that for some $\overline{\lambda}(\omega) > 0$ for all $\lambda \geq \overline{\lambda}(\omega)$ the operator $\mathscr{H}^\omega - \lambda$ is invertible, and $(\mathscr{H}^\omega - \lambda)^{-1}$ is a compact operator on L^2. Hence the spectrum of $\mathscr{H}^\omega$ is discrete and the eigenvalues converge to $-\infty$. By a classical result, see [Paz83, Theorem 3.3], the semigroup generated by $\mathscr{H}^\omega$, denoted by $e^{t\mathscr{H}^\omega}$, is compact. Moreover, as a consequence of strong maximum principle (in $d = 2$ such a result for singular stochastic PDEs is proven in [CFG17, Theorem 5.1 and Remark 5.2]), the semigroup $e^{t\mathscr{H}^\omega}$ is strictly positive: that is, for any non-zero continuous function f that is positive (i.e. $f(x) \geq 0$, $\forall x \in \mathbf{T}^d$), it holds that $e^{t\mathscr{H}^\omega} f(x) > 0$, $\forall x \in \mathbf{T}^d$. Therefore since $e^{t\mathscr{H}^\omega}$ is a compact, strictly positive operator, the Krein-Rutman Theorem ([Dei85, Theorem 19.3]) implies that the largest eigenvalue of $\mathscr{H}^\omega$ has multiplicity one and the associated eigenfunction is strictly positive.

□

Lemma IV.2.2. *Fix $\omega \in \Omega$ and consider the Anderson Hamiltonian $\mathscr{H}^\omega$ as in the previous lemma. Define the domain:*

$$\mathscr{D}_\omega = \{\textit{Finite linear combinations of } \{e_k(\omega)\}_{k \in \mathbf{N}}\}.$$

The domain $\mathscr{D}_\omega$ is dense in $C(\mathbf{T}^d)$. Moreover, for arbitrary $\zeta \in (0,1)$ and all $\varphi \in C^\infty$, there exists a sequence $\varphi^k \in \mathscr{D}_\omega$ with $\lim_{k\to\infty} \varphi^k = \varphi$ in $\mathscr{C}^\zeta$.

Proof. Since $\omega \in \Omega$ is fixed, we avoid writing the dependence on it to lighten the notation. As the statement regarding the approximation of φ in $\mathscr{C}^\zeta$ implies density in $C(\mathbf{T}^d)$ we restrict to proving the approximation. First, we require some better understanding of the parabolic Anderson semigroup. Here we make use of some known regularization results.

Step 1. Consider the operator $\mathscr{H}$ as in the previous lemma and the associated semigroup:

$$e^{t\mathscr{H}} : L^2(\mathbf{T}^d) \to L^2(\mathbf{T}^d).$$

This semigroup inherits some of the regularizing properties of the heat semigroup, namely, for $T > 0$ and $p \in [1, \infty]$ it can be extended so that:

$$\sup_{0<t\leq T} t^\gamma \|e^{t\mathscr{H}} \varphi\|_{\mathscr{C}^\alpha_p} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p}, \tag{IV.1}$$

for α, β and γ satisfying:

$$\gamma > \frac{\alpha - \beta}{2}, \qquad \beta + 2 > \frac{d}{2}, \qquad \alpha < 2 - \frac{d}{2}, \qquad \alpha > \beta.$$

The first constraint is essentially identical to the one appearing in Schauder estimates (cf. Proposition III.6.7), the second one guarantees that the product $e^{t\Delta}\varphi \cdot \xi$ is a well-defined product of distributions, while the third constraint is due to the fact that $\int_0^t e^{(t-s)\Delta}\xi \, \mathrm{d}s$ has always worse regularity than $2 - \frac{d}{2}$. Similarly, for $\beta > 2 - \frac{d}{2}$ and $\zeta < 2 - \frac{d}{2}$ one has:

$$\sup_{0\leq t\leq T} \|e^{t\mathscr{H}} \varphi\|_{\mathscr{C}^\zeta_p} \lesssim \|\varphi\|_{\mathscr{C}^\beta_p}. \tag{IV.2}$$

We will not prove these results. Instead we refer to [GP17, Section 6] for the study of singular SPDEs with irregular initial conditions (see also Propositions II.3.1 and Theorem IV.3.4 for similar statements).

Step 2. Applying iteratively Equation (IV.1) and Besov embedding implies that $e_k \in \mathscr{C}^{2-\frac{d}{2}-\kappa}$ for any $\kappa > 0$. Hence the embedding $\mathscr{D}_\omega \subseteq \mathscr{C}^{2-\frac{d}{2}-\kappa}$ holds. Now we prove the statement regarding the approximability of φ. For any $\varphi \in C^\infty$ and $\zeta = 1-\kappa < 1$ (for some $\kappa > 0$) one has:

$$\lim_{t \to 0^+} \frac{1}{t} \int_0^t e^{s\mathscr{H}} \varphi \, \mathrm{d}s = \varphi \quad \text{in} \quad \mathscr{C}^\zeta.$$

This result can be seen as follows: Equation (IV.2) implies that

$$\sup_{0 \le t \le T} \left\| \frac{1}{t} \int_0^t e^{s\mathscr{H}} \varphi \, \mathrm{d}s \right\|_{\mathscr{C}^{\zeta'}} < \infty,$$

for $\zeta < \zeta' < 2 - \frac{d}{2}$. The estimate above implies compactness in $\mathscr{C}^\zeta$. Projecting on the eigenfunctions e_k one sees that any limit point is necessarily φ. Hence fix any $\varepsilon > 0$ and choose $t(\varepsilon)$ such that

$$\left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathscr{H}} \varphi \, \mathrm{d}s - \varphi \right\|_{\mathscr{C}^\zeta} < \frac{\varepsilon}{2}.$$

Define $\Pi_{\le N} \varphi = \sum_{k=0}^N \langle \varphi, e_k \rangle e_k$. Since the projection commutes with the operator, the proof is complete if we can show that there exists an $N(\varepsilon)$ such that:

$$\left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathscr{H}} (\Pi_{\le N(\varepsilon)} \varphi - \varphi) \, \mathrm{d}s \right\|_{\mathscr{C}^\zeta} \le \frac{\varepsilon}{2}.$$

Here we use (IV.1) to bound for general $\psi \in L^2$:

$$\begin{aligned}
\left\| \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} e^{s\mathscr{H}} \psi \, \mathrm{d}s \right\|_{\mathscr{C}^\zeta} &\lesssim \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} \left(\frac{s}{2}\right)^{-\left(\frac{1}{2}-\frac{\kappa}{4}\right)} \| e^{\frac{s}{2}\mathscr{H}} \psi \|_{\mathscr{C}^{-\frac{\kappa}{2}}} \, \mathrm{d}s \\
&\lesssim \frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} \left(\frac{s}{2}\right)^{-\left(\frac{1}{2}-\frac{\kappa}{4}\right)} \| e^{\frac{s}{2}\mathscr{H}} \psi \|_{\mathscr{C}_2^{\frac{d}{2}-\frac{\kappa}{2}}} \, \mathrm{d}s \\
&\lesssim \left(\frac{1}{t(\varepsilon)} \int_0^{t(\varepsilon)} s^{-1+\frac{\kappa}{4}+\frac{\kappa}{8}} \, \mathrm{d}s \right) \|\psi\|_{L^2} \lesssim t(\varepsilon)^{-1+\frac{3\kappa}{8}} \|\psi\|_{L^2},
\end{aligned}$$

where we additionally applied Besov embedding. Choosing $N(\varepsilon)$ such that $\|\Pi_{\le N}\varphi - \varphi\|_{L^2} \lesssim t(\varepsilon)^{1-\frac{3\kappa}{8}} \frac{\varepsilon}{2}$, the proof is complete. □

IV.2.2 Convergence of eigenfunctions

Before we move on to study semidiscrete approximations of the Anderson Hamiltonian, we recall and adapt a result by Kato concerning the convergence of eigenvalues and (in a generalized sense) the convergence of eigenfunctions of a sequence of closed linear operators. In this subsection we will restrict to closed linear operators on a Hilbert space H (with norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$). We will denote with

$$\sigma(A), \rho(A) \subseteq \mathbf{C}$$

the spectrum and the resolvent sets of a closed linear operator A on H respectively. If A is bounded, we denote with $\|A\| = \sup_{\|x\|=1} \|Ax\|$ its operator norm. We write $B(H)$ for the

space of bounded operators, endowed with operator norm.
Now, consider a bounded set $\Omega \subseteq \mathbf{C}$ such that the boundary $\Gamma = \partial\Omega$ is a smooth curve satisfying $\Gamma \subseteq \varrho(A)$. We write $R(A,\zeta) = (A-\zeta)^{-1}$ for the resolvent of A at $\zeta \in \varrho(A)$. Then we introduce the Riesz projection

$$P(\Omega, A) = -\frac{1}{2\pi\imath}\int_{\Gamma} R(A,\zeta)\,\mathrm{d}\zeta,$$

which for all our purposes coincides with the projection on certain eigenspaces, as described in the following lemma.

Lemma IV.2.3. *Let A be a selfadjoint operator on H. Suppose that Ω (with boundary Γ as above) contains only isolated points of the spectrum: $\Omega \bigcap \sigma(A) = \{\lambda_i\}_{i=1}^m$. Then $P(\Omega, A)$ coincides with the orthogonal projection on the space:*

$$\bigcup_{i=1}^{m} \mathrm{Ker}(A-\lambda_i).$$

This result is proven for example in [HS96, Proposition 6.3]. We provide the main steps of the proof for clarity, highlighting the salient points but omitting technical steps, such as motivating that a given function is holomorphic.

Idea of proof. Write P instead of $P(\Omega, A)$ and assume first that $m = 1$. To see that P is an orthogonal projection we have to show that $P^2 = P$ and that P is self-adjoint. For the first point observe that

$$\begin{aligned} P^2 =& \left(-\frac{1}{2\pi\imath}\right)^2 \int_{\Gamma}\int_{\Gamma} R(\zeta')R(\zeta) d\zeta d\zeta' \\ =& \left(-\frac{1}{2\pi\imath}\right)^2 \int_{\Gamma_\varepsilon}\int_{\Gamma} \frac{\{R(\zeta')-R(\zeta)\}}{(\zeta'-\zeta)} d\zeta d\zeta', \end{aligned}$$

where we assume that $\Gamma_\varepsilon = \partial\Omega_\varepsilon \subseteq \varrho(A)$, with Ω_ε open, satisfies $\Gamma, \Omega \subseteq \Omega_\varepsilon$ and $\Omega_\varepsilon \setminus \Omega \subseteq \varrho(A)$, so that the resolvent is holomorphic in $\Omega_\varepsilon \setminus \Omega$ (in this way changing the curve from Γ to Γ_ε does not affect the integral). Then by Cauchy's integral formula:

$$\begin{aligned} P^2 =& \left(-\frac{1}{2\pi\imath}\right)^2 \int_{\Gamma_\varepsilon}\int_{\Gamma} \frac{-R(\zeta)}{(\zeta'-\zeta)} d\zeta d\zeta' \\ =& -\frac{1}{2\pi\imath}\int_{\Gamma} R(\zeta) d\zeta \\ =& P, \end{aligned}$$

so that indeed P is a projection (but not yet an orthogonal one). Since $m = 1$ we can also assume that Γ is a circle of radius δ around $\lambda_1 \in \mathbf{R}$ (the eigenvalue is real, since the operator is self-adjoint). In this way we see that the adjoint of P satisfies:

$$\begin{aligned} P^* &= -\frac{1}{2\pi\imath}\int_{-\pi}^{\pi} R(A, \lambda + \delta e^{-i\theta})\delta\,\mathrm{d}\theta \\ &= -\frac{1}{2\pi\imath}\int_{-\pi}^{\pi} R(A, \lambda + \delta e^{i\theta})\delta\,\mathrm{d}\theta \\ &= P, \end{aligned}$$

through a change of variables. Hence the operator is self-adjoint, so that P is an orthogonal projection. Denote with $\mathrm{Rng}(P)$ the space onto which P projects. We want to show that $\mathrm{Rng}(P) = \mathrm{Ker}(A - \lambda_1)$. If $h \in \mathrm{Ker}(A - \lambda_1)$, then

$$Ph = -\frac{1}{2\pi\iota}\int_\Gamma (\lambda_1 - \zeta)^{-1} h \,\mathrm{d}\zeta = h.$$

Hence $\mathrm{Ker}(P) \subseteq \mathrm{Rng}(P)$. If instead $h \in \mathrm{Rng}(P)$, then:

$$\begin{aligned}(A-\lambda_1)h = (A-\lambda_1)Ph &= -\frac{1}{2\pi\iota}\int_\Gamma (A-\lambda_1)(A-\zeta)^{-1} h\,\mathrm{d}\zeta \\ &= -\frac{1}{2\pi\iota}\int_\Gamma (\zeta-\lambda_1)(A-\zeta)^{-1} h\,\mathrm{d}\zeta.\end{aligned}$$

Now we use that A is self-adjoint, which implies that for every $n \in \mathbf{N}$, $\|R(A,\zeta)^{2^n}\| = \|R(A,\zeta)\|^{2^n}$, so that:

$$\|R(A,\zeta)\| = \lim_{n\to\infty} \|R(A,\zeta)^{2^n}\|^{\frac{1}{2^n}} = \sup_{\lambda\in\sigma((A-\zeta)^{-1})} |\lambda| = |\lambda_1 - \zeta|^{-1}.$$

Here the first equality is a consequence of Gelfand's formula; the last equality holds by choosing Γ to be a sufficiently small circle about λ_1. In particular

$$\|(\zeta-\lambda_1)(A-\zeta)^{-1}\| \le 1$$

uniformly over $\zeta \in \Omega \setminus \{\lambda_1\}$ and the map $\zeta \mapsto (\zeta - \lambda_1)(A-\zeta)^{-1}$ can thus be extended to a holomorphic function (with values in the space of bounded operators) on the entire Ω (in particular it is defined also in λ_1). So one obtains

$$(A-\lambda_1)Ph = 0.$$

The case $m > 1$ follows analogously by considering Γ a union of small circles around each point of the spectrum. □

The last step in the proof uses very indirectly the self-adjointness of the operator A. It can therefore be useful to observe that the speed of the blowup of $R(A,\zeta)$ near λ_1 is connected to the *algebraic multiplicity* of the eigenvalue. As a prototypical example, in the case of the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

the resolvents explode at order: $\|R(A,\zeta)\| = \frac{1}{2|\zeta|^2}(1 + o(\zeta))$ as $\zeta \to 0$. Also, in this example the identity $\|A^{2^n}\| = \|A\|^{2^n}$ fails completely, because $A^2 = 0$. In conclusion, if A is not self-adjoint one has only the inclusion $\mathrm{Ker}(A - \lambda_1) \subseteq \mathrm{Ran}(P)$. The projection P then projects onto the *generalized eigenspace* associated to λ_1.

The following result states that Riesz projections are continuous with respect to convergence in the resolvent sense. This is a weaker version of a result by Kato, that holds for operators that are not necessarily selfadjoint and parts of the spectrum that are not necessarily isolated eigenvalues: [Kat95, Theorem IV.3.16].

Proposition IV.2.4. *Let A_n be a sequence of closed self-adjoint operators on H. Let A be a closed self-adjoint operator such that, for some $\zeta_0 \in \rho(A)$:*

$$\zeta_0 \in \rho(A_n), \ \forall n \in \mathbf{N}, \qquad \text{and} \qquad \lim_{n\to\infty} \|R(A^n,\zeta_0) - R(A,\zeta_0)\| = 0.$$

Let λ be an isolated eigenvalue of A and consider a smooth curve $\Gamma = \partial\Omega$ around λ, such that $\Gamma \subseteq \rho(A)$, $\Omega \bigcap \sigma(A) = \{\lambda\}$. Then

$$\lim_{n\to\infty} \|P(\Omega, A_n) - P(\Omega, A)\| = 0.$$

Idea of proof. The integral representation of the Riesz projections allows to reduce the problem to one of passage of the limit under integral sign. Namely, we would like to prove that:

$$\lim_{n\to\infty} -\frac{1}{2\pi\iota}\int_\Gamma R(A^n,\zeta)\,\mathrm{d}\zeta = -\frac{1}{2\pi\iota}\int_\Gamma R(A,\zeta)\,\mathrm{d}\zeta.$$

Then the problem is to prove the continuous dependence of R on A and ζ. [Kat95, Theorem IV.2.25] guarantees that if $\|R(A^n,\zeta_0) - R(A,\zeta_0)\| \to 0$ for one $\zeta_0 \in \rho(A)$, then the same holds for all $\zeta \in \rho(A)$ (and n sufficiently large). In addition, this implies convergence of A^n in a so-called *generalized sense* (with respect to a distance named $\hat{\delta}$). The main point of the proof is then the continuity of R jointly with respect to ζ and A, using the latter distance: see [Kat95, Theorem IV.3.15]. □

The previous result allows us to deduce the following.

Corollary IV.2.5. *In the setting of the previous proposition, let $\{e_j\}_{j=1}^{m(\lambda)}$ be orthonormal eigenfunctions associated to the eigenvalue λ of the operator A (here $m(\lambda)$ is the multiplicity of λ). There exists an $n(\lambda) \in \mathbf{N}$ such that for all $n \geq n(\lambda)$ the following statements hold.*

i $\dim(\mathrm{Rng}(P(\Omega, A_n))) = \dim(\mathrm{Rng}(P(\Omega, A))) = m(\lambda)$.

ii For every $j \in \{1,\dots m(\lambda)\}$ there exists an $e_j^n \in \mathscr{D}(A_n)$ (the domain of A_n) satisfying:

$$e_j^n \to e_j \quad \text{in} \quad H, \qquad A_n e_j^n \to \lambda e_j \quad \text{in} \quad H.$$

iii For every $j \in \{1,\dots,m(\lambda)\}$, e_j^n has a representation of the form

$$e_j^n = \sum_{i=1}^{m(\lambda)} \alpha_{ij}^n \overline{e}_i^n, \qquad \sum_{i=1}^{m(\lambda)} (\alpha_{ij}^n)^2 = 1,$$

with $\{\overline{e}_i^n\}_{i=1,\dots,m(\lambda)}$ a set of eigenfunctions of A_n. That is, for every $i \in \{1,\dots,m(\lambda)\}$:

$$A_n \overline{e}_i^n = \lambda_i^n \overline{e}_i^n, \quad \text{for some} \quad \lambda_i^n \in \mathbf{R} \text{ s.t. } \lim_{n\to\infty} \lambda_i^n = \lambda.$$

iv If λ is a simple eigenvalue, then e_1^n can be chosen to be an eigenfunction of A_n, with eigenvalue $\lambda_1^n \to \lambda$.

Proof. Consider $m_n(\lambda) = \dim(\mathrm{Rng}(P(\Omega, A_n)))$ and $\{\bar{e}_j^n\}_{j=1}^{m_n(\lambda)}$ an orthonormal basis for the subspace on which $P(\Omega, A_n)$ projects. In particular, in view of Lemma IV.2.3, we can choose $\bar{e}_j^n$ to be eigenfunctions for A_n, each associated to an eigenvalue λ_j^n. According to the same lemma, one has:

$$P(\Omega, A_n)v = \sum_{i=1}^{m_n(\lambda)} \langle v, \bar{e}_j^n \rangle \bar{e}_j^n, \qquad \forall v \in H.$$

Define for $j = 1, \dots, m(\lambda)$:

$$\tilde{e}_j^n = \sum_{i=1}^{m_n(\lambda)} \langle e_j, \bar{e}_i^n \rangle \bar{e}_i^n = P(\Omega, A_n) e_j.$$

From the convergence

$$\|P(\Omega, A_n) - P(\Omega, A)\| \to 0,$$

which is the content of the previous proposition, we obtain that for $j = 1, \dots, m(\lambda)$:

$$\lim_{n\to\infty} \tilde{e}_j^n := \lim_{n\to\infty} \sum_{i=1}^{m_n(\lambda)} \langle e_j, \bar{e}_i^n \rangle \bar{e}_i^n = e_j \quad \text{in} \quad H.$$

Hence we can assume that $n(\lambda)$ is sufficiently large, so that

$$\left| \|\tilde{e}_j^n - \sum_{i=1}^{j-1} \langle \tilde{e}_j^n, \tilde{e}_i^n \rangle \tilde{e}_i^n\| - 1 \right| \geq \frac{1}{2} > 0, \qquad \forall j = 1, \dots, m(\lambda).$$

Then we can define (via a Gram-Schmidt procedure)

$$e_j^n = \frac{\tilde{e}_j^n - \sum_{i=1}^{j-1} \langle \tilde{e}_j^n, \tilde{e}_i^n \rangle \tilde{e}_i^n}{\left\| \tilde{e}_j^n - \sum_{i=1}^{j-1} \langle \tilde{e}_j^n, \tilde{e}_i^n \rangle \tilde{e}_i^n \right\|},$$

and we obtain a set $\{e_j^n\}_{j=1}^{m(\lambda)}$ of orthonormal functions with

$$\lim_{n\to\infty} e_j^n = e_j \quad \text{in} \quad H, \qquad P(\Omega, A_n) e_j^n = e_j^n.$$

In particular, $m_n(\lambda) \geqslant m(\lambda)$. Suppose $m_n(\lambda) > m(\lambda)$ on a subsequence n_k of n that converges to ∞. Choose, along that subsequence, a unit element $e_{m(\lambda)+1}^{n_k} \in H$ with

$$P(\Omega, A_{n_k}) e_{m(\lambda)+1}^{n_k} = e_{m(\lambda)+1}^{n_k}, \qquad \langle e_{m(\lambda)+1}^{n_k}, e_j^{n_k} \rangle = 0, \qquad \forall j = 1, \dots, m(\lambda).$$

We can then assume for arbitrary δ (provided n_k is large enough), that

$$\sum_{j=1}^{m(\lambda)} \|e_j^{n_k} - e_j\| < \delta.$$

Then

$$\|P(\Omega, A_{n_k})(e^{n_k}_{m(\lambda)+1}) - P(\Omega, A)(e^{n_k}_{m(\lambda)+1})\| \geqslant 1 - \left\| \sum_{j=1}^{m(\lambda)} \langle e^{n_k}_{m(\lambda)+1}, e^{n_k}_j - e_j \rangle e_j \right\|$$
$$\geqslant 1 - \delta.$$

Since δ is arbitrarily small this contradicts the convergence of the projections.
Let us pass to the convergence of $A_n e^n_j$. Observe that:

$$A_n \tilde{e}_j = \sum_{i=1}^{m(\lambda)} \lambda^n_i \langle e_j, \overline{e}^n_i \rangle \overline{e}^n_i.$$

Hence

$$\begin{aligned}\|A_n \tilde{e}^n_j - \lambda e_j\| &\leqslant \|A_n \tilde{e}^n_j - \lambda \tilde{e}^n_j\| + \lambda \|\tilde{e}^n_j - e_j\| \\ &\leqslant \sqrt{\sum_{i=1}^{m(\lambda)} (\lambda - \lambda^n_i)^2 \langle e_j, \overline{e}^n_i \rangle^2} + \lambda \|\tilde{e}^n_j - e_j\| \\ &\leqslant \sqrt{\sum_{i=1}^{m_n} (\lambda - \lambda^n_i)^2} + \lambda \|\tilde{e}^n_j - e_j\|,\end{aligned}$$

and the last two terms converge to zero, provided that for each i:

$$\lim_{n\to\infty} \lambda^n_i = \lambda.$$

This follows from the upper semicontinuity of the spectrum proven in [Kat95, Theorem IV.3.1]. If we now use the definition of e^n_j we obtain similarly that:

$$\begin{aligned}\lim_{n\to\infty} A_n e^n_j &= \lim_{n\to\infty} \frac{A_n \tilde{e}^n_j - \sum_{i=1}^{j-1} \langle \tilde{e}^n_j, \tilde{e}^n_i \rangle A_n \tilde{e}^n_i}{\left\| \tilde{e}^n_j - \sum_{i=1}^{j-1} \langle \tilde{e}^n_j, \tilde{e}^n_i \rangle \tilde{e}^n_i \right\|} \\ &= \lambda e_j.\end{aligned}$$

To conclude the proof, note that the representation of e^n_j in terms of the basis $\{\overline{e}^n_i\}$ follows from the fact that the latter consists of orthonormal functions and that $\|e^n_j\| = 1$. Clearly, if $m(\lambda) = 1$ we can choose $e^n_1 = \overline{e}^n_1$. □

IV.2.3 Convergence in resolvent sense

This section describes the general idea behind the convergence that we will prove in the upcoming subsection. As before, we denote with Rng(A) the image $A(H)$ of a bounded operator on a Hilbert space H.

Proposition IV.2.6. *Consider a sequence of selfadjoint operators A_n on a Hilbert space H. Assume there exists a $\lambda_0 \in \mathbf{R}$ and an operator $B_{\lambda_0} \in B(H)$ such that:*

$$\lambda_0 \in \varrho(A_n)\ \forall n \in \mathbf{N}, \qquad \lim_{n\to\infty} \|R(A_n, \lambda_0) - B_{\lambda_0}\| = 0,$$

and satisfying

$$\mathrm{Ker}(B_{\lambda_0}) = \{0\}.$$

There exists a unique selfadjoint operator A on H defined by:

$$D(A) = \mathrm{Rng}(B_{\lambda_0}), \qquad \textit{and} \qquad A = B_{\lambda_0}^{-1}x + \lambda_0 x, \quad x \in D(A).$$

The domain $D(A)$ and the operator A do not depend on the choice of λ_0. Moreover, A satisfies:

$$B_{\lambda_0} = R(A, \lambda_0).$$

Proof. First, note that if $x \in D(A) = \mathrm{Rng}(B_{\lambda_0})$, then the preimage $B_{\lambda_0}^{-1}x$ is uniquely defined, since we assumed that $\mathrm{Ker}(B_{\lambda_0}) = \{0\}$. It remains to check that A is a self-adjoint operator: for this we refer, for example, to [Tay11, Proposition 8.2]. By construction we have that $B_{\lambda_0} = R(A, \lambda_0)$ and through the resolvent identity (for all $\lambda \in \varrho(A)$):

$$R(A, \lambda) = R(A, \lambda_0) + (\lambda - \lambda_0)R(A, \lambda_0)R(A, \lambda),$$

we see that the domain does not depend on the choice of λ_0. □

At this point, we can describe the structure of the proof of Theorem III.3.15 as follows:

i The crux of the argument is to show that for a fixed $\lambda \in \mathbf{R}$ the resolvents $R(\mathscr{H}_n, \lambda)$ converge:

$$\lim_{n \to \infty} R(\mathscr{H}_n, \lambda) = B_\lambda,$$

for some bounded injective B_λ.

ii The previous proposition then guarantees the existence of a selfadjoint operator $\mathscr{H}$ such that $B_\lambda = R(\mathscr{H}, \lambda)$.

iii Finally, the convergence of eigenfunctions and eigenvalues follows from Corollary IV.2.5.

Remark IV.2.7. *This argument does not require an explicit construction of the operator $\mathscr{H}$ or of its domain $D(\mathscr{H})$. It will appear clearly from the proof that the limiting resolvent $R(\mathscr{H}, \lambda)$ coincides with the resolvent constructed in [AC15] (although the article treats only the case $d = 2$, a similar but simpler construction works also in $d = 1$). In particular, the latter article explicitly describes the range of the resolvent (i.e. the domain of the operator $\mathscr{H}$), as a space of* strongly paracontrolled *distributions and it provides an explicit representation of $\mathscr{H}$ on this domain.*

IV.2.4 Proof of Theorem III.3.15

The paracontrolled approach in [AC15] to construct the Anderson Hamiltonian in $d = 2$ follows the Ansatz that the solution ψ to the resolvent equation we introduced above is of the form $\psi = \psi' \otimes X + \psi^\sharp$, the previous being a paraproduct as defined in Lemma I.1.3, with X_λ solving $(-\nu_0 \Delta + 1)X = \xi$, and $\psi^\sharp \in \mathscr{C}^{1+2\kappa}$ (we will call a ψ of this form *paracontrolled*).

This should be interpreted as a "Taylor expansion" in terms of functionals of the noise, and the reason why the rest term is expected to be of better regularity is encoded in the concept of subcriticality, introduced in [Hai14]. Now, for paracontrolled ψ the previously ill-defined product can be rewritten as $\psi\xi = (\psi' \circledcirc X)\xi + \psi^\sharp\xi$. While the last term is now well-defined (recall that if $d = 2$, $\xi \in \mathscr{C}^{-1-\kappa}$), a commutator estimate (see Lemma IV.2.15) guarantees that the resonant product can be approximated as $(\psi' \circledcirc X) \odot \xi \simeq \psi'(X \odot \xi)$. The latter resonant product $X \odot \xi$ remains still ill-defined in terms of regularity, but one can make sense of it through some Gaussian computations (since X_λ and ξ are both Gaussian fields), up to renormalisation. By this we mean that the product lives in two levels of the Wiener chaos. While the second chaos part turns out to be well-defined, the zeroth chaos is diverging. Eventually, one can rigorously define a distribution $X \diamond \xi$ that formally can be written as $X \odot \xi - \infty = X \odot \xi - \mathbb{E}\big[X \odot \xi\big]$, which lives in the second Wiener chaos and explains the ∞ appearing in the equation. This explains why in $d = 2$ the Hamiltonian is sometimes written as:

$$\nu_0 \Delta + \xi - \infty,$$

where the latter "∞" comes from the renormalisation.

In the cartoon we have just sketched, we hope to explain that theories for singular stochastic PDEs have two critical ingredients. First, some stochastic computations guarantee the existence of certain products of random distributions. Second, given a realization of these distributions, an purely analytic argument, based on regularity estimates and a Taylor-like expansion guarantees the existence of a solution to the PDE.

In the present setting we concentrate on semidiscrete approximations of the Anderson Hamiltonian, that is we will prove that ψ as a above is the limit $\psi = \lim_{n\to\infty} \psi_n$, with $(-\mathscr{A}_n + \lambda)\psi_n = \Pi_n^2(\xi^n - c_n 1_{\{d=2\}})\Pi_n^2\psi_n - \varphi$, with $\mathscr{A}_n$ as in Chapter III. Following the previous explanation we will first state some stochastic estimates and then pass to the main analytic result. The next definition introduces the space in which we will control the stochastic terms.

Definition IV.2.8. *Consider $d = 2$ and fix any $\kappa \in (0, \frac{1}{2})$. For any $n \in \mathbf{N}$ we will call an enhanced noise a vector of distributions*

$$\boldsymbol{\xi}_n = (\xi^n, Y_n) \in \mathscr{S}'(\mathbf{T}^2) \times C([1,\infty); \mathscr{S}'(\mathbf{T}^2)),$$

where Y_n is a map

$$[1,\infty) \ni \lambda \mapsto Y_{n,\lambda} \in \mathscr{S}'(\mathbf{T}^2).$$

For $\boldsymbol{\xi}_n$ we introduce the following norm, with $X_{n,\lambda} = (-\mathscr{A}_n + \lambda)^{-1}\xi^n$:

$$\begin{aligned} \|\!|\!|\boldsymbol{\xi}_n|\!|\!|_{n,\kappa} := &\sup_{\zeta\in[0,1]} \Big\{ n^{-\zeta} \|\xi^n\|_{\mathscr{C}^{-(1-\zeta)-\frac{\kappa}{2}}} \Big\} + n^{-1}\|\xi^n\|_{L^\infty} + n^{-1-\kappa}\|\xi^n\|_{\mathscr{C}^{\kappa}_{\frac{1}{2\kappa}}} \\ &+ \sup_{\lambda\geqslant 1} \Big\{ n\|\mathscr{Q}_n X_{n,\lambda}\|_{L^\infty} + \lambda^{-\frac{\kappa}{4}}\|Y_{n,\lambda}\|_{\mathscr{C}^{-\frac{\kappa}{2}}} \Big\}. \end{aligned}$$

We observe that we can immediately bound some further quantities related to $\boldsymbol{\xi}_n$.

Lemma IV.2.9. *For $n \in \mathbf{N}$ and $\lambda \geqslant 1$ consider an enhanced noise $\boldsymbol{\xi}_n$ as in Definition IV.2.8. Then we can bound, for any $\kappa \in (0, \frac{1}{2}), \delta \in [0,1]$ and uniformly over n, λ:*

$$\sup_{\zeta \in [0,1]} \lambda^{\delta} n^{-\zeta} \Big\{ \|\mathscr{P}_n X_{n,\lambda}\|_{\mathscr{C}^{-(1-\zeta)+2(1-\delta)-\frac{\kappa}{2}}} + n^{2(1-\delta)} \|\mathscr{Q}_n X_{n,\lambda}\|_{\mathscr{C}^{-(1-\zeta)-\frac{\kappa}{2}}} \Big\} \lesssim |||\boldsymbol{\xi}_n|||_{n,\kappa}.$$

Proof. This is a consequence of the elliptic Schauder estimates of Proposition III.6.6. □

The following stochastic estimates hold true and give meaning to the definition of the norm $|||\boldsymbol{\xi}_n|||_{n,\kappa}$.

Proposition IV.2.10. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space supporting a sequence of random functions $\xi^n \colon \mathbf{T}^d \to \mathbf{R}$ as in Assumption III.3.2. In dimension $d = 2$, for $\lambda \geqslant 1$, define*

$$X_{n,\lambda} = (-\mathscr{A}_n + \lambda)^{-1} \xi^n, \qquad \xi^n \diamond \Pi_n^2 X_{n,\lambda} = \xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n,$$

where

$$c_n = \sum_{k \in \mathbf{Z}^2} \frac{\widehat{\chi}^2(n^{-1}k)\widehat{\chi}_Q(n^{-1}k)}{-\vartheta_n(k) + 1}, \quad \text{with} \quad c_n \simeq \log n.$$

If $d = 1$ one can bound for any $\kappa \in (0, \frac{1}{2})$:

$$\sup_{n \in \mathbf{N}} \mathbb{E}\Big[\sup_{\zeta \in [0,1]} n^{-\frac{\zeta}{2}} \|\xi^n\|_{\mathscr{C}^{-\frac{1}{2}(1-\zeta)-\frac{\kappa}{2}}} + n^{-1} \|\xi^n\|_{L^\infty} \Big] < \infty.$$

If $d = 2$ define the enhanced noise

$$\boldsymbol{\xi}_n = (\xi^n, (\xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n)_{\lambda \geqslant 1}),$$

taking values in the space of Definition IV.2.8. One can bound, for any $\kappa > 0$:

$$\sup_{n \in \mathbf{N}} \mathbb{E}\Big[|||\boldsymbol{\xi}_n|||_{n,\kappa} \Big] < \infty.$$

Moreover, for any fixed $\kappa \in (0, \frac{1}{2})$ there exists a probability space $(\overline{\Omega}, \overline{\mathscr{F}}, \overline{\mathbb{P}})$, supporting space white noise ξ on $\mathbf{T}^d$, and a sequence of random functions $\overline{\xi}^n \colon \mathbf{T}^d \to \mathbf{R}$ such that $\xi^n = \overline{\xi}^n$ in distribution and such that for almost all $\omega \in \overline{\Omega}$:

$$\overline{\xi}^n(\omega) \to \xi(\omega) \ \text{in} \ \mathscr{C}^{-\frac{d}{2}-\kappa}.$$

In dimension $d = 2$, for any $\lambda \geqslant 1$ there exists also a random distribution $\xi \diamond X_\lambda$ such that:

$$\begin{aligned} \mathscr{P}_n(-\mathscr{A}_n + \lambda)^{-1} \overline{\xi}^n(\omega) &\to (-\Delta + \lambda)^{-1} \xi(\omega) && \text{in} \ \mathscr{C}^{2-\frac{d}{2}-\kappa}, \\ \overline{\xi}^n \diamond \Pi_n^2 X_{n,\lambda} &\to \xi \diamond X_\lambda(\omega) && \text{in} \ \mathscr{C}^{-\kappa}. \end{aligned}$$

Finally, again in $d = 2$ and for almost all $\omega \in \overline{\Omega}$, one an bound:

$$\sup_{n \in \mathbf{N}} |||\boldsymbol{\xi}_n(\omega)|||_{n,\kappa} < \infty.$$

The proof of this result is mostly technical, and for the sake of readability deferred to after the proof of the theorem, in Section IV.2.5. In view of the previous result we will work under the following assumption.

Assumption IV.2.11. *Consider $\kappa \in (0, \frac{1}{2})$ fixed. Up to changing the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we assume that for all $\omega \in \Omega$ outside a null-set N the convergences in Proposition IV.2.10 hold true. If $d = 2$ in addition:*

$$\sup_{n \in \mathbf{N}} \|\!|\xi_n(\omega)|\!\|_{n,\kappa} < \infty.$$

Having fixed the correct probability space and having explained our method, we are now in position to prove Theorem III.3.15. The next result proves that the operators $\mathscr{H}_n$ converge in resolvent sense.

Proposition IV.2.12. *Under Assumption IV.2.11 fix $\omega \in \Omega \setminus N$. Consider, for $n \in \mathbf{N}$, the bounded selfadjoint operators*

$$\mathscr{H}_n^\omega \colon L^2 \to L^2, \qquad \mathscr{H}_n^\omega \psi = (\mathscr{A}_n + \Pi_n^2(\xi^n - c_n)\Pi_n^2)\psi.$$

There exists a $\overline{\lambda}(\omega) \in [1, \infty)$ such that $-\mathscr{H}_n^\omega + \lambda(\omega)$ is invertible for all $n \in \mathbf{N}$ and $\lambda(\omega) \geqslant \overline{\lambda}(\omega)$, and there exists an operator $B_\lambda(\omega) \in B(H)$ such that

$$\lim_{n \to \infty} (-\mathscr{H}_n^\omega + \lambda(\omega))^{-1} = B_\lambda(\omega) \ \text{ in } \ B(L^2(\mathbf{T}^d)).$$

Proof. The strategy of the proof is a perturbation of the proof in [AC15] and is based on a fixed point argument. In Step 1 we describe the space in which we can solve the resolvent equation through a fixed point argument, uniformly over n and λ large enough (throughout the proof the realization ω is fixed and omitted to keep the notation clean). The estimates that will allow us to apply Banach's fixed point theorem are discussed in Steps 2 through 4. The convergence as $n \to \infty$ is established in Steps 5 and 6. Throughout the proof the parameter $\kappa \in (0, \frac{1}{2})$ will be chosen small enough, so that all computations hold.

Step 1. Fix $p \in [1, \infty]$ as well as $\varphi \in \mathscr{C}_p^{-1+2\kappa}$. In dimension $d = 1$, solving the resolvent equation $(-\mathscr{H}_n + \lambda)\psi = \varphi$ is equivalent to solving (with $c_n = 0$) the fixed point problem

$$\psi = M_{\varphi,\lambda}(\psi) := (-\mathscr{A}_n + \lambda)^{-1}[\Pi_n^2[\xi^n - c_n]\Pi_n^2\psi + \varphi]. \tag{IV.3}$$

In dimension $d = 2$ we will not prove directly that $M_{\varphi,\lambda}$ is a contraction (while in $d = 1$ this is possible: the arguments that follow are then superfluous and Proposition III.6.6 allows to find a fixed point ψ). Instead, to find the fixed point we look for a paracontrolled solution. Consider a space $\mathscr{D}_n^\lambda \subseteq \mathscr{S}'(\mathbf{T}^d) \times \mathscr{S}'(\mathbf{T}^d)$ which consists of pairs $(\psi', \psi^\sharp)$ and is characterized by the norm

$$\|(\psi', \psi^\sharp)\|_{\mathscr{D}_n^\lambda} := \|\psi'\|_{\mathscr{C}_p^{1-\kappa}} + \|\mathscr{P}_n\psi^\sharp\|_{\mathscr{C}_p^{1+\kappa}} + n^{2-\kappa}\|\mathscr{Q}_n\psi^\sharp\|_{\mathscr{C}_p^{-1+2\kappa}},$$

where we used the operators $\mathscr{P}_n, \mathscr{Q}_n$ as in Definition III.6.3. The norm does not depend on λ, but to every pair $(\psi', \psi^\sharp) \in \mathscr{D}_n^\lambda$ we associate a function ψ by

$$\psi = \Pi_n^2\Big\{\psi' \varolessthan [(-\mathscr{A}_n + \lambda)^{-1}\xi^n]\Big\} + \psi^\sharp.$$

With an abuse of notation, we identify the pair $(\psi', \psi^\sharp)$ with the function ψ and write $\|\psi\|_{\mathscr{D}_n^\lambda} = \|(\psi', \psi^\sharp)\|_{\mathscr{D}_n^\lambda}$. Define the map $\overline{M}_{\varphi,\lambda} \colon \mathscr{D}_n^\lambda \to L^p$ as

$$\overline{M}_{\varphi,\lambda}(\psi) := (-\mathscr{A}_n + \lambda)^{-1}[\Pi_n^2\xi^n\Pi_n^2\psi - c_n\Pi_n^2\psi' + \varphi].$$

The map $\overline{M}_{\varphi,\lambda}$ can be extended to a map from $\mathscr{D}_n^\lambda$ into itself by defining:

$$\begin{aligned}\mathscr{M}_{\varphi,\lambda}(\psi) &= (M'_{\varphi,\lambda}(\psi),\ M^\sharp_{\varphi,\lambda}(\psi))\\ &:= (\Pi_n^2\psi,\ \overline{M}_{\varphi,\lambda}(\psi) - \Pi_n^2\{(\Pi_n^2\psi)\otimes[(-\mathscr{A}_n+1)^{-1}\xi^n]\}) \in \mathscr{D}_n^\lambda.\end{aligned}$$

Any fixed point of $\mathscr{M}_{\varphi,\lambda}$ is also a fixed point for $\overline{M}_{\varphi,\lambda}$ and since the fixed point satisfies

$$\psi' = \Pi_n^2\psi,$$

it solves also the fixed point equation (IV.3) for $M_{\varphi,\lambda}$. Similarly, if $\psi \in L^p$ solves Equation (IV.3), then $\psi \in \mathscr{D}_n^\lambda$ (for fixed $n \in \mathbf{N}$ the embedding $L^p \subseteq \mathscr{D}_n^\lambda$ is continuous) and ψ is a fixed point for $\mathscr{M}_{\varphi,\lambda}$. We conclude that solutions $\psi \in L^p$ to $(-\mathscr{H}_n+\lambda)\psi = \varphi$ are equivalent to fixed points of $\mathscr{M}_{\varphi,\lambda}$. We will show that for λ sufficiently large $\mathscr{M}_{\varphi,\lambda}$ admits a unique fixed point for all $\varphi \in \mathscr{C}_p^{-1+2\kappa}$. In the course of the proof we repeatedly make use of the elliptic Schauder estimates of Proposition III.6.6, the regularization properties of Π_n of Corollary III.7.4, the estimates on $X_{n,\lambda}$ of Lemma IV.2.9 and the paraproduct estimates of Lemma I.1.3, without stating them explicitly every time.

Step 2. Our aim is to control (paying particular attention to the dependence on λ and the uniformity over n) the quantity:

$$\|\mathscr{M}_{\varphi,\lambda}(\psi)\|_{\mathscr{D}_n^\lambda} = \|\Pi_n^2\psi\|_{\mathscr{C}_p^{1-\kappa}} + \|\mathscr{P}_n\, M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathscr{C}_p^{1+\kappa}} + \|\mathscr{Q}_n\, M^\sharp_{\varphi,\lambda}(\psi)\|_{\mathscr{C}_p^{-1+2\kappa}},$$

in terms on $\|\psi\|_{\mathscr{D}_n^\lambda}$ and $\|\varphi\|_{\mathscr{C}_p^{-1+2\kappa}}$. As for the first term, $\|\Pi_n^2\psi\|_{\mathscr{C}_p^{1-\kappa}}$, we observe that

$$\begin{aligned}\|\Pi_n^2\psi\|_{\mathscr{C}_p^{1-\kappa}} &= \left\|\Pi_n^4\left\{\psi'\otimes X_{n,\lambda}\right\} + \Pi_n^2\psi^\sharp\right\|_{\mathscr{C}_p^{1-\kappa}}\\ &\lesssim \|\psi'\|_{\mathscr{C}_p^{1-\kappa}}\left(\|\mathscr{P}_nX_{n,\lambda}\|_{\mathscr{C}_p^{1-\kappa}} + n^{2-\frac{\kappa}{2}}\|\mathscr{Q}_nX_{n,\lambda}\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}}\right)\\ &\quad + \|\mathscr{P}_n\psi^\sharp\|_{\mathscr{C}_p^{1+\kappa}} + n^{2-3\kappa}\|\mathscr{Q}_n\psi^\sharp\|_{\mathscr{C}^{-1+2\kappa}}\\ &\lesssim \lambda^{-\frac{\kappa}{4}}\|\psi\|_{\mathscr{D}_n^\lambda}(1+|||\boldsymbol{\xi}|||_{n,\kappa}) + \|\mathscr{P}_n\psi^\sharp\|_{\mathscr{C}_p^{1+\kappa}} + n^{2-\kappa}\|\mathscr{Q}_n\psi^\sharp\|_{\mathscr{C}^{-1+2\kappa}}.\end{aligned} \tag{IV.4}$$

To tackle the norms involving $M^\sharp$, first rewrite

$$M^\sharp_\varphi(\psi) = M^{\sharp,1}_\varphi(\psi) + M^{\sharp,2}_\varphi(\psi),$$

with

$$M^{\sharp,1}_\varphi(\psi) = (-\mathscr{A}_n+\lambda)^{-1}\left\{\varphi + \Pi_n^2[\xi^n\odot\Pi_n^2\psi^\sharp] + \Pi_n^2\left\{\xi^n\odot[\Pi_n^2(\psi'\otimes X_{n,\lambda})] - c_n\psi'\right\} + \Pi_n^2\left\{\xi^n\otimes\Pi_n^2\psi\right\}\right\}$$

and

$$M^{\sharp,2}_\varphi(\psi) = \Pi_n^2 C_{n,\lambda}(\Pi_n^2\psi,\xi^n),$$

where $C_{n,\lambda}(\Pi_n^2\psi,\xi^n)$ is the commutator

$$C_{n,\lambda}(\Pi_n^2\psi,\xi^n) = (-\mathscr{A}_n+\lambda)^{-1}[(\Pi_n^2\psi)\otimes\xi^n] - [(\Pi_n^2\psi)\otimes(-\mathscr{A}_n+\lambda)^{-1}(\xi^n)].$$

For clarity we divide the estimates for the two terms $M^{\sharp,1}_\varphi, M^{\sharp,2}_\varphi$ in two distinct steps.

Step 3: Estimates for $M^{\sharp,1}_{\varphi}$. Combining the Schauder estimates with the smoothing properties of Π_n and the paraproduct estimates one finds that

$$\lambda^{\frac{\kappa}{2}}\Big(\|\mathscr{P}_n M^{\sharp,1}_{\varphi}(\psi)\|_{\mathscr{C}^{1+\kappa}_p} + n^{2-\kappa}\|\mathscr{Q}_n M^{\sharp,1}_{\varphi}(\psi)\|_{\mathscr{C}^{-1+2\kappa}_p}\Big) \lesssim \|\varphi\|_{\mathscr{C}^{-1+2\kappa}_p} + \|\Pi_n^2\psi^{\sharp}\|_{\mathscr{C}^{1+\kappa}_p}\|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}} + \|\xi^n \odot [\Pi_n^2(\psi' \otimes X_{n,\lambda})] - c_n\psi'\|_{\mathscr{C}^{-1+2\kappa}_p}.$$

To treat $\|\xi^n \odot [\Pi_n^2(\psi' \otimes X_{n,\lambda}) - c_n\psi']\|_{\mathscr{C}^{-1+2\kappa}_p}$, we introduce (cf. Lemma IV.2.14) the commutators

$$C_n^{\Pi}(f,g) = \Pi_n^2(f \otimes g) - f \otimes \Pi_n^2 g, \qquad C^{\odot}(f,g,h) = f \odot (g \otimes h) - g(f \odot h).$$

Then the previous resonant product can be split into:

$$\|\xi^n \odot [\Pi_n^2(\psi' \otimes X_{n,\lambda}) - c_n\psi']\|_{\mathscr{C}^{-1+2\kappa}_p} \leq \|\xi^n \odot C_n^{\Pi}(\psi', X_{n,\lambda})\|_{\mathscr{C}^{-1+2\kappa}_p} + \|C^{\odot}(\xi^n, \psi', \Pi_n^2 X_{n,\lambda})\|_{\mathscr{C}^{-1+2\kappa}_p} + \|\psi'(\xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n)\|_{\mathscr{C}^{-1+2\kappa}_p}. \tag{IV.5}$$

Starting with the first term, by Lemma IV.2.16

$$\begin{aligned}\|\xi^n \odot C_n^{\Pi}(\psi', X_{n,\lambda})\|_{\mathscr{C}^{-1+2\kappa}_p} &\lesssim \|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}}\|\mathscr{P}_n C_n^{\Pi}(\psi', X_{n,\lambda})\|_{\mathscr{C}^{1+\kappa}_p} + \|\xi^n\|_{\mathscr{C}^{-1+\kappa}}\|\mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{P}_n X_{n,\lambda})\|_{\mathscr{C}^{1-\frac{\kappa}{2}}_p} \\ &\quad + \|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_{n,\lambda})\|_{\mathscr{C}^{-1+2\kappa}_p} \\ &\lesssim \|\psi'\|_{\mathscr{C}^{1-\kappa}_p}\|\!|\!|\xi_n|\!|\!|^2_{n,\kappa} + \|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_{n,\lambda})\|_{\mathscr{C}^{-1+2\kappa}_p}.\end{aligned}$$

The last quantity requires a bit of attention, since at first sight none of the two terms involved in the product has positive regularity: while the commutator guarantees us powers of n, it does not guarantee regularization on small scales. For this we need the estimate of ξ^n in spaces of positive regularity. Since ξ^n is constant on boxes this is not possible in the L^∞ scale of spaces, so we have to introduce an additional integrability parameter. For this we assume that κ is small enough so that $\frac{1}{r} = \frac{1}{p} + 2\kappa \leq 1$ and $-1 + 2\kappa \leq -\frac{7\kappa}{2}$. Then:

$$\begin{aligned}\|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_n)\|_{\mathscr{C}^{-1+2\kappa}_p} &\leq \|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_n)\|_{\mathscr{C}^{-\frac{7\kappa}{2}}_p} \\ &\lesssim \|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_n)\|_{\mathscr{C}^{\frac{\kappa}{2}}_r} \\ &\lesssim \|\xi^n\|_{\mathscr{C}^{\kappa}_{\frac{1}{2\kappa}}}\|\mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_n)\|_{\mathscr{C}^{-\frac{\kappa}{2}}_p}\end{aligned}$$

where in the second step we used Besov embedding and in the last step we used the resonant product estimate with arbitrary integrability parameters from Lemma I.1.3. Overall:

$$\begin{aligned}\|\xi^n \odot \mathscr{Q}_n C_n^{\Pi}(\psi', \mathscr{Q}_n X_n)\|_{\mathscr{C}^{-1+2\kappa}_p} &\lesssim n^{1+\kappa}|\!|\!|\xi_n|\!|\!| n^{-(1-2\kappa)}\|\psi'\|_{\mathscr{C}^{1-\kappa}_p}\|\mathscr{Q}_n X_n\|_{\mathscr{C}^{-\frac{\kappa}{2}}_p} \\ &\lesssim \|\psi'\|_{\mathscr{C}^{1-\kappa}_p}|\!|\!|\xi_n|\!|\!|^2.\end{aligned}$$

As for the second term in (IV.5), by Lemma IV.2.15

$$\begin{aligned}\|C^{\odot}(\xi^n,\psi',\Pi_n^2 X_n)\|_{\mathscr{C}_p^{-1+2\kappa}} &\le \|C^{\odot}(\xi^n,\psi',\Pi_n^2 X_n)\|_{\mathscr{C}_p^{-\kappa}} \\ &\lesssim \|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}}\|\psi'\|_{\mathscr{C}_p^{1-\kappa}}\|\Pi_n^2 X_n\|_{\mathscr{C}^{1-\frac{\kappa}{2}}} \lesssim \|\psi'\|_{\mathscr{C}_p^{1-\kappa}}\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa}^2.\end{aligned}$$

Here we estimated, via Lemma IV.2.9:

$$\begin{aligned}\|\Pi_n^2 X_{n,\lambda}\|_{\mathscr{C}^{1-\frac{\kappa}{2}}} &\leqslant \|\Pi_n^2 \mathscr{P}_n X_{n,\lambda}\|_{\mathscr{C}^{1-\frac{\kappa}{2}}} + \|\Pi_n^2 \mathscr{Q}_n X_{n,\lambda}\|_{\mathscr{C}^{1-\frac{\kappa}{2}}} \\ &\lesssim \vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa} + n\|X_{n,\lambda}\|_{\mathscr{C}^{-\frac{\kappa}{2}}} \lesssim \vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa}.\end{aligned}$$

Similarly for the last term in (IV.5). Here we recall that in the norm $\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa}$ the term $Y_{n,\lambda} = \xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n$ is allowed to mildly explode for $\lambda \to \infty$. We obtain:

$$\begin{aligned}\|\psi'(\xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n)\|_{\mathscr{C}_p^{-1+2\kappa}} &\lesssim \|\psi'\|_{\mathscr{C}_p^{1-\kappa}}\|\xi^n \odot \Pi_n^2 X_n - c_n\|_{\mathscr{C}^{-1+2\kappa}} \\ &\lesssim \lambda^{\frac{\kappa}{4}}\|\psi'\|_{\mathscr{C}_p^{1-\kappa}}\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa}.\end{aligned} \tag{IV.6}$$

Step 4: Estimates for $M_\varphi^{\sharp,2}$. Here we apply the commutator estimate for $C_{n,\lambda}(\Pi_n^2\psi,\xi^n)$ from Lemma IV.2.17. We start by estimating the large scales:

$$\begin{aligned}\|\mathscr{P}_n M_\varphi^{\sharp,2}(\psi)\|_{\mathscr{C}_p^{1+\kappa}} &= \|\Pi_n^2 \mathscr{P}_n C_{n,\lambda}(\Pi_n^2\psi,\xi^n)\|_{\mathscr{C}_p^{1+\kappa}} \\ &\lesssim \|\mathscr{P}_n C_{n,\lambda}(\Pi_n^2\psi,\xi^n)\|_{\mathscr{C}_p^{1+\kappa}} \\ &\lesssim \lambda^{-\frac{\kappa}{2}}\|\Pi_n^2\psi\|_{\mathscr{C}_p^{1-\kappa}}\|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}} \\ &\lesssim \lambda^{-\frac{\kappa}{2}}\|\psi\|_{\mathscr{D}_n^\lambda}(1+\vert\kern-0.25ex\vert\kern-0.25ex\vert \xi_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa})^2,\end{aligned}$$

where we used that, provided κ is sufficiently small, $(1-\kappa)+(-1-\kappa/2)+2(1-\kappa/2) > 1+\kappa$ together with the estimate (IV.4) for $\Pi_n^2\psi$. On small scales we find:

$$\begin{aligned}\lambda^{\frac{\kappa}{2}} n^{2-\kappa}\|\mathscr{Q}_n M_\varphi^{\sharp,2}(\psi)\|_{\mathscr{C}_p^{-1+2\kappa}} &\lesssim \lambda^{\frac{\kappa}{2}} n^{2-\kappa}\|\mathscr{Q}_n C_{n,\lambda}(\Pi_n^2\psi,\xi^n)\|_{\mathscr{C}_p^{-1+2\kappa}} \\ &\lesssim n^{-1}\Big(\lambda^{\frac{\kappa}{2}} n^{1+2(1-\kappa/2)}\|\mathscr{Q}_n C_{n,\lambda}(\Pi_n^2\psi,\xi^n)\|_{\mathscr{C}_p^{-1+2\kappa}}\Big) \\ &\lesssim \|\Pi_n^2\psi\|_{\mathscr{C}_p^{1-\kappa}}(n^{-1}\|\xi^n\|_{\mathscr{C}^{-1+3\kappa}}) \\ &\lesssim \|\psi\|_{\mathscr{D}_n^\lambda}(1+\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa})^2,\end{aligned}$$

where we once again used the estimates on $\Pi_n^2\psi$ from (IV.4).

Step 5: Collecting the estimates. The estimates of step 2 guarantee that there exists an increasing map

$$\mathfrak{c}\colon [0,\infty) \to [1,\infty)$$

such that

$$\|M'_\varphi(\psi)\|_{\mathscr{C}_p^{1-\kappa}} \leqslant \mathfrak{c}(\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa})\Big(\lambda^{-\frac{\kappa}{2}}\|\psi'\|_{\mathscr{C}_p^{1-\kappa}} + \|\mathscr{P}_n\psi^\sharp\|_{\mathscr{C}_p^{1+\kappa}} + \|\mathscr{Q}_n\psi^\sharp\|_{\mathscr{C}_p^{-1+2\kappa}}\Big). \tag{IV.7}$$

In addition, estimates of steps 3 and 4 guarantee that (up to choosing a larger $\mathfrak{c}$):

$$\|\mathscr{P}_n M_\varphi^\sharp(\psi)\|_{\mathscr{C}_p^{1+\kappa}} + n^{2-\kappa}\|\mathscr{Q}_n M_\varphi^\sharp(\psi)\|_{\mathscr{C}_p^{-1+2\kappa}} \leqslant \lambda^{-\frac{\kappa}{4}}\mathfrak{c}(\vert\kern-0.25ex\vert\kern-0.25ex\vert \boldsymbol{\xi}_n \vert\kern-0.25ex\vert\kern-0.25ex\vert_{n,\kappa})\Big(\|\varphi\|_{\mathscr{C}_p^{-1+2\kappa}} + \|\psi\|_{\mathscr{D}_n^\lambda}\Big). \tag{IV.8}$$

Observe that the factor $\lambda^{-\frac{\kappa}{4}}$, instead of $\lambda^{-\frac{\kappa}{2}}$, is not a typo: it follows from (IV.6), where we pay a factor $\lambda^{\frac{\kappa}{4}}$ to control the product $\xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n$. Combined with the linearity of the map $\mathscr{M}_\varphi$ we find that:

$$\|\mathscr{M}_\varphi(\psi)\|_{\mathscr{D}_n} \leq \mathfrak{c}(\|\!|\!|\xi_n|\!|\!\|_{n,\kappa})\Big[\|\varphi\|_{\mathscr{C}_p^{-1+2\kappa}} + \|\psi\|_{\mathscr{D}_n}\Big]$$
$$\Big\|\Big[\mathscr{M}_\varphi(\psi) - \mathscr{M}_\varphi(\tilde{\psi})\Big]^2\Big\|_{\mathscr{D}_n} \leq \mathfrak{c}^2(\|\!|\!|\xi_n|\!|\!\|_{n,\kappa})\Big[\lambda^{-\frac{\kappa}{4}}\|\psi - \tilde{\psi}\|_{\mathscr{D}_n}\Big].$$

Note that we take the second power of the map in the last estimate, because in (IV.7) we do not have a small factor $\lambda^{-\frac{\kappa}{4}}$ in front of the rest term with $\psi^\sharp$.
In particular, we finally can conclude that there exists a $\bar{\lambda} = \bar{\lambda}(\sup_n \|\!|\!|\xi_n|\!|\!\|_{n,\kappa})$ (so it is independent of n) such that for $\lambda > \bar{\lambda}$ the map $\mathscr{M}_\varphi$ admits a unique fixed point, which we denote by $\mathscr{H}_{n,\lambda}^{-1}\varphi$. Moreover, by the Banach fixed point theorem

$$\|\mathscr{H}_{n,\lambda}^{-1}\varphi\|_{\mathscr{D}_n} \lesssim \|\mathscr{M}_\varphi^2(0)\|_{\mathscr{D}_n} \lesssim \mathfrak{c}^2(\|\!|\!|\xi_n|\!|\!\|_{n,\kappa})\|\varphi\|_{\mathscr{C}_p^{-1+2\kappa}}, \tag{IV.9}$$

implying that $\mathscr{H}_{n,\lambda}^{-1} \in B(\mathscr{C}_p^{-1+2\kappa}, \mathscr{D}_n^\lambda)$, with the norm bounded uniformly in n. Similar, but less involved calculations lead to a construction of the resolvent $\mathscr{H}_\lambda^{-1} = (\mathscr{H} - \lambda)^{-1}$ in the continuum for $\lambda \geq \bar{\lambda}$ (in the continuous case no division of scales is required). The resolvent is then a bounded operator $\mathscr{H}_\lambda^{-1} \in B(\mathscr{C}_p^{-1+2\kappa}, \mathscr{D}^\lambda)$, where the latter is the Banach space defined by the norm (for $\psi = \psi' \otimes (-\Delta + \lambda)^{-1}\xi + \psi^\sharp$):

$$\|\psi\|_{\mathscr{D}^\lambda} = \|\psi'\|_{\mathscr{C}_p^{1-\kappa}} + \|\psi^\sharp\|_{\mathscr{C}_p^{1+\kappa}}.$$

By linearity and computations on the line of those in the previous steps one can then show that:

$$\lim_{n\to\infty} \sup_{\|\varphi\|_{\mathscr{C}_p^{-1+2\kappa}} \leq 1} \Big\|(\mathscr{H}_{n,\lambda}^{-1}\varphi)' - (\mathscr{H}_\lambda^{-1}\varphi)'\Big\|_{\mathscr{C}_p^{1-\kappa}} + \Big\|\mathscr{P}_n(\mathscr{H}_{n,\lambda}^{-1}\varphi)^\sharp - (\mathscr{H}_\lambda^{-1}\varphi)^\sharp\Big\|_{\mathscr{C}_p^{1+\kappa}} = 0. \tag{IV.10}$$

Since $\mathscr{H}_{n,\lambda}^{-1} \in B(\mathscr{C}_p^{-1+2\kappa}, \mathscr{D}_n^\lambda)$, to prove convergence of the resolvents in $B(L^2, L^2)$ it would be sufficient to show, in the particular case $p = 2$, that $\mathscr{D}_n^\lambda \hookrightarrow L^p$, in the sense that $\|\psi\|_{L^p} \lesssim \|\psi\|_{\mathscr{D}_n^\lambda}$. Unfortunately, this is not the case, because a priori $\mathscr{Q}_n\psi^\sharp \in \mathscr{C}_p^{-1+2\kappa}$. So we need a better control on the regularity of $\psi^\sharp$, which we will obtain by using that $\varphi \in L^2$.

Step 6: L^2 estimates. Let us fix $p = 2$. We want to improve our previous bound by showing that if $\varphi \in L^2$, then for every $\psi \in \mathscr{D}_n^\lambda$:

$$\|\mathscr{Q}_n M_\varphi^\sharp(\psi)\|_{L^2} \lesssim n^{-\kappa}\mathfrak{c}(\|\!|\!|\xi_n|\!|\!\|_{n,\kappa})(\|\varphi\|_{L^2} + \|\psi\|_{\mathscr{D}_n^\lambda}). \tag{IV.11}$$

Let us start with estimating by Plancherel (using the same notation as in Section III.6):

$$\begin{aligned}\|(-\mathscr{A}_n + \lambda)^{-1}\mathscr{Q}_n\varphi\|_{L^2}^2 &\simeq \sum_{k\in\mathbf{Z}^d}\left|\frac{(1-\daleth)(n^{-1}k)}{\lambda + n^2(1-\hat{\chi}^4(n^{-1}k))}\right|^2|\hat{\varphi}(k)|^2 \\ &\lesssim \frac{1}{n^4}\sum_{k\in\mathbf{Z}^d}|(1-\daleth)(n^{-1}k)\hat{\varphi}(k)|^2 \lesssim \frac{1}{n^4}\|\mathscr{Q}_n\varphi\|_{L^2}^2,\end{aligned}$$

where we used that $\hat{\chi}^4(n^{-1}k) < 1 \forall k \neq 0$, together with the support properties of $(1 - \urcorner)(n^{-1}k)$. Hence we conclude that

$$\|\mathscr{Q}_n M^{\sharp}_{\varphi}(\psi)\|_{L^2} \lesssim n^{-2}\|\mathscr{Q}_n \varphi\|_{L^2} + \|\mathscr{Q}_n \widetilde{M}^{\sharp,1}_{\varphi}(\psi)\|_{L^2} + \|\mathscr{Q}_n M^{\sharp,2}_{\varphi}(\psi)\|_{L^2},$$

with $M^{\sharp,2}_{\varphi}(\psi)$ as in step 2 and

$$\widetilde{M}^{\sharp,1}_{\varphi}(\psi) = (-\mathscr{A}_n + \lambda)^{-1}\Pi^2_n\Big\{\xi^n \odot \Pi^2_n \psi^{\sharp} + \xi^n \odot [\Pi^2_n(\psi' \otimes X_{n,\lambda})] - c_n\psi' + \xi^n \otimes \Pi^2_n \psi\Big\}.$$

The smoothing effect of Π^2_n and the elliptic Schauder estimates guarantee that

$$\begin{aligned}
n^{\frac{\kappa}{2}}&\|\mathscr{Q}_n \widetilde{M}^{\sharp,1}_{\varphi}(\psi)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\
&= n^{\frac{\kappa}{2}}\Big\|\Pi^2_n\mathscr{Q}_n(-\mathscr{A}_n+\lambda)^{-1}\Big(\xi^n \odot \Pi^2_n\psi^{\sharp} + \xi^n \odot [\Pi^2_n(\psi'\otimes X_{n,\lambda})] - c_n\psi' + \xi^n \otimes \Pi^2_n\psi\Big)\Big\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\
&\lesssim n^{2-\kappa}\Big\|\mathscr{Q}_n(-\mathscr{A}_n+\lambda)^{-1}\Big(\xi^n \odot \Pi^2_n\psi^{\sharp} + \xi^n \odot [\Pi^2_n(\psi'\otimes X_{n,\lambda})] - c_n\psi' + \xi^n \otimes \Pi^2_n\psi\Big)\Big\|_{\mathscr{C}_2^{-1+2\kappa}} \\
&\lesssim \Big\|\xi^n \odot \Pi^2_n\psi^{\sharp} + \xi^n \odot [\Pi^2_n(\psi'\otimes X_{n,\lambda})] - c_n\psi' + \xi^n \otimes \Pi^2_n\psi\Big\|_{\mathscr{C}_2^{-1+2\kappa}}.
\end{aligned}$$

Now we can follow verbatim the estimates of step 3 to obtain, up to slightly increasing $\mathfrak{c}$:

$$n^{\kappa}\|\mathscr{Q}_n \widetilde{M}^{\sharp,1}_{\varphi}(\psi)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \leqslant \mathfrak{c}(\|\!|\xi_n|\!\|_{n,\kappa})\|\psi\|_{\mathscr{D}^{\lambda}_n}. \tag{IV.12}$$

Similarly for $M^{\sharp,2}_{\varphi}(\psi)$, where we find:

$$\begin{aligned}
n^{\frac{\kappa}{2}}\|\mathscr{Q}_n M^{\sharp,2}_{\varphi}(\psi)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} &= n^{\frac{\kappa}{2}}\|\mathscr{Q}_n\Pi^2_n C_{n,\lambda}(\Pi^2_n\psi, \xi^n)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\
&\lesssim n^{2-\kappa}\|\mathscr{Q}_n C_{n,\lambda}(\Pi^2_n\psi, \xi^n)\|_{\mathscr{C}_2^{-1+2\kappa}} \\
&\leqslant \mathfrak{c}(\|\!|\xi_n|\!\|_{n,\kappa})\|\psi\|_{\mathscr{D}^{\lambda}_n},
\end{aligned} \tag{IV.13}$$

where in the last step we followed verbatim the calculations in step 4. In particular, we have concluded the proof of (IV.11). The bound (IV.11) allows us in particular to conclude that

$$\lim_{n\to\infty}\sup_{\|\varphi\|_{L^2}\leqslant 1}\|\mathscr{Q}_n\mathscr{H}^{-1}_{n,\lambda}\varphi\|_{L^2} = 0.$$

Together with (IV.10) we conclude that

$$\begin{aligned}
\lim_{n\to\infty}&\sup_{\|\varphi\|_{L^2}\leqslant 1}\|\mathscr{H}^{-1}_{n,\lambda}\varphi - \mathscr{H}^{-1}_{\lambda}\varphi\|_{L^2} \\
&\leq \lim_{n\to\infty}\sup_{\|\varphi\|_{L^2}\leqslant 1}\Big\{\|(\mathscr{H}^{-1}_{n,\lambda}\varphi)'\otimes X_{n,\lambda} - (\mathscr{H}^{-1}_{\lambda}\varphi)'\otimes(-\Delta+\lambda)^{-1}\xi\|_{L^2} \\
&\qquad + \|\mathscr{P}_n(\mathscr{H}^{-1}_{n,\lambda}\varphi)^{\sharp} - (\mathscr{H}^{-1}_{\lambda}\varphi)^{\sharp}\|_{L^2} + \|\mathscr{Q}_n(\mathscr{H}^{-1}_{n,\lambda}\varphi)^{\sharp}\|_{L^2}\Big\} \\
&= 0,
\end{aligned}$$

thus proving the convergence of the resolvents.

□

Having established convergence in resolvent sense of the operator $\mathscr{H}_n$ we complete the proof of Theorem III.3.15 by showing that the eigenfunctions of the operators converge in an appropriate sense.

Proof of Theorem III.3.15. As usual, let us fix $\omega \in \Omega$, the latter satisfying Assumption IV.2.11 and to lighten the notation we avoid writing explicitly the dependence on ω in what follows. Also, as in the previous proof we restrict to discussing the case $d = 2$, which is more complicated.

To complete the proof of the theorem we collect all the previous results. Proposition IV.2.12 guarantees that $\mathscr{H}_n$ converges to $\mathscr{H}$ in the resolvent sense, as operators on $L^2(\mathbf{T}^d)$. In particular, Corollary IV.2.5 guarantees that, for any eigenvalue λ of $\mathscr{H}$ with multiplicity $m(\lambda) \in \mathbf{N}$ and associated orthogonal eigenfunctions $\{e_j\}_{j=1}^{m(\lambda)}$, there exists a sequence $\{e_j^n\}_{j=1}^{m(\lambda)} \subseteq L^2(\mathbf{T}^d)$, for $n \geqslant n(\lambda)$ with $n(\lambda)$ sufficiently large, such that:

$$e_j^n \to e_j, \qquad \mathscr{H}_n e_j^n \to \mathscr{H} e_j, \qquad \text{in } L^2(\mathbf{T}^d).$$

Moreover any e_j^n can be represented as

$$e_j^n = \sum_{i=1}^{m(\lambda)} \alpha_{ij}^n \overline{e}_i^n, \qquad \sum_{i=1}^{m(\lambda)} (\alpha_{ij}^n)^2 = 1,$$

where $\overline{e}_i^n$ are eigenfunctions for $\mathscr{H}_n$ with eigenvalue λ_i^n such that $\lim_{n\to\infty} \lambda_i^n = \lambda$. To conclude the proof we will show the following additional convergences, for any $\kappa \in (0, \frac{1}{2})$ sufficiently small:

$$\Pi_n e_j^n \to e_j, \qquad \Pi_n \mathscr{H}_n e_j^n \to \lambda e_j \qquad \text{in } \mathscr{C}^\kappa(\mathbf{T}^d),$$

for $\kappa > 0$ sufficiently small. In the previous discussion we already have explained the convergences above in $L^2(\mathbf{T}^d)$. By compact embedding $\mathscr{C}^\kappa(\mathbf{T}^d) \subseteq \mathscr{C}^{\kappa'}(\mathbf{T}^d)$ for $\kappa' < \kappa$, and since κ is arbitrary, it thus suffices to prove the bounds:

$$\sup_{n \geqslant n(\lambda)} \left\{ \|\Pi_n e_j^n\|_{\mathscr{C}^\kappa} + \|\Pi_n \mathscr{H}_n e_j^n\|_{\mathscr{C}^\kappa} \right\} < \infty.$$

By our previous considerations, observing that

$$\mathscr{H}_n e_j^n = \sum_{i=1}^{m(\lambda)} \lambda_i^n \alpha_{ij} \overline{e}_i^n,$$

we can further reduce the problem to proving that

$$\sup_{n \geqslant n(\lambda)} \|\Pi_n \overline{e}_i^n\|_{\mathscr{C}^\kappa} < \infty, \qquad \forall i = 1, \ldots, m(\lambda). \tag{IV.14}$$

Now we fix i and make use of the fact that $\overline{e}_i^n$ is an eigenfunction of $\mathscr{H}_n$ with eigenvalue $\lambda_i^n \to \lambda$. To lighten the notation, since i is fixed, let us write

$$e^n = \overline{e}_i^n.$$

We find that for $\mu > 1$ sufficiently large such that Proposition IV.2.12 applies (with λ replaced by μ, and following the notations introduced by the proposition and its proof) and defining $v^n = (\mu - \lambda_i^n) e^n$:

$$\begin{aligned} e^n &= \mathscr{H}_{n,\mu}^{-1} v^n \\ &= \Pi_n^2 \Big\{ (e^n)' \otimes X_{n,\mu} \Big\} + (e^n)^\sharp \end{aligned}$$

The bound (IV.9) now guarantees that

$$\|e^n\|_{\mathscr{D}_n^\mu} = \|(e^n)'\|_{\mathscr{C}_2^{1-\kappa}} + \|\mathscr{P}_n (e^n)^\sharp\|_{\mathscr{C}_2^{1+\kappa}} + n^{2-\kappa} \|\mathscr{Q}_n (e^n)^\sharp\|_{\mathscr{C}_2^{-1+2\kappa}} \lesssim \|e^n\|_{L^2} \lesssim 1.$$

This bound is sufficient for large scales, but small scales need more care. Here we observe that

$$(e^n)^\sharp = (-\mathscr{A}_n + \mu)^{-1} v^n + \widetilde{M}_{v^n}^{\sharp,1}(e^n) + M_{v^n}^{\sharp,2}(e^n),$$

where $\widetilde{M}^{\sharp,1}, M^{\sharp,2}$ have been introduced in Step 6 of Proposition IV.2.12 and satisfy, following (IV.12) and (IV.13):

$$\|\mathscr{Q}_n \widetilde{M}_{v^n}^{\sharp,1}(e^n)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|\mathscr{Q}_n M_{v^n}^{\sharp,2}(e^n)\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \lesssim \|e^n\|_{\mathscr{D}_n^\mu} \lesssim 1.$$

Now we are in position to conclude our estimate. By Besov embedding, since we are considering the case $d = 2$ (note that in $d = 1$ we loose less regularity, so the estimates simplify) we have

$$\|\varphi\|_{\mathscr{C}^{\alpha-1}} \lesssim \|\varphi\|_{\mathscr{C}_2^\alpha}, \qquad \forall \varphi \in \mathscr{C}_2^\alpha.$$

In particular we find that

$$\sup_{n \geqslant n(\lambda)} \|(e^n)'\|_{\mathscr{C}^{-\kappa}} \lesssim \sup_{n \geqslant n(\lambda)} \|e^n\|_{\mathscr{D}_n^\mu} < \infty,$$

so that (note that the term Π_n^3 appears because we want to estimate the norm of $\Pi_n e^n$: for this estimate the presence of the additional Π_n does not matter):

$$\begin{aligned} \sup_{n \geqslant n(\lambda)} \Big\| \Pi_n^3 \Big\{ (e^n)' \otimes X_{n,\mu} \Big\} \Big\|_{\mathscr{C}^{1-3\kappa}} &\lesssim \sup_{n \geqslant n(\lambda)} n^{2-\kappa} \|(e^n)' \otimes X_{n,\mu}\|_{\mathscr{C}^{-1-2\kappa}} \\ &\lesssim \sup_{n \geqslant n(\lambda)} \|(e^n)'\|_{\mathscr{C}^{-\kappa}} \|X_{n,\mu}\|_{\mathscr{C}^{-1-\kappa}} \\ &\lesssim \sup_{n \geqslant n(\lambda)} \|e^n\|_{\mathscr{D}_n^\mu} |||\xi_n|||_{n,\kappa} < \infty. \end{aligned}$$

Next we control the rest term:

$$\begin{aligned} \|\Pi_n (e^n)^\sharp\|_{\mathscr{C}^{\frac{\kappa}{2}}} &\lesssim \|\Pi_n (e^n)^\sharp\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\ &\lesssim \|\mathscr{P}_n \Pi_n (e^n)^\sharp\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|\mathscr{Q}_n \Pi_n (-\mathscr{A}_n + \mu)^{-1} v^n\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|\mathscr{Q}_n \Pi_n (\widetilde{M}_{v^n}^{\sharp,1}(e^n) + M_{v^n}^{\sharp,2}(e^n))\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\ &\lesssim \|\mathscr{Q}_n \Pi_n (-\mathscr{A}_n + \mu)^{-1} v^n\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|\mathscr{P}_n (e^n)^\sharp\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|\mathscr{Q}_n (\widetilde{M}_{v^n}^{\sharp,1}(e^n) + M_{v^n}^{\sharp,2}(e^n))\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \\ &\lesssim \|\mathscr{Q}_n \Pi_n (-\mathscr{A}_n + \mu)^{-1} v^n\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} + \|e^n\|_{\mathscr{D}_n^\mu}, \end{aligned}$$

where in the last step we used all the previous estimates. Observe that so far we did non use the smoothing effect of the additional term Π_n. We use this effect in the following last step, where we estimate the only remaining term:

$$\|\mathscr{Q}_n\Pi_n(-\mathscr{A}_n+\mu)^{-1}v^n\|_{\mathscr{C}_2^{1+\frac{\kappa}{2}}} \lesssim \Big\|(1+|\cdot|^2)^{\frac{d+1}{4}}\mathscr{F}_{\mathbf{T}^d}\Big(\mathscr{Q}_n\Pi_n(-\mathscr{A}_n+\mu)^{-1}e^n\Big)\Big\|_{L^2(\mathbf{Z}^d)}.$$

Here we used that for κ sufficiently small and since $d=2$: $1+\frac{\kappa}{2}\leqslant\frac{d+1}{2}$. Then we used one of the many definitions of fractional Sobolev spaces, via the norm (for $\alpha>0$):

$$\|\varphi\|_{H^\alpha} = \|(1-\Delta)^{\frac{\alpha}{2}}\varphi\|_{L^2(\mathbf{T}^d)} \simeq \|(1+|\cdot|^2)^{\frac{\alpha}{2}}\mathscr{F}_{\mathbf{T}^d}\varphi\|_{L^2},$$

together with the embedding (see for example [Tri10, Section 2.3.5]):

$$\|\varphi\|_{\mathscr{C}_2^\alpha} \lesssim \|\varphi\|_{H^\alpha}, \qquad \forall\varphi\in H^\alpha.$$

Hence we conclude with the following estimate (here we follow the notations of Section III.6):

$$\begin{aligned}
&\Big\|(1+|\cdot|^2)^{\frac{d+1}{4}}\mathscr{F}_{\mathbf{T}^d}\Big(\mathscr{Q}_n\Pi_n(-\mathscr{A}_n+\mu)^{-1}e^n\Big)\Big\|^2_{L^2(\mathbf{Z}^d)}\\
&\qquad = \sum_{k\in\mathbf{Z}^d}\left|\frac{(1+|k|^2)^{\frac{d+1}{4}}}{\mu+n^2(1-\hat{\chi}^4(n^{-1}k)))}\right|^2 |\hat{\chi}(n^{-1}k)(1-\urcorner)(n^{-1}k))|^2|\hat{e}^n(k)|^2\\
&\qquad \lesssim \left(\frac{1}{\mu+n^2}\right)^2\sum_{k\in\mathbf{Z}^d}(1+|k|)^{d+1}|\hat{\chi}(n^{-1}k)(1-\urcorner)(n^{-1}k))|^2|\hat{e}^n(k)|^2\\
&\qquad \lesssim \left(\frac{1}{\mu+n^2}\right)^2\sum_{k\in\mathbf{Z}^d}\frac{(1+|k|)^{d+1}}{(1+|n^{-1}k|)^{d+1}}|(1-\urcorner)(n^{-1}k))|^2|\hat{e}^n(k)|^2\\
&\qquad \lesssim \left(\frac{1}{\mu+n^2}\right)^2 n^{d+1}\sum_{k\in\mathbf{Z}^d\setminus\{0\}}\frac{(1+|k|)^{d+1}}{|k|^{d+1}}|\hat{e}^n(k)|^2\\
&\qquad \lesssim n^{(d+1)-4}\|e^n\|^2_{L^2} \lesssim \|e^n\|^2_{L^2} \lesssim 1.
\end{aligned}$$

Here we used the fact that $\hat{\chi}(k)<1$ for $k\neq0$ together with the support properties of $\urcorner$ to bound

$$\frac{1}{\mu+n^2(1-\hat{\chi}^4(n^{-1}k))} \lesssim \frac{1}{n^2}$$

uniformly over n and k such that $(1-\urcorner)(n^{-1}k)\neq0$. We also applied the bound

$$|\hat{\chi}(k)| \lesssim \frac{1}{(1+|k|)^{\frac{d+1}{2}}}$$

from Lemma III.6.1. This concludes the proof of the theorem, since we have proven (IV.14) with κ replaced by $\frac{\kappa}{2}$ (but this does not matter since κ is arbitrarily small).

Before we conclude, let us observe that in the last bound we used that $d=2$ to bound $1+\frac{\kappa}{2}\leqslant\frac{d+1}{2}$. If $d=1$ this fails, but we actually need less, since by Besov embedding

$$\|\Pi_n e^n\|_{\mathscr{C}^\kappa(\mathbf{T}^1)} \lesssim \|\Pi_n e^n\|_{\mathscr{C}_2^{\frac{1}{2}+\kappa}(\mathbf{T}^1)}.$$

In particular, following all the previous steps we can bound, for κ sufficiently small such that $\frac{1}{2}+\kappa \leqslant \frac{d+1}{2} = 1$:

$$
\begin{aligned}
\|\mathscr{Q}_n \Pi_n(-\mathscr{A}_n+\mu)v^n\|_{\mathscr{C}^\kappa} &\lesssim \|\mathscr{Q}_n\Pi_n(-\mathscr{A}_n+\mu)^{-1}e^n\|_{\mathscr{C}_2^{\frac{1}{2}+\frac{\kappa}{2}}} \\
&\lesssim \Big\|(1+|\cdot|^2)^{\frac{d+1}{4}}\mathscr{F}_{\mathbf{T}^d}\Big(\mathscr{Q}_n\Pi_n(-\mathscr{A}_n+\mu)^{-1}e^n\Big)\Big\|_{L^2(\mathbf{Z}^d)},
\end{aligned}
$$

and from here we can follow, for example, the same calculations as above. □

Remark IV.2.13. *We observe that in the last bound for $(e^n)^\sharp$ we used Π_n to gain $\frac{d+1}{2}$ regularity. In dimension $d=2$ this is crucially larger than 1. This statement is in apparent contraddiction with Corollary III.7.4, where we show a possible regularity gain of at most 1. While the latter corollary works for any integrability parameter p and extends to other characteristics functions (than just those of balls), the improvement we see in the proof depends on the choice $p=2$ and our exact computations for the decay of the Fourier transform $\hat{\chi}$.*

IV.2.5 Stochastic Estimates

Before concluding, we provide the proof of the stochastic bounds we stated at the beginning of the section.

Proof of Proposition IV.2.10. First we will prove the bounds for $\xi^n, X_{n,\lambda}$ and $\xi^n \diamond \Pi_n^2 X_{n,\lambda}$. Eventually we address the convergence of these terms. Although only in the first case the dimension is allowed to be both $d=1$ and $d=2$, we will keep d as a parameter throughout the proof, for the sake of clarity. For convenience, let us indicate sums on $\mathbf{Z}^d$ with integrals (for $m\in\mathbf{N}$):

$$
\int_{(\mathbf{Z}^d)^m} f(k_1,\dots,k_m)\,\mathrm{d}k_1\cdots\mathrm{d}k_m = \sum_{k_1,\dots,k_m\in\mathbf{Z}^d} f(k_1,\dots,k_m).
$$

Step 1: Bounds on ξ^n. First, observe that by Assumption III.4.1:

$$
|\xi^n(x)| \le 2n^{\frac{d}{2}}.
$$

This explains both the L^∞ bounds on ξ^n and the bound in $\mathscr{C}^{-\frac{\kappa}{2}}$ (i.e. for $\zeta=1$). If we show that

$$
\sup_{n\in\mathbf{N}} \mathbb{E}\Big[\|\xi^n\|_{\mathscr{C}^{-\frac{d}{2}-\frac{\kappa}{2}}}\Big] < \infty,
$$

the bound for arbitrary ζ follows, since by interpolation, from the definition of Besov spaces, for any $\zeta\in[0,1]$ and $\alpha,\beta\in\mathbf{R}$:

$$
\|\varphi\|_{\mathscr{C}^{\zeta\alpha+(1-\zeta)\beta}} \le \|\varphi\|_{\mathscr{C}^\alpha}^{\zeta}\|\varphi\|_{\mathscr{C}^\beta}^{1-\zeta}.
$$

Hence let us consider the case $\zeta=0$. By Besov embedding, the required inequality follows if one can show that for any $p\in[2,\infty)$:

$$
\sup_{n\in\mathbf{N}} \mathbb{E}\|\xi^n\|^p_{B_{p,p}^{-\frac{d}{2}-\frac{\kappa}{4}}} < \infty.
$$

Here in view of Assumption III.3.2, and by the discrete Burkholder-Davis-Gundy inequality as well as Jensen's inequality one finds that:

$$\int_{\mathbf{T}^d} \mathbb{E}[|\Delta_j n^{\frac{d}{2}} s_n|^p(x)]\,\mathrm{d}x \lesssim \int_{\mathbf{T}^d} \Big(\sum_{z\in\mathbf{Z}^d_n} n^{-d} |\Delta_j \chi_{Q_n}|^2(z+x)\Big)^{p/2} \mathrm{d}x$$
$$\leq \int_{\mathbf{T}^d} \Big(\int_{\mathbf{T}^d} \mathrm{d}z\, |K_j(x+z)|^2\Big)^{p/2} \mathrm{d}x \lesssim \|K_j\|^p_{L^2} \lesssim 2^{j\frac{dp}{2}},$$

which is a bound of the required order.

Now, let us pass to the bound in $\mathscr{C}^{\kappa}_{\frac{1}{2\kappa}}$. In fact we will prove that for any $p\in[1,\infty)$, $\zeta\in[0,\frac{1}{p})$ we have a bound on $\|\xi^n\|_{\mathscr{C}^\zeta_p}$. We use the Sobolev-Slobodeckij norm of Proposition III.7.1 for the Besov space $B^\zeta_{p,p}$ (which embeds in $\mathscr{C}^\zeta_p$, so finding a bound in the latter space is sufficient). Let us start by computing:

$$\|\xi^n\|_{B^\zeta_{p,p}} \simeq \|\xi^n\|_{L^p} + \Big(\int_{\mathbf{T}^d}\int_{\mathbf{T}^d} \frac{|\xi^n(x)-\xi^n(y)|^p}{|x-y|^{d+\zeta p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}$$
$$\lesssim n^{\frac{d}{2}} + \Big(\sum_{z\in\mathbf{Z}^d_n\cap\mathbf{T}^d}\int_{Q_n(z)}\int_{\mathbf{T}^d} \frac{|\xi^n(x)-\xi^n(y)|^p}{|x-y|^{d+\zeta p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}$$
$$\lesssim n^{\frac{d}{2}} + \Big(\sum_{z\in\mathbf{Z}^d_n\cap\mathbf{T}^d}\int_{Q_n(z)}\int_{\mathbf{T}^d\setminus Q_n(z)} \frac{|\xi^n(x)-\xi^n(y)|^p}{|x-y|^{d+\zeta p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}}$$
$$\lesssim n^{\frac{d}{2}} + n^{\frac{d}{2}} n^{\frac{d}{p}}\Big(\int_{Q_n(0)}\int_{\mathbf{T}^d\setminus Q_n(0)} \frac{1}{|x-y|^{d+\zeta p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}},$$

where we have used in the last step that $\|\xi^n\|_\infty \lesssim n^{\frac{d}{2}}$. Now we can follow the same calculations as in the proof of Lemma III.7.3 to obtain

$$\Big(\int_{Q_n(0)}\int_{\mathbf{T}^d\setminus Q_n(0)} \frac{1}{|x-y|^{d+\zeta p}}\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac{1}{p}} \lesssim n^{\zeta-\frac{d}{p}}.$$

Hence, overall for any $p\in[1,\infty)$ and $\zeta\in[0,1/p)$:

$$\|\xi^n\|_{\mathscr{C}^\zeta_p} \lesssim n^{\frac{d}{2}+\zeta}.$$

Step 2: Bounds for $X_{n,\lambda}$. As for $X_{n,\lambda}$, we need to bound $n\|\mathscr{Q}_n X_{n,\lambda}\|_{L^\infty}$. Here:

$$\begin{aligned}\|\mathscr{Q}_n X_{n,\lambda}\|_{L^\infty(\mathbf{T}^d)} &= \|\mathscr{F}^{-1}_{\mathbf{T}^d}[(1-\urcorner)(n^{-1}\cdot)(-\vartheta_n+\lambda)^{-1}(\cdot)\widehat{\xi}_n(\cdot)]\|_{L^\infty(\mathbf{T}^d)}\\ &\leq \|\mathscr{F}^{-1}_{\mathbf{T}^d}[(1-\urcorner)(n^{-1}\cdot)(-\vartheta_n+\lambda)^{-1}(\cdot)]\|_{L^1(\mathbf{T}^d)}\|\xi^n\|_{L^\infty(\mathbf{T}^d)} \\ &\lesssim n^{-2}\|\mathscr{F}^{-1}_{\mathbf{R}^d}[(1-\urcorner)(n^{-1}\cdot)(-\widehat{\chi}^2+1+n^{-2}\lambda)^{-1}(n^{-1}\cdot)]\|_{L^1(\mathbf{R}^d)}\|\xi^n\|_{L^\infty(\mathbf{T}^d)}\end{aligned} \tag{IV.15}$$

where we applied the Poisson summation formula of Lemma I.1.1. Note that

$$\begin{aligned}&\|\mathscr{F}^{-1}_{\mathbf{R}^d}[(1-\urcorner)(n^{-1}\cdot)(-\widehat{\chi}^2+1+n^{-2}\lambda)^{-1}(n^{-1}\cdot)]\|_{L^1(\mathbf{R}^d)}\\ &\leq \Big\|\mathscr{F}^{-1}_{\mathbf{R}^d}\Big[\frac{1-\urcorner(n^{-1}\cdot)}{1+n^{-2}\lambda} + (1-\urcorner)(n^{-1}\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\Big](n^{-1}\cdot)\Big]\Big\|_{L^1(\mathbf{R}^d)}\\ &\leq \Big\|\mathscr{F}^{-1}_{\mathbf{R}^d}\Big[\frac{1-\urcorner(n^{-1}\cdot)}{1+n^{-2}\lambda}\Big]\Big\|_{L^1(\mathbf{R}^d)} + \Big\|\mathscr{F}^{-1}_{\mathbf{R}^d}\Big[(1-\urcorner)(n^{-1}\cdot)\Big[\frac{1}{-\widehat{\chi}^2+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\Big](n^{-1}\cdot)\Big]\Big\|_{L^1(\mathbf{R}^d)}\end{aligned}$$

The first summand is bounded in $L^1(\mathbf{R}^d)$ uniformly over n and λ (with some abuse of notation for the Dirac δ function). As for the second summand observe that, for some $c > 0$:

$$\begin{aligned}
&\left\| \mathscr{F}^{-1}_{\mathbf{R}^d}\left[(1-\daleth)(n^{-1}\cdot)\left[\frac{1}{-\widehat{\chi}^2+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\right](n^{-1}\cdot)\right]\right\|_{L^1(\mathbf{R}^d)} \\
&\quad \le \sup_{x\in\mathbf{R}^d}(1+|x|^2)^{\frac{d+1}{2}}\left|\int_{\mathbf{R}^d} e^{2\pi\iota\langle x,k\rangle}(1-\daleth(k))\left[\frac{1}{-\widehat{\chi}^2(k)+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\right]\mathrm{d}k\right| \\
&\quad \lesssim \sum_{0\le|\alpha|\le 2d}\int_{\mathbf{R}^d}\left|\left[\partial^\alpha\left(\frac{1}{-\widehat{\chi}^2(k)+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\right)\right]1_{\{|k|\ge c\}}\right|\mathrm{d}k,
\end{aligned}$$

where with the sum we indicate all partial derivatives up to order $2d$. Now this term can be bounded by Lemma III.6.1. Let us show this for $\alpha = 0$ (the other cases are similar), where by a Taylor expansion:

$$\begin{aligned}
\int_{\mathbf{R}^d}\left|\frac{1}{-\widehat{\chi}^2(k)+1+n^{-2}\lambda}-\frac{1}{1+n^{-2}\lambda}\right|1_{\{|k|\ge c\}}\,\mathrm{d}k &\lesssim_c \left(\frac{1}{1+n^{-2}\lambda}\right)^2\int_{\mathbf{R}^d}\hat{\chi}^2(k)1_{\{|k|\ge c\}}\,\mathrm{d}k \\
&\lesssim \int_{\mathbf{R}^d}\frac{1}{1+|k|^{d+1}}\,\mathrm{d}k < \infty.
\end{aligned}$$

Combining the last two observations with (IV.15) leads to

$$\sup_{\lambda\geqslant 1}\|\mathscr{Q}_n X_{n,\lambda}\|_{L^\infty(\mathbf{T}^d)} \lesssim n^{-2}\|\xi^n\|_{L^\infty(\mathbf{T}^d)} \lesssim n^{-2+\frac{d}{2}},$$

which is of the required order.

Step 3: Bounds on $\xi^n \odot \Pi_n^2 X_{n,\lambda}$. We now consider the bound on $\xi^n \odot \Pi^2_n X_{n,\lambda}$, starting with $\lambda = 1$: at the end of this step we explain how to obtain a bound uniformly over λ at the cost of a small explosion in λ. In this computation it is important to note that $d = 2$.

Define $\psi_0(k_1,k_2)$ and $\widehat{\xi}_n(k)$ as

$$\psi_0(k_1,k_2) := \sum_{|i-j|\le 1}\varrho_i(k_1)\varrho_j(k_2), \qquad \widehat{\xi}_n(k) := \mathscr{F}_{\mathbf{T}^d}\xi^n(k).$$

Then

$$\begin{aligned}
\mathbb{E}\left[\widehat{\xi}_n(k_1)\widehat{\xi}_n(k_2)\right] &= \int_{(\mathbf{T}^2)^2} e^{-2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\chi_{Q_n(x_1)}(x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2 \\
&= \int_{\mathbf{T}^2} e^{-2\pi\iota(k_1+k_2)\cdot x_1}\widehat{\chi}_Q(n^{-1}k_2)\,\mathrm{d}x_1 = \widehat{\chi}_Q(n^{-1}k_1)1_{\{k_1+k_2=0\}}.
\end{aligned}$$

Hence to compute the renormalisation constant observe that

$$\begin{aligned}
c_n = \mathbb{E}\left[\xi^n\odot\Pi_n^2 X_{n,1}(x)\right] &= \int_{(\mathbf{Z}^2)^2} e^{2\pi\iota(k_1+k_2)\cdot x}\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(n^{-1}k_2)}{-\vartheta_n(k_2)+1}\mathbb{E}\left[\widehat{\xi}_n(k_1)\widehat{\xi}_n(k_2)\right]\,\mathrm{d}k_1\,\mathrm{d}k_2 \\
&= \int_{\mathbf{Z}^2}\frac{\widehat{\chi}^2(n^{-1}k)\widehat{\chi}_Q(n^{-1}k)}{-\vartheta_n(k)+1}\,\mathrm{d}k.
\end{aligned}$$

A similar calculation shows that actually $c_n = \mathbb{E}\Big[\xi^n \Pi_n^2 X_{n,1}\Big]$ and the asymptotic $c_n \simeq \log n$ follows from a manipulation of the sum.

We turn our attention to a bound for $\|\xi^n \diamond \Pi_n^2 X_{n,1}\|_{\mathscr{C}^{-\frac{\kappa}{2}}}$. As before, for $p \geq 2$, consider

$$\begin{aligned}\mathbb{E}\|\xi^n \odot X_{n,1} - c_n\|_{B^\alpha_{p,p}}^p &= \sum_{j\geq -1} 2^{\alpha j p}\mathbb{E}\|\Delta_j(\xi^n \odot X_{n,1} - c_n 1_{j=-1})\|^p_{L^p(\mathbf{T}^d)} \\ &= \sum_{j\geq -1} 2^{\alpha j p}\int_{\mathbf{T}^d} \mathbb{E}|\Delta_j(\xi^n \odot X_{n,1} - c_n 1_{j=-1})|^p(x)\,\mathrm{d}x.\end{aligned} \tag{IV.16}$$

It is now convenient to introduce the notation:

$$\mathscr{K}^n_m(x) = \mathscr{F}^{-1}_{\mathbf{T}^2}\bigg(\varrho_m(\cdot)\frac{\widehat{\chi}^2(n^{-1}\cdot)}{-\vartheta_n(\cdot)+1}\bigg)(x).$$

Then the integrand in (IV.16) can be written as

$$\begin{aligned}&\mathbb{E}\Big[|\Delta_j(\xi^n \odot \Pi^2_n X_{n,1})(x) - c_n 1_{\{j=-1\}}|^p\Big] \\ &\quad = \mathbb{E}\Big[|\Delta_j(\xi^n \odot \Pi^2_n X_{n,1})(x) - \mathbb{E}\Delta_j(\xi^n \odot \Pi^2_n X_{n,1})(x)|^p\Big] \\ &\quad = \mathbb{E}\bigg|\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\bigg(\int_{(\mathbf{T}^2)^2} K_l(y-z_1)\mathscr{K}^n_m(y-z_2)\xi^n(z_1)\diamond\xi^n(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\bigg)\mathrm{d}y\bigg|^p,\end{aligned} \tag{IV.17}$$

where, conveniently:

$$\xi^n(z_1)\diamond\xi^n(z_2) = \xi^n(z_1)\xi^n(z_2) - \mathbb{E}\Big[\xi^n(z_1)\xi^n(z_2)\Big].$$

Now we can write (IV.17) as a discrete stochastic integral and apply [MP19, Lemma 5.1] to obtain

$$\begin{aligned}&\mathbb{E}\bigg|\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\sum_{x_1,x_2\in\mathbf{Z}^2_n\cap\mathbf{T}^2}\bigg(\int_{Q_n(x_1)\times Q_n(x_2)} K_l(y-z_1)\mathscr{K}^n_m(y-z_2)\mathrm{d}z_1\,\mathrm{d}z_2\bigg)\xi^n(x_1)\diamond\xi^n(x_2)\,\mathrm{d}y\bigg|^p \\ &\lesssim \bigg[\sum_{x_1,x_2\in\mathbf{Z}^2_n\cap\mathbf{T}^2} n^{-2d}\bigg|\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1}\bigg(\fint_{Q_n(x_1)\times Q_n(x_2)} K_l(y-z_1)\mathscr{K}^n_m(y-z_2)\mathrm{d}z_1\mathrm{d}z_2\bigg)\mathrm{d}y\bigg|^2\bigg]^{p/2} \\ &= \bigg[\sum_{x_1,x_2\in\mathbf{Z}^2_n\cap\mathbf{T}^2} n^{-2d}\bigg|\fint_{Q_n(x_1)\times Q_n(x_2)}\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1} K_l(y-z_1)\mathscr{K}^n_m(y-z_2)\,\mathrm{d}y\,\mathrm{d}z_1\,\mathrm{d}z_2\bigg|^2\bigg]^{p/2} \\ &\leq \bigg[\int_{(\mathbf{T}^2)^2}\bigg|\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1} K_l(y-z_1)\mathscr{K}^n_m(y-z_2)\,\mathrm{d}y\bigg|^2\mathrm{d}z_1\,\mathrm{d}z_2\bigg]^{p/2},\end{aligned}$$

where the last step is an application of Jensen's inequality. Now, via Parseval's Theorem, the latter is bounded by

$$\begin{aligned}&\bigg[\int_{(\mathbf{Z}^2)^2}\bigg|\int_{\mathbf{T}^2} K_j(x-y)\sum_{|l-m|\leq 1} e^{2\pi\iota k_1\cdot y}\varrho_l(k_1)e^{2\pi\iota k_2\cdot y}\varrho_m(k_2)\frac{\widehat{\chi}^2(n^{-1}k_2)}{-\vartheta_n(k_2)+1}\,\mathrm{d}y\bigg|^2\mathrm{d}k_1\,\mathrm{d}k_2\bigg]^{p/2} \\ &\qquad = \bigg[\int_{(\mathbf{Z}^2)^2}\bigg|e^{2\pi\iota(k_1+k_2)\cdot x}\varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{\widehat{\chi}^2(n^{-1}k_2)}{-\vartheta_n(k_2)+1}\bigg|^2\mathrm{d}k_1\,\mathrm{d}k_2\bigg]^{p/2}.\end{aligned}$$

By Lemma III.6.1:

$$\frac{\widehat{\chi}^2(n^{-1}k)}{-\vartheta_n(k)+1} \lesssim \frac{\widehat{\chi}^2(n^{-1}k)}{|k|^2+1} 1_{\{|k \lesssim n|\}} + \frac{|k|^{-3}}{1} 1_{\{|k| \gtrsim n\}} \lesssim \frac{1}{1+|k|^2}.$$

Finally, taking into account the supports of the functions,

$$\left[\int_{(\mathbf{Z}^2)^2} \left| \varrho_j(k_1+k_2)\psi_0(k_1,k_2)\frac{1}{1+|k_2|^2}\right|^2 \mathrm{d}k_1\,\mathrm{d}k_2 \right]^{p/2} \lesssim \left[2^{j2d}2^{-4j}\right]^{p/2} \leq 1,$$

which provides a bound of the required order. This concludes the proof of the required bound in the case $\lambda = 1$. For general $\lambda \geqslant 1$ we observe that

$$\xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n = \xi^n \odot \Pi_n^2(X_{n,\lambda} - X_{n,1}) + \xi^n \odot \Pi_n^2 X_{n,\lambda} - c_n.$$

To complete the proof of our result it now suffices to show that

$$\mathbb{E}\sup_{\lambda \geqslant 1} \lambda^{-\frac{\kappa}{4}} \|\xi^n \odot \Pi_n^2(X_{n,\lambda} - X_{n,1})\|_{\mathscr{C}^{-\frac{\kappa}{2}}} < \infty.$$

For this purpose we observe that by a resolvent identity:

$$\begin{aligned} X_{n,\lambda} - X_{n,1} &= \left[(-\mathscr{A}_n+\lambda)^{-1} - (-\mathscr{A}_n+1)^{-1}\right]\xi^n \\ &= (1-\lambda)(-\mathscr{A}_n+\lambda)^{-1}(-\mathscr{A}_n+1)^{-1}\xi^n. \end{aligned}$$

Now we can apply the elliptic Schauder estimates of Proposition III.6.6 to obtain:

$$\begin{aligned} \lambda^{-\frac{\kappa}{4}}\|\Pi_n^2\mathscr{P}_n(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{1+\frac{\kappa}{4}}} &\lesssim \lambda^{1-\frac{\kappa}{4}}\|(-\mathscr{A}_n+\lambda)^{-1}X_{n,1}\|_{\mathscr{C}^{1+\frac{\kappa}{4}}} \\ &\lesssim \|X_{n,1}\|_{\mathscr{C}^{1-\frac{\kappa}{4}}} \\ &\lesssim \|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{4}}}. \end{aligned}$$

And on small scales, using the regularizing properties of Π_n^2 from Corollary III.7.4:

$$\begin{aligned} \lambda^{-\frac{\kappa}{4}}\|\Pi_n^2\mathscr{Q}_n(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{1+\frac{\kappa}{4}}} &\lesssim \lambda^{-\frac{\kappa}{4}} n^{2-\frac{\kappa}{4}}\|\mathscr{Q}_n(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}} \\ &\lesssim \lambda^{1-\frac{\kappa}{4}} n^{2-\frac{\kappa}{4}}\|\mathscr{Q}_n(-\mathscr{A}_n+\lambda)^{-1}X_{n,1}\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}} \\ &\lesssim n^{2-\frac{3\kappa}{4}}\|\mathscr{Q}_{n-k_0}X_{n,1}\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}} \\ &\lesssim n^{-\frac{3\kappa}{4}}\|\xi^n\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}}. \end{aligned}$$

Here we have chosen a deterministic $k_0 \in \mathbf{N}$ (uniformly over n) such that $\mathscr{Q}_n\mathscr{Q}_{n-k_0} = \mathscr{Q}_n$. Hence overall we obtain:

$$\begin{aligned} \mathbb{E}\sup_{\lambda\geqslant 1}\lambda^{-\frac{\kappa}{4}}\|\xi^n \odot \Pi_n^2(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{-\frac{\kappa}{2}}} &\lesssim \mathbb{E}\sup_{\lambda\geqslant 1}\lambda^{-\frac{\kappa}{4}}\|\xi^n \odot \Pi_n^2(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{\frac{\kappa}{8}}} \\ &\lesssim \mathbb{E}\|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{8}}} \sup_{\lambda\geqslant 1}\lambda^{-\frac{\kappa}{4}}\|\Pi_n^2(X_{n,\lambda}-X_{n,1})\|_{\mathscr{C}^{1+\frac{\kappa}{4}}} \\ &\lesssim \mathbb{E}\left[\|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{8}}}\left(\|\xi^n\|_{\mathscr{C}^{-1-\frac{\kappa}{4}}} + n^{-\frac{\kappa}{2}-\frac{\kappa}{4}}\|\xi^n\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}}\right)\right] \\ &\lesssim 1, \end{aligned}$$

where the last averages is bounded by the arguments presented in Step 1 (up to changing κ). With this we have concluded the proof of the regularity bound. We are left with a discussion of the convergence.

Step 4. What we established so far implies tightness of the following sequences of random variables in their respective spaces:

$$\xi^n \in \mathscr{C}^{-\frac{d}{2}-\kappa}, \qquad \mathscr{P}_n X_{n,\lambda} \in \mathscr{C}^{1-\kappa}, \qquad \xi^n \diamond \Pi_n^2 X_{n,\lambda} \in \mathscr{C}^{-\kappa}.$$

The next step is to show that the limiting points of ξ^n and $\xi^n \diamond \Pi_n^2 X_{n,\lambda}$ are unique in distribution. In particular, in view of Proposition III.6.6, this would imply weak convergence also of $\mathscr{P}_n X_{n,\lambda}$. In the last step we will address the almost sure convergence and the almost sure uniform bound.

Convergence of ξ^n to space time white noise ξ is an instance of central limit theorem (notice the normalization of variance in Assumption III.3.2). We therefore focus our attention on the more involved Wick product $\xi^n \diamond X_{n,\lambda}$. Now, the deterministic bounds at the end of Step 3 show that the convergences

$$\xi^n \to \xi \quad \text{in} \quad \mathscr{C}^{-1-\kappa}, \qquad \xi^n \diamond \Pi_n^2 X_{n,1} \to \xi \diamond X_1 \quad \text{in} \quad \mathscr{C}^{-\kappa}$$

for any $\kappa > 0$ imply also the convergence of $\xi^n \diamond \Pi_n^2 X_{n,\lambda}$ for general $\lambda \geqslant 1$. Hence we can restrict to discussing the case $\lambda = 1$. For fixed $\varphi \in \mathscr{S}(\mathbf{T}^2)$

$$\begin{aligned}
&\langle \varphi, \xi^n \diamond X_{n,1} \rangle \\
&= \int_{\mathbf{T}^2} \varphi(y) \sum_{|l-m|\le 1} \sum_{x_1,x_2 \in \mathbf{Z}_n^2 \cap \mathbf{T}^2} \left(\int_{Q_n(x_1)\times Q_n(x_2)} K_l(y-z_1)\mathscr{K}_m^n(y-z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2 \right) \xi^n(x_1) \diamond \xi^n(x_2)\,\mathrm{d}y \\
&= \sum_{x_1,x_2 \in \mathbf{Z}_n^2} \Big\langle \varphi(\cdot), \sum_{|l-m|\le 1} \Pi_n K_l(\cdot - x_1) \Pi_n \mathscr{K}_m^n(\cdot - x_2) \Big\rangle \xi^n(x_1) \diamond \xi^n(x_2).
\end{aligned}$$

Consider a map $L_n : \left(\mathbf{Z}_n^2\right)^2 \to \mathbf{R}$ defined by

$$L_n(x_1,x_2) := \langle \varphi(\cdot), \sum_{|l-m|\le 1} \Pi_n^Q K_l(\cdot - x_1) \Pi_n^Q \mathscr{K}_m(\cdot - x_2) \rangle 1_{\{(x_1,x_2)\in \mathbf{T}^2 \times \mathbf{T}^2\}}.$$

This definition naturally extends to $n = \infty$, where L maps $(\mathbf{R}^2)^2$ to $\mathbf{R}$. Our goal is to show that

$$\sum_{(x_1,x_2)\in(\mathbf{Z}_n^2)^2} L_n(x_1,x_2)\xi^n(x_1) \diamond \xi^n(x_2) \to \int_{(\mathbf{R}^2)^2} L(x_1,x_2)\xi(\mathrm{d}x_1) \diamond \xi(\mathrm{d}x_2), \tag{IV.18}$$

where convergence holds in distribution and the limit is interpreted as an iterated stochastic integral in the second Wiener-Itô chaos. It is sufficient to verify the assumptions of [MP19, Lemma 5.4]. That is, we have to show that there exists a $g \in L^2((\mathbf{R}^2)^2)$ such that:

$$\sup_{n\in\mathbf{N}} |1_{(n\mathbf{T}^2)^2} \mathscr{F}_{(\mathbf{Z}_n^2)^2} L_n| \le g, \qquad \lim_{n\to\infty} \|1_{(n\mathbf{T}^2)^2} \mathscr{F}_{(\mathbf{Z}_n^2)^2} L_n - \mathscr{F}_{(\mathbf{R}^2)^2} L\|_{L^2((\mathbf{R}^2)^2)} = 0$$

For this purpose we calculate

$$
\begin{aligned}
&1_{(n\mathbf{T}^2)^2}\mathscr{F}_{(\mathbf{Z}_n^2)^2}L_n(k_1,k_2)\\
&= 1_{(n\mathbf{T}^2)^2}(k_1,k_2)\int\limits_{(\mathbf{Z}_n^2\cap\mathbf{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\langle\varphi(\cdot),\sum_{|l-m|\le 1}\Pi_n^Q K_l(\cdot-x_1)\Pi_n^Q\mathscr{K}_m(\cdot-x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= 1_{(n\mathbf{T}^2)^2}(k_1,k_2)\int\limits_{(\mathbf{T}^2)^2} e^{2\pi\iota(k_1\cdot x_1+k_2\cdot x_2)}\langle\varphi(\cdot),\sum_{|l-m|\le 1}K_l(\cdot-x_1)\mathscr{K}_m(\cdot-x_2)\rangle\,\mathrm{d}x_1\,\mathrm{d}x_2\\
&= 1_{(n\mathbf{T}^2)^2}(k_1,k_2)\int_{\mathbf{T}^2}\varphi(y)e^{2\pi\iota(k_1+k_2)\cdot y}\sum_{|l-m|\le 1}\varrho_l(-k_1)\varrho_m(-k_2)\frac{\widehat{\chi}^2(-n^{-1}k_2)}{-\vartheta_n(-k_2)+1}\,\mathrm{d}y\\
&= 1_{(n\mathbf{T}^2)^2}(k_1,k_2)(\mathscr{F}_{\mathbf{T}^2}\varphi)(k_1+k_2)\sum_{|l-m|\le 1}\varrho_l(k_1)\varrho_m(k_2)\frac{\widehat{\chi}^2(n^{-1}k_2)}{-\vartheta_n(k_2)+1},
\end{aligned}
$$

so that the required assumptions are naturally satisfied. Since φ is smooth, the latter term is bounded in L^2, uniformly over n. In particular (IV.18) follows. Hence the distribution of any limit point of $\langle\varphi,\xi^n\diamond\Pi_n^2X_{n,1}\rangle$ is uniquely characterized and since φ is arbitrary this implies convergence in distribution of $\xi^n\diamond\Pi_n^2X_{n,1}$.

Step 5. Above we have proven that ξ^n and $\xi^n\diamond\Pi_n^2X_{n,\lambda}$ converge in distribution in $\mathscr{C}^{-1-\kappa}$ and $\mathscr{C}^{-\kappa}$ respectively. Now let us prove almost sure convergence up to changing probability space (we discuss only the case of ξ^n, since the other term can be treated similarly). We would like to apply Skorohod's representation theorem, which requires the underlying space to be separable. Unfortunately the space $\mathscr{C}^{-1-\kappa}$ is not separable, but we can embed

$$
\mathscr{C}^{-1-\kappa}\subseteq B^{-1-\kappa}_{p(\kappa),p(\kappa)}\subseteq\mathscr{C}^{-1-2\kappa}
$$

for some $p(\kappa)\in(1,\infty)$ sufficiently large. Now the space $B^{-1-\kappa}_{p(\kappa),p(\kappa)}$ is separable, so we can apply Skorohod's representation theorem to obtain almost sure convergence in $\mathscr{C}^{-1-2\kappa}$. Since κ is arbitrary this is sufficient for the required result.
The last statement we have to prove is that in this new probability space (that we call $(\overline{\Omega},\overline{\mathscr{F}},\overline{\mathbb{P}})$) we have a uniform bound for almost all $\omega\in\overline{\Omega}$:

$$
\sup_{n\in\mathbf{N}}|\!|\!|\xi_n(\omega)|\!|\!|_{n,\kappa}<\infty.
$$

Recall that

$$
\begin{aligned}
|\!|\!|\xi_n(\omega)|\!|\!|_{n,\kappa}:=\sup_{\zeta\in[0,1]}&\Big\{n^{-\zeta}\|\xi^n\|_{\mathscr{C}^{-(1-\zeta)-\frac{\kappa}{2}}}\Big\}+n^{-1}\|\xi^n\|_{L^\infty}+n^{-1-\kappa}\|\xi^n\|_{\mathscr{C}^{\kappa}_{\frac{1}{2\kappa}}}\\
&+\sup_{\lambda\geqslant 1}\Big\{n\|\mathscr{Q}_nX_{n,\lambda}\|_{L^\infty}+\lambda^{-\frac{\kappa}{4}}\|Y_{n,\lambda}\|_{\mathscr{C}^{-\frac{\kappa}{2}}}\Big\}.
\end{aligned}
$$

Now following Steps 1 and 2 we see that the bounds on $\|\xi^n(\omega)\|_\infty,\|\xi^n(\omega)\|_{\mathscr{C}^{\kappa}_{\frac{1}{2\kappa}}}$ and $\|X_{n,\lambda}(\omega)\|_\infty$ depend only on the deterministic bound $|\xi^n(\omega,x)|\leqslant 2n$ (in $d=2$), so we are left with proving:

$$
\begin{aligned}
&\sup_{n\in\mathbf{N}}\left\{\sup_{\zeta\in[0,1]} n^{-\zeta}\|\xi^n(\omega)\|_{\mathscr{C}^{-(1-\zeta)-\frac{\kappa}{2}}}+\sup_{\lambda\geqslant 1}\lambda^{-\frac{\kappa}{4}}\|\xi^n\diamond\Pi_n^2X_{n,\lambda}(\omega)\|_{\mathscr{C}^{-\frac{\kappa}{2}}}\right\}\\
&\lesssim\sup_{n\in\mathbf{N}}\left\{\sup_{\zeta\in[0,1]}(n^{-1}\|\xi^n(\omega)\|_\infty)^\zeta\|\xi^n(\omega)\|^{1-\zeta}_{\mathscr{C}^{-1-\frac{\kappa}{2(1-\zeta)}}}+\|\xi^n\diamond\Pi_n^2X_{n,1}(\omega)\|_{\mathscr{C}^{-\frac{\kappa}{2}}}\right.\\
&\qquad\left.+\sup_{\lambda\geqslant 1}\lambda^{-\frac{\kappa}{4}}\|\xi^n\odot\Pi_n^2(X_{n,\lambda}-X_{n,1})(\omega)\|_{\mathscr{C}^{-\frac{\kappa}{2}}}\right\}\\
&\lesssim\sup_{n\in\mathbf{N}}\left\{1+\|\xi^n(\omega)\|_{\mathscr{C}^{-1-\frac{\kappa}{2}}}+\|\xi^n\diamond\Pi_n^2X_{n,1}(\omega)\|_{\mathscr{C}^{-\frac{\kappa}{2}}}\right.\\
&\qquad\left.+\|\xi^n(\omega)\|_{\mathscr{C}^{-1-\frac{\kappa}{8}}}\left(\|\xi^n(\omega)\|_{\mathscr{C}^{-1-\frac{\kappa}{4}}}+n^{-\frac{\kappa}{2}-\frac{\kappa}{4}}\|\xi^n(\omega)\|_{\mathscr{C}^{-1+\frac{\kappa}{2}}}\right)\right\},
\end{aligned}
$$

where we used interpolation for the first term, as in Step 1, and the same bounds as in Step 3 for the last term. In particular now the uniform bound is a consequence of the convergence of ξ^n and $\xi^n\diamond\Pi_n^2X_{n,1}$ in the correct spaces. □

IV.2.6 Commutator estimates

This section is devoted to products of distributions and commutator estimates. Recall from Section I.1.1 that we can decompose a product of distributions in paraproducts (through the symbol $\varolessthan$) and resonant products ($\odot$). For $\varphi,\psi\in\mathscr{S}'(\mathbf{T}^d)$ set

$$
S_i\varphi:=\sum_{j=-1}^{i-1}\Delta_j\varphi,\qquad \varphi\varolessthan\psi:=\sum_{i\geq -1}S_{i-1}\varphi\Delta_i\psi,\qquad \varphi\odot\psi:=\sum_{|i-j|\leq 1}\Delta_j\varphi\Delta_i\psi,
$$

where the latter sum might not be well defined. Then, an a-priori ill-posed product of φ and ψ can be written as

$$
\varphi\cdot\psi=\varphi\varolessthan\psi+\varphi\odot\psi+\varphi\varogreaterthan\psi.
$$

The aim of this section is to deal with the following commutators.

Definition IV.2.14. *For distributions $\varphi,\psi,\sigma\in\mathscr{S}'(\mathbf{T}^d)$ we define the (a-priori ill-posed) commutators*

$$
\begin{aligned}
C^{\odot}(\varphi,\psi,\sigma)&:=\varphi\odot(\psi\varolessthan\sigma)-\psi(\varphi\odot\sigma),\\
C^{\Pi}_n(\varphi,\psi)&:=\Pi_n^2(\varphi\varolessthan\psi)-\varphi\varolessthan\Pi_n^2\psi,\\
C_{n,\lambda}(\varphi,\psi)&:=(-\mathscr{A}_n+\lambda)^{-1}(\varphi\varolessthan\psi)-\varphi\varolessthan(-\mathscr{A}_n+\lambda)^{-1}\psi.
\end{aligned}
$$

The first commutator estimate is crucial, but by now well-known (we already used it in Lemma II.3.4).

Lemma IV.2.15 ([GP15], Lemma 14)**.** *For $\varphi,\psi,\sigma\in\mathscr{S}'(\mathbf{T}^d)$, $\alpha,\beta,\gamma\in\mathbf{R}$ with $\alpha+\beta+\gamma>0$ and $p\in[1,\infty]$:*

$$
\|C^{\odot}(\varphi,\psi,\sigma)\|_{\mathscr{C}_p^{\alpha+\gamma}}\lesssim\|\varphi\|_{\mathscr{C}^\alpha}\|\psi\|_{\mathscr{C}_p^\beta}\|\sigma\|_{\mathscr{C}^\gamma}.
$$

We pass to the second commutator. Recall the operators $\mathscr{P}_n,\mathscr{Q}_n$ as in Definition III.6.3.

Lemma IV.2.16. *For $\varphi, \psi \in \mathscr{S}'(\mathbf{T}^d)$ and $\alpha \in \mathbf{R}, \beta > 0, p \in [1,\infty]$ it holds for every $\delta \in [0, \beta \wedge 1)$:*

$$\|\mathscr{P}_n C_n^{\Pi}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha+\delta}} \lesssim \|\varphi\|_{\mathscr{C}_p^{\beta}} \|\psi\|_{\mathscr{C}^{\alpha}}, \qquad \|\mathscr{Q}_n C_n^{\Pi}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha}} \lesssim n^{-\delta} \|\varphi\|_{\mathscr{C}_p^{\beta}} \|\psi\|_{\mathscr{C}^{\alpha}}.$$

Proof. Note that for any $i \geq 0$ there exists an annulus $\mathscr{A}$ (that is a set of the form $\{k \in \mathbf{R}^d \mid r \leq |k| \leq R\}$ for some $0 < r < R$) such that the Fourier transform of

$$\Pi_n^2[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi_n^2 \Delta_i \varphi$$

is supported in $2^i \mathscr{A}$. It is therefore sufficient to show that

$$\left\|\Pi_n^2[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi_n^2 \Delta_i \varphi\right\|_{L^p} \lesssim n^{-\delta} \|\varphi\|_{\mathscr{C}_p^{\beta}} \|\Delta_i \psi\|_{L^\infty}, \tag{IV.19}$$

since this implies the required bound by estimating $n^{-\delta} \lesssim 2^{-\delta i}$ for i such that $\mathscr{P}_n \Delta_i \neq 0$. To obtain (IV.19), recall the Sobolev-Slobodeckij characterization of fractional spaces of Proposition III.7.1, which implies that for $\delta \in [0, \beta \wedge 1)$

$$\begin{aligned}
\|\Pi_n^2[S_{i-1}\varphi \Delta_i \psi] - S_{i-1}\varphi \Pi_n^2 \Delta_i \varphi\|_{L^p} &\leq \left(\int_{\mathbf{T}^d} \left| \fint_{B_n(x)} [S_{i-1}\varphi(y) - S_{i-1}\varphi(x)] \Delta_i \psi(y) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{1/p} \\
&\lesssim n^{-\delta} \left(\int_{\mathbf{T}^d} \left| \fint_{B_n(x)} \frac{[S_{i-1}\varphi(y) - S_{i-1}\varphi(x)]}{|y-x|^{\delta}} \Delta_i \psi(y) \, \mathrm{d}y \right|^p \mathrm{d}x \right)^{1/p} \\
\lesssim n^{-\delta} \left(\int_{\mathbf{T}^d} \int_{B_n(x)} \frac{|S_{i-1}\varphi(y) - S_{i-1}\varphi(x)|^p}{|y-x|^{d+\delta p}} \, \mathrm{d}y \, \mathrm{d}x \right)^{1/p} & \|\Delta_i \psi\|_\infty \lesssim n^{-\delta} \|S_{i-1}\varphi\|_{\mathscr{C}_p^{\beta}} 2^{-\alpha i} \|\psi\|_{\mathscr{C}^{\alpha}},
\end{aligned}$$

where the first inequality follows by Jensen and we have used the embedding $B^{\beta}_{p,\infty} \subset B^{\delta}_{p,p}$. Now the result follows since:

$$\|S_{i-1}\varphi\|_{\mathscr{C}_p^{\beta}} \lesssim \|\varphi\|_{\mathscr{C}_p^{\beta}}.$$

This concludes the proof. □

Lemma IV.2.17. *For $\alpha \in (0,1), \beta \in \mathbf{R}, \lambda \geq 1$ and $p \in [1,\infty]$ it holds that:*

$$\|\mathscr{P}_n C_{n,\lambda}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha+\beta+2}} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}} \|\psi\|_{\mathscr{C}^{\beta}}, \qquad \forall \varphi \in \mathscr{C}_p^{\alpha}, \psi \in \mathscr{C}^{\beta}.$$

In addition there exists a $k \in \mathbf{N}$ such that for $n \geq k$

$$n^3 \|\mathscr{Q}_n C_{n,\lambda}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha+\beta-1}} \lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}} \|\mathscr{Q}_{n-k}\psi\|_{\mathscr{C}^{\beta}}, \qquad \forall \varphi \in \mathscr{C}_p^{\alpha}, \psi \in \mathscr{C}^{\beta}.$$

Proof. By the elliptic Schauder estimates in Proposition III.6.6, it is sufficient to prove that

$$\begin{aligned}
\|(-\mathscr{A}_n + \lambda)\mathscr{P}_n C_{n,\lambda}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha+\beta}} &\lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}} \|\psi\|_{\mathscr{C}^{\beta}}, \\
n\|(-\mathscr{A}_n + \lambda)\mathscr{Q}_n C_{n,\lambda}(\varphi,\psi)\|_{\mathscr{C}_p^{\alpha+\beta-1}} &\lesssim \|\varphi\|_{\mathscr{C}_p^{\alpha}} \|\mathscr{Q}_{n-k}\psi\|_{\mathscr{C}^{\beta}}.
\end{aligned}$$

In turn to obtain this bound, since the quantities below are supported in an annulus $2^i \mathscr{A}$, it suffices to estimate for a given sequence $i(n)$ such that $2^{i(n)} \simeq n$:

$$\|S_{i-1}\varphi \Delta_i \psi - (-\mathscr{A}_n + \lambda)[S_{i-1}\varphi(-\mathscr{A}_n + \lambda)^{-1} \Delta_i \psi]\|_{L^p} \lesssim 2^{-i(\alpha+\beta)} \|\varphi\|_{\mathscr{C}_p^{\alpha}} \|\psi\|_{\mathscr{C}^{\beta}}, \tag{IV.20}$$

if $i \le i(n)$, and similarly

$$\|S_{i-1}\varphi\Delta_i\psi - (-\mathscr{A}_n+\lambda)[S_{i-1}\varphi(-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi]\|_{L^p} \lesssim n^{-1}2^{-i(\alpha+\beta-1)}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\mathscr{Q}_{n-k}\psi\|_{\mathscr{C}^\beta}, \tag{IV.21}$$

if $i > i(n)$. Moreover, we can choose k such that

$$\mathscr{Q}_{n-k}\Delta_i = \Delta_i, \forall i \ge i(n), n \in \mathbf{N},$$

so that we may replace ψ by $\mathscr{Q}_{n-k}\psi$ on small scales (hence we will no longer discuss the appearance of $\mathscr{Q}_{n-k}$). To obtain these estimates, let $B_n(\varphi,\psi)$ be defined as

$$B_n(\varphi,\psi)(x) = n^2 \fint_{B_n(x)}\fint_{B_n(y)}\fint_{B_n(z)}\fint_{B_n(r)}(\varphi(s)-\varphi(x))(\psi(s)-\psi(x))\,\mathrm{d}s\,\mathrm{d}r\,\mathrm{d}z\,\mathrm{d}y.$$

Then $\mathscr{A}_n$ acting on a product can be decomposed as

$$\mathscr{A}_n(\varphi\cdot\psi) = \mathscr{A}_n(\varphi)\cdot\psi + \varphi\cdot\mathscr{A}_n(\psi) + B_n(\varphi,\psi),$$

Hence proving Equations (IV.20) and (IV.21) reduces to finding a bound for

$$\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi](-\mathscr{A}_n+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} + \|B_n(S_{i-1}\varphi,(-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi)\|_{L^p}.$$

Starting with the first term, one has:

$$\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi](-\mathscr{A}_n+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} \lesssim \|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi]\|_{L^p}\|(-\mathscr{A}_n+\lambda)^{-1}[\Delta_i\psi]\|_{L^\infty}.$$

If $i \le i(n)$, since $\alpha < 2$, one can estimate via Proposition III.6.2:

$$\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi]\|_{L^p} \le \sum_{j=-1}^{i-1}\|(-\mathscr{A}_n+\lambda)[\Delta_j\varphi]\|_{L^p} \lesssim \sum_{j=-1}^{i-1}2^{j(2-\alpha)}\|\varphi\|_{\mathscr{C}^\alpha_p} \lesssim 2^{i(2-\alpha)}\|\varphi\|_{\mathscr{C}^\alpha_p}.$$

If $i > i(n)$, following the previous calculations and using that $\alpha > 0$:

$$\begin{aligned}\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi]\|_{L^p} &\le \sum_{j=-1}^{i(n)-1}\|(-\mathscr{A}_n+\lambda)[\Delta_j\varphi]\|_{L^p} + \sum_{j=i(n)}^{i-1}\|(-\mathscr{A}_n+\lambda)[\Delta_j\varphi]\|_{L^p} \\ &\lesssim n^{(2-\alpha)}\|\varphi\|_{\mathscr{C}^\alpha_p}.\end{aligned}$$

By Proposition III.6.7 moreover

$$\|(-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi\|_{L^\infty} \lesssim \left(2^{-2i}1_{\{i\le i(n)\}} + n^{-2}1_{\{i>i(n)\}}\right)2^{-\beta i}\|\psi\|_{\mathscr{C}^\beta}.$$

Together with the previous bounds we have proven that for $i \le i(n)$:

$$\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi](-\mathscr{A}_n+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} \lesssim 2^{-i(\alpha+\beta)}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\psi\|_{\mathscr{C}^\beta},$$

and similarly (using that $\alpha < 1$) for $i > i(n)$:

$$\begin{aligned}\|(-\mathscr{A}_n+\lambda)[S_{i-1}\varphi](-\mathscr{A}_n+\lambda)^{-1}[\Delta_i\psi]\|_{L^p} &\lesssim n^{-\alpha}2^{-i\beta}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\mathscr{Q}_{n-k}\psi\|_{\mathscr{C}^\beta} \\ &\lesssim n^{-1}2^{-i(\beta+\alpha-1)}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\mathscr{Q}_{n-k}\psi\|_{\mathscr{C}^\beta},\end{aligned}$$

which are bounds of the required order for (IV.20) and (IV.21). Finally, we have to bound the term containing B_n. If $i \le i(n)$, using $\alpha < 1$ we find

$$\begin{aligned}\|B(S_{i-1}\varphi, (-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi)\|_{L^p} &\lesssim \|\nabla S_{i-1}\varphi\|_{L^p}\|\nabla(-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi\|_{L^\infty}\\ &\lesssim 2^{-i(\alpha-1)}\|\varphi\|_{\mathscr{C}^\alpha_p}2^{-i(1+\beta)}\|\psi\|_{\mathscr{C}^\beta} \lesssim 2^{-i(\alpha+\beta)}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\psi\|_{\mathscr{C}^\beta},\end{aligned}$$

whereas if $i > i(n)$

$$\begin{aligned}\|B_n(S_{i-1}\varphi, (-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi)\|_{L^p} &\lesssim n\|\nabla S_{i-1}\varphi\|_{L^p}\|(-\mathscr{A}_n+\lambda)^{-1}\Delta_i\psi\|_{L^\infty}\\ &\lesssim n^{-1}2^{-i(\alpha-1)}2^{-\beta i}\|\varphi\|_{\mathscr{C}^\alpha_p}\|\psi\|_{\mathscr{C}^\beta}.\end{aligned}$$

These bounds are again of the correct order for (IV.20), (IV.21) and hence the proof is complete. □

IV.3 Dirichlet boundary conditions

In this section we study PAM with Dirichlet boundary conditions on the box $[0,L]^d$, for $d = 1,2$ and arbitrary $L \in \mathbf{N}$. The continuous case was already studied in [Cv19], so we do not devote much attention to it. Our aim is rather to develop a suitable lattice discretization of the approach in [Cv19] and to recover the continuous case as the lattice become coarser.

To ease the upcoming notation we write:

$$N = 2L.$$

Consider $n \in \mathbf{N}$. Write respectively Λ_n for the discretized box and Θ_n for the discretized torus

$$\Lambda_n = \frac{1}{n}(\mathbb{Z}^d \cap [0,Ln]^d), \qquad \Theta_n = \mathbf{Z}^d_n / N\mathbf{Z}^d.$$

Define the "dual lattice" to Θ_n:

$$\Xi_n = \mathbf{Z}^d_N / n\mathbf{Z}^d, \qquad \Xi^+_n = \frac{1}{N}(\mathbb{Z}^d \cap [0,Ln]^d).$$

Then we can introduce the discrete boundaries

$$\partial\Lambda_n = \{k \in \Lambda_n \ : \ k_i = 0 \text{ for some } i \in \{1,\dots,d\}\}$$

and similarly $\partial\Xi^+_n$ (observe we only consider boundary points on the axes). Write

$$A^n_{\mathfrak{d}} = \Xi^+_n \setminus \partial\Xi^+_n, \qquad A^n_{\mathfrak{n}} = \Xi^+_n.$$

For $p \ge 1$ and any functions $\varphi, \psi \colon \Theta^n \to \mathbf{R}$, we write

$$\|\varphi\|_{L^p(\Theta^n)} = \left(\frac{1}{n^d}\sum_{x\in\Theta^n}|f(x)|^p\right)^{\frac{1}{p}}, \qquad \langle\varphi,\psi\rangle = \frac{1}{n^d}\sum_{x\in\Theta_n}\varphi(x)\psi(x).$$

As we are interested in $n \to \infty$, note that formally:

$$\Lambda_n \to [0,L]^d, \qquad \Theta_n \to \mathbf{T}_N = [-L,L]/\sim, \qquad \Xi_n \to \mathbf{Z}^d_N, \qquad \Xi^+_n \to \mathbf{N}^d_0.$$

In this sense all above notations extend to the case $n = \infty$ and we may use this notation when it is convenient.

IV.3.1 The Analytic Setting

The idea behind the approach of [Cv19] is to consider even and odd extensions of functions on Λ_n to periodic functions on Θ_n, and then to work with the more common tools from periodic paracontrolled distributions on Θ_n. It will be convenient to write, for $x \in \Lambda_n, \mathfrak{q} \in \{-1,1\}^d$:

$$\mathfrak{q} \circ x = (\mathfrak{q}_i x_i)_{i=1,\dots,d} \qquad \Pi \mathfrak{q} = \prod_{i=1}^d \mathfrak{q}_i.$$

So for $u, v \colon \Lambda_n \to \mathbf{R}$ such that $u|_{\partial \Lambda_n} \equiv 0$ we define the even and odd extensions:

$$\Pi_o u \colon \Theta_n \to \mathbf{R}, \quad \Pi_o u(\mathfrak{q} \circ x) = (\Pi \mathfrak{q}) u(x), \qquad \Pi_e v \colon \Theta_n \to \mathbf{R}, \quad \Pi_e v(\mathfrak{q} \circ x) = v(x).$$

Once we extend functions to the discrete torus, we can work with the discrete periodic Fourier transform, defined for $\varphi \colon \Theta_n \to \mathbf{R}$ by

$$\mathscr{F}_{\Theta_n} \varphi(k) = \frac{1}{n^d} \sum_{x \in \Theta_n} \varphi(x) e^{-2\pi \iota \langle x, k \rangle}, \qquad k \in \Xi_n.$$

We have a periodic, a Dirichlet and a Neumann basis, which we indicate with $\{\mathfrak{e}_k\}_{k \in \Xi_n}, \{\mathfrak{d}_k\}_{k \in \Xi_n^+ \setminus \partial \Xi_n^+}, \{\mathfrak{n}_k\}_{k \in \Xi_n^+}$ respectively. Here $\mathfrak{e}_k$ is the classical Fourier basis:

$$\mathfrak{e}_k(x) = \frac{e^{2\pi \iota \langle x, k \rangle}}{N^{\frac{d}{2}}}, \quad \text{so that} \quad \mathscr{F}_{\Theta_n} \varphi(k) = N^{\frac{d}{2}} \langle \varphi, \mathfrak{e}_k \rangle, \ \ k \in \Xi_n,$$

while the Dirichlet and Neumann bases consist of sine and cosine functions respectively:

$$\mathfrak{d}_k(x) = \frac{1}{N^{\frac{d}{2}}} \prod_{i=1}^d 2 \sin(2\pi k_i x_i), \ k \in A^n_{\mathfrak{d}} \ \ \mathfrak{n}_k(x) = \frac{1}{N^{\frac{d}{2}}} \prod_{i=1}^d 2^{1 - 1_{\{k_i = 0\}}/2} \cos(2\pi k_i x_i), \ k \in A^n_{\mathfrak{n}}.$$

To the previous explicit expressions we will prefer the following alternative characterization, with $\nu_k = 2^{-\#\{i : k_i = 0\}/2}$:

$$\Pi_o \mathfrak{d}_k = \iota^d \sum_{\mathfrak{q} \in \{-1,1\}^d} \Pi \mathfrak{q} \cdot \mathfrak{e}_{\mathfrak{q} \circ k}, \quad \forall k \in A^n_{\mathfrak{d}}, \quad \Pi_e \mathfrak{n}_k = \nu_k \sum_{\mathfrak{q} \in \{-1,1\}^d} \mathfrak{e}_{\mathfrak{q} \circ k}, \quad \forall k \in A^n_{\mathfrak{n}}.$$

For $\mathfrak{l} \in \{\mathfrak{d}, \mathfrak{n}\}$ and $n < \infty$ write $\mathscr{S}'_{\mathfrak{l}}(\Lambda_n) = \mathrm{span}\{\mathfrak{l}_k\}_{k \in A^n_{\mathfrak{l}}}$ for the space of discrete distributions. In the continuous case we define distributions via formal Fourier series:

$$\mathscr{S}'_{\mathfrak{l}}([0,L]^d) = \Big\{ \sum_{k \in A^\infty_{\mathfrak{l}}} \alpha_k \mathfrak{l}_k \ : \ |\alpha_k| \leq C(1 + |\kappa|^\gamma), \text{ for some } C, \gamma \geq 0 \Big\}.$$

Now let us introduce Littlewood-Paley blocks on the lattice with Dirichlet boundary conditions, in order to control products between distributions on Λ_n uniformly in n. Consider an *even* function $\sigma \colon \Xi_n \to \mathbb{R}$. Then for $\varphi \in \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$ we define the *Fourier multiplier*:

$$\sigma(D)\varphi = \sum_{k \in A^n_{\mathfrak{l}}} \sigma(k) \langle \varphi, \mathfrak{l}_k \rangle \mathfrak{l}_k.$$

Upon extending φ in an even or odd fashion we recover the classical notion of Fourier multiplier (namely on a torus: $\sigma(D)\varphi = \mathscr{F}^{-1}_{\Theta_n}(\sigma \mathscr{F}_{\Theta_n} \varphi)$), since $\Pi_o\big(\sigma(D)\varphi\big) = \sigma(D)\Pi_o \varphi$ and verbatim for Π_e: here we use that σ is an even function. Consider the dyadic partition of the unity $\{\varrho_j\}_{j\geq -1}$ we fixed at the beginning of this work and define $j_n = \min\{j \geq -1 \colon \mathrm{supp}(\varrho_j) \not\subseteq (-\frac{n}{2}, \frac{n}{2})^d\}$ (by our assumption on the support of ϱ_{-1} and ϱ_0 we have $j_n \geq 1$), so as to define for $\varphi \in \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$:

$$\Delta^n_j \varphi = \varrho_j(D)\varphi \text{ for } j < j_n, \qquad \Delta^n_{j_n}\varphi = \Big(1 - \sum_{-1\leq j<j_n} \varrho_j(D)\Big)\varphi.$$

This allows one to define the *paraproduct* and the *resonant product* of two distributions respectively:

$$\varphi \varolessthan \psi = \sum_{-1\leq j\leq j_n} \sum_{-1\leq i\leq j-1} \Delta^n_i \varphi \Delta^n_j \psi, \qquad \varphi \odot \psi = \sum_{\substack{|i-j|\leq 1 \\ -1\leq i,j\leq j_n}} \Delta^n_i \varphi \Delta^n_j \psi.$$

In view of the previous calculations this is coherent with the definition on the lattice we used in Chapter II (with the difference that here we consider periodic functions, whereas in Chapter II we consider distributions on $\mathbf{R}^d$), in the sense that:

$$\Pi_o\Big(\Delta^n_j\varphi\Big) = \Delta^n_j \Pi_o \varphi, \qquad \Pi_e\Big(\Delta^n_j\varphi\Big) = \Delta^n_j \Pi_e \varphi, \ \ -1 \leq j \leq j_n.$$

We then define Dirichlet and Neumann Besov spaces via the following norms:

$$\|u\|_{B^{\mathfrak{d},\alpha}_{p,q}(\Lambda_n)} = \|\Pi_o u\|_{B^{\alpha}_{p,q}(\Theta_n)} = \|(2^{\alpha j}\|\Delta_j \Pi_o u\|_{L^p(\Theta_n)})_j\|_{\ell^q(\leq j_n)} \qquad u \in \mathscr{S}'_{\mathfrak{d}}(\Lambda_n)$$

and similarly for $\mathfrak{n}$ upon replacing Π_o with Π_e. For brevity we write

$$\mathscr{C}^{\alpha}_{\mathfrak{l},p}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{p,\infty}(\Lambda_n) \qquad \mathscr{C}^{\alpha}_{\mathfrak{l}}(\Lambda_n) = B^{\mathfrak{l},\alpha}_{\infty,\infty}(\Lambda_n), \qquad \mathfrak{l} \in \{\mathfrak{n},\mathfrak{d}\}.$$

We also write $\|u\|_{L^p_{\mathfrak{d}}(\Lambda_n)} = \|\Pi_o u\|_{L^p(\Theta_n)}$ and $\|u\|_{L^p_{\mathfrak{n}}(\Lambda_n)} = \|\Pi_e u\|_{L^p(\Theta_n)}$: this is coherent with the definition of the Besov spaces, but the scaling is slightly unnatural, since $\|1\|_{L^p_{\mathfrak{l}}(\Lambda_n)} = (2L)^{\frac{d}{p}}$. Having introduced Besov spaces we can define the spaces of time-dependent functions $\mathscr{M}^{\gamma}\mathscr{C}^{\alpha}_{\mathfrak{l},p}$ and $\mathscr{L}^{\gamma,\alpha}_{\mathfrak{l},\alpha}$ for $\mathfrak{l} \in \{\mathfrak{d},\mathfrak{n}\}$ as in I. The last ingredient to understand products of distributions with Dirichlet boundary conditions is the following pair of identities:

$$\Pi_e(\varphi\psi) = \Pi_e\varphi\Pi_e\psi, \qquad \Pi_o(\varphi\psi) = \Pi_o\varphi\Pi_e\psi. \tag{IV.22}$$

To solve equations with Dirichlet boundary conditions, introduce the Laplace operators for $\varphi \colon \Lambda_n \to \mathbb{R}$, $\psi \colon \Theta_n \to \mathbf{R}$:

$$\Delta^n \psi(x) = n^2 \sum_{|x-y|=n^{-1}} \psi(y) - \psi(x), \qquad \Delta^n_{\mathfrak{d}}\varphi = (\Delta^n \Pi_o \varphi)|_{\Lambda_n}, \qquad \Delta^n_{\mathfrak{n}}\varphi = (\Delta^n \Pi_e \varphi)|_{\Lambda_n}.$$

The latter two operators are defined only on the domain $\mathrm{Dom}(\Delta^n_{\mathfrak{l}}) = \mathscr{S}'_{\mathfrak{l}}(\Lambda_n)$. A direct computation (cf. [MP19, Section 3]) then shows that one can represent both laplacians as Fourier multipliers:

$$-\Delta^n_{\mathfrak{l}} \mathfrak{l}_k = l^n(k)\mathfrak{l}_k, \qquad l^n(k) = -\sum_{j=1}^{d} 2n^2\Big(\cos(2\pi k_j/n) - 1\Big), \qquad \text{for } \mathfrak{l} \in \{\mathfrak{d},\mathfrak{n}\}.$$

Note that l^n is an even function in k, so all the remarks from the previous discussion apply. In the continuous case we use the classical Laplacian: the boundary condition is encoded in the domain. We write $\Delta_{\mathfrak{l}}$ for the Laplacian on $\mathscr{S}'_{\mathfrak{l}}([0,L]^d)$. We introduce Dirichlet and Neumann extension operators as follows:

$$\mathscr{E}^n_{\mathfrak{d}} u = \mathscr{E}^n(\Pi_o u)\Big|_{[0,L]^d}, \qquad \mathscr{E}^n_{\mathfrak{n}} u = \mathscr{E}^n(\Pi_e u)\Big|_{[0,L]^d}, \qquad \text{for } n < \infty,$$

where the periodic extension operator $\mathscr{E}^n$ is defined as in Chapter I. These functions are well-defined since for fixed n the extension $\mathscr{E}^n(\cdot)$ is a smooth function. Moreover a simple calculation shows that

$$\Pi_o(\mathscr{E}^n_{\mathfrak{d}} u) = \mathscr{E}^n(\Pi_o u), \qquad \Pi_e(\mathscr{E}^n_{\mathfrak{n}} u) = \mathscr{E}^n(\Pi_e u). \tag{IV.23}$$

IV.3.2 Solving the Equation

We now study Equation (8) in dimension $d = 1,2$ on a box. We recall Assumption II.2.1 on the random environment.

Assumption IV.3.1. *For every $n \in \mathbf{N}$, $\{\xi^n(x)\}_{x \in \mathbf{Z}^d_n}$ is a set of i.i.d random variables with:*

$$n^{-d/2}\xi^n(x) \sim \Phi, \tag{IV.24}$$

for a probability distribution Φ on $\mathbf{R}$ with finite moments of every order and which satisfies

$$\mathbb{E}[\Phi] = 0, \;\; \mathbb{E}[\Phi^2] = 1.$$

These probabilistic assumptions guarantee certain analytical properties which are highlighted in the next lemma. For convenience, in the remainder of this work we shift Λ_n to be centered around the origin and identify it with a subset of $[-L/2, L/2]^d$, naturally extending the results of the previous section to this set. To be precise, for $L \in 2\mathbf{N}$ we redefine $\Lambda_n = \{x \in \mathbf{Z}^d_n \; : \; x \in [-L/2, L/2]^d\}$. Moreover, in the following let χ be the same cut-off function as in Assumption II.2.4. In dimension $d = 2$ define the renormalisation constant:

$$\kappa_n = \int_{\mathbf{T}^2_n} \frac{\chi(k)}{l^n(k)} \,\mathrm{d}k \simeq \log(n). \tag{IV.25}$$

Remark IV.3.2. *The renormalisation constant κ_n is identical to the one in* (II.2), *that is used for the renormalisation of PAM on $\mathbf{R}^2$. This is very important to us: if the renormalisation constants were different we would not be able to compare the rSBM on $\mathbf{R}^2$ with the one on $(-L/2, L/2)^2$ with Dirichlet boundary conditions.*

Lemma IV.3.3. *Let $\{\overline{\xi}^n(x)\}_{x \in \mathbf{Z}^d_n, n \in \mathbf{N}}$ satisfy Assumption IV.3.1. There exists a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ supporting for all $n \in \mathbf{N}$ random variables c_n, ν and $\{\xi^n(x)\}_{x \in \mathbf{Z}^d_n}, \xi \in \mathscr{S}'(\mathbf{R}^d)$ such that ξ is space white noise on $\mathbf{R}^d$ and $\xi^n = \overline{\xi}^n$ in distribution.*

Such random variables satisfy the following requirements. Let $X^n_{\mathfrak{n}}$ be the (random) solution to the equation $-\Delta^n_{\mathfrak{n}} X^n_{\mathfrak{n}} = \chi(D)\xi^n$. For every $\omega \in \Omega$ and α satisfying

$$\alpha \in (1, \tfrac{3}{2}) \text{ in } d = 1, \qquad \alpha \in (\tfrac{2}{3}, 1) \text{ in } d = 2, \tag{IV.26}$$

the following holds for all $L \in 2\mathbf{N}$:

(i) $\xi(\omega) \in \mathscr{C}_{\mathfrak{n}}^{\alpha-2}([-L/2,L/2]^d)$ *as well as* $\sup_n \|\xi^n(\omega)\|_{\mathscr{C}_{\mathfrak{n}}^{\alpha-2}(\Lambda_n)} < +\infty$ *and* $\mathscr{E}_{\mathfrak{n}}^n \xi^n(\omega) \to \xi(\omega)$ *in* $\mathscr{C}_{\mathfrak{n}}^{\alpha-2}([-L/2,L/2]^d)$.

(ii) *For any* $\varepsilon > 0$ *(with* $(\cdot)_+ = \max\{0,\cdot\}$*):*

$$\sup_n \|n^{-d/2}\xi_+^n(\omega)\|_{\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)} + \sup_n \|n^{-d/2}|\xi^n(\omega)|\|_{\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)} + \sup_n \|n^{-d/2}\xi_+^n(\omega)\|_{L^2_{\mathfrak{n}}(\Lambda_n)} < \infty.$$

Moreover, $\nu(\omega) \geq 0$ *and* $\mathscr{E}_{\mathfrak{n}}^n n^{-d/2}\xi_+^n(\omega) \to \nu(\omega)$, $\mathscr{E}_{\mathfrak{n}}^n n^{-d/2}|\xi^n(\omega)| \to 2\nu(\omega)$ *in* $\mathscr{C}_{\mathfrak{n}}^{-\varepsilon}(\Lambda_n)$.

(iii) *If* $d = 2$, *in addition,* $n^{-d/2}c_n(\omega) \to 0$ *and there exist distributions* $X_{\mathfrak{n}}(\omega), X_{\mathfrak{n}} \diamond \xi(\omega)$ *in* $\mathscr{C}_{\mathfrak{n}}^{\alpha}([-L/2,L/2]^d)$ *and* $\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([-L/2,L/2]^d)$ *respectively, such that:*

$$\sup_n \|X_{\mathfrak{n}}^n(\omega)\|_{\mathscr{C}_{\mathfrak{n}}^{\alpha}(\Lambda_n)} + \sup_n \|(X_{\mathfrak{n}}^n \odot \xi^n)(\omega) - c_n(\omega)\|_{\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}(\Lambda_n)} < \infty$$

and $\mathscr{E}_{\mathfrak{n}}^n X_{\mathfrak{n}}^n(\omega) \to X_{\mathfrak{n}}(\omega)$ *in* $\mathscr{C}_{\mathfrak{n}}^{\alpha}([-L/2,L/2]^d)$, $\mathscr{E}_{\mathfrak{n}}^n\Big((X_{\mathfrak{n}}^n \odot \xi^n)(\omega) - c_n(\omega)\Big) \to X_{\mathfrak{n}} \diamond \xi(\omega)$ *in* $\mathscr{C}_{\mathfrak{n}}^{2\alpha-2}([-L/2,L/2]^d)$.

Finally, $\mathbb{P}(c_n(\omega) = \kappa_n, \forall n \in \mathbf{N}$ *and* $\nu(\omega) = \mathbb{E}\Phi_+) = 1$ *and for all* $\omega \in \Omega$, $\xi^n(\omega)$ *satisfies Assumption II.2.4, with the same renormalisation constant* $c_n(\omega)$ *as above if* $d = 2$.

The proof of this lemma is postponed to the next subsection. For clarity, observe that the first point is a CLT, the second point a LLN, while the third one is essentially the convergence of a Wick product in a second Wiener-Itô chaos. We say "essentially", because we are not defining exactly $\kappa_n = \mathbb{E}[X_{\mathfrak{n}}^n \odot \xi^n]$, as the latter average is a function, not a constant. Instead, we choose κ_n such that the limit $\lim_n\Big(\kappa_n - \mathbb{E}[X_{\mathfrak{n}}^n \odot \xi^n]\Big)$ exists in some appropriate function space.

Theorem IV.3.4. *Consider* ξ^n *as in Lemma IV.3.3 and* α *as in* (IV.26), *any* $T > 0$, $p \in [1,+\infty]$, $\gamma_0 \in [0,1)$ *and* $\vartheta, \zeta, \alpha_0$ *satisfying:*

$$\vartheta \in \begin{cases} (2-\alpha,\alpha), & d=1, \\ (2-2\alpha,\alpha), & d=2, \end{cases} \qquad \zeta > (\vartheta-2)\vee(-\alpha), \qquad \alpha_0 > (\vartheta-2)\vee(-\alpha), \tag{IV.27}$$

and let $w_0^n \in \mathscr{C}_{\mathfrak{d},p}^{\zeta}(\Lambda_n)$ *and* $f^n \in \mathscr{M}^{\gamma_0}\mathscr{C}_{\mathfrak{d},p}^{\alpha_0}(\Lambda_n)$ *be such that*

$$\mathscr{E}_{\mathfrak{d}}^n w_0^n \to w_0 \text{ in } \mathscr{C}_{\mathfrak{d},p}^{\zeta}([-L/2,L/2]^d), \qquad \mathscr{E}_{\mathfrak{d}}^n f^n \to f \text{ in } \mathscr{M}^{\gamma_0}\mathscr{C}_{\mathfrak{d},p}^{\alpha_0}([-L/2,L/2]^d).$$

For every $\omega \in \Omega$ *let* $w^n \colon [0,T]\times\Lambda_n \to \mathbf{R}$ *be the unique solution to the finite-dimensional linear ODE:*

$$\partial_t w^n = (\Delta_{\mathfrak{d}}^n + \xi^n(\omega) - c_n(\omega)1_{\{d=2\}})w^n + f^n, \ \ w^n(0) = w_0^n, \ \ w(t,x) = 0 \ \ \forall (t,x) \in (0,T]\times\partial\Lambda_n. \tag{IV.28}$$

There exist a unique (paracontrolled in the sense of [Cv19] in $d = 2$*) solution* w *to the equation*

$$\partial_t w = (\Delta_{\mathfrak{d}} + \xi)w + f, \ \ w(0) = w_0, \ \ w(t,x) = 0 \ \ \forall (t,x) \in (0,T]\times\partial[-L/2,L/2]^d, \tag{IV.29}$$

and for all $\gamma > (\vartheta-\zeta)_+/2 \vee \gamma_0$ the sequence w^n is uniformly bounded in $\mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}(\Lambda_n)$:

$$\sup_n \|w^n\|_{\mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}(\Lambda_n)} \lesssim \sup_n \|w^n_0\|_{\mathscr{C}^{\zeta}_{\mathfrak{d},p}(\Lambda_n)} + \sup_n \|f^n\|_{\mathscr{M}^{\gamma_0}\mathscr{C}^{\alpha_0}_{\mathfrak{d},p}(\Lambda_n)},$$

where the proportionality constant depends on the time horizon T and the magnitude of the norms in Lemma IV.3.3. Moreover,

$$\mathscr{E}^n_{\mathfrak{d}} w^n \to w \text{ in } \mathscr{L}^{\gamma,\vartheta}_{\mathfrak{d},p}([-L/2,L/2]^d).$$

Proof. In view of (IV.22) we can take $\tilde{w} = \Pi_o w$, so that solving Equation (IV.28) is equivalent to solving on the discrete torus Θ_n the equation:

$$\partial_t \tilde{w}^n = \Delta^n \tilde{w}^n + \Pi_e(\xi^n(\omega) - c_n(\omega)1_{\{d=2\}})\tilde{w}^n + \Pi_o f^n, \qquad \tilde{w}^n(0) = \Pi_o w_0, \tag{IV.30}$$

and then restricting the solution to the cube Λ_n with $w^n = \tilde{w}^n|_{\Lambda_n}$. Via the bounds in Lemma IV.3.3 this equation can be solved for all $\omega \in \Omega$ through Schauder estimates and (in dimension $d = 2$) paracontrolled theory following the arguments of [MP19] (without considering weights, which make the problem only more complex). From the arguments of the same article and Equation (IV.23) we can also deduce the convergence of the extensions. Note that the solution theories in [Cv19] and [MP19] coincide, although the former concentrates on the construction of the Hamiltonian rather than the solutions to the parabolic equation (confront also with Proposition II.3.1). □

For every $\omega \in \Omega$, define the Anderson Hamiltonian $\mathscr{H}^{\omega}_{\mathfrak{d},L} = \Delta_{\mathfrak{d}} + \xi(\omega)$ with Dirichlet boundary conditions:

$$\mathscr{H}^{\omega}_{\mathfrak{d},L} \colon \mathscr{D}(\mathscr{H}^{\omega}_{\mathfrak{d},L}) \subseteq L^2([-L/2,L/2]) \longrightarrow L^2([-L/2,L/2]).$$

The domain and spectral decomposition for this operator are rigorously constructed in [Cv19] with the help of the resolvent equation for $d = 2$ and [Gau19] via Dirichlet forms in $d = 1$. At the discrete level, write $\mathscr{H}^{n,\omega}_{\mathfrak{d},L}$ for the operator $\Delta^n_{\mathfrak{d}} + \xi^n(\omega) - c_n(\omega)1_{\{d=2\}}$. These operators generate semigroups of compact operators $T^{n,\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^{n,\omega}_{\mathfrak{d},L}}$ and $T^{\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^{\omega}_{\mathfrak{d},L}}$. In particular, the following result is a simple consequence of the just quoted works.

Lemma IV.3.5. *For a given null set $N_0 \subseteq \Omega$ and all $\omega \in N_0^c$, for all $L \in \mathbf{N}$ the operator $\mathscr{H}^{\omega}_{\mathfrak{d},L}$ has a discrete, bounded from above, spectrum and admits an eigenfunction $e_{\lambda_1(\omega,L)}$ associated to its largest eigenvalue $\lambda_1(\omega,L)$, such that $e_{\lambda_1(\omega,L)}(x) > 0$ for all $x \in (-\frac{L}{2},\frac{L}{2})^d$.*

Proof. That the spectrum is discrete and bounded from above can be found in the works quoted above. For $\varphi, \psi \in L^2((-\frac{L}{2},\frac{L}{2})^d)$ we write $\psi \geq \varphi$ if $\psi(x) - \varphi(x) \geq 0$ for Lebesgue-almost all x and we write $\psi \gg \varphi$ if $\psi(x) - \varphi(x) > 0$ for Lebesgue-almost all x. By the strong maximum principle of [CFG17, Theorem 5.1] (which easily extends to our setting, see Remark 5.2 of the same paper) we know that for the semigroup $T^{\mathfrak{d},L,\omega}_t = e^{t\mathscr{H}^{\omega}_{\mathfrak{d},L}}$ of the PAM we have $T^{\mathfrak{d},L,\omega}_t \varphi \gg 0$ whenever $\varphi \geq 0$ and $\varphi \neq 0$; we even get $T^{\mathfrak{d},L,\omega}_t \varphi(x) > 0$ for all x in the interior $(-\frac{L}{2},\frac{L}{2})^d$. So by a consequence of the Krein-Rutman theorem, see [Dei85, Theorem 19.3], there exists an eigenfunction $e_{\lambda_1(\omega,L)} \gg 0$. And since $e_{\lambda_1(\omega,L)} = e^{-t\lambda_1(\omega,L)} T^{\mathfrak{d},L,\omega}_t e_{\lambda_1(\omega,L)}$, we have $e_{\lambda_1(\omega,L)}(x) > 0$ for all $x \in (-\frac{L}{2},\frac{L}{2})^d$. □

IV.3.3 Stochastic Estimates

Here we prove Lemma IV.3.3. The following bounds are essentially an adaptation of [CGP17, Section 4.2] to Dirichlet boundary conditions (see [Cv19] for the spatially continuous setting). The key issue is to bound the resonant product $X^n_{\mathfrak{n}} \odot \xi^n$, that can be decomposed in its zeroth and second Wiener-Itô chaos. The main difference with respect to the periodic case, and the central point of the following proof, is that the zeroth chaos is not a constant, yet our calculations will show that up to a *constant* blow up κ_n this term is well-defined.

Proof of Lemma IV.3.3. Step 0. We shall prove the lemma for fixed L, α, ε. The convergence happens simultaneously over all parameter choices in view of similar arguments as in the proof of Corollary II.5.13. Instead of proving the path-wise convergences of the lemma, it is sufficient to show the convergences in distribution. The results then follows by Skorohod's representation theorem, by setting $\nu(\omega) = c_n(\omega) = \xi^n(\omega) = 0$ on a null set. Let us write ξ^n instead of $\overline{\xi}^n$. We will show that there exists a space white noise ξ on $\mathbf{R}^d$ and (if $d = 2$) a random distribution $X_{\mathfrak{n}} \diamond \xi$ such that (all convergences being in distribution):

$$\sup_n \mathbb{E}[\|\xi^n\|^q_{\mathscr{C}^{\alpha-2}_{\mathfrak{n}}(\Lambda_n)}] < +\infty, \qquad \mathscr{E}^n_{\mathfrak{n}} \xi^n \to \xi \text{ in } \mathscr{C}^{\alpha-2}_{\mathfrak{n}}([0,L]^d), \tag{IV.31}$$

as well as:

$$\sup_n \mathbb{E}[\|n^{-d/2}(\xi^n)_+\|_{\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}(\Lambda_n)} + \|n^{-d/2}(\xi^n)_+\|_{L^2(\Lambda_n)}] < +\infty, \tag{IV.32}$$

with $\mathscr{E}^n_{\mathfrak{n}} n^{-d/2}(\xi^n)_+ \to \mathbb{E}\Phi_+$ in $\mathscr{C}^{-\varepsilon}_{\mathfrak{n}}([0,L]^d)$. Moreover, in dimension $d = 2$, we have (recall κ_n from (IV.25)):

$$\sup_n \mathbb{E}[\|X^n_{\mathfrak{n}}\|_{\mathscr{C}^{\alpha}_{\mathfrak{n}}(\Lambda_n)} + \|(X^n_{\mathfrak{n}} \odot \xi^n) - \kappa_n\|_{\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}(\Lambda_n)}] < +\infty \tag{IV.33}$$

as well as $\mathscr{E}^n_{\mathfrak{n}} X^n_{\mathfrak{n}} \to X_{\mathfrak{n}}$ in $\mathscr{C}^{\alpha}_{\mathfrak{n}}([0,L]^d)$, and $\mathscr{E}^n_{\mathfrak{n}}(X^n_{\mathfrak{n}} \odot \xi^n - \kappa_n) \to X_{\mathfrak{n}} \diamond \xi$ in $\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}([0,L]^d)$. Once these bounds and convergences are established, the proof is concluded. Note that ξ^n satisfies Assumption II.2.4 in view of Lemma II.2.5.

Step 1. The bound and the convergence from (IV.31) for ξ^n and (IV.33) for X^n are simpler than the bound for $X^n_{\mathfrak{n}} \odot \xi^n$. Also, Equation (IV.32) and the following convergences are analogous to the ones in Section II.7. Hence we restrict to proving the bound and the convergence of $X^n_{\mathfrak{n}} \odot \xi^n$ as in (IV.33).

Step 2. We establish first the uniform bound in $B^{\mathfrak{n},2\alpha-2}_{p,p}(\Lambda_n)$ (instead of $\mathscr{C}^{2\alpha-2}_{\mathfrak{n}}$) for any $p \geq 1$ and α such that $2\alpha - 2 < 0$. The results on the Hölder scale follow by Besov embedding. In order to avoid confusion, we omit the subindex $\mathfrak{n}$ in the noise terms and write sums as discrete integrals against scaled measures with the following definitions:

$$\int_{\Theta_n} f(x)\,\mathrm{d}x = \sum_{x \in \Theta_n} \frac{f(x)}{n^d}, \quad \int_{\Xi_n} f(k)\,\mathrm{d}k = \sum_{k \in \Xi_n} \frac{f(k)}{N^d}, \quad \int_{\{-1,1\}^d} f(\mathfrak{q})\,\mathrm{d}\mathfrak{q} = \sum_{\mathfrak{q} \in \{-1,1\}^d} f(\mathfrak{q}).$$

Then, observing that $\nu_k^{-2} = \#\{\mathfrak{q} \in \{-1,1\}^d \ : \ \mathfrak{q} \circ k = k\}$, one has for $f : \Xi_n \to \mathbf{C}$:

$$\int_{\Xi_n} f(k)\,\mathrm{d}k = \int_{\{-1,1\}^d \times \Xi_n^+} \nu_k^2\, f(\mathfrak{q} \circ k)\,\mathrm{d}\mathfrak{q}\,\mathrm{d}k. \tag{IV.34}$$

In this setting, our aim to estimate uniformly over n the following quantity:

$$\mathbb{E}\Big[\|(X^n \odot \xi^n) - c_n\|^p_{B^{\mathfrak{n},2\alpha-2}_{p,p}(\Lambda_n)}\Big] = \sum_{j=-1}^{j_n} 2^{(2\alpha-2)jp} \int_{\Theta_n} \mathbb{E}\Big[|\Pi_e \Delta_j (X^n \odot \xi^n - \kappa_n)|^p(x)\Big] \mathrm{d}x.$$

For $k_1, k_2 \in \Xi_n$ and $\mathfrak{q}_1, \mathfrak{q}_2 \in \{-1,1\}^d$ we adopt the notation:

$$k_{[12]} = k_1 + k_2, \mathfrak{q}_{[12]} = \mathfrak{q}_1 + \mathfrak{q}_2, \qquad (\mathfrak{q} \circ k)_{[12]} = \mathfrak{q}_1 \circ k_1 + \mathfrak{q}_2 \circ k_2$$

and $\psi^n_0(k_1,k_2) = \sum_{\substack{|i-j|\le 1 \\ -1 \le i,j \le j_n}} \varrho_i(k_1)\varrho_j(k_2)$. Hence via (IV.34):

$$\begin{aligned}
\Delta_j \Pi_e (X^n \odot \xi^n)(x) = & \int_{(\{-1,1\}^d \times \Xi_n^+)^2} N^d \nu_{k_1} \nu_{k_2} e^{2\pi\iota\langle x, (\mathfrak{q}\circ k)_{[12]}\rangle} \cdot \\
& \cdot \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi^n_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1}\rangle \langle \xi^n, \mathfrak{n}_{k_2}\rangle \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \\
= & \int_{(\{-1,1\}^d \times \Xi_n^+)^2} 1_{\{k_1 \neq k_2\}} N^d \nu_{k_1} \nu_{k_2} e^{2\pi\iota\langle x, (\mathfrak{q}\circ k)_{[12]}\rangle} \cdot \\
& \cdot \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi^n_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \langle \xi^n, \mathfrak{n}_{k_1}\rangle \langle \xi^n, \mathfrak{n}_{k_2}\rangle \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} + \mathrm{Diag}
\end{aligned}$$

where Diag indicates the integral over the set $\{k_1 = k_2\}$. First, since Φ has all moments finite, we apply a generalized discrete BDG inequality [CGP17, Proposition 4.3] and the same calculations as in [CGP17, Corollary 4.7] to find:

$$\begin{aligned}
& \mathbb{E}[|\Delta_j(\Pi_e(X^n \odot \xi^n)(x) - \kappa_n)|^p] \\
& \quad \lesssim \mathbb{E}[|\Delta_j(\Pi_e(X^n \odot \xi^n)(x) - \mathbb{E}[\mathrm{Diag}]|^p] + |\mathbb{E}[\mathrm{Diag}] - 1_{\{j=-1\}}\kappa_n|^p \\
& \quad \lesssim \left[\int \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} \left| \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi^n_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 \right]^{\frac{p}{2}} + |\mathbb{E}[\mathrm{Diag}] - 1_{\{j=-1\}}\kappa_n|^p.
\end{aligned}$$

For the first term on the right hand side we have:

$$\begin{aligned}
& \int_{(\{-1,1\}^d \times \Xi_n^+)^2} \left| \varrho_j((\mathfrak{q}\circ k)_{[12]}) \psi^n_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k_{12} = \int_{\Xi_n^2} \left| \varrho_j(k_{[12]}) \psi^n_0(k_1,k_2) \frac{\chi(k_2)}{l^n(k_2)} \right|^2 \mathrm{d}k_{12} \\
& \quad \lesssim \sum_{i \ge j-\ell} \int_{\Xi_n^2} 1_{\{|k_1+k_2| \sim 2^j\}} 1_{\{|k_2| \sim 2^i\}} 2^{-4i}\, \mathrm{d}k_{12} \lesssim \sum_{i \ge j-\ell} 2^{jd} 2^{i(d-4)} \lesssim 2^{2j(d-2)},
\end{aligned}$$

which is of the required order (and we used that $d < 4$). Let us pass to the diagonal, term. Since $\{\langle \xi^n, \mathfrak{n}_k\rangle\}_{k \in A^n_{\mathfrak{n}}}$ are uncorrelated we rewrite the term as:

$$\int_{\Xi_n^+ \times (\{-1,1\}^d)^2} \nu_k^2 e^{2\pi\iota\langle x, \mathfrak{q}_{[12]}\circ k\rangle} \varrho_j(\mathfrak{q}_{[12]} \circ k) \frac{\chi(k)}{l^n(k)} \mathrm{d}\mathfrak{q}_{12}\, \mathrm{d}k \;-\; 1_{\{j=-1\}}\kappa_n.$$

We split up this sum in different terms according to the relative values of $\mathfrak{q}_1, \mathfrak{q}_2$. If $\mathfrak{q}_1 = -\mathfrak{q}_2$ (there are 2^d such terms) the sum does not depend on x and it disappears for $j \ge 0$. Let us assume $j = -1$. Via (IV.34) and parity we are then left with the constant:

$$2^d \int_{\Xi_n^+} \nu_k^2 \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k - \kappa_n = \int_{\Xi_n} \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k - \kappa_n.$$

The sum on the left-hand side diverges logarithmically in n, just as κ_n. We show that the difference converges to a constant. To clarify our computation let us introduce an auxiliary constant $\bar{\kappa}_n = \int_{\Xi_n} \mathrm{d}k \, \overline{\nu}_k^2 \frac{\chi(k)}{l^n(k)}$, where $\overline{\nu}_k = 2^{-\#\{i \colon k_i = \pm n\}/2}$. For $x \in \mathbf{R}^d, r \geq 0$, indicate with $Q_r^n(x) \subseteq \mathbf{T}_n^d$ the box $Q_r^n(x) = \{y \in \mathbf{T}_n^d \colon |y - x|_\infty \leq r/2\}$ ($|\cdot|_\infty$ being the maximum of the component-wise distances in $\mathbf{T}_n^d$). Then we bound uniformly over n and N:

$$\begin{aligned} |\kappa_n - \bar{\kappa}_n| &= \left| \int_{\mathbf{T}_n^d} \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k - \int_{\Xi_n} \overline{\nu}_k^2 \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k \right| = \left| \sum_{k \in \Xi_n} \int_{Q^n_{\frac{1}{N}}(k)} \frac{\chi(k+k')}{l^n(k+k')} - \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k' \right| \\ &\lesssim \frac{1}{N} \left(1 + \frac{1}{N^d} \sum_{k \in \Xi_n} \sup_{\vartheta \in Q_{\frac{1}{N}}(k)} \frac{\chi(k)}{(l^n(\vartheta))^2} |\nabla l^n(\vartheta)| \right) \lesssim \frac{1}{N} \left(1 + \frac{1}{N^d} \sum_{k \in \frac{1}{N}\mathbf{Z}^d} \frac{\chi(k)}{|k|^3} \right) \lesssim \frac{1}{N}, \end{aligned}$$

where we have used that $d = 2$, $|l^n(\vartheta)| \gtrsim |\vartheta|^2$ on $[-n/2, n/2]^d$ as well as $|\nabla l^n(\vartheta)| \lesssim |\theta|$ on $[-n/2, n/2]^d$. Similar calculations show that the difference converges: $\lim_{n \to \infty} \kappa_n - \bar{\kappa}_n \in \mathbf{R}$. We are now able to estimate:

$$\left| \int_{\Xi_n} \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k - \kappa_n \right| \lesssim 1 + |\bar{\kappa}_n - \kappa_n| \lesssim 1$$

where we used that the sum on the boundary $\partial \Xi_n$ converges to zero and is thus uniformly bounded in n. For the same reason, the above difference converges to the limit $\lim_{n \to \infty} \overline{\kappa}_n - \kappa_n \in \mathbf{R}$.

For all other possibilities of $\mathfrak{q}_1, \mathfrak{q}_2$ we show boundedness in a distributional sense. The same calculations show that in fact these terms converge to a deterministic distribution for $n \to \infty$. If $\mathfrak{q}_1 = \mathfrak{q}_2$ we have:

$$\left| \int_{\Xi_n^+} \nu_k^2 e^{2\pi\iota \langle x, 2\mathfrak{q}_1 k \rangle} \varrho_j(2k) \frac{\chi(k)}{l^n(k)} \right| \lesssim 2^{j(d-2)} \, \mathrm{d}k.$$

Finally, if only one of the two components of $\mathfrak{q}_1, \mathfrak{q}_2$ differs (let us suppose it is the first one) we find (with $x = (x_1, x_2)$ and $k = (k_1, k_2)$):

$$\left| \int_{\Xi_n^+} \nu_k^2 e^{2\pi\iota 2 x_2 k_2} \varrho_j(2k_2) \frac{\chi(k)}{l^n(k)} \, \mathrm{d}k \right| \lesssim \left(\sum_{k_1 \geq 1} \frac{1}{|k_1|^{2\theta}} \right) \left(\sum_{k_2 \geq 1} \frac{\varrho_j(2k_2)}{|k_2|^{2(1-\theta)}} \right) \lesssim 2^{j\varepsilon}$$

for any $\varepsilon > 0$, up to choosing $\theta \in (1/2, 1)$ sufficiently close to $1/2$.

Step 3. Now we address the convergence in distribution. The previous calculations and compact embedding of Hölder-Besov spaces guarantee tightness of the sequence $X_{\mathfrak{n}}^n \odot \xi^n - \kappa_n$ in the required Hölder spaces for any $\alpha < 0$. We have to identify uniquely the distribution of any limit point. Whereas for ξ and $X_{\mathfrak{n}}^n$ the limit points are Gaussian and uniquely identified as white noise ξ and $\Delta_{\mathfrak{n}}^{-1} \chi(D) \xi$ respectively, the resonant product requires more care. Here we can use methods developed in [CSZ17] (also used in

[MP19, Section 5]) for discrete Gaussian chaos decomposition. Pick any smooth function $\varphi\colon [-L,L]^d \to \mathbf{R}$ and consider the quantity:

$$\begin{aligned}\langle \varphi, \Pi_e(X^n \odot \xi^n - \kappa_n)\rangle &= \frac{1}{n^d}\sum_{x\in\Theta_n}\varphi(x)\Big(\Pi_e(X^n\odot\xi^n)(x)-\kappa_n\Big)\\ &= \frac{1}{n^d}\sum_{x\in\Theta_n}\varphi(x)\Big((\Pi_e X^n\odot\Pi_e\xi^n)(x)-\kappa_n\Big).\end{aligned}$$

The latter quantity (omitting the integral against φ at first) can be rewritten with (IV.34) as:

$$\begin{aligned}&\sum_{\substack{|i-j|\le 1\\ 1\le i,j\le j_n}}\int_{(\Theta_n)^2}\widetilde{K}_i^n(x-x_1)\Pi_e\xi^n(x_1)K_j^n(x-x_2)\Pi_e\xi^n(x_2)\,\mathrm{d}x_{12}-\kappa_n\\ &=\sum_{\substack{|i-j|\le 1\\ -1\le i,j\le j_n}}\int\limits_{(\Lambda_n\times\{-1,1\}^d)^2}\nu_{x_1}^2\nu_{x_2}^2\widetilde{K}_i^n(x-\mathfrak{q}_1\circ x_1)K_j^n(x-\mathfrak{q}_2\circ x_2)\xi^n(x_1)\xi^n(x_2)\,\mathrm{d}x_{12}\,\mathrm{d}\mathfrak{q}_{12}-\kappa_n,\end{aligned}$$

where $\widetilde{K}_i^n(x) = \mathscr{F}_{\Theta_n}^{-1}\Big[\varrho_i(\cdot)\frac{\chi(\cdot)}{l^n(\cdot)}\Big](x)$ and $K_j^n(x) = \mathscr{F}_{\Theta_n}^{-1}[\varrho_j(\cdot)](x)$. Now in the previous calculations we could show that $\mathbb{E}\Big[X^n\odot\xi^n\Big]-\kappa_n$ converges in the sense of distributions. Hence, when tested against φ is suffices to study the convergence in distribution of

$$\int\limits_{(\Lambda_n)^2} L_n^{\varphi}(x_1,x_2)\Big(\xi^n(x_1)\xi^n(x_2)-\mathbb{E}[\xi^n(x_1)\xi^n(x_2)]\Big)\mathrm{d}x_{12},$$

with

$$L_n^{\varphi}(x_1,x_2)=\int_{\Theta_n}\varphi(x)\sum_{\substack{|i-j|\le 1\\ -1\le i,j\le j_n}}\int_{(\{-1,1\}^d)^2}\nu_{x_1}^2\nu_{x_2}^2\widetilde{K}_i^n(x-\mathfrak{q}_1\circ x_1)K_j^n(x-\mathfrak{q}_2\circ x_2)\,\mathrm{d}\mathfrak{q}_{12}\,\mathrm{d}x.$$

Now we can apply [CSZ17, Theorem 2.3] to obtain that

$$\int\limits_{(\Lambda_n)^2} L_n^{\varphi}(x_1,x_2)\Big(\xi^n(x_1)\xi^n(x_2)-\mathbb{E}[\xi^n(x_1)\xi^n(x_2)]\Big)\mathrm{d}x_{12}\longrightarrow\int\limits_{([0,L]^2)^2}L^{\varphi}(x_1,x_2)\xi(\mathrm{d}x_1)\diamond\xi(\mathrm{d}x_2),$$

in distribution (and where the latter is an integral in the second Wiener chaos), with

$$L^{\varphi}(x_1,x_2)=\int_{\mathbf{T}_N^2}\varphi(x)\sum_{\substack{|i-j|\le 1\\ -1\le i,j\le j_n}}\int_{(\{-1,1\}^d)^2}\nu_{x_1}^2\nu_{x_2}^2\widetilde{K}_i(x-\mathfrak{q}_1\circ x_1)K_j(x-\mathfrak{q}_2\circ x_2)\,\mathrm{d}\mathfrak{q}_{12}\,\mathrm{d}x,$$

(the definition of $\widetilde{K}_i^n$ and K_j^n naturally extend to). The theorem can be applied, provided that $\lim_{n\to\infty}\|L_n^{\varphi}-L^{\varphi}\|_{L^2(([0,L]^2)^2)}=0$, where we extend L_n^{φ} to $([0,L]^2)^2$ by defining it as a constant on any square centered at lattice points. To see this we can follow the calculations that brought us until here. To improve the readability let us just prove that L_n^{φ} is uniformly bounded in $L^2(([0,L]^2)^2)$ (the convergence follows similarly):

$$\begin{aligned}\|L_n^{\varphi}\|^2_{L^2(([0,L]^2)^2)} &\lesssim \int_{(\Theta_n)^2} \left| \int_{\Theta_n} \varphi(x) \sum_{\substack{|i-j|\le 1\\ -1\le i,j\le j_n}} \widetilde{K}_i^n(x-x_1) K_j^n(x-x_2)\,\mathrm{d}x \right|^2 \mathrm{d}x_{12} \\ &\lesssim \int_{(\Xi_n)^2} \left| \mathscr{F}_{\Theta_n}\varphi(k_1-k_2) \sum_{\substack{|i-j|\le 1\\ -1\le i,j\le j_n}} \varrho_i(k_1) \frac{\chi(k_1)}{l^n(k_2)} \varrho_j(k_2) \right|^2 \mathrm{d}k_{12},\end{aligned}$$

by Parseval. Using the smoothness of φ and $|l^n(k)| \simeq |k|^2$ one obtains the uniform bound. This concludes the proof. □

V

Synchronization for KPZ

V.1 Introduction

In this chapter, we prove synchronization and a one force, one solution (1F1S) principle for KPZ-like equations (the true KPZ equation is driven by space-time white noise in $d = 1$):

$$(\partial_t - \Delta)h(t,x) = |\nabla h|^2(t,x) + \eta(t,x), \qquad h(0,x) = h_0(x), \qquad t \in (0,\infty), \ \ x \in \mathbf{T}^d, \tag{V.1}$$

where η is some ergodic noise. At least formally, this equation is linked to two avatars. The first one is Burgers equation, satisfied by $v = \nabla h$:

$$(\partial_t - \Delta)v(t,x) = \nabla|v|^2(t,x) + \nabla\eta(t,x), \quad v(0,x) = v_0(x), \quad t \in (0,\infty), \ \ x \in \mathbf{T}^d. \tag{V.2}$$

The second avatar is the heat equation with multiplicative noise, solved by $u = \exp(h)$:

$$(\partial_t - \Delta)u(t,x) = \eta(t,x)u(t,x), \qquad u(0,x) = u_0(x), \qquad t \in (0,\infty), \ \ x \in \mathbf{T}^d. \tag{V.3}$$

We will mainly work under the assumption that the last equation is well-posed and the solution map generates a linear random dynamical system that satisfies certain properties. In Section V.3 we introduce Hilbert's projective metric and a related contraction principle for positive operators (see Theorem V.3.1). This allows us to formulate a random variant of the Krein-Rutman (see Theorem V.4.4). Assuming a random dynamical systems satisfies some properties that we consider natural for solution maps to Equation V.3, we show that synchronization and 1F1S holds (see Theorem V.5.3). We then show in Section V.6 how to apply this result in the case of space-time white noise and in the case of a fractional noise.

V.2 Notations

In this chapter the notation differs slightly from previous chapters, so we introduce again some definitions, at the cost of slightly repeating ourselves.

For $\alpha > 0$ let $\lfloor \alpha \rfloor$ be the smallest integer beneath α and for a multiindex $k \in \mathbf{N}^d$ write $|k| = \sum_{i=1}^d k_i$. Denote with $C(\mathbf{T}^d)$ the space of continuous real-valued functions on $\mathbf{T}^d$, and, for $\alpha > 0$, with $C^{\alpha}(\mathbf{T})$ the space of $\lfloor \alpha \rfloor$– differentiable functions f such that $\partial^k f$ is $(\alpha - \lfloor \alpha \rfloor)$–Hölder continuous for every multiindex $k \in \mathbf{N}^d$ such that $|k| = \lfloor \alpha \rfloor$, if $\alpha - \lfloor \alpha \rfloor > 0$, or simply continuous if $\alpha \in \mathbf{N}_0$. For $\alpha \in \mathbf{R}_+$ we obtain the following seminorms on $C^{\alpha}(\mathbf{T}^d)$:

$$[f]_{\alpha} = \max_{|k|=\lfloor \alpha \rfloor} \|\partial^k f\|_{\infty} 1_{\{|k|>0\}} + \sup_{x,y \in \mathbf{T}^d} \frac{|\partial^k f(x) - \partial^k f(y)|}{|x-y|^{\alpha - \lfloor \alpha \rfloor}}.$$

We write $C^{\infty}(\mathbf{T}^d) = \bigcap_{k \in \mathbf{N}} C^k(\mathbf{T}^d)$.

Now, let X be a Banach space. We denote with $\mathscr{B}(X)$ the Borel σ–algebra on X. Let $[a,b] \subseteq \mathbf{R}$ be an interval, then define $C([a,b];X)$ the space of continuous functions $f : [a,b] \to X$. For any $O \subseteq \mathbf{R}$, we write $C_{\mathrm{loc}}(O;X)$ for the space of continuous functions with the topology of uniform convergence on all compact subsets of O. Given two Banach spaces X, Y denote with $\mathscr{L}(X;Y)$ the space of linear bounded operators $A : X \to Y$ with the classical operator norm. If $X = Y$ we write simply $\mathscr{L}(X)$.

Next we introduce Besov spaces which, unlike those in Section I.1.1, are weighted in time. Following [BCD11, Section 2.2] choose a smooth dyadic partition of the unity on $\mathbf{R}^d$ (resp. $\mathbf{R}^{d+1}$) $(\chi, \{\varrho_j\}_{j \geq 0})$ and define $\varrho_{-1} = \chi$ and define the Fourier transforms for $f : \mathbf{T}^d \to \mathbf{R}$ and $g : \mathbf{R} \times \mathbf{T}^d \to \mathbf{R}$:

$$\begin{aligned} \mathscr{F}_{\mathbf{T}^d} f(k) &= \int_{\mathbf{T}^d} e^{-2\pi\iota\langle k,x \rangle} f(x) \, \mathrm{d}x, \qquad k \in \mathbf{Z}^d, \\ \mathscr{F}_{\mathbf{R} \times \mathbf{T}^d} g(\tau, k) &= \int_{\mathbf{R} \times \mathbf{T}^d} e^{-2\pi\iota(\tau t + \langle k,x \rangle)} g(t,x) \, \mathrm{d}t \, \mathrm{d}x, \qquad (\tau, k) \in \mathbf{R} \times \mathbf{Z}^d. \end{aligned}$$

These definitions extend naturally to spatial (resp. space-time) tempered distributions $\mathscr{S}'(\mathbf{T}^d)$ (resp. $\mathscr{S}'(\mathbf{R} \times \mathbf{T}^d)$), which are the topological duals of Schwartz functions: $\mathscr{S}(\mathbf{T}^d) = C^{\infty}(\mathbf{T}^d)$ and

$$\mathscr{S}(\mathbf{R} \times \mathbf{T}^d) = \Big\{ \varphi \ : \ \sup_{t \in \mathbf{R}, x \in \mathbf{T}^d} \Big\{ (1 + |t|)^p |\partial_x^{\mu} \varphi(t,x)| \Big\} < \infty, \ \forall p \geqslant 0, \mu \in \mathbf{N}_0^{d+1} \Big\}.$$

Similarly one defines the respective inverse Fourier transforms $\mathscr{F}_{\mathbf{T}^d}^{-1}$ and $\mathscr{F}_{\mathbf{R} \times \mathbf{T}^d}^{-1}$. Then define the spatial (resp. space-time) Paley blocks:

$$\Delta_j f(x) = \mathscr{F}_{\mathbf{T}^d}^{-1} [\varrho_j \cdot \mathscr{F}_{\mathbf{T}^d} f](x), \qquad \Delta_j g(t,x) = \mathscr{F}_{\mathbf{R} \times \mathbf{T}^d}^{-1} [\varrho_j \cdot \mathscr{F}_{\mathbf{R} \times \mathbf{T}^d} g](t,x).$$

Eventually one defines, for $\alpha \in \mathbf{R}$, $a > 0$, $p, q \in [1, \infty]$, the spaces $B_{p,q}^{\alpha}(\mathbf{T}^d)$ and $B_{p,q}^{\alpha,a}(\mathbf{R} \times \mathbf{T}^d)$ as the set of tempered distributions such that, respectively, the following norms are finite:

$$\begin{aligned} \|f\|_{B_{p,q}^{\alpha}(\mathbf{T}^d)} &= \|(2^{j\alpha} \|\Delta_j f\|_{L^p(\mathbf{T})})_{j \geq -1}\|_{\ell^q}, \\ \|g\|_{B_{p,q}^{\alpha,a}(\mathbf{R} \times \mathbf{T}^d)} &= \|(2^{j\alpha} \|\Delta_j f(\cdot)/\langle \cdot \rangle^a\|_{L^p(\mathbf{R} \times \mathbf{T}^d)})_{j \geq -1}\|_{\ell^q}, \end{aligned}$$

where we denote with $\langle (t,x) \rangle$ the weight $\langle (t,x) \rangle = 1 + |t|$. For $p = q = 2$ one obtains the Hilbert spaces $H^{\alpha}(\mathbf{T}^d) = B_{2,2}^{\alpha}(\mathbf{T}^d)$ and

$$H_a^{\alpha}(\mathbf{R} \times \mathbf{T}^d) = B_{2,2}^{\alpha,a}(\mathbf{R} \times \mathbf{T}^d). \tag{V.4}$$

One can also consider functions that depend on time only and introduce, for the same range of parameters, the spaces $B^{\alpha,a}_{p,q}(\mathbf{R})$ via the norm:

$$\|f\|_{B^{\alpha,a}_{p,q}(\mathbf{R})} = \|(2^{j\alpha}\|\Delta_j f(\cdot)/\langle\cdot\rangle^a\|_{L^p(\mathbf{R})})_{j\geq -1}\|_{\ell^q}, \qquad \langle t\rangle = 1+|t|.$$

Here the Paley blocks are defined by $\Delta_j f(t) = \mathscr{F}^{-1}_{\mathbf{R}}(\varrho_j \cdot \mathscr{F}_{\mathbf{R}} f)(t)$, for a dyadic partition of the unity $\{\varrho_j\}_{j\geqslant -1}$ on $\mathbf{R}$. As above we then define

$$H^\alpha_a(\mathbf{R}) = B^{\alpha,a}_{2,2}(\mathbf{R}). \tag{V.5}$$

Finally, recall that for $p = q = \infty$ and $\alpha \in \mathbf{R}_+ \setminus \mathbf{N}_0$: $B^\alpha_{\infty,\infty}(\mathbf{T}^d) = C^\alpha(\mathbf{T}^d)$ (see e.g. [Tri10, Chapter 2]).

V.3 Setting

This section, based on [Bus73], introduces the projective space of positive continuous functions and a related contraction principle for strictly positive operators. Let X be a Banach space and $K \subseteq X$ a closed cone such that $K \cap (-K) = \{0\}$. Denote with $\mathring{K}$ the interior of K and write $K^+ = K \setminus \{0\}$. Such cone induces a partial order in X by defining for $x, y \in X$:

$$x \leq y \Leftrightarrow y - x \in K \quad \text{and} \quad x < y \Leftrightarrow y - x \in \mathring{K}.$$

Consider for $x, y \in K^+$:

$$M(x,y) = \inf\{\lambda \geq 0 : x \leq \lambda y\}, \qquad m(x,y) = \sup\{\mu \geq 0 : \mu y \leq x\},$$

with the convention $\inf \emptyset = \infty$. Then $M(x,y) \in (0,\infty]$ and $m(x,y) \in [0,\infty)$ so that one can define Hilbert's projective distance:

$$d_H(x,y) = \log(M(x,y)) - \log(m(x,y)) \in [0,\infty], \qquad \forall\, x,y \in K^+.$$

This metric is only semidefinite positive on K^+, and may be infinite. A remedy for the first issue is to consider an affine space $U \subseteq X$ which intersects transversely K^+, that is:

$$\forall x \in K^+, \quad \exists! \lambda > 0 \quad \text{s.t.} \quad \lambda x \in U.$$

Write $\lambda(x)$ for the normalization constant above. As for the second issue, one can observe that the distance is finite on the interior of K, cf. [Bus73, Theorem 2.1], and thus, defining $E = \mathring{K} \cap U$, one has that (E, d_H) is a metric space.

Consider now $\mathscr{L}(X)$ the set of linear bounded operators on X, and for an operator $A \in \mathscr{L}(X)$ the following conventions define different concepts of positivity:

$$\begin{aligned} A(K) \subseteq K \quad &\Rightarrow \quad A \text{ nonnegative.} \\ A(\mathring{K}) \subseteq \mathring{K} \quad &\Rightarrow \quad A \text{ positive.} \\ A(K^+) \subseteq \mathring{K} \quad &\Rightarrow \quad A \text{ strictly positive.} \end{aligned}$$

The projective action of a positive operator A on X is then defined by: $A^{\pi}x = Ax \cdot \lambda(Ax)$. One can view A^{π} as a map $A^{\pi}\colon E \to E$ and one then denotes with $\tau(A)$ the projective norm associated to A:

$$\tau(A) = \sup_{\substack{x,y\in E\\ x\neq y}} \frac{d_H(A^{\pi}x, A^{\pi}y)}{d_H(x,y)}. \tag{V.6}$$

The backbone of our approach is Birkhoff's theorem for positive operators [Bus73, Theorem 3.2], which is stated below.

Theorem V.3.1. *Let $\Delta(F)$ denote the diameter of a set $F \subseteq E$:*

$$\Delta(F) = \sup_{x,y\in F}\{d_H(x,y)\}.$$

The following identity holds:

$$\tau(A) = \tanh\Big(\frac{1}{4}\Delta(A^{\pi}(E))\Big) \leq 1.$$

Then denote with $\mathscr{L}_{\mathrm{cp}}(X)$ the space of positive operators A which are contractive in (E, d_H):

$$A \in \mathscr{L}_{\mathrm{cp}}(X) \quad \Leftrightarrow \quad A \in \mathscr{L}(X),\ \ A \text{ positive},\ \ \tau(A) < 1.$$

The only example considered in this work is $X = C(\mathbf{T}^d)$ the space of real-valued continuous functions on the torus, where K is the cone of positive functions. Here the following holds.

Lemma V.3.2. *Let $X = C(\mathbf{T}^d)$ and $K = \{f \in X :\ f(x) \geq 0,\ \ \forall x \in \mathbf{T}^d\}$, and consider:*

$$U = \Big\{f \in X\colon \int_{\mathbf{T}^d} f(x)\,\mathrm{d}x = 1\Big\}.$$

For the associated metric space (E, d_H) the following inequality holds:

$$\|\log(f) - \log(g)\|_{\infty} \leq d_H(f,g) \leq 2\|\log(f) - \log(g)\|_{\infty}, \qquad \forall f, g \in E. \tag{V.7}$$

In particular, (E, d_H) is a complete metric space. In addition, if a strictly positive operator A can be represented by a kernel, i.e. there exists $K \in C(\mathbf{T}^d \times \mathbf{T}^d)$ such that:

$$A(f)(x) = \int_{\mathbf{T}^d} K(x,y) f(y)\ \mathrm{d}y, \quad \forall x \in \mathbf{T}^d$$

and there exits constants $0 < \alpha \leq \beta < \infty$ such that

$$\alpha \leq K(x,y) \leq \beta, \quad \forall x, y \in \mathbf{T}^d,$$

then A is contractive, i.e. $A \in \mathscr{L}_{\mathrm{cp}}(X)$.

Proof. As for the inequality, since $f, g \in U$ (and hence $\int f(x)\,\mathrm{d}x = \int g(x)\,\mathrm{d}x = 1$), there exists a point x_0 such that $f(x_0) = g(x_0)$. In particular if we rewrite the distance

$$\begin{aligned} d_H(f,g) &= \log M(f,g) - \log m(f,g) \\ &= \max(\log(f/g)) - \min(\log(f/g)) \\ &= \max(\log(f/g)) + \max(\log(g/f)) \end{aligned}$$

we see that in the sum both terms are positive an bounded by $\|\log(f) - \log(g)\|_\infty$. Conversely we have that:

$$\|\log(f) - \log(g)\|_\infty \le \max(\log(f) - \log(g)) + \max(\log(g) - \log(f)).$$

Completeness of (E, d_H) is a consequence of Inequality (V.7): for a given Cauchy sequence $f_n \in E$ the sequence $\log(f_n)$ is a Cauchy sequence in $C(\mathbf{T}^d)$. By completeness of the latter there exists a $g \in C(\mathbf{T}^d)$ such that $\log(f_n) \to g$ in the uniform topology. By dominated convergence $\int_{\mathbf{T}^d} \exp(g)(x)\,\mathrm{d}x = 1$, so that $\exp(g) \in E$, and hence $f_n \to \exp(g)$ in E.

The result regarding the kernel can be found in [Bus73, Section 6].

□

Remark V.3.3. *For the sake of simplicity we did not address the general question of completeness of the space (E, d_H), since in the case of interest to us completeness follows from* (V.7). *Yet general criteria for completeness are known, see for example [Bus73, Section 4] and the references therein.*

Remark V.3.4. *In view of* (V.6), *an application of Banach's fixed point theorem in (E, d_H) to operators satisfying the conditions of Lemma V.3.2 delivers the existence of a unique positive eigenfunction for A. This is a variant of the Krein-Rutman theorem. The formulation we propose here is convenient because of its natural extension to random dynamical systems.*

V.4 A Random Krein-Rutman Theorem

In this section we reformulate the results of [AGD94, Hen97], which refer to the case of positive random matrices, for positive operators on Banach spaces.

An *invertible metric discrete dynamical system* (IDS) $(\Omega, \mathscr{F}, \mathbb{P}, \vartheta)$ is a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ together with a measurable map $\vartheta \colon \mathbf{Z} \times \Omega \to \Omega$ such that $\vartheta(z+z', \cdot) = \vartheta(z, \vartheta(z', \cdot))$ and $\vartheta(0, \omega) = \omega$ for all $\omega \in \Omega$, and such that $\mathbb{P}$ is invariant under $\vartheta(z, \cdot)$ for $z \in \mathbf{Z}$. For brevity we write $\vartheta^z(\cdot)$ for the map $\vartheta(z, \cdot)$. A set $\widetilde{\Omega} \subseteq \Omega$ is said to be *invariant* for ϑ if $\vartheta^z \widetilde{\Omega} = \widetilde{\Omega}$, for all $z \in \mathbf{Z}$. An IDS is said to be *ergodic* if any invariant set $\widetilde{\Omega}$ satisfies $\mathbb{P}(\widetilde{\Omega}) \in \{0, 1\}$ (cf. [Arn98, Appendix A]).

Consider X, E as in the previous section and, for a given IDS, a random variable $A \colon \Omega \to \mathscr{L}(X)$. This generates a measurable, linear, discrete random dynamical system (RDS) (see [Arn98, Definition 1.1.1]) φ on X by defining:

$$\varphi_n(\omega)x = A(\vartheta^n \omega) \cdots A(\omega)x, \qquad n \in \mathbf{N}_0. \tag{V.8}$$

If $A(\omega)$ is in addition positive for every $\omega \in \Omega$ (we then simply say that A is positive), we can interpret φ as an RDS on E via the projective action:

$$\varphi_n^\pi(\omega)x = A^\pi(\vartheta^n \omega) \circ \cdots \circ A^\pi(\omega)x, \qquad n \in \mathbf{N}_0.$$

Before we move on, let us recall the definition of invariant measures for random dynamical systems, cf. [Arn98, Section 1.4].

Definition V.4.1. *In the same setting as above, we say that a measure μ on $\Omega \times E$ is invariant for φ^{π} if:*

i The marginal μ_{Ω} of μ on Ω satisfies

$$\mu_{\Omega} = \mathbb{P}.$$

ii The measure μ is Θ_n–invariant, where Θ_n is the skew-product

$$\Theta_n(\omega, x) = (\vartheta^n \omega, \varphi_n^{\pi}(\omega)x).$$

Remark V.4.2. *In most cases an invariant measure μ for a random dynamical system φ admits a factorization of the form*

$$\mu(A \times B) = \int_{A \times B} \mu_{\omega}(\mathrm{d}x) \mathbb{P}(\mathrm{d}\omega),$$

where $A \subseteq \Omega$ and $B \subseteq X$ are measurable sets, and $\omega \mapsto \mu_{\omega}(C)$ is a measurable function for every measurable $C \subseteq X$. We then identify the measure μ with its factor μ_{ω}. In the setting of this article we will only deal with random Dirac measures, of the form

$$\mu_{\omega}(\mathrm{d}x) = \delta_{x_0(\omega)},$$

for a measurable map $x_0 \colon \Omega \to X$.

Assumption V.4.3. *Assume we are given X, K, U, E as in the previous section and that (E, d_H) is a* complete *metric space. Assume in addition that there exists an* ergodic *IDS $(\Omega, \mathscr{F}, \mathbb{P}, \vartheta)$. Let φ_n be a RDS defined via a random* positive *operator A as above, such that:*

$$\mathbb{P}\Big(A \in \mathscr{L}_{cp}(X)\Big) > 0.$$

In this setting the following is a random version of the Krein-Rutman theorem.

Theorem V.4.4. *Under Assumption V.4.3 there exists a ϑ–invariant set $\widetilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}$–measure and a random variable $u \colon \Omega \to E$ such that:*

i For all $\omega \in \widetilde{\Omega}$ and $f, g \in E$:

$$\limsup_{n \to \infty} \left[\frac{1}{n} \sup_{f,g \in E} \Big(\log d_H(\varphi_n^{\pi}(\omega) f, \varphi_n^{\pi}(\omega) g) \Big) \right] \leq \mathbb{E} \log \Big(\tau(A) \Big) < 0.$$

ii u is measurable w.r.t. to the σ–field $\mathscr{F}^{-} = \sigma((A(\vartheta^{-n} \cdot))_{n \in \mathbf{N}})$ and:

$$\varphi_n^{\pi}(\omega) u(\omega) = u(\vartheta^n \omega).$$

iii For all $\omega \in \widetilde{\Omega}$:

$$\limsup_{n \to \infty} \left[\frac{1}{n} \sup_{f \in E} \Big(\log d_H(\varphi_n^{\pi}(\vartheta^{-n}\omega) f, u(\omega)) \Big) \right] \leq \mathbb{E} \log \Big(\tau(A) \Big) < 0$$

as well as:

$$\limsup_{n \to \infty} \left[\frac{1}{n} \sup_{f \in E} \Big(\log d_H(\varphi_n^{\pi}(\omega) f, u(\vartheta^n \omega)) \Big) \right] \leq \mathbb{E} \log \Big(\tau(A) \Big) < 0.$$

iv The measure $\delta_{u(\omega)}$ is the unique invariant measure for the RDS φ^π on E.

Notation V.4.5. *We refer to the first property as* asymptotic synchronization *and to the third property as* one force, one solution *principle.*

Remark V.4.6. *Theorem V.4.4 can be stated also in continuous time. Suppose that $\vartheta\colon \mathbf{R}\times\Omega \to \Omega$ generates an invertible, measure-preserving and ergodic dynamical system over $(\Omega,\mathscr{F},\mathbb{P})$ and*

$$\varphi\colon \mathbf{R}_+ \times \Omega \times X \to X$$

defines a linear (i.e. $\varphi_t(\omega) \in \mathscr{L}(X)$, $\forall t \geq 0, \omega \in \Omega$) random dynamical system (see [Arn98, Definition 1.1.1]). Assume in addition that

$$\varphi_t(\omega) \text{ is positive } \forall t \geq 0,\ \omega \in \Omega, \qquad \mathbb{P}(\varphi_1 \in \mathscr{L}_{\mathrm{cp}}(X)) > 0.$$

Then there exists a ϑ–invariant set $\widetilde{\Omega}$ of full $\mathbb{P}$–measure, such that for all $\omega \in \widetilde{\Omega}$:

$$\limsup_{t\to\infty}\left[\frac{1}{t}\sup_{f,g\in E}\Big(\log\Big(d_H(\varphi^\pi_t(\omega)f,\varphi^\pi_t(\omega)g)\Big)\Big)\right] \leqslant \mathbb{E}\log\Big(\tau(\varphi_1)\Big) < 0.$$

And similarly one can adapt the properties at the points $(ii)-(iv)$ of Theorem V.4.4. This extension follows directly from the discrete case, observing that for $n = \lfloor t \rfloor$, since $\tau(\cdot) \leqslant 1$:

$$\begin{aligned}\log\big(d_H\big(\varphi^\pi_t(\omega)f,\varphi^\pi_t(\omega)g\big)\big) &\leq \log\Big(\tau(\varphi_{t-n}(\vartheta^{-n}\omega))d_H(\varphi^\pi_n(\omega)f,\varphi^\pi_n(\omega)g)\Big)\\ &\leq \log\Big(d_H(\varphi^\pi_n(\omega)f,\varphi^\pi_n(\omega)g)\Big).\end{aligned}$$

Then one can apply Theorem V.4.4 since any discrete random dynamical system has the form (V.8)*, with $A(\omega) = \varphi_1(\omega)$.*

The proof of Theorem V.4.4 will rely on the following lemma.

Lemma V.4.7. *There exists a ϑ–invariant set $\widetilde{\Omega} \subseteq \Omega$ of full $\mathbb{P}$–measure and an $\mathscr{F}^-$–adapted random variable $u : \Omega \to E$ such that:*

$$\varphi^\pi_n(\omega)u(\omega) = u(\vartheta^n\omega), \qquad \forall \omega \in \widetilde{\Omega}, n \in \mathbf{N}.$$

Moreover for all $\omega \in \widetilde{\Omega}$:

$$\limsup_{n\to\infty}\left[\frac{1}{n}\sup_{f\in E}\Big(\log d_H(\varphi^\pi_n(\vartheta^{-n}\omega)f,u(\omega))\Big)\right] \leq \mathbb{E}\log\Big(\tau(A)\Big) < 0.$$

Proof. We start by observing (as in [Hen97, Proof of Lemma 3.3]) that the sequence of sets $F_n(\omega) = \varphi^\pi_n(\vartheta^{-n}\omega)(E)$ is decreasing, i.e. $F_{n+1} \subseteq F_n$. Let us write $F(\omega) = \bigcap_{n\geq 1} F_n(\omega)$. Hence by Theorem V.3.1:

$$\Delta(F) \leq \lim_{n\to\infty}\Delta(F_n) = \lim_{n\to\infty} 4\ \mathrm{arctanh}\ (\tau(\varphi_n(\vartheta^{-n}\omega))).$$

Now, by the ergodic theorem and Assumption V.4.3 there exists a ϑ–invariant set $\widetilde{\Omega}$ of full $\mathbb{P}$–measure such that for all $\omega \in \widetilde{\Omega}$:

$$\limsup_{n\to\infty}\frac{1}{n}\log\Big(\tau(\varphi_n(\vartheta^{-n}\omega))\Big) \leq \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n}\log\tau(A(\vartheta^{-i}\omega)) = \mathbb{E}\log\Big(\tau(A)\Big) < 0. \qquad \text{(V.9)}$$

In particular $\lim_{n\to\infty}\tau\Big(\varphi_n(\theta^{-n}\omega)\Big)=0$, and since $\operatorname{arctanh}(0)=0$ we have that $\Delta(F)=0$. By completeness of E it follows that F is a singleton. Let us write $F(\omega)=\{u(\omega)\}$ and extend u trivially outside of $\widetilde{\Omega}$: it is clear that u is measurable with respect to $\mathscr{F}^-$. Since for $k\in\mathbf{N}$ and $n\geq k$

$$\varphi_n(\vartheta^{-n}\vartheta^k\omega)=\varphi_k(\omega)\circ\varphi_{n-k}(\vartheta^{-(n-k)}\omega),$$

passing to the limit with $n\to\infty$ we have: $u(\vartheta^k\omega)=\varphi_k^\pi(\omega)u(\omega)$.

Finally, as in the former result, a Taylor expansion guarantees that there exists a constant $c(\omega)>0$ such that:

$$\Delta(\varphi_n^\pi(\vartheta^{-n}\omega)(E))=4\ \operatorname{arctanh}\left(\tau(\varphi_n(\vartheta^{-n}\omega))\right)\leq 4(1+c(\omega))\tau(\varphi_n(\vartheta^{-n}\omega)).$$

This estimate, combined with the fact that

$$\sup_{f\in E}d_H(\varphi_n^\pi(\vartheta^{-n}\omega)f,u(\omega))=\sup_{f\in E}d_H(\varphi_n^\pi(\vartheta^{-n}\omega)f,\varphi_n^\pi(\vartheta^{-n}\omega)u(\vartheta^{-n}\omega))\leq\Delta(\varphi_n^\pi(\vartheta^{-n}\omega)(E))$$

and (V.9) provides the required convergence result.

□

Proof of Theorem V.4.4. As for the first property, compute:

$$\begin{aligned}d_H(\varphi_n^\pi(\omega)f,\varphi_n^\pi(\omega)g)&\leq\tau(A(\vartheta^n\omega))d_H(\varphi_{n-1}^\pi(\omega)f,\varphi_{n-1}^\pi(\omega)g)\\&\leq\prod_{i=0}^n\tau(A(\vartheta^i\omega))d_H(f,g).\end{aligned}$$

Then, applying the logarithm and Birkhoff's ergodic theorem we find:

$$\limsup_{n\to\infty}\frac{1}{n}\log\left(\tau(\varphi_n(\omega))\right)\leq\mathbb{E}\log\left(\tau(A)\right)<0.$$

If $\mathbb{E}\log\Big(\tau(A)\Big)=-\infty$ we can instead follow the previous computation with $\tau(A(\vartheta^i\omega))$ replaced by $\tau(A(\vartheta^i\omega))\vee e^{-M}$ and eventually pass to the limit $M\to\infty$. To obtain the result uniformly over f,g first observe that via Theorem V.3.1:

$$\sup_{f,g\in E}\Big(\log d_H(\varphi_n^\pi(\omega)f,\varphi_n^\pi(\omega)g)\Big)=\log\Big(\Delta(\varphi_n^\pi(\omega)(E))\Big)=\log\Big(4\ \operatorname{arctanh}(\tau(\varphi_n(\omega)))\Big),$$

and by a Taylor approximation, since $\lim_{n\to\infty}\tau(\varphi_n(\omega))=0$, there exists a constant $c(\omega)>0$ such that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{n}\log\Big(4\ \operatorname{arctanh}(\tau(\varphi_n(\omega)))\Big)&\leq\limsup_{n\to\infty}\frac{1}{n}\log\Big((1+c(\omega))\tau(\varphi_n(\omega))\Big)\\&=\limsup_{n\to\infty}\frac{1}{n}\log\Big(\tau(\varphi_n(\omega))\Big)\leq\mathbb{E}\log\tau(A).\end{aligned}$$

Point (ii) as well as the first property of (iii) follow from Lemma V.4.7. As for the second property of (iii) we observe that

$$\begin{aligned}\sup_{f\in E}\Big(\log d_H(\varphi_n^\pi(\omega)f,u(\vartheta^n\omega))\Big)&=\sup_{f\in E}\Big(\log d_H(\varphi_n^\pi(\omega)f,\varphi_n^\pi(\omega)u(\omega))\Big)\\&\leq\sup_{f,g\in E}\Big(\log d_H(\varphi_n^\pi(\omega)f,\varphi_n^\pi(\omega)g)\Big),\end{aligned}$$

so that the estimate is now a consequence of point (i). As for (iv), we have that for any two measurable $A \subseteq \Omega, B \subseteq E$:

$$\begin{aligned}\int_{\Omega\times E} 1_A(\vartheta^n\omega)1_B(\varphi^\pi_n(\omega)f)\delta_{u(\omega)}(\mathrm{d}f)\mathbb{P}(\mathrm{d}\omega) &= \int_{\Omega\times E} 1_A(\vartheta^n\omega)1_B(u(\vartheta^n\omega))\mathbb{P}(\mathrm{d}\omega)\\ &= \int_{\Omega\times E} 1_A(\omega)1_B(u(\omega))\mathbb{P}(\mathrm{d}\omega),\end{aligned}$$

which implies that $\delta_{u(\omega)}$ is invariant (see Definition V.4.1). Finally, to see that $\delta_{u(\omega)}$ is the unique invariant measure, let μ be any invariant measure. Then

$$\begin{aligned}\int_{\Omega\times E} \min\{1, d_H(f, u(\omega))\}\mu(\mathrm{d}\omega, \mathrm{d}f) &= \lim_{n\to\infty}\int_{\Omega\times E} \min\{1, d_H(\varphi^\pi_n(\omega)f, u(\vartheta^n\omega))\}\mu(\mathrm{d}\omega, \mathrm{d}f)\\ &\leq \lim_{n\to\infty}\int_{\Omega} \sup_{f\in E}\min\{1, d_H(\varphi^\pi_n(\omega)f, u(\vartheta^n\omega))\}\mathbb{P}(\mathrm{d}\omega)\\ &\leq 0,\end{aligned}$$

where in the last line we used dominated convergence and the results of point (iii). In particular, we have found that

$$\mu(\{(\omega, f) \in \Omega\times E \,:\, f \neq u(\omega)\}) = 0,$$

implying that $\mu(\mathrm{d}\omega, \mathrm{d}f) = \delta_{u(\omega)}(\mathrm{d}f)\mathbb{P}(\mathrm{d}\omega)$. Note that the invariant sets in all points can be chosen equal to the same $\widetilde{\Omega}$ up to taking intersections of invariant sets, which are still invariant.

□

V.5 Synchronization for linear SPDEs

In this section we discuss how to apply the previous results to stochastic PDEs. Concrete examples will be covered in the next section. For clarity, nonetheless, the reader should keep in mind that we want to study ergodic properties of solutions to Equation (V.1). Since the associated heat equation with multiplicative noise (V.3) is linear and the solution map is expected to be strictly positive (because the defining differential operator is parabolic), we may assume that the solution map generates a continuous, linear, strictly positive random dynamical system φ.

Definition V.5.1. *A* continuous *RDS over a* discrete *IDS* $(\Omega, \mathscr{F}, \mathbb{P}, \vartheta)$ *and on a measurable space* $(X, \mathscr{B})$ *is a map*

$$\varphi\colon \mathbf{R}_+ \times \Omega \times X \to X$$

such that the following two properties hold:

i Measurability: φ *is* $\mathscr{B}(\mathbf{R}_+)\otimes\mathscr{F}\otimes\mathscr{B}$*-measurable.*

ii Cocycle property: $\varphi(0, \omega) = \mathrm{Id}_X$*, for all* $\omega \in \Omega$ *and:*

$$\varphi(t+n, \omega) = \varphi(t, \vartheta^n\omega)\circ\varphi(n, \omega), \qquad \forall t \in \mathbf{R}_+, n \in \mathbf{N}_0, \omega \in \Omega.$$

We then formulate the following assumptions, under which our main result will hold.

Assumption V.5.2. *Let $d \in \mathbf{N}$ and $\beta > 0$. Let $(\Omega_{\text{kpz}}, \mathscr{F}, \mathbb{P}, \vartheta)$ be a discrete ergodic IDS, over which is defined a continuous RDS φ:*

$$\varphi : \mathbf{R}_+ \times \Omega_{\text{kpz}} \to \mathscr{L}(C(\mathbf{T}^d)).$$

There exists a ϑ–invariant set $\widetilde{\Omega} \subseteq \Omega_{\text{kpz}}$ of full $\mathbb{P}$–measure such that the following properties are satisfied for all $\omega \in \widetilde{\Omega}$ and any $T > S > 0$:

i There exists a kernel $K : \Omega_{\text{kpz}} \to C_{\text{loc}}((0,\infty); C(\mathbf{T}^d \times \mathbf{T}^d))$ such that for all $S \le t \le T$:

$$\varphi_t(\omega) f(x) = \int_{\mathbf{T}^d} K(\omega, t, x, y) f(y) \, \mathrm{d}y, \qquad \forall f \in C(\mathbf{T}^d), x \in \mathbf{T}^d.$$

ii There exist $0 < \gamma(\omega, S, T) \le \delta(\omega, S, T)$ such that:

$$\gamma(\omega, S, T) \le K(\omega, t, x, y) \le \delta(\omega, S, T), \qquad \forall x, y \in \mathbf{T}^d, \ S \le t \le T,$$

which implies that $\mathbb{P}\Big(\varphi_t \in \mathscr{L}_{\text{cp}}(C(\mathbf{T}^d)), \forall t \in (0,\infty)\Big) = 1$.

iii There exists a constant $C(\beta, \omega, S, T)$ such that:

$$\|\varphi_t f\|_\beta \le C(\beta, \omega, S, T) \|f\|_\infty, \qquad \forall f \in C(\mathbf{T}^d), \ S \le t \le T.$$

iv Consider (E, d_H) as in Lemma V.3.2. The following moment estimates are satisfied for any $f \in E$:

$$\mathbb{E} \log\Big(C(\beta, S, T)\Big) + \mathbb{E} \sup_{S \le t \le T} d_H(\varphi_t^\pi f, f) < +\infty,$$

where φ_t^π is defined to be the identity outside of $\widetilde{\Omega}$.

The first two assumptions allow us to use the results from the previous section. The last two will guarantee convergence in appropriate Hölder spaces. In view of the motivating example and in the setting of the previous assumption, we say that for $z \in \mathbf{Z}$ and $h_0 \in C(\mathbf{T}^d)$ the map

$$[z, +\infty) \times \mathbf{T}^d \ni (t, x) \mapsto h^z(\omega, t, x), \qquad h^z(\omega, z, x) = h_0(x)$$

solves Equation (V.1) if $h^z(\omega, t) = \log\Big(\varphi_t(\vartheta^z \omega) \exp(h_0)\Big)$ for φ_t as in the previous assumption.

Theorem V.5.3. *Under Assumption V.5.2, for $i = 1, 2$, $h_0^i \in C(\mathbf{T}^d)$ and $n \in \mathbf{N}$, let $h_i(t) \in C(\mathbf{T}^d)$ be the random solution to Equation* (V.1) *started at time* 0 *with initial data h_0^i and evaluated at time $t \ge 0$. Similarly, let $h_i^{-n}(t) \in C(\mathbf{T}^d)$ be the solution started at time $-n$ with initial data h_0^i and evaluated at time $t \ge -n$. There exists an invariant set $\overline{\Omega} \subseteq \Omega_{\text{kpz}}$ of full $\mathbb{P}$–measure such that for any $0 < \alpha < \beta$, for any $T > 0$ and any $\omega \in \overline{\Omega}$:*

i *There exists a map* $c(h_0^1, h_0^2)\colon \Omega_{\mathrm{kpz}} \times \mathbf{R}_+ \to \mathbf{R}$ *such that:*

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{t\in[n,n+T]}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_{C^\alpha(\mathbf{T}^d)}\Big)\right] \\ \leq \Big(1-\frac{\alpha}{\beta}\Big)\mathbb{E}\log\Big(\tau(\varphi_1)\Big)<0,$$

as well as:

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{t\in[n,n+T]}[h_i(\omega,t)]_\beta\Big)\right]\leq 0.$$

And uniformly over h_0^i*:*

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{\substack{h_0^i\in C(\mathbf{T}^d),\\ t\in[n,n+T]}}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Big)\right]\leq\mathbb{E}\log\Big(\tau(\varphi_1)\Big)<0,$$

ii *There exists a random function* $h_\infty\colon \Omega_{\mathrm{kpz}}\to C_{\mathrm{loc}}((-\infty,\infty);C^\alpha(\mathbf{T}^d))$ *and a sequence of maps* $c^{-n}(h_0^1)\colon \Omega_{\mathrm{kpz}}\times\mathbf{R}_+\to\mathbf{R}$ *for which:*

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{\substack{h_0^1\in C(\mathbf{T}^d),\\ t\in[(-T)\vee(-n),T]}}\|h_1^{-n}(\omega,t)-h_\infty(\omega,t)-c^{-n}(\omega,t,h_0^1)\|_{C^\alpha(\mathbf{T}^d)}\Big)\right] \\ \leq\Big(1-\frac{\alpha}{\beta}\Big)\mathbb{E}\log\Big(\tau(\varphi_1)\Big)<0.$$

Passing to the gradient one can omit all constants and find the following for Burgers' Equation.

Corollary V.5.4. *In the same setting as before, it immediately follows that also:*

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{t\in[n,n+T]}\|\nabla_x h_1(\omega,t)-\nabla_x h_2(\omega,t)\|_{C^{\alpha-1}(\mathbf{T}^d)}\Big)\right]\leq\Big(1-\frac{\alpha}{\beta}\Big)\mathbb{E}\log\Big(\tau(\varphi_1)\Big)<0,$$

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\Big(\sup_{\substack{h_0^1\in C(\mathbf{T}^d),\\ t\in[(-T)\vee(-n),T]}}\|\nabla_x h_1^{-n}(\omega,t)-\nabla_x h_\infty(\omega,t)\|_{C^{\alpha-1}(\mathbf{T}^d)}\Big)\right]\leq\Big(1-\frac{\alpha}{\beta}\Big)\mathbb{E}\log\Big(\tau(\varphi_1)\Big)<0,$$

where the space $C^{\alpha-1}(\mathbf{T}^d)$ *is understood as the Besov space* $B^{\alpha-1}_{\infty,\infty}(\mathbf{T}^d)$ *for* $\alpha\in(0,1)$.

Proof of Theorem V.5.3. Let us first define $\overline{\Omega}=\widetilde{\Omega}$, the latter as in Assumption V.5.2, and consider $\omega\in\overline{\Omega}$ and $T>0$. In the course of the proof, where necessary (for example to apply the ergodic theorem) we will redefine $\overline{\Omega}$ to be a smaller ϑ–invariant set of full $\mathbb{P}$–measure.

Step 1. Define:

$$u_0^i=\exp(h_0^i)/\|\exp(h_0^i)\|_{L^1}\in E,$$

so that $h_i(\omega,t)=\log\Big(\varphi_t^\pi(\omega)u_0^i\Big)+c_i(\omega,t)$, where $c_i(\omega,t)\in\mathbf{R}$ is the normalization constant:

$$c_i(\omega,t)=\log\Big(\int_{\mathbf{T}^d}(\varphi_t(\omega)u_0^i)(x)\,\mathrm{d}x\Big)+\log\Big(\int_{\mathbf{T}^d}\exp(h_0^i)(x)\,\mathrm{d}x\Big).$$

Let us write $c(\omega,t,h_0^1,h_0^2)=c_1(\omega,t)-c_2(\omega,t)$. Similarly, for $-n\le t\le 0$ one has:

$$h_i^{-n}(\omega,t)=\log\Big(\varphi^\pi_{n+t}(\vartheta^{-n}\omega)u_0^i\Big)+c_i^{-n}(\omega,t)=h_i(\vartheta^{-n}\omega,n+t),$$

where $c_i^{-n}(\omega,t)=c_i(\vartheta^{-n}\omega,n+t)$. Also, write $c^{-n}(\omega,t,h_0^1,h_0^2)=c_1^{-n}(\omega,t)-c_2^{-n}(\omega,t)$. Now we prove the following simpler version of the required result:

$$\begin{aligned}\limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{\substack{h_0^i\in C(\mathbf{T}^d),\\ t\in[n,n+T]}}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Bigg)\Bigg]&\\ \le\mathbb{E}\log\Big(\tau(\varphi_1)\Big),&\\ \limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{\substack{h_0^i\in C(\mathbf{T}^d)\\ t\in[(-T)\vee(-n),T]}}\|h_1^{-n}(\omega,t)-h_2^{-n}(\omega,t)-c^{-n}(\omega,t,h_0^1,h_0^2)\|_\infty\Bigg)\Bigg]&\\ \le\mathbb{E}\log\Big(\tau(\varphi_1)\Big).&\end{aligned}\tag{V.10}$$

First we eliminate the time supremum, since in view of Inequality (V.7):

$$\begin{aligned}\limsup_{n\to\infty}&\Bigg[\frac{1}{n}\log\Bigg(\sup_{\substack{h_0^i\in C(\mathbf{T}^d)\\ t\in[n,n+T]}}\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty\Bigg)\Bigg]\\ &\le\limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{\substack{h_0^i\in C(\mathbf{T}^d)\\ t\in[n,n+T]}}d_H(\varphi^\pi_t(\omega)u_0^1,\varphi^\pi_t(\omega)u_0^2)\Bigg)\Bigg]\\ &\le\limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{h_0^i\in C(\mathbf{T}^d)}d_H(\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_n(\omega)u_0^2)\Bigg)\Bigg]\end{aligned}$$

where we used the definition of the contraction constant $\tau(\cdot)$ together with the fact that $\tau(\cdot)\le 1$ (cf. Theorem V.3.1) to obtain

$$\begin{aligned}d_H(\varphi^\pi_t(\omega)u_0^1,\varphi^\pi_t(\omega)u_0^2)&=d_H(\varphi^\pi_{t-n}(\vartheta^n\omega)\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_{t-n}(\vartheta^n\omega)\varphi^\pi_n(\omega)u_0^2)\\ &\le\tau(\varphi^\pi_{t-n}(\vartheta^n\omega))d_H(\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_n(\omega)u_0^2)\\ &\le d_H(\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_n(\omega),u_0^2)\end{aligned}$$

so that one can estimate:

$$\sup_{t\in[n,n+T]}d_H(\varphi^\pi_t(\omega)u_0^1,\varphi^\pi_t(\omega)u_0^2)\le d_H(\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_n(\omega)u_0^2).$$

Similarly, also for the backwards case. At this point, in view of Assumption V.5.2, we can apply Theorem V.4.4 in the setting of Lemma V.3.2 with $A(\omega)=\varphi_1(\omega)$ to see that there exists a $u_\infty=\exp(h_\infty)\colon\Omega_{\mathrm{kpz}}\to C(\mathbf{T}^d)$ such that:

$$\begin{aligned}\limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{u_0^i\in E}d_H(\varphi^\pi_n(\omega)u_0^1,\varphi^\pi_n(\omega)u_0^2)\Bigg)\Bigg]&\le\mathbb{E}\log\Big(\tau(\varphi_1)\Big),\\ \limsup_{n\to\infty}\Bigg[\frac{1}{n}\log\Bigg(\sup_{u_0^1\in E}d_H(\varphi^\pi_n(\vartheta^{-n}\omega)u_0^1,u_\infty(\omega))\Bigg)\Bigg]&\le\mathbb{E}\log\Big(\tau(\varphi_1)\Big),\end{aligned}$$

up to choosing ω in a possibly smaller $\overline{\Omega}$, which via the previous calculation implies (V.10). In particular, this also proves the bound uniformly over h_0^i at point (i) of the theorem.

Step 2. We pass to prove convergence in $C^\alpha(\mathbf{T}^d)$ for $0 < \alpha < \beta$. Hence consider $\alpha < \beta$ fixed and define $\theta \in (0,1)$ by $\alpha = \beta\theta$. As convergence in $C(\mathbf{T}^d)$ is already established, to prove convergence in $C^\alpha(\mathbf{T}^d)$ one has to control the α–seminorm $[\cdot]_\alpha$ of $h_1 - h_2$. We treat the forwards and backwards in time cases differently, starting with the first case. Let us recall the bound

$$[f]_\alpha \leq C(\alpha,\beta)\|f\|_\infty^{1-\theta}[f]_\beta^\theta,$$

which is proven in Lemma V.8.5. With this bound one can estimate the Hölder seminorm via:

$$\begin{aligned}[h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)]_\alpha &\leq C(\alpha,\beta)\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty^{1-\theta}\cdot \\ &\quad\cdot\Big([\log(\varphi_t^\pi(\omega)u_0^1)]_\beta+[\log(\varphi_t^\pi(\omega)u_0^2)]_\beta\Big)^\theta.\end{aligned} \tag{V.11}$$

Since we already proved that the first factor in the product vanishes exponentially fast, our aim will be to prove that the second factor does not explode exponentially fast. This amounts to proving the second bound at point (i). To this end, fix $n \in \mathbf{N}, T > 0$ and $t \in [n, n+T]$, and define τ by $t = n-1+\tau$. We can use Lemma V.8.6 to bound the last terms by:

$$\begin{aligned}[h_1(\omega,t)]_\beta &= [\log(\varphi_t^\pi(\omega)u_0^1)]_\beta \\ &\leq \overline{C}(\beta)\left(\frac{1+[\varphi_t^\pi(\omega)u_0^1]_\beta}{m(\varphi_t^\pi(\omega)u_0^1)}\right)^{\lfloor\beta\rfloor+1} \\ &\leqslant \overline{C}(\beta)\left(\frac{1+[\varphi_\tau^\pi(\vartheta^{n-1}\omega)\circ\varphi_{n-1}^\pi(\omega)u_0^1]_\beta}{m(\varphi_t^\pi(\omega)u_0^1)}\right)^{\lfloor\beta\rfloor+1} \\ &\leq \overline{C}(\beta)\left(\frac{1+C(\beta,\vartheta^{n-1}\omega,1,T+1)\|\varphi_{n-1}^\pi(\omega)u_0^1\|_\infty}{m(\varphi_t^\pi(\omega)u_0^1)}\right)^{\lfloor\beta\rfloor+1}\end{aligned}$$

where $m(\cdot)$ indicates the minimum of a function and $\overline{C}(\beta)$ is the deterministic constant of Lemma V.8.6. We can plug this estimate into Equation (V.11) to obtain for some deterministic $\widetilde{C}(\alpha,\beta) > 0$:

$$\begin{aligned}&\log[h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)]_\alpha \\ &\qquad\leq(1-\theta)\log\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty \\ &\qquad\quad+\theta(\lfloor\beta\rfloor+1)\log\Big(\sum_{i=1,2}\frac{1+C(\beta,\vartheta^{n-1}\omega,1,T+1)\|\varphi_{n-1}^\pi(\omega)u_0^i\|_\infty}{m(\varphi_t^\pi(\omega)u_0^i)}\Big) \\ &\qquad\quad+\widetilde{C}(\alpha,\beta) \\ &\qquad\leq(1-\theta)\log\|h_1(\omega,t)-h_2(\omega,t)-c(\omega,t,h_0^1,h_0^2)\|_\infty \\ &\qquad\quad+\sum_{i=1,2}\theta(\lfloor\beta\rfloor+1)\log\Big(2\frac{(1+C(\beta,\vartheta^{n-1}\omega,1,T+1))\|\varphi_{n-1}^\pi(\omega)u_0^i\|_\infty}{m(\varphi_t^\pi(\omega)u_0^i)}\Big) \\ &\qquad\quad+\widetilde{C}(\alpha,\beta),\end{aligned}$$

where in the last line we used that $\log \max_i x_i = \max_i \log x_i$ and that $\|\varphi^\pi_{n-1}(\omega)u^i_0\|_\infty \geqslant 1$, since $\varphi^\pi_{n-1}(\omega)u^i_0 \in E$ and hence $\int_{\mathbf{T}^d} \varphi^\pi_{n-1}(\omega)u^i_0(x)\,\mathrm{d}x = 1$. To conclude, in view of Equation (V.10), we have to prove that for $i = 1,2$:

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\left(\sup_{n\le t\le n+T}\left(\frac{1+C(\beta,\vartheta^{n-1}\omega,1,T+1)}{m(\varphi^\pi_t(\omega)u^i_0)}\|\varphi^\pi_{n-1}(\omega)u^i_0\|_\infty\right)\right)\right]\le 0. \tag{V.12}$$

In particular, the latter inequality also implies the β-Hölder norm bound of h_i at point (i) of the theorem. Now we observe that for $n\in\mathbf{N}, n\ge 1,\ T>0, t=n-1+\tau\in[n,n+T]$ and for any $f\in E$:

$$\begin{aligned}\log\Big(1+C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big)+\log\Big(\|\varphi^\pi_{n-1}(\omega)f\|_\infty\Big)-\log\Big(m\Big(\varphi^\pi_t(\omega)f\Big)\Big)\\ \le \log\Big(1+C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big)+2\sup_{1\le\tau\le T+1} d_H(\varphi^\pi_{n-1+\tau}(\omega)f,f).\end{aligned}$$

Here we used again that $\varphi^\pi_s(\omega)f$ lies in E for all ω and s, and that for $g\in E$ we have $m(g)\leqslant 1\leqslant \|g\|_\infty$, since it holds that $\int_{\mathbf{T}^d} g(x)\,\mathrm{d}x = 1$. In fact this implies

$$\log\Big(m(\varphi^\pi_s(\omega)f)\Big)\leqslant 0\leqslant \log\Big(\|\varphi^\pi_s(\omega)f\|_\infty\Big),$$

so that

$$\begin{aligned}\log\Big(\|\varphi^\pi_{n-1}(\omega)f\|_\infty\Big)-\log\Big(m\Big(\varphi^\pi_t(\omega)f\Big)\Big) &\leqslant \log\Big(\|\varphi^\pi_{n-1}(\omega)f\|_\infty\Big)-\log\Big(m(\varphi^\pi_{n-1}(\omega)f)\Big)\\ &\quad+\log\Big(\|\varphi^\pi_t(\omega)f\|_\infty\Big)-\log\Big(m\Big(\varphi^\pi_t(\omega)f\Big)\Big)\\ &\leqslant 2\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi^\pi_{n-1+\tau}(\omega)f,f).\end{aligned}$$

Hence we have reduced (V.12) to proving the following:

$$\limsup_{n\to\infty}\frac{1}{n}\left[\log\Big(C(\beta,\vartheta^{n-1}\omega,1,T+1)\Big)+\sup_{1\le\tau\le T+1} d_H(\varphi^\pi_{n-1+\tau}(\omega)f,f)\right]\le 0. \tag{V.13}$$

Let us start with the last term and bound:

$$\begin{aligned}&d_H(\varphi^\pi_{n-1+\tau}(\omega)f,f)\\ &\quad\le \tau(\varphi_\tau(\vartheta^{n-1}\omega))d_H(\varphi^\pi_{n-1}(\omega)f,f)+d_H(\varphi^\pi_\tau(\vartheta^{n-1}\omega)f,f)\\ &\quad\le \sum_{i=0}^{n-1}\prod_{j=i+1}^{n-1}\tau(\varphi_1(\vartheta^j\omega))d_H(\varphi^\pi_1(\vartheta^i\omega)f,f)+\sup_{1\le\tau\le T+1} d_H(\varphi^\pi_\tau(\vartheta^{n-1}\omega)f,f).\end{aligned} \tag{V.14}$$

By Assumption V.5.2 $\mathbb{E}[\sup_{1\le\tau\le T+1} d_H(\varphi^\pi_\tau 1,1)]<\infty$, hence by the ergodic theorem, up to further reducing the set $\overline{\Omega}$, for all $\omega\in\overline{\Omega}$:

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi^\pi_\tau(\vartheta^i\omega)1,1)=\mathbb{E}\Big[\sup_{1\le\tau\le T+1} d_H(\varphi^\pi_\tau 1,1)\Big]<\infty.$$

Now by the triangle inequality we have, for any $f\in E$:

$$d_H(\varphi^\pi_\tau(\vartheta^n\omega)f,f)\leqslant d_H(\varphi^\pi_\tau(\vartheta^n\omega)f,\varphi^\pi_\tau(\vartheta^n\omega)1)+d_H(\varphi^\pi_\tau(\vartheta^n\omega)1,1)+d_H(1,f).$$

Hence, in particular, by our previous results and Lemma V.8.4 applied to the central term above, we find that for any $j \in \mathbf{N}$:

$$\lim_{n\to\infty} \frac{1}{n} \sup_{1\le\tau\le T+1} d_H(\varphi^\pi_\tau(\vartheta^{n-j}\omega)f, f) = 0. \tag{V.15}$$

Here we used that $\overline{\Omega}$ is invariant under ϑ. So if $\omega \in \overline{\Omega}$, then also $\vartheta^{-j}\omega \in \overline{\Omega}$. Now observe that by Lebesgue dominated convergence, since $d_H(\varphi^\pi_1(\omega)f, f) \in L^1(\Omega)$ and since $\tau(\cdot) \leqslant 1$ as well as $\lim_{c\to\infty}\prod_{j=1}^c \tau(\varphi_1(\vartheta^j\omega)) = 0$, $\forall \omega \in \overline{\Omega}$, it holds that:

$$\lim_{c\to\infty} \mathbb{E}\left[\prod_{j=1}^{c} \tau(\varphi_1(\vartheta^j\cdot))d_H(\varphi^\pi_1(\cdot)f, f)\right] = 0.$$

Hence fix any $\varepsilon > 0$ and choose a deterministic $c(\varepsilon) \in \mathbf{N}$ so that:

$$\mathbb{E}\left[\prod_{j=1}^{c(\varepsilon)} \tau(\varphi_1(\vartheta^j\cdot))d_H(\varphi^\pi_1(\cdot)f, f)\right] \leqslant \varepsilon.$$

Now we use the bound (V.14) together with (V.15) and the ergodic theorem to obtain:

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{1}{n} \sup_{1\le\tau\le T+1} d_H(\varphi^\pi_{n-1+\tau}(\omega)f, f) \\
&\quad\le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1-c(\varepsilon)} \prod_{j=i+1}^{i+c(\varepsilon)} \tau(\varphi_1(\vartheta^j\omega))d_H(\varphi^\pi_1(\vartheta^i\omega)f, f) \\
&\qquad+ \limsup_{n\to\infty} \frac{1}{n} \sum_{i=n-1-c(\varepsilon)}^{n} \sup_{1\leqslant\tau\leqslant T+1} d_H(\varphi^\pi_\tau(\vartheta^{i-1}\omega)f, f) \\
&\quad\le \varepsilon.
\end{aligned}$$

As ε is arbitrarily small we have proven that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{1\le\tau\le T+1} d_H(\varphi^\pi_{n-1+\tau}(\omega)f, f) \leqslant 0,$$

which is of the required order for (V.13). To complete the proof of (V.13) we are left with the term containing $C(\beta, \vartheta^n\omega)$. Once more Assumption V.5.2 together with the ergodic theorem and Lemma V.8.4 imply that:

$$\lim_{n\to\infty} \frac{1}{n} \log C(\beta, \vartheta^n\omega, 1, T+1) = 0,$$

thus completing the proof of (V.13) and hence of point (i) of our theorem.

Step 3. Now, let us pass to the convergence in C^α backwards in time, which completes the proof of point (ii). The proof is analogous to, but simpler than the one we presented in Step 2. Since in Equation (V.10) we proved convergence in the $\|\cdot\|_\infty$ norm, we now have to consider the $[\cdot]_\alpha$ seminorm. Up to replacing T with $\lceil T\rceil$ assume $T \in \mathbf{N}$. Then, consider $n \in \mathbf{N}$ with $T < n-1$ and $-T \leq t \leq T$ so that $t = -T-1+\tau$ with $1 \leq \tau \leq 2T+1$. As in (V.11) we define $\theta = \frac{\alpha}{\beta} \in (0,1)$ and use the interpolation bound of Lemma V.8.5:

$$[h_1^{-n}(\omega,t)-h_\infty(\omega,t)-c(\omega,t,h_0^1)]_\alpha \leq C(\alpha,\beta)\Big(\|h_1^{-n}(\omega,t)-h_\infty(\omega,t)-c(\omega,t,h_0^1)\|_\infty\Big)^{1-\theta}\cdot$$
$$\cdot\Big([\log(\varphi_t^\pi(\vartheta^{-n}\omega)u_0^1)]_\beta+[\log(u_\infty(\omega,t))]_\beta\Big)^\theta.$$

In view of Equation (V.10) it now suffices to prove that

$$\limsup_{n\to\infty}\left[\frac{1}{n}\log\left(\sup_{-T\leq t\leq T}\Big([\log(\varphi_t^\pi(\vartheta^{-n}\omega)u_0^1)]_\beta+[\log(u_\infty(\omega,t))]_\beta\Big)\right)\right] \leqslant 0. \tag{V.16}$$

Since the $[\cdot]_\beta$ seminorm is invariant under constant shifts (i.e. $[f+\zeta]_\beta = [f]_\beta$ for any $f\colon \mathbf{T}^d \to \mathbf{R}$, $\zeta \in \mathbf{R}$) we can rewrite the terms inside the limit as

$$\Big[\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\circ\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\Big)\Big]_\beta + \Big[\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)u_\infty(\omega,-T-1)\Big)\Big]_\beta.$$

In particular, since $\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1 \to u_\infty(\omega,-T-1)$ in $C(\mathbf{T}^d)$ uniformly over u_0^1, there exists a $c(\omega)>0$ such that

$$u_\infty(\omega,-T-1)(x) \geqslant c(\omega), \quad \varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1(x) \geq c(\omega), \quad \forall x \in \mathbf{T}^d,\ n\in\mathbf{N},\ u_0^1 \in C(\mathbf{T}^d).$$

By Assumption V.5.2 and Lemma V.8.6 this implies that:

$$\sup_{-T\leqslant t\leqslant T}[\log(u_\infty(\omega,t))]_\beta < \infty,$$

as well as:

$$\inf_{u_0^1\in C(\mathbf{T}^d)}\ \inf_{n>T+1}\ \inf_{\substack{1\leq\tau\leq 2T+1\\ x\in\mathbf{T}^d}} \varphi_\tau(\vartheta^{-T-1}\omega)\Big(\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\Big)(x) \geq \overline{c}(\omega),$$

with

$$\overline{c}(\omega) = c(\omega)\gamma(\vartheta^{-T-1}\omega,1,2T+1).$$

Hence, applying again Lemma V.8.6, we obtain:

$$\begin{aligned}\Big[\log\Big(\varphi_\tau(\vartheta^{-T-1}\omega)\circ\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\Big)\Big]_\beta &\leqslant \left(\frac{1+\Big[\varphi_\tau(\vartheta^{-T-1}\omega)\circ\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\Big]_\beta}{\overline{c}(\omega)}\right)^{\lfloor\beta\rfloor+1}\\ &\leqslant \left(\frac{1+C(\beta,\vartheta^{-T-1}\omega,1,2T+1)\|\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\|_\infty}{\overline{c}(\omega)}\right)^{\lfloor\beta\rfloor+1}\\ &\leqslant \left(\frac{1+C(\beta,\vartheta^{-T-1}\omega,1,2T+1)M(\omega)}{\overline{c}(\omega)}\right)^{\lfloor\beta\rfloor+1},\end{aligned}$$

with $M(\omega) = \sup_n \|\varphi_{n-T-1}^\pi(\vartheta^{-n}\omega)u_0^1\|_\infty < \infty$ in view of (V.10). All the calculations so far show that we can bound both terms in (V.16) uniformly over n. So (V.16) is proven, and this concludes the proof of (ii). □

V.6 Examples

We treat two prototypical examples, which show the range of applicability of the previous results. First, we consider the KPZ equation driven by a noise that is fractional in time but smooth in space. In a second example, we consider the KPZ equation driven by space-time white noise.

V.6.1 KPZ driven by fractional noise

Fix a Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$ and consider the noise $\eta(t,x) = \xi^H(t)V(x)$ for some $V \in C^\infty(\mathbf{T})$ and where $\xi^H(t) = \partial_t \beta^H(t)$ for a fractional Brownian motion β^H of Hurst parameter H. We restrict to $H > \frac{1}{2}$ because the case $H = \frac{1}{2}$ is identical to the setting in [Sin91], while for $H < \frac{1}{2}$ one encounters difficulties with fractional stochastic calculus that lie beyond the scopes of this work. For convenience, we let us define the noise ξ^H via its spectral covariance function, see [PT00, Section 3], namely as the Gaussian process indexed by functions $f\colon \mathbf{R} \to \mathbf{R}$ such that $\int_{\mathbf{R}} |\tau|^{1-2H} |\hat{f}(\tau)|^2 \,\mathrm{d}\tau < \infty$ (with $\hat{f}$ being the temporal Fourier transform), with covariance:

$$\mathbb{E}\Big[\xi^H(f)\xi^H(g)\Big] = c_H \int_{\mathbf{R}} |\tau|^{1-2H} \hat{f}(\tau)\overline{\hat{g}(\tau)}\,\mathrm{d}\tau, \qquad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}. \tag{V.17}$$

For the statement of the following lemma, recall the definition of $H^\alpha_a(\mathbf{R})$ given in (V.5).

Lemma V.6.1. *Fix any $H \in (\frac{1}{2}, 1)$, $\alpha < H - 1$, $a > \frac{1}{2}$. Let ξ^H be the Gaussian process as defined by* (V.17)*. Then, almost surely ξ^H takes values in $H^\alpha_a(\mathbf{R})$. Next, define $\Omega_{\mathrm{kpz}} = H^\alpha_a(\mathbf{R})$ and $\mathscr{F} = \mathscr{B}(H^\alpha_a(\mathbf{R}))$ and let $\mathbb{P}$ be the law of ξ^H on Ω_{kpz}. Furthermore, let $\{\vartheta^z\}_{z\in\mathbf{Z}}$ be the integer translation group, which acts on smooth functions $\varphi \in \mathscr{S}(\mathbf{R})$ by:*

$$\vartheta^z\varphi(t) = \varphi(t+z), \quad \forall\, t \in \mathbf{R},$$

and which is extended by duality to all distributions $\omega \in \Omega_{\mathrm{kpz}}$:

$$\langle \vartheta^z\omega, \varphi\rangle = \langle \omega, \vartheta^{-z}\varphi\rangle, \quad \forall \varphi \in \mathscr{S}(\mathbf{R}).$$

Then the space $(\Omega_{\mathrm{kpz}}, \mathscr{F}, \mathbb{P}, \vartheta)$ forms an ergodic IDS. In addition, up to modifying ξ^H on a ϑ–invariant null-set N_0, for any $\omega \in \Omega_{\mathrm{kpz}}$ there exists a $\beta^H(\omega) \in C^{\alpha+1}_{\mathrm{loc}}(\mathbf{R})$ with:

$$\xi^H(\omega) = \partial_t\beta^H(\omega) \ \text{ in the sense of distributions}, \qquad \beta^H_0(\omega) = 0.$$

Moreover, $(\beta^H_t)_{t\geq 0}$ has the law of a fractional Brownian motion of parameter H.

Proof. To show that ξ^H takes values in H^α_a almost surely, observe that:

$$\mathbb{E}\|\xi^H\|^2_{H^\alpha_a(\mathbf{R})} = \sum_{j\geq -1} 2^{2\alpha j}\mathbb{E}\|\Delta_j\xi(\cdot)/\langle\cdot\rangle^a\|_{L^2(\mathbf{R})}.$$

Then one can bound:

$$\begin{aligned}\mathbb{E}\Big[\|\Delta_j\xi^H(\cdot)/\langle\cdot\rangle^a\|^2_{L^2}\Big] &= \int_{\mathbf{R}} \frac{1}{(1+|t|)^{2a}}\mathbb{E}\Big[|\Delta_j\xi^H(t)|^2\Big]\mathrm{d}t \lesssim_a \sup_{t\in\mathbf{R}}\mathbb{E}\Big[|\Delta_j\xi^H(t)|^2\Big] \\ &= c_H\int_{\mathbf{R}}|\tau|^{1-2H}\varrho_j^2(\tau)\,\mathrm{d}\tau \lesssim 2^{j2(1-H)},\end{aligned}$$

where in the first line we used that $2a > 1$. In the second line, we used that for $j \geq 0$ $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a function ϱ with support in an annulus (i.e. a set of the form $\mathscr{A} = \{\tau : A < |\tau| < B\}$ for some $0 < A < B$). This provides the required regularity estimate:

$$\mathbb{E}\|\xi^H\|^2_{H^\alpha_a(\mathbf{R})} < \infty.$$

The ergodicity is a consequence of the criterion in Proposition V.7.1 with $\mathbf{B} = H^\alpha_a(\mathbf{R})$, provided that we can verify condition (V.36) on the covariances. Observe that $H^\alpha_a(\mathbf{R})$ is a separable Banach space with dual $(H^\alpha_a(\mathbf{R}))^* = H^{-\alpha}_{-a}(\mathbf{R})$ (this result follows with the same calculations of [Tri10, Theorem 2.11.2] for the unweighted case, see also the discussion in [Tri10, Section 7.2]), and that the space $\mathscr{S}(\mathbf{R})$ of Schwartz functions, i.e. smooth functions with polynomial decay at infinity of any order, is dense in $H^\beta_b(\mathbf{R})$ for any value of $\beta \in \mathbf{R}$ and $b > 0$ (see [Tri10, Remark 7.2.2]).
In view of these facts, and since we have shown that $\mathbb{E}\|\xi^H\|^2_{\mathbf{B}} < \infty$, by condition (V.37) of Proposition V.7.1 it suffices to prove that for any $\varphi, \varphi' \in \mathscr{S}(\mathbf{R})$:

$$\lim_{n\to\infty} \mathrm{Cov}(\langle \xi^H, \varphi\rangle, \langle \vartheta^n \xi^H, \varphi'\rangle) = 0.$$

Here we can compute as follows:

$$\begin{aligned}\lim_{n\to\infty} \mathrm{Cov}(\langle \xi^H, \varphi\rangle, \langle \vartheta^n \xi^H, \varphi'\rangle) &\simeq \lim_{n\to\infty} \int_{\mathbf{R}} |\tau|^{1-2H} e^{in\tau} \hat{\varphi}(\tau)\overline{\hat{\varphi}}'(\tau)\,\mathrm{d}\tau \\ &= 0.\end{aligned}$$

To obtain the last line we made use of the Riemann-Lebesgue lemma, since $f(\tau) = |\tau|^{1-2H}\hat{\varphi}(\tau)\hat{\varphi}'(\tau)$ satisfies $f \in L^1(\mathbf{R})$. In fact, f is integrable near $\tau = 0$ because $H \in (1/2, 1)$ while $f(\tau)$ decays polynomially fast for $\tau \to \pm\infty$ since $\varphi, \varphi' \in \mathscr{S}(\mathbf{R})$. Hence, ergodicity is proven.

Now, one can define the primitive $\beta^H(\omega)$ through

$$\beta^H_t = \xi^H(1_{[0,t]}), \text{ in } L^2(\mathbb{P}),$$

so that following [PT00, Section 3] $(\beta^H_t)_{t\geq 0}$ has the law of a fractional Brownian motion. In particular, almost surely, the process $\beta^H_t(\omega)$ has the required regularity. The null-set $\overline{N}_0$ on which the result does not hold can be chosen to be ϑ−invariant, by defining $N_0 = \bigcup_{z\in\mathbf{Z}} \vartheta^z \overline{N}_0$. Then one can set $\xi^H = 0$ on N_0.

□

The next step is to show wellposedness of the SPDE:

$$(\partial_t - \partial_x^2)u(t,x) = \xi^H(t)V(x)u(t,x), \qquad u(0,x) = u_0(x), \qquad (t,x) \in \mathbf{R}_+ \times \mathbf{T}. \tag{V.18}$$

We will work pathwise: since our noise is sufficiently regular, i.e. $H > \frac{1}{2}$ we can use Young integrals to make sense of the solution (for $H = \frac{1}{2}$, we would need Itô integration instead). We will use the following result:

Lemma V.6.2. *For any $\alpha, \beta, T > 0$ such that $\alpha + \beta > 1$ and $f \in C^\alpha([0,T])$, $g \in C^\beta([0,T])$ one can define the Young integral*

$$\mathscr{I}_t(f,g) = \int_0^t f(s)\,\mathrm{d}g(s).$$

The map $\mathscr{I}$ is continuous between the spaces:

$$\mathscr{I}\colon C^{\alpha}([0,T])\times C^{\beta}([0,T])\to C^{\beta}([0,T]),$$

satisfying the bound

$$\|\mathscr{I}(f,g)\|_{C^{\beta}([0,T])}\lesssim\|f\|_{C^{\alpha}([0,T])}\|g\|_{C^{\beta}([0,T])}.$$

If $g\in C^{1}([0,T])$ the integral coincides with

$$\mathscr{I}_{t}(f,g)=\int_{0}^{t}f(s)\partial_{s}g(s)\,\mathrm{d}s.$$

An instructive proof of this result is given in [FV11, Proposition 6.11] (for $\frac{1}{\alpha}$−variation spaces instead of Hölder spaces), or in [FH20, Chapter 4].

Definition V.6.3. *Consider $H\in(\frac{1}{2},1)$ and let P_t be the periodic heat semigroup:*

$$P_{t}f(x)=\sum_{z\in\mathbf{Z}}(4\pi t)^{-\frac{d}{2}}\int_{\mathbf{T}}f(y)e^{-\frac{|x-y-z|^{2}}{4t}}\,\mathrm{d}y.$$

Fix $\omega\in\Omega_{\mathrm{kpz}}$ and ξ^{H} as in Lemma V.6.1. We say that $u\colon\Omega_{\mathrm{kpz}}\times\mathbf{R}_{+}\times\mathbf{T}\to\mathbf{R}$ is a mild solution to Equation (V.18) *if for any $\alpha<H$ and $S>0$*

$$s\mapsto P_{t-s}[u(\omega,s,\cdot)V(\cdot)](x)\in C^{\alpha}([S,t]),\qquad\forall t\geqslant S,\ x\in\mathbf{T}$$

and if u satisfies:

$$\begin{aligned}&u(\omega,t,x)=P_{t-S}u(\omega,S)(x)+\int_{S}^{t}P_{t-s}[u(\omega,s,\cdot)V(\cdot)](x)\,\mathrm{d}\beta^{H}_{s}(\omega),\quad\forall t\geqslant S,\ x\in\mathbf{T},\\&\lim_{S\to0}u(\omega,S,\cdot)=u_{0}(\cdot),\ \ in\ \ C^{-\zeta}(\mathbf{T}),\quad\forall\zeta>0,\end{aligned}\tag{V.19}$$

where, since the time regularities $\alpha<H$ of the integrand and $\alpha'<H$ of $t\mapsto\beta^{H}_{t}(\omega)$ can be chosen so that $\alpha+\alpha'>1$, because $H\in(1/2,1)$, the integral in (V.19) *is well-defined as a Young integral: see Lemma V.6.2.*

We can now prove the following result.

Lemma V.6.4. *Consider $H\in(\frac{1}{2},1)$ and $\Omega_{\mathrm{kpz}},\xi^{H}$ as in Lemma V.6.1. For all $\omega\in\Omega_{\mathrm{kpz}}$, for every $u_{0}\in C(\mathbf{T})$ there exists a unique mild solution u to Equation* (V.18) *such that for any $\alpha<H,k\in\mathbf{N},0<S<T<\infty$:*

$$(t,x)\mapsto\partial_{x}^{k}u(\omega,t,x)\in C^{\alpha}([S,T]\times\mathbf{T}).$$

Moreover, the solution u can be represented as:

$$u(\omega,t,x)=e^{X(\omega,t,x)}w(\omega,t,x),$$

with

$$X(\omega,t,x)=\int_{0}^{t}P_{t-s}V(x)\,\mathrm{d}\beta^{H}_{s}(\omega),\tag{V.20}$$

and w a solution to

$$\begin{aligned}(\partial_{t}-\partial_{x}^{2})w(t,x)&=2\partial_{x}X(t,x)\partial_{x}w(t,x)+(\partial_{x}X)^{2}(t,x)w(t,x),\\w(0,x)&=u_{0}(x).\end{aligned}\tag{V.21}$$

The solution map $(\varphi_{t}(\omega)u_{0})(x):=u(\omega,t,x)$ defines a continuous linear RDS on $C(\mathbf{T})$.

Proof. Let us fix any $\omega \in \Omega_{\mathrm{kpz}}$. Since all the following arguments work pathwise, we will henceforth omit writing the dependence on ω. To solve Equation (V.18), observe that $(s,x) \mapsto P_{t-s}V(x) \in C^{\infty}([0,t]\times \mathbf{T})$, since V is smooth. We can then use Lemma V.6.2 to define $X(t,x)$ by Equation (V.20), so that formally $X(t,x)$ solves:

$$(\partial_t - \partial_x^2)X(t,x) = \xi^H(t)V(x), \qquad X(0,x) = 0, \qquad \forall (t,x) \in \mathbf{R}_+ \times \mathbf{T}.$$

We will require a bound on the temporal regularity of X. To this end, let us write by integration by parts

$$X(t,x) = -\int_0^t \beta_s^H (P_{t-s}\partial_x^2 V)(x)\,\mathrm{d}s + V(x)\beta_t^H,$$

so that taking spatial derivatives in the above representation we obtain the following regularity:

$$(t,x) \mapsto \partial_x^k X(t,x) \in C^{\alpha}([0,T]\times\mathbf{T}) \tag{V.22}$$

for any $\alpha \in (\frac{1}{2}, H)$, $T > 0, k \in \mathbf{N}_0$. We also observe that for any other path $f \in C^{\alpha}([0,T];\mathbf{R})$, by Lemma V.6.2 (taking smooth approximations of β^H and using the continuity of the Young integral)

$$\int_0^t f_s\,\mathrm{d}X(s,x) = \int_0^t f_s \partial_x^2 X(s,x)\,\mathrm{d}s + \int_0^t f_s V(x)\,\mathrm{d}\beta_s^H. \tag{V.23}$$

Now, as a consequence of Lemma V.8.1 there exists a unique mild solution w to Equation (V.21) and the same result implies that the solution w satisfies:

$$(t,x) \mapsto \partial_x^k w(\omega,t,x) \in C^1_{\mathrm{loc}}((0,T]\times\mathbf{T}), \tag{V.24}$$

for any $T > 0, k \in \mathbf{N}_0$. At this point, let us define u as $u = e^X w$. For any fixed $S > 0$ we find that, by the chain rule (which holds in view of Lemma V.6.2, by taking smooth approximations of the integrand and integrator)

$$\begin{aligned} u(t,x) &= u(S,x) + \int_S^t e^{X(s,x)} w(s,x)\,\mathrm{d}X(s,x) + \int_S^t e^{X(s,x)} w(s,x)\partial_s w(s,x)\,\mathrm{d}s \\ &= u(S,x) + \int_S^t \partial_x^2 u(s,x)\,\mathrm{d}s + \int_S^t u(s,x)V(x)\,\mathrm{d}\beta_s^H, \end{aligned}$$

where we used (V.23) and (V.21). Now by (V.22) and (V.24)

$$(t,x) \mapsto \partial_x^k u(t,x) \in C^{\alpha}([S,T]\times\mathbf{T})$$

for any $k \in \mathbf{N}_0, \alpha \in (\frac{1}{2}, H)$ and $0 < S < T$. In particular, we find that

$$(s,x) \mapsto \partial_x^k P_{t-s}[u(s,\cdot)V(\cdot)](x) \in C^{\alpha}([S,t]\times\mathbf{T}), \quad \forall\, 0 < S \leqslant t,\ \ \alpha \in \left(\frac{1}{2}, H\right).$$

Then we can define $\widetilde{u}$ via the Young integral:

$$\widetilde{u}(t,x) = P_{t-S}u(S,x) + \int_S^t P_{t-s}[u(s,\cdot)V(\cdot)](x)\,\mathrm{d}\beta_s^H.$$

An application of the chain rule show that $u-\widetilde{u}$ is a smooth solution to $(\partial_t-\partial_x^2)(u-\widetilde{u})=0$, and hence $u=\widetilde{u}$. To conclude that u satisfies Equation (V.19) we need that

$$\lim_{S\to 0} u(S,\cdot)=u_0, \quad \text{in } \ C^{-\zeta}(\mathbf{T}),\ \forall \zeta>0,$$

which follows since $\lim_{S\to 0} w(S,\cdot)=u_0$. Conversely, one can follow the steps of this proof backwards to find that every mild solution is of the required form $u=e^X w$.

Finally, Lemma V.8.1 also implies that the solution map is, for fixed $t\geq 0$, an element of $\mathscr{L}(C(\mathbf{T}))$. To conclude we have to show that the cocycle property holds for φ, namely that for $n\in\mathbf{N}_0$:

$$\varphi_{t+n}(\omega)u_0=\varphi_t(\vartheta^n\omega)\circ\varphi_n(\omega)u_0.$$

First observe that $X_{t+n}(\omega)-P_tX_n(\omega)=X_t(\vartheta^n\omega)$. Hence, recalling the decomposition of φ:

$$\varphi_{t+n}(\omega)u_0=e^{X_t(\vartheta^n\omega)}(e^{P_tX_n(\omega)}w_{t+n}(\omega)),$$

so that the cocycle property is proven since one can check that $\overline{w}_t(\omega)=e^{P_tX_n(\omega)}w_{t+n}(\omega)$ solves Equation (V.21) with $X(\omega)$ replaced by $X(\vartheta^n\omega)$ and $\overline{w}_0=\varphi_t(\omega)u_0$. □

We can now prove that Equation (V.18) falls in the framework of the theory developed in the previous sections.

Proposition V.6.5. *The RDS φ introduced in Lemma V.6.4 satisfies, for any $\beta>0$, Assumption V.5.2. In particular, for all $\omega\in\Omega_{\mathrm{kpz}}$, for any $u_0\in C(\mathbf{T}), u_0>0$, the function $t\mapsto\log(\varphi_t(\omega)u_0)=:h_t(\omega)$ is the unique mild solution to*

$$(\partial_t-\partial_x^2)h(\omega,t,x)=(\partial_x h(\omega,t,x))^2+V(x)\xi^H(\omega,t), \qquad h(\omega,0,x)=\log(u_0(x)), \tag{V.25}$$

meaning that for any $\alpha<H, k\in\mathbf{N}, 0<S<T<\infty$:

$$(t,x)\mapsto\partial_x^k h(\omega,t,x)\in C^\alpha((S,T)\times\mathbf{T})$$

and for all $0<S\leqslant t$, $\zeta>0$ and $x\in\mathbf{T}$:

$$h(\omega,t,x)=P_{t-S}h(\omega,S)(x)+\int_S^t P_{t-s}[(\partial_x h(\omega,s))^2](x)\,\mathrm{d}s+\int_S^t P_{t-s}[V](x)\,\mathrm{d}\beta_s^H,$$
$$\lim_{S\to 0} h(\omega,S,\cdot)=h_0(\cdot) \quad \textit{in} \quad C^{-\zeta}.$$

Such solution satisfies all the results of Theorem V.5.3.

Proof. Let us fix $\omega\in\Omega_{\mathrm{kpz}}$ and to lighten the notation we will henceforth not write explicitly the dependence on it. The first step is to prove that for such ω, points $(i)-(iii)$ of Assumption V.5.2 are satisfied. Let us start with the kernel representation. Formally, one can write:

$$K(t,x,y)=\varphi_t(\delta_y)(x). \tag{V.26}$$

This can be made rigorous, if one can start Equation (V.18) in δ_y. In Lemma V.8.2 we show that that for any $\gamma>0$, $\{\delta_y\}_{y\in\mathbf{T}}\subseteq B^{-\gamma}_{1,\infty}$, and $\|\delta_x-\delta_y\|_{B^{-\gamma}_{1,\infty}}\lesssim|x-y|^\gamma$. In addition, by

Lemma V.6.4 the solution $\varphi_t u_0 = e^{X_t} w_t$, where $X(t,x) = \int_0^t P_{t-s}[V](x)\,\mathrm{d}\beta_s^H$ does not depend on u_0 and w is the solution to:

$$(\partial_t - \partial_x^2)w = 2\partial_x X \partial_x w + (\partial_x X)^2 w, \qquad w(0) = u_0.$$

As the coefficients $(\partial_x X)^2$ and $\partial_x X$ are smooth in space and continuous in time, Lemma V.8.1 implies that the equation for w can be started also in $u_0 = \delta_y$. Let us denote with w^y such solution. The same Lemma V.8.1 implies the following bound, for any $\eta \in [0,2)$, $t \in [S,T]$ and some $q > 0$:

$$\|w^y(t,\cdot) - w^z(t,\cdot)\|_{B^{\eta-\gamma}_{1,\infty}} \leqslant \|\delta_y - \delta_z\|_{B^{-\gamma}_{1,\infty}} e^{C(S,T)(1+\sum_{k=0}^{\lfloor \eta \rfloor +1} \sup_{0\le t\le T} \|\partial_x^k X_t(\omega)\|_\infty)^q}.$$

We can choose η, γ so that $\eta - \gamma > 1$. In this case, by Besov embeddings

$$\begin{aligned}\|w^y(t,\cdot) - w^z(t,\cdot)\|_{C(\mathbf{T}^d)} &\lesssim \|w^y(t,\cdot) - w^z(t,\cdot)\|_{C^{\eta-\gamma-1}} \simeq \|w^y(t,\cdot) - w^z(t,\cdot)\|_{B^{\eta-\gamma-1}_{\infty,\infty}} \\ &\lesssim \|w^y(t,\cdot) - w^z(t,\cdot)\|_{B^{\eta-\gamma}_{1,\infty}}.\end{aligned}$$

Hence K in Equation (V.26) is rigorously defined as $K(t,x,y) = e^{X(t,x)} w^y(t,x)$. In particular, putting together the previous bounds, we have that

$$\sup_{S\leqslant t\leqslant T} \|K(t,\cdot,y) - K(t,\cdot,z)\|_\infty \leqslant |y-z|^\gamma e^{C(S,T)(1+\sum_{k=0}^{\lfloor \eta \rfloor +1} \sup_{0\le t\le T} \|\partial_x^k X_t(\omega)\|_\infty)^q},$$

which implies that for any $t > 0, K(t) \in C(\mathbf{T}\times\mathbf{T})$. That K is a fundamental solution for the PDE follows by linearity, thus concluding the proof of (i) in Assumption V.5.2. The fact that K is strictly positive, as required in point (ii) of the assumptions is instead the consequence of a strong maximum principle (cf. [Lie96, Theorem 2.7]) applied to w, since $e^X > 0$. The smoothing effect of point (iii) in Assumption V.5.2 follows again from the representation $\varphi_t u_0 = e^{X_t} w_t$ and spatial smoothness of both X and w we already showed in the proof of Lemma V.6.4. In particular, the smoothing effect can be made quantitative, via the estimate of Lemma V.8.1, to obtain that for $0 < S < T < \infty$ there exists constants $C(S,T), q \ge 0$ such that:

$$\sup_{S\le t\le T} \|\varphi_t(\omega) u_0\|_{C^\beta} \le \|u_0\|_\infty e^{C(S,T)(1+\sum_{k=0}^{\lfloor \beta \rfloor +1} \sup_{0\le t\le T} \|\partial_x^k X_t(\omega)\|_\infty)^q}.$$

Note that at first Lemma V.8.1 allows to regularize at most by $\eta < 2$, but splitting the interval $[0,S]$ into small pieces and applying iteratively the result on every piece provides the result for arbitrary β. Now observe that in view of (V.20) for any $k \in \mathbf{N}$:

$$\begin{aligned}|\partial_x^k X(t,x)| &= \left| \int_0^t P_{t-s}[\partial_x^k V](x)\,\mathrm{d}\beta_s^H \right| \\ &\lesssim \|s \mapsto P_{t-s}[\partial_x^k V](x)\|_{C^\alpha([0,T])} \|s \mapsto \beta_s^H\|_{C^\alpha([0,T])},\end{aligned}$$

for any $\alpha \in (\frac{1}{2}, H)$, as an application of Lemma V.6.2. Since $s \mapsto P_{t-s}[\partial_x V](x)$ is smooth (since V is smooth), we have obtained:

$$\sup_{0\leqslant t\leqslant T} \|\partial_x^k X(t)\|_\infty \leqslant C(T,V) \|\beta^H\|_{C^\alpha([0,T])}.$$

Now, for any $q \geqslant 0$
$$\mathbb{E}\|\beta^H\|^q_{C^\alpha([0,T])} < \infty.$$
This follows from Kolmogorov's continuity criterion, or via calculations similar to those in Lemma V.6.1 (note that we show $\mathbb{E}\|\xi^H\|_{H^\alpha_a} < \infty$, but the similar calculations show that $\mathbb{E}\|\xi^H\|^q_{B^{\alpha,a}_{\infty,\infty}} < \infty$, for any $q \geqslant 0$). We can conclude that:
$$\sum_{k=1}^{\lfloor\beta\rfloor+1} \mathbb{E} \sup_{0\leq t\leq T} \|\partial_x^k X_t\|^q_\infty < \infty,$$
thus proving the first average bound of point (iv) in Assumption V.5.2. As for the second bound, in view of Lemma V.3.2, one has:
$$\begin{aligned} d_H(\varphi_t^\pi(\omega)f, f) &\lesssim \|\log(\varphi_t(\omega)f) - \log\int_{\mathbf{T}} (\varphi_t(\omega)f)(x)\,\mathrm{d}x\|_\infty + \|\log f - \log\int_{\mathbf{T}} f(x)\,\mathrm{d}x\|_\infty \\ &\lesssim \|\log(\varphi_t(\omega)f)\|_\infty + \|\log f\|_\infty, \end{aligned}$$
so that our aim is to bound
$$\mathbb{E}\sup_{S\leq t\leq T}\|\log(\varphi_t f)\|_\infty.$$
On one side, one has the upper bound:
$$\log(\varphi_t(\omega)f) \leq \log\|\varphi_t(\omega)f\|_\infty \lesssim_{S,T} \log\|f\|_\infty + \left(1 + \sum_{k=0}^{1} \sup_{0\leq t\leq T}\|\partial_x^k X_t(\omega)\|_\infty\right)^q,$$
which is integrable. As for the lower bound, observe that $\log(\varphi_t(\omega)f) = X_t(\omega) + \log w_t(\omega)$. One can check that $v_t(\omega) = \log w_t(\omega)$ is a solution to the equation:
$$(\partial_t - \partial_x^2)v = 2(\partial_x X)\partial_x v + (\partial_x X)^2 + (\partial_x v)^2, \qquad v(0) = \log f. \tag{V.27}$$
By comparison (cf. [Lie96, Theorem 2.7]), one has: $v(t,x) \geq -\|\log f\|_\infty, \forall t \geq 0, x \in \mathbf{T}$. So assuming that $q \geq 1$, one has overall:
$$\|\log(\varphi_t(\omega)f)\|_\infty \lesssim_f 1 + \left(1 + \sum_{k=0}^{1} \sup_{0\leq t\leq T}\|\partial_x^k X_t(\omega)\|_\infty\right)^q,$$
which is once again integrable. Hence the required assumptions are satisfied and we can apply Theorem V.5.3.

Finally, that h_t satisfies the smoothness assumption and is a mild solution to the KPZ equation driven by fractional noise follows by the same steps of the proof of Lemma V.6.4. □

Remark V.6.6. *In the same setting as in Proposition V.6.5, for any $h_0^1, h_0^2 \in C(\mathbf{T})$ the constant $c(\omega, t, h_0^1, h_0^2)$ in Theorem V.5.3 can be chosen independent of time.*

Proof. Observe that it is sufficient to prove that there exists a constant $\bar{c}(\omega, h_0^1, h_0^2)$ such that for every $\omega \in \widetilde{\Omega}$ (for an invariant set $\widetilde{\Omega}$ of full $\mathbb{P}$−measure) and any $T > 0$:
$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{t\in[n,n+T]} |c(\omega, t, h_0^1, h_0^2) - \bar{c}(\omega, h_0^1, h_0^2)| \leq \mathbb{E}\log\Big(\tau(\varphi_1)\Big).$$

As a simple consequence of Theorem V.5.3 one has:

$$\limsup_{n\to\infty}\frac{1}{n}\sup_{t\in[n,n+T]}\log\|\Pi_\times(h_1(\omega,t)-h_2(\omega,t))\|_\alpha \le \mathbb{E}\log\Big(\tau(\varphi_1)\Big),$$

for any $\alpha > 0$, where $\Pi_\times$ is defined for $f \in C(\mathbf{T})$ as $\Pi_\times f = f - \int_{\mathbf{T}} f(x)\,\mathrm{d}x$, and without loss of generality one can choose the constants to be defined as:

$$c(\omega,t,h_0^1,h_0^2) = \int_{\mathbf{T}} h_1(\omega,t,x) - h_2(\omega,t,x)\,\mathrm{d}x.$$

Since by Proposition V.6.5, h_i is a solution to the KPZ Equation one has that:

$$\partial_t c(\omega,t,h_0^1,h_0^2) = \int_{\mathbf{T}} \partial_x(h_1-h_2)\partial_x(h_1+h_2)(\omega,t,x)\,\mathrm{d}x. \tag{V.28}$$

Now, (i) of Theorem V.5.3 implies that:

$$\limsup_{n\to\infty}\frac{1}{n}\log\|\partial_x h_1 - \partial_x h_2\|_\infty < 0,$$
$$\limsup_{n\to\infty}\frac{1}{n}\log\|\partial_x h_1\|_\infty + \|\partial_x h_2\|_\infty \leqslant 0.$$

Hence we find that:

$$\limsup_{n\to\infty}\frac{1}{n}\sup_{t\in[n,n+1]}\log|\partial_t c(\omega,t,h_0^1,h_0^2)| < 0.$$

In particular this implies that there exists a constant $\overline{c}(\omega,h_0^1,h_0^2) := \lim_{t\to\infty} c(\omega,t,h_0^1,h_0^2)$ and in addition

$$|\overline{c}(\omega,h_0^1,h_0^2) - c(\omega,t,h_0^1,h_0^2)| \le \int_t^\infty |\partial_s c(\omega,s,h_0^1,h_0^2)|\,\mathrm{d}s \lesssim e^{-\delta t},$$

for some $\delta > 0$, which proves the required result.

□

V.6.2 KPZ driven by space-time white noise

In this section we consider the random force η in (V.1) to be space-time white noise ξ in one spatial dimension. That is, a Gaussian processes indexed by functions in $L^2(\mathbf{R}\times\mathbf{T})$ such that:

$$\mathbb{E}\Big[\xi(f)\xi(g)\Big] = \int_{\mathbf{R}\times\mathbf{T}} f(t,x)g(t,x)\,\mathrm{d}t\,\mathrm{d}x. \tag{V.29}$$

For the next result recall the definition of $H_a^\alpha(\mathbf{R}\times\mathbf{T})$ from (V.4).

Lemma V.6.7. *Fix any $\alpha < -1$ and $a > \frac{1}{2}$. Let ξ be a Gaussian process as defied in* (V.29). *Then, almost surely ξ takes values in $H_a^\alpha(\mathbf{R}\times\mathbf{T})$. In particular*

$$\mathbb{E}\|\xi\|^2_{H_a^\alpha(\mathbf{R}\times\mathbf{T})} < \infty.$$

Next, define $\Omega_{\mathrm{kpz}} = H^{\alpha}_{a}(\mathbf{R}\times\mathbf{T}), \mathscr{F} = \mathscr{B}(H^{\alpha}_{a}(\mathbf{R}\times\mathbf{T}))$ *and let* $\mathbb{P}$ *be the law of* ξ *on* Ω_{kpz}. *Furthermore, let* $\{\vartheta^z\}_{z\in\mathbf{Z}}$ *be the integer translation group, which acts on smooth functions* $\varphi \in \mathscr{S}(\mathbf{R}\times\mathbf{T})$ *by:*

$$\vartheta^z\varphi(t,x) = \varphi(t+z,x), \quad \forall\ (t,x)\in\mathbf{R}\times\mathbf{T},$$

and which is extended by duality to all distributions $\omega\in\Omega_{\mathrm{kpz}}$*:*

$$\langle\vartheta^z\omega,\varphi\rangle = \langle\omega,\vartheta^{-z}\varphi\rangle, \quad \forall\ \varphi\in\mathscr{S}(\mathbf{R}\times\mathbf{T}).$$

Then the space $(\Omega_{\mathrm{kpz}},\mathscr{F},\mathbb{P},\vartheta)$ *forms an ergodic IDS.*

Proof. We start by showing that ξ takes values in $H^{\alpha}_{a}(\mathbf{R}\times\mathbf{T})$ almost surely. By definition:

$$\mathbb{E}\|\xi\|^2_{H^{\alpha}_{a}} = \sum_{j\geq -1} 2^{2\alpha j}\mathbb{E}\|\Delta_j\xi(\cdot)/\langle\cdot\rangle^a\|_{L^2},$$

and for the latter one has:

$$\begin{aligned}\mathbb{E}\Big[\|\Delta_j\xi(\cdot)/\langle\cdot\rangle^a\|^2_{L^2}\Big] &= \int_{\mathbf{R}\times\mathbf{T}}\frac{1}{(1+|t|)^{2a}}\mathbb{E}\Big[|\Delta_j\xi(t,x)|^2\Big]\mathrm{d}t\,\mathrm{d}x \lesssim_a \sup_{(t,x)\in\mathbf{R}\times\mathbf{T}}\mathbb{E}\Big[|\Delta_j\xi(t,x)|^2\Big]\\ &= \int_{\mathbf{R}\times\mathbf{T}}|\mathscr{F}^{-1}_{\mathbf{R}\times\mathbf{T}}\varrho_j|^2(t,x)\,\mathrm{d}t\,\mathrm{d}x \simeq \sum_{k\in\mathbf{Z}}\int_{\mathbf{R}}\varrho_j^2(k,\tau)\,\mathrm{d}\tau \lesssim 2^{2j},\end{aligned}$$

where we used that $2a>1$ and that for $j\geq 0$ $\varrho_j(\cdot) = \varrho(2^{-j}\cdot)$ for a function ϱ with support in an annulus. We can conclude that

$$\mathbb{E}\|\xi\|^2_{H^{\alpha}_{a}(\mathbf{R}\times\mathbf{T})} < \infty.$$

The last step in the proof is to show ergodicity of the IDS. Here we apply Proposition V.7.1, so we have to check that condition (V.36). We have proven that $\mathbb{E}\|\xi\|^2_{H^{\alpha}_{a}} < \infty$, and (as in the proof of Lemma V.6.1) let us note that $(H^{\alpha}_{a}(\mathbf{R}\times\mathbf{T}))^* = H^{-\alpha}_{a}(\mathbf{R}\times\mathbf{T})$ and $\mathscr{S}(\mathbf{R}\times\mathbf{T})$ is dense in $H^{\beta}_{b}(\mathbf{R}\times\mathbf{T})$ for every $\beta\in\mathbf{R}, b>0$. Hence we can deduce ergodicity from the simplified criterion (V.37), namely we have to prove that for $\varphi,\varphi'\in\mathscr{S}(\mathbf{R}\times\mathbf{T})$:

$$\lim_{n\to\infty}\mathrm{Cov}(\langle\xi,\varphi\rangle,\langle\vartheta^n\xi,\varphi'\rangle) = \lim_{n\to\infty}\int_{\mathbf{R}\times\mathbf{T}}\varphi(t,x)\varphi'(t-n,x)\,\mathrm{d}t\,\mathrm{d}x = 0,$$

which is true because of the rapid decay at infinity of φ,φ'. This concludes the proof. □

Now we will consider h,u the respective solutions to the KPZ and stochastic heat equation driven by space-time white noise:

$$(\partial_t-\partial_x^2)h = (\partial_x h)^2+\xi-\infty, \qquad h(0,x) = h_0(x), \qquad (t,x)\in\mathbf{R}_+\times\mathbf{T}, \tag{V.30}$$

$$(\partial_t-\partial_x^2)u = u(\xi-\infty), \qquad u(0,x) = u_0(x), \qquad (t,x)\in\mathbf{R}_+\times\mathbf{T}, \tag{V.31}$$

in the sense of [GP17, Theorem 6.15]. Here the presence of the infinity "∞" indicates the necessity of renormalisation to make sense of the solution. Well posedness of the stochastic heat equation (V.31) can be proven also with martingale techniques, which do not provide a solution theory for the KPZ equation, though. Instead, here we make use of pathwise approaches to solving the above equations [Hai13, Hai14, GIP15], that require tools such as regularity structures or paracontrolled distributions. The main reference for us will be [GP17], which provides both a comprehensible introduction (see for example Chapter 3) to the tools available in paracontrolled analysis. Such theories consider smooth approximations ξ_ε of the noise ξ, for which the equations are well-posed, and study the convergence of the solutions as $\varepsilon \to 0$. The renormalisation can then be understood as a Stratonovich-Itô correction term. We refer to the mentioned works as *pathwise* approaches, since they are completely deterministic, given the realization of the noise and some functionals thereof. These functionals are collected in a random variable called the *enhanced noise* $\mathbf{Y}(\omega)$.

In Lemma V.8.3 we recall the construction of the enhanced noise and record its transformation under ϑ^z. Lemma V.8.3 together with the existing solution theory for the equation guarantee that the solution map forms a random dynamical system. This is the content of the following result, which stands in analogy to Lemma V.6.4 for fractional noise.

Lemma V.6.8. *Consider* $(\Omega_{\mathrm{kpz}}, \mathscr{F}, \mathbb{P})$ *as in Lemma V.6.7. Then for every* $\omega \in \Omega_{\mathrm{kpz}}$ *and* $u_0 \in C(\mathbf{T})$ *there exists a unique solution* u *to Equation* (V.31) *in the sense of [GP17, Theorem 6.15], associated to the enhanced noise* $\mathbf{Y}(\omega)$ *as in Lemma V.8.3. In addition, the solution map* $\varphi_t(\omega)u_0 = u_t$ *defines a continuous linear RDS on* $C(\mathbf{T})$.

Proof. Fix $\omega \in \Omega$. The existence result [GP17, Theorem 6.15] builds a unique solution that depends continuously on the initial condition and the enhanced noise $\mathbf{Y}(\omega)$, and is linear with respect to the initial condition. What is left to show is that the solution map satisfies the cocycle property: $\varphi_{n+t}(\omega)u_0 = \varphi_t(\vartheta^n\omega) \circ \varphi_n(\omega)u_0$. From [GP17, Theorem 4.5] (see the arguments that precede the theorem for a proof), the solution $\varphi_{t+n}(\omega)u_0$ can be represented as:

$$\varphi_{t+n}(\omega)u_0 = e^{Y_{t+n}(\omega) + Y^{\vee}_{t+n}(\omega) + 2Y^{\vee_{\vee}}_{t+n}(\omega)} w^P,$$

where the terms inside the exponential are recalled in Lemma V.8.3, and with w^P solving

$$\begin{aligned}(\partial_t - \partial_x^2)w^P = 4\Big[&(\partial_t - \partial_x^2)(Y^{\vee\vee} + Y^{\vee_{\vee_{\vee}}}) + (\partial_x Y \partial_x Y^{\vee_{\vee}} - \partial_x Y \odot \partial_x Y^{\vee_{\vee}}) \\ &+ \partial_x Y^{\vee} \partial_x Y^{\vee_{\vee}} + (\partial_x Y^{\vee_{\vee}})^2\Big](\omega)w^P + 2\partial_x(Y + Y^{\vee} + Y^{\vee_{\vee}})(\omega)\partial_x w^P, \\ w^P(0) = e^{-Y_0(\omega)}u_0,&\end{aligned} \tag{V.32}$$

in the paracontrolled sense of [GP17, Theorem 6.15]. Now one can use Equation (V.38) of Lemma V.8.3 to obtain:

$$\varphi_{t+n}(\omega) = e^{Y_t(\vartheta^n\omega) + Y_t^{\vee}(\vartheta^n\omega) + 2Y_t^{\vee_{\vee}}(\vartheta^n\omega)} \overline{w}^P_t(\omega),$$

where

$$\overline{w}^P_t(\omega) = e^{P_t Y_n^{\vee}(\omega) + 2P_t Y_n^{\vee_{\vee}}(\omega)} w^P_{t+n}(\omega).$$

In turn, $\overline{w}^P(\omega)$ satisfies $\overline{w}^P_0(\omega) = e^{-Y_0(\vartheta^n\omega)}\varphi_n(\omega)u_0$ and a formal calculation that can be made rigorous using the solution theory of [GP17, Theorem 6.15] can be seen to satisfy

Equation (V.32) with ω replaced be $\vartheta^n\omega$ and initial condition $e^{-Y_0(\vartheta^n\omega)}\varphi_n(\omega)u_0$. So the cocycle property is satisfied and the proof concluded. □

The RDS φ introduced in the previous lemma falls into the framework of Section V.5. To prove this, we follow the same approach of Proposition V.6.9, which addresses the fractional noise case. First, we will construct the random kernel $K(\omega,t,x,y)$ for the solution map $\varphi_t(\omega)$. Here the key point is to use results from [GP17] to start Equation (V.31) in $u_0(x)=\delta_y(x)$. Then points $(i)-(iii)$ of Assumption V.5.2 follow by treating (V.31) as a pathwise perturbation of the heat equation: these results have been already established, see e.g. [CFG17]. The most challenging part of the proof is to prove the moments bounds of point (iv) of Assumption V.5.2. As in Proposition V.6.5 the proof of these bounds relies on an appropriate decomposition $\varphi_t(\omega)u_0 = e^{Z_t(\omega)}w_t(\omega)$, where Z_t is a functional of the noise, together with a lower bound on w_t (first established in [PR19a]), which is the consequence of a comparison principle.

Proposition V.6.9. *The RDS φ be defined as in Lemma V.6.8 satisfies Assumption V.5.2 for any $\beta<\frac{1}{2}$. In particular, the results of Theorem V.5.3 apply.*

Proof. Fix $\omega\in\Omega_{\mathrm{kpz}}$. Let us start by checking the first property of Assumption V.5.2. We can define the kernel by $K(\omega,t,x,y)=\varphi_t(\omega)(\delta_y)(x)$, where δ_y indicates a Dirac δ centered at y. Here $\varphi_t(\omega)(\delta_y)$ is the solution to (V.31) with $u_0=\delta_y$. This solution exists in view of [GP17, Theorem 6.15]: in fact, this result shows that for any $0<\beta,\zeta<\frac{1}{2}$, and any $p\in[1,\infty]$ the solution map $\varphi(\omega)$ can be extended to a map

$$\varphi(\omega)\in C_{\mathrm{loc}}((0,\infty);\mathscr{L}(B^{-\gamma}_{p,\infty};B^{\beta}_{p,\infty})),$$

where we used that, in the language of [GP17], the space $\mathscr{D}^{\exp,\delta}_{\mathrm{rhe}}$ of paracontrolled distributions, in which the solution lives, embeds in $C_{\mathrm{loc}}((0,\infty);B^{\beta}_{p,\infty})$, for suitable values of δ as described in the quoted theorem. Near $t=0$ one expects that $\|\varphi_t(\omega)u_0\|_{B^{\beta}_{p,\infty}}$ blows up, if $u_0\in B^{-\zeta}_{p,\infty}$. The exact speed of this blow-up is provided as well in the theorem, but since we are not interested in quantifying the blow-up, we can exploit the result we wrote to deduce the apparently stronger:

$$\varphi(\omega)\in C_{\mathrm{loc}}((0,\infty);\mathscr{L}(B^{-\gamma}_{p,\infty};C^{\beta})). \tag{V.33}$$

This follows by Besov embedding: for $p\le q$: $B^{\alpha}_{p,\infty}(\mathbf{T}^d)\subseteq B^{\alpha-d(\frac{1}{p}-\frac{1}{q})}_{q,\infty}(\mathbf{T}^d)$. Assuming without loss of generality that $\beta,\zeta>\frac{1}{4}$, uniformly over $0<S\le t\le T<\infty$ one can bound:

$$\|\varphi_t u_0\|_{C^\beta}\lesssim\|\varphi_{S/2}u_0\|_{C^{-\zeta}}\lesssim\|\varphi_{S/2}u_0\|_{B^{\beta}_{2,\infty}}\lesssim\|\varphi_{S/4}u_0\|_{B^{-\zeta}_{2,\infty}}\lesssim\|\varphi_{S/4}u_0\|_{B^{\beta}_{1,\infty}}\lesssim\|u_0\|_{B^{-\zeta}_{p,\infty}}.$$

So overall we obtain (V.33), and in particular:

$$\sup_{S\le t\le T}\|\varphi_t(\omega)u_0\|_{C^\beta}\le C(\omega,\beta,\zeta,p,S,T)\|u_0\|_{B^{-\zeta}_{p,\infty}},\ \text{ for any }\ 0<S<T<\infty. \tag{V.34}$$

Now since $\{\delta_y\}_{y\in\mathbf{T}}\subseteq B^{-\zeta}_1$ for any $\zeta>0$, as proven in Lemma V.8.2, the kernel $K(\omega,t,x,y)$ is well-defined. The continuity in t,x follows from the previous estimates, while the continuity in y follows from (V.34) together with Lemma V.8.2.

We can pass to the second property of Assumption V.5.2. The upper bound $\delta(\omega, S, T)$ is a consequence of the continuity of the kernel K. The lower bound $\gamma(\omega, S, T)$ is instead a consequence of a strong maximum principle which, implies that $K(\omega, t, x, y) > 0, \forall t > 0, x, y \in \mathbf{T}$. In this pathwise setting, the strong maximum principle is proven in [CFG17, Theorem 5.1] (it was previously established in [Mue91] with probabilistic techniques).

The third property is a consequence of Equation (V.34), by defining $C(\omega, \beta, S, T) := C(\omega, \beta, \frac{1}{4}, \infty, S, T)$, so we are left with only the last property to check. We start with the fact that

$$\mathbb{E} \log C(\beta, S, T) < \infty.$$

To see this, observe that there exists some deterministic $A(\beta, S, T), q \geq 1$ such that:

$$\sup_{t \in [S,T]} \|\varphi_t(\omega) f\|_\beta \leq e^{A(\beta,S,T)(1+\|\mathbf{Y}(\omega)\|_{\mathscr{Y}_{\mathrm{kpz}}})^q} \|f\|_{C^{-\frac{1}{4}}}, \tag{V.35}$$

that is we can choose $C(\omega, \beta, S, T)$:

$$C(\omega, \beta, S, T) = e^{A(\beta,S,T)(1+\|\mathbf{Y}(\omega)\|_{\mathscr{Y}_{\mathrm{kpz}}})^q}.$$

Inequality (V.35) is implicit in the proof of [GP17, Theorem 6.15], since the proof relies on a Picard iteration and a Gronwall argument. The bound can be found explicitly in [PR19a, Theorem 5.5 and Section 5.2]: here the equation in set on the entire line $\mathbf{R}$, which is a more general setting, since one can always extend the noise periodically. Thus we have $\mathbb{E} \log C(\beta, S, T) \lesssim_{\beta,S,T} 1 + \mathbb{E}\Big[\|\mathbf{Y}\|^q_{\mathscr{Y}_{\mathrm{kpz}}}\Big]$, so that the result is proven if one shows that for any $q \geq 0$: $\mathbb{E}\|\mathbf{Y}\|^q_{\mathscr{Y}_{\mathrm{kpz}}} < \infty$, which is the content of [GP17, Theorem 9.3].

We then pass to the second bound. Since by the triangle inequality the bound does not depend on the choice of f, set $f = 1$. It is thus enough to prove that:

$$\mathbb{E} \sup_{S \leq t \leq T} \|\log(\varphi_t(\omega) 1)\|_\infty < \infty.$$

We proceed as in the proof of Proposition V.6.9. On one side one has the upper bound:

$$\log(\varphi_t(\omega)1) \leq \log \|\varphi_t(\omega)1\|_\infty \leq \log C(\omega, \beta, S, T),$$

which is integrable by the arguments we just presented. As for the lower bound, the approach of Proposition V.6.9 has to be adapted to the present singular setting. One way to perform a similar calculation has been already developed [PR19a, Lemma 3.10]. We sketch again the argument here for clarity, assuming that the elements of $\mathbf{Y}(\omega)$ are smooth. We will eventually refer to the appropriate well posedness results to complete the proof. Recall that $\varphi_t(\omega) u_0 = e^{Y_t(\omega) + Y_t^{\vee}(\omega) + 2Y_t^{\vee\vee}(\omega)} w_t^P$, where w^P solves Equation (V.32). Then define:

$$\begin{aligned}
b(\mathbf{Y}) &= 2(\partial_x Y + \partial_x Y^{\vee} + \partial_x Y^{\vee\vee}) \\
c(\mathbf{Y}) &= 4[(\partial_t - \partial_x^2)(Y^{\vee\vee} + Y^{\vee\vee\vee}) + (\partial_x Y \partial_x Y^{\vee\vee} - \partial_x Y \odot \partial_x Y^{\vee\vee}) + \partial_x Y^{\vee} \partial_x Y^{\vee\vee} + (\partial_x Y^{\vee\vee})^2].
\end{aligned}$$

Assuming that $b(\mathbf{Y}), c(\mathbf{Y})$ are smooth one sees that $h^P = \log w^P$ solves:

$$(\partial_t - \partial_x^2) h^P = b(\mathbf{Y}) \partial_x h^P + c(\mathbf{Y}) + (\partial_x h^P)^2, \qquad h^P(0) = \log w^P(0).$$

By comparison, $h^P \geq -\widetilde{h}^P$, with the latter solving:

$$(\partial_t - \partial_x^2)\widetilde{h}^P = b(\mathbf{Y})\partial_x \widetilde{h}^P - c(\mathbf{Y}) + (\partial_x \widetilde{h}^P)^2, \qquad h^P(0) = -\log w^P(0).$$

In particular

$$h^P \geq -\log \widetilde{w}^P \geq -\log \|\widetilde{w}^P\|_\infty,$$

where $\widetilde{w}^P$ solves:

$$(\partial_t - \partial_x^2)\widetilde{w}^P = b(\mathbf{Y})\partial_x \widetilde{w}^P - c(\mathbf{Y})\widetilde{w}^P, \qquad \widetilde{w}^P(0) = \frac{1}{w^P(0)}.$$

Note that with respect to the equation in the proof of [PR19a, Lemma 3.10] some factors 2 are out of place: this is because here we consider the operator ∂_x^2 instead of $\frac{1}{2}\partial_x^2$. The equation for $\widetilde{w}^P$ is almost identical to the one for w^P and admits a paracontrolled solution as an application of [PR19a, Proposition 5.6]. In particular the quoted result implies that:

$$\sup_{S \leq t \leq T} \|\widetilde{w}_t\|_\infty \leq e^{\overline{C}(S,T)(1+\|\mathbf{Y}\|_{\mathscr{Y}_{\mathrm{kpz}}})^{\overline{q}}},$$

for some $\overline{C}, \overline{q} \geq 1$. Since $\|Y\|_\infty + \|Y^{\vee}\|_\infty + \|Y^{\vee\!\vee}\|_\infty \lesssim \|\mathbf{Y}\|_{\mathscr{Y}_{\mathrm{kpz}}}$, one has overall that:

$$\log \varphi_t(\omega)1 \gtrsim_{S,T} -1 - \|\mathbf{Y}\|_{\mathscr{Y}_{\mathrm{kpz}}}^{\overline{q}}.$$

Together with the previous results and the moment bound on $\mathbb{E}\|\mathbf{Y}\|^{\overline{q}}$ we already recalled, this proves that:

$$\mathbb{E} \sup_{S \leq t \leq T} d_H(\varphi_t \cdot f, f) < \infty.$$

Hence the proof is complete.

□

Remark V.6.10. *As an alternative to our proof of a lower bound to $h_t = \log \varphi_t(\omega)u_0$, it seems possible to use an optimal control representation of h, see [GP17, Theorem 7.13]. Both approaches rely crucially on the pathwise solution theory for the KPZ equation.*

Remark V.6.11. *In the previous proposition we have proven that we can apply Theorem V.5.3. The latter guarantees synchronization up to subtracting time-dependent constants $c(\omega, t)$. In fact it seems possible to choose $c(\omega, t) \equiv \overline{c}(\omega)$ for a time-independent $\overline{c}(\omega)$. For fractional noise we could show this in Remark V.6.6, but in the argument we made use of the spatial smoothness of the noise to write an ODE for the constant $c(\omega, t)$: Equation* (V.28).

It seems reasonable to expect that the approach of Remark V.6.6 can be lifted to the space-time white noise setting by defining the product which appears in the ODE for example in a paracontrolled way. To complete the argument one would need to control the paracontrolled, and not only the Hölder norms in the convergences of Theorem V.5.3. This appears feasible, but falls beyond the aims of the present paper.

V.7 Mixing of Gaussian fields

Let us state a general criterion which ensures that a possibly infinite-dimensional Gaussian field is mixing (and hence ergodic). This is a generalization of a classical result for one-dimensional processes, cf. [CFS82, Chapter 14]. We indicate with $\mathbf{B}^*$ the dual of a Banach space $\mathbf{B}$ and write $\langle \cdot, \cdot \rangle$ for the dual pairing.

Proposition V.7.1. *Let $\mathbf{B}$ be a separable Banach space. Let μ be a Gaussian measure on $(\mathbf{B}, \mathscr{B}(\mathbf{B}))$ and $\vartheta \colon \mathbf{N}_0 \times \mathbf{B} \to \mathbf{B}$ a dynamical system which leaves μ invariant. Let ξ be any random variable with values in $\mathbf{B}$ and law μ. The condition*

$$\lim_{n \to \infty} \mathrm{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi' \rangle) = 0, \qquad \forall \varphi, \varphi' \in \mathbf{B}^* \tag{V.36}$$

implies that the system is mixing, that is for all $A, B \in \mathscr{B}(\mathbf{B})$:

$$\lim_{n \to \infty} \mu(A \cap \vartheta^{-n} B) = \mu(A)\mu(B).$$

If in addition μ satisfies that

$$\mathbb{E}[\|\xi\|_{\mathbf{B}}^2] < \infty,$$

and $S \subseteq \mathbf{B}^*$ is a dense subset then

$$\lim_{n \to \infty} \mathrm{Cov}(\langle \xi, \varphi \rangle, \langle \vartheta^n \xi, \varphi \rangle) = 0, \qquad \forall \varphi, \varphi' \in S \tag{V.37}$$

implies condition (V.36).

Proof. First, we reduce ourselves to the finite-dimensional case. Indeed, note that the sequence $\{\vartheta^n \xi\}_{n \in \mathbf{N}}$ is tight in $\mathbf{B}$, because ϑ leaves μ invariant. Tightness implies that the sequence is flatly concentrated (cf. [dA70, Theorem 2.1 and Definition 2.1]), that is for every $\varepsilon > 0$ there exists a finite-dimensional linear space $S^\varepsilon \subseteq \mathbf{B}$ such that:

$$\mathbb{P}\Big(\vartheta^n \xi \in S^\varepsilon\Big) \geq 1 - \varepsilon, \qquad \forall n \in \mathbf{N},$$

which in turn implies that

$$\mathbb{P}\Big((\xi, \vartheta^n \xi) \in S^\varepsilon \times S^\varepsilon\Big) \geq 1 - 2\varepsilon, \qquad \forall n \in \mathbf{N}.$$

Hence, it is sufficient to check the mixing property for all $A, B \in \mathscr{B}(S^\varepsilon)$ and $\varepsilon > 0$. In fact, assuming the property holds in S^ε, then for general $A, B \in \mathscr{B}(\mathbf{B})$ we have for any $\varepsilon > 0$

$$\begin{aligned} |\mu(A \cap \vartheta^{-n} B) - \mu((A \cap S^\varepsilon) \cap (\vartheta^{-n} B \cap S^\varepsilon)| &\lesssim \varepsilon \\ |\mu(A \cap S^\varepsilon) - \mu(A)| &\lesssim \varepsilon, \end{aligned}$$

so that for any $\varepsilon > 0$ and some $C > 0$:

$$|\lim_{n \to \infty} \mu(A \cap \vartheta^{-n} B) - \mu(A)\mu(B)| \leqslant C\varepsilon,$$

which proves the claim.

This means that there exists an $n \in \mathbf{N}$ and $\varphi_i \in \mathbf{B}^*$ for $i = 1, \dots, n$ such that we have to check the mixing property for the vector:

$$((\langle \xi, \varphi_i \rangle)_{i=1,\dots,n}, \ \ (\langle \vartheta^n \xi, \varphi_i \rangle)_{i=1,\dots,n}).$$

In this setting and in view of our assumptions the result on the mixing property follows from [FS13, Theorem 2.3].

Finally, we have to prove that if $\mathbb{E}\|\xi\|^2_{\mathbf{B}} < \infty$, then it suffices to check condition (V.37) for $\varphi, \varphi' \in S$. Indeed take any $\psi, \psi' \in \mathbf{B}^*$. Since S is dense, consider for any $\varepsilon \in (0,1)$ a pair $\varphi_\varepsilon, \varphi'_\varepsilon \in S$ such that

$$\|\psi - \varphi_\varepsilon\|_{\mathbf{B}^*} + \|\psi' - \varphi'_\varepsilon\|_{\mathbf{B}^*} \leqslant \varepsilon.$$

Then define $M > 0$ by

$$M = \sup_{\varepsilon \in (0,1)} (\|\varphi_\varepsilon\|_{\mathbf{B}^*} + \|\varphi'_\varepsilon\|_{\mathbf{B}^*}) < \infty.$$

We can bound, for every $n \in \mathbf{N}$:

$$\begin{aligned}
|\mathrm{Cov}(\langle \xi, \psi \rangle, \langle \vartheta^n \xi, \psi' \rangle) - &\mathrm{Cov}(\langle \xi, \varphi_\varepsilon \rangle, \langle \vartheta^n \xi, \varphi'_\varepsilon \rangle)| \\
&\leqslant \mathbb{E}|\langle \xi, \psi - \varphi_\varepsilon \rangle \cdot \langle \vartheta^n \xi, \psi' \rangle| + \mathbb{E}|\langle \xi, \varphi_\varepsilon \rangle, \langle \vartheta^n \xi, \psi' - \varphi'_\varepsilon \rangle)| \\
&\leqslant \varepsilon \|\psi'\|_{\mathbf{B}^*} \mathbb{E}\|\xi\|_{\mathbf{B}} \|\vartheta^n \xi\|_{\mathbf{B}} + \varepsilon \|\varphi_\varepsilon\|_{\mathbf{B}^*} \mathbb{E}\|\xi\|_{\mathbf{B}} \|\vartheta^n \xi\|_{\mathbf{B}} \\
&\leqslant \varepsilon \cdot 2M \cdot \mathbb{E}\|\xi\|^2_{\mathbf{B}}.
\end{aligned}$$

In particular, since by assumption $\varphi_\varepsilon, \varphi'_\varepsilon$ satisfy condition (V.37), we have proven that:

$$\limsup_{n \to \infty} |\mathrm{Cov}(\langle \xi, \psi \rangle, \langle \vartheta^n \xi, \psi' \rangle)| \leqslant \varepsilon \cdot 2M \cdot \mathbb{E}\|\xi\|^2_{\mathbf{B}}.$$

As ε is arbitrary this proves that condition (V.36) is true. □

V.8 Some analytic results

Lemma V.8.1. *Let P_t be the heat semigroup. One can estimate, for $\alpha \in \mathbf{R}, \beta \in [0,2), p \in [1, \infty]$ and any $T > 0$:*

$$\sup_{0 \le t \le T} t^{\frac{\beta}{2}} \|P_t f\|_{B^{\alpha+\beta}_{p,\infty}(\mathbf{T}^d)} \lesssim \|f\|_{B^\alpha_{p,\infty}(\mathbf{T}^d)}.$$

If one additionally chooses $b \in L^\infty([0,T]; B^\gamma_{\infty,\infty}(\mathbf{T}^d; \mathbf{R}^d)), c \in L^\infty([0,T]; B^\gamma_{\infty,\infty}(\mathbf{T}^d))$, such that:

$$\zeta := \gamma \wedge \alpha + \beta, \qquad \gamma + \zeta - 1 > 0, \qquad \beta \ge 1$$

there exists a unique mild solution w to:

$$(\partial_t - \Delta) w(t,x) = b(t,x) \cdot \nabla w(t,x) + c(t,x) w(t,x), \qquad w(0,x) = w_0(x),$$

meaning that

$$w(t,x) = P_t w_0(x) + \int_0^t P_{t-s}[b(s) \cdot \nabla w(s) + c(s) w(s)](x) \, \mathrm{d}s.$$

Moreover, there exists a $q \ge 0$ and $C(T) \ge 0$ such that:

$$\sup_{0 \le t \le T} t^{\frac{\beta}{2}} \|w_t\|_{B^\zeta_{p,\infty}} \lesssim \|w_0\|_{B^\alpha_{p,\infty}} e^{C(T)(1 + \|b\|_{L^\infty([0,T]; B^\gamma_{\infty,\infty})} + \|c\|_{L^\infty([0,T]; B^\gamma_{\infty,\infty})})^q}.$$

Proof. The estimate regarding the heat kernel is classical. For a reference from the field of singular SPDEs see [GIP15, Lemma A.7]. Let us pass to the PDE. Here consider any w such that $M := \sup_{0\le t\le T} t^{\frac{\beta}{2}}\|w\|_{B^{\zeta}_{p,\infty}} < \infty$, and let $N := \sup_{0\le t\le T}\Big\{\|b_t\|_{B^{\gamma}_{\infty,\infty}(\mathbf{T}^d;\mathbf{R}^d)} + \|c_t\|_{B^{\gamma}_{\infty,\infty}(\mathbf{T}^d)}\Big\}$. Then consider:

$$\mathscr{I}(w)_t = P_t w_0 + \int_0^t P_{t-s}\Big[b_s\cdot\nabla w_s + c_s w_s\Big]\mathrm{d}s.$$

It follows from the smoothing effect of the heat kernel that:

$$\sup_{0\le t\le T} t^{\frac{\beta}{2}}\|\mathscr{I}(w)_t\|_{B^{\zeta}_{p,\infty}} \lesssim \|w_0\|_{B^{\alpha}_{p,\infty}} + \sup_{0\le t\le T} t^{\frac{\beta}{2}}\int_0^t (t-s)^{-\frac{\beta}{2}}\Big(\|b_s\cdot\nabla w_s\|_{B^{(\zeta-1)\wedge\gamma}_{p,\infty}} + \|c_s w_s\|_{B^{\zeta\wedge\gamma}_{p,\infty}}\Big)\,\mathrm{d}s$$

Now from our condition on the coefficient and estimates on products of distributions (see [BCD11, Theorem 2.82 and 2.85]) the latter term can in turn be bounded by:

$$\begin{aligned}\sup_{0\le t\le T} t^{\frac{\beta}{2}}\|\mathscr{I}(w)_t\|_{B^{\zeta}_{p,\infty}} &\lesssim |w_0\|_{B^{\alpha}_{p,\infty}} + MN\sup_{0\le t\le T} t^{\frac{\beta}{2}}\int_0^t (t-s)^{-\frac{\beta}{2}}s^{-\frac{\beta}{2}}\,\mathrm{d}s\\ &\lesssim \|w_0\|_{B^{\alpha}_{p,\infty}} + MNT^{1-\frac{\beta}{2}}.\end{aligned}$$

It follows that for small $T>0$ the map $\mathscr{I}$ is a contraction providing the existence of solutions for small times. By linearity and a Gronwall-type argument, this estimate also provides the required a-priori bound. □

Lemma V.8.2. *For any $\gamma>0$, the inclusion $\{\delta_y\}_{y\in\mathbf{T}^d}\subseteq B^{-\gamma}_{1,\infty}$ holds. Moreover, there exists an $L>0$ such that:*

$$\|\delta_x-\delta_y\|_{B^{-\gamma}_{1,\infty}} \le L|x-y|^{\gamma}.$$

Proof. We divide the proof in two steps. Recall that by definition we have to bound $\sup_{j\ge -1} 2^{-\gamma j}\|\Delta_j(\delta_x-\delta_y)\|_{L^1}$. Hence we choose j_0 as the smallest integer such that $2^{-j_0}\le|x-y|$. We first look at small scales $j\ge j_0$ and then at large scales $j<j_0$. For small scales, by the Poisson summation formula, since $\varrho_j(k)=\varrho_0(2^{-j}k)$, and by defining $K_j(x)=\mathscr{F}^{-1}_{\mathbf{R}}\varrho_j(x)=2^jK(2^jx)$ for some $K\in\mathscr{S}(\mathbf{R})$ (the space of tempered distributions):

$$\begin{aligned}2^{-\gamma j}\big\|\Delta_j(\delta_x-\delta_y)\big\|_{L^1} &\le |x-y|^{\gamma}\int_{\mathbf{R}} 2^j|K(2^j(z-x))-K(2^j(z-y))|\,\mathrm{d}z\\ &\lesssim |x-y|^{\gamma}\int_{\mathbf{R}} 2^j|K(2^jz)|\,\mathrm{d}z \lesssim |x-y|^{\gamma}.\end{aligned}$$

While for large scales, since we have $|2^j(x-y)|\le 1$, applying the Poisson summation formula, by the mean value theorem and since $K\in\mathscr{S}(\mathbf{R})$ (the Schwartz space of functions):

$$\begin{aligned}2^{-\gamma j}\big\|\Delta_j(\delta_x-\delta_y)\big\|_{L^1} &\le 2^{-\gamma j}\int |K(z)-K(z+2^j(x-y))|\,\mathrm{d}z\\ &\le |x-y|^{\gamma}\int \max_{|\xi-z|\le 1}\frac{|K(\xi)-K(z)|}{|\xi-z|^{\alpha}}\,\mathrm{d}z \lesssim |x-y|^{\gamma}.\end{aligned}$$

The result follows.

□

Lemma V.8.3. *Fix any $\alpha < \frac{1}{2}$. Consider the space*

$$\mathscr{Y}_{\text{kpz}} \subseteq C_{\text{loc}}([0,\infty); C^{\alpha} \times C^{2\alpha} \times C^{\alpha+1} \times C^{2\alpha+1} \times C^{2\alpha+1} \times C^{2\alpha-1}),$$

with the norm $\|\cdot\|_{\mathscr{Y}_{\text{kpz}}}$ as in [GP17, Definition 4.1]. There exists a random variable $\mathbf{Y}\colon \Omega_{\text{kpz}} \to \mathscr{Y}_{\text{kpz}}$ which coincides almost surely with the random variable constructed in [GP17, Theorem 9.3] and is given by:

$$\mathbf{Y}(\omega) = (Y(\omega), Y^{\vee}(\omega), Y^{\vee\!\!\!\vee}(\omega), Y^{\vee\!\!\!\vee\!\!\!\vee}(\omega), Y^{\vee\!\vee}(\omega), \partial_x \mathscr{P} \odot \partial_x Y(\omega)),$$

where the latter solve (formally):

$$\begin{aligned}
(\partial_t - \partial_x^2) Y &= \Pi_{\times} \xi, \\
(\partial_t - \partial_x^2) Y^{\vee} &= (\partial_x Y)^2 - \infty, \\
(\partial_t - \partial_x^2) Y^{\vee\!\!\!\vee} &= \partial_x Y \partial_x Y^{\vee}, \\
(\partial_t - \partial_x^2) Y^{\vee\!\!\!\vee\!\!\!\vee} &= \partial_x Y^{\vee\!\!\!\vee} \odot \partial_x Y - \infty, \\
(\partial_t - \partial_x^2) Y^{\vee\!\vee} &= (\partial_x Y^{\vee})^2 - \infty \\
(\partial_t - \partial_x^2) \mathscr{P} &= \partial_x Y.
\end{aligned}$$

Here $\Pi_{\times} f = f - \int f(x)\, \mathrm{d}x$ and $f \odot g = \sum_{|i-j| \le 1} \Delta_i f \Delta_j g$ is the resonant product between two distributions (which is a-priori ill-defined). Finally, the presence of infinity indicates the necessity of Wick renormalisation, in the sense of [GP17, Theorem 9.3]. Y is started in invariance, that is:

$$Y_t = \int_{-\infty}^{t} P_{t-s} \Pi_{\times} \xi \, \mathrm{d}s,$$

while all other elements are started in $Y^{\tau}(0) = 0$. In particular $\mathbf{Y}$ changes as follows under the action of ϑ^n, for $n \in \mathbf{N}_0, t \ge 0, \omega \in \Omega_{\text{kpz}}$:

$$\begin{aligned}
\mathbf{Y}_t(\vartheta^n \omega) = (&Y_{t+n}, Y^{\vee}_{t+n} - P_t Y_n, Y^{\vee\!\!\!\vee}_{t+n} - P_t Y^{\vee\!\!\!\vee}_n, \\
&Y^{\vee\!\!\!\vee\!\!\!\vee}_{t+n} - P_t Y^{\vee\!\!\!\vee\!\!\!\vee}_n, Y^{\vee\!\vee}_{t+n} - P_t Y^{\vee\!\vee}_n, \partial_x(\mathscr{P}_{t+n} - P_t \mathscr{P}_n) \odot \partial_x (Y_{t+n} - P_t Y_n))(\omega).
\end{aligned} \tag{V.38}$$

Proof. The only point that requires a proof is the action of the translation operator. By taking into account the initial conditions and using [GP17, Theorem 9.3], Equation (V.38) holds for fixed n, for all $\omega \notin N_n$ and all $t \ge 0$, for a given null-set N_n (since the random variables are constructed in $L^2(\Omega_{\text{kpz}}; \mathscr{Y}_{\text{kpz}})$). Considering $N = \bigcup_{n \in \mathbf{N}} N_n$ and setting $\mathbf{Y}(\omega) = 0$ for $\omega \in N$, one obtains the result for all $\omega \in \Omega_{\text{kpz}}$. □

Lemma V.8.4. *Consider a sequence $\{a_k\}_{k \in \mathbf{N}}$ of positive ($a_k \geqslant 0$) real numbers. Suppose that*

$$S_n = \frac{1}{n} \sum_{k=1}^{n} a_k$$

converges. Namely, there exists a $\sigma \in [0, \infty)$ such that

$$\lim_{n \to \infty} S_n = \sigma.$$

Then

$$\lim_{n \to \infty} \frac{1}{n} a_n = 0.$$

Proof. Since S_n is convergent fix any $\varepsilon > 0$ and let $n(\varepsilon) \in \mathbf{N}$ be such that

$$|S_n - S_m| \leqslant \varepsilon, \quad \forall n, m \geqslant n(\varepsilon).$$

We can assume, up to taking a larger $n(\varepsilon)$, that $n(\varepsilon) \geqslant \frac{\sigma+\varepsilon}{\varepsilon}$. Now consider $n \geqslant n(\varepsilon)+1$. We can compute

$$\begin{aligned} \varepsilon \geqslant |S_n - S_{n-1}| &= \left| \frac{a_n}{n} - \left(\frac{1}{n-1} - \frac{1}{n} \right) \sum_{k=1}^{n-1} a_k \right| \\ &= \left| \frac{a_n}{n} - \frac{1}{n} S_{n-1} \right| \\ &\geqslant \frac{a_n}{n} - \frac{S_{n-1}}{n} \geqslant \frac{a_n}{n} - \varepsilon \frac{\sigma+\varepsilon}{\sigma+\varepsilon}, \end{aligned}$$

which implies that $\frac{a_n}{n} \leqslant 2\varepsilon$ for all $n \geqslant n(\varepsilon)+1$. Since ε is arbitrary this completes the proof. □

Lemma V.8.5. *Consider any $\beta \in (0,\infty)$, $\alpha \in (0,\beta)$ and let $\theta = \frac{\alpha}{\beta} \in (0,1)$. Then there exists a constant $C(\alpha,\beta) > 0$ such that for every $f \in C^\beta$:*

$$[f]_\alpha \leqslant C(\alpha,\beta) \|f\|_\infty^{1-\theta} [f]_\beta^\theta.$$

Proof. We start by recalling, for $k \in \{1,\ldots,d\}$, the one-dimensional Landau-Kolmogorov inequality (see for example [Kol] or many online resources):

$$\|\partial_{x_k} f\|_\infty \lesssim \|f\|_\infty^{1-\frac{1}{n}} \|\partial_{x_k}^n f\|_\infty^{\frac{1}{n}},$$

Iterating this inequality one obtains that for any $n, l \in \mathbf{N}$ and $k_i \in \{1,\ldots,d\}$, $\forall i = 1,\ldots,l$:

$$\begin{aligned} \|\partial_{x_{k_1}} \cdots \partial_{x_{k_l}} f\|_\infty &\lesssim \|\partial_{x_{k_2}} \cdots \partial_{x_{k_l}} f\|_\infty^{1-\frac{1}{n+1}} [f]_{n+l}^{\frac{1}{n+1}} \\ &\lesssim \|f\|_\infty^{\prod_{i=1}^l \left(1-\frac{1}{n+i}\right)} [f]_{n+l}^{\sum_{i=1}^l \frac{1}{n+i} \prod_{j=1}^{i-1} \left(1-\frac{1}{n+j}\right)}. \end{aligned}$$

Since (both identities can be proven by induction over l):

$$\prod_{i=1}^{l} \left(1 - \frac{1}{n+i}\right) = 1 - \frac{l}{n+l}, \qquad \sum_{i=1}^{l} \frac{1}{n+i} \prod_{j=1}^{i-1} \left(1 - \frac{1}{n+j}\right) = \frac{l}{n+l},$$

we have proven that

$$[f]_l \leqslant C(l, n+l) \|f\|_\infty^{1-\frac{l}{n+l}} [f]_{n+l}^{\frac{l}{n+l}}, \tag{V.39}$$

which is the desired inequality for integer α, β. To pass to the fractional case we will first prove that for $\beta \geqslant n, n \in \mathbf{N}$:

$$[f]_n \lesssim \|f\|_\infty^{1-\frac{n}{\beta}} [f]_\beta^{\frac{n}{\beta}}. \tag{V.40}$$

We can further simplify this by considering $\beta \in (1,2)$ and proving:

$$\|\partial_{x_k} f\|_\infty \leqslant 2\|f\|_\infty^{1-\frac{1}{\beta}} [f]_\beta^{\frac{1}{\beta}}$$

To obtain this let e_k be the unit vector in the k−th direction, and consider for $h > 0, x \in \mathbf{T}^d$:

$$\partial_{x_k} f(x) = \frac{f(x+he_k) - f(x)}{h} + R(x,h).$$

Since

$$\frac{f(x+he_k) - f(x)}{h} = \partial_{x_k} f(\xi),$$

for some $\xi \in [x, x+he_k]$ (were $[x, x+he_k]$ is the line between x and $x+he_k$), we can bound the rest term by:

$$|R(x,h)| \leqslant \sup_{\xi \in [x,x+he_k]} |\partial_{x_k} f(x) - \partial_{x_k} f(\xi)| \leqslant h^{\beta-1} [f]_\beta.$$

Hence we have

$$\begin{aligned} \|\partial_{x_k} f\|_\infty &\leqslant h^{-1} \|f\|_\infty + h^{\beta-1}[f]_\beta \\ &\leqslant 2\|f\|_\infty^{1-\frac{1}{\beta}} [f]_\beta^{\frac{1}{\beta}} \end{aligned}$$

by setting $h = (\|f\|_\infty / [f]_\beta)^{\frac{1}{\beta}}$. Next, we deduce (V.40) for any $\beta > 1$ and $n = \lfloor \beta \rfloor$. Using all the estimates we already derived:

$$\begin{aligned} [f]_n &\lesssim [f]_{n-1}^{1-\frac{1}{\beta-n+1}} [f]_\beta^{\frac{1}{\beta-n+1}} \\ &\lesssim \|f\|_\infty^{\left(1-\frac{n-1}{n}\right)\left(1-\frac{1}{\beta-n+1}\right)} [f]_n^{\left(1-\frac{1}{\beta-n+1}\right)\frac{n-1}{n}} [f]_\beta^{\frac{1}{\beta-n+1}} \\ \iff [f]_n^\zeta &\lesssim \|f\|_\infty^{\left(1-\frac{n-1}{n}\right)\left(1-\frac{1}{\beta-n+1}\right)} [f]_\beta^{\frac{1}{\beta-n+1}}, \end{aligned}$$

for

$$\zeta = 1 - \left(\frac{\beta-n}{\beta-n+1}\right)\left(\frac{n-1}{n}\right) = \frac{\beta}{n(\beta-n+1)},$$

so that the last estimate implies (V.40) for the chosen β and n. To conclude the proof of (V.40) we have to consider the case $\beta > 1, n \leqslant \lfloor \beta \rfloor$. We find that:

$$\begin{aligned} [f]_n &\lesssim \|f\|_\infty^{1-\frac{n}{\lfloor\beta\rfloor}} [f]_{\lfloor\beta\rfloor}^{\frac{n}{\lfloor\beta\rfloor}} \\ &\lesssim \|f\|_\infty^{1-\frac{n}{\lfloor\beta\rfloor}} \left(\|f\|_\infty^{1-\frac{\lfloor\beta\rfloor}{\beta}} [f]_\beta^{\frac{\lfloor\beta\rfloor}{\beta}} \right)^{\frac{n}{\lfloor\beta\rfloor}} \\ &\lesssim \|f\|_\infty^{1-\frac{n}{\beta}} [f]_\beta^{\frac{n}{\beta}}. \end{aligned}$$

At this point, we can collect all our results to complete the proof. Consider $k, n \in \mathbf{N}_0$ such that $\alpha \in [k, k+1)$ and $\beta \in [n, n+1)$. Of course $n \geqslant k$. Furthermore, define

$$\alpha' = \alpha - k, \qquad \beta' = \beta - n.$$

Step 1: $n = k$. Note that one can bound

$$[f]_\alpha \simeq \sum_{|\mu|=k} \sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x-y|^{\alpha'}}.$$

In fact, if $n \geqslant 1, f \in C^{\alpha}$, for every μ with $|\mu| = n$ there exists an $x_0 \in \mathbf{T}^d$ such that $\partial^\mu f(x_0) = 0$, so that

$$\|\partial^\mu f\|_\infty \lesssim \sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x-y|^{\alpha'}}.$$

Hence, using (V.40) we can compute (defining $[f]_0 = \|f\|_\infty$ if $n = 0$):

$$\begin{aligned}
[f]_\alpha &\lesssim \sum_{|\mu|=n} \left(\sup_{x\neq y} |\partial^\mu f(x) - \partial^\mu f(y)| \right)^{1-\frac{\alpha'}{\beta'}} \left(\sup_{x \neq y} \frac{|\partial^\mu f(x) - \partial^\mu f(y)|}{|x-y|^{\beta'}} \right)^{\frac{\alpha'}{\beta'}} \\
&\lesssim [f]_n^{1-\frac{\alpha'}{\beta'}} [f]_\beta^{\frac{\alpha'}{\beta'}} \\
&\lesssim \|f\|_\infty^{\left(1-\frac{\alpha'}{\beta'}\right)\left(1-\frac{n}{\beta}\right)} [f]_\beta^{\frac{\alpha'}{\beta'}+\frac{n}{\beta}\left(1-\frac{\alpha'}{\beta'}\right)} \\
&\lesssim \|f\|_\infty^{1-\frac{\alpha}{\beta}} [f]_\beta^{\frac{\alpha}{\beta}},
\end{aligned}$$

which is the required result.
Step 2: $k < n$. Here we compute, following the same steps as above:

$$\begin{aligned}
[f]_\alpha &\lesssim [f]_k^{1-\alpha'} [f]_{k+1}^{\alpha'} \\
&\lesssim \|f\|_\infty^{(1-\alpha')\left(1-\frac{k}{\beta}\right)+\alpha'\left(1-\frac{k+1}{\beta}\right)} [f]_\beta^{(1-\alpha')\frac{k}{\beta}+\alpha'\frac{k+1}{\beta}} \\
&\lesssim \|f\|_\infty^{1-\frac{\alpha}{\beta}} [f]_\beta^{\frac{\alpha}{\beta}},
\end{aligned}$$

which completes the proof of the result. □

Lemma V.8.6. *Consider $\alpha > 0$ and $f \in C^\alpha$ with $f(x) > 0$, $\forall x \in \mathbf{T}^d$ and $\int_{\mathbf{T}^d} f(x)\,\mathrm{d}x = 1$. Then defining $m(f) = \min_{x \in \mathbf{T}^d} f(x)$ one can bound for some $\overline{C}(\alpha) > 0$:*

$$[\log(f)]_\alpha \leqslant \overline{C}(\alpha) \left(\frac{1 + [f]_\alpha}{m(f)} \right)^{\lfloor \alpha \rfloor + 1}.$$

Proof. First, observe that for any multiindex μ with $|\mu| = k \in \mathbf{N}$ and f sufficiently smooth we have a decomposition of the form

$$\partial^\mu \log(f) = \sum_{1 \leqslant p \leqslant k} \frac{\sum_{i=1}^{\zeta(p,\mu)} C(i,p,\mu) \prod_{\lambda \in A^i(p,\mu)} (\partial^\lambda f)}{f^p}, \tag{V.41}$$

where $A^i(p,\mu) \subseteq \mathbf{N}^d$ are finite sets of multiindices such that

$$|A^i(p,\mu)| \leqslant p, \qquad \text{and} \qquad \lambda \in A^i(p,\mu) \Rightarrow |\lambda| \leqslant |\mu|,$$

and $C(i,p,\mu) \in \mathbf{R}$ are some coefficients (here $|A^i(p,\mu)|$ indicates the cardinality of the set). One can check by hand that this decomposition holds true if $|\mu| = 1$. In addition, assuming the decomposition holds true for some $\mu \in \mathbf{N}^d$, one has for any $i \in \{1,\dots,d\}$

$$\partial_{x_i}\partial^\mu \log(f) = \sum_{1\leqslant p\leqslant k} -p\frac{\sum_{i=1}^{\zeta(p,\mu)} C(i,p,\mu)\Big(\prod_{\lambda\in A^i(p,\mu)}(\partial^\lambda f)\Big)\partial_{x_i} f}{f^{p+1}} + \frac{\sum_{i=1}^{\zeta(p,\mu)} C(i,p,\mu)\sum_{\lambda'\in A^i(p,\mu)}\Big(\prod_{\lambda\in A^i(p,\mu)\setminus\{\lambda'\}}(\partial^\lambda f)\Big)(\partial_{x_i}\partial^{\lambda'} f)}{f^p},$$

which is again of the required form. Hence by induction the decomposition holds true for all μ.

To conclude the proof of our result we will now need the following to inequalities. Fix any $\alpha' \in (0,1)$, $f,g \in C(\mathbf{T}^d)$ as well as any smooth function $\varphi\colon \mathscr{U} \to \mathbf{R}$, where $\mathscr{U} \subseteq \mathbf{R}$ is an open set such that $f(\mathbf{T}^d) \subseteq \mathscr{U}$. Then:

$$[\varphi(f)]_{\alpha'} \leqslant \sup_{x\in\mathbf{T}^d} |\varphi'(f(x))|[f]_{\alpha'}, \qquad [f\cdot g]_{\alpha'} \leqslant \|f\|_\infty [g]_{\alpha'} + [f]_{\alpha'}\|g\|_\infty. \tag{V.42}$$

Both inequalities are immediate consequences of the definition of the Hölder seminorm. For the first one:

$$[\varphi(f)]_{\alpha'} = \sup_{x\neq y\in\mathbf{T}^d} \frac{|\varphi(f)(x)-\varphi(f)(y)|}{|x-y|^{\alpha'}} \leqslant \sup_{x\in\mathbf{T}^d}|\varphi'(f(x))|[f]_{\alpha'},$$

while for the second

$$[f\cdot g]_{\alpha'} \leqslant \sup_{x\neq y\in\mathbf{T}^d} \frac{|f(x)-f(y)||g(x)|+|g(x)-g(y)||f(y)|}{|x-y|^{\alpha'}} \leqslant \|f\|_\infty [g]_{\alpha'} + [f]_{\alpha'}\|g\|_\infty.$$

Now we can complete the proof. We find via (V.41) that for $\alpha \geqslant 1, \alpha' = \alpha - \lfloor \alpha \rfloor$:

$$\begin{aligned}[\log(f)]_\alpha &\lesssim \sum_{|\mu|=\lfloor\alpha\rfloor}\sum_{1\leqslant p\leqslant\lfloor\alpha\rfloor}\sum_{i=1}^{\zeta(p,\mu)} \left\|\frac{\prod_{\lambda\in A^i(p,\mu)}(\partial^\lambda f)}{f^p}\right\|_\infty + \left[\frac{\prod_{\lambda\in A^i(p,\mu)}(\partial^\lambda f)}{f^p}\right]_{\alpha'} \\ &\lesssim \sum_{|\mu|=\lfloor\alpha\rfloor}\sum_{1\leqslant p\leqslant\lfloor\alpha\rfloor}\sum_{i=1}^{\zeta(p,\mu)} \frac{[f]_\alpha^{|A^i(p,\mu)|}}{m(f)^p} + \frac{[f]_\alpha^{|A^i(p,\mu)|+1}}{m(f)^{p+1}}.\end{aligned}$$

where in the last step we used (V.42). Now, since $\int f(x)\,\mathrm{d}x = 1$ we have that $m(f) \leqslant 1$. In addition we have that $|A^i(p,\mu)| \leqslant \lfloor\alpha\rfloor$, so overall we have that:

$$[\log(f)]_\alpha \lesssim \left(\frac{1+[f]_\alpha}{m(f)}\right)^{\lfloor\alpha\rfloor+1},$$

which is the required inequality. The case $\alpha \in (0,1)$ is much simpler and follows directly from (V.42). □

Appendix A

A.1 Construction of the BRWRE

This section is dedicated to a rigorous construction of the Branching Random Walk in a Random Environment (BRWRE) of Chapter II. For simplicity and without loss of generality we will work with $n = 1$. Since the space $\mathbf{N}_0^{\mathbf{Z}^d}$ is harder to deal with and we do not need it, we consider the countable subspace $E = \left(\mathbf{N}_0^{\mathbf{Z}^d}\right)_0$ of functions $\eta\colon \mathbf{Z}^d \to \mathbf{N}_0$ with $\eta(x) = 0$, except for finitely many $x \in \mathbf{Z}^d$. We endow E with the distance $d(\eta, \eta') = \sum_{x\in\mathbf{Z}^d} |\eta(x) - \eta'(x)|$, under which E is a discrete and hence locally compact separable metric space. Recall the notations from Section II.2. Below we will construct the "semidirect product measure" $\mathbb{P} \ltimes \mathbb{P}^\omega$ on $\Omega \times \mathbb{D}([0,+\infty);E)$, by which we mean that there exists a Markov kernel κ such that for $A \subset \mathscr{F}, B \subset \mathscr{B}(\mathbb{D}([0,+\infty);E))$:

$$\mathbb{P} \ltimes \mathbb{P}^\omega(A \times B) = \int_A \kappa(\omega, B)\,\mathrm{d}\mathbb{P}(\omega). \tag{A.1}$$

By "Markov kernel" we mean a map $\kappa\colon \Omega \times \mathscr{B}(\mathbb{D}([0,\infty);E)) \to [0,1]$ such that $\kappa(\omega,\cdot)$ is a probability measure on $\mathbb{D}([0,\infty);E)$ for every $\omega \in \Omega$ and

$$\omega \mapsto \kappa(\omega, A)$$

is a measurable map for every $A \subseteq \mathscr{B}(\mathbb{D}([0,\infty);E))$.

Lemma A.1.1. *Assume that for any $\omega \in \Omega$ the potential $\xi(\omega)$ is uniformly bounded and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}^{\mathrm{tot}}_\pi$ on $\Omega = \Omega \times \mathbb{D}([0,+\infty);E)$ endowed with the product sigma algebra, such that $\mathbb{P}^{\mathrm{tot}}_\pi = \mathbb{P} \ltimes \mathbb{P}^\omega_\pi$, with $\mathbb{P}^\omega_\pi$ being the unique measure on $\mathbb{D}([0,+\infty);E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ whose generator is given by $\mathscr{L}^\omega\colon \mathscr{D}(\mathscr{L}^\omega) \to C_b(E)$, with*

$$\begin{aligned}\mathscr{L}^\omega(F)(\eta) \\ = \sum_{x\in\mathbf{Z}^d} \eta_x \cdot \Big[\Delta_x F(\eta) + \xi_+(\omega,x) d_x^+ F(\eta) + \xi_-(\omega,x) d_x^- F(\eta)\Big],\end{aligned}$$

where the domain $\mathscr{D}(\mathscr{L}^\omega)$ is the set of functions $F \in C_b(E)$ such that the right-hand side lies in $C_b(E)$.

Proof. The construction for fixed $\omega \in \Omega$ is classical. Indeed, the generator has the form of [EK86, (4.2.1)], with $\lambda(\eta) = \sum_{x\in\mathbf{Z}^d} \eta_x(2d + |\xi|(\omega,x))$, and we only need to rule out explosions by verifying that almost surely $\sum_{k\in\mathbf{N}} \frac{1}{\lambda(Y_k)} = +\infty$, where Y is the associated discrete time Markov chain. This is the case, since ξ is bounded and thus

$$\sum_{k\in\mathbf{N}} \frac{1}{\lambda(Y_k)} \gtrsim \sum_{k\in\mathbf{N}} \frac{1}{\sum_x Y_k(x)} \geq \sum_{k\in\mathbf{N}} \frac{1}{c+k} = +\infty$$

with $c = \sum_x \pi(x)$. It follows that $\mathscr{L}^\omega$ is the generator associated to the process u. This allows us to define for fixed ω the law $\kappa(\omega,\cdot)$ of our process on $\mathbb{D}([0,+\infty);E)$. To construct the measure $\mathbb{P}^{\mathrm{tot}}_\pi$ we have to show that κ is a Markov kernel, which amounts to proving measurability in the ω coordinate. But κ depends continuously on ξ, which we can verify by coupling the processes for ξ and $\tilde{\xi}$ through a construction based on Poisson jumps at rate $K > \|\xi\|_\infty, \|\tilde{\xi}\|_\infty$ and then rejecting the jumps if an independent uniform $[0,K]$ variable is not in $[0,|\xi(x)|]$ respectively in $[0,|\tilde{\xi}(x)|]$. Since ξ is measurable in ω, also κ is measurable in ω. □

Next, we extend the construction to potentials of sub-polynomial growth:

Lemma A.1.2. *Let $\xi(\omega) \in \bigcap_{a>0} L^\infty(\mathbf{Z}^d; p(a))$ for all $\omega \in \Omega$ and consider $\pi \in E$. There exists a unique probability measure $\mathbb{P}^{\mathrm{tot}}_\pi = \mathbb{P} \ltimes \mathbb{P}^\omega_\pi$ on $\Omega \times \mathbb{D}([0,+\infty);E)$ endowed with the product sigma algebra, where $\mathbb{P}^\omega_\pi$ is the unique measure on $\mathbb{D}([0,+\infty);E)$ under which the canonical process u is a Markov jump process with $u(0) = \pi$ and with generator $\mathscr{L}^\omega$ and $\mathscr{D}(\mathscr{L}^\omega)$ defined as in the previous lemma.*

Proof. Let us fix $\omega \in \Omega$. Consider the Markov jump processes u^k started in π with generator $\mathscr{L}^{\omega,k}$ associated to $\xi^k(x) = (\xi(x) \wedge k) \vee (-k)$ whose existence follows from the previous result. The sequence $\{u^k\}_{k\in\mathbf{N}}$ is tight (this follows as in Lemma II.4.2 and Corollary II.4.3, keeping n fixed but letting k vary) and converges weakly to a Markov process u. Indeed, for $k, R \in \mathbf{N}$ let τ^k_R be the first time with $\mathrm{supp}(u^k(\tau^k_R)) \not\subset Q(R)$, where $Q(R)$ is the square of radius R around the origin, and let τ_R be the corresponding exit time for u. Then we get for all $k > \max_{x\in Q(R)} |\xi(x)|$, for all $T > 0$, and all $F \in C_b(\mathbb{D}([0,T];E))$:

$$\mathbb{E}^\omega_\pi[F((u^k(t))_{t\in[0,T]})1_{\{\tau^k_R \leq T\}}] = \mathbb{E}^\omega_\pi[F((u(t))_{t\in[0,T]})1_{\{\tau_R \leq T\}}],$$

where we used that the exit time τ_R is continuous because E is a discrete space. Moreover, from the tightness of $\{u^k\}_{k\in\mathbf{N}}$ it follows that for all $\varepsilon > 0$ and $T > 0$ there exists $R \in \mathbf{N}$ with $\sup_k \mathbb{P}(\tau^k_R \leq T) < \varepsilon$. This proves the uniqueness in law and that u is the limit (rather than subsequential limit) of $\{u^k\}_{k\in\mathbf{N}}$. Similarly we get the Markov property of u from the Markov property of the $\{u^k\}_{k\in\mathbf{N}}$ and from the convergence of the transition functions.

It remains to verify that $\mathscr{L}^\omega$ is the generator of u. But for large enough R we have $\mathbb{P}^\omega_\pi(\tau_R \leq h) = O(h^2)$ as $h \to 0^+$, because on the event $\{\tau_R \leq h\}$ at least two transitions must have happened (recall that π is compactly supported). We can thus compute for any $F \in C_b(E)$:

$$\mathbb{E}^\omega_\pi\Big[F(u(h))\Big] = \mathbb{E}^\omega_\pi\Big[F(u^k(h))\Big] + O(h^2).$$

The result on the generator then follows from the previous lemma. As before, we now have constructed a collection of probability measures $\kappa(\omega,\cdot)$ as the limit of the Markov kernels $\kappa^k(\omega,\cdot)$. Since measurability is preserved when passing to the limit, this concludes the proof. □

A.2 Tightness criteria

We recall two classical tightness criteria.

Proposition A.2.1. *[Jak86, Theorem 3.1] Let X be a separable metric space and fix $T > 0$. Let F be a family of real, continuous functions on X which separates points and is closed under addition. Then a sequence of probability measures $\{\mathbb{P}_n\}_{n\in\mathbf{N}}$ on $\mathbb{D}([0,T];X)$ is tight if the following two conditions are satisfied:*

i For each $\varepsilon > 0$ there exists a compact set $K \subset X$ such that

$$\inf_{n\in\mathbf{N}} \mathbb{P}_n\Big(X_t \in K,\ \forall t \in [0,T]\Big) \geq 1-\varepsilon,$$

where X_t is the canonical process on $\mathbb{D}([0,T];X)$.

ii For each $f \in F$ sequence $\mathbb{P}_n \circ f^{-1}$ is tight as a measure on $\mathbb{D}([0,T];\mathbf{R})$.

In the next criterion, the space $W^{2,\zeta}([0,T];Y) \subset L^2([0,T];Y)$ is defined by the Sobolev-Slobodeckij norm

$$\|f\|_{W^{2,\zeta}([0,T];Y)} = \|f\|_{L^2([0,T];Y)} + \left(\int_0^T \int_0^T \frac{\|f(t)-f(r)\|_Y^2}{|t-r|^{2\zeta+1}}\,\mathrm{d}t\,\mathrm{d}r\right)^{1/2}.$$

Proposition A.2.2 (Corollary 5, [Sim87])**.** *Let X, Y, Z be three Banach spaces such that $X \subset Y \subset Z$ with the embedding $X \subset Y$ being compact and fix $T > 0$. Then the following embedding is compact, for any $s > 0$:*

$$L^p([0,T];X) \cap W^{s,p}([0,T];Z) \subseteq L^p([0,T];Y).$$

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