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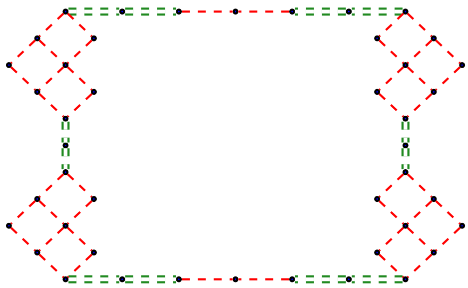
Convex Geometry of Subword Complexes of Coxeter Groups



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"[. . .] l'important no és acabar una obra, sinó deixar entreveure que un dia serà possible començar alguna cosa."

Anotacions, quadern FJM (*1941*)
Joan Miró

Summary

This monography presents results related to the convex geometry of a family of simplicial complexes called “subword complexes”. These simplicial complexes are defined using the Bruhat order of Coxeter groups. Despite a simple combinatorial definition much of their combinatorial properties are still not understood. In contrast, many of their known connections make use of specific geometric realizations of these simplicial complexes. When such realizations are missing, many connections can only be conjectured to exist.

This monography lays down a framework using an alliance of algebraic combinatorics and discrete geometry to study further subword complexes. It provides an abstract, though transparent, perspective on subword complexes based on linear algebra and combinatorics on words. The main contribution is the presentation of a universal partial oriented matroid whose realizability over the real numbers implies the realizability of subword complexes as oriented matroids.

Zusammenfassung

Diese Monographie präsentiert Ergebnisse im Zusammenhang mit einer Familie von simplizialen Komplexen, die "Subwortkomplexe" genannt werden. Diese Simplizialkomplexe werden mit Hilfe der Bruhat-Ordnung von Coxeter-Gruppen definiert. Trotz einer einfachen kombinatorischen Definition werden viele ihrer kombinatorischen Eigenschaften immer noch nicht verstanden. Spezifische geometrische Realisierungen dieser Simplizialkomplexe machen neue Herangehensweisen an Vermutungen des Gebiets möglich. Wenn solche Verbindungen fehlen, können viele Zusammenhänge nur vermutet werden.

Diese Monographie legt einen Rahmen fest, in dem eine Allianz aus algebraischer Kombinatorik und diskreter Geometrie verwendet wird, um weitere Subwortkomplexe zu untersuchen. Es bietet eine abstrakte und transparente Perspektive auf Teilwortkomplexe, die auf linearer Algebra und Kombinatorik von Wörtern basiert. Der Hauptbeitrag ist die Darstellung eines universellen, nur teilweise orientierten Matroids, dessen Realisierbarkeit über den reellen Zahlen die Realisierbarkeit von Teilwortkomplexen als orientierte Matroide impliziert.

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Contents

Notation

$\mathbb{N}, \mathbb{R}$	the nonnegative integers, the real numbers
n, $[n]$	an element of $\mathbb{N} \setminus \{0\}$ and $\{1, 2, \ldots, n\}$
$\binom{[n]}{k}$	the collection of k-elements subsets of $[n]$
$\#S$	cardinality of a set S
$\mathfrak{S}_n$ or $\mathfrak{S}_S$	The symmetric group on n objects where S is a set of cardinality n
$\mathbb{Z}_2$	the multiplicative group $(\{\pm 1\}, \times)$
V_d, V_d^*	a real vector space of dimension d and its dual vector space
$\mathrm{Vander}(d)$	Vandermonde matrix of size d
λ	a partition with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$
$\text{ш}_{\lambda,J}$	Schur function on variables labeled by J for the partition λ
ш_{λ}	Schur function on variables labeled by $[n]$ for the partition λ
Λ	a sequence of partition of length n
$\text{Ш}_{\Lambda,P}$	partial Schur function with respect to an ordered set partition P and a sequence of partitions Λ
$\lambda_{\mathfrak{z},i}$	standard partition
$\Lambda_{\mathfrak{z}}$	sequence of standard partitions
S	a finite alphabet of letters ordered $s_1 < s_2 < \cdots < s_n$
S^*	free monoid over S
e	identity of S^*, the empty word
$\{w\}_i$	set of occurrences of the letter s_i in w
Ω_w	the ordered partition $(\{w\}_1, \ldots, \{w\}_n)$ for a word $w \in S^*$
$\lvert w\rvert_i$	the number of occurrences of the letter s_i in w
$\mathrm{rev}(w)$	the reverse word of a word $w \in S^*$
$\alpha(w)$	the abelian vector $(\lvert w\rvert_1, \ldots, \lvert w\rvert_n)$ of $w \in S^*$
$\overline{w}$	lexicographic normal form of $\alpha(w)$
$\mathrm{std}(w)$	the standard permutation of w
$\mathrm{inv}(w)$	the number of inversions of $\mathrm{std}(w)$
ϕ	alphabetic order reversing map on S
$\overleftarrow{\mathrm{inv}}(w)$	the number of swaps of w in $\phi(S)$

(W, S)	a finite irreducible Coxeter system: a Coxeter group W with generators S
ℓ	length function of Coxeter groups
$w_\circ$, $N := \ell(w_\circ)$	the longest element of a finite Coxeter group W, the length of $w_\circ$
$\mathcal{R}(w)$	the set of reduced words of an element $w \in W$
ν	the höchstfrequenz of the group W
$\mathcal{G}(w)$	graph of reduced words and braid moves of $w \in W$
$\mathcal{G}^{\text{even}}(w)$	graph of even braid moves of w
$\mathcal{G}^{\text{odd}}(w)$	graph of odd braid moves of w
$\mathcal{G}^{\text{comm}}$	graph of commutation classes of w
$\mathcal{G}^{\text{braid}}$	graph of braid classes of w
$\mathcal{G}^{2}(w)$	graph of only commutation moves of w
$\Delta_W(p)$	subword complex of type W for the word $p \in S^*$

$\mathbf{A}, \mathbf{B}$	vector configurations indexed by J
$\text{cone}_{\mathbf{A}}(C)$	non-negative span of a set of vectors indexed by C in $\mathbf{A}$
$\mathcal{F}$	a vector fan
$\mathcal{F}_{p,\mathbf{A}}$	collection of cones spanned by sets of $\mathbf{A}$ corresponding to faces of $\Delta_W(p)$
$\Delta_{\mathcal{F}}$	spherical simplicial complex arising from a complete simplicial fan $\mathcal{F}$
$\text{Gale}(A)$	set of Gale transforms of $\mathbf{A}$
C^*	dual simplex $J \setminus C$
$\text{cone}_{\mathbf{B}}(C^*)$	dual simplicial cone of C

F_{odd}	odd-sign function
$F_{\text{even}}, \tau(w)$	even-sign function, the T-sign of a word w
$\sigma(w)$	the S-sign of a word w
⧖	the punctual sign function

$\mathcal{T}_W$	the variables tensor of W
${\mathcal{P}^i}_{s,k}(N, n, d)$	a parameter tensor
${\mathcal{C}^i}_{j,k}(v, \mathcal{P})$	coefficient tensor of the word v with respect to $\mathcal{P}$
${M^i}_l(v, \mathcal{P})$	model matrix of the word v with respect to $\mathcal{P}$
$\mathcal{V}(v)$	Vandermonde divisor of a word v

Conventions. Words in the alphabet S are written as a concatenation of letters. Vectors and linear functions are denoted using bold letters such as $\mathbf{e}, \mathbf{g}, \mathbf{v}, \mathbf{x}, \mathbf{y} \ldots$. Tensors are denoted using capitalized calligraphic letters such as $\mathcal{M}, \mathcal{T}, \mathcal{U}, \mathcal{V}, \mathcal{X} \ldots$. Exponents on tensors designate vector spaces while indices designate dual vector spaces. Letters of words are considered with their embedding in the word: two letters representing the same generators are considered different, since they are at different positions. To write classes of reduced words, we use the following convention that $12\{13\}21 := \{121321, 123121\}$.

Introduction

Let Δ be a simplicial complex homeomorphic to a $(d-1)$-sphere.

Is there a d-dimensional simplicial polytope P whose face lattice is isomorphic to the one of Δ?

In the affirmative case, the simplicial sphere Δ is *polytopal*. Steinitz showed that every 2-dimensional simplicial sphere is polytopal [SR76][Grü03, Section 13]. In higher dimension, this question is part of the larger "Steinitz Problem" asking to determine polytopal spheres among all simplicial spheres. A lot of work has been done towards brute force enumeration of simplicial and polytopal spheres in low dimensions, and spheres with fews vertices, see [Fir17, Fir18] and [Bri16, BZ18] to get an overview of the most recent results. The determination of the polytopality of a simplicial sphere is known to be NP-hard [RG96, Mnë88, RGZ95], making progress in this direction continually limited. In general, approaches trying to find combinatorial local conditions are bound to fail [Stu87]. Kalai's *squeezed spheres* [Kal88] and further constructions by Pfeifle and Ziegler [PZ04] and by Nevo, Santos, and Wilson [NSW16] show that asymptotically "most" simplicial spheres are not polytopal. Further, simplicial polytopes are even "rare" among *geodesic* simplicial sphere, i.e. simplicial spheres with a realization on the sphere where edges are geodesic arcs [NSW16]. Equivalently, $(d-1)$-geodesic simplicial spheres correspond to complete simplicial fans in $\mathbb{R}^d$; one obtains the geodesic arcs by intersecting the fan with the unit sphere.

Facing this situation, an approach to study Steinitz problem consists in finding flexible polytopal constructions or combinatorial obstructions to polytopality. One famous example of polytope with many constructions is that of the associahedron [Sta63]. Among the notable constructions are the fiber polytope realization [GKZ94, Chapter 7], the cluster algebra approach [CFZ02], and the simple combinatorial construction using planar binary trees [Lod04]. A myriad of variations and descriptions are possible [MHPS12]. The fact that it is related to so many areas of mathematics opens the door to approaches to Steinitz problem stemming from other areas of research.

Multi-triangulations offer a wide generalization of the simplicial sphere dual to the associahedron [PP12]. The simplicial complex whose faces are multi-triangulations is conjectured to be a polytopal sphere. This conjecture first appeared in writing in the Oberwolfach Book of Abstract of Jonsson in 2003 [Jon03]. Currently, the only known polytopal construction is for the 2-triangulations of the 8-gon [BP09, Ceb12, BCL15], and certain cases are known to be realizable as geodesic spheres [BCL15, Man18].

The complex of multi-triangulations turns out to be an example of subword complexes, a family of simplicial complexes related to the Bruhat order of Coxeter groups [KM04, KM05]. Introduced in the context of Gröbner geometry of Schubert varieties, these simplicial complexes offered a rich crossroads to a variety of research avenues: cluster algebras [CLS14], toric geometry [Esc16], root polytopes [EM18], Hopf algebras [BC17], among many others. Optimists may wish that a notion from these areas is key to determine the polytopality of subword complexes and therefore determine if multi-associahedra exist.

In the article [BCL15], Bergeon, Ceballos and the author lay down necessary conditions for the polytopality of subword complexes. The first step consisted in showing the existence of a certain sign function, which is then used to formulate certain sign conditions on minors of matrices to obtain *signature matrices*. Then, a combinatorial construction is given that provides signature matrices and it was possible to prove that they lead to complete simplicial fans for subword complexes of type A_3 and for certain cases in type A_4. In spite of these positive results, the reason *why* the construction works is still mysterious. The general knowledge on subword complexes is still scarce. Namely, certain combinatorial aspects of reduced words that lay at the center of the problem are still not explored in details. The geometric interpretation of these aspects is hence inexistant.

In this thesis, we lay down a framework to study the relevant combinatorial and geometric properties surrounding the *convexity* of subword complexes. We present the necessary notions from linear algebra, algebraic combinatorics and discrete geometry in Chapter 1. In Chapter 2, we present a *theory of sign functions* that unifies the usual sign function of permutations and the sign function presented in [BCL15]. In Chapter 3, we present a *theory of Model matrices* based on multilinear algebra using tensors to give a flexible factorization of "partial alternant matrices". Finally, in Chapter 4, we combine both theories and present the *Universality of parameter matrices*. The universality result shows that for each finite irreducible Coxeter group W, there exists a "partial oriented matroid" $\mathcal{P}$ such that

If $\mathcal{P}$ is realizable, then every subword complex of type W is realizable as an "oriented matroid polytope".

Chapter 1

Preliminaries

1.1 Multilinear algebra

Let $d \geq 1$ and V_d be a d-dimensional real vector space and denote its dual space by V_d^*. As usual, vectors in V_d are represented as column vectors, while linear functions V_d^* are represented as row vectors. We denote the transpose of vectors and of linear functions by $*^\top$. We use Einstein summation convention for tensors, described as follows. Given a basis $\{\mathbf{e}^1, \dots, \mathbf{e}^d\}$ of V_d and a basis $\{\mathbf{f}_1, \dots, \mathbf{f}_d\}$ of V_d^*, a $(d \times d)$-matrix $M^i{}_j = (m_{i,j}) = (m^i_j)$ represents the tensor

$$\mathcal{M}^i{}_j := \sum_{i=1}^{d} \sum_{j=1}^{d} m^i_j \mathbf{e}^i \otimes \mathbf{f}_j \in V_d \otimes V_d^*.$$

Given a tensor $\mathcal{T} \in (V_d)^a \times (V_d^*)^b$, a *row* of $\mathcal{T}$ is given by the restriction of $\mathcal{T}$ to a basis element of $(V_d)^a$, i.e. a row is indexed by a tuple $(i_1, \dots, i_a) \in [d]^a$. Similarly, *columns* of $\mathcal{T}$ are obtained by restricting $\mathcal{T}$ to basis elements of $(V_d^*)^b$, and are labeled by tuples in $[d]^b$. We view the product of $(d_1 \times d_2)$-matrices with $(d_2 \times d_3)$-matrices using tensors using the following linear map:

$$\begin{aligned} \left(V_{d_1} \otimes V_{d_2}^*\right) \times \left(V_{d_2} \otimes V_{d_3}^*\right) &\to V_{d_1} \otimes V_{d_3}^* \\ ((\mathbf{x} \otimes \mathbf{f}), (\mathbf{y} \otimes \mathbf{g})) &\mapsto \mathbf{f}(\mathbf{y}) \cdot (\mathbf{x} \otimes \mathbf{g}). \end{aligned} \tag{1.1}$$

More generally, given a tensor $\mathcal{T}^i{}_j \in V_{d_1} \otimes V_{d_2}^*$ and a tensor $\mathcal{U}^j{}_k \in V_{d_2} \otimes V_{d_3}^*$, we write the tensor contraction as

$$\mathcal{V}^i{}_k := \mathcal{T}^i{}_j \cdot \mathcal{U}^j{}_k,$$

using the rule given in Equation (1.1). Contraction of higher rank tensors is defined similarly, by matching the appropriate pairs of indices.

1.2 Vandermonde matrix and Binet–Cauchy Formula

The Vandermonde matrix of size d is

$$\mathrm{Vander}(d) := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_d \\ x_1^2 & x_2^2 & \cdots & x_d^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{d-1} & x_2^{d-1} & \cdots & x_d^{d-1} \end{pmatrix} = \sum_{i=1}^{d}\sum_{j=1}^{d} x_j^{i-1}\mathbf{e}^i \otimes \mathbf{f}_j = (x_j^{i-1})_{(i,j)\in[d]\times[d]},$$

and its determinant is

$$\det \mathrm{Vander}(d) = \prod_{1\le i<j\le d}(x_j - x_i) = \sum_{\pi\in\mathfrak{S}_d} \mathrm{sign}(\pi) x_1^{\pi(1)-1}\cdots x_d^{\pi(d)-1}. \tag{1.2}$$

The Vandermonde matrix also is obtained as the product of two rectangular matrices as follows. Let $W_d := \mathcal{W}^i{}_{j,k} := \bigoplus_{i=1}^d \mathrm{Id}_d$ be the augmentation of d identity matrices by concatenating them columnwise. This matrix can be rewritten as a tensor in $V_d \otimes V_d^* \otimes V_d^*$ as $\sum_{i=1}^d \left(\mathbf{e}^i \otimes \left(\sum_{j=1}^d \mathbf{f}_j\right) \otimes \mathbf{f}_i\right)$. Further, we define the tensor $\mathcal{X}^{k,j}{}_l \in V_d \otimes V_d \otimes V_d^*$ as

$$\mathcal{X}^{k,j}{}_l := \sum_{j=1}^{d}\left(\sum_{k=1}^{d} x_j^{k-1}\right)\mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{f}_j.$$

By flattening it, the tensor $\mathcal{X}^{k,j}{}_l$ can be written as a matrix as

$$X_d = \begin{pmatrix} \begin{matrix}1\\x_1\\x_1^2\\\vdots\\x_1^{d-1}\end{matrix} & 0 & \cdots & 0 \\ 0 & \begin{matrix}1\\x_2\\x_2^2\\\vdots\\x_2^{d-1}\end{matrix} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & \begin{matrix}1\\x_N\\x_N^2\\\vdots\\x_N^{d-1}\end{matrix} \end{pmatrix}.$$

From the definitions of W_d and X_d, and the properties of product of tensors, we get

$$\mathrm{Vander}(d) = W_d X_d = \mathcal{W}^i{}_{j,k} \cdot \mathcal{X}^{k,j}{}_l.$$

Indeed, the data is split in order for the product to give back the Vandermonde matrix.

Later on, we look at variations of the Vandermonde matrices given in this product form. In order to get hold of the signs of determinants of these variations, we use the Binet–Cauchy formula. If M is a matrix, we denote by $[M]_Z$ the submatrix of M formed by the rows (or columns) indexed by the set Z.

Theorem 1.1 (Binet–Cauchy formula, see [RS13, Section 10.5, p.377] or [AZ14, Chapter 31])**.** *If P is an $(r \times s)$-matrix, Q an $(s \times r)$-matrix, and $r \leq s$, then*

$$\det(PQ) = \sum_{Z \in \binom{[s]}{r}} (\det[P]_Z)(\det[Q]_Z), \tag{1.3}$$

where $[P]_Z$ is the $(r \times r)$-submatrix of P with column-set Z, and $[Q]_Z$ the $(r \times r)$-submatrix of Q with the corresponding row-set Z.

In our case, since W_d and X_d enjoy a simple structure, it is easy to reobtain the formula for the Vandermonde determinant:

$$\det \mathrm{Vander}(d) = \det(W_d X_d) = \sum_{Z \in \binom{[s]}{r}} (\det[W_d]_Z)(\det[X_d]_Z).$$

- The matrices $[W_d]_Z$ either have two equal columns or are permutation matrices. Therefore, the determinants $\det[W_d]_Z$ are 0, 1, or -1. They are zero if at least two columns are equal, and equal to the sign of the permutation associated to the permutation matrix otherwise.
- The determinant of $[X_d]_Z$ is zero if two rows use the same variable x_i. Otherwise, the determinant is a monomial with variables in $\{x_1, \dots, x_d\}$.

Combining the two conditions for the determinants to be non-zero, we get back the fact that the determinant of the Vandermonde matrix is the sum of the signed monomials in exactly $d-1$ variables which have all distinct powers in $\{1, \dots, d-1\}$ as in Equation (1.2).

1.3 Partial Schur functions

It is worth noting that altering W_d and X_d defined above in particular ways lead to generalizations of the Vandermonde matrix. For example, Schur functions can be defined this way.

Definition 1.2 (Schur functions using Vandermonde matrices [Mac15, Section 1.3], [Sta99, Chapter 7.15], and [Sag01, Section 4.6])**.** Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ be a partition with

$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$, and J be an ordered set of indices of cardinality d. The *Schur function* $ш_{\lambda,J}$ in the variables $\{x_j : j \in J\}$ is the quotient

$$ш_{\lambda,J} := \frac{\det(x_j^{i-1+\lambda_{d-i+1}})_{(i,j)\in[d]\times J}}{\det \mathrm{Vander}_J(d)},$$

where $\mathrm{Vander}_J(d)$ is the Vandermonde matrix $\mathrm{Vander}(d)$ with variables indexed by J. When $J = [n]$ for some $n \geq 1$ we omit the subscript J and simply write $ш_\lambda$.

Example 1.3.

1. Let $d = 3$ and consider the partition $(4, 1, 0)$. The Schur function $ш_{(4,1,0)}$ is the determinant of the matrix below divided by the Vandermonde determinant $\det \mathrm{Vander}(3)$:

$$\begin{aligned} ш_{(4,1,0)} &= \frac{1}{\det \mathrm{Vander}(3)} \begin{vmatrix} x_1^{0+0} & x_2^{0+0} & x_3^{0+0} \\ x_1^{1+1} & x_2^{1+1} & x_3^{1+1} \\ x_1^{2+4} & x_2^{2+4} & x_3^{2+4} \end{vmatrix} \\ &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2)(x_1 + x_3)(x_2 + x_3). \end{aligned}$$

   The corresponding matrices $W_{(4,1,0)}$ and $X_{(4,1,0)}$—whose product gives the above matrix—are

$$W_{(4,1,0)} = \bigoplus_{i=1}^{3} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \text{ and } X_{(4,1,0)} = \textstyle\sum_{j=1}^{3} \left(\sum_{k=1}^{3} x_j^{k-1}\right) \mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{f}_j.$$

2. If we set each part in the partition λ to 0, we get back the Vandermonde matrix, which corresponds to the degree 0 symmetric polynomial 1.

The following definition becomes natural. We have not encountered an occurrence in this specific form in the litterature, nevertheless it has most likely been considered before.

Definition 1.4 (Partial Schur functions)**.** Let $m \geq 1$, $P = (p_i)_{i=1}^n$ be an ordered set partition of $[m]$ where parts may be empty, and $\Lambda = (\lambda_i)_{i=1}^n$ be a sequence of partitions such that the number of parts of λ_i is the cardinality of p_i. The *partial Schur function* with respect to P and Λ is

$$Ш_{\Lambda,P} := \prod_{i=1}^{n} ш_{\lambda_i,p_i}.$$

This function is symmetric with respect to the action of $\prod_{i=1}^n \mathfrak{S}_{p_i}$ as a subgroup of $\mathfrak{S}_m$.

Example 1.5. Let $m = 4$, $P = (\{1,3\},\{2,4\})$, $\Lambda_1 = ((0,0),(1,0))$, $\Lambda_2 = ((1,0),(0,0))$, and $\Lambda_3 = ((3,1),(2,0))$. The partial Schur functions with respect to P and the sequences of partitions are

$$\text{Ш}_{\Lambda_1,P} = \text{ш}_{(0,0),\{1,3\}} \cdot \text{ш}_{(1,0),\{2,4\}} = 1 \cdot (x_2 + x_4),$$
$$\text{Ш}_{\Lambda_2,P} = \text{ш}_{(1,0),\{1,3\}} \cdot \text{ш}_{(0,0),\{2,4\}} = (x_1 + x_3) \cdot 1,$$
$$\text{Ш}_{\Lambda_3,P} = \text{ш}_{(3,1),\{1,3\}} \cdot \text{ш}_{(2,0),\{2,4\}} = x_1x_3(x_1^2 + x_1x_3 + x_3^2) \cdot (x_2^2 + x_2x_4 + x_4^2).$$

1.4 Combinatorics on words

For basic notions on combinatorics on words and monoids, we refer to the books [Lot97, Chapter 1] and [Die90, Chapter 1]. Let $S = \{s_1, \dots, s_n\}$ be a finite alphabet of *letters* or *generators* equipped with the lexicographic order $s_1 < s_2 < \cdots < s_n$. Let $S^* := \bigoplus_{i\in\mathbb{N}} S^i$ be the free monoid generated by elements in S by concatenation, and call its elements *words* or *expressions* and denote by e the *identity element* or *empty word*. Given a word $w \in S^m$, it is usually written $w = w_1 \cdots w_m$, where w_i denotes its i-th letter, and the *length* of w is m. A word w of length m is equivalently defined as a function $w : [m] \to S$; then w_i represents the image $w(i) \in S$. When there exists two words $u, v \in S^*$ such that $w = ufv$, the word $f \in S^*$ is called a *factor* of w. An *occurrence* of a factor f of length k in a word $w = w_1 \cdots w_m$ is a set of positions $\{i, \dots, i+(k-1)\}$ with $i \in [m-k+1]$ such that $w = w_1 \cdots w_{i-1} f w_{i+k} \cdots w_m$, i.e. $f = w_i \cdots w_{i+(k-1)}$. In particular, a factor of length 1 gives an occurrence of some letter $s \in S$. We denote by $\{w\}_i$ the *set of occurrences of the letter* $s_i \in S$ in w. Given a word $w \in S^m$, we hence associate the ordered set partition $\Omega_w = (\{w\}_1, \dots, \{w\}_n)$ of $[m]$ to w. Further we denote the cardinality of the set $\{w\}_i$ by $|w|_i := \#\{w\}_i$. Let $k \leq m$ and $w : [m] \to S$ be a word of length m, a *subword* v of length k of w is a word obtained by the composition $v := w \circ u$, for some increasing function $u : [k] \to [m]$. A subword is therefore a concatenation of a sequence of factors. By extension, we define an occurrence of a subword as the union of the occurrences of the factors whose concatenation give the subword. Consequently, the same word $v \in S^*$ may give rise to several subword occurrences in a longer word w, which are obtained by distinct sequences of factors of w. Given a subword $v = w \circ u$ of length k of a word w, its *complement word* $w \setminus v$ is the subword of w obtained by $w \circ u'$ where $u' : [m-k] \to [m]$ is the increasing function such that $u'(i)$ is the i-th smallest element in $[m] \setminus u([k])$. Given a word $w = w_1 \cdots w_m$, its *reverse word* $\mathrm{rev}(w)$ is the word $w_m \cdots w_1$.

Define the monoid morphism α from S^* to the free commutative monoid $\mathbb{N}^S$ by its image on the set S as follows

$$\alpha : S^* \to \mathbb{N}^S$$
$$s \mapsto \mathbb{1}_s(t) := \begin{cases} 1, & \text{if } t = s, \\ 0, & \text{else.} \end{cases}$$

Given an ordering of the alphabet of generators S and a word $w \in S^*$, the image

$$\alpha(w) = (|w|_1, \ldots, |w|_n)$$

is a weak composition (i.e. an ordered partition where zeros can appear) called the *abelian vector* of w. The morphism α records the number of occurrences of letters in words, and can be thought as an *abelianization* of the monoid S^*. Consequently, the sum of the entries in $\alpha(w)$ is the *length* of the word w. To lighten the notation, we shall write α_w for the abelian vector $\alpha(w)$.

1.5 Coxeter groups

For basic notions on Coxeter groups, we refer to the books [Hum90] and [BB05]. Let $(W, S = \{s_1, \ldots, s_n\})$ be a *finite irreducible Coxeter system* with Coxeter matrix $M = (m_{i,j})_{i,j\in[n]}$. We denote by $R = \{(s_is_j)^{m_{i,j}} : s_i, s_j \in S\}$ the associated set of *braid relations*. Some choices of lexicographic orders are more natural; hence, when influencial, we specify the ordering by indicating M and R explicitly. The set of braid relations R generates a free submonoid R^* of S^* and the quotient monoid S^*/R^* consisting of left-cosets of R^* in S^* has a left- and right-inverse and thus forms a group which is isomorphic to W [BB05, p.3]. From this standpoint, the elements of a Coxeter group W are equivalence classes of expressions and the representative expressions with shortest length are called *reduced*. Bearing this in mind, we henceforth represent an element w of the monoid S^* and of the group W *both* using concatenation of letters. Whenever a distinction is pertinent we emphasize if the word or its equivalence class is meant by writing $w \in S^*$, or $w \in W$, respectively.

Throughout this text, we adopt the following notations. The function $\ell : W \to \mathbb{N}$ denotes the *length function* sending an element $w \in W$ to the length of its reduced expressions, the symbol $w_\circ$ denotes the *longest element* of W, and $N := \ell(w_\circ)$. Given an element $w \in W$, we denote the *set of reduced expressions* of w by $\mathcal{R}(w)$ which is a finite subset of S^*.

Problem 1.6. Let $w \in W$.

1. Characterize the set of abelian vectors $\{\alpha_v : v \in \mathcal{R}(w)\}$.

2. Give precise bounds on the number

$$\nu(w) := \max\big\{\max\{\alpha_v(s) : s \in S\} : v \in \mathcal{R}(w)\big\},$$

which is *the maximum number of occurrences of a letter in any reduced expression of* w.

3. Describe the vector

$$\mu(w) := \big(\min\{\alpha_v(s) : v \in \mathcal{R}(w)\}\big)_{s\in S},$$

which gives *the minimum number of occurrences of each letter in any reduced expression of* w.

The abelian vectors for the reduced words of the longest element of types A, B, D, for small rank and H are gathered in Tables A.1 to A.4 of Appendix A.

In the symmetric group case $\mathfrak{S}_{n+1} = A_n$ and taking $w = w_\circ$, Problem 1.6(2) is related to the *k-set problem* via duality between points and pseudolines on the plane, see [Mat02, Chapter 11], [GOT18, Chapter 5], and [PP12, Section 3.1] for a contextual explanation. Currently, the best lower bound we know for $\nu(w_\circ)$ in this case is $ne^{\Omega(\sqrt{\log n/2})}$ [Tó1], and the best upper bound is $O(n^{4/3})$ [Dey98]. For a nice recent book on related topics, see [Epp18, Section 3.5]. Further, in type A, Problem 1.6(3) is answered by

$$\mu(w_\circ) = \begin{cases} (1, 2, \dots, \lceil \frac{n}{2} \rceil, \dots, 2, 1) & \text{if } n \text{ is odd,} \\ (1, 2, \dots, \frac{n}{2}, \frac{n}{2}, \dots, 2, 1) & \text{if } n \text{ is even.} \end{cases}$$

Indeed, this can be proved directly using the minimal number of inversions used at each position necessary to obtain the reverse permutation $[n+1, n, \dots, 2, 1]$.

We denote $\nu := \nu(w_\circ)$ and refer to this value as the *höchstfrequenz* of the group W. As Section 3.5 reveals, the höchstfrequenz of the group W is an important parameter for the genericity of vector configurations geometrically realizing subword complexes.

1.6 Graphs on reduced expressions

Given two words $u, v \in S^*$ representing an element $w \in W$, they are related by a sequence of insertion or deletion of factors contained in R^*. Based on this fact, one defines *braid moves* in a word by replacing a factor $s_is_js_i\dots$ of length $m_{i,j}$ by a factor $s_js_is_j\dots$ of length $m_{i,j}$, where $i \neq j$. It is a well-known property of Coxeter groups that reduced expressions in $\mathcal{R}(w)$ are connected via finite sequences of braid moves, in particular that no reductions $s_i^2 = e$ are necessary [Mat64, Tit69] (see [BB05, Theorem 3.3] for a textbook version). The graph $\mathcal{G}(w)$ whose vertices are reduced expressions of w and edges represent braid moves between expressions is hence connected. The diameter of $\mathcal{G}(w)$ has been studied in [AD10, RR13] and other closely related enumerative properties in [Ten17]. Certain minors of $\mathcal{G}(w)$ are of particular interest here. They are represented in Figure 1.1 using a Hasse diagram of the graph minor containment ordering. For example, $\mathcal{G}^{\text{comm}}(w)$ is obtained from $\mathcal{G}(w)$ by contracting edges of $\mathcal{G}(w)$ representing braid moves of length 2. In these minors, the resulting multiple edges are fusionned into a unique edge.

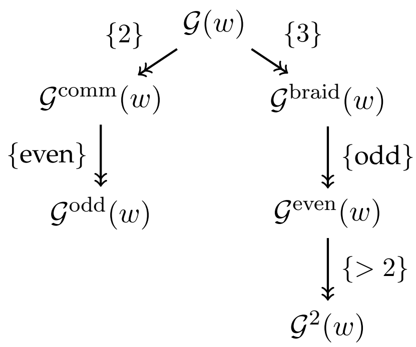


FIGURE 1.1: Minors of $\mathcal{G}(w)$ obtained by contracting the edges indicated by the arrow labels.

The symmetric group case has received more scrutiny: the vertices of $\mathcal{G}^{\text{braid}}$ are called *braid classes* and the vertices of $\mathcal{G}^{\text{comm}}$ are called *commutation classes*. Bounds on the number of vertices of $\mathcal{G}(w)$ in terms of the number of vertices of $\mathcal{G}^{\text{braid}}(w)$ and $\mathcal{G}^{\text{comm}}(w)$ have been obtained in [FMPT18]. An asymptotic study of expected number of commutations in reduced words was done in type A [Rei05] and type B [Ten15] and it is possible to determine the number of elements that have a unique reduced word [Har17]. Further, it is possible to define a metric on this graph which relates naturally to balanced tableaux [Ass19].

Remark 1.7. In the *simply laced* cases (types A_n, D_n, E_6, E_7, or E_8), there are only two types of braid moves, therefore $\mathcal{G}^{\text{braid}} = \mathcal{G}^{\text{even}}$ and $\mathcal{G}^{\text{comm}} = \mathcal{G}^{\text{odd}}$.

Remark 1.8. In the symmetric group case A_n, the graph $\mathcal{G}^{\text{odd}}(w_\circ)$ is the underlying graph of the Hasse diagram of the higher Bruhat order $B(n+1,2)$, see [MS89, Zie93, FW00]. It is also studied using rhombic tilings [Eln97], and is used to study intersections of Schubert cells [SSV97]. In other finite types, the relation between $\mathcal{G}^{\text{odd}}(w_\circ)$ and potential higher Bruhat orders remains unclear.

Example 1.9 (Dihedral Group $I_2(m)$)**.** Let $W = I_2(m)$, with $m \geq 2$. The graph $\mathcal{G}(w_\circ)$ has two vertices and one edge between them, see Figure 1.2.

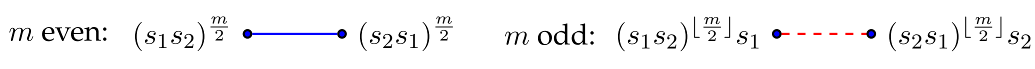


FIGURE 1.2: The graphs $\mathcal{G}(w_\circ)$ for the dihedral group $I_2(m)$

Remark 1.10. To lighten figures, we write reduced words using the indices of the letters in S and write classes of reduced words with more than one element as $12\{13\}21 := \{121321, 123121\}$.

Example 1.11 (Symmetric group $\mathfrak{S}_4 = A_3$)**.** Let $W = A_3$ and $S = \{s_1, s_2, s_3\}$, such that $(s_1s_2)^3 = (s_1s_3)^2 = (s_2s_3)^3 = e$, see Figure 1.3.

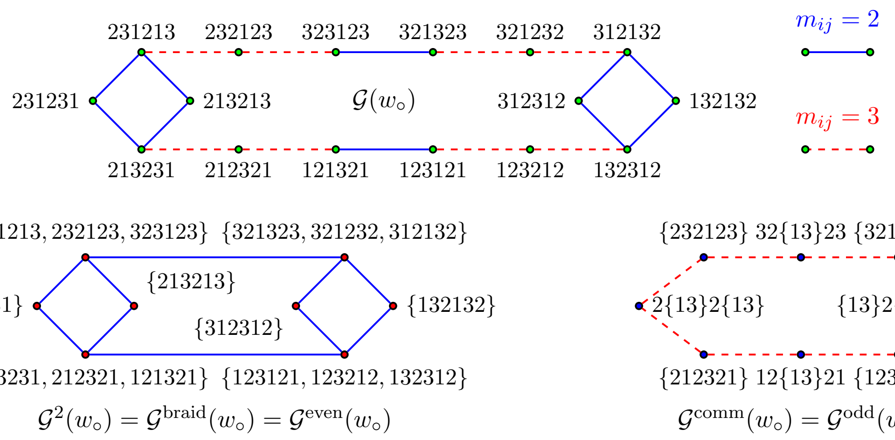


FIGURE 1.3: The graphs $\mathcal{G}(w_\circ)$, $\mathcal{G}^2(w_\circ) = \mathcal{G}^{\text{braid}}(w_\circ) = \mathcal{G}^{\text{even}}(w_\circ)$, and $\mathcal{G}^{\text{comm}}(w_\circ) = \mathcal{G}^{\text{odd}}(w_\circ)$ for the symmetric group $\mathfrak{S}_4 = A_3$

Example 1.12 (The hyperoctahedral group B_3)**.** Let $W = B_3$ and $S = \{s_1, s_2, s_3\}$, such that $(s_1s_2)^4 = (s_1s_3)^2 = (s_2s_3)^3 = e$. The commutation classes of $w_\circ$ are illustrated in Figure 1.4. Since every odd-length braid moves has length 3, we get $\mathcal{G}^{\text{braid}}(w_\circ) = \mathcal{G}^{\text{even}}(w_\circ)$. There are 42 reduced expressions for $w_\circ$. The impatient reader is invited to see Figure 2.5 on page 24 for an illustration.

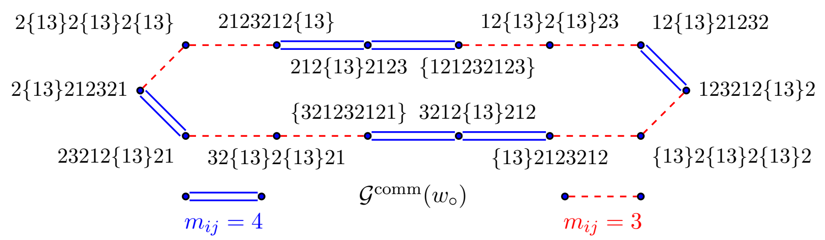


FIGURE 1.4: The commutation classes of the longest word of the group B_3

Example 1.13 (Icosahedral group H_3)**.** Let $W = H_3$ and $S = \{s_1, s_2, s_3\}$, such that $(s_1s_2)^5 = (s_1s_3)^2 = (s_2s_3)^3 = e$. There are 44 commutation classes in $\mathcal{G}^{comm}(w_\circ)$; see Figure 1.5 for an illustration.

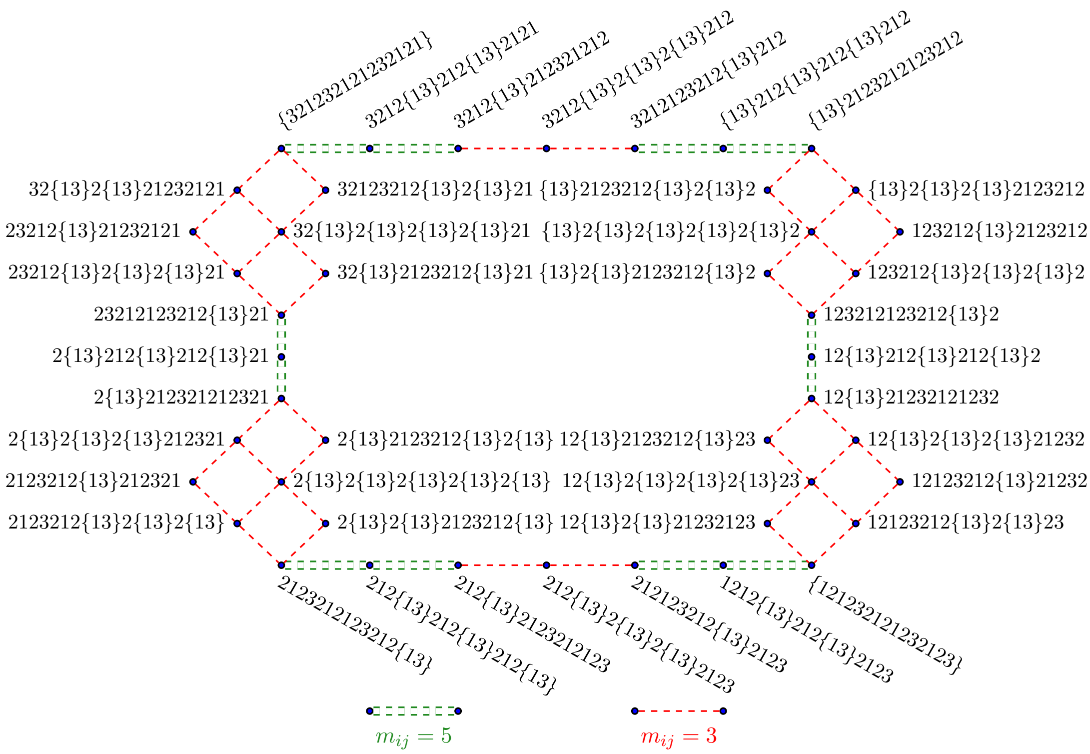


FIGURE 1.5: The commutation classes of the longest word of the group H_3

Of particular importance for us is the fact that the graphs $\mathcal{G}(w_\circ)$, $\mathcal{G}^{\text{even}}(w)$, and $\mathcal{G}^{\text{odd}}(w)$ are bipartite graphs. The following theorem was proved for finite Coxeter group by Bergeron, Ceballos, and the author using a geometric argument in [BCL15] and generalized to infinite Coxeter groups and extended to a finer description by Grinberg and Postnikov in [GP17] using only conjugations instead of automorphisms.

Theorem 1.14 ([BCL15, Theorem 3.1],[GP17, Theorem 2.0.3])**.** *Let W be a Coxeter group and $w \in W$. For any set $\mathcal{E}$ of braid moves closed under automorphism of W, the minor of $\mathcal{G}(w)$ obtained by contracting edges not contained in $\mathcal{E}$ is a bipartite graph. In particular, $\mathcal{G}(w)$, $\mathcal{G}^{\text{even}}(w)$, and $\mathcal{G}^{\text{odd}}(w)$ are bipartite graphs.*

1.7 Subword complexes of Coxeter groups

For each finite Coxeter group (W, S), Knutson and Miller introduced a family of simplicial complexes called *subword complexes* which reveals the Bruhat order within words in S^* [KM04, KM05]. Subword complexes form one of the tools they used to connect the algebra and combinatorics of Schubert polynomials to the geometry of Schubert varieties. In [DHM19], Davis, Hersh and Miller explain how to use subword complexes to study fibers of maps parametrizing totally nonnegative matrices. We present here an adaptation of the original definition, using results from [CLS14, Section 3] and [BCL15, Section 2], that particularly suits our purposes.

Given a word $p \in S^*$, we can order the occurrences of all its subwords by set-inclusion to obtain a Boolean lattice $(2^{[\ell(p)]}, \subseteq)$ and subword complexes are certain lower ideals of such Boolean lattices determined using reduced words. More precisely:

Definition 1.15 (Subword complexes, see [KM04, Definition 2.1])**.** Let (W, S) be a finite Coxeter group, $w_\circ$ be its longest element and $p \in S^m$. The *subword complex* $\Delta_W(p)$ is the simplicial complex on the set $[m]$ whose facets are complements of occurrences of reduced words for $w_\circ$ in the word p.

Subword complexes possess a particularly nice combinatorial and topological structure: they are vertex-decomposable and homeomorphic to sphere or balls [KM04, Theorem 2.5 and 3.7]. Knutson and Miller originally asked whether they can be realized as the boundary of a convex polytope [KM04, Question 6.4]. So far, the realized subword complexes include famous polytopes: simplices, even-dimensional cyclic polytopes, polar dual of generalized associahedra, see [CLS14, Section 6] for a survey on the related conjectures and the references therein. The only "non-classical" instance which is realized is a 6-dimensional polytope with 12 vertices realizing the simplicial complex of 2-triangulations of the 8-gon, see [BP09, Ceb12, BCL15], which is a type A_3 subword complex. Further, fan realizations have been provided for type A_3 and two cases in A_4 [BCL15] and for 2-triangulations (type A) with rank $5, 6, 7$ and 8 [Man18].

Due to their combinatorial provenance, we can attribute a combinatorial type to each facet.

Definition 1.16 (Combinatorial type and abelian vector of facet)**.** Let $\Delta_W(p)$ be a subword complex. The *combinatorial type* of a facet f of $\Delta_W(p)$ is the complement subword $p \setminus f$ of f in p. Two facets are *combinatorially equivalent* when their combinatorial types are the same. The *abelian vector* of a facet f is the abelian vector $\alpha_{p\setminus f}$ of its combinatorial type.

Example 1.17. Let $W = A_2$ and $p = p_1p_2p_3p_4p_5 = s_1s_2s_1s_2s_2$. The subword complex $\Delta_{A_2}(p)$ has two combinatorial types of facets. The facets $\{1, 4\}$ and $\{1, 5\}$ have type $s_2s_1s_2$ and the facet $\{4, 5\}$ has type $s_1s_2s_1$. The abelian vector of $\{1, 4\}$ and $\{1, 5\}$ is $(1, 2)$ and the abelian vector of $\{4, 5\}$ is $(2, 1)$. Notice that the letters p_2 and p_3 are contained in every occurrences of reduced word for $w_\circ$ in p, so 2 and 3 are non-vertices of the subword complex $\Delta_{A_2}(p)$, see Figure 1.6:

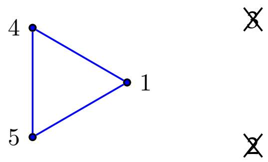


FIGURE 1.6: A small subword complex with two non-vertices

1.8 Fans and Gale duality

For notions related to vector configurations, simplicial fans, and Gale duality, we rely on the reference books [Grü03, Section 5.4], [Zie95, Chapter 6] and [DLRS10, Section 2.5, 4.1, and 5.4]. We assume the reader's familiarity with the elementary objects from convex geometry apart from the following essential notions, that we recall below.

Definition 1.18 (Vector configuration, see [DLRS10, Definition 2.5.1])**.** Given a finite totally ordered label set J of cardinality m, a *vector configuration* in $\mathbb{R}^d$ is a finite set $\mathbf{A} := \{\mathbf{p}_j \mid j \in J\}$ of labeled vectors $\mathbf{p}_j \in \mathbb{R}^d$. The *rank* of $\mathbf{A}$ is its rank as a set of vectors.

We assume vector configurations to have maximal rank, i.e. $r = d$, and we write them as a matrices in $\mathbb{R}^{d\times m}$. We denote the non-negative span of a set of vectors given by a subset $C \subseteq J$ by $\mathrm{cone}_{\mathbf{A}}(C)$. A vector configuration is *acyclic* if there is a linear function that is positive in all the elements of the configuration. It is *totally cyclic* if $\mathrm{cone}_{\mathbf{A}}(J)$ is equal to the vector space spanned by $\mathbf{A}$.

Definition 1.19 (Complete simplicial fan, see e.g. [Zie95, Section 7.1])**.** Let $\mathbf{A}$ be a vector configuration in $\mathbb{R}^d$ consisting of m labeled vectors. A *fan* supported by $\mathbf{A}$ is a family $\mathcal{F} = \{K_1, K_2, \dots, K_k\}$ of nonempty polyhedral cones generated by vectors in $\mathbf{A}$ such that:

- Every nonempty face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.
- The intersection of any two cones in $\mathcal{F}$ is a face of both.

A fan $\mathcal{F}$ is *simplicial* if every $K \in \mathcal{F}$ is a simplicial cone, and it is *complete* if the union $K_1 \cup \cdots \cup K_k$ is $\mathbb{R}^d$. The 1-dimensional cones of a fan are called *rays*, while the $(d-1)$-dimensional cones are called *ridges*. The spherical simplicial complex on $[m]$ whose faces are index sets of cones in a complete simplicial fan $\mathcal{F}$ is denoted by $\Delta_{\mathcal{F}}$.

To each vector configuration corresponds dual objects called *Gale transforms*.

Definition 1.20 (Gale transform [DLRS10, Definition 4.1.35] [Zie95, Section 6.4])**.** Let $\mathbf{A} \in \mathbb{R}^{d\times m}$ be a rank d vector configuration with m elements. A *Gale transform* $\mathbf{B} \in$

$\mathbb{R}^{(m-d)\times m}$ of $\mathbf{A}$ is vector configuration of rank $m-r$ whose rowspan equals the right-kernel of $\mathbf{A}$. We denote by $\mathrm{Gale}(\mathbf{A})$ the set of all Gale transforms of $\mathbf{A}$.

Let $\mathbf{A} \in \mathbb{R}^{d\times m}$ be a vector configuration supporting a complete simplicial fan $\mathcal{F}$. Gale duality implies that $\mathbf{A}$ is totally cyclic and that its Gale duals are acyclic. Furthermore, full-dimensional cones in $\mathcal{F}$ are complements of full-dimensional subconfigurations of any $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. Given such a full-dimensional cone in $\mathcal{F}$ spanned by a set of vectors $C \subseteq J$, its *dual simplex* C^* is $J \setminus C$. The cone generated by C^* in $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$ is called the *dual simplicial cone* of C and is denoted by $\mathrm{cone}_{\mathbf{B}}(C^*)$.

Following [DLRS10, Section 9.5], to obtain a realization of a simplicial sphere Δ as the boundary of a convex simplicial polytope one possibility is to proceed in two steps:

(T) Obtain a vector configuration $\mathbf{A}$ supporting a complete simplicial fan $\mathcal{F}_\Delta$ whose cone lattice is isomorphic to the face lattice of the simplicial sphere Δ, (Triangulation) and

(R) prove that the underlying triangulation of the space is regular. Equivalently, find one point on each ray, so that taking the convex hull of these points yields a simplicial polytope whose boundary complex is isomorphic to Δ. (Regularity)

The first step relies heavily on the combinatorial structure of the sphere, whereas the success of the second step relies heavily on the geometry of the obtained simplicial fan. We recall the following lemma that gives conditions to form a complete simplicial fan and fulfill step (T).

Lemma 1.21 (see [BCL15, Lemma 3])**.** *Let Δ be a simplicial complex on $J = [m]$ homeomorphic to a sphere of dimension $d-1$. A totally cyclic vector configuration $\mathbf{A} \in \mathbb{R}^{d\times m}$ supports a complete simplicial fan realization of Δ if and only if the following conditions on $\mathbf{A}$ and a Gale transform $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$ are satisfied.*

(B) *Dual simplicial cones $\mathrm{cone}_{\mathbf{B}}(C^*)$ are independant in $\mathbb{R}^{m-r}$. (Basis)*

(F) *If I and J are two facets of Δ intersecting along a ridge, then the interior of the corresponding dual simplicial cones intersect. (Flip)*

(I) *There is a cone in $\mathbf{A}$ whose interior is not intersected by any other cones. (Injectivity)*

Now, assume that we have completed step (T) with a vector configuration $\mathbf{A}$ and a Gale dual $\mathbf{B}$. The next step is to prove that the induced triangulation of $\mathbb{R}^d$ by $\mathcal{F}_\Delta$ is regular. Having a fan $\mathcal{F}_\Delta$ already guarantee that the dual simplicial cones of adjacent facets intersect in their interior. To have a regular triangulation, the common intersection of *all* dual simplicial cones should have non-empty interior:

Proposition 1.22 (see [DLRS10, Theorem 5.4.5 and 5.4.7]). *Let $\mathcal{F}$ be a complete simplicial fan in $\mathbb{R}^d$ supported by a configuration $\mathbf{A}$ of m vectors and $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. The triangulation of $\mathbb{R}^d$ induced by $\mathcal{F}$ is regular if and only if the intersection of all dual simplicial cones in $\mathbf{B}$ is a full-dimensional cone in $\mathbb{R}^{m-d}$.*

Chapter 2

Sign functions of words

In the symmetric group $\mathfrak{S}_{n+1} \cong A_n$, the sign of a permutation is defined using the parity of its number of pairwise inversions; even permutations being "$+$" and odd permutations being "$-$". This definition shows directly that the Hasse diagram of the weak order of the symmetric group (and more generally of Coxeter groups) is bipartite. In this section, we present an extension of this notion on subsets of words of S^* defined using sign functions.

Definition 2.1 (Sign functions on words)**.** Let $M \subseteq S^*$. A function from M to the multiplicative group $\mathbb{Z}_2$ is a *sign function* on M.

Lemma 2.2. *Let $M \subseteq S^*$. The set of sign functions on M with the binary operation*

$$\begin{aligned} {\mathbb{Z}_2}^M \times {\mathbb{Z}_2}^M &\to {\mathbb{Z}_2}^M \\ (\phi, \psi) \mapsto \phi\psi(m) &:= \begin{cases} 1, & \text{if } \phi(m) = \psi(m), \\ -1, & \text{else,} \end{cases} \end{aligned}$$

form a group: the group of sign functions on M.

2.1 Sign function on minors of $\mathcal{G}(w)$

Given an element $w \in W$, Theorem 1.14 gives a way to define a close cousin of signs of permutations where now the ground set is $\mathcal{R}(w)$ and even and odd expressions are then defined in various ways, as Theorem 1.14 permits. Two cases are more relevant:

Sign function F_{odd} Changing the sign when an odd-length braid move is done, and leaving the sign unchanged when an even-length braid move is done. Since the minor $\mathcal{G}^{\text{odd}}(w)$ is bipartite, we can assign one part to have positive sign and the other to have negative sign. This way, along every edge representing a braid move of odd length, the sign changes and along the contracted edges, the sign remains unchanged.

Sign function F_{even} Changing the sign when an even-length braid move is done, and leaving the sign unchanged when an odd-length braid move is done. Similarly, since $\mathcal{G}^{\text{even}}(w)$ is bipartite we can assign positive and negative signs to the reduced expressions.

As illustrated in Figure 1.1, since $\mathcal{G}^{\text{odd}}$ is a minor of $\mathcal{G}^{\text{comm}}$ and $\mathcal{G}^{\text{even}}$ is a minor of $\mathcal{G}^{\text{braid}}$, the sign functions F_{even} and F_{odd} are class functions on braid classes and commutation classes, respectively. These sign functions are unique up to a global multiplication by "-1".

Example 2.3 (F_{even} and F_{odd} sign functions on braid and commutation classes in type A_3)**.** We can give "$+$" and "$-$" signs to the vertices of $\mathcal{G}^{\text{even}}(w_\circ)$ and $\mathcal{G}^{\text{odd}}(w_\circ)$ and by Remark 1.7, we get functions on braid and commutation classes that change along edges in $\mathcal{G}^{\text{even}}(w_\circ)$ and $\mathcal{G}^{\text{odd}}(w_\circ)$, see Figure 2.1.

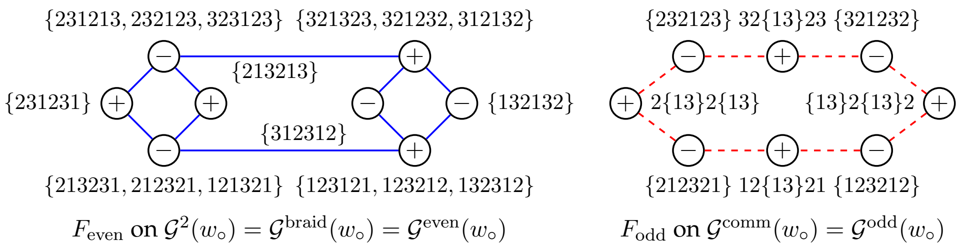


FIGURE 2.1: F_{even} and F_{odd} sign functions on braid and commutation classes of $w_\circ$ in type A_3

Example 2.4 (F_{even} sign function on braid classes of type B_3)**.** We can give "$+$" and "$-$" signs to the vertices of $\mathcal{G}^{\text{even}}(w_\circ)$ and get a class function on braid classes, illustrated in Figure 2.2.

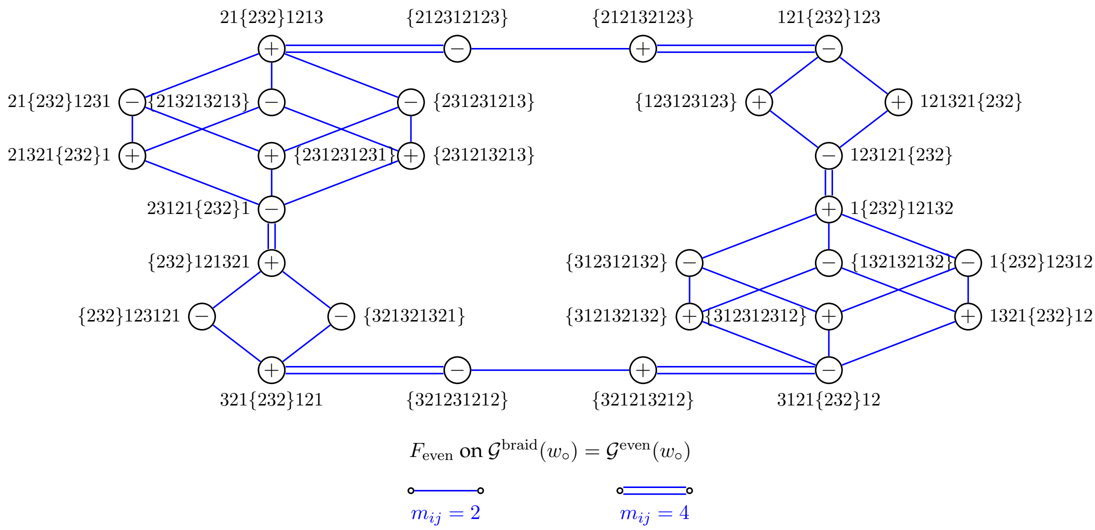


FIGURE 2.2: F_{even} sign function on braid classes for $w_\circ$ in type B_3

The T-signature of reduced expressions is the sign function where even-length braid moves change the sign, i.e., the sign function F_{even} above on the graph $\mathcal{G}^{\text{even}}(w_\circ)$:

Definition 2.5 (T-sign function, [BCL15, Definition 3.5])**.** Let $w_\circ$ be the longest element of W. The *T-sign function* is defined as

$$\begin{aligned}\tau : \mathcal{R}(w_\circ) &\to \{+1, -1\}\\ w &\mapsto \tau(w),\end{aligned}$$

whereby if w and w' are two reduced expressions of $w_\circ$ related by a braid move of length $m_{i,j}$, then $\tau(w) = (-1)^{m_{i,j}-1}\tau(w')$.

Remark 2.6. The letter T is used to hint at a usual notation for the set of reflections of W, see [BCL15, Remark 3.7] for an equivalent formulation in type A using T. The T-sign function is well-defined by Theorem 1.14 and unique up to a global multiplication by "-1". A certain choice is well-suited for our purpose and is specified in Section 2.4.

Example 2.7. In Figures 2.1 and 2.2, we can determine the sign of all reduced expressions using the bipartiteness of $\mathcal{G}^{\text{even}}(w_\circ)$, namely every element in the same braid class (of length 3 in those cases) receive the same sign.

The T-sign function is central to a combinatorial and geometric condition that led to the construction of complete simplicial fans for subword complexes in [BCL15]. It helped to deliver complete fans with cone lattices corresponding to subword complexes of type A_3 and some of type A_4. This condition is based on signature matrices:

Definition 2.8 (Coxeter signature matrices [BCL15, Definition 9])**.** Let (W, S) be a finite irreducible Coxeter system and $p \in S^m$. A matrix $\mathbf{B} \in \mathbb{R}^{N\times m}$ is a *signature matrix* of type W for p, if for every occurrence Z of every reduced expression v of $w_\circ$ in p, the equality

$$\operatorname{sign}(\det[\mathbf{B}]_Z) = \tau(v)$$

holds.

Let $p \in S^m$ and $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. We denotes by $\mathcal{F}_{p,\mathbf{A}}$ the collection of cones spanned by sets of columns of $\mathbf{A}$ that correspond to faces of the subword complex $\Delta_W(p)$. The following theorem originally from Ceballos' dissertation demonstrates the important role of the T-sign function, and thus of signatures matrices, for $\mathcal{F}_{p,\mathbf{A}}$ to form a complete simplicial fan.

Theorem 2.9 ([Ceb12, Section 3.1, Theorem 3.7], see also [BCL15, Theorem 3]). *Let $p \in S^m$ and $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. The collection of cones $\mathcal{F}_{p,\mathbf{A}}$ is a complete simplicial fan realizing the subword complex $\Delta_W(p)$ if and only if*

- (S) *the Gale dual $\mathbf{B} \in \operatorname{Gale}(\mathbf{A})$ is a signature matrix for p, (Signature) and*
- (I) *there is a facet of $\Delta_W(p)$ such that the interior of its associated cone in $\mathcal{F}_{p,\mathbf{A}}$ does not intersect any other cone of $\mathcal{F}_{p,\mathbf{A}}$. (Injectivity)*

2.2 The S-sign function

The monoid morphism from S^* to $\mathbb{Z}_2$ sending a word w to $(-1)^{\ell(w)}$ is the *alternating sign function*. This function does not take the lexicographic order on S into account. In contrast, the S-sign function defined below intrinsically makes use of the lexicographic order. As it turns out, the S-sign function is an integral part of the sign of $\det[\mathbf{B}]_Z$ in Definition 2.8. Before defining the sign function, we first give basic concepts, see e.g. [Die90, Chapter 1] for more details.

Definition 2.10 (Lexicographic normal form of an abelian vector). Let $w \in S^*$ be a word with abelian vector $\alpha_w = (c_i)_{s_i \in S}$. The word $\overline{w} := s_1^{c_1} \dots s_n^{c_n}$ is the *lexicographic normal form* of α_w.

Definition 2.11 (Standard permutation and inversion number of a word). Let $w \in S^m$ be a word with abelian vector α_w. Permutations in $\mathfrak{S}_m$ acts on the letters of w as

$$\pi \cdot w := w_{\pi(1)} \cdots w_{\pi(m)},$$

where $\pi \in \mathfrak{S}_m$. The permutation of $\mathfrak{S}_m$ with exactly the same inversions as w is called its *standard permutation* and is denoted $\operatorname{std}(w)$. The standard permutation is the minimal length permutation whose inverse sorts w: $\mathrm{std}(w)^{-1}\cdot w = \overline{w}$. The *inversion number* $\mathrm{inv}(w)$ of w is the number of inversions of $\mathrm{std}(w)$. Equivalently, the inversion number $\mathrm{inv}(w)$ is the smallest number of swaps of two consecutive letters of w required to obtain the lexicographic normal form of α_w.

Remark 2.12. The lexicographic normal form appears as the *nondecreasing rearrangement* of a word, see e.g. [HR01, Section 2]. Therein, the authors refer to an article of Schensted [Sch61] where the standardization of a word was introduced.

Lemma 2.13. *Let $v \in S^m$. The number of swaps $\mathrm{inv}(v)$ equals the number of pairs $(i,j) \in [m]^2$ such that $i < j$ and the letter $v_j \in S$ is smaller than the letter $v_i \in S$ in the lexicographic order.*

Definition 2.14 (S-sign of a word)**.** Let $w \in S^*$. The *S-sign* $\sigma(w)$ of w is

$$\sigma(w) := (-1)^{\mathrm{inv}(w)},$$

where $\mathrm{inv}(w)$ is the inversion number of w.

The following proposition predicts the behavior of the S-sign depending on the abelian vector of words.

Proposition 2.15. *Let $w \in S^m$ and let $\alpha_w = (c_i)_{s_i \in S}$ be its abelian vector. The inversion number of w and its reverse $\mathrm{rev}(w)$ satisfy*

$$\mathrm{inv}(w) + \mathrm{inv}(\mathrm{rev}(w)) = \binom{m}{2} - \sum_{s\in S}\binom{c_i}{2}.$$

Therefore, $\mathrm{inv}(w)$ and $\mathrm{inv}(\mathrm{rev}(w))$ have the same parity if and only if $\binom{m}{2} - \sum_{s\in S}\binom{c_i}{2}$ is even.

Proof. We give a bijective proof. There are $\binom{m}{2}$ pairs of distinct positions in the word w. These pairs split into three exclusive cases.

- The two letters are the same in S,
- the two letters are in lexicographic order from left-to-right, or
- the two letters are in lexicographic order from right-to-left.

There are $\sum_{s\in S}\binom{c_i}{2}$ pairs in the first case. The two other cases are exactly the inversions of w and $\mathrm{rev}(w)$ by Lemma 2.13. □

Example 2.16 (Sign of permutations)**.** The S-signature of words in S^* is an extension of the usual sign function on permutation. Let $w \in S^*$ with abelian vector $\alpha_w = (1,\dots,1)$. Reading the indices of the letters of the word w from left to right gives a permutation of $[n]$. The inversion number $\mathrm{inv}(w)$ counts the number of pairs of (necessarily distinct) numbers in $[n]$ which are unordered in w and hence the S-sign of w is equal to the usual sign of the permutation that w represents.

Proposition 2.17. *Let $w \in S^*$ and $\phi : S \to S$ be the lexicographic order reversing map such that $\phi(s_i) = s_{n-i+1}$. Denote by $\overleftarrow{\mathrm{inv}}(w)$ the number of swaps of w in the reversed ordering of the alphabet. Then*

$$\overleftarrow{\mathrm{inv}}(w) = \mathrm{inv}(\mathrm{rev}(w)) = \mathrm{inv}(\phi(w)).$$

Proof. When permuting the letters of w to obtain the lexicographic normal form with respect to the reverse lexicographic order we obtain the reverse of the lexicographic normal form obtained with the usual ordering of the alphabet. Therefore, reversing w and ordering alphabetically gives the same number of inversions by symmetry, proving $\overleftarrow{\mathrm{inv}}(w) = \mathrm{inv}(\mathrm{rev}(w))$.

By Lemma 2.13, the number $\overleftarrow{\mathrm{inv}}(w)$ is equal to the number of pairs $(i, j) \in [m]^2$ such that $i < j$ and the letter $w_j \in S$ is *larger* than the letter $w_i \in S$ in the lexicographic order. By applying ϕ to w these pairs (i, j) become exactly the inversions of $\phi(w)$, proving that $\overleftarrow{\mathrm{inv}}(w) = \mathrm{inv}(\phi(w))$. □

The S-sign of words behaves differently from the T-sign along braid moves as the following theorem shows.

Theorem 2.18. *Let $1 \leq i < j \leq n$, and $u, v \in S^*$ be two words. Further, define*

$$b_{i,j} := s_i s_j s_i \ldots \textit{ of length } m_{i,j}, \quad \kappa := \sum_{i<k\leq j} |u|_k, \quad \textit{and } \mu := \sum_{i\leq k<j} |v|_k.$$

In other words, the number κ is the number of occurrences of letters s_k in u such that $i < k \leq j$ and μ is the number of occurrences of letters s_k in v such that $i \leq k < j$. The S-sign function σ satisfies

$$\sigma(ub_{i,j}v) = \begin{cases} (-1)^{\frac{m_{i,j}}{2}} \sigma(ub_{j,i}v), & \textit{if } m_{i,j} \textit{ is even,} \\ (-1)^{\kappa+\mu} \sigma(ub_{j,i}v), & \textit{if } m_{i,j} \textit{ is odd.} \end{cases}$$

Proof. By Lemma 2.13, we have to track the change in the number of inversions in the word after doing a braid move.

Suppose that $m_{i,j}$ is even. Since the abelian vector of the words $ub_{i,j}v$ and $ub_{j,i}v$ are the same, it suffices to examine the changes in the number of swaps involving two letters that are contained in $b_{i,j}$. Indeed, the ordering of any other pair of positions stay inchanged. The number of swaps in $b_{i,j}$ is $m_{i,j}(m_{i,j} - 2)/8$ and the number of swaps in $b_{j,i}$ is $m_{i,j}(m_{i,j} + 2)/8$ hence their difference is $m_{i,j}/2$.

Suppose that $m_{i,j}$ is odd. It suffices to consider the change in the number of swaps involving at least one position in $b_{i,j}$. The number of swaps in $b_{i,j}$ and $b_{j,i}$ are the same, since the first occurrence of s_i is not swapped with any occurrence of s_j in $b_{i,j}$, and putting it at the end and simultaneously replacing it by s_j to obtain $b_{j,i}$ does not create any new swap. Therefore, we only need to count the number of swaps involving the first occurrence of s_i in $b_{i,j}$ with letters in u, which are not swaps once the occurrence of s_i is moved at the end of $b_{i,j}$ and replaced by s_j. This number is exactly κ. Further, after removing s_i at the beginning of $b_{i,j}$ and putting s_j at its end, we create swaps with the letters in v which did not need to be swapped with s_i, this number of swaps is exactly μ. □

Corollary 2.19. *Let $w \in W$ and $u, v \in \mathcal{R}(w)$ be two reduced words for w that are related by k commutations, i.e. braid moves of length 2. The S- and T-sign function satisfy*

$$\sigma(u) = (-1)^k \sigma(v) \text{ and } \tau(u) = (-1)^k \tau(v).$$

In other words, both the S-sign and the T-sign change along braid moves of length 2.

2.3 S-sign functions on reduced expressions for small rank Coxeter groups

Example 2.20 (Dihedral Group $I_2(m)$)**.** Let $W = I_2(m)$, with $m \geq 2$. The S-sign function for the reduced expressions is determined by the residue of $m \bmod 4$, see Figure 2.3.

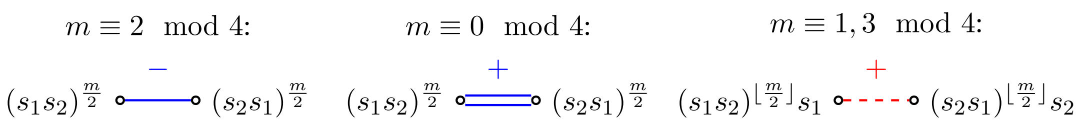


FIGURE 2.3: The S-sign for reduced expressions of $w_\circ$ for the dihedral group $I_2(m)$. Since the S-sign values vary within the same residue class, we label the edge by the product of the S-signs of its vertices.

Example 2.21 (Symmetric group $\mathfrak{S}_4 = A_3$)**.** Let $W = A_3$. The S-sign function for the reduced expressions of $w_\circ$ *is not* equal to the T-sign, see Figure 2.4. For example, the underlined braid move of length 3 between $\underline{s_1s_2s_1}s_3s_2s_1$ and $\underline{s_2s_1s_2}s_3s_2s_1$ changes the S-sign: by computing the values of κ and μ in Theorem 2.18, we get $\kappa = 0$ and $\mu = 1$. In contrast, the T-sign does not change since 3 is odd.

Example 2.22. Let $W = B_3$. The S-sign function on reduced expressions of $w_\circ$ is illustrated in Figure 2.5. Observe that the S-sign does not change on all braid moves of

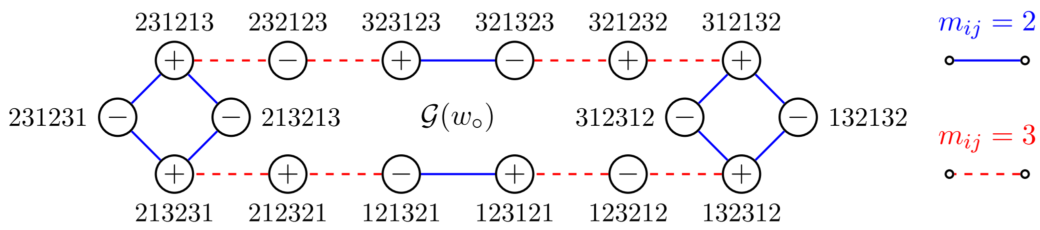


FIGURE 2.4: The S-sign for reduced expressions of $w_\circ$ in type A_3

length 3, hence in this case, the S-sign is a well-defined class function on braid classes $\mathcal{G}^{\text{even}}(w_\circ) = \mathcal{G}^{\text{braid}}(w_\circ)$. Nevertheless, it is not equal to the T-sign function on $\mathcal{G}^{\text{even}}(w_\circ)$ since the T-sign function changes along braid moves of length 4.

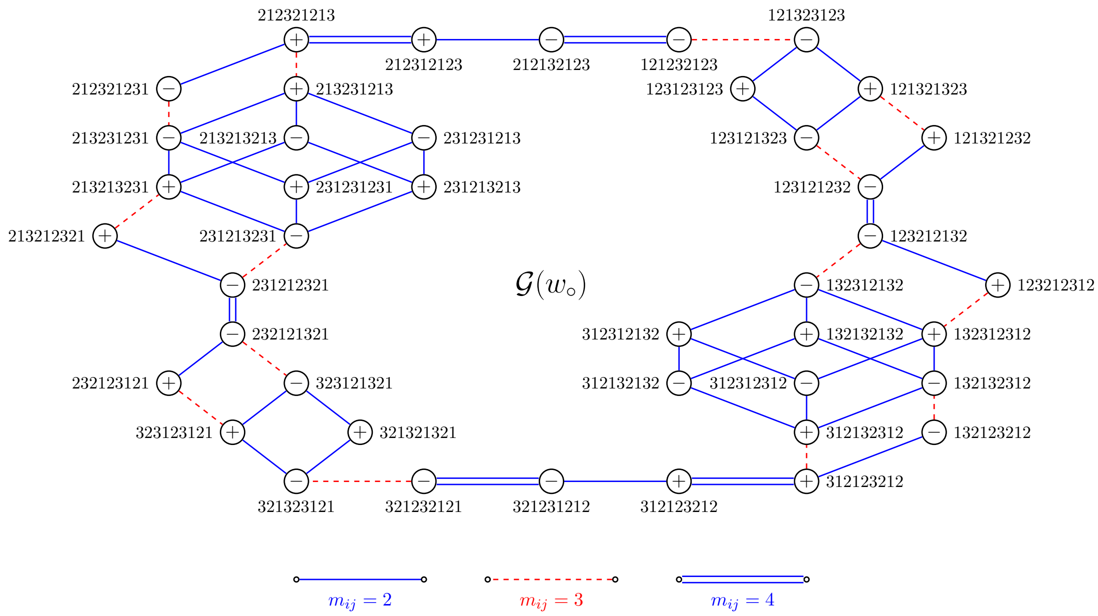


FIGURE 2.5: The S-sign function on reduced words of $w_\circ$ in type B_3

2.4 The punctual sign function

The existence of polytopal realizations of subword complexes requires the existence of the T-sign function, as first observed in [Ceb12, Proposition 3.4]. As we have seen in Definition 2.8, the T-sign prescribes the orientation of dual simplicial cones in order to get a complete simplicial fan. The S-sign further takes care of the intrinsic ordering related to a word and contributes to determine the orientation of dual simplicial cones. We investigate these relations further in Sections 3 and 4. Here, we introduce the *punctual sign function* which considers both signs and give some examples.

Definition 2.23 (Punctual sign function ⧖ [1])**.** The *punctual sign function* ⧖ is defined as

$$\begin{aligned} \text{⧖} : \mathcal{R}(w_\circ) &\to \{+1, -1\} \\ w &\mapsto \sigma(w) \cdot \tau(w), \end{aligned}$$

where σ is the S-sign function on words in S^* and τ is the T-sign function on reduced words $\mathcal{R}(w_\circ)$.

Since T is defined up to a global multiplication by "-1", the punctual sign function is also well-defined up to a global multiplication by "-1". For this reason, we henceforth fix the T-sign of the lexicographically first reduced subword of $w_\circ$ occuring in $(s_1 \cdots s_n)^\infty$ to have positive sign. The definition of product of sign functions allows to interpret the values of the ⧖-sign function: it is positive when the S and T functions are equal, and negative otherwise. Further, its behavior along braid moves is determined as follows. Set $w = ub_{i,j}v$ and $w' = ub_{j,i}v$ with $\ell(b_{i,j}) = m_{i,j}$ as in Theorem 2.18, then

$$\text{⧖}(w) = \begin{cases} \text{⧖}(w') & \text{if } m_{i,j} \equiv 2 \mod 4, \\ -\text{⧖}(w') & \text{if } m_{i,j} \equiv 0 \mod 4, \\ (-1)^{\kappa+\mu}\text{⧖}(w') & \text{if } m_{i,j} \equiv 1 \text{ or } 3 \mod 4. \end{cases}$$

Example 2.24 (Dihedral Group $I_2(m)$)**.** Let $W = I_2(m)$, with $m \geq 2$. The punctual sign function for the reduced expressions of $w_\circ$ is determined by the residue of $m \bmod 4$, see Figure 2.6.

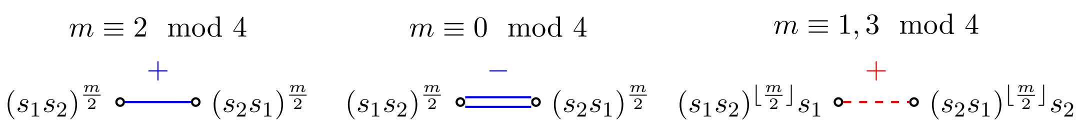


FIGURE 2.6: The punctual signs for reduced expressions of the dihedral group $I_2(m)$. Since the punctual sign values varies within the same residue class, we label the edge by the product of the punctual signs of its vertices.

Example 2.25 (Symmetric group $\mathfrak{S}_4 = A_3$)**.** Let $W = A_3$. The punctual sign function is illustrated in Figure 2.7.

[1] By multiplying the sign functions S and T, we can see the abbreviation "s.t." (*sine tempore*), which describes academic events starting punctually. The symbol ⧖ can be pronounced using IPA as ˈʃt.

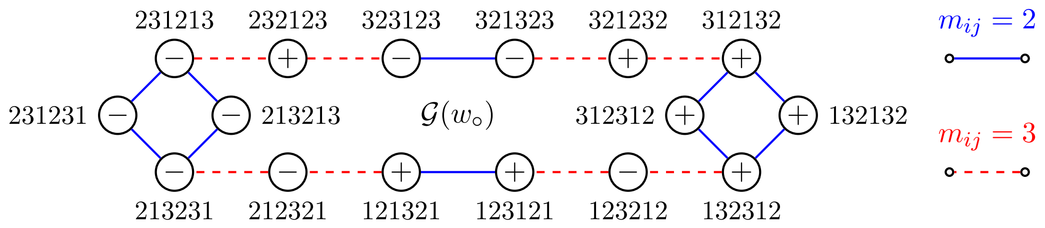


FIGURE 2.7: The punctual sign function for reduced expressions of the group A_3

Example 2.26 (Hyperoctahedral group B_3)**.** Let $W = B_3$. The punctual sign function is illustrated in Figure 2.8.

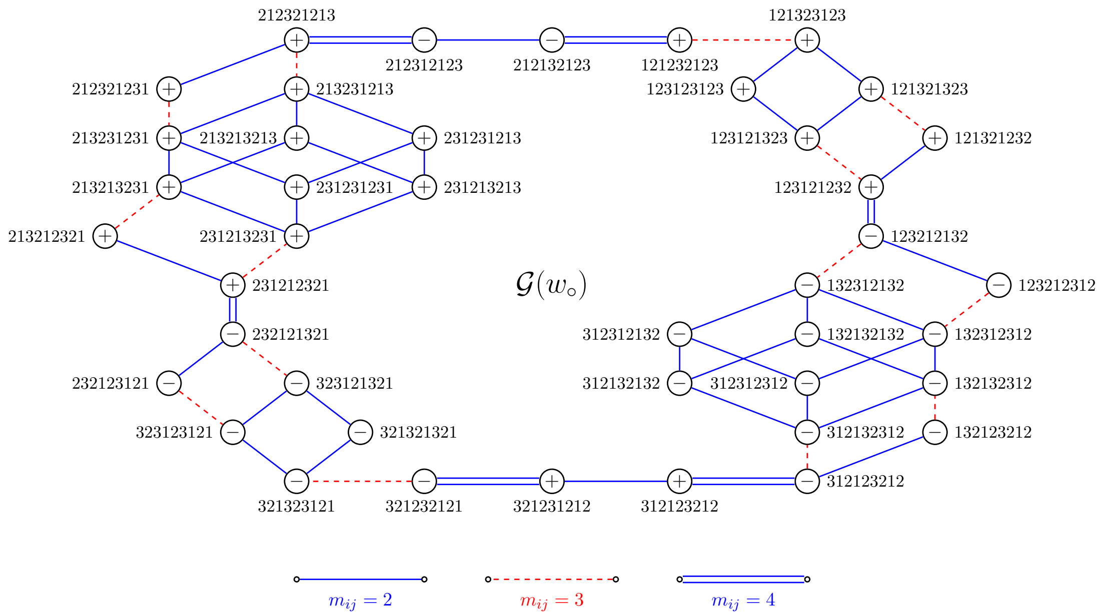


FIGURE 2.8: The punctual sign function on reduced words of $w_\circ$ in type B_3.

Chapter 3

Model matrices

In this section, we give a factorization formula for the determinant of matrices

$$\Big(f_{i,j}(x_j)\Big)_{i,j\in[k]},$$

where $f_{i,j}(x_j)$ is a polynomial in $\mathbb{R}[x_j]$ of bounded degree. Matrices of this form include the so-called *alternant matrices*. They were considered already in the XIXth century, if not earlier, see [Sta99, Chapter 7, Notes]. The case when $f_{i,j}$ does not depend on the index j (equivalently, if interchanging variables is equivalent to permuting the columns) is classical [Ait39, Chapter 6] and [Mui60, Chapter XI]. The Vandermonde matrix is the case when $f_{i,j}(x_j) = x_j^{i-1}$, for $i, j \in [k]$. Since all the columns of the Vandermonde matrix are equal up to change of variables, its determinant is an alternating polynomial in the variables $x_1, \ldots, x_k$ with respect to the group action of $\mathfrak{S}_k$ by permuting the variable indices. As Definition 1.2 shows, the same holds for the columns of the matrix used to define Schur functions, leading to the fact that its determinant is divisible by the Vandermonde determinant and thus the quotient is a symmetric polynomial in the variables $x_1, \ldots, x_k$. At the opposite end, if no two columns are equal up to a change of variables, no non-trivial permutation action acts canonically on the determinant.

We are particularly interested in the case when subsets of columns are equal up to a change of variable. When the columns are partitioned into subsets of columns that are equal up to permuting the variable indices, we get a *partially symmetric polynomial* expressible as a product of symmetric polynomials, see Theorem 3.15 and Corollary 3.16. The Binet–Cauchy formula is at the center of the approach below.

3.1 Definitions

In order to study matrices with polynomial entries, we define certain tensors. They allow to dissect the data into smaller pieces, that are then easier to control and analyze as done in Sections 1.2 and 1.3. The variables tensor is used to provide polynomials of degree at most $d-1$ in a $(N \times N)$-matrix:

Definition 3.1 (Variables tensor)**.** Let $d \geq N \geq 1$. The *variables tensor* $\mathcal{T}^{k,j}{}_l(d,N)$ is the tensor in $V_d \otimes V_N \otimes V_N{}^*$ over $\mathbb{R}[x_1, \ldots, x_N]$ defined as

$$\mathcal{T}^{k,j}{}_l(d,N) := \sum_{j=1}^{N} \sum_{l=j}^{j} \left(\sum_{k=1}^{d} x_l^{k-1} \right) \mathbf{e}^k \otimes \mathbf{e}^j \otimes \mathbf{f}_l.$$

The parameter tensor encodes the coefficients of the polynomials that appear:

Definition 3.2 (Parameter tensor)**.** Let $d \geq N \geq 1$ and S be an alphabet of cardinality n. A *parameter tensor* $\mathcal{P}^i{}_{s,k}(N,n,d)$ is a tensor in $V_N \otimes V_n{}^* \otimes V_d{}^*$ over $\mathbb{R}$.

We index the basis tensors of $V_N \otimes V_n{}^* \otimes V_d{}^*$ with the set $[N] \times S \times \{0, 1, \ldots, d-1\}$. In particular, the columns of a parameter tensor are indexed by couples in $S \times \{0, 1, \ldots, d-1\}$.

Definition 3.3 (Coefficients tensor of a word)**.** Let $d \geq 1$, $v = v_1 v_2 \ldots v_N$ be a word in S^*, and $\mathcal{P} = (p^i{}_{s,j})_{(i,j,s) \in [N] \times S \times \{0,\ldots,d-1\}}$ be a parameter tensor. The *coefficients tensor* of v with respect to $\mathcal{P}$ is the tensor in $V_N \otimes V_N{}^* \otimes V_d{}^*$ defined as

$$\mathcal{C}^i{}_{j,k}(v,\mathcal{P}) := \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{d} p^i{}_{v_j,k-1} \, \mathbf{e}^i \otimes \mathbf{f}_j \otimes \mathbf{f}_k.$$

Multiplying the coefficients tensor with the variables tensor and flattening, we get a matrix that models square matrices where certain groups of columns are equal up to a relabeling of variables, according to occurrences of letters in the chosen word v.

Definition 3.4 (Model matrix of a word)**.** Let $d \geq 1$, $v = v_1 v_2 \ldots v_N$ be a word in S^*, and $\mathcal{P} = (p^i{}_{s,j})_{(i,s,j) \in [N] \times S \times \{0,\ldots,d-1\}}$ be a parameter tensor. Denote by $\mathbb{R}[\mathcal{P}]$ the real polynomial ring whose variables are the non-zero coefficients of $\mathcal{P}$. The *model matrix* of v with respect to $\mathcal{P}$ is the $(N \times N)$-matrix

$$M^i{}_l(v,\mathcal{P}) := \mathcal{C}^i{}_{j,k}(v,\mathcal{P}) \cdot \mathcal{T}^{k,j}{}_l(d,N),$$

whose entries in column l are polynomials of degree $d-1$ in the variable x_l with coefficients taken in the parameter tensor $\mathcal{P}$ with second index v_l.

The entries of $M(v,\mathcal{P})$ in column l are polynomials in $(\mathbb{R}[\mathcal{P}_{v_l}])[x_l]$. Further, whenever $v_i = v_j$ and $i \neq j$, the columns i and j are equal up to relabeling their variables. We have already seen such examples: the Vandermonde matrix in Section 1.2, and in Example 1.3. Here is another example that we examine further later on.

Example 3.5. Consider the matrix

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -x_1 & x_2 & -x_3 & x_4 \\ x_1^2 & -x_2^2 & x_3^2 & -x_4^2 \end{pmatrix}.$$

In this case, $d-1=2$, $N=4$, and $v = s_1s_2s_1s_2$. The corresponding parameter tensor is

$$\mathcal{P} := (\mathbf{e}^1 \otimes \mathbf{f}_{s_1} \otimes \mathbf{f}_1) - (\mathbf{e}^3 \otimes \mathbf{f}_{s_1} \otimes \mathbf{f}_2) + (\mathbf{e}^4 \otimes \mathbf{f}_{s_1} \otimes \mathbf{f}_3) + (\mathbf{e}^2 \otimes \mathbf{f}_{s_2} \otimes \mathbf{f}_1) + (\mathbf{e}^3 \otimes \mathbf{f}_{s_2} \otimes \mathbf{f}_2) - (\mathbf{e}^4 \otimes \mathbf{f}_{s_2} \otimes \mathbf{f}_3).$$

The matrix can be written using the corresponding coefficients and variables matrices as:

$$M = \bigoplus_{k=1}^{2} \overbrace{\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right)}^{\mathcal{P}} \times \begin{pmatrix} 1 & x_1 & x_1^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & x_2^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 & x_3^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_4 & x_4^2 \end{pmatrix}^{\top}.$$

3.2 Model matrices for reduced words

For the remainder of Section 3, we present the results with the combinatorics of Coxeter groups in mind. The general result about the factorization of determinants of matrices of polynomials can be deduced directly by removing the restrictions coming from the Coxeter group in play. We remind the reader of the definitions of the following objects used throughout this section.

$$\begin{array}{rcl} (W,S) & := & \text{a finite irreducible Coxeter system,} \\ n & := & \#S\text{, the cardinality of } S, \\ N & := & \ell(w_\circ)\text{, the length of the longest element,} \\ \nu & := & \text{the höchstfrequenz of } W\text{, defined in Section 1.5.} \end{array}$$

Definition 3.6 (Variables tensor of a Coxeter system)**.** The *variables tensor* of (W,S) is the variables tensor $\mathcal{T}_W := \mathcal{T}^{k,j}{}_l(\nu,N)$.

Definition 3.7 (Model matrix of a reduced word)**.** Let $\mathcal{P} = ({p^i}_{s,j})_{(i,j,s)\in[N]\times S\times\{0,\dots,\nu-1\}}$ be a parameter tensor and $v = v_1v_2\dots v_N \in \mathcal{R}(w_\circ)$. The *model matrix* of v with respect to $\mathcal{P}$ is the $(N\times N)$-matrix

$$M^i{}_l(v,\mathcal{P}) := \mathcal{C}^i{}_{j,k}(v,\mathcal{P})\cdot \mathcal{T}^{k,j}{}_l(\nu,N),$$

Example 3.8 (Symmetric group $\mathfrak{S}_3 = A_2$)**.** We have $n=2, \nu=2, \mathcal{R}(w_\circ) = \{s_1s_2s_1, s_2s_1s_2\}$, and $N=3$. The model matrix for $s_1s_2s_1$ is

$$M(s_1s_2s_1,\mathcal{P}) = \begin{pmatrix} {p^1}_{s_1,0}+{p^1}_{s_1,1}x_1 & {p^1}_{s_2,0}+{p^1}_{s_2,1}x_2 & {p^1}_{s_1,0}+{p^1}_{s_1,1}x_3 \\ {p^2}_{s_1,0}+{p^2}_{s_1,1}x_1 & {p^2}_{s_2,0}+{p^2}_{s_2,1}x_2 & {p^2}_{s_1,0}+{p^2}_{s_1,1}x_3 \\ {p^3}_{s_1,0}+{p^3}_{s_1,1}x_1 & {p^3}_{s_2,0}+{p^3}_{s_2,1}x_2 & {p^3}_{s_1,0}+{p^3}_{s_1,1}x_3 \end{pmatrix}.$$

The model matrix for $s_2s_1s_2$ is

$$M(s_2s_1s_2,\mathcal{P}) = \begin{pmatrix} {p^1}_{s_2,0}+{p^1}_{s_2,1}x_1 & {p^1}_{s_1,0}+{p^1}_{s_1,1}x_2 & {p^1}_{s_2,0}+{p^1}_{s_2,1}x_3 \\ {p^2}_{s_2,0}+{p^2}_{s_2,1}x_1 & {p^2}_{s_1,0}+{p^2}_{s_1,1}x_2 & {p^2}_{s_2,0}+{p^2}_{s_2,1}x_3 \\ {p^3}_{s_2,0}+{p^3}_{s_2,1}x_1 & {p^3}_{s_1,0}+{p^3}_{s_1,1}x_2 & {p^3}_{s_2,0}+{p^3}_{s_2,1}x_3 \end{pmatrix}.$$

3.3 Binet–Cauchy on model matrices

We use Binet–Cauchy's formula (1.3) from Section 1.2 to give a description of the determinants of model matrices. Before giving a first description, we set some useful notations and give two lemmas. Let $Z \subseteq \{0,\dots,\nu N-1\}$ and $\#Z = N$, we write

$$Z = \{z_1, z_2, \dots, z_N\},$$

such that $z_1 < z_2 \dots < z_N$, and

$$z_i = q_i\nu + r_i, \text{ with } 0 \le r_i < \nu, \text{ for all } i \in [N].$$

We use the set Z to index columns of the coefficients tensor and the rows of the variables tensor. The correspondance between indices in Z and columns of the coefficients tensor is described as follows. The columns of the coefficients tensor are labeled by couples in $[N]\times[\nu]$. Given a couple (j,k) indexing a column of the coefficients tensor, we define $q := j-1$, $r := k-1$, and $z := q\nu + r$. This way, the couple (j,k) corresponds to a unique element z in $\{0,\dots,\nu N-1\}$, and vice-versa. The index $z = q\nu + r$ correspond to the index $(q,r) \in \{0,1,\dots,N-1\}\times\{0,1,\dots\nu-1\}$, and $(j,k)\in[N]\times[\nu]$. The correspondance with rows of the variables tensor works similarly.

The following two lemmas can be checked using the definition of parameter and variables tensors and properties of the determinant.

Lemma 3.9. *Let $Z \subseteq \{0, \dots, \nu N - 1\}$ with $\#Z = N$. The determinant of the variables tensor $\mathcal{T}_W$ restricted to the rows in Z is*

$$\det[\mathcal{T}_W]_Z = \begin{cases} 0 & \text{if } q_i = q_j \text{ for some } i \neq j, \\ x_1^{r_1} \cdots x_N^{r_N} & \text{else.} \end{cases}$$

Lemma 3.10. *Let $\mathcal{P}$ be a parameter matrix for (W, S), $v = v_1 v_2 \cdots v_N \in \mathcal{R}(w_\circ)$, and $Z \subseteq \{0, \dots, \nu N - 1\}$ with $\#Z = N$. If $r_i = r_j$ and $v_i = v_j$ with $1 \leq i < j \leq N$, then the determinant of the coefficients tensor $\mathcal{C}(v, \mathcal{P})$ restricted to the columns in Z is 0.*

Given a reduced expression $v = v_1 v_2 \cdots v_N \in \mathcal{R}(w_\circ)$, the previous lemmas motivate the definition of the following set of N-subsets of $\{0, \dots, \nu N - 1\}$:

$$\mathcal{Z}_v := \{Z \subset \{0, \dots, \nu N - 1\} : \#Z = N,\ q_i \neq q_j \text{ for all } i \neq j, \text{ and if } v_i = v_j, \text{ then } r_i \neq r_j\}.$$

The subsets in $\mathcal{Z}_v$ are precisely those whose summand are not implied to be equal to zero in the Binet–Cauchy formula for the determinant of the model matrix. The following proposition is a consequence of Lemmas 3.9 and 3.10, and Theorem 1.1, and is improved in Theorem 3.15.

Proposition 3.11. *Let $\mathcal{P}$ be a parameter matrix for (W, S), and $v \in \mathcal{R}(w_\circ)$. The determinant of the model matrix $\mathcal{M}(v, \mathcal{P})$ is*

$$\det \mathcal{M}(v, \mathcal{P}) = \sum_{Z \in \mathcal{Z}_v} \det[\mathcal{C}(v, \mathcal{P})]_Z \cdot x_1^{r_1} \cdots x_N^{r_N}.$$

3.4 Formula for the determinant of model matrices via parameter matrices

We proceed to express $\det M(v, \mathcal{P})$ for some reduced word $v \in \mathcal{R}(w_\circ)$ in terms of maximal minors of $\mathcal{P}$. We begin by a bijection to relabel the columns appropriately using a permutation and a tuple of partitions. Given some set of indices $Z \in \mathcal{Z}_v$, if $v_i = v_j$ and $i \neq j$, then $r_i \neq r_j$. Consequently the matrix $[\mathcal{C}(v, \mathcal{P})]_Z$ formed by concatenating the columns in Z is equal to a column permutation π_Z of the matrix $[\mathcal{P}]_{\mathfrak{z}}$ of $\mathcal{P}$ formed by concatenating the columns in the set

$$\mathfrak{z} := \{(s_i, r_j) : j \in [N],\ v_j = s_i \text{ and } z_j = q_j \nu + r_j \in Z\}$$

increasingly with respect to the lexicographic order. Given the abelian vector $\alpha_v = (c_i)_{s_i \in S}$ of v, this motivates the definition of the following subsets of columns of $\mathcal{P}$:

$$\mathfrak{Z}_{\alpha_v} := \{\mathfrak{z} \subseteq S \times \{0,,\ldots,\nu-1\} : \mathfrak{z} \text{ contains exactly } c_i \text{ elements } (s_i,\cdot), \forall s_i \in S\}.$$

For each letter $s_i \in S$, we write the values r_j where $v_j = s_i$ in a list of length c_i:

$$R_i := [r_j : \text{if } v_j = s_i]_{j=1}^{N}.$$

Since all entries in R_i are distinct, the list R_i corresponds canonically to a permutation π_i in $\mathfrak{S}_{\{v\}_i}$. The permutations $\{\pi_i\}_{i=1}^{n}$ act on disjoint sets and can be seen as permutations in $\mathfrak{S}_N$, so we define $\pi_Z := \pi_1 \cdots \pi_n \in \mathfrak{S}_N$. Observe that this permutation $\pi_Z \in \mathfrak{S}_N$ is such that $\pi \cdot v = v$. Further, the map

$$\begin{aligned} \mathcal{Z}_v &\to \mathfrak{Z}_{\alpha_v} \times \prod_{i=1}^{n} \mathfrak{S}_{\{v\}_i} \\ Z &\mapsto (\mathfrak{z}, \pi_Z) \end{aligned}$$

is a bijection.

Lemma 3.12. *Let $v \in \mathcal{R}(w_\circ)$. If $Z \in \mathcal{Z}_v$, then*

$$\det[\mathcal{C}(v,\mathcal{P})]_Z = \sigma(v)\sigma(\pi_Z)\det[\mathcal{P}]_{\mathfrak{z}}.$$

Proof. From the definition of $\mathcal{C}(v,\mathcal{P})$ and $\mathcal{Z}_v$, the matrix $[\mathcal{C}(v,\mathcal{P})]_Z$ is a permutation of the columns of a column-submatrix of $\mathcal{P}$. The column of $\mathcal{C}(v,\mathcal{P})$ indexed by $z_j = q_j\nu + r_j$ correspond to the column of $\mathcal{P}$ indexed by the ordered pair (s_i, r_j) where $v_j = s_i$. We use the latter labeling to obtain the permutation of the columns of $[\mathcal{C}(v,\mathcal{P})]_Z$ in two steps as follows. First permute the columns using π_Z^{-1}. This permutation orders increasingly the labels r_j while keeping the labels s_i unchanged. Then, permute the columns using $\mathrm{std}(v)^{-1}$. Since the standard permutation of v has shortest length, it does not change the ordering whenever two columns have the same first label coordinate. □

Before giving the factorization formula, we give two last definitions. The first one is related to a common bijection between subsets $\binom{[n]}{k}$ and partitions with exactly k parts (that may be empty) of size at most $n-k$.

Definition 3.13 (Standard partition $\Lambda_{\mathfrak{z}}$)**.** Let $v \in \mathcal{R}(w_\circ)$ with abelian vector $\alpha_v = (c_i)_{s_i \in S}$ and $\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}$. For $i \in [n]$, order decreasingly the elements of R_i and substract $c_i - j$ to the element at position j (starting at $j = 1$) to obtain the *standard partition* $\lambda_{\mathfrak{z},i}$. The sequence of partitions $\Lambda_{\mathfrak{z}}$ is defined as $(\lambda_{\mathfrak{z},i})_{i=1}^{n}$.

Definition 3.14 (Vandermonde divisor)**.** Given $v \in \mathcal{R}(w_\circ)$ with abelian vector $\alpha_v = (c_i)_{s_i \in S}$, let

$$\mathcal{V}(v) := \prod_{\substack{s_i \in S \\ c_i \geq 2}} \prod_{\substack{v_j = v_k = s_i \\ j < k}} (x_k - x_j)$$

be the *Vandermonde divisor* of v. The degree of $\mathcal{V}(v)$ is $\sum_{s_i \in S} \binom{c_i}{2}$.

Theorem 3.15. *Let $\mathcal{P}$ be a parameter matrix for a Coxeter system (W, S), $v \in \mathcal{R}(w_\circ)$, and $\Omega_v := (\{v\}_1, \dots, \{v\}_n)$ be the ordered set partition of $[N]$ determined by v. The determinant of the model matrix $M(v, \mathcal{P})$ of v with respect to $\mathcal{P}$ is the multivariate polynomial*

$$\det M(v, \mathcal{P}) = \sigma(v)\mathcal{V}(v) \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}]_{\mathfrak{z}} Ш_{\Lambda_{\mathfrak{z}}, \Omega_v},$$

where $\Lambda_{\mathfrak{z}} = (\lambda_{\mathfrak{z},1}, \dots, \lambda_{\mathfrak{z},n})$, and $Ш_{\Lambda_{\mathfrak{z}}, \Omega_v}$ is the partial Schur function with respect to $\Lambda_{\mathfrak{z}}$ and Ω_v, as defined in Section 1.3.

Proof. Let $v = v_1 v_2 \cdots v_N$, with $v_i \in S$. Since setting $x_k = x_j$ whenever $v_j = v_k = s_i$ for some $s_i \in S$ in $M(v, \mathcal{P})$ makes its determinant vanish, we know from Hilbert's Nullstellensatz that $(x_k - x_j)$ divides the determinant, see e.g. [Hum90, Lemma 3.3]. Hence, we know that $\mathcal{V}(v)$ divides $\det M(v, \mathcal{P})$, and need to determine the quotient of the division.

Yet by Proposition 3.11,

$$\det M(v, \mathcal{P}) = \sum_{Z \in \mathcal{Z}_v} \det[\mathcal{C}(v, \mathcal{P})]_Z \cdot x_1^{r_1} \cdots x_N^{r_N}.$$

By Lemma 3.12

$$\det[\mathcal{C}(v, \mathcal{P})]_Z = \sigma(v)\sigma(\pi_Z) \det[\mathcal{P}]_{\mathfrak{z}},$$

and $M(v, \mathcal{P})$ now becomes

$$\begin{aligned} \det M(v, \mathcal{P}) &= \sum_{Z \in \mathcal{Z}_v} \sigma(v)\sigma(\pi_Z) \det[\mathcal{P}]_{\mathfrak{z}} \cdot x_1^{r_1} \cdots x_N^{r_N}, \\ &= \sigma(v) \sum_{Z \in \mathcal{Z}_v} \det[\mathcal{P}]_{\mathfrak{z}} \sigma(\pi_Z) \cdot x_1^{r_1} \cdots x_N^{r_N}. \end{aligned}$$

Since Z is uniquely determined by $(\mathfrak{z}, \pi_Z)$ and vice-versa, we rewrite the sum as

$$\begin{aligned} \det \mathcal{M}(v, \mathcal{P}) &= \sigma(v) \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[P]_{\mathfrak{z}} \sum_{\pi_1 \in \mathfrak{S}_{\{v\}_1}} \sum_{\pi_2 \in \mathfrak{S}_{\{v\}_2}} \cdots \sum_{\pi_n \in \mathfrak{S}_{\{v\}_n}} \sigma(\pi_Z) \cdot x_1^{\pi(1)} \cdots x_N^{\pi(N)}, \\ &= \sigma(v) \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[P]_{\mathfrak{z}} \sum_{\pi_1 \in \mathfrak{S}_{\{v\}_1}} \sigma(\pi_1) \sum_{\pi_2 \in \mathfrak{S}_{\{v\}_2}} \sigma(\pi_2) \cdots \sum_{\pi_n \in \mathfrak{S}_{\{v\}_n}} \sigma(\pi_n) \cdot x_1^{\pi(1)} \cdots x_N^{\pi(N)}. \end{aligned}$$

The powers of the variables x_j such that $j \in [N]$ and $v_j \neq s_n$ stay constant in the last sum, so we factor their product to get

$$\det \mathcal{M}(v, \mathcal{P}) = \sigma(v) \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[P]_{\mathfrak{z}} \sum_{\pi_1 \in \mathfrak{S}_{\{v\}_1}} \sigma(\pi_1) \sum_{\pi_2 \in \mathfrak{S}_{\{v\}_2}} \sigma(\pi_2) \cdots \prod_{i \in [N] \setminus \{v\}_n} x_i^{\pi(i)} \sum_{\pi_n \in \mathfrak{S}_{\{v\}_n}} \sigma(\pi_n) \cdot \prod_{j \in \{v\}_n} x_j^{\pi(j)}.$$

By Definition 1.2, we get

$$\sum_{\pi_n \in \mathfrak{S}_{\{v\}_n}} \sigma(\pi_n) \cdot \prod_{j \in \{v\}_n} x_j^{\pi(j)} = \det \mathrm{Vander}_{\{v\}_n}(c_n) \text{Ш}_{\lambda_{\mathfrak{z},n}, \{v\}_n}.$$

The latter equality leads to the equation

$$\det \mathcal{M}(v, \mathcal{P}) = \sigma(v) \det \mathrm{Vander}_{\{v\}_n}(c_n) \\ \times \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[P]_{\mathfrak{z}} \text{Ш}_{\lambda_{\mathfrak{z},n}, \{v\}_n} \sum_{\pi_1 \in \mathfrak{S}_{\{v\}_1}} \sigma(\pi_1) \sum_{\pi_2 \in \mathfrak{S}_{\{v\}_2}} \sigma(\pi_2) \cdots \sum_{\pi_{n-1} \in \mathfrak{S}_{\{v\}_{n-1}}} \sigma(\pi_{n-1}) \prod_{i \in [N] \setminus \{v\}_n} x_i^{\pi(i)}.$$

Repeating the last step $n-1$ times, we get

$$\det M(v, \mathcal{P}) = \sigma(v) \mathcal{V}(v) \sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}]_{\mathfrak{z}} \text{Ш}_{\lambda_{\mathfrak{z},1}, \{v\}_1} \text{Ш}_{\lambda_{\mathfrak{z},2}, \{v\}_2} \cdots \text{Ш}_{\lambda_{\mathfrak{z},n}, \{v\}_n}. \qquad \square$$

Corollary 3.16. *If the abelian vector of v is $\alpha_v = (c_i)_{s_i \in S}$, then the polynomial*

$$\sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[P]_{\mathfrak{z}} \text{Ш}_{\Lambda_{\mathfrak{z}}, \Omega_v},$$

is symmetric with respect to the group $\prod_{i=1}^n \mathfrak{S}_{c_i}$ acting on $\{x_1, \ldots, x_N\}$ by permutation of indices such that $v_{\pi(j)} = v_j$, for all $j \in [N]$.

Corollary 3.17. *If $x_i > 0$ for all $i \in [N]$ and $x_j > x_i$ whenever $i < j$ and $v_i = v_j$, then*

$$\operatorname{sign}(\det M(v, \mathcal{P})) = \sigma(v) \operatorname{sign}\left(\sum_{\mathfrak{z} \in \mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}]_{\mathfrak{z}} \text{Ш}_{\Lambda_{\mathfrak{z}}, \Omega_v} \right).$$

Example 3.18 (Example 3.5 continued)**.** The S-sign of $v = s_1 s_2 s_1 s_2$ is -1 and the Vandermonde divisor is $(x_3 - x_1)(x_4 - x_2)$. To obtain the determinant of M, we should compute the minors of $\mathcal{P}$ with 2 columns in the first block corresponding to the letter s_1 and 2 columns in the second block corresponding to the letter s_2. There are 9 minors in total, of which only two are non-zero: when $\mathfrak{z}_1 = \{(s_1, 0), (s_1, 1), (s_2, 0), (s_2, 2)\}$, and $\mathfrak{z}_2 = \{(s_1, 0), (s_1, 2), (s_2, 0), (s_2, 1)\}$. We get $\det[\mathcal{P}]_{\mathfrak{z}_1} = -1$ and $\det[\mathcal{P}]_{\mathfrak{z}_2} = 1$. By Theorem 3.15, the determinant of M is

$$\det M = -(x_3 - x_1)(x_4 - x_2) \left((-1) \left(\text{Ш}_{(0,0),\{1,3\}} \text{Ш}_{(1,0),\{2,4\}} \right) + (1) \left(\text{Ш}_{(1,0),\{1,3\}} \text{Ш}_{(0,0),\{2,4\}} \right) \right).$$

We get the values of the Schur polynomials from Example 1.5. Thus

$$\det M = -(x_3 - x_1)(x_4 - x_2)(x_1 - x_2 + x_3 - x_4).$$

The sum in the formula for the determinant is in fact a tensor in $V \otimes V$, where V is the vector space over the partitions of length 2 with parts of size at most 1:

$$\begin{array}{c} \\ \text{Ш}_{(0,0),\{1,3\}} \\ \text{Ш}_{(1,0),\{1,3\}} \\ \text{Ш}_{(1,1),\{1,3\}} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Ш}_{(0,0),\{2,4\}} & \text{Ш}_{(1,0),\{2,4\}} & \text{Ш}_{(1,1),\{2,4\}} \end{array} \\ \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array} = -\text{Ш}_{(0,0),\{1,3\}} \otimes \text{Ш}_{(1,0),\{2,4\}} + \text{Ш}_{(1,0),\{1,3\}} \otimes \text{Ш}_{(0,0),\{2,4\}}.$$

Example 3.19. Consider the matrix

$$M = \begin{pmatrix} x_1 + 1 & x_2 + 1 & x_3 + 1 \\ x_1^2 + x_1 & x_2^2 + x_2 & x_3^2 + x_3 \\ x_1^2 + 1 & x_2^2 + 1 & x_3^2 + 1 \end{pmatrix}.$$

It can be written using the coefficients and variables matrices as:

$$M = \bigoplus_{i=1}^{3} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & x_1 & x_1^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & x_2 & x_2^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & x_3 & x_3^2 \end{pmatrix}^{\top}.$$

Since interchanging variables is equivalent to permuting columns, the parameter tensor is a usual matrix

$$P = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and there is only one subset $\mathfrak{z} \in \mathfrak{Z}_{(3)}$, i.e. all the columns of P. The S-sign of the word $s_1 s_1 s_1$ is $+1$. The Vandermonde part is $(x_3 - x_1)(x_3 - x_2)(x_2 - x_1)$. The determinant of P is 2 and $\text{Ш}_{(0,0,0)} = 1$. Using Theorem 3.15 we get

$$\det M = 2(x_3 - x_1)(x_3 - x_2)(x_2 - x_1).$$

3.5 The parameter matrices behind Bergeron–Ceballos–Labbé's counting matrices

In the article [BCL15], the construction of fans is based on a matrix called *counting matrix*, whose entries enumerated occurrences of certain subwords contained in a fixed word. As it turns out, these counting matrices are signature matrices. The factorizations of the determinants of counting matrices were critical in order to prove the correctness of the construction. In view of the intricate description in the previous section, the fact that counting matrices of type A_3—obtained via a simple combinatorial rule—are signature matrices should be regarded as a highly exceptional and yet not fully explained behavior.

The theory of parameter matrices allows to shed some new light on the factorization formulas of counting matrices. Indeed, Theorem 3.15 gives a complete description of the factorizations presented in the article [BCL15]. Furthermore, for any finite irreducible Coxeter group, it precisely dictates how one may obtain signature matrices through parameter matrices. We revisit here these counting matrices using parameter matrices.

3.5.1 Type A_1

We have $n = N = \nu = 1$. Parameter matrices are (1×1)-matrices $P = (p)$ containing the real number p and the variables matrix $\mathcal{T}_W$ is (1). This way, given the only reduced word $v = s_1$, the model matrix $M(v, P)$ is the (1×1)-matrix (p). In order to be a signature matrix, the real number p should be non-zero. In [BCL15, Appendix], the counting matrix is obtained by setting $p = 1$.

3.5.2 Type A_2

We have $n = 2$, $N = 3$, and $\nu = 2$. Parameter tensors are $(3 \times 2 \times 2)$-dimensional, compare with Example 3.8. For some given positive integer m, the counting matrix $D_{s_1s_2,m}$ gives rise to the parameter matrix

$$P_{s_1s_2,m} = \begin{array}{c} \begin{matrix} (s_1,0) & (s_1,1) & (s_2,0) & (s_2,1) \end{matrix} \\ \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ m & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right) \end{array}.$$

This parameter matrix has two non-zero minors $\{(s_1, 0), (s_1, 1), (s_2, 0)\}$ and $\{(s_1, 0), (s_2, 0), (s_2, 1)\}$, which are both equal to -1. Further, the corresponding partial Schur functions $Ш_{((0,0),(0)),(\{1,3\},\{2\})}$ and $Ш_{((0),(0,0)),(\{2\},\{1,3\})}$ are both equal to 1. For $v = s_1s_2s_1$, the model matrix $M(s_1s_2s_1, P_{s_1s_2,m})$ is

$$M(s_1s_2s_1, P_{s_1s_2,m}) = \begin{pmatrix} 1 & 0 & 1 \\ -x_1+m & x_2 & -x_3+m \\ 0 & 1 & 0 \end{pmatrix}.$$

For $v = s_2s_1s_2$, the model matrix $M(s_2s_1s_2, P_{s_1s_2,m})$ is

$$M(s_2s_1s_2, P_{s_1s_2,m}) = \begin{pmatrix} 0 & 1 & 0 \\ x_1 & -x_2+m & x_3 \\ 1 & 0 & 1 \end{pmatrix}.$$

To get back the counting matrix, one has to set the parameter x_i to be the position of the factor s_1s_2 in which v_i appears in $(s_1s_2)^m$, and remove 1 if $v_i = s_1$. This number fits exactly with how the counting matrix is defined in this case.

From this, we get that

$$\begin{aligned} \det M(s_1s_2s_1, P_{s_1s_2,m}) &= \sigma(s_1s_2s_1)(x_3 - x_1)\cdot(-1\cdot 1) \\ &= (-1)(x_3-x_1)(-1) = (x_3 - x_1). \end{aligned}$$

The determinant of the model matrix $M(s_2s_1s_2, P_{s_1s_2,m})$ is similar.

3.5.3 Type A_3

We have $n = 2$, $N = 6$, and $\nu = 3$. Parameter tensors are $(6\times 3\times 3)$-dimensional. For some given positive integer m, the counting matrix $D_{s_1s_2s_3,m}$ gives rise to the parameter matrix

$$P_{s_1s_2s_3,m} = \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (s_1,0) & (s_1,1) & (s_1,2) & (s_2,0) & (s_2,1) & (s_2,2) & (s_3,0) & (s_3,1) & (s_3,2) \end{array} \\ \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & m+1 & -1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & m+1 & -1 & \binom{m+2}{2} & -m-\frac{3}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & m+1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right) \end{array}.$$

Whereas, the counting matrix $D_{s_2 s_1 s_3,m}$ gives rise to the parameter matrix

$$P_{s_2s_1s_3,m} = \begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ (s_1,0) & (s_1,1) & (s_1,2) & (s_2,0) & (s_2,1) & (s_2,2) & (s_3,0) & (s_3,1) & (s_3,2) \end{matrix} \\ \left(\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ m+1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & m+1 & -1 & 0 \\ \binom{m+1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 1 & \binom{m+1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \end{array}.$$

Although the two parameter matrices look different, their non-zero minors are almost all equal. In order to specify the columns of the parameter matrices compactly, we label the nine columns from 0 to 8 from left-to-right:

$$\begin{aligned} \det[P_*]_{\{0,1,2|3,4|6\}} &= \tfrac{1}{2}, \quad \det[P_*]_{\{0,1,2|3|6,7\}} = \tfrac{1}{2}, \quad \det[P_*]_{\{0,1|3,4,5|6\}} = -1 \\ \det[P_*]_{\{0,1|3,4|6,8\}} &= -\tfrac{1}{2}, \quad \det[P_*]_{\{0,1|3,5|6,7\}} = 1, \quad \det[P_*]_{\{0,1|3|6,7,8\}} = -\tfrac{1}{2} \\ \det[P_*]_{\{0,2|3,4|6,7\}} &= -\tfrac{1}{2}, \quad \det[P_*]_{\{0|3,4,5|6,7\}} = 1, \quad \det[P_*]_{\{0|3,4|6,7,8\}} = -\tfrac{1}{2} \end{aligned}$$

The only minor which is different is $\det[P_*]_{\{0,1|3,4|6,7\}}$ which is 0 for $P_{s_1s_2s_3,m}$ and 1 for $P_{s_2s_1s_3,m}$. This explains the very small differences in the formulas for the determinants in [BCL15, Table 2 and 3] although the way they are obtained are different. The difference appears in a factor for words with abelian vector $(2,2,2)$. For example, consider the reduced word $v = s_2s_1s_3s_2s_3s_1$. Then $\Omega_v = (\{2,6\},\{1,4\},\{3,5\})$ and there are three minors of $P_{s_1s_2s_3,m}$ in $\mathfrak{Z}_{\alpha_v}$ that are non-zero: $\mathfrak{z}_1 := \{0,1,3,4,6,8\}, \mathfrak{z}_2 := \{0,1,3,5,6,7\}$, and $\mathfrak{z}_3 := \{0,2,3,4,6,7\}$. By Theorem 3.15 and Example 2.21 the determinant of the model matrix $M(v, P_{s_1s_2s_3,m})$ is

$$\begin{aligned} \det M(v, P_{s_1s_2s_3,m}) &= (1)(x_4-x_1)(x_6-x_2)(x_5-x_3)\left(\det[P]_{\mathfrak{z}_1}\text{Ш}_{\Lambda_{\mathfrak{z}_1,\Omega_v}} + \det[P]_{\mathfrak{z}_2}\text{Ш}_{\Lambda_{\mathfrak{z}_2,\Omega_v}} + \det[P]_{\mathfrak{z}_3}\text{Ш}_{\Lambda_{\mathfrak{z}_3,\Omega_v}}\right), \\ &= (x_4-x_1)(x_6-x_2)(x_5-x_3)\left(-\frac{1}{2}\cdot(x_3+x_5) + 1\cdot(x_1+x_4) - \frac{1}{2}\cdot(x_2+x_6)\right), \\ &= -\frac{1}{2}(x_4-x_1)(x_6-x_2)(x_5-x_3)\left(x_3+x_5-2\cdot(x_1+x_4)+x_2+x_6\right), \\ &= -\frac{1}{2}(x_1-x_4)(x_2-x_6)(x_3-x_5)\left(2\cdot(x_1+x_4)-x_2-x_6-x_3-x_5\right). \end{aligned}$$

The last expression is written as in [BCL15, Table 3].

3.5.4 Type A_4

As noticed in [BCL15, Section 9], the construction using counting matrices in type A_4 delivered a parameter matrix which was not generic enough. Indeed, the reduced word $w = s_2s_1s_2s_3s_4s_2s_3s_2s_1s_2$ has abelian vector $(2,5,2,1)$ and the counting matrix only gives points lying on curves of degree 3 in $\mathbb{R}^{10}$. Hence taking 5 points on the curve corresponding to the letter s_2 does not span a 5-dimensional space, which is necessary for a dual simplicial cone corresponding to the reduced word w. Let $c = s_2s_4s_1s_3$ and $w_\circ(c) := s_2s_4s_1s_3s_2s_4s_1s_3s_2s_4$. The reduced word w appears first as a subword of $c^k w_\circ(c)$ when $k \geq 2$. This explains the non-zero numbers in the fourth column of Table 7 of [BCL15], which represented (in particular) the word w. This case epitomizes the *fundamental* difference between realizing the *cluster complex* as a simplicial fan and realizing the *multi*-cluster complex as a simplicial fan. When k increases in the above word, certain reduced words which *never* appear as subword in the cluster complex, suddenly appear and require a higher genericity. In fact, the höchstfrequenz is at least $ne^{\Omega(\sqrt{\log n/2})}$ and the parameter tensor should have at least this degree of genericity in order to produce a signature matrix.

Chapter 4

Universality of parameter matrices

In this section, we show that parameter tensors are universal in the following sense:

> Given a complete simplicial fan $\mathcal{F}$ supported by a vector configuration $\mathbf{A}$ realizing a subword complex $\Delta_W(p)$ and a Gale dual $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$, there exists a parameter tensor $\mathcal{P}_{\mathbf{A}}$ that parametrizes $\mathbf{B}$. Equivalently, $\mathbf{B}$ is the product of a variables tensor and a coefficient tensor $\mathcal{C}(p, \mathcal{P})$ given by some parameter tensor $\mathcal{P}$.

Concretely, consider some matrix $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$, whose column i correspond to the i-th letter p_i of p and a Gale dual $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$. For each letter $s_j \in S$, proceed as follows. Consider the columns $i \in [m]$ of $\mathbf{B}$ such that $p_i = s_j$, i.e. the occurrences of s_j in p. For each coordinate $k \in [N]$ of the columns, it is possible to find a polynomial of degree at most $|p|_j - 1$ that interpolate the values at these occurrences. There are many possibilities to do so; to get a specific choice, we consider the two-dimensional points $(i, \mathbf{B}(k, i))$ for the k-th coordinate, where $\mathbf{B}(k, i)$ denotes the the k-th entry of the i-th column of $\mathbf{B}$. This way, as i increases, so does the first entry $x_i := i$. Doing this for each letter $s_j \in S$, we get a parameter tensor $\mathcal{P}_{\mathbf{B}}$ with $d := \max\{|p|_j - 1 \ : \ j \in [n]\}$. In order to know if $\mathbf{B}$ is a signature matrix for p, we use Corollary 3.17.

Theorem 4.1. *Let $p \in S^m$ and $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. Further let $\mathcal{P}_{\mathbf{B}}$ denote the parameter tensor associated to a Gale dual $\mathbf{B} \in \mathrm{Gale}(\mathbf{A})$ as above. In particular, assume that $x_i > 0$ for all $i \in [N]$ and $x_j > x_i$ whenever $i < j$ and $v_i = v_j$. The matrix $\mathbf{B}$ is a signature matrix for p if and only if*

$$\mathrm{sign}\left(\sum_{\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}_{\mathbf{B}}]_{\mathfrak{z}} \text{Ш}_{\Lambda_{\mathfrak{z}},\Omega_v}\right) = \Xi(v) = \sigma(v)\tau(v)$$

for every reduced word v of $w_\circ$ which is a subword of p.

The definition of signature matrix involves checking *every* occurrence of *every* reduced expression directly on the Gale dual $\mathbf{B}$. The previous theorem shows that it suffices to check the parameter matrix *once* for each commutation classes under the condition that it was constructed via the above procedure, which yields increasing input numbers $\{x_i\}_{i\in[r]}$. Indeed, both sides of the equation remain invariant after doing a braid move of length 2 on v, thanks to Corollary 2.19. Moreover, given two reduced subwords with the same combinatorial type, or even with the same abelian vector, the left-hand side are equal up to a relabeling of variables. Thus, the previous theorem reduces significantly the amount of minors to check in a Gale dual matrix in order to verify if it is a signature matrix:

it suffices to examine the signs of minors of the parameter matrix given by the abelian vectors of reduced words in $\mathcal{R}(w_\circ)$.

The following universality result follows from the above discussion.

Theorem 4.2 (Universality of parameter tensors)**.** *Let $p \in S^m$ and $\mathcal{F}_{p,\mathbf{A}}$ be a complete simplicial fan realizing the subword complex $\Delta_W(p)$ for some matrix $\mathbf{A} \in \mathbb{R}^{(m-N)\times m}$. There exist a parameter tensor $\mathcal{P}_{\mathbf{A}}$, and m real numbers $x_i > 0$, with $i \in [m]$, such that*

- *$i < j$ and $p_i = p_j$ implies $x_i < x_j$, and*
- *for every reduced word v of $w_\circ$ which is a subword of p, the following equality holds*

$$\operatorname{sign}\left(\sum_{\mathfrak{z}\in\mathfrak{Z}_{\alpha_v}} \det[\mathcal{P}_{\mathbf{A}}]_{\mathfrak{z}} \text{Ш}_{\Lambda_{\mathfrak{z}},\Omega_v}\right) = \mathbb{X}(v) = \sigma(v)\tau(v).$$

This theorem illustrates how both the S-sign and T-sign functions *lay at the heart of geometrical realizations of subword complexes.*

Example 4.3 (Example 3.18 continued, $2k$-dimensional cyclic polytopes on $2k+4$ vertices)**.** Let $k \geq 1$, $c = s_1s_2$, $w_\circ(c) = s_1s_2s_1s_2$, and $p = c^k w_\circ(c)$. We now consider the curves $f_1(x) = (1, 0, -x, x^2)$ and $f_2(x) = (0, 1, x, -x^2)$ and we assign a number $x_i > 0$ to each letter p_i of p, such that $x_j > x_i$ whenever $p_i = p_j$ and $j < i$. If $p_i = s_1$, we evaluate f_1 at x_i, otherwise $p_i = s_2$ and we evaluate f_2 at x_i to assign a vector in $\mathbb{R}^4$ to each letter of p. There are two reduced words $s_1s_2s_1s_2$ and $s_2s_1s_2s_1$. Using the computation in Example 3.18, we get the following conditions

$$-1 = \mathbb{X}(s_1s_2s_1s_2) = \operatorname{sign}(x_{i_1} - x_{i_2} + x_{i_3} - x_{i_4}) \text{ if } p_{i_1}p_{i_2}p_{i_3}p_{i_4} = s_1s_2s_1s_2,$$

$$1 = \mathbb{X}(s_2s_1s_2s_1) = \operatorname{sign}(-x_{i_1} + x_{i_2} - x_{i_3} + x_{i_4}) \text{ if } p_{i_1}p_{i_2}p_{i_3}p_{i_4} = s_2s_1s_2s_1,$$

to get a signature matrix. These conditions are equivalent to $x_1 < x_2 < \cdots < x_{2k+3} < x_{2k+4}$. This comes as no surprise, since this is an instance of the $2k$-dimensional cyclic polytope on $2k + 4$ vertices. Taking $x_i = i$, we can verify that all conditions are satisfied and we get a signature matrix for the (dual of the oriented matroid of the) cyclic polytope.

Appendix A

Some Abelian vectors of reduced words of the longest elements

Here we give the possible abelian vectors of the longest element $w_\circ$ for the finite irreducible Coxeter groups of small rank. The computations used `Sage`'s implementation of Coxeter groups to generate all reduced word [SageMath]. The generation of the reduced word proceeds without much difficulty; the current bottleneck being that in types A, B, D, H of higher ranks and other types, the computations all require more than 256GB of RAM memory.

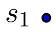
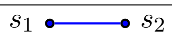
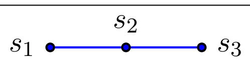
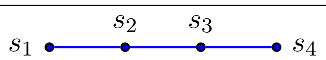
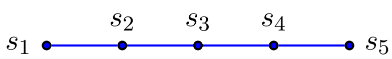
Type A_n	Abelian vectors of $w_\circ$
	$\{(1)\}$
	$\{(2,1),(1,2)\}$
	$\{(3,2,1),(2,3,1),(2,2,2),(1,3,2),(1,2,3)\}$
	$\{(4,3,2,1),(3,4,2,1),(3,3,3,1),$ $(3,3,2,2),(3,2,4,1),(3,2,3,2),$ $(2,5,2,1),(2,4,3,1),(2,4,2,2),$ $(2,3,4,1),(2,3,3,2),(2,3,2,3),$ $(2,2,4,2),(2,2,3,3),(1,4,3,2),$ $(1,4,2,3),(1,3,4,2),(1,3,3,3),$ $(1,2,5,2),(1,2,4,3),(1,2,3,4)\}$
	97 abelian vectors with coordinatewise minimum $(1,2,3,2,1)$ and maximum $(5,6,6,6,5)$.

Table A.1: Abelian vectors of the reduced words for the longest element in type A

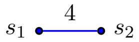
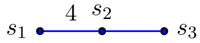
Type B_n	Abelian vectors of $w_\circ$
	$\{(2,2)\}$
	$\{(3,4,2),(3,3,3)\}$
	$\{(4,6,4,2),(4,6,3,3),(4,5,5,2),(4,5,4,3),(4,4,6,2),(4,4,5,3),(4,4,4,4)\}$

Table A.2: Abelian vectors of the reduced words for the longest element in type B

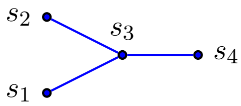
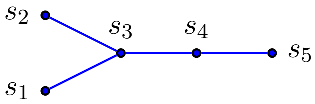
Type D_n	Abelian vectors of $w_\circ$
	$\{(4,2,4,2),(3,3,4,2),(3,3,3,3),$ $(3,2,5,2),(3,2,4,3),(2,4,4,2),$ $(2,3,5,2),(2,3,4,3),(2,2,6,2),$ $(2,2,5,3),(2,2,4,4)\}$
	111 abelian vectors with coordinatewise minimum $(2,2,4,3,2)$ and maximum $(6,6,9,7,5)$

Table A.3: Abelian vectors of the reduced words for the longest element in type D_4 and D_5

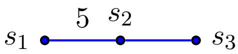
Type H_n	Abelian vectors of $w_\circ$
	$\{(6,6,3),(5,7,3),(5,6,4),(5,5,5)\}$

Table A.4: Abelian vectors of the reduced words for the longest element in type H_3

Declaration of Authorship

I, Jean-Philippe Labbé, declare that this monography titled, “Convex Geometry of Subword Complexes of Coxeter groups” and the work presented in it are my own. I confirm that:

- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this monography is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed: ________________________

Date: ________________________

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