



The infinitesimal generator of the stochastic Burgers equation

Massimiliano Gubinelli¹ · Nicolas Perkowski² 

Received: 29 October 2018 / Revised: 13 August 2020 / Published online: 26 August 2020
© The Author(s) 2020

Abstract

We develop a martingale approach for a class of singular stochastic PDEs of Burgers type (including fractional and multi-component Burgers equations) by constructing a domain for their infinitesimal generators. It was known that the domain must have trivial intersection with the usual cylinder test functions, and to overcome this difficulty we import some ideas from paracontrolled distributions to an infinite dimensional setting in order to construct a domain of *controlled functions*. Using the new domain, we are able to prove existence and uniqueness for the Kolmogorov backward equation and the martingale problem. We also extend the uniqueness result for “energy solutions” of the stochastic Burgers equation of Gubinelli and Perkowski (J Am Math Soc 31(2):427–471, 2018) to a wider class of equations. As applications of our approach we prove that the stochastic Burgers equation on the torus is exponentially L^2 -ergodic, and that the stochastic Burgers equation on the real line is ergodic.

Mathematics Subject Classification 60H17

Financial support by DFG via the CRC 1060 and partially by EPSRC Grant Number EP/R014604/1 is gratefully acknowledged.

Financial support by DFG via the Heisenberg program and via Research Unit FOR 2402 is gratefully acknowledged.

✉ Nicolas Perkowski
perkowski@math.fu-berlin.de

Massimiliano Gubinelli
gubinelli@iam.uni-bonn.de

¹ Hausdorff Center for Mathematics and Institute for Applied Mathematics, Universität Bonn, Bonn, Germany

² Institut für Mathematik, Freie Universität Berlin, Berlin, Germany

1 Introduction

The (conservative) stochastic Burgers equation $u : \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{R}$ (or $u : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$)

$$\partial_t u = \Delta u + \partial_x u^2 + \sqrt{2} \partial_x \xi, \quad (1)$$

where ξ is a space-time white noise, is one of the most prominent *singular stochastic PDEs*, a class of equations that are ill posed due to the interplay of very irregular noise and nonlinearities. The difficulty is that u has only distributional regularity (under the stationary measure it is a white noise in space for all times), and therefore the meaning of the nonlinearity $\partial_x u^2$ is dubious.

In recent years, new solution theories like regularity structures [20,40] or paracontrolled distributions [26,33] were developed for singular SPDEs, see [38] for an up-to-date and fairly exhaustive review. These theories are based on analytic (as opposed to probabilistic) tools. In the example of the stochastic Burgers equation we roughly speaking use that u is not a generic distribution, but it is a local perturbation of a Gaussian (obtained from ξ). We construct the nonlinearity and some higher order terms of the Gaussian by explicit computation, and then we freeze the realization of ξ and of the nonlinear terms we just constructed and use pathwise and analytic tools to control the nonlinearity for the (better behaved) remainder. This requires the introduction of new function spaces of *modelled* (resp. *paracontrolled*) distributions, which are exactly those distributions that are given as local perturbations as described before, and for which the nonlinearity can be constructed.

This point of view was first developed for rough paths, which provide a pathwise solution theory for SDEs by writing the solutions as local perturbations of the Brownian motion [37,47]. Rough paths provide a new topology in which the solution depends continuously on the driving noise, and this is useful in a range of applications. But of course there are also probabilistic solution theories for SDEs, based for example on Itô or Stratonovich integration (strong solutions) or on the martingale problem (weak solutions), and depending on the aim it may be easier to work with the pathwise approach or with the probabilistic one.

For singular SPDEs the situation is somewhat unsatisfactory because while the pathwise approach applies to a wide range of equations, it seems completely unclear how to set up a general probabilistic solution theory. There are some exceptions, for example martingale techniques tend to work in the “not-so-singular” case when the equation is singular but can be handled via a simple change of variables and does not require regularity structures (sometimes this is called the *Da Prato-Debussche regime* [12,13]); see [52,53] and also [22,23] for an example where the change of variable trick does not work but still the equation is not too singular. For truly singular equations there exist only very few probabilistic results. Röckner et al. [56] constructed a Dirichlet form for the Φ_3^4 equation and used the pathwise results to show that the form is closable, but it is unclear if the process corresponding to this form is the same as the one that is constructed via regularity structures or even if it is unique.

Maybe the strongest probabilistic results exist for the stochastic Burgers Eq. (1): First results, on which we comment more below, are due to Assing [1]. In Gonçalves and Jara [28] construct so called *energy solutions* to Burgers equation, roughly speak-

ing by requiring that u solves the martingale problem associated to

$$\partial_t u = \Delta u + \lim_{\varepsilon \rightarrow 0} \partial_x (u * \rho^\varepsilon)^2 + \sqrt{2} \partial_x \xi,$$

where ρ^ε is an approximation of the identity. This notion of solution is refined in [27] where the authors additionally impose a structural condition for the time-reversed process $(u_{T-t})_{t \in [0, T]}$, and they assume that u is stationary. These two assumptions allow them to derive strong estimates for additive functionals $\int_0^t F(u_s) ds$ of u via the *Itô trick*. They obtain the existence of solutions in this stronger sense by Galerkin approximation. The uniqueness of the refined solutions is shown in [34], leading to the first probabilistic well-posedness result for a truly singular SPDE. Extensions to non-stationary initial conditions that are absolutely continuous with respect to the invariant measure are given in [30,35], and in [55] some singular initial conditions are considered; see also [36] for Burgers equation with Dirichlet boundary condition.

The reason why the uniqueness proofs work is that we can linearize the equation via the *Cole–Hopf transform*: By formally applying Itô’s formula, we get $u = \partial_x \log w$, where w solves the stochastic heat equation $\partial_t w = \Delta w + \sqrt{2} w \xi$, a well posed equation which can be handled with classical SPDE approaches as in [17,46,54]. The proof of uniqueness in [34] shows that the formal application of Itô’s formula is allowed for the refined energy solutions of [27], and it heavily uses the good control of additive functionals from the Itô trick. Since the Cole–Hopf transform breaks down for essentially all other singular SPDEs, there is no hope of extending this approach to other equations.

The aim of the present paper is to provide a new and intrinsic (without transformation) martingale approach to some singular SPDEs. For simplicity we lead the main argumentation on the example of the Burgers equation, but later we also treat multi-component and fractional generalizations. The starting point is the observation that u is a Markov process, and therefore it must have an infinitesimal generator. The problem is that typical test functions on the state space of u (the space of Schwartz distributions) are not in the domain of the generator; this includes the test functions that are used in the energy solution approach, where the term

$$\lim_{\varepsilon \rightarrow 0} \int_0^t [\partial_x (u_s * \rho^\varepsilon)^2](f) ds$$

for a test function f is not of finite variation, which means that for $\varphi(u) = u(f)$ the process $(\varphi(u_t))_t$ is not a semimartingale, and therefore φ cannot be in the domain of the generator. This was already noted by Assing [1], who defined the formal generator on *cylinder test functions* but with image in the space of *Hida distributions*. Our aim is to find a (more complicated) domain of functions that are mapped to functions and not distributions under a formal extension of Assing’s operator.

For this purpose we take inspiration from recent developments in *singular diffusions*, i.e. diffusions with distributional drift. Indeed, Assing’s results show that we can interpret the Burgers drift as a distribution in an infinite-dimensional space, see also the discussion in [35]. In finite-dimensions the papers [6,9,24,25] all follow a similar strategy for solving $dX_t = b(X_t)dt + dW_t$ for distributional b : They identify a

domain for the formal infinitesimal generator $\mathcal{L} = \frac{1}{2} \Delta + b \cdot \nabla$ and then show existence and uniqueness of solutions for the corresponding martingale problem. So far this is very classical, but the key observation is that for distributional b the domain does not contain any smooth functions and instead one has to identify a class of non-smooth test functions with a special structure, adapted to b . Roughly speaking they must be local perturbations of a linear functional constructed from b . This is very reminiscent of the rough path/regularity structure philosophy, and in fact [6,9] even use tools from rough paths resp. paracontrolled distributions.

We would like to use the same strategy for the stochastic Burgers equation. But rough paths and controlled distributions are finite-dimensional theories, and here we are in an infinite-dimensional setting. To set up a theory of function spaces and distributions we need a reference measure (in finite dimensions this is typically Lebesgue measure), and we will work with the stationary measure of u , the law μ of the white noise. This is a Gaussian measure, and by the chaos decomposition we can identify $L^2(\mu)$ with the Fock space $\bigoplus_{n=0}^{\infty} L^2(\mathbb{T}^n)$, which has enough structure so that we can do analysis on it. In that way we construct a domain of *controlled functions* which are mapped to $L^2(\mu)$ by the generator of u , and this allows us to define a martingale problem for u . By Galerkin approximation we easily obtain the existence of solutions to the martingale problem. To see uniqueness, we use the duality with the Kolmogorov backward equation: Existence for the backward equation yields uniqueness for the martingale problem, and existence for the martingale problem yields uniqueness for the backward equation. We construct solutions to the backward equation by a compactness argument, relying on energy estimates in spaces of controlled functions. In that way we obtain a self-contained probabilistic solution theory for Burgers equation and fractional and multi-component generalizations. As a simple application we obtain the exponential L^2 -ergodicity of u on the torus, and the ergodicity of the stochastic Burgers equation on \mathbb{R} .

Finally we study the connection of our new approach with the Gonçalves–Jara energy solutions. One of the main motivations for studying the martingale problem for singular SPDEs is that it is a convenient tool for deriving the equations as scaling limits: The *weak KPZ universality conjecture* [8,50,51] says that a wide range of interface growth models converge in the weakly asymmetric or the weak noise regime to the Kardar–Parisi–Zhang (KPZ) equation h , for which $u = \partial_x h$. Energy solutions are a powerful tool for proving this convergence, see e.g. [10,19,28,30,32]. For that purpose it is crucial to work with nice test functions, and since there seems to be no easy way of identifying the complicated functions in the domain of the generator of u with test functions on the state space of a given particle system, our new martingale problem is probably not so useful for deriving convergence theorems. This motivates us to show that the notion of energy solution is in fact stronger than our martingale problem: Every energy solution solves the martingale problem for our generator, and thus it is unique in law.

All this also works for the fractional and multi-component Burgers equations. For the fractional Burgers equation we treat the entire *locally subcritical* regime (in the language of Hairer [40]), which in regularity structures would lead to very complicated expansions, while for us a first order expansion is sufficient. Although by now there are very sophisticated and powerful black box type tools available in regularity structures that should handle the complicated expansion automatically [2,4,7].

Our approach is somewhat related to the recent advances in regularization by noise for SPDEs [14,15], where unique strong solutions for SPDEs with bounded measurable drift are constructed by solving infinite-dimensional resolvent type equations. Of course our drift is unbounded and not even a function.

The lynchpin of our arguments is the Gaussian invariant measure μ , and in principle our methods should extend to other equations with Gaussian invariant measures, like the singular stochastic Navier Stokes equations studied in [27]. It would even suffice to have a Gaussian quasi-invariant measure, i.e. a process which stays absolutely continuous (or rather *incompressible* in the sense of Definition 4.2) with respect to a Gaussian reference measure. But for general singular SPDEs we would have to work with more complicated measures like the Φ_3^4 measure for which we cannot reduce the analysis to the Fock space. Currently it is not clear how to extend our methods to such problems, so while we provide a probabilistic theory of some singular SPDEs that actually tackles the problem at hand and does not shift the singularity away via the Cole–Hopf transform, it is still much less general than regularity structures and it remains an important and challenging open problem to find more general probabilistic methods for singular SPDEs.

Structure of the paper Below we introduce some commonly used notation. In Sect. 2 we derive the explicit representation of the Burgers generator on Fock space and we introduce a space of controlled functions which are in the domain of the generator. In Sect. 3 we study the Kolmogorov backward equation and show the existence of solutions with the help of energy estimates for the Galerkin approximation and a compactness principle in controlled spaces, while uniqueness is easy. Section 4 is devoted to the martingale problem: We show existence via tightness of the Galerkin approximations and uniqueness via duality with the backward equation. As an application of our results we give a short proof of exponential L^2 -ergodicity. Finally we formulate a cylinder function martingale problem in the spirit of energy solutions, and we show that it is stronger than the martingale problem and therefore also has unique solutions. In Sect. 5 we briefly discuss extensions to multi-component and fractional Burgers equations. We do all the analysis on the torus, but with minor changes it carries over to the real line, as we explain in Sect. 5.3, where we also prove the ergodicity of Burgers equation on the real line. The appendix collects some auxiliary estimates.

Notation We work on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the Fourier transform of $\varphi \in L^2(\mathbb{T}^n)$ is

$$\mathcal{F}\varphi(k_1, \dots, k_n) = \hat{\varphi}(k_1, \dots, k_n) = \int_{\mathbb{T}^n} e^{-2\pi i k \cdot x} \varphi(x) dx, \quad k \in \mathbb{Z}^n.$$

To shorten the formulas we usually write

$$k_{1:n} := (k_1, \dots, k_n), \quad x_{1:n} := (x_1, \dots, x_n)$$

and

$$\int_x (\dots) := \int (\dots) dx$$

Moreover, we set $\mathbb{Z}_0 := \mathbb{Z} \setminus \{0\}$ and we mostly restrict our attention to the subspace

$$L^2_0(\mathbb{T}^n) := \left\{ \varphi \in L^2(\mathbb{T}^n) : \hat{\varphi}(k_{1:n}) = 0, \forall k \in \mathbb{Z}^n \setminus \mathbb{Z}^n_0 \right\}.$$

The space $C^k_p(\mathbb{R}^n)$ consists of all C^k functions whose partial derivatives of order up to k have polynomial growth.

We write $a \lesssim b$ or $b \gtrsim a$ if there exists a constant $c > 0$, independent of the variables under consideration, such that $a \leq c \cdot b$, and we write $a \simeq b$ if $a \lesssim b$ and $b \lesssim a$.

2 A domain for the Burgers generator

2.1 The generator of the Galerkin approximation

Consider the solution $u^m : \mathbb{R}_+ \times \mathbb{T} \rightarrow \mathbb{R}$ to the Galerkin approximation of the conservative stochastic Burgers equation

$$\partial_t u^m = \Delta u^m + B_m(u^m) + \sqrt{2} \partial_x \xi := \Delta u^m + \partial_x \Pi_m (\Pi_m u^m)^2 + \sqrt{2} \partial_x \xi, \tag{2}$$

where ξ is a space-time white noise and

$$\Pi_m u(x) = \sum_{|k| \leq m} e^{2\pi i k x} \hat{u}(k)$$

is the projection onto the first $2m + 1$ Fourier modes. Throughout the paper we write μ for the law of the average zero white noise on \mathbb{T} , i.e. the centered Gaussian measure on $H^{-1/2-}(\mathbb{T}) := \bigcup_{\varepsilon > 0} H^{-1/2-\varepsilon}(\mathbb{T})$ with covariance

$$\int u(f)u(g)\mu(du) = \langle f - \hat{f}(0), g - \hat{g}(0) \rangle_{L^2(\mathbb{T})}$$

for all $f, g \in \bigcup_{\varepsilon > 0} H^{1/2+\varepsilon}(\mathbb{T})$.

Lemma 2.1 *Equation (2) has a unique strong solution $u^m \in C(\mathbb{R}_+, H^{-1/2-}(\mathbb{T}))$ for every deterministic initial condition in $H^{-1/2-}(\mathbb{T})$. The solution is a strong Markov process and it is invariant under μ . Moreover, for all $\alpha > 1/2$ and $p \in [1, \infty)$, there exists $C = C(m, t, p, \alpha) > 0$ such that*

$$\mathbb{E} \left[\sup_{s \in [0, t]} \|u^m_s\|_{H^{-\alpha}}^p \right] \leq C \left(1 + \|u^m_0\|_{H^{-\alpha}}^p \right).$$

Proof Local existence and uniqueness and the strong Markov property follow from standard theory because written in Fourier coordinates we can decouple $u^m = v^m + Z^m := \Pi_m u^m + (1 - \Pi_m)u^m$, where v^m solves a finite-dimensional SDE with locally

Lipschitz continuous coefficients and Z^m solves an infinite-dimensional but linear SDE. Global existence and invariance of μ are shown in Section 4 of [27]. It is well known and easy to check that Z^m has trajectories in $C(\mathbb{R}_+, H^{-1/2-}(\mathbb{T}))$, see e.g. [31, Chapter 2.3], and v^m has compact spectral support and therefore even $v^m \in C(\mathbb{R}_+, C^\infty(\mathbb{T}))$. Thus u^m has trajectories in $C(\mathbb{R}_+, H^{-1/2-}(\mathbb{T}))$. The moment bound can be derived using similar arguments as in [27]. The reason why v^m behaves nicely is that B_m leaves the $L^2(\mathbb{T})$ norm invariant since

$$\langle u, B_m(u) \rangle_{L^2(\mathbb{T})} = -\langle \partial_x \Pi_m u, (\Pi_m u)^2 \rangle_{L^2(\mathbb{T})} = -\frac{1}{3} \langle \partial_x (\Pi_m u)^3, 1 \rangle_{L^2(\mathbb{T})} = 0$$

by the periodic boundary conditions. To see the invariance of μ we also need that B_m is divergence free when written in Fourier coordinates. See Section 4 of [27] or Lemma 5 of [32] for details. \square

We define the semigroup of u^m for all bounded and measurable $\varphi: H^{-1/2-} \rightarrow \mathbb{R}$ as $T_t^m \varphi(u) := \mathbb{E}_u[\varphi(u_t^m)]$, where under \mathbb{P}_u the process u^m solves (2) with initial condition u .

Lemma 2.2 *For all $p \in [1, \infty]$ the family of operators $(T_t^m)_{t \geq 0}$ can be uniquely extended to a contraction semigroup on $L^p(\mu)$, which is continuous for $p \in [1, \infty)$.*

Proof This uses the invariance of μ and follows by approximating L^p functions with bounded measurable functions. To see the continuity for $p \in [1, \infty)$ we use that in this case continuous bounded functions are dense in $L^p(\mu)$. \square

Our next aim is to derive the generator of the semigroup T^m on $L^2(\mu)$. For that purpose let $f_1, \dots, f_n \in C^\infty(\mathbb{T})$, let $\Phi \in C_p^2(\mathbb{R}^n, \mathbb{R})$, the C^2 functions with polynomially growing partial derivatives of order up to 2, and let $\varphi \in \mathcal{C}$ be a cylinder function of the form $\varphi(u) = \Phi(u(f_1), \dots, u(f_n))$. Let us introduce the notation

$$\begin{aligned} \mathcal{L}_0 \varphi(u) &:= \sum_{i=1}^n \partial_i \Phi(u(f_1), \dots, u(f_n)) u(\Delta f_i) \\ &\quad + \sum_{i,j=1}^n \partial_{ij}^2 \Phi(u(f_1), \dots, u(f_n)) \langle \partial_x f_i, \partial_x f_j \rangle_{L^2(\mathbb{T})}, \\ \mathcal{G}^m \varphi(u) &:= \sum_{i=1}^n \partial_i \Phi(u(f_1), \dots, u(f_n)) \langle B_m(u), f_i \rangle_{L^2(\mathbb{T})} = \int_{\mathbb{T}} B_m(u)(x) D_x \varphi(u) dx, \end{aligned}$$

where

$$D_x \varphi(u) = \sum_{i=1}^n \partial_i \Phi(u(f_1), \dots, u(f_n)) f_i(x)$$

is the Malliavin derivative with respect to μ , and

$$\mathcal{L}^m := \mathcal{L}_0 + \mathcal{G}^m.$$

Then Itô’s formula gives

$$d\varphi(u_t^m) = \mathcal{L}^m \varphi(u_t^m) dt + \sum_{i=1}^n \partial_i \Phi(u_t^m(f_1), \dots, u_t^m(f_n)) dM_t(f_i),$$

where $M(f_i)$ is a continuous martingale under \mathbb{P}_u , with quadratic variation $\langle M(f_i) \rangle_t = 2\|\partial_x f_i\|_{L^2(\mathbb{T})}^2 t$ and therefore $\int_0^\cdot \sum_{i=1}^n \partial_i \Phi(u_t^m(f_1), \dots, u_t^m(f_n)) dM_t(f_i)$ is a martingale under \mathbb{P}_u . Consequently, we have

$$T_t^m \varphi(u) - \varphi(u) = \int_0^t T_s^m (\mathcal{L}^m \varphi)(u) ds$$

for all $u \in H^{-1/2-}$.

To extend this to more general functions φ and to obtain suitable bounds for \mathcal{L}_0 and \mathcal{G}^m we work with the chaos expansion: Every function $\varphi \in L^2(\mu)$ can be written uniquely as $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$, where $\varphi_n \in L^2_0(\mathbb{T}^n)$ is symmetric in its n arguments and W_n is an n -th order Wiener–Itô integral; recall that $L^2_0(\mathbb{T}^n) = \{\varphi \in L^2(\mathbb{T}^n) : \hat{\varphi}(k) = 0 \forall k \in \mathbb{Z}^n \setminus \mathbb{Z}^n_0\}$. Moreover, we have

$$\|\varphi\|_{L^2(\mu)}^2 = \sum_{n \geq 0} n! \|\varphi_n\|_{L^2(\mathbb{T}^n)}^2,$$

see [42, 49] for details. If $\varphi_n \in L^2_0(\mathbb{T}^n)$ is not symmetric, then we define $W_n(\varphi_n) := W_n(\tilde{\varphi}_n)$, where

$$\tilde{\varphi}_n(x_1, \dots, x_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \varphi_n(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

is the symmetrization of φ_n . Here Σ_n denotes the group of permutations of $\{1, \dots, n\}$, and $\|\tilde{\varphi}_n\|_{L^2(\mathbb{T}^n)} \leq \|\varphi_n\|_{L^2(\mathbb{T}^n)}$ by the triangle inequality.

Convention In what follows, a norm $\|\cdot\|$ without subscript always denotes the $L^2(\mu)$ norm, and an inner product $\langle \cdot, \cdot \rangle$ without subscript denotes the $L^2(\mu)$ inner product.

Lemma 2.3 Let $\varphi \in \mathcal{C}$ with chaos expansion $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$. Then

$$\mathcal{L}_0 \varphi = \sum_{n \geq 0} W_n(\Delta \varphi_n) := \sum_{n \geq 0} W_n \left(\left(\partial_{11}^2 + \dots + \partial_{nn}^2 \right) \varphi_n \right).$$

Proof The proof is the same as for [34, Lemma 3.7]. □

Let us write ρ^m for the inverse Fourier transform of $\mathbb{1}_{|\cdot| \leq m}$, and $f_x := f(x - \cdot)$.

Lemma 2.4 For $\varphi \in \mathcal{C}$ with chaos expansion $\varphi = \sum_{n \geq 0} W_n(\varphi_n)$ we define

$$\mathcal{G}_+^m W_n(\varphi_n) = n W_{n+1} \left(\int_{x,s} \partial_x \rho_x^m(s) \rho_s^m \otimes \rho_s^m \otimes \varphi_n(x, \cdot) \right), \tag{3}$$

$$\mathcal{G}_-^m W_n(\varphi_n) = 2n(n - 1)W_{n-1} \left(\int_{x,y,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m \otimes \varphi_n(x, y, \cdot) \right), \quad (4)$$

for which $\mathcal{G}^m \varphi = \mathcal{G}_+^m \varphi + \mathcal{G}_-^m \varphi$. Moreover, we have for all (symmetric) $\varphi_{n+1} \in L_0^2(\mathbb{T}^{n+1})$ and $\varphi_n \in L_0^2(\mathbb{T}^n)$

$$\langle W_{n+1}(\varphi_{n+1}), \mathcal{G}_+^m W_n(\varphi_n) \rangle = -\langle \mathcal{G}_-^m W_{n+1}(\varphi_{n+1}), W_n(\varphi_n) \rangle.$$

Proof Since $\|\rho_s^m\|_{L^2(\mathbb{T})}^2 = \|\rho^m\|_{L^2(\mathbb{T})}^2$ does not depend on s and thus vanishes under differentiation, we have

$$\begin{aligned} B_m(u)(x) &= W_2 \left(\int \partial_x \rho_x^m(s) \rho_s^m \otimes \rho_s^m ds \right) + \int \partial_x \rho_x^m(s) \|\rho_s^m\|_{L^2(\mathbb{T})}^2 ds \\ &= W_2 \left(\int \partial_x \rho_x^m(s) \rho_s^m \otimes \rho_s^m ds \right) \end{aligned}$$

and then, since $D_x W_n(\varphi_n) = nW_{n-1}(\varphi_n(x, \cdot))$ [49, Proposition 1.2.7] and by the contraction rules for Wiener–Itô integrals [49, Proposition 1.1.3],

$$\begin{aligned} &\int_x B_m(u)(x) D_x W_n(\varphi_n) \\ &= n \int_x W_2 \left(\int_s \partial_x \rho_x^m(s) (\rho_s^m)^{\otimes 2} \right) W_{n-1}(\varphi_n(x, \cdot)) \\ &= nW_{n+1} \left(\int_{x,s} \partial_x \rho_x^m(s) (\rho_s^m)^{\otimes 2} \otimes \varphi_n(x, \cdot) \right) \\ &\quad + 2n(n - 1)W_{n-1} \left(\int_{x,y,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m \otimes \varphi_n(x, y, \cdot) \right) \\ &\quad + n(n - 1)(n - 2)W_{n-3} \left(\int_{x,y,z,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m(z) \varphi_n(x, y, z, \cdot) \right). \end{aligned}$$

Let us look more carefully at the last term on the right hand side. Note that $\partial_x \rho_x^m(s) = -\partial_s \rho_s^m(x)$ and φ_n is symmetric under exchange of its arguments. Therefore, by symmetrisation,

$$\begin{aligned} &\int_{x,y,z,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m(z) \varphi_n(x, y, z, \cdot) \\ &= \int_{x,y,z,s} (-\partial_s \rho_s^m(x)) \rho_s^m(y) \rho_s^m(z) \varphi_n(x, y, z, \cdot) \\ &= -\frac{1}{3} \int_{x,y,z,s} \partial_s (\rho_s^m(x) \rho_s^m(y) \rho_s^m(z)) \varphi_n(x, y, z, \cdot) = 0 \end{aligned}$$

by the periodic boundary conditions. We deduce that the last term in the decomposition of $\int_x B_m(u)(x) D_x W_n(\varphi_n)$ vanishes.

It remains to show that $-\mathcal{G}_m^+$ is the adjoint of \mathcal{G}_m^- : Since φ_{n+1} is symmetric in its $(n + 1)$ arguments, we have $\langle \varphi_{n+1}, \psi \rangle_{L^2(\mathbb{T}^{n+1})} = \langle \varphi_{n+1}, \tilde{\psi} \rangle_{L^2(\mathbb{T}^{n+1})}$ for all ψ , where $\tilde{\psi}$ is the symmetrization of ψ , and therefore we do not need to symmetrize the kernel of $\mathcal{G}_+^m W_n(\varphi_n)$ in the following computations:

$$\begin{aligned} & \langle W_{n+1}(\varphi_{n+1}), \mathcal{G}_+^m W_n(\varphi_n) \rangle \\ &= (n + 1)! \int_{r_{1:n+1}} \varphi_{n+1}(r_{1:n+1}) n \int_{x,s} \partial_x \rho_x^m(s) \rho_s^m(r_1) \rho_s^m(r_2) \varphi_n(x, r_{3:n+1}) \\ &= (n + 1)! \int_{r_{1:n+1}} \varphi_{n+1}(r_{1:n+1}) n \int_{x,s} \rho_x^m(s) 2 \partial_s \rho_s^m(r_1) \rho_s^m(r_2) \varphi_n(x, r_{3:n+1}) \\ &= n! 2(n + 1)n \int_{r_{1:n+1}, x, s} \varphi_{n+1}(r_{1:n+1}) \rho_x^m(s) \partial_s \rho_s^m(r_1) \rho_s^m(r_2) \varphi_n(x, r_{3:n+1}) \\ &= n! 2(n + 1)n \int_{r_{1:n}, x, y, s} \varphi_{n+1}(x, y, r_{2:n}) \rho_{r_1}^m(s) \partial_s \rho_s^m(x) \rho_s^m(y) \varphi_n(r_{1:n}), \end{aligned}$$

where in the last step we renamed the variables as follows: $r_1 \leftrightarrow x, r_2 \rightarrow y, r_i \rightarrow r_{i-1}$ for $i \geq 3$. The claim now follows by noting that $\rho_{r_1}^m(s) = \rho_s^m(r_1)$ and $\partial_s \rho_s^m(x) = -\partial_x \rho_x^m(s)$, and thus

$$\begin{aligned} & \langle W_{n+1}(\varphi_{n+1}), \mathcal{G}_+^m W_n(\varphi_n) \rangle \\ &= -n! 2(n + 1)n \int_{r_{1:n}} \int_{x,y,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m(r_1) \varphi_{n+1}(x, y, r_{2:n}) \varphi_n(r_{1:n}) \\ &= -\langle \mathcal{G}_-^m W_{n+1}(\varphi_{n+1}), W_n(\varphi_n) \rangle. \end{aligned}$$

□

Remark 2.5 Note that the proof did not use the specific form of ρ^m and the same arguments work as long as ρ^m is an even function.

For $m \rightarrow \infty$, the kernel for $\mathcal{G}_-^m W_n(\varphi_n)$ formally converges to

$$\begin{aligned} \int_{x,y} \partial_x (\delta_x(y) \delta_x(r_1)) \varphi_n(x, y, r_{2:n-1}) &= - \int_{x,y} \delta_x(y) \delta_x(r_1) \partial_1 \varphi_n(x, y, r_{2:n-1}) \\ &= -\partial_1 \varphi_n(r_1, r_1, r_{2:n-1}), \end{aligned}$$

where δ denotes the Dirac delta. For sufficiently nice φ_n this kernel is in $L^2_0(\mathbb{T}^{n-1})$. On the other hand, the formal limit $\mathcal{G}_+ W_n(\varphi_n)$ has the kernel

$$\begin{aligned} \int_x \partial_x (\delta_x(r_1) \delta_x(r_2)) \varphi_n(x, r_{3:n+1}) &= - \int_x \delta_x(r_1) \delta_x(r_2) \partial_x \varphi_n(x, r_{3:n+1}) \\ &= -\delta_{r_1}(r_2) \partial_1 \varphi_n(r_{2:n+1}), \end{aligned}$$

which is never in $L^2_0(\mathbb{T}^{n+1})$, no matter how nice φ_n is. The idea is therefore to construct (non-cylinder) functions for which suitable cancellations happen between \mathcal{L}_0 and the limit \mathcal{G} of \mathcal{G}^m and whose image under the Burgers generator \mathcal{L} belongs to $L^2(\mu)$.

It will be easier for us to work on the Fock space: For $n \in \mathbb{N}$ let $L^2_{0,s}(\mathbb{T}^n)$ be the symmetric functions in $L^2_0(\mathbb{T}^n)$ and let

$$\Gamma L^2 = \Gamma L^2(\mathbb{T}) = \bigoplus_{n=0}^{\infty} \left(L^2_0(\mathbb{T}^n) / L^2_{0,s}(\mathbb{T}^n) \right),$$

where $L^2_0(\mathbb{T}^n) / L^2_{0,s}(\mathbb{T}^n)$ are the equivalence classes in $L^2_0(\mathbb{T}^n)$ for the equivalence relation that identifies two functions with the same symmetrization. We equip ΓL^2 with the norm

$$\|\varphi\|^2_{\Gamma L^2} = \sum_n n! \|\tilde{\varphi}_n\|^2_{L^2(\mathbb{T}^n)} = \sum_n n! \sum_{k \in \mathbb{Z}^n} |\mathcal{F}\tilde{\varphi}_n(k)|^2,$$

where we applied Parseval’s identity. The space ΓL^2 is isomorphic to $L^2(\mu)$, so in what follows we will often identify $\varphi \in \Gamma L^2$ with an element of $L^2(\mu)$, and vice versa, without explicitly mentioning it. For simplicity we will usually write $\varphi_n \in L^2_0(\mathbb{T}^n)$ for the n -th kernel of an element in the Fock space and $\Gamma L^2 = \bigoplus_{n=0}^{\infty} L^2_0(\mathbb{T}^n)$, etc., omitting from the notation that we actually mean equivalence classes.

Definition 2.6 The number operator (or Ornstein-Uhlenbeck operator) \mathcal{N} acts on Fock space as $(\mathcal{N}\varphi)_n := n\varphi_n$. With a small abuse of notation, we denote with the same symbols $\mathcal{L}, \mathcal{L}_0, \mathcal{G}^m_+, \mathcal{G}^m_-$ the Fock version of the operators introduced above in such a way that on smooth cylinder functions we have:

$$\mathcal{L}_0 \sum_{n \geq 0} W_n(\varphi_n) = \sum_{n \geq 0} W_n((\mathcal{L}_0\varphi)_n), \quad \mathcal{G}^m_{\pm} \sum_{n \geq 0} W_n(\varphi_n) = \sum_{n \geq 0} W_n((\mathcal{G}^m_{\pm}\varphi)_n). \tag{5}$$

Lemma 2.7 In Fourier variables the operators $\mathcal{L}_0, \mathcal{G}^m_+, \mathcal{G}^m_-$ are given by

$$\begin{aligned} \mathcal{F}(\mathcal{L}_0\varphi)_n(k_{1:n}) &= -(|2\pi k_1|^2 + \dots + |2\pi k_n|^2)\hat{\varphi}_n(k_{1:n}), \\ \mathcal{F}(\mathcal{G}^m_+\varphi)_n(k_{1:n}) &= -(n-1)\mathbb{1}_{|k_1|, |k_2|, |k_1+k_2| \leq m} 2\pi i(k_1+k_2)\hat{\varphi}_{n-1}(k_1+k_2, k_{3:n}), \\ \mathcal{F}(\mathcal{G}^m_-\varphi)_n(k_{1:n}) &= -2\pi i k_1 n(n+1) \sum_{p+q=k_1} \mathbb{1}_{|k_1|, |p|, |q| \leq m} \hat{\varphi}_{n+1}(p, q, k_{2:n}), \end{aligned} \tag{6}$$

respectively, where the functions on the right hand side might not be symmetric, so strictly speaking we still have to symmetrize them.

Proof The Fourier representation for \mathcal{L}_0 is obvious. In the following we often use without comment that ρ^m is an even function, i.e. $\rho^m_s(x) = \rho^m_x(s)$. The kernel for $(\mathcal{G}^m_+\varphi)_{n+1}$ has the Fourier transform

$$\begin{aligned} n \int_{r_{1:n+1}} e^{-2\pi i k \cdot r} \int_{x,s} \partial_x \rho^m_x(s) \rho^m_s(r_1) \rho^m_s(r_2) \varphi_n(x, r_{3:n+1}) \\ = n \mathbb{1}_{|k_1|, |k_2| \leq m} \int_{r_{3:n+1}} \int_{x,s} \partial_x \rho^m_x(s) e^{-2\pi i(k_1+k_2)s - 2\pi i k_{3:n+1} \cdot r_{3:n+1}} \end{aligned}$$

$$\begin{aligned} & \times \sum_{\ell} e^{2\pi i(\ell_1 x + \ell_{2:n} \cdot r_{3:n+1})} \hat{\varphi}_n(\ell) \\ &= -n \mathbb{1}_{|k_1|, |k_2|, |k_1+k_2| \leq m} 2\pi i(k_1 + k_2) \sum_{\ell_1} \int_x e^{-2\pi i(k_1+k_2)x} e^{2\pi i\ell_1 x} \hat{\varphi}_n(\ell_1, k_{3:n+1}) \\ &= -n \mathbb{1}_{|k_1|, |k_2|, |k_1+k_2| \leq m} 2\pi i(k_1 + k_2) \hat{\varphi}_n(k_1 + k_2, k_{3:n+1}). \end{aligned}$$

To derive $\mathcal{F}(\mathcal{G}_-^m \varphi)_{n-1}$, note that

$$\begin{aligned} & \int_{r_{1:n-1}} e^{-2\pi i k_{1:n-1} \cdot r_{1:n-1}} \int_{x,y,s} \partial_x \rho_x^m(s) \rho_s^m(y) \rho_s^m(r_1) \varphi_n(x, y, r_{2:n-1}) \\ &= \mathbb{1}_{|k_1| \leq m} \int_{r_{2:n-1}} e^{-2\pi i k_{2:n-1} \cdot r_{2:n-1}} \int_{x,y,s} e^{-2\pi i k_1 s} \partial_x \rho_x^m(s) \rho_s^m(y) \varphi_n(x, y, r_{2:n-1}) \\ &= \mathbb{1}_{|k_1| \leq m} \int_{r_{2:n-1}} e^{-2\pi i k_{2:n-1} \cdot r_{2:n-1}} \int_{x,y} \\ & \quad \sum_{p+q=k_1} \mathbb{1}_{|p|, |q| \leq m} (-2\pi i p) e^{-2\pi i(p x + q y)} \varphi_n(x, y, r_{2:n-1}) \\ &= - \sum_{p+q=k_1} \mathbb{1}_{|k_1|, |p|, |q| \leq m} 2\pi i p \hat{\varphi}_n(p, q, k_{2:n-1}) \\ &= - \sum_{p+q=k_1} \mathbb{1}_{|k_1|, |p|, |q| \leq m} \pi i(p + q) \hat{\varphi}_n(p, q, k_{2:n-1}) \\ &= - \sum_{p+q=k_1} \mathbb{1}_{|k_1|, |p|, |q| \leq m} \pi i k_1 \hat{\varphi}_n(p, q, k_{2:n-1}), \end{aligned}$$

from where our representation for \mathcal{G}_-^m follows. □

2.2 Estimates for the Burgers drift

Here we derive some estimates for the Burgers drift. We work with weighted norms on the Fock space. We define for suitable functions f the operators $f(\mathcal{N})$ and $f(\mathcal{L}_0)$ by spectral calculus, which is a complicated way of saying that

$$(f(\mathcal{N})\varphi)_n = f(n)\varphi_n, \quad \mathcal{F}(f(\mathcal{L}_0)\varphi)_n(k) = f(-|2\pi k|^2) \hat{\varphi}_n(k).$$

Lemma 2.8 Fix $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ and let $\varphi \in \Gamma L^2$. Then the following two bounds hold uniformly in m :

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-3/4} \mathcal{G}_-^m \varphi\| \lesssim \|w(\mathcal{N} - 1) \mathcal{N}(-\mathcal{L}_0)^\gamma \varphi\|, \tag{7}$$

for all $\gamma > 1/4$, and

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-3/4} \mathcal{G}_+^m \varphi\| \lesssim \|w(\mathcal{N} + 1) \mathcal{N}(-\mathcal{L}_0)^\gamma \varphi\|, \tag{8}$$

for all $\gamma < 1/2$. Moreover, we have the following m -dependent bound for $\gamma \in [0, 1/2]$:

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-1/2}\mathcal{G}^m\varphi\| \lesssim m^{1/2}\|(w(\mathcal{N} + 1) + w(\mathcal{N} - 1))\mathcal{N}(-\mathcal{L}_0)^\gamma\varphi\|. \tag{9}$$

Proof 1. We start by estimating \mathcal{G}^m uniformly in m . Observe that, by the Cauchy–Schwarz inequality together with Lemma A.1 (which we can apply since $\gamma > 1/4$),

$$\begin{aligned} & \left| \sum_{p+q=k_1} 1_{|k_1|,|p|,|q|\leq m} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \\ & \leq \sum_{p+q=k_1} (p^2 + q^2 + k_2^2 + \dots + k_n^2)^{-2\gamma} \\ & \quad \sum_{p+q=k_1} (p^2 + q^2 + k_2^2 + \dots + k_n^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ & \lesssim (k_1^2 + \dots + k_n^2)^{-2\gamma+1/2} \sum_{p+q=k_1} (p^2 + q^2 + k_2^2 + \dots + k_n^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} & \sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} |\mathcal{F}(\mathcal{G}^m\varphi)_n(k_{1:n})|^2 \\ & \lesssim \sum_{k_{1:n}} \frac{k_1^2}{(k_1^2 + \dots + k_n^2)^{3/2-2\gamma}} n^4 \left| \sum_{p+q=k_1} 1_{|k_1|,|p|,|q|\leq m} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \\ & \lesssim n^4 \sum_{k_{1:n}} \sum_{p+q=k_1} \frac{k_1^2}{k_1^2 + \dots + k_n^2} (p^2 + q^2 + k_2^2 + \dots + k_n^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2. \end{aligned}$$

For $C \geq 0$ the function $x \mapsto \frac{x}{x+C}$ is increasing, and since $k_1^2 \leq 2p^2 + 2q^2$, we have

$$\frac{k_1^2}{k_1^2 + \dots + k_n^2} \leq \frac{2p^2 + 2q^2}{2p^2 + 2q^2 + k_2^2 + \dots + k_n^2} \lesssim \frac{p^2 + q^2}{p^2 + q^2 + k_2^2 + \dots + k_n^2}.$$

This leads to

$$\begin{aligned} & \sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} |\mathcal{F}(\mathcal{G}^m\varphi)_n(k_{1:n})|^2 \\ & \lesssim n^4 \sum_{k_{1:n}} \sum_{p+q=k_1} \frac{p^2 + q^2}{p^2 + q^2 + k_2^2 + \dots + k_n^2} (p^2 + q^2 + k_2^2 + \dots + k_n^2)^{2\gamma} \\ & \quad |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \end{aligned}$$

$$\begin{aligned}
 &= n^4 \sum_{k_{1:n+1}} \frac{k_1^2 + k_2^2}{k_1^2 + \dots + k_{n+1}^2} (k_1^2 + \dots + k_{n+1}^2)^{2\gamma} |\hat{\varphi}_{n+1}(k_{1:n+1})|^2 \\
 &\lesssim n^3 \sum_{k_{1:n+1}} (k_1^2 + \dots + k_{n+1}^2)^{2\gamma} |\hat{\varphi}_{n+1}(k_{1:n+1})|^2,
 \end{aligned}$$

where in the last step we used the symmetry of $\hat{\varphi}_{n+1}$ in the variables $k_{1:n+1}$. Therefore, uniformly in m , we have that

$$\begin{aligned}
 \|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-3/4} \mathcal{G}_-^m \varphi\|^2 &\simeq \sum_{n \geq 0} n! w(n)^2 \sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} |\mathcal{F}(\mathcal{G}_-^m \varphi)_n(k_{1:n})|^2 \\
 &\lesssim \sum_{n \geq 0} n! w(n)^2 n^3 \sum_{k_{1:n+1}} (k_1^2 + \dots + k_{n+1}^2)^{2\gamma} |\hat{\varphi}_{n+1}(k_{1:n+1})|^2 \\
 &\lesssim \sum_{n \geq 1} n! w(n-1)^2 n^2 \sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma} |\hat{\varphi}_n(k_{1:n})|^2 \\
 &= \|w(\mathcal{N}-1) \mathcal{N}(-\mathcal{L}_0)^\gamma \varphi\|^2.
 \end{aligned}$$

- To derive the corresponding bound for \mathcal{G}_+^m , we apply Lemma A.1 in the fourth line below (using that $2\gamma - 3/2 < -1/2$ since $\gamma < 1/2$):

$$\begin{aligned}
 &\sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} |\mathcal{F}(\mathcal{G}_+^m \varphi)_n(k_{1:n})|^2 \\
 &\lesssim \sum_{k_{1:n}} \mathbb{1}_{|k_1|, |k_2|, |k_1+k_2| \leq m} n^2 (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} |k_1 + k_2|^2 |\hat{\varphi}_{n-1}(k_1 + k_2, k_{3:n})|^2 \\
 &\lesssim n^2 \sum_{\ell, k_{3:n}} \sum_{k_1+k_2=\ell} (k_1^2 + \dots + k_n^2)^{2\gamma-3/2} \ell^2 |\hat{\varphi}_{n-1}(\ell, k_{3:n})|^2 \\
 &\lesssim n^2 \sum_{\ell, k_{3:n}} (\ell^2 + k_3^2 + \dots + k_n^2)^{2\gamma-1} \ell^2 |\hat{\varphi}_{n-1}(\ell, k_{3:n})|^2 \\
 &\lesssim n \sum_{k_{1:n-1}} (k_1^2 + \dots + k_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(k_{1:n-1})|^2.
 \end{aligned}$$

Since $\mathcal{G}_+^m \varphi_0 = 0$ we only have to consider $n \geq 2$ and thus we can bound the factor n by $2(n-1)$ and we obtain the following estimate:

$$\|w(\mathcal{N})(-\mathcal{L}_0)^{\gamma-3/4} \mathcal{G}_+^m \varphi\| \lesssim \|w(\mathcal{N}+1) \mathcal{N}(-\mathcal{L}_0)^\gamma \varphi\|.$$

- If we use the cutoff in m to gain regularity in k , we get for $\gamma \leq 1/2$ (so $2\gamma - 1 \leq 0$):

$$\begin{aligned}
 &\sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma-1} |\mathcal{F}(\mathcal{G}_+^m \varphi)_n(k_{1:n})|^2 \\
 &\lesssim n^2 \sum_{k_{1:n}} \mathbb{1}_{|k_1|, |k_2|, |k_1+k_2| \leq m} (k_1^2 + \dots + k_n^2)^{2\gamma-1} |k_1 + k_2|^2 |\hat{\varphi}_{n-1}(k_1 + k_2, k_{3:n})|^2
 \end{aligned}$$

$$\begin{aligned}
 &= n^2 \sum_{k_{1:n-1}} \sum_{\ell_1+\ell_2=k_1} \mathbb{1}_{|\ell_1|,|\ell_2|,|\ell_1+\ell_2|\leq m} (\ell_1^2 + \ell_2^2 + k_2^2 + \dots + k_{n-1}^2)^{2\gamma-1} \\
 &\quad k_1^2 |\hat{\varphi}_{n-1}(k_{1:n-1})|^2 \\
 &\lesssim n^2 m \sum_{k_{1:n-1}} (k_1^2 + \dots + k_{n-1}^2)^{2\gamma-1} k_1^2 |\hat{\varphi}_{n-1}(k_{1:n-1})|^2 \\
 &\lesssim nm \sum_{k_{1:n-1}} (k_1^2 + \dots + k_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(k_{1:n-1})|^2,
 \end{aligned}$$

and thus $\|w(\mathcal{N})(-\mathcal{L}_0)^{-\gamma} \mathcal{G}_+^m \varphi\| \lesssim m^{1/2} \|w(\mathcal{N} + 1)\mathcal{N}(-\mathcal{L}_0)^{-\gamma+1/2} \varphi\|$.

By making similar use of the cutoff $\mathbb{1}_{|p|,|q|\leq m}$, we also obtain the claimed bound for \mathcal{G}_-^m : We estimate

$$\left| \sum_{p+q=k_1} \mathbb{1}_{|k_1|,|p|,|q|\leq m} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \lesssim m \sum_{p+q=k_1} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2,$$

and then proceed with the same arguments as in Step 1 above; note that for $C \geq 0$ the function $x \mapsto \frac{x}{(x+C)^{1-2\gamma}}$ is increasing provided that $1 - 2\gamma \leq 1$, i.e. $\gamma \geq 0$. \square

Remark 2.9 For later reference we note that a slight variation of the first estimate in Step 1 of the proof gives for all $\beta > 1/4$ and $k \in \mathbb{Z}_0^n$:

$$\left| \sum_{p+q=k_1} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \lesssim (k_1^2)^{1/2-2\beta} \sum_{p+q=k_1} (p^2 + q^2)^{2\beta} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2. \tag{10}$$

Remark 2.10 When studying fluctuations of Markov processes, the *graded sector condition* is sometimes useful. This condition assumes that there exists a grading of orthogonal subspaces $L^2(\mu) = \bigoplus_{n \geq 0} \mathcal{A}_n$, such that on each \mathcal{A}_n the quadratic form associated with the full generator can be controlled by the one associated with its symmetric part, see [43, Chapter 2.7.4] for a precise definition. At first glance this might seem tailor made to describe our situation. However, for the graded sector condition we would need

$$|\langle \varphi_n, \mathcal{G}_-^m \varphi_{n+1} \rangle| \lesssim (1+n)^\beta \|(-\mathcal{L}_0)^{1/2} \varphi_n\| \|(-\mathcal{L}_0)^{1/2} \varphi_{n+1}\|$$

for some $\beta < 1$, see [43, eq. (2.45)] while by Lemma 2.8 we can only take $\beta = 1$. Therefore, the condition just barely fails in our setting. On the other hand, we can take $\|(-\mathcal{L}_0)^{1/4} \varphi_n\|$ on the right hand side, and we will leverage this gain in regularity. And while we can allow $\beta = 1$, for $\beta > 1$ the computations in Sect. 3.1 would not work.

Corollary 2.11 *Let*

$$\mathcal{D}_{\text{naive}}(\mathcal{L}^m) := \{\psi \in \Gamma L^2 : \|\mathcal{N}(-\mathcal{L}_0)^{1/2} \varphi\| + \|(-\mathcal{L}_0) \varphi\| < \infty\},$$

and let $\hat{\mathcal{L}}^m$ be the infinitesimal generator of the continuous contraction semi-group $(T_t^m)_{t \geq 0}$ on ΓL^2 , with domain $\mathcal{D}(\hat{\mathcal{L}}^m)$. Then $\mathcal{D}_{\text{naive}}(\mathcal{L}^m) \subset \mathcal{D}(\hat{\mathcal{L}}^m)$ and $\hat{\mathcal{L}}^m|_{\mathcal{D}_{\text{naive}}(\mathcal{L}^m)} = \mathcal{L}^m$.

Proof Let u^m be the process from Lemma 2, with initial condition $u_0^m = u$. If $\varphi \in \mathcal{C} \cap \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$, then

$$T_t^m \varphi(u) - \varphi(u) = \mathbb{E}_u \left[\int_0^t \mathcal{L}^m \varphi(u_s^m) ds \right] = \int_0^t T_s^m (\mathcal{L}^m \varphi)(u) ds.$$

For general $\varphi \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ the identity $T_t^m \varphi - \varphi = \int_0^t T_s^m (\mathcal{L}^m \varphi) ds$ holds by approximation (with a Bochner integral in $L^2(\mu)$ on the right hand side), using our m -dependent estimate (9) for \mathcal{G}^m . By Lemma 2.2 the map $s \mapsto T_s^m \mathcal{L}^m \varphi \in L^2(\mu)$ is continuous, and thus $t^{-1}(T_t^m \varphi - \varphi) \rightarrow \mathcal{L}^m \varphi$ in $L^2(\mu)$ as $t \rightarrow 0$. It follows that $\varphi \in \mathcal{D}(\hat{\mathcal{L}}^m)$ and $\hat{\mathcal{L}}^m \varphi = \mathcal{L}^m \varphi$. □

As the notation $\mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ suggests, we will later introduce another, smaller domain $\mathcal{D}(\mathcal{L}^m)$ on which we have better estimates.

2.3 Controlled functions

Lemma 2.8 gives bounds for $\mathcal{G}^m \varphi$ that are either in distributional spaces, or they diverge with m . Therefore, we can only define the limiting operator \mathcal{G} with values in distributional spaces:

Definition 2.12 Fix $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ and $\gamma > 1/4$. We define the bounded linear operator

$$\mathcal{G}_- : w(\mathcal{N})^{-1}(1 - \mathcal{L}_0)^{-\gamma} \Gamma L^2 \rightarrow w(\mathcal{N})^{-1} \mathcal{N}(-\mathcal{L}_0)^{3/4-\gamma} \Gamma L^2, \quad \mathcal{G}_- \varphi = \lim_{m \rightarrow \infty} \mathcal{G}_-^m \varphi,$$

where with the dominated convergence theorem it is not difficult to see that the convergence holds in $w(\mathcal{N})^{-1} \mathcal{N}(-\mathcal{L}_0)^{3/4-\gamma} \Gamma L^2$. For $\gamma < 1/2$, we also define the bounded linear operator

$$\mathcal{G}_+ : w(\mathcal{N})^{-1}(1 - \mathcal{L}_0)^{-\gamma} \Gamma L^2 \rightarrow w(\mathcal{N})^{-1} \mathcal{N}(-\mathcal{L}_0)^{3/4-\gamma} \Gamma L^2, \quad \mathcal{G}_+ \varphi = \lim_{m \rightarrow \infty} \mathcal{G}_+^m \varphi,$$

again with convergence in $w(\mathcal{N})^{-1} \mathcal{N}(-\mathcal{L}_0)^{3/4-\gamma} \Gamma L^2$. In particular, we get for $\delta > 0$ and $\mathcal{G} := \mathcal{G}_- + \mathcal{G}_+$ and $\varphi \in \Gamma L^2$:

$$\|(-\mathcal{L}_0)^{-1/4-\delta} \mathcal{G} \varphi\| \lesssim \|\mathcal{N}(-\mathcal{L}_0)^{1/2} \varphi\|.$$

The problem is that $\mathcal{G} \varphi$ lives in a distributional space and not in ΓL^2 . But on the other hand we do not care so much about \mathcal{G} itself and we are mainly interested in the sum $\mathcal{L} := \mathcal{L}_0 + \mathcal{G}$. To construct a domain that is mapped to ΓL^2 by \mathcal{L} we will consider functions φ for which $\mathcal{G} \varphi$ and $\mathcal{L}_0 \varphi$ have some cancellations, so in particular $\mathcal{L}_0 \varphi$ will be a distribution and φ will be non-smooth.

This problem has some similarities to the problem of constructing a finite-dimensional diffusion with distributional drift b and additive noise. In that case the formal generator is $\frac{1}{2}\Delta + b \cdot \nabla$, and we can construct a domain by solving the resolvent equation $(\lambda - \frac{1}{2}\Delta)u = b \cdot \nabla u + v$ for suitable v and for $\lambda > 0$.

In our case we could start with a nice function $\psi \in \Gamma L^2$ and try to solve the resolvent equation for $\lambda > 0$:

$$(\lambda - \mathcal{L}_0)\varphi = \mathcal{G}\varphi + \psi \Leftrightarrow \varphi = (\lambda - \mathcal{L}_0)^{-1}\mathcal{G}\varphi + (\lambda - \mathcal{L}_0)^{-1}\psi.$$

Then we would get $\mathcal{L}\varphi = \lambda\varphi - \psi$, and the right hand side is in ΓL^2 for $\varphi, \psi \in \Gamma L^2$. If we only consider the regularity with respect to \mathcal{L}_0 and for now we ignore the behavior with respect to \mathcal{N} , then the resolvent equation is actually in the ‘‘Young regime’’: $\mathcal{G}\varphi$ is well defined whenever $\varphi \in (-\mathcal{L}_0)^{-1/2}\Gamma L^2$, and then \mathcal{G} loses $(-\mathcal{L}_0)^{3/4+\delta}$ ‘‘derivatives’’, for any $\delta > 0$. So if $\delta \leq 1/4$, then $(\lambda - \mathcal{L}_0)^{-1}$ gains enough regularity to map back to $(-\mathcal{L}_0)^{-1/2}\Gamma L^2$. But in this formal discussion we ignored the behavior with respect to \mathcal{N} , and we are unable to close the estimates because \mathcal{G} introduces growth in \mathcal{N} which cannot be cured by applying $(\lambda - \mathcal{L}_0)^{-1}$: Indeed, we actually have $\mathcal{G}\varphi \in \mathcal{N}(-\mathcal{L}_0)^{1/4+\delta}\Gamma L^2$ and not $\mathcal{G}\varphi \in (-\mathcal{L}_0)^{1/4+\delta}\Gamma L^2$.

To overcome this problem, we introduce an approximation \mathcal{G}^γ of \mathcal{G} which captures the singular part of the small scale behavior of \mathcal{G} by letting

$$\mathcal{F}(\mathcal{G}^\gamma \varphi)_n(k_{1:n}) := \mathbb{1}_{|k_{1:n}|_\infty \geq N_n} \mathcal{F}(\mathcal{G}\varphi)_n(k_{1:n})$$

for a suitable (\mathcal{N} -dependent) cutoff N_n to be determined. The advantage of \mathcal{G}^γ is that the cutoff $\mathbb{1}_{|k_{1:n}|_\infty \geq N_n}$ allows us to ‘‘trade spare regularity’’ in $(-\mathcal{L}_0)$ against regularity in \mathcal{N} . Using \mathcal{G}^γ we introduce a controlled Ansatz of the form

$$\varphi = (-\mathcal{L}_0)^{-1}\mathcal{G}^\gamma \varphi + \varphi^\sharp, \tag{11}$$

where φ^\sharp will be chosen sufficiently regular. Note that this is essentially the resolvent equation for $\lambda = 0$ and $\psi = (-\mathcal{L}_0)\varphi^\sharp$, except that we replaced \mathcal{G} with \mathcal{G}^γ . A useful intuition about the Ansatz (11) is that, starting from a given test function φ^\sharp , it ‘‘prepares’’ functions φ which have the right small scale behavior compatible with the operator \mathcal{L} .

We start by showing that for an appropriate cutoff N_n we can solve Eq. (11) and express φ as a function of φ^\sharp .

Definition 2.13 A *weight* is a map $w: \mathbb{N}_0 \rightarrow (0, \infty)$ such that there exists $C > 0$ with $w(n) \leq Cw(n+i)$, for $i \in \{-1, 1\}$, uniformly in n . In that case we write $|w|$ for the smallest such constant C .

Lemma 2.14 *Let w be a weight, let $\gamma \in (1/4, 1/2]$, and let $L \geq 1$. For $N_n = L(1+n)^3$ we have*

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1}\mathcal{G}^\gamma \varphi\| \lesssim |w|L^{-1/2}\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\|. \tag{12}$$

Thus there exists $L_0 = L_0(|w|)$ such that for all $L \geq L_0$, and all φ^\sharp with $\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\| < \infty$, there is a unique solution $\mathcal{K}\varphi^\sharp$ to

$$\mathcal{K}\varphi^\sharp = (-\mathcal{L}_0)^{-1} \mathcal{G}^\succ \mathcal{K}\varphi^\sharp + \varphi^\sharp$$

in $w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$, and $\mathcal{K}\varphi^\sharp$ satisfies

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \mathcal{K}\varphi^\sharp\| + |w|^{-1} L^{1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (\mathcal{K}\varphi^\sharp - \varphi^\sharp)\| \lesssim \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\|. \tag{13}$$

We also write $\varphi^\succ := \mathcal{K}\varphi^\sharp - \varphi^\sharp = (-\mathcal{L}_0)^{-1} \mathcal{G}^\succ \mathcal{K}\varphi^\sharp$.

Proof 1. We start by estimating \mathcal{G}_+^\succ (which is defined like \mathcal{G}^\succ , only with \mathcal{G}_+ in place of \mathcal{G}):

$$\begin{aligned} & \sum_{k_{1:n}} |\mathcal{F}((-\mathcal{L}_0)^\gamma \mathcal{G}_+^\succ \varphi)_n(k_{1:n})|^2 \\ & \lesssim n^2 \sum_{k_{1:n}} \mathbb{1}_{|k_{1:n}|_\infty \geq N_n} \frac{(k_1 + k_2)^2}{(k_1^2 + \dots + k_n^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(k_1 + k_2, k_{3:n})|^2 \\ & \leq n^2 \sum_{\ell_{1:n-1}, p} \mathbb{1}_{|\ell_{1:n-1}|_\infty \vee |p| \geq N_n/2} \frac{\ell_1^2}{((\ell_1 - p)^2 + p^2 + \ell_2^2 + \dots + \ell_{n-1}^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2, \end{aligned}$$

where we used the change of variables $\ell_1 = k_1 + k_2$, $p = k_2$, and $\ell_{2+i} = k_{1+i}$ for $i \geq 0$, and we used that $|p| \vee |\ell_1 - p| \geq N_n$ implies $|p| \vee |\ell_1| \geq N_n/2$. Since $(\ell_1 - p)^2 + p^2 \simeq \ell_1^2 + p^2$ we can replace $((\ell_1 - p)^2 + p^2 + \ell_2^2 + \dots + \ell_{n-1}^2)^{-(2-2\gamma)}$ by $(p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{-(2-2\gamma)}$. And since $1 - 2\gamma \geq 0$, we have

$$\ell_1^2 + \dots + \ell_{n-1}^2 \leq (\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma} (p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{1-2\gamma}. \tag{14}$$

We now use the symmetry of $\hat{\varphi}_{n-1}(\ell_{1:n-1})$ in $\ell_{1:n-1}$, before applying (14) and Lemma A.1, to derive the estimate

$$\begin{aligned} & n^2 \sum_{\ell_{1:n-1}, p} \mathbb{1}_{|\ell_{1:n-1}|_\infty \vee |p| \geq N_n/2} \frac{\ell_1^2}{(p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ & \lesssim n \sum_{\ell_{1:n-1}, p} \mathbb{1}_{|\ell_{1:n-1}|_\infty \vee |p| \geq N_n/2} \frac{\ell_1^2 + \dots + \ell_{n-1}^2}{(p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ & \leq n \sum_{\ell_{1:n-1}, p} (\mathbb{1}_{|p| \geq N_n/2} + \mathbb{1}_{|\ell_{1:n-1}|_\infty \geq N_n/2}) \frac{(\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma}}{p^2 + \ell_1^2 + \dots + \ell_{n-1}^2} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ & \lesssim n \sum_{\ell_{1:n-1}} \left(\sum_{|p| \geq N_n/2} \frac{1}{p^2} + \frac{\mathbb{1}_{|\ell_{1:n-1}|_\infty \geq N_n/2}}{(\ell_1^2 + \dots + \ell_{n-1}^2)^{1/2}} \right) (\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ & \lesssim n \sum_{\ell_{1:n-1}} N_n^{-1} (\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2. \end{aligned}$$

Thus, with our choice of $N_n = L(1 + n)^3$,

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}_+^\gamma \varphi\| \lesssim |w|L^{-1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\|. \tag{15}$$

2. Next, we bound \mathcal{G}_-^γ . We apply (10) with $\beta = \gamma$ (here we need $\gamma > 1/4$) to estimate

$$\begin{aligned} & \sum_{k_{1:n}} |\mathcal{F}((-\mathcal{L}_0)^\gamma)^{-1} \mathcal{G}_-^\gamma \varphi)_n(k_{1:n})|^2 \\ & \lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 k_1^2}{(k_1^2 + \dots + k_n^2)^{2-2\gamma}} \left| \sum_{p+q=k_1} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \\ & \lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 k_1^2 (k_1^2)^{3/2-2\gamma-1}}{(k_1^2 + \dots + k_n^2)^{2-2\gamma}} \sum_{p+q=k_1} (p^2 + q^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ & \lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 (k_1^2)^{3/2}}{(k_1^2 + \dots + k_n^2)^2} \sum_{p+q=k_1} (p^2 + q^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ & \leq N_n^{-1} n^4 \sum_{\ell_{1:n+1}} (\ell_1^2 + \dots + \ell_{n+1}^2)^{2\gamma} |\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2, \end{aligned}$$

which together with $N_n = L(1 + n)^3$ leads to the bound

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}_-^\gamma \varphi\| \lesssim |w|L^{-1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\|. \tag{16}$$

The claimed inequality (12) now follows by combining (15) and (16).

3. Consequently, for given $\varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$, the map

$$\Psi : w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2 \ni \psi \mapsto (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \psi + \varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$$

satisfies for some $K > 0$

$$\begin{aligned} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \Psi(\psi)\| & \leq \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \psi\| + \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\| \\ & \leq K|w|L^{-1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \psi\| + \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\| < \infty. \end{aligned}$$

In particular, Ψ is well defined, and if L is large enough so that $K|w|L^{-1/2} \leq 1/2$, then Ψ is a contraction leaving the ball with radius $2\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\|$ invariant. Therefore, it has a unique fixed point $\mathcal{K}\varphi^\sharp$ which satisfies

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \mathcal{K}\varphi^\sharp\| \leq 2\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\|.$$

Since $\mathcal{K}\varphi^\sharp$ is a fixed point, we also get

$$\begin{aligned} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (\mathcal{K}\varphi^\sharp - \varphi^\sharp)\| & = \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \mathcal{K}\varphi^\sharp\| \\ & \lesssim |w|L^{-1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\|. \end{aligned}$$

□

Remark 2.15 The lemma shows that for all $\varphi \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$ we can define $\varphi^\sharp := \varphi - (-\mathcal{L}_0)^{-1} \mathcal{G}^\succ \varphi$ and then

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi^\sharp\| \lesssim \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\|.$$

However, this only works up to $\gamma = 1/2$, so no matter how regular φ is, the (spatial) regularity of φ^\sharp is limited in general. The key point of Lemma 2.14 is that it identifies a class of φ for which φ^\sharp has arbitrarily good regularity.

Remark 2.16 The cutoff N_n for which we can construct $\mathcal{K}\varphi^\sharp$ depends on the weight w via $|w|$; we say that the cutoff is *adapted to the weight w* if the construction of Lemma 2.14 works. If we consider weights $w(n) = (1 + n)^\alpha$, then $|w|$ is uniformly bounded in $|\alpha| \leq K$, for any fixed K , and we can find one cutoff which is adapted to all those weights. This is the situation that we are mostly interested in.

Remark 2.17 The bound (12) also holds for $\mathcal{G}^{m,\succ}$, which is defined analogously to \mathcal{G}^\succ . Therefore, we can also construct a map $\mathcal{K}^m : w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2 \rightarrow w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$ that associates to every $\varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$ a unique $\mathcal{K}^m \varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\gamma} \Gamma L^2$ with

$$\mathcal{K}^m \varphi^\sharp = (-\mathcal{L}_0)^{-1} \mathcal{G}^{m,\succ} \mathcal{K}^m \varphi^\sharp + \varphi^\sharp.$$

Let us write $\mathcal{G}^\prec = \mathcal{G} - \mathcal{G}^\succ$. The following proposition gives a bound for $\mathcal{L}\mathcal{K}\varphi^\sharp$ in terms of φ^\sharp . By Remark 2.17, similar bounds hold for $\mathcal{L}^m \mathcal{K}^m \varphi^\sharp$, uniformly in m .

Proposition 2.18 *Let w be a weight, let $\gamma \geq 0$, and let the cutoff N_n be adapted to w and $(w(n)(1 + n)^{9/2+7\gamma})_n$, and let $\delta > 0$. Consider*

$$\varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1} \Gamma L^2 \cap w(\mathcal{N})^{-1}(1 + \mathcal{N})^{-9/2-7\gamma} (-\mathcal{L}_0)^{-1/4-\delta} \Gamma L^2.$$

We set $\varphi := \mathcal{K}\varphi^\sharp$. Then $\mathcal{L}\varphi := \mathcal{L}_0\varphi^\sharp + \mathcal{G}^\prec \varphi$ is a well defined operator, and we have

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \mathcal{G}^\prec \varphi\| \lesssim \|w(\mathcal{N})(1 + \mathcal{N})^{9/2+7\gamma} (-\mathcal{L}_0)^{1/4+\delta} \varphi^\sharp\|. \tag{17}$$

Proof We treat \mathcal{G}_+^\prec and \mathcal{G}_-^\prec separately (both with their obvious definition). We also assume that $\delta \in (0, 1/4]$, but once we established the bound (17) for such δ it holds of course also for $\delta > 1/4$.

1. To control $\mathcal{G}_+^\prec \varphi$, we bound

$$\begin{aligned} & \sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma} |\mathcal{F}(\mathcal{G}_+^\prec \varphi)_n(k_{1:n})|^2 \\ & \lesssim n^2 \sum_{k_{1:n}} \mathbb{1}_{|k_{1:n}|_\infty < N_n} (k_1^2 + \dots + k_n^2)^{2\gamma} |k_1 + k_2|^2 |\hat{\varphi}_{n-1}(k_1 + k_2, k_{3:n})|^2 \end{aligned}$$

$$\begin{aligned} &\lesssim n^2 \sum_{k_{1:n}} \mathbb{1}_{|k_{1:n}|_\infty < N_n} N_n^{4\gamma} n^{2\gamma} (|k_1 + k_2|^2)^{1/2+2\delta} N_n^{1-4\delta} |\hat{\varphi}_{n-1}(k_1 + k_2, k_{2:n-1})|^2 \\ &\lesssim n^{2+2\gamma} N_n^{2+4\gamma-4\delta} \sum_{k_{1:n-1}} (k_1^2 + \dots + k_{n-1}^2)^{1/2+2\delta} |\hat{\varphi}_{n-1}(k_{1:n-1})|^2, \end{aligned}$$

and since $N_n \simeq (n+1)^3$ we get $\|w(\mathcal{N})\mathcal{G}_\pm^\leq\varphi\| \lesssim \|w(\mathcal{N})(1+\mathcal{N})^{9/2+7\gamma}(-\mathcal{L}_0)^{1/4+\delta}\varphi\|$. Applying Lemma 2.14, we can estimate the right hand side by $\|w(\mathcal{N})(1+\mathcal{N})^{9/2+7\gamma}(-\mathcal{L}_0)^{1/4+\delta}\varphi^\sharp\|$, because we assumed that $\delta \in (0, 1/4]$.

2. Next, let us estimate $\mathcal{G}_\pm^\leq\varphi$. As usual we apply (10), this time with $\beta = 1/4 + \delta > 1/4$, to bound

$$\begin{aligned} &\sum_{k_{1:n}} (k_1^2 + \dots + k_n^2)^{2\gamma} |\mathcal{F}(\mathcal{G}_\pm^\leq\varphi)_n(k_{1:n})|^2 \\ &\lesssim (n+1)^4 \sum_{k_{1:n}} \mathbb{1}_{|k_{1:n}|_\infty < N_n} k_1^2 (k_1^2 + \dots + k_n^2)^{2\gamma} \left| \sum_{p+q=k_1} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \\ &\lesssim (n+1)^4 \sum_{k_{1:n}} \mathbb{1}_{|k_{1:n}|_\infty < N_n} k_1^2 (k_1^2 + \dots + k_n^2)^{2\gamma} |k_1|^{-4\delta} \\ &\quad \sum_{p+q=k_1} (p^2 + q^2)^{1/2+2\delta} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ &\lesssim (n+1)^{4+2\gamma} N_n^{2+4\gamma} \sum_{\ell_{1:n+1}} (\ell_1^2 + \dots + \ell_{n+1}^2)^{1/2+2\delta} |\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2, \end{aligned}$$

from where we deduce as before that $\|w(\mathcal{N})\mathcal{G}_\pm^\leq\varphi\| \lesssim \|w(\mathcal{N})(\mathcal{N}+1)^{9/2+7\gamma}(-\mathcal{L}_0)^{1/4+\delta}\varphi^\sharp\|$.

□

To simplify the notation we write from now for $\gamma \geq 0$

$$\alpha(\gamma) := 9/2 + 7\gamma. \tag{18}$$

Lemma 2.19 *For a given weight w and a cutoff as in Proposition 2.18 (for $\gamma = 0$), we set*

$$\mathcal{D}_w(\mathcal{L}) := \{\mathcal{K}\varphi^\sharp : \varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1}\Gamma L^2 \cap w(\mathcal{N})^{-1}(1+\mathcal{N})^{-9/2}(-\mathcal{L}_0)^{-1/2}\Gamma L^2\}.$$

Then $\mathcal{D}_w(\mathcal{L})$ is dense in $w(\mathcal{N})^{-1}\Gamma L^2$. More precisely, for all

$$\psi \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1}\Gamma L^2 \cap w(\mathcal{N})^{-1}(1+\mathcal{N})^{-9/2}(-\mathcal{L}_0)^{-1/2}\Gamma L^2,$$

and for all $M \geq 1$ there exists $\varphi^M \in \mathcal{D}_w(\mathcal{L})$ such that

$$\begin{aligned} \|w(\mathcal{N})(-\mathcal{L}_0)^{1/2}(\varphi^M - \psi)\| &\lesssim M^{-1/2}\|w(\mathcal{N})(-\mathcal{L}_0)^{1/2}\psi\|, \\ \|w(\mathcal{N})(-\mathcal{L}_0)^{1/2}\varphi^M\| &\lesssim \|w(\mathcal{N})(-\mathcal{L}_0)^{1/2}\psi\|, \\ \|w(\mathcal{N})\mathcal{L}\varphi^M\| &\lesssim M^{1/2}(\|w(\mathcal{N})(-\mathcal{L}_0)\psi\| \\ &\quad + \|w(\mathcal{N})(\mathcal{N} + 1)^{9/2}(-\mathcal{L}_0)^{1/2}\psi\|). \end{aligned} \tag{19}$$

If $w \equiv 1$, we simply write $\mathcal{D}(\mathcal{L})$.

Proof Let ψ be as in the statement of the lemma. Since such ψ are dense in $w(\mathcal{N})^{-1}\Gamma L^2$ it suffices to construct φ^M such that the inequalities (19) hold. For this purpose we apply Lemma 2.14 to find a unique function $\varphi^M \in w(\mathcal{N})^{-1}\Gamma L^2$ that satisfies

$$\hat{\varphi}_n^M(k_{1:n}) = \mathbb{1}_{|k|_\infty \geq MN_n} \mathcal{F}((-\mathcal{L}_0)^{-1}\mathcal{G}\varphi^M)_n(k_{1:n}) + \hat{\psi}_n(k_{1:n}),$$

and for which the first two estimates in (19) hold by Lemma 2.14. To see that $\varphi^M \in \mathcal{D}_w(\mathcal{L})$ note that

$$\hat{\varphi}_n^M(k_{1:n}) = \mathcal{F}((-\mathcal{L}_0)^{-1}\mathcal{G}^\succ\varphi^M)_n(k_{1:n}) + \hat{\varphi}_n^{M,\sharp}(k_{1:n}),$$

where

$$\hat{\varphi}_n^{M,\sharp}(k_{1:n}) = \hat{\psi}_n(k_{1:n}) - \mathbb{1}_{N_n \leq |k|_\infty < MN_n} \mathcal{F}((-\mathcal{L}_0)^{-1}\mathcal{G}\varphi^M)_n(k_{1:n}).$$

In particular we have $\mathcal{L}\varphi^M = \mathcal{G}^\prec\varphi^M + \mathcal{L}_0\varphi^{M,\sharp}$, and by Proposition 2.18 it suffices to estimate $\varphi^{M,\sharp}$ in $w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1}\Gamma L^2 \cap w(\mathcal{N})^{-1}(1 + \mathcal{N})^{-9/2}(-\mathcal{L}_0)^{-1/2}\Gamma L^2$. The first contribution ψ satisfies the required bounds by assumption, so it suffices to show that the second contribution, denote it as ψ^M , satisfies

$$\begin{aligned} \|w(\mathcal{N})(-\mathcal{L}_0)\psi^M\| &\lesssim M^{1/2}\|w(\mathcal{N})(1 + \mathcal{N})^{9/2}(-\mathcal{L}_0)^{1/2}\psi\|, \\ \|w(\mathcal{N})(\mathcal{N} + 1)^{9/2}(-\mathcal{L}_0)^{1/2}\psi^M\| &\lesssim \|w(\mathcal{N})(1 + \mathcal{N})^{9/2}(-\mathcal{L}_0)^{1/2}\psi\|. \end{aligned} \tag{20}$$

But

$$\mathcal{F}((-\mathcal{L}_0)\psi^M)_n(k_{1:n}) = -\mathbb{1}_{N_n \leq |k|_\infty < MN_n} \mathcal{F}(\mathcal{G}\varphi^M)_n(k_{1:n}),$$

so that we can estimate this term similarly as in (9). If the cutoff MN_n was independent of n , we would get $\|w(\mathcal{N})(-\mathcal{L}_0)\psi^M\| \lesssim (MN_n)^{1/2}\|w(\mathcal{N})(1 + \mathcal{N})(-\mathcal{L}_0)^{1/2}\varphi^M\|$ from (9), so after including the factor $N_n \simeq (1 + n)^3$ into the weight we get

$$\|w(\mathcal{N})(-\mathcal{L}_0)\psi^M\| \lesssim M^{1/2}\|w(\mathcal{N})(1 + \mathcal{N})^{5/2}(-\mathcal{L}_0)^{1/2}\varphi^M\|,$$

and then the first estimate of (20) follows from (13). Similarly

$$\begin{aligned} |\mathcal{F}((-\mathcal{L}_0)^{1/2}\psi^M)_n(k_{1:n})|^2 &\simeq (k_1^2 + \dots + k_n^2)^{-1} \mathbb{1}_{N_n \leq |k|_\infty < MN_n} |\mathcal{F}(\mathcal{G}\varphi^M)_n(k_{1:n})|^2 \\ &\lesssim N_n^{-1} |\mathcal{F}((-\mathcal{L}_0)^{-1/4}\mathcal{G}_-\varphi^M)_n(k_{1:n})|^2 \\ &\quad + N_n^{-2/3} |\mathcal{F}((-\mathcal{L}_0)^{-1/3}\mathcal{G}_+\varphi^M)_n(k_{1:n})|^2, \end{aligned}$$

and since $N_n \simeq (1 + n)^3$ we get with (7), (8) that

$$\begin{aligned} &\|w(\mathcal{N})(1 + \mathcal{N})^{9/2}(-\mathcal{L}_0)^{1/2}\psi^M\| \\ &\lesssim \|w(\mathcal{N})[(1 + \mathcal{N})^4(-\mathcal{L}_0)^{1/2} + (1 + \mathcal{N})^{9/2}(-\mathcal{L}_0)^{-5/12}]\varphi^M\|, \end{aligned}$$

which together with (13) yields (20) and then (19). □

Remark 2.20 As discussed before, our analysis also works for \mathcal{L}^m and we define

$$\mathcal{D}_w(\mathcal{L}^m) := \{\mathcal{K}^m\varphi^\sharp : \varphi^\sharp \in w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1}\Gamma L^2 \cap w(\mathcal{N})^{-1}(1 + \mathcal{N})^{-9/2}(-\mathcal{L}_0)^{-1/2}\Gamma L^2\}.$$

For $w \equiv 1$ we also write $\mathcal{D}(\mathcal{L}^m)$, and in that case $\mathcal{D}(\mathcal{L}^m) \subset \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ for the domain $\mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ of Corollary 2.11: Indeed, we have $\|(-\mathcal{L}_0)^{1/2}\mathcal{N}\mathcal{K}^m\varphi^\sharp\| \lesssim \|(-\mathcal{L}_0)^{1/2}\mathcal{N}\varphi^\sharp\|$ by (13). Moreover,

$$\|(-\mathcal{L}_0)\mathcal{K}^m\varphi^\sharp\| \leq \|(-\mathcal{L}_0)(-\mathcal{L}_0)^{-1}\mathcal{G}^{m,\succ}\mathcal{K}^m\varphi^\sharp\| + \|(-\mathcal{L}_0)\varphi^\sharp\|,$$

and the second term on the right hand side is finite by assumption. For first term on the right hand side we apply the m -dependent estimate (9) (which also holds for $\mathcal{G}^{m,\succ}$) and then (13) to bound

$$\|\mathcal{G}^{m,\succ}\mathcal{K}^m\varphi^\sharp\| \lesssim m^{1/2}\|(-\mathcal{L}_0)^{1/2}\mathcal{K}^m\varphi^\sharp\| \lesssim m^{1/2}\|(-\mathcal{L}_0)^{1/2}\varphi^\sharp\|.$$

Remark 2.21 The same construction works for the operator $\mathcal{L}^{(\lambda)} = \mathcal{L}_0 + \lambda\mathcal{G}$ for $\lambda \in \mathbb{R}$. For $\lambda \neq 1$ the intersection of the resulting domain $\mathcal{D}(\mathcal{L}^{(\lambda)})$ with $\mathcal{D}(\mathcal{L})$ consists only of constants.

Lemma 2.22 For any $\varphi \in \mathcal{D}(\mathcal{L})$, we have

$$\langle \varphi, \mathcal{L}\varphi \rangle = -\|(-\mathcal{L}_0)^{1/2}\varphi\|^2 \leq 0,$$

and therefore the operator $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ is dissipative.

Proof For $\varphi \in \mathcal{D}(\mathcal{L})$ we have $\mathcal{L}_0\varphi \in \Gamma L^2$ and $\varphi \in (-\mathcal{L}_0)^{-1/2}(1 + \mathcal{N})^{-1}\Gamma L^2$ by assumption. So Definition 2.12 with $\delta = 0$ (for \mathcal{G}_-) respectively $\delta \in (0, 1/4]$ (for \mathcal{G}_+) gives $\mathcal{G}\varphi \in (-\mathcal{L}_0)^{1/2}\Gamma L^2$. Therefore, we can conclude by approximation in the chain of equalities

$$\langle \varphi, \mathcal{L}\varphi \rangle = -\langle \varphi, (-\mathcal{L}_0)\varphi \rangle + \langle \varphi, \mathcal{G}\varphi \rangle = -\langle \varphi, (-\mathcal{L}_0)\varphi \rangle = -\|(-\mathcal{L}_0)^{1/2}\varphi\|^2,$$

since all the inner products are well defined. Here we used the antisymmetry of the form associated to \mathcal{G} (see Lemma 2.4):

$$\langle \varphi, \mathcal{G}\varphi \rangle = \lim_{m \rightarrow \infty} \langle \varphi, \mathcal{G}^m \varphi \rangle = - \lim_{m \rightarrow \infty} \langle \mathcal{G}^m \varphi, \varphi \rangle = -\langle \mathcal{G}\varphi, \varphi \rangle.$$

□

Remark 2.23 We can introduce another dissipative operator \mathcal{L}^- given by $\mathcal{L}^- = \mathcal{L}_0 - \mathcal{G} = \mathcal{L}^{(-1)}$ on the domain $\mathcal{D}(\mathcal{L}^-)$. Then if $\varphi \in \mathcal{D}(\mathcal{L})$ and $\psi \in \mathcal{D}(\mathcal{L}^-)$ we have $\mathcal{L}_0\varphi, \mathcal{G}\varphi, \mathcal{L}_0\psi, \mathcal{G}\psi \in (-\mathcal{L}_0)^{1/2}\Gamma L^2$ and $\varphi, \psi \in (\mathcal{N} + 1)^{-1}(-\mathcal{L}_0)^{-1/2}\Gamma L^2$ so the identities $\mathcal{L}\varphi = \mathcal{L}_0\varphi + \mathcal{G}\varphi, \mathcal{L}^-\psi = \mathcal{L}_0\psi - \mathcal{G}\psi$ hold (as distributions) and

$$\langle \psi, \mathcal{L}\varphi \rangle = \langle \psi, \mathcal{L}_0\varphi \rangle + \langle \psi, \mathcal{G}\varphi \rangle = \langle \psi, \mathcal{L}_0\varphi \rangle - \langle \mathcal{G}\psi, \varphi \rangle = \langle \mathcal{L}^-\psi, \varphi \rangle.$$

As a consequence $\mathcal{L}^- \subseteq \mathcal{L}^*$ and symmetrically $\mathcal{L} \subseteq (\mathcal{L}^-)^*$. The closed operators $\mathcal{L}^*, (\mathcal{L}^-)^*$ are dissipative and satisfy

$$\mathcal{L}^*, (\mathcal{L}^-)^* \leq \mathcal{L}_0$$

in the sense of quadratic forms and on their respective domains.

For finite m the operator $(\mathcal{L}^m)^- := \mathcal{L}_0 - \mathcal{G}^m$ is defined on $\mathcal{D}_{\text{naive}}((\mathcal{L}^m)^-) := \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ and it is the restriction of the adjoint $(\hat{\mathcal{L}}^m)^*$ of \mathcal{L}^m to this domain. Indeed, $(\hat{\mathcal{L}}^m)^*$ is the infinitesimal generator of the time-reversed process $(u^m_{T-t})_{t \in [0, T]}$ for $T > 0$, and this time-reversed process solves Burgers equation with a minus sign in front of the nonlinearity (see [27,34]). So the claim $(\mathcal{L}^m)^- \subset (\hat{\mathcal{L}}^m)^*$ follows by the same arguments as for the forward equation.

3 The Kolmogorov backward equation

So far we constructed a dense domain $\mathcal{D}(\mathcal{L})$ for the operator \mathcal{L} . In this section we will analyze the Kolmogorov backward equation $\partial_t \varphi = \mathcal{L}\varphi$. More precisely we consider the backward equation for the Galerkin approximation (2) with generator $\hat{\mathcal{L}}^m$, and we derive uniform estimates in controlled spaces for the solution. By compactness, this gives the existence of strong solutions to the backward equation after removing the cutoff. Uniqueness easily follows from the dissipativity of \mathcal{L} .

3.1 A priori bounds

Recall that T^m is the semigroup generated by the Galerkin approximation u^m , the solution to (2). First, we consider $\varphi^m(t) = T_t^m \varphi_0^m$, for $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$, and we derive some basic a priori estimates without using our controlled structure. Roughly speaking our aim is to gain some control of the growth in the chaos variable n by making use of the antisymmetry of \mathcal{G} . In the following Sect. 3.2 we then estimate the regularity with respect to $(-\mathcal{L}_0)$ by using the controlled structure.

So let $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \subset \mathcal{D}(\hat{\mathcal{L}}^m)$ and let $\varphi^m(t) = T_t^m \varphi_0^m$ be the solution to the backward equation $\partial_t \varphi^m(t) = \hat{\mathcal{L}}^m \varphi^m(t)$. Unfortunately, we do not know yet if $\varphi^m(t) \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ for $t > 0$, and we have no explicit expression for $\hat{\mathcal{L}}^m \varphi^m(t)$, which makes it difficult to derive estimates.

To circumvent this problem we introduce suitable cutoffs: Let $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ be compactly supported and let $K > 0$, so that $w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K}$ is a bounded linear operator and it commutes with the Fréchet derivative. Then

$$\begin{aligned} \frac{1}{2} \partial_t \|w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m(t)\|^2 &= \langle w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m(t), w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \hat{\mathcal{L}}^m \varphi^m(t) \rangle \\ &= \langle w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m(t), \hat{\mathcal{L}}^m \varphi^m(t) \rangle \\ &= \langle (\hat{\mathcal{L}}^m)^*(w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m(t)), \varphi^m(t) \rangle, \end{aligned}$$

where $(\hat{\mathcal{L}}^m)^*$ is the adjoint of $\hat{\mathcal{L}}^m$. By the discussion in Remark 2.23 we have $(\hat{\mathcal{L}}^m)^* \psi = (\mathcal{L}^m)^- \psi$ for all $\psi \in \mathcal{D}_{\text{naive}}((\mathcal{L}^m)^-) := \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$. And since $w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m(t)$ has only finitely many Fourier modes it is of course in $\mathcal{D}_{\text{naive}}(\mathcal{L}^m)$. This leads to

$$\begin{aligned} \frac{1}{2} \partial_t \|w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m\|^2 &= \langle (\mathcal{L}_0 - \mathcal{G}^m)(w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m), \varphi^m \rangle \\ &= -\|(-\mathcal{L}_0)^{1/2} w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m\|^2 \\ &\quad - \langle \mathcal{G}^m(w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m), \varphi^m \rangle, \end{aligned}$$

where we used that $-\mathcal{L}_0$ is a positive operator which commutes with $w(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K}$. Let now $K > m$, so in particular $\mathcal{G}^m \mathbb{1}_{|\mathcal{L}_0| \leq K} = \mathcal{G}^m$, and let \tilde{w} be compactly supported and such that $\tilde{w}(n)w(n+i) = w(n+i)$ for $i \in \{-1, 0, 1\}$ (i.e. $\tilde{w} \equiv 1$ on a set that is slightly larger than the support of w). Then we get, using the “skew-symmetry” of \mathcal{G}^m (Lemma 2.4),

$$\begin{aligned} &\langle \mathcal{G}^m(w^2(\mathcal{N})\mathbb{1}_{|\mathcal{L}_0| \leq K} \varphi^m), \varphi^m \rangle \\ &= \langle \mathcal{G}_+^m(w^2(\mathcal{N})\varphi^m), \varphi^m \rangle + \langle \mathcal{G}_-^m(w^2(\mathcal{N})\varphi^m), \varphi^m \rangle \\ &= \langle \mathcal{G}_+^m(w^2(\mathcal{N})\varphi^m), \varphi^m \rangle + \langle w^2(\mathcal{N} + 1)\mathcal{G}_-^m(\tilde{w}(\mathcal{N})\varphi^m), \varphi^m \rangle \\ &= \langle \mathcal{G}_+^m(w^2(\mathcal{N})\varphi^m), \varphi^m \rangle - \langle (\tilde{w}(\mathcal{N})\varphi^m), \mathcal{G}_+^m(w^2(\mathcal{N} + 1)\varphi^m) \rangle \\ &= -\langle \mathcal{G}_+^m(h(\mathcal{N} + 1)\varphi^m), \varphi^m \rangle, \end{aligned}$$

where $h(n) = w^2(n) - w^2(n - 1)$. So letting $K \rightarrow \infty$, we see that

$$\frac{1}{2} \partial_t \|w(\mathcal{N})\varphi^m\|^2 = -\|(-\mathcal{L}_0)^{1/2} w(\mathcal{N})\varphi^m\|^2 + \langle \mathcal{G}_+^m(h(\mathcal{N} + 1)\varphi^m), \varphi^m \rangle. \tag{21}$$

Lemma 3.1 *For all $\alpha \in \mathbb{R}$, there exists $C = C(\alpha) > 0$ such that for all $\varphi_0^m \in (1 + \mathcal{N})^{-\alpha} \Gamma L^2$ and for $\varphi^m(t) = T_t^m \varphi_0^m$:*

$$\|(1 + \mathcal{N})^\alpha \varphi^m(t)\|^2 \leq C e^{tC} \|(1 + \mathcal{N})^\alpha \varphi_0^m\|^2, \tag{22}$$

as well as

$$\int_0^\infty e^{-tC} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^m(t)\|^2 dt \leq C \|(1 + \mathcal{N})^\alpha \varphi_0^m\|^2. \tag{23}$$

Proof 1. Unfortunately we cannot directly take $w(n) = (1+n)^\alpha$ in the considerations above, because this w is not compactly supported. But there is an alternative representation of the norm $\|(1 + \mathcal{N})^\alpha \cdot\|$: Let $(\rho_i)_{i \geq -1}$ be a *dyadic partition of unity*, i.e. there are radial functions $\rho_{-1}, \rho \in C_c^\infty(\mathbb{R})$ such that with $\rho_i := \rho(2^{-i} \cdot)$, for $i \geq 0$, we have $\text{supp}(\rho_i) \cap \text{supp}(\rho_j) = \emptyset$ for $|i - j| > 1$, and such that $\sum_{i \geq -1} \rho_i(x) \equiv 1$. We also assume that ρ is supported in $\{|x| \in (\frac{3}{4}, \frac{8}{3})\}$ and that $\sum_{i \geq -1} \rho_i^2(x) \simeq 1$; see [3, Chapter 2.2] for a construction of such a dyadic partition of unity. In what follows we write $i \sim j$ if $2^i \simeq 2^j$, i.e. if $|i - j| \leq L$ for some fixed $L > 0$. Then we have for $\alpha \in \mathbb{R}$ and $\varphi \in \Gamma L^2$:

$$\begin{aligned} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})\varphi\|^2 &= \sum_{i \geq -1} 2^{2i\alpha} \sum_{n \geq 0} n! \rho_i(n)^2 \|\varphi_n\|_{L^2(\mathbb{T}^n)}^2 \\ &\simeq \sum_{n \geq 0} n!(1+n)^{2\alpha} \sum_{i \geq -1} \rho_i(n)^2 \|\varphi_n\|_{L^2(\mathbb{T}^n)}^2 \\ &\simeq \sum_{n \geq 0} n!(1+n)^{2\alpha} \|\varphi_n\|_{L^2(\mathbb{T}^n)}^2 = \|(1 + \mathcal{N})^\alpha \varphi\|^2, \end{aligned}$$

where we used that $\sum_i \rho_i^2(n) \simeq 1$. In other words, it suffices to show the claimed bounds for the norm $\sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N}) \cdot\|^2$.

2. First assume that $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \cap (1 + \mathcal{N})^{-\alpha} \Gamma L^2$. The starting point of our estimate is the identity (21) whose right hand side we have to control. We use the uniform bound (8) for \mathcal{G}_+^m and get for $g: \mathbb{N}_0 \rightarrow \mathbb{R}_+$ which satisfies $g(n) \neq 0$ whenever $h(n+i) \neq 0, i \in \{-1, 0, 1\}$:

$$\begin{aligned} &| \langle \mathcal{G}_+^m(h(\mathcal{N} + 1)\varphi^m), \varphi^m \rangle | \\ &\leq \|g(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1/2} \mathcal{G}_+^m(h(\mathcal{N} + 1)\varphi^m)\| \times \|g(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m\| \\ &\lesssim \|g(\mathcal{N} + 1)^{-1} \mathcal{N}(-\mathcal{L}_0)^{1/4}(h(\mathcal{N} + 1)\varphi^m)\| \times \|g(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m\| \\ &= \left\| \frac{h(\mathcal{N} + 1)}{g(\mathcal{N} + 1)} \mathcal{N}(-\mathcal{L}_0)^{1/4} \varphi^m \right\| \times \|g(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m\|. \end{aligned}$$

Young’s inequality for products bounds the first term on the right hand side by

$$\begin{aligned} &\left\| \frac{h(\mathcal{N} + 1)}{g(\mathcal{N} + 1)} \mathcal{N}(-\mathcal{L}_0)^{1/4} \varphi^m \right\| \\ &\lesssim \delta \|g(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m\| + \delta^{-1} \left\| \left(\frac{h(\mathcal{N} + 1)\mathcal{N}}{g(\mathcal{N} + 1)g(\mathcal{N})^{1/2}} \right)^2 \varphi^m \right\|, \end{aligned}$$

for all $\delta > 0$. With another application of Young’s inequality this yields

$$\begin{aligned} & |\langle \mathcal{G}_+^m \varphi^m, h(\mathcal{N})\varphi^m \rangle| \\ & \leq \delta \|g(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m\|^2 + C(\delta) \left\| \left(\frac{h(\mathcal{N} + 1)\mathcal{N}}{g(\mathcal{N} + 1)g(\mathcal{N})^{1/2}} \right)^2 \varphi^m \right\|^2, \end{aligned}$$

for all $\delta > 0$, where δ is not the same as in the previous inequality, and $C(\delta) > 0$. Now we take $w = \rho_i$ and $g = \sum_{j:|j-i|\leq 2} \rho_j$ and obtain for $n \simeq 2^i$

$$\begin{aligned} \left| \frac{h(n + 1)n}{g(n + 1)g(n)^{1/2}} \right| &= |h(n + 1)n| = |(\rho_i(n + 1)^2 - \rho_i(n)^2)n| \\ &\leq (\rho_i(n) + \rho_i(n + 1))|\rho_i(n + 1) - \rho_i(n)|n \\ &\lesssim \sum_{j \sim i} \rho_j(n) \max\{\|\rho'_{-1}\|_\infty, \|\rho'\|_\infty\} 2^{-i} n \lesssim \sum_{j \sim i} \rho_j(n). \end{aligned}$$

For $n \not\simeq 2^i$ we simply have $h(n + 1)n/(g(n + 1)g(n)^{1/2}) = 0$. So together with (21) we obtain that for all $\delta > 0$ there exists $C = C(\delta) > 0$, independent of i , such that

$$\begin{aligned} & \frac{1}{2} \|\rho_i(\mathcal{N})\varphi^m(t)\|^2 + \int_0^t \|\rho_i(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds \\ & \leq \frac{1}{2} \|\rho_i(\mathcal{N})\varphi_0^m\|^2 + \int_0^t \delta \sum_{j \sim i} \|\rho_j(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds \\ & \quad + \int_0^t C \sum_{j \sim i} \|\rho_j(\mathcal{N})\varphi^m(s)\|^2 ds. \end{aligned}$$

Consequently, for any $\delta > 0$ and $\alpha \in \mathbb{R}$ there exists a new $C = C(\delta, \alpha) > 0$ such that

$$\begin{aligned} & \frac{1}{2} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})\varphi^m(t)\|^2 + \int_0^t \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds \\ & \leq \frac{1}{2} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})\varphi_0^m\|^2 + \delta \int_0^t \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})(-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds \\ & \quad + C \int_0^t \frac{1}{2} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N})\varphi^m(s)\|^2 ds. \tag{24} \end{aligned}$$

Now we take $\delta = 1/2$ to bring the second term on the right hand side to the left hand side.

3. For $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \cap (1 + \mathcal{N})^{-\alpha} \Gamma L^2$ the first bound (22) now follows from (24) and Gronwall’s lemma (and the equivalence of the norms addressed in point 1. above). For general $\varphi_0^m \in (1 + \mathcal{N})^{-\alpha} \Gamma L^2$ let $\mathcal{F}((\varphi_0^{m,N})_n)(k) := \mathbb{1}_{n,|k|\leq N} \hat{\varphi}_0^m(k)$,

so $\varphi_0^{m,N} \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \cap (1 + \mathcal{N})^{-\alpha} \Gamma L^2$ and now we use the fact that T_t^m is a continuous linear operator and apply Fatou’s lemma to deduce that

$$\begin{aligned} \|(1 + \mathcal{N})^\alpha T_t^m \varphi_0^m\|^2 &\leq \lim_{N \rightarrow \infty} \|(1 + \mathcal{N})^\alpha T_t^m \varphi_0^{m,N}\|^2 \\ &\leq \lim_{N \rightarrow \infty} C e^{tC} \|(1 + \mathcal{N})^\alpha \varphi_0^{m,N}\|^2 \\ &\leq C e^{tC} \|(1 + \mathcal{N})^\alpha \varphi_0^m\|^2. \end{aligned}$$

4. To derive the second bound (23) for $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \cap (1 + \mathcal{N})^{-\alpha} \Gamma L^2$, observe that

$$\begin{aligned} \partial_t \left(e^{-tC} \frac{1}{2} \|\rho_i(\mathcal{N}) \varphi^m(t)\|^2 \right) &= e^{-tC} \frac{1}{2} \partial_t \|\rho_i(\mathcal{N}) \varphi^m(t)\|^2 \\ &\quad - C e^{-tC} \frac{1}{2} \|\rho_i(\mathcal{N}) \varphi^m(t)\|^2, \end{aligned}$$

and thus (24) yields

$$\begin{aligned} e^{-tC} \frac{1}{2} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N}) \varphi^m(t)\|^2 &+ \int_0^t e^{-sC} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N}) (-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds \\ &\leq \frac{1}{2} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N}) \varphi_0^m\|^2 + \delta \int_0^t e^{-sC} \sum_{i \geq -1} 2^{2i\alpha} \|\rho_i(\mathcal{N}) (-\mathcal{L}_0)^{1/2} \varphi^m(s)\|^2 ds. \end{aligned}$$

Now we take again $\delta = 1/2$, bring the integral term from the right hand side to the left, and send $t \rightarrow \infty$ to deduce (23). The extension to general $\varphi_0^m \in (1 + \mathcal{N})^{-\alpha} \Gamma L^2$ follows from Fatou’s lemma, as in step 3. □

Corollary 3.2 For α and C as in Lemma 3.1 and for $\varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ with $(1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m \in \Gamma L^2$ we have both

$$\|(1 + \mathcal{N})^\alpha \partial_t \varphi^m(t)\|^2 = \|(1 + \mathcal{N})^\alpha \hat{\mathcal{L}}^m \varphi^m(t)\|^2 \leq C e^{tC} \|(1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m\|^2, \tag{25}$$

as well as

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^m(t)\|^2 \lesssim C t e^{tC} \|(1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m\|^2 + \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m\|^2. \tag{26}$$

Proof We use that $\partial_t \varphi^m(t) = \partial_t T_t^m \varphi_0^m = \hat{\mathcal{L}}^m T_t^m \varphi_0^m = T_t^m \mathcal{L}^m \varphi_0^m$. Then (25) directly follows from (22), while (23) gives

$$\int_0^\infty e^{-tC} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \partial_t \varphi^m(t)\|^2 dt \leq C \|(1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m\|^2,$$

and therefore

$$\begin{aligned} & \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^m(t) \|^2 \\ & \lesssim \left\| \int_0^t (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \partial_s \varphi^m(s) ds \right\|^2 + \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m \|^2 \\ & \leq t \int_0^t \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \partial_s \varphi^m(s) \|^2 ds + \| (\mathcal{N} + 1)^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m \|^2 \\ & \leq t e^{tC} \int_0^t e^{-sC} \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \partial_s \varphi^m(s) \|^2 ds + \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m \|^2 \\ & \leq C t e^{tC} \| (1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m \|^2 + \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m \|^2. \end{aligned}$$

This is the claimed bound (26). □

3.2 Controlled solutions

The estimates (25) and (26) give bounds for $\varphi^m(t)$, $\partial_t \varphi^m(t)$, and $\hat{\mathcal{L}}^m \varphi^m(t)$ that are uniform in m and locally uniform in t . This allows us to show that $\varphi^m(t) \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ for all $t \geq 0$:

Lemma 3.3 *Let $\alpha \geq 1$ and let*

$$\varphi_0^m \in (1 + \mathcal{N})^{-\alpha} (1 - \mathcal{L}_0)^{-1} \Gamma L^2 \cap (1 + \mathcal{N})^{-\alpha-1} (1 - \mathcal{L}_0)^{-1/2} \Gamma L^2 \subset \mathcal{D}_{\text{naive}}(\mathcal{L}^m).$$

Then $\varphi^m(t) := T_t^m \varphi_0^m \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ for all $t \geq 0$, and in particular $\hat{\mathcal{L}}^m \varphi^m(t) = \mathcal{L}^m \varphi^m(t)$.

Proof With the decomposition $\mathcal{L}^m = \mathcal{L}_0 + \mathcal{G}^m$ and the m -dependent bound (9) for \mathcal{G}^m we get

$$\| (1 + \mathcal{N}) \mathcal{L}^m \varphi_0^m \| \lesssim \| (1 + \mathcal{N}) \mathcal{L}_0 \varphi_0^m \| + \| (1 + \mathcal{N})^2 (-\mathcal{L}_0)^{1/2} \varphi_0^m \| < \infty$$

and therefore (25) shows that $(1 + \mathcal{N}) \hat{\mathcal{L}}^m \varphi^m(t) \in \Gamma L^2$. Another application of (9) yields

$$\begin{aligned} \| \mathcal{G}^m \varphi^m(t) \| & \lesssim m^{1/2} \| \mathcal{N} (-\mathcal{L}_0)^{1/2} \varphi^m(t) \| \\ & \lesssim \| (1 + \mathcal{N}) \mathcal{L}^m \varphi_0^m \| + \| (1 + \mathcal{N}) (-\mathcal{L}_0)^{1/2} \varphi_0^m \| < \infty \end{aligned}$$

where the second estimate follows from (26) and we hid the factor $(C t e^{tC})^{1/2}$ inside the implicit constant. So far we showed that $\hat{\mathcal{L}}^m \varphi^m(t), \mathcal{G}^m \varphi^m(t) \in \Gamma L^2$. Moreover, we know for any test function $\psi \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m) = \mathcal{D}_{\text{naive}}((\mathcal{L}^m)^-) \subset \mathcal{D}((\hat{\mathcal{L}}^m)^*)$:

$$\langle (\hat{\mathcal{L}}^m - \mathcal{G}^m) \varphi^m(t), \psi \rangle = \langle \varphi^m(t), ((\hat{\mathcal{L}}^m)^* + \mathcal{G}^m) \psi \rangle = \langle \varphi^m(t), \mathcal{L}_0 \psi \rangle.$$

Therefore, $\varphi^m(t) \in \mathcal{D}(\mathcal{L}_0^*)$ and $\mathcal{L}_0^* \varphi^m(t) = (\hat{\mathcal{L}}^m - \mathcal{G}^m) \varphi^m(t)$. But using the Fourier representation of \mathcal{L}_0 it is easy to see that this is a self-adjoint operator, and therefore

we have $\varphi^m(t) \in (1 - \mathcal{L}_0)^{-1}\Gamma L^2$. Since we already saw that $\varphi^m(t) \in (1 + \mathcal{N})^{-2}(1 - \mathcal{L}_0)^{-1/2}\Gamma L^2$, we indeed have $\varphi^m(t) \in \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ and the proof is complete. \square

Our aim is now to use our a priori estimates on $\varphi^m(t)$ to construct solutions of the limiting backward equation $\partial_t \varphi = \mathcal{L}\varphi$ that are in the domain $\mathcal{D}(\mathcal{L})$ from Sect. 2.3. Therefore, let us define

$$\varphi^{m,\sharp} := \varphi^m - (-\mathcal{L}_0)^{-1}\mathcal{G}^{m,>}\varphi^m, \tag{27}$$

so that $\varphi^m = \mathcal{K}^m\varphi^{m,\sharp}$.

Convention Throughout this section we consider a cutoff N_n in Lemma 2.14 that is adapted to the weight $(1 + \mathcal{N})^\beta$ for any β that we encounter below.

Lemma 3.4 The a priori estimates from the previous section give for $\varphi_0^m \in \mathcal{D}(\mathcal{L}^m) \subset \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$:

$$\begin{aligned} & \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^{m,\sharp}(t) \| \\ & \lesssim (te^{tC} + 1)^{1/2} (\| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0) \varphi_0^{m,\sharp} \| + \| (1 + \mathcal{N})^{\alpha+9/2} (-\mathcal{L}_0)^{1/2} \varphi_0^{m,\sharp} \|). \end{aligned} \tag{28}$$

Proof It follows from (26) and Lemma 2.14 that

$$\begin{aligned} & \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^{m,\sharp}(t) \|^2 \\ & \lesssim \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi^m(t) \|^2 + \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{-1/2} \mathcal{G}^{m,>}\varphi^m(t) \|^2 \\ & \lesssim te^{tC} \| (1 + \mathcal{N})^\alpha \mathcal{L}^m \varphi_0^m \|^2 + \| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1/2} \varphi_0^m \|^2 \\ & \lesssim (te^{tC} + 1) (\| (1 + \mathcal{N})^\alpha (-\mathcal{L}_0) \varphi_0^{m,\sharp} \|^2 + \| (1 + \mathcal{N})^{\alpha+9/2} (-\mathcal{L}_0)^{1/2} \varphi_0^{m,\sharp} \|^2), \end{aligned}$$

where in the last step we applied Proposition 2.18. \square

Unfortunately this estimate is not enough to show that $\varphi^m \in \mathcal{D}(\mathcal{L}^m)$, which requires a bound on $\| (-\mathcal{L}_0) \varphi^{m,\sharp} \| + \| (1 + \mathcal{N})^{9/2} (-\mathcal{L}_0)^{1/2} \varphi^{m,\sharp} \|$. And in fact we will need even more regularity to deduce compactness in the right spaces. So let us analyze the equation for $\varphi^{m,\sharp}$. For that purpose we want to commute the time derivative ∂_t with $(-\mathcal{L}_0)^{-1}\mathcal{G}^{m,>}$, so let us first show that $(-\mathcal{L}_0)^{-1}\mathcal{G}^{m,>}$ is a continuous linear operator: Since $|k_{1:n}|_2 \geq |k_{1:n}|_\infty$ we can bound $(-\mathcal{L}_0)^{-1/2} \mathbb{1}_{|k_{1:n}|_\infty \geq N_n}$ by $N_n^{-1} \leq (1 + n)^{-3}$, and thus

$$\begin{aligned} \| (-\mathcal{L}_0)^{-1} \mathcal{G}^{m,>}\varphi \| & \leq \| (1 + \mathcal{N})^{-3} (-\mathcal{L}_0)^{-1/2} \mathcal{G}^{m,>}\varphi \| \\ & \lesssim m^{1/2} \| (1 + \mathcal{N})^{-2}\varphi \| \leq m^{1/2} \|\varphi\|, \end{aligned}$$

where the second inequality is a variation of (9) with $\gamma = 1/2$: Since we did not make use of any cancellations when proving (9), we can simply ignore the additional indicator function in $\mathcal{G}^{m,>}$ compared to \mathcal{G}^m and in that way we get the same bound

for $\mathcal{G}^{m, >}$. Consequently, $(-\mathcal{L}_0)^{-1}\mathcal{G}^{m, >}$ commutes with the (Fréchet) time derivative and we get

$$\begin{aligned} \partial_t \varphi^{m, \sharp} &= \mathcal{L}^m \varphi^m - (-\mathcal{L}_0)^{-1} \mathcal{G}^{m, >} \partial_t \varphi^m \\ &= \mathcal{L}_0 \varphi^{m, \sharp} + \mathcal{G}^{m, <} \varphi^m - (-\mathcal{L}_0)^{-1} \mathcal{G}^{m, >} \partial_t \varphi^m. \end{aligned} \tag{29}$$

The second term on the right hand side can be controlled with (17), which gives for $\gamma \geq 0$ and $\delta > 0$

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\gamma \mathcal{G}^{m, <} \varphi^m\| \lesssim \|(1 + \mathcal{N})^{\alpha + \alpha(\gamma)} (-\mathcal{L}_0)^{1/4 + \delta} \varphi^{m, \sharp}\|,$$

so together with our a priori bound (28) we get

$$\begin{aligned} \sup_{t \in [0, T]} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\gamma \mathcal{G}^{m, <} \varphi^m(t)\| &\lesssim_T \|(1 + \mathcal{N})^{\alpha + \alpha(\gamma)} (-\mathcal{L}_0) \varphi_0^{m, \sharp}\| \\ &\quad + \|(1 + \mathcal{N})^{\alpha + \alpha(\gamma) + 9/2} (-\mathcal{L}_0)^{1/2} \varphi_0^{m, \sharp}\|. \end{aligned} \tag{30}$$

The remaining term $(-\mathcal{L}_0)^{-1} \mathcal{G}^{m, >} \partial_t \varphi^m$ is more tricky. We can plug in the explicit form of the time derivative, $\partial_t \varphi^m = \mathcal{G}^{m, <} \varphi^m + \mathcal{L}_0 \varphi^{m, \sharp}$, but then we have a problem with the term $\mathcal{L}_0 \varphi^{m, \sharp}$ because it is of the same order as the leading term of the equation for $\varphi^{m, \sharp}$. Therefore, we would like to gain a bit of regularity in $(-\mathcal{L}_0)$ from $(-\mathcal{L}_0)^{-1} \mathcal{G}^{m, >}$, and indeed this is possible by slightly adapting the proof of Lemma 2.14; see Lemma A.2 in the appendix for details. This gives for $\gamma \in (1/2, 3/4)$

$$\begin{aligned} &\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^{m, >} \partial_t \varphi^m\| \\ &\lesssim \|(1 + \mathcal{N})^{\alpha + 3/2} (-\mathcal{L}_0)^{\gamma - 1/4} (\mathcal{G}^{m, <} \varphi^m + \mathcal{L}_0 \varphi^{m, \sharp})\| \\ &\lesssim \|(1 + \mathcal{N})^{\alpha + 3/2 + \alpha(\gamma - 1/4)} (-\mathcal{L}_0)^{1/4 + \delta} \varphi^{m, \sharp}\| \\ &\quad + \|(1 + \mathcal{N})^{\alpha + 3/2} (-\mathcal{L}_0)^{\gamma + 3/4} \varphi^{m, \sharp}\|. \end{aligned}$$

Recall that $\alpha(\gamma) = 9/2 + 7\gamma$, and therefore $3/2 + \alpha(\gamma - 1/4) \leq \alpha(\gamma)$ and the first term on the right hand side is bounded by the same expression as in (30). For the remaining term we apply Young’s inequality: There exists $p > 0$ such that for all $\varepsilon \in (0, 1)$

$$\begin{aligned} &\|(1 + \mathcal{N})^{\alpha + 3/2} (-\mathcal{L}_0)^{\gamma + 3/4} \varphi^{m, \sharp}\| \\ &\lesssim \varepsilon^{-p} \|(1 + \mathcal{N})^p (-\mathcal{L}_0)^{1/2} \varphi^{m, \sharp}\| + \varepsilon \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{\gamma + 7/8} \varphi^{m, \sharp}\|. \end{aligned} \tag{31}$$

The first term on the right hand side is under control by our a priori estimates, and as the following lemma shows the second term on the right hand side can be estimated using the regularizing effect of the semigroup $(S_t)_{t \geq 0} = (e^{t\mathcal{L}_0})_{t \geq 0}$ generated by \mathcal{L}_0 .

Lemma 3.5 *Let $\gamma \in (3/8, 5/8)$. There exists $p = p(\alpha, \gamma)$ such that for all $T > 0$*

$$\begin{aligned} &\sup_{t \in [0, T]} (\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} \varphi^{m, \sharp}(t)\| + \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\gamma \partial_t \varphi^{m, \sharp}(t)\|) \\ &\lesssim_T \|(1 + \mathcal{N})^p (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m, \sharp}\|. \end{aligned} \tag{32}$$

Proof The variation of constants formula gives $\varphi^{m,\sharp}(t) = S_t \varphi_0^{m,\sharp} + \int_0^t S_{t-s} (\partial_s - \mathcal{L}_0) \varphi^{m,\sharp}(s) ds$, and by writing the explicit representation of S_t and \mathcal{L}_0 in Fourier variables we easily see that

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\beta S_t \psi\| \lesssim t^{-\beta} \|(1 + \mathcal{N})^\alpha \psi\|$$

for all $\beta \geq 0$. Since $\gamma + 1/8 \in (1/2, 3/4)$ we can combine this with our previous estimates, and in that way we obtain for some $K, K_T > 0$ and for $t \in [0, T]$

$$\begin{aligned} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} \varphi^{m,\sharp}(t)\| &\lesssim \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m,\sharp}\| \\ &+ \int_0^t (t-s)^{-1+1/8} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{\gamma+1/8} (\partial_s - \mathcal{L}_0) \varphi^{m,\sharp}(s)\| ds \\ &\leq K \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m,\sharp}\| + K_T (1 + \varepsilon^{-p}) \|(1 + \mathcal{N})^p (-\mathcal{L}_0) \varphi_0^{m,\sharp}\| \\ &+ K T^{1/8} \varepsilon \sup_{s \in [0, T]} \|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} \varphi^{m,\sharp}(s)\|. \end{aligned}$$

The right hand side does not depend on t , and therefore we can take the supremum over $t \in [0, T]$, and then we choose $\varepsilon > 0$ small enough so that $K T^{1/8} \varepsilon \leq 1/2$ and we bring the last term on the right hand side to the left and thus we obtain the claimed bound for the spatial regularity. For the temporal regularity, i.e. for $\partial_t \varphi^{m,\sharp}$, we simply use that

$$\partial_t \varphi^{m,\sharp} = \mathcal{L}_0 \varphi^{m,\sharp} + (\partial_t - \mathcal{L}_0) \varphi^{m,\sharp},$$

and then we apply the previous bounds to the two terms on the right hand side. □

For $s, t \in [0, T]$ we now interpolate the two estimates

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^\gamma (\varphi^{m,\sharp}(t) - \varphi^{m,\sharp}(s))\| \lesssim_T |t-s| \times \|(1 + \mathcal{N})^p (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m,\sharp}\|$$

and

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma} (\varphi^{m,\sharp}(t) - \varphi^{m,\sharp}(s))\| \lesssim_T \|(1 + \mathcal{N})^p (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m,\sharp}\|$$

to obtain some $\kappa \in (0, 1)$ such that

$$\|(1 + \mathcal{N})^\alpha (-\mathcal{L}_0)^{1+\gamma/2} (\varphi^{m,\sharp}(t) - \varphi^{m,\sharp}(s))\| \lesssim |t-s|^\kappa \times \|(1 + \mathcal{N})^p (-\mathcal{L}_0)^{1+\gamma} \varphi_0^{m,\sharp}\|.$$

For $\alpha \geq 0$ we introduce the space

$$\mathcal{U}_\alpha := \bigcup_{\gamma \in (3/8, 5/8)} \mathcal{K} (1 + \mathcal{N})^{-p(\alpha, \gamma)} (-\mathcal{L}_0)^{-1-\gamma} \Gamma L^2 \subseteq \Gamma L^2, \tag{33}$$

where $p(\alpha, \gamma)$ is as above, and $\mathcal{U} := \mathcal{U}_{9/2+} := \bigcup_{\alpha > 9/2} \mathcal{U}_\alpha$:

Theorem 3.6 *Let $\alpha \geq 0$ and $\varphi_0 \in \mathcal{U}_\alpha$. Then there exists a solution*

$$\varphi \in \bigcup_{\delta > 0} C(\mathbb{R}_+, (1 + \mathcal{N})^{-\alpha+\delta}(-\mathcal{L}_0)^{-1}\Gamma L^2)$$

of the backward equation

$$\partial_t \varphi = \mathcal{L}\varphi, \quad \varphi(0) = \varphi_0. \tag{34}$$

For $\varphi_0 \in \mathcal{U}$ we have $\varphi \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L})) \cap C^1(\mathbb{R}, \Gamma L^2)$ and by dissipativity of \mathcal{L} the solution φ is unique in this space.

Proof Take $\varphi_0 \in \mathcal{U}_\alpha$ and denote $\varphi_0^\sharp = \mathcal{K}^{-1}\varphi_0 \in (1 + \mathcal{N})^{-p}(-\mathcal{L}_0)^{-1-\gamma}\Gamma L^2$ for some $\gamma \in (3/8, 5/8)$ and $p = p(\alpha, \gamma)$. Consider for $m \in \mathbb{N}$ the solution φ^m to $\partial_t \varphi^m = \mathcal{L}^m \varphi^m$ with initial condition $\varphi^m(0) = \mathcal{K}^m \varphi_0^\sharp$. It follows from a diagonal sequence argument that bounded sets in $(1 + \mathcal{N})^{-\alpha}(-\mathcal{L}_0)^{-1-\gamma/2}\Gamma L^2$ are relatively compact in $(1 + \mathcal{N})^{-\alpha+\delta}(-\mathcal{L}_0)^{-1}\Gamma L^2$ for $\delta > 0$. Therefore, $(\varphi^{m,\sharp})_m$ is relatively compact in $C(\mathbb{R}_+, (1 + \mathcal{N})^{-\alpha+\delta}(-\mathcal{L}_0)^{-1}\Gamma L^2)$ (equipped with the topology of uniform convergence on compacts) by the Arzelà–Ascoli theorem. Let φ^\sharp be a limit point and define $\varphi = \mathcal{K}\varphi^\sharp$. To see that $\partial_t \varphi = \mathcal{L}\varphi$, note that (along a convergent subsequence, which we omit from the notation for simplicity)

$$\begin{aligned} \varphi(t) - \varphi(0) &= \lim_{m \rightarrow \infty} (\varphi^m(t) - \varphi^m(0)) = \lim_{m \rightarrow \infty} \int_0^t \mathcal{L}^m \varphi^m(s) ds \\ &= \lim_{m \rightarrow \infty} \int_0^t (\mathcal{L}_0 \varphi^{m,\sharp}(s) + \mathcal{G}^{m,\prec} \mathcal{K}^m \varphi^{m,\sharp}(s)) ds \\ &= \lim_{m \rightarrow \infty} \int_0^t (\mathcal{L}_0 \varphi^\sharp(s) + \mathcal{G}^{m,\prec} \mathcal{K}^m \varphi^\sharp(s)) ds \\ &= \int_0^t (\mathcal{L}_0 \varphi^\sharp(s) + \mathcal{G}^\prec \mathcal{K} \varphi^\sharp(s)) ds, \end{aligned}$$

where the second-to-last step follows from our uniform bounds on $\mathcal{L}_0, \mathcal{G}^{m,\prec}, \mathcal{K}^m$ and the convergence of $\varphi^{m,\sharp}$ to φ^\sharp , and the last step follows from our bounds for $\mathcal{G}^\prec, \mathcal{K}$ together with the dominated convergence theorem. If $\alpha > 9/2$, then $\varphi \in \mathcal{D}(\mathcal{L})$ by definition, see Lemma 2.19. Moreover, in that case $\mathcal{L}\varphi \in C(\mathbb{R}_+, \Gamma L^2)$ and since $\varphi(t) - \varphi(s) = \int_s^t \mathcal{L}\varphi(r) dr$ we get $\varphi \in C^1(\mathbb{R}_+, \Gamma L^2)$. In this case we have

$$\partial_t \|\varphi(t)\|^2 = 2\langle \varphi(t), \mathcal{L}\varphi(t) \rangle \leq 0,$$

by the dissipativity of \mathcal{L} (Lemma 2.22). Therefore, any solution ψ satisfies $\|\psi(t)\| \leq \|\varphi_0\|$, which together with the linearity of the equation gives uniqueness. \square

4 The martingale problem

Our next aim is to construct a process $(u_t)_{t \geq 0}$ with infinitesimal generator given by (an extension of) \mathcal{L} . We will do so by solving the martingale problem for \mathcal{L} .

Let \mathcal{S}' denote the Schwartz distributions on \mathbb{T} .

Definition 4.1 Let $u = (u_t)_{t \geq 0}$ be a stochastic process with trajectories in $C(\mathbb{R}_+, \mathcal{S}')$ and such that $\text{law}(u_t) \ll \mu$ for all $t \geq 0$. We say that u solves the martingale problem for \mathcal{L} with initial distribution ν if

- i. $u_0 \sim \nu$, and
- ii. for all $\varphi \in \mathcal{D}(\mathcal{L})$ and $t \geq 0$ we have $\int_0^t |\mathcal{L}\varphi(u_s)| ds < \infty$ almost surely and the process

$$\varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}\varphi(u_s) ds, \quad t \geq 0,$$

is a martingale in the filtration generated by (u_t) .

Note that, since φ and $\mathcal{L}\varphi$ are not cylinder functions, we need the condition $\text{law}(u_t) \ll \mu$ in order for $\varphi(u_t)$ and $\mathcal{L}\varphi(u_t)$ to be well defined.

The following class of processes will play an important role in our study of the martingale problem.

Definition 4.2 We say that a process $(u_t)_{t \geq 0}$ with values in \mathcal{S}' is *incompressible* if $\text{law}(u_t) \ll \mu$ for all $t \geq 0$ and for all $T > 0$ there exists $C(T) > 0$ such that for all $\varphi \in \Gamma L^2$

$$\sup_{t \leq T} \mathbb{E}[|\varphi(u_t)|] \leq C(T) \|\varphi\|.$$

We will establish the existence of incompressible solutions to the martingale problem by a compactness argument. The duality of martingale problem and backward equation yields the uniqueness of incompressible solutions to the martingale problem. Since the domain of \mathcal{L} is rather complicated, we then study a ‘‘cylinder function martingale problem’’, a generalization of the energy solutions of [27,28,34], and we show that every solution to the cylinder function martingale problem solves the martingale problem for \mathcal{L} and in particular its law is unique.

4.1 Existence of solutions

Here we show that under ‘‘near-stationary’’ initial conditions the Galerkin approximations $(u^m)_m$ solving (2) are tight in $C(\mathbb{R}_+, \mathcal{S}')$, and that any weak limit is an incompressible solution to the martingale problem for the generator \mathcal{L} in the sense of Definitions 4.1 and 4.2. The following elementary inequality will be used throughout this section.

Lemma 4.3 Let u^m be a solution to (2) with $d \text{law}(u_0^m)/d\mu = \eta \in L^2(\mu)$. Then we have for any measurable and bounded or positive $\Psi: C(\mathbb{R}_+, \mathcal{S}') \rightarrow \mathbb{R}$

$$|\mathbb{E}[\Psi(u^m)]| \leq \|\eta\| \mathbb{E}_\mu[\Psi(u^m)^2]^{1/2},$$

where \mathbb{P}_μ denotes the distribution of u^m under the stationary initial condition $u_0^m \sim \mu$. In particular, u^m is incompressible.

Proof The Cauchy–Schwarz inequality and Jensen’s inequality yield

$$\begin{aligned} \mathbb{E}[\Psi(u^m)] &= \int \mathbb{E}_u[\Psi(u^m)]\eta(u)\mu(du) \leq \|\eta\| \left(\int \mathbb{E}_u[\Psi(u^m)]^2\mu(du) \right)^{1/2} \\ &\leq \|\eta\|\mathbb{E}_\mu[\Psi(u^m)^2]^{1/2}. \end{aligned}$$

□

Recall that D_x denotes the Malliavin derivative with respect to μ .

Lemma 4.4 *Let u^m be a solution to (2) with $d \text{ law}(u_0^m)/d\mu = \eta \in L^2(\mu)$. Let $\varphi \in \mathcal{D}(\mathcal{L}^m)$ and consider $M_t^{m,\varphi} := \varphi(u_t^m) - \varphi(u_0^m) - \int_0^t \mathcal{L}^m \varphi(u_s^m) ds$. Then $M^{m,\varphi}$ is a continuous martingale with quadratic variation*

$$\langle M^{m,\varphi} \rangle_t = \int_0^t \mathcal{E}\varphi(u_m(s)) ds, \quad \text{where} \quad \mathcal{E}\varphi = 2 \int_{\mathbb{T}} |\partial_x D_x \varphi|^2 dx. \tag{35}$$

Moreover, for $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ we have

$$\|w(\mathcal{N})(\mathcal{E}\varphi)^{1/2}\| = \sqrt{2}\|w(\mathcal{N} - 1)(-\mathcal{L}_0)^{1/2}\varphi\|. \tag{36}$$

Proof Let φ be a cylinder function with finite chaos expansion (i.e. $\varphi_n = 0$ for all n and all sufficiently large n) and finitely many Fourier modes (i.e. $\hat{\varphi}_n(k) = 0$ for all sufficiently large $|k|$). Then it follows from Itô’s formula that $M^{m,\varphi}$ is a martingale with quadratic variation given by (35), and the Burkholder–Davis–Gundy inequality gives for all $T > 0$

$$\mathbb{E}[\sup_{t \leq T} |M_t^{m,\varphi}|] \lesssim \mathbb{E}[\langle M^{m,\varphi} \rangle_T^{1/2}] \leq \|\eta\|\mathbb{E}_\mu[\langle M^{m,\varphi} \rangle_T]^{1/2} = \|\eta\|T^{1/2}\|(\mathcal{E}\varphi)^{1/2}\|. \tag{37}$$

Moreover, for any $w : \mathbb{N}_0 \rightarrow \mathbb{R}_+$ we get the following equality for the “weighted energy”:

$$\begin{aligned} \|w(\mathcal{N})(\mathcal{E}\varphi)^{1/2}\|^2 &= 2 \int_{\mathbb{T}} \|w(\mathcal{N})\partial_x D_x \varphi\|^2 dx \\ &= 2 \int_{\mathbb{T}} \left(\sum_{n=1}^{\infty} (n-1)!w(n-1)^2n^2 \|\partial_x \varphi_n(x, r_{2:n})\|_{L^2_r(\mathbb{T}^{n-1})}^2 \right) dx \\ &= 2 \sum_{n=1}^{\infty} n!w(n-1)^2n \sum_{k_{1:n}} |2\pi k_1|^2 |\hat{\varphi}_n(k_{1:n})|^2 \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{n=1}^{\infty} n! w(n-1)^2 \sum_{k_{1:n}} (|2\pi k_1|^2 + \dots + |2\pi k_n|^2) |\hat{\varphi}_n(k_{1:n})|^2 \\
 &= 2 \|w(\mathcal{N}-1)(-\mathcal{L}_0)^{1/2} \varphi\|^2.
 \end{aligned}
 \tag{38}$$

If $\varphi \in \mathcal{D}(\mathcal{L}^m) \subset \mathcal{D}_{\text{naive}}(\mathcal{L}^m) \subset (-\mathcal{L}_0)^{-1/2}(1 + \mathcal{N})^{-1} \Gamma L^2$, where the first inclusion holds by Remark 2.20, then we consider the function φ^M with finite chaos expansion and finitely many Fourier modes, given by

$$\mathcal{F}(\varphi^M)_n(k) := \mathbb{1}_{n, |k| \leq M} \hat{\varphi}_n(k).$$

Then φ^M converges to φ in $(1 - \mathcal{L}_0)^{-1/2}(1 + \mathcal{N})^{-1} \Gamma L^2 \cap (1 - \mathcal{L}_0)^{-1} \Gamma L^2 = \mathcal{D}_{\text{naive}}(\mathcal{L}^m)$. So (37) together with (38) (for $w \equiv 1$) shows that the continuous martingales M^{m, φ^M} converge uniformly on compacts in $L^1(\mathbb{P})$ to a continuous martingale $M^{m, \varphi}$ with quadratic variation given by (35). Moreover, we have for fixed $t \geq 0$

$$\begin{aligned}
 M_t^{m, \varphi} &= L^1 - \lim_{M \rightarrow \infty} M_t^{m, \varphi^M} \\
 &= L^1 - \lim_{M \rightarrow \infty} \left(\varphi^M(u_t^m) - \varphi^M(u_0^m) - \int_0^t \mathcal{L}^m \varphi^M(u_s^m) ds \right) \\
 &= \varphi(u_t^m) - \varphi(u_0^m) - \int_0^t \mathcal{L}^m \varphi(u_s^m) ds,
 \end{aligned}$$

where the last step follows from the incompressibility of u^m and because φ^M converges in $\mathcal{D}_{\text{naive}}(\mathcal{L}^m)$ to φ . Since $M^{m, \varphi}$ and the process on the right hand side are both continuous, they are indistinguishable.

It remains to show that (38) also holds for the limit φ of φ^M . This follows from two applications of the monotone convergence theorem because on both sides of (38) the number of positive terms that are summed up increases if we increase M . □

We need to control higher moments to prove tightness, and the following classical result is useful for this purpose.

Remark 4.5 Let $p \geq 2$ and define $c_p := \sqrt{p-1}$. It follows from the hypercontractivity of the Ornstein–Uhlenbeck semigroup that $\| |\varphi|^{p/2} \|^2 \leq \|c_p^{\mathcal{N}} \varphi\|^p$ for all $\varphi \in \Gamma L^2$; see [49, Theorem 1.4.1].

In Lemma 2.19 we defined a domain $\mathcal{D}_w(\mathcal{L})$ of functions that are mapped to $w(\mathcal{N})^{-1} \Gamma L^2$ by \mathcal{L} . From now on, we write $\mathcal{D}_p(\mathcal{L}) := \mathcal{D}_w(\mathcal{L})$ for $w(n) = c_p^n$ with the constant $c_p > 0$ of Remark 4.5.

Theorem 4.6 Let $\eta \in L^2(\mu)$ and let u^m be the solution to (2) with $\text{law}(u_0^m) \sim \eta d\mu$. Then $(u^m)_{m \in \mathbb{N}}$ is tight in $C(\mathbb{R}_+, \mathcal{S}')$, and any weak limit is incompressible and it solves the martingale problem for \mathcal{L} with initial distribution $\eta d\mu$.

Proof 1. We first consider $p \geq 2$ and $\varphi \in \mathcal{D}_{2p}(\mathcal{L}^m)$ and we derive an estimate for $\mathbb{E}[|\varphi(u_t^m) - \varphi(u_s^m)|^p]$. For that purpose we split $\varphi(u_t^m) - \varphi(u_s^m) = \int_s^t \mathcal{L}^m \varphi(u_r^m) dr + M_t^{m,\varphi} - M_s^{m,\varphi}$, and observe that by Lemma 4.3 and Remark 4.5

$$\begin{aligned} \mathbb{E} \left[\left| \int_s^t \mathcal{L}^m \varphi(u_r^m) dr \right|^p \right] &\lesssim \mathbb{E}_\mu \left[\left| \int_s^t \mathcal{L}^m \varphi(u_r^m) dr \right|^{2p} \right]^{1/2} \\ &\leq |t - s|^p \| \mathcal{L}^m \varphi \|^p \leq |t - s|^p \| c_{2p}^{\mathcal{N}} \mathcal{L}^m \varphi \|^p. \end{aligned}$$

Next, we bound the martingale term with the Burkholder–Davis–Gundy inequality and (36):

$$\begin{aligned} \mathbb{E}[|M_t^{m,\varphi} - M_s^{m,\varphi}|^p] &\lesssim \mathbb{E} \left[\left(\int_s^t \mathcal{E} \varphi(u_s^m) ds \right)^{p/2} \right] \lesssim \mathbb{E}_\mu \left[\left(\int_s^t \mathcal{E} \varphi(u_s^m) ds \right)^p \right]^{1/2} \\ &\lesssim |t - s|^{p/2} \| (\mathcal{E} \varphi)^{p/2} \| \leq |t - s|^{p/2} \| c_{2p}^{\mathcal{N}} (\mathcal{E} \varphi)^{1/2} \|^p \\ &\lesssim |t - s|^{p/2} \| c_{2p}^{\mathcal{N}} (-\mathcal{L}_0)^{1/2} \varphi \|^p. \end{aligned}$$

2. Let now $\varphi \in c_{2p}^{-\mathcal{N}}(-\mathcal{L}_0)^{-1} \Gamma L^2 \cap c_{2p}^{-\mathcal{N}}(1 + \mathcal{N})^{-9/2} (-\mathcal{L}_0)^{-1/2} \Gamma L^2$. We apply Step 1 and (19) to find for all $M \geq 1$ a function $\varphi^M \in \mathcal{D}_{2p}(\mathcal{L}^m)$ with

$$\begin{aligned} &\mathbb{E}[|\varphi(u_t^m) - \varphi(u_s^m)|^p] \\ &\lesssim \mathbb{E}[|\varphi(u_t^m) - \varphi^M(u_t^m)|^p] + \mathbb{E}[|\varphi(u_s^m) - \varphi^M(u_s^m)|^p] \\ &\quad + \mathbb{E}[|\varphi^M(u_t^m) - \varphi^M(u_s^m)|^p] \\ &\lesssim \| \varphi - \varphi^M \|^p + |t - s|^{p/2} \| c_{2p}^{\mathcal{N}} (-\mathcal{L}_0)^{1/2} \varphi^M \|^p + |t - s|^p \| c_{2p}^{\mathcal{N}} \mathcal{L}^m \varphi^M \|^p \\ &\lesssim \| c_{2p}^{\mathcal{N}} (\varphi - \varphi^M) \|^p + |t - s|^{p/2} \| c_{2p}^{\mathcal{N}} (-\mathcal{L}_0)^{1/2} \varphi^M \|^p + |t - s|^p \| c_{2p}^{\mathcal{N}} \mathcal{L}^m \varphi^M \|^p \\ &\lesssim M^{-p/2} \| c_{2p}^{\mathcal{N}} \varphi \|^p + |t - s|^{p/2} \| c_{2p}^{\mathcal{N}} (-\mathcal{L}_0)^{1/2} \varphi \|^p \\ &\quad + |t - s|^p M^{p/2} (\| c_{2p}^{\mathcal{N}} (-\mathcal{L}_0) \varphi \|^p + \| c_{2p}^{\mathcal{N}} (1 + \mathcal{N})^{9/2} (-\mathcal{L}_0)^{1/2} \varphi \|^p). \end{aligned}$$

For $|t - s| \leq 1$ we choose $M = |t - s|^{-1}$ and see that the right hand side is of order $|t - s|^{p/2}$. The law of the initial condition $\varphi(u_0^m)$ does not depend on m , so for $p > 2$ and $\varphi \in c_{2p}^{-\mathcal{N}}(-\mathcal{L}_0)^{-1} \Gamma L^2 \cap c_{2p}^{-\mathcal{N}}(1 + \mathcal{N})^{-9/2} (-\mathcal{L}_0)^{-1/2} \Gamma L^2$ it follows from Kolmogorov’s continuity criterion that the sequence of real valued processes $(\varphi(u^m))_m$ is tight in $C(\mathbb{R}_+, \mathbb{R})$. This space contains all functions of the form $\varphi(u) = u(f)$ with $f \in C^\infty(\mathbb{T})$, where $u(f)$ denotes the application of the distribution $u \in \mathcal{S}'$ to the test function f . Therefore, we can apply Mitoma’s criterion [48] to deduce that the sequence (u^m) is tight in $C(\mathbb{R}_+, \mathcal{S}')$.

3. It remains to show that any weak limit u of (u^m) is incompressible and it solves the martingale problem for \mathcal{L} with initial distribution $\eta d\mu$. As $u_0^m \sim \eta d\mu$, also any weak limit has initial distribution $\eta d\mu$. To see that u is incompressible, note

that for $\varphi \in \Gamma L^2$:

$$\mathbb{E}[|\varphi(u_t)|] \leq \liminf_{m \rightarrow \infty} \mathbb{E}[|\varphi(u_t^m)|] \leq \|\eta\| \|\varphi\|.$$

implies that for $\varphi \in \Gamma L^2$ and for any bounded cylinder function ψ

$$\begin{aligned} \limsup_{m \rightarrow \infty} |\mathbb{E}[\varphi(u_t)] - \mathbb{E}[\varphi(u_t^m)]| &\leq \mathbb{E}[(\varphi - \psi)(u_t)] \\ &\quad + \limsup_{m \rightarrow \infty} \{ |\mathbb{E}[\psi(u_t)] - \mathbb{E}[\psi(u_t^m)]| + \mathbb{E}[(\varphi - \psi)(u_t^m)] \} \\ &\lesssim \|\varphi - \psi\|. \end{aligned}$$

Since the bounded cylinder functions are dense in ΓL^2 , the left hand side must equal zero. The same argument also shows that $\limsup_{m \rightarrow \infty} \left| \mathbb{E} \left[\int_s^t \varphi(u_r) dr \right] - \mathbb{E} \left[\int_s^t \varphi(u_r^m) dr \right] \right| = 0$ and then that for $\varphi \in \mathcal{D}(\mathcal{L})$ and for bounded and continuous $G: C([0, s], \mathcal{S}^t) \rightarrow \mathbb{R}$:

$$\begin{aligned} &\mathbb{E} \left[\left(\varphi(u_t) - \varphi(u_s) - \int_s^t \mathcal{L}\varphi(u_r) dr \right) G((u_r)_{r \in [0, s]}) \right] \\ &= \lim_{m \rightarrow \infty} \mathbb{E} \left[\left(\varphi(u_t^m) - \varphi(u_s^m) - \int_s^t \mathcal{L}\varphi(u_r^m) dr \right) G((u_r^m)_{r \in [0, s]}) \right]. \end{aligned}$$

This is not quite sufficient to prove that the left hand side equals zero, because u^m solves the martingale problem for \mathcal{L}^m and not for \mathcal{L} . But for $\varphi \in \mathcal{D}(\mathcal{L})$ there exists φ^\sharp with $\varphi = \mathcal{K}\varphi^\sharp$, so let us define $\varphi^m = \mathcal{K}^m\varphi^\sharp$. It follows from the dominated convergence theorem and the proof of Lemma 2.14 that $\|\varphi^m - \varphi\| \rightarrow 0$ as $m \rightarrow \infty$. Moreover, $\mathcal{L}^m\varphi^m = \mathcal{L}_0\varphi^\sharp + \mathcal{G}^{m, \prec}\mathcal{K}^m\varphi^\sharp$, and therefore another application of the dominated convergence theorem in the proof of Proposition 2.18 shows that $\|\mathcal{L}^m\varphi^m - \mathcal{L}\varphi\| \rightarrow 0$. Hence, the incompressibility of u^m yields

$$\begin{aligned} &\lim_{m \rightarrow \infty} \mathbb{E} \left[\left(\varphi(u_t^m) - \varphi(u_s^m) - \int_s^t \mathcal{L}\varphi(u_r^m) dr \right) G((u_r^m)_{r \in [0, s]}) \right] \\ &= \lim_{m \rightarrow \infty} \mathbb{E} \left[\left(\varphi^m(u_t^m) - \varphi^m(u_s^m) - \int_s^t \mathcal{L}^m\varphi^m(u_r^m) dr \right) G((u_r^m)_{r \in [0, s]}) \right] = 0, \end{aligned}$$

which concludes the proof. □

Remark 4.7 For simplicity we restricted our attention to $\eta \in L^2(\mu)$. But the same arguments show the existence of solutions to the martingale problem for initial conditions $\eta d\mu$ with $\eta \in L^q(\mu)$ for $q > 1$. The key requirement is that we can control expectations of u^m in terms of higher moments under the stationary measure \mathbb{P}_μ , and this also works for $\eta \in L^q(\mu)$. For $q < 2$ we would simply have to adapt the definition of incompressibility and to restrict our domain in the martingale problem from $\mathcal{D}(\mathcal{L})$ to $\mathcal{D}_{q'}(\mathcal{L})$, where q' is the conjugate exponent of q . On the other hand the uniqueness

proof below really needs $\eta \in L^2$ because we only control the solution to the backward equation in spaces with polynomial weights, but not with exponential weights.

4.2 Uniqueness of solutions

Let $\eta \in \Gamma L^2$ be a probability density (with respect to μ). Let the process $(u_t)_{t \geq 0} \in C(\mathbb{R}_+, S')$ be incompressible and solve the martingale problem for \mathcal{L} with initial distribution $u_0 \sim \eta d\mu$. Here we use the duality of martingale problem and backward equation to show that the law of u is unique and that it is a Markov process with invariant measure μ .

In Lemma A.3 in the appendix we show that for $\varphi \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L})) \cap C^1(\mathbb{R}_+, \Gamma L^2)$ the process $\varphi(t, u_t) - \varphi(0, u_0) - \int_0^t (\partial_s + \mathcal{L})\varphi(s, u_s) ds$, for $t \geq 0$, is a martingale. This will be important in the proof of the next theorem.

Theorem 4.8 *Let $\eta \in \Gamma L^2$ with $\eta \geq 0$ and $\int \eta d\mu = 1$. Let u be an incompressible solution to the martingale problem for \mathcal{L} with initial distribution $u_0 \sim \eta d\mu$. Then u is a Markov process and its law is unique. Moreover, μ is a stationary measure for u .*

Proof Let $\varphi_0 \in \mathcal{U}$ and let $\varphi \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L})) \cap C^1(\mathbb{R}_+, \Gamma L^2)$ be the solution to $\partial_t \varphi = \mathcal{L}\varphi$ with initial condition $\varphi(0) = \varphi_0$, see Theorem 3.6. Then Lemma A.3 shows that

$$\begin{aligned} \mathbb{E}[\varphi_0(u_t)] &= \mathbb{E}[\varphi(t - t, u_t)] \\ &= \mathbb{E} \left[\varphi(t - 0, u_0) + \int_0^t (-\partial_t \varphi(t - s, u_s) + \mathcal{L}\varphi(t - s, u_s)) ds \right] \\ &= \mathbb{E}[\varphi(t, u(0))] = \langle \varphi(t), \eta \rangle \end{aligned}$$

is uniquely determined. Here we used that if $\| -\partial_t \varphi(t - s) + \mathcal{L}\varphi(t - s) \| = 0$, then by assumption also $\mathbb{E}[-\partial_t \varphi(t - s, u_s) + \mathcal{L}\varphi(t - s, u_s)] = 0$. It is easy to see that \mathcal{U} is dense in $\mathcal{D}(\mathcal{L})$, and since $\mathcal{D}(\mathcal{L})$ is dense in ΓL^2 and $\mathbb{E}[|\psi(u_t) - \tilde{\psi}(u_t)|] \lesssim \|\psi - \tilde{\psi}\|$, the law of u_t is unique.

Next, let ψ_1 be bounded and measurable and let $\psi_2 \in \mathcal{U}$. Let $0 \leq t_1 < t_2$ and let $\partial_t \varphi_2 = \mathcal{L}\varphi_2$ with initial condition $\varphi_2(0) = \psi_2$. Then

$$\begin{aligned} \mathbb{E}[\psi_1(u_{t_1})\psi_2(u_{t_2})] &= \mathbb{E}[\psi_1(u_{t_1})\varphi_2(t_2 - t_2, u_{t_2})] \\ &= \mathbb{E} \left[\psi_1(u_{t_1}) \left\{ \varphi_2(t_2 - t_1, u_{t_1}) + \int_{t_1}^{t_2} (-\partial_t + \mathcal{L})\varphi_2(t_2 - s, u_s) ds \right\} \right] \\ &= \mathbb{E}[\psi_1(u_{t_1})\varphi_2(t_2 - t_1, u_{t_1})]. \end{aligned}$$

Since we already saw that the law of $u(t_1)$ is unique, also the law of (u_{t_1}, u_{t_2}) is unique (by a monotone class argument). Iterating this, we get the uniqueness of law $(u_{t_1}, \dots, u_{t_n})$ for all $0 \leq t_1 < \dots < t_n$, and therefore the uniqueness of law $(u_t : t \geq 0)$.

To see the Markov property, let $0 \leq t < s$, let X be an $\mathcal{F}_t = \sigma(u_r : r \leq t)$ measurable bounded random variable, and let $\varphi_0 \in \mathcal{U}$. Let φ be the solution to the

backward equation with initial condition $\varphi(0) = \varphi_0$. Then

$$\begin{aligned}\mathbb{E}[X\varphi_0(u_s)] &= \mathbb{E}[X\varphi(s-s, u_s)] \\ &= \mathbb{E}\left[X\left(\varphi(s-t, u_t) + \int_t^s (-\partial_t + \mathcal{L})\varphi(s-r, u_r)dr\right)\right] \\ &= \mathbb{E}[X\varphi(s-t, u_t)],\end{aligned}$$

which shows that $\mathbb{E}[\varphi_0(u_s)|\mathcal{F}_t] = \varphi(s-t, u_t) = \mathbb{E}[\varphi_0(u_s)|u_t]$. Now the Markov property follows by another density argument.

To see that u is stationary with respect to μ it suffices to consider the Galerkin approximation with initial distribution law $(u_0^m) = \mu$. This is a stationary process and it converges to the solution of the martingale problem, which therefore is a stationary process with initial distribution μ . \square

Remark 4.9 The strong Markov property seems difficult to obtain with our tools: If τ is a stopping time, then there is no reason why the law of u_τ should be absolutely continuous with respect to μ , regardless of the initial distribution of u . Since such absolute continuity is crucial for our method, it is not clear how to deal with $(u_{\tau+t})_{t \geq 0}$.

Definition 4.10 For $t \geq 0$ we define T_t as the continuous extension to ΓL^2 of the map

$$\mathcal{U} \ni \varphi_0 \mapsto \varphi(t) \in \Gamma L^2,$$

where φ solves the backward equation with initial condition φ_0 . Since (T_t^m) is a contraction semigroup on ΓL^2 for all m , Fatou's lemma yields that $\|\varphi(t)\| \leq \|\varphi_0\|$. So T_t indeed exists and is unique.

Proposition 4.11 *The operators $(T_t)_{t \geq 0}$ define a strongly continuous contraction semigroup on ΓL^2 and*

$$T_t\varphi = \varphi + \int_0^t T_s\mathcal{L}\varphi ds, \quad t \geq 0,$$

for all $\varphi \in \mathcal{D}(\mathcal{L})$. The Hille–Yosida generator $\hat{\mathcal{L}}$ of $(T_t)_t$ is an extension of \mathcal{L} , and $\mathcal{D}(\mathcal{L})$ is a core for $\hat{\mathcal{L}}$ (i.e. $\hat{\mathcal{L}}$ is the closure of \mathcal{L}).

Proof To see the semigroup property, let $\eta \in \Gamma L^2$ be such that $\eta d\mu$ is a probability measure. Let u be the solution to the martingale problem for \mathcal{L} with initial distribution $\eta d\mu$. We showed in the proof of Theorem 4.8 that for $\varphi \in \mathcal{U}$ we have $\langle T_t\varphi, \eta \rangle = \mathbb{E}[\varphi(u_t)]$ and almost surely $\mathbb{E}[\varphi(u_{t+s})|\mathcal{F}_s] = T_t\varphi(u_s)$, and thus

$$\langle T_{t+s}\varphi, \eta \rangle = \mathbb{E}[\varphi(u_{t+s})] = \mathbb{E}[\mathbb{E}[\varphi(u_{t+s})|\mathcal{F}_s]] = \mathbb{E}[T_t\varphi(u_s)] = \langle T_s(T_t\varphi), \eta \rangle.$$

Since T_{t+s} , T_t , and T_s are contractions, and since $\mathcal{U} \subset \Gamma L^2$ is dense, the equality holds for all $\varphi \in \Gamma L^2$. By linearity it extends to all $\eta \in \Gamma L^2$, and therefore (T_t) is a semigroup.

It also follows from the martingale problem that for $\varphi \in \mathcal{D}(\mathcal{L})$

$$T_t\varphi = \varphi + \int_0^t T_s\mathcal{L}\varphi ds, \quad t \geq 0$$

and this also proves the strong continuity of $t \mapsto T_t\varphi$. By approximation the continuity extends to $t \mapsto T_t\psi$ for all $\psi \in \Gamma L^2$.

We conclude that $\partial_t T_t\varphi|_{t=0} = \mathcal{L}\varphi$ for $\varphi \in \mathcal{D}(\mathcal{L})$, and thus $\hat{\mathcal{L}}$ is an extension of \mathcal{L} . Moreover, Theorem 3.6 shows that $T_t : \mathcal{U} \rightarrow \mathcal{D}(\mathcal{L})$ for all $t \geq 0$. Since $\mathcal{U} \subset \mathcal{D}(\mathcal{L})$ and \mathcal{U} is dense, $\mathcal{D}(\mathcal{L})$ is a core for $\hat{\mathcal{L}}$ by Proposition 1.3.3 in [18]. □

4.3 Exponential ergodicity

The Burgers generator formally satisfies a spectral gap inequality and thus it should be exponentially L^2 -ergodic (see e.g. [39, Chapter 2] for the definition of the spectral gap inequality and its relation to exponential ergodicity). Indeed, the symmetric part of \mathcal{L} is \mathcal{L}_0 for which the spectral gap is known, and its antisymmetric part \mathcal{G} should not contribute to the spectral gap inequality. Having identified a domain for \mathcal{L} , we can make this formal argument rigorous. We remark that the ergodicity of Burgers equation was already shown in [41], even in a stronger sense. The only new result here is the exponential speed of convergence (and our proof is very simple).

Consider $\varphi \in \mathcal{U}$ and let $(\varphi(t))$ be the unique solution to the backward equation with $\varphi(0) = \varphi$ that we constructed in Theorem 3.6. From Proposition 4.11 we know that $T_t\varphi = \varphi(t)$ for the Burgers semigroup, and from Lemma 2.22 we obtain

$$\frac{1}{2}\partial_t \|\varphi(t)\|^2 = -\|(-\mathcal{L}_0)^{1/2}\varphi(t)\|^2.$$

Assume that $\int \varphi d\mu = \varphi_0 = 0$ for the zero-th chaos component, which by construction holds whenever $(\mathcal{K}^{-1}\varphi)_0 = 0$. Using the stationarity of (u_t) with respect to μ we see that then also $(\varphi(t))_0 = 0$. Recall that $\mathcal{F}(\varphi(t))_n(k_{1:n}) = 0$ whenever $k_i = 0$ for some i , which leads to

$$\|(-\mathcal{L}_0)^{1/2}\varphi(t)\|^2 \geq |2\pi|^2 \|\varphi(t)\|^2,$$

and thus $\partial_t \|\varphi(t)\|^2 \leq -8\pi^2 \|\varphi(t)\|^2$. Therefore, Gronwall’s inequality yields

$$\|T_t\varphi\| \leq e^{-4\pi^2 t} \|\varphi\|. \tag{39}$$

This holds for all $\varphi \in \mathcal{U}$ with $\int \varphi d\mu = 0$, but since the left and right hand side can both be controlled in terms of $\|\varphi\|$ it extends to all $\varphi \in \Gamma L^2$ with $\int \varphi d\mu = 0$.

There are two interesting consequences:

1. The measure μ is ergodic: Recall that the set of invariant distributions of a Markov process is convex, and the extremal points are the mutually singular ergodic measures. Moreover, μ is ergodic if and only if for all $A \subset \mathcal{S}'$ with $T_t \mathbb{1}_A \stackrel{\mu\text{-a.s.}}{=} \mathbb{1}_A$ for

$t \geq 0$ we have $\mu(A) \in \{0, 1\}$, see [16, Theorem 3.2.4]. But from (39) we know that $T_t \mathbb{1}_A \rightarrow \mu(A)$ in $L^2(\mu)$ as $t \rightarrow \infty$, so if $T_t \mathbb{1}_A \stackrel{\mu\text{-a.s.}}{=} \mathbb{1}_A$ we get $\mathbb{1}_A \stackrel{\mu\text{-a.s.}}{=} \mu(A)$ and thus $\mu(A) \in \{0, 1\}$. Therefore, μ is ergodic and in particular there exists no invariant distribution that is absolutely continuous with respect to μ , other than μ itself.

2. We can solve the Poisson equation $\hat{\mathcal{L}}\varphi = \psi$ for all $\psi \in \Gamma L^2$ with $\int \psi d\mu = 0$ by setting $\varphi = \int_0^\infty T_t \psi dt$, which is well defined by (39). Here $\hat{\mathcal{L}}$ is the Hille–Yosida generator and we do not necessarily have $\varphi \in \mathcal{D}(\mathcal{L})$.

4.4 Martingale problem with cylinder functions

The martingale approach to Burgers equation is particularly useful for proving that the equation arises as scaling limit of particle systems. The disadvantage of the martingale problem based on controlled functions is that, given a microscopic system for which we want to prove convergence to Burgers equation, it may be difficult to find similar controlled functions before passing to the limit. Instead it is often more natural to derive a characterization of the scaling limit based on cylinder test functions. Here we show that in some cases this characterization already implies that the limit solves our martingale problem for the controlled domain of the generator, and therefore it is unique in law. The biggest restriction is that we have to assume that the process allows for the *Itô trick*:

Definition 4.12 A process $(u_t)_{t \geq 0}$ with trajectories in $C(\mathbb{R}_+, S')$ solves the cylinder function martingale problem for \mathcal{L} with initial distribution ν if $u_0 \sim \nu$, and if the following conditions are satisfied:

- i. $\mathbb{E}[|\varphi(u_t)|] \lesssim \|\varphi\|$ locally uniformly in t , namely u is incompressible;
- ii. There exists an approximation of the identity (ρ^ε) such that for all $f \in C^\infty(\mathbb{T})$ the process

$$M_t^f = u_t(f) - u_0(f) - \lim_{\varepsilon \rightarrow 0} \int_0^t \mathcal{L}^\varepsilon u_s(f) ds$$

is a continuous martingale in the filtration generated by (u_t) , where

$$\mathcal{L}^\varepsilon u(f) = \mathcal{L}_0 u(f) + \langle \partial_x (u * \rho^\varepsilon)^2, f \rangle_{L^2(\mathbb{T})};$$

moreover M^f has quadratic variation $\langle M^f \rangle_t = 2t \|\partial_x f\|_{L^2}^2$.

- iii. The Itô trick works: for all cylinder functions φ and all $p \geq 1$ we have

$$\mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \varphi(u_s) ds \right|^p \right] \lesssim T^{p/2} \|c_{2p}^{\mathcal{N}}(-\mathcal{L}_0)^{-1/2} \varphi\|^p.$$

Remark 4.13 In [27,28] so called *stationary energy solutions* to the Burgers equation are defined. The definition in [27] makes the following alternative assumptions:

- i'. For all times $t \geq 0$ the law of u_t is μ ;
- ii'. the conditions in ii. above hold, and additionally the process $\lim_{\varepsilon \rightarrow 0} \int_0^t \mathcal{L}^\varepsilon u_s(f) ds$ has vanishing quadratic variation;
- iii'. for $T \geq 0$ let $\hat{u}_t = u_{T-t}$; then $\hat{M}_t^f = \hat{u}_t(f) - \hat{u}_0(f) + \lim_{\varepsilon \rightarrow 0} \int_0^t \mathcal{L}^\varepsilon \hat{u}_s(f) ds$ is a continuous martingale in the filtration generated by (\hat{u}_t) , with quadratic variation $\langle \hat{M}^f \rangle_t = 2t \|\partial_x f\|_{L^2}^2$.

Clearly i'. and ii'. are stronger than i. and ii., and it is shown [34, Proposition 3.2] that any process satisfying i'. , ii'. , iii'. also satisfies the first inequality in

$$\begin{aligned} \mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \mathcal{L}_0 \varphi(u_s) ds \right|^p \right] &\lesssim T^{p/2} \|(\mathcal{E}\varphi)^{p/4}\|^2 \lesssim T^{p/2} \|c_p^{\mathcal{N}}(\mathcal{E}\varphi)^{1/2}\|^p \\ &\simeq T^{p/2} \|c_p^{\mathcal{N}}(-\mathcal{L}_0)^{1/2}\varphi\|^p, \end{aligned} \tag{40}$$

where the second inequality uses Remark 4.5, and the third inequality is from (36). If $\int \varphi d\mu = 0$, we can solve the equation $-\mathcal{L}_0\psi = \varphi$ and then (40) applied to ψ gives

$$\mathbb{E} \left[\sup_{t \leq T} \left| \int_0^t \varphi(u_s) ds \right|^p \right] \lesssim T^{p/2} \|c_p^{\mathcal{N}}(-\mathcal{L}_0)^{-1/2}\varphi\|^p,$$

i.e. a stronger version of iii. Therefore, we also have uniqueness in law for any process which satisfies i'. , ii'. and iii'. , or alternatively i., ii., and (40).

Note that the constant $c_{2p}^{\mathcal{N}}$ in iii. is not a typo. This is what we get if we consider a non-stationary process whose initial condition has an L^2 -density with respect to μ and we apply Lemma 4.3 to pass to a stationary process that has the properties above.

In what follows we fix the filtration $\mathcal{F}_t = \sigma(u_s : s \in [0, t])$, $t \geq 0$, and we assume that u solves the cylinder function martingale problem for \mathcal{L} with initial distribution ν .

Lemma 4.14 *Let $\varphi(u) = \Phi(u(f_1), \dots, u(f_k)) \in \mathcal{C}$ be a cylinder function. Then the process*

$$M_t^\varphi = \varphi(u_t) - \varphi(u_0) - \lim_{m \rightarrow \infty} \int_0^t \mathcal{L}^m \varphi(u_s) ds$$

is a continuous martingale with respect to (\mathcal{F}_t) , where for $B_m(u) := \partial_x \Pi_m(\Pi_m u)^2$:

$$\mathcal{L}^m \varphi(u) = \mathcal{L}_0 \varphi(u) + \sum_{i=1}^k \partial_i \Phi(u(f_1), \dots, u(f_k)) \langle B_m(u), f_i \rangle_{L^2(\mathbb{T})}.$$

Proof Let us write

$$\begin{aligned} u_t^m(f) &:= u_0(f) + \int_0^t u_s(\Delta f) ds + A_t^{m,f} + M_t^f \\ &:= u_0(f) + \int_0^t u_s(\Delta f) ds + \int_0^t \langle B_m(u_s), f \rangle_{L^2(\mathbb{T})} ds + M_t^f, \end{aligned}$$

for $f \in C^\infty(\mathbb{T})$. Then by Itô’s formula the process

$$\varphi(u_t^m) - \varphi(u_0^m) - \int_0^t \mathcal{L}_0 \varphi(u_s^m) ds - \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s^m(f_1), \dots, u_s^m(f_k)) dA_s^{m, f_i}$$

is a martingale. In [34, Corollary 3.17] it is shown that for all $\alpha < 3/4$ and all $T > 0$ and $p \in [1, \infty)$ we have $\mathbb{E}[\|A^{m, f_i} - A^{f_i}\|_{C^\alpha([0, T], \mathbb{R})}^p] \rightarrow 0$ for the limit A^{f_i} of A^{m, f_i} . Here $C^\alpha([0, T], \mathbb{R})$ is the space of α -Hölder continuous functions. Strictly speaking [34] only consider the approximation $\partial_x(\Pi_m u)^2$ of the nonlinearity, but it is not difficult to generalize the analysis to $\partial_x \Pi_m(\Pi_m u)^2$. In particular, we have

$$\lim_{m \rightarrow \infty} \mathbb{E} \left[\left| \varphi(u_t^m) - \varphi(u_0^m) - \int_0^t \mathcal{L}_0 \varphi(u_s^m) ds - \left(\varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}_0 \varphi(u_s) ds \right) \right|^p \right] = 0.$$

Moreover, we can interpret $\int_0^t \sum_{i=1}^k \partial_i \Phi(u_s^m(f_1), \dots, u_s^m(f_k)) dA_s^{m, f_i}$ as a Young integral. Therefore, Theorem 1.16 in [45] together with the Cauchy–Schwarz inequality yields

$$\begin{aligned} & \mathbb{E} \left[\left| \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s^m(f_1), \dots, u_s^m(f_k)) dA_s^{m, f_i} - \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s(f_1), \dots, u_s(f_k)) dA_s^{m, f_i} \right| \right] \\ & \lesssim \sum_{i=1}^k \mathbb{E}[\|\partial_i \Phi(u^m(f_1), \dots, u^m(f_k)) - \partial_i \Phi(u(f_1), \dots, u(f_k))\|_{C^\beta([0, T], \mathbb{R})}^2]^{1/2} \mathbb{E}[\|A^{m, f_i}\|_{C^\alpha([0, T], \mathbb{R})}^2]^{1/2}, \end{aligned}$$

for $\beta > 1 - \alpha$ and $\alpha < 3/4$. Since $\partial_i \Phi$ is locally Lipschitz continuous with polynomial growth of the derivative and we may take $\beta < \alpha$, and since u^m converges to u in $L^p(C^\alpha([0, T], \mathbb{R}))$, the first expectation on the right hand side converges to zero. The second expectation $\mathbb{E}[\|A^{m, f_i}\|_{C^\alpha([0, T], \mathbb{R})}^2]$ is uniformly bounded in m , and therefore the left hand side converges to zero. Similar arguments show that

$$\begin{aligned} & \lim_{m \rightarrow \infty} \mathbb{E} \left[\left| \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s(f_1), \dots, u_s(f_k)) dA_s^{m, f_i} - \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s(f_1), \dots, u_s(f_k)) dA_s^{f_i} \right| \right] = 0, \end{aligned}$$

and since all the convergences are in L^1 we get that

$$\begin{aligned}
 M_t^\varphi &= \varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}_0\varphi(u_s)ds - \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s(f_1), \dots, u_s(f_k))dA_s^{f_i} \\
 &= \varphi(u_t) - \varphi(u_0) - \int_0^t \mathcal{L}_0\varphi(u_s)ds - \lim_{m \rightarrow \infty} \int_0^t \sum_{i=1}^k \partial_i \Phi(u_s(f_1), \dots, u_s(f_k))dA_s^{m, f_i}
 \end{aligned}$$

is a continuous martingale. □

While it may not be obvious from the proof, here we already used that the Itô trick works for (u_t) . Indeed, Corollary 3.17 of [34] crucially relies on it.

Theorem 4.15 *Let u solve the cylinder function martingale problem for \mathcal{L} with initial distribution ν . Then u solves the martingale problem for \mathcal{L} in the sense of Sect. 4.1, and in particular its law is unique by Theorem 4.8.*

Proof Let $\varphi \in \mathcal{D}(\mathcal{L})$ and define φ^M via $\mathcal{F}(\varphi_n^M)(k) = \mathbb{1}_{n \leq M} \mathbb{1}_{|k| \leq M} \hat{\varphi}_n(k)$. In particular, $\varphi^M \in \mathcal{C}$ and by Lemma 4.14 the process

$$M_t^{\varphi^M} = \varphi^M(u_t) - \varphi^M(u_0) - \lim_{m \rightarrow \infty} \int_0^t \mathcal{L}^m \varphi^M(u_s)ds$$

is a martingale. By construction $\mathbb{E}[\varphi^M(u_t) - \varphi^M(u_0) - \varphi(u_t) - \varphi(u_0)] \rightarrow 0$ as $M \rightarrow \infty$, so if we can show that

$$\lim_{M \rightarrow \infty} \mathbb{E} \left[\left| \lim_{m \rightarrow \infty} \int_0^t \mathcal{L}^m \varphi^M(u_s)ds - \int_0^t \mathcal{L}\varphi(u_s)ds \right| \right] = 0,$$

then the proof is complete. We saw in the proof of Lemma 4.14 that the integral $\int_0^t \mathcal{L}^m \varphi^M(u_s)ds$ converges in L^1 as $m \rightarrow \infty$, and therefore we can exchange the limit in m with the expectation. So it suffices to show that the right hand side of the following inequality is zero:

$$\begin{aligned}
 &\lim_{M \rightarrow \infty} \lim_{m \rightarrow \infty} \mathbb{E} \left[\left| \int_0^t (\mathcal{L}^m \varphi^M - \mathcal{L}\varphi)(u_s)ds \right| \right] \\
 &\lesssim_t \lim_{M \rightarrow \infty} \lim_{m \rightarrow \infty} \|(-\mathcal{L}_0)^{-1/2}(\mathcal{L}^m \varphi^M - \mathcal{L}\varphi)\| \\
 &\lesssim \lim_{M \rightarrow \infty} \lim_{m \rightarrow \infty} [\|(-\mathcal{L}_0)^{1/2}(\varphi^M - \varphi)\| + \|(-\mathcal{L}_0)^{-1/2}(\mathcal{G}^m \varphi^M - \mathcal{G}\varphi)\|].
 \end{aligned}$$

For the first term on the right hand side this follows from the fact that $\|(-\mathcal{L}_0)^{1/2}\varphi\| \lesssim \|(-\mathcal{L}_0)^{1/2}\varphi^\sharp\|$ by Lemma 2.14 and from the dominated convergence theorem. For the second term on the right hand side we have by the triangle inequality and Lemma 2.8

$$\begin{aligned} \|(-\mathcal{L}_0)^{-1/2}(\mathcal{G}^m \varphi^M - \mathcal{G}\varphi)\| &\leq \|(-\mathcal{L}_0)^{-1/2}\mathcal{G}^m(\varphi^M - \varphi)\| \\ &\quad + \|(-\mathcal{L}_0)^{-1/2}(\mathcal{G}^m - \mathcal{G})\varphi\| \\ &\lesssim \|(-\mathcal{L}_0)^{1/2}\mathcal{N}(\varphi^M - \varphi)\| + \|(-\mathcal{L}_0)^{-1/2}(\mathcal{G}^m - \mathcal{G})\varphi\|. \end{aligned}$$

The first term vanishes as $M \rightarrow \infty$. The second term vanishes by the uniform estimates of Lemma 2.8 together with the dominated convergence theorem. □

5 Extensions

The uniqueness in law of solutions to the cylinder function martingale problem is not new, the stationary case was previously treated in [34] and a non-stationary case (even slightly more general than the one covered here) in [35]. This was extended to Burgers equation with Dirichlet boundary conditions in [36]. However, these works are crucially based on the Cole–Hopf transform that linearizes the equation, and they do not say anything about the generator \mathcal{L} . In the following we show that our arguments adapt to some variants of Burgers equation, none of which can be linearized via the Cole–Hopf transform. In that sense our new approach is much more robust than the previous works.

5.1 Multi-component Burgers equation

Let us consider the multi-component Burgers equation studied in [21,44]. This equation reads for $u \in C(\mathbb{R}_+, (S')^d)$ as

$$\partial_t u^i = \Delta u^i + \sum_{j,j'=1}^d \Gamma_{jj'}^i \partial_x(u^j u^{j'}) + \sqrt{2} \partial_x \xi^i, \quad i = 1, \dots, d,$$

where (ξ^1, \dots, ξ^d) are independent space-time white noises and we assume the so called *trilinear condition* of [21]:

$$\Gamma_{jj'}^i = \Gamma_{j'j}^i = \Gamma_{ji}^j,$$

i.e. that Γ is symmetric in its three arguments (i, j, j') . Under this condition the product measure $\mu^{\otimes d}$ is invariant for u , also at the level of the Galerkin approximation, see Proposition 5.5 of [21]. We can interpret $\mu^{\otimes d}$ as a white noise on $L_0^2(\{1, \dots, d\} \times \mathbb{T}) \simeq L_0^2(\mathbb{T}, \mathbb{R}^d)$, equipped with the inner product

$$\langle f, g \rangle_{L^2(\mathbb{T} \times \{1, \dots, d\})} := \sum_{i=1}^d \langle f^i, g^i \rangle_{L^2(\mathbb{T})} := \sum_{i=1}^d \langle f(i, \cdot), g(i, \cdot) \rangle_{L^2(\mathbb{T})}$$

and where we assume that $\hat{f}(i, 0) := \hat{f}^i(0) = 0$ for all i , and similarly for g ; see also Example 1.1.2 of [49]. To simplify notation we write $\mathbb{T}_d = \mathbb{T} \times \{1, \dots, d\}$

from now on, not to be confused with \mathbb{T}^d . Cylinder functions now take the form $\varphi(u) = \Phi(u(f_1), \dots, u(f_J))$ for $\Phi \in C_p^2(\mathbb{R}^J)$ and $f_j \in C^\infty(\mathbb{T}_d) \simeq C^\infty(\mathbb{T}, \mathbb{R}^d)$, where the duality pairing $u(f)$ is defined as

$$u(f) = \sum_{i=1}^d u^i(f^i) = \sum_{i=1}^d u^i(f(i, \cdot)),$$

and in what follows we switch between the notations $f^i(x) = f(i, x)$ depending on what is more convenient. The chaos expansion takes symmetric kernels $\varphi_n \in L_0^2(\mathbb{T}_d^n)$ as input, and the Malliavin derivative acts on the cylinder function $\varphi(u) = \Phi(u(f_1), \dots, u(f_J))$ with $f_j \in C^\infty(\mathbb{T}_d) \simeq C^\infty(\mathbb{T}, \mathbb{R}^d)$ and $\Phi \in C_p^2(\mathbb{R}^J)$ as

$$\begin{aligned} D_\zeta \varphi &= D_{(ix)} \varphi = \sum_{j=1}^J \partial_j \Phi(u(f_1), \dots, u(f_J)) f_j^i(x) \\ &= \sum_{j=1}^J \partial_j \Phi(u(f_1), \dots, u(f_J)) f_j(\zeta), \end{aligned}$$

where from now on we write ζ for the elements of \mathbb{T}_d . As for $d = 1$, we also have $D_\zeta W_n(\varphi_n) = nW_{n-1}(\varphi_n(\zeta, \cdot))$. Let us write formally

$$\begin{aligned} B(u)(\zeta) &= B(u)(i, x) = \sum_{j, j'=1}^d \Gamma_{jj'}^i \partial_x (u^j u^{j'})(x) \\ &= W_2 \left(\sum_{j, j'=1}^d \Gamma_{jj'}^i \partial_x (\delta_{(jx)} \otimes \delta_{(j'x)}) \right), \end{aligned}$$

where $\delta_{(jx)}(iy) = \mathbb{1}_{i=j} \delta(x - y)$. Then the Burgers part of the generator is formally given by

$$\mathcal{G}\varphi(u) = \langle B(u), D\varphi(u) \rangle_{L^2(\mathbb{T}_d)} =: \int_{\zeta} B(u)(\zeta) D_\zeta \varphi(u) d\zeta.$$

This becomes rigorous if we consider the Galerkin approximation with cutoff Π_m , but for simplicity we continue to formally argue for $m = \infty$. We have the following generalization of Lemma 2.4:

Lemma 5.1 *We have $\mathcal{G} = \mathcal{G}_+ + \mathcal{G}_-$, where*

$$\mathcal{G}_+ W_n(\varphi_n) = nW_{n+1} \left(\int_{(ix)} \sum_{j, j'=1}^d \Gamma_{jj'}^i \partial_x (\delta_{(jx)} \otimes \delta_{(j'x)}) \otimes \varphi_n((ix), \cdot) \right),$$

$$\mathcal{G}_- W_n(\varphi_n) = 2n(n-1)W_{n-1} \left(\int_{(i_1x_1), (i_2x_2)} \sum_{j, j'=1}^d \Gamma_{jj'}^{i_1} \partial_{x_1} (\delta_{(jx_1)}(i_2x_2)\delta_{(j'x_1)}) \otimes \varphi_n((i_1x_1), (i_2x_2), \cdot) \right).$$

Moreover we have for all $\varphi_{n+1} \in L^2(\mathbb{T}^{n+1})$ and $\varphi_n \in L^2(\mathbb{T}^n)$:

$$\langle W_{n+1}(\varphi_{n+1}), \mathcal{G}_+ W_n(\varphi_n) \rangle = -\langle \mathcal{G}_- W_{n+1}(\varphi_{n+1}), W_n(\varphi_n) \rangle.$$

Proof This follows similarly as in Lemma 2.4, making constant use of the trilinear condition for Γ . □

The Fourier variables now are indexed by $\mathbb{Z}_0 \times \{1, \dots, d\} =: \mathbb{Z}_d$, and we write (ik) or κ for the elements of \mathbb{Z}_d , and

$$\hat{f}(\kappa) = \hat{f}(ik) = \int_{\mathbb{T}} e^{-2\pi i \kappa x} f(i, x) dx, \quad \kappa = (ik) \in \mathbb{Z}_d.$$

We have for $\varphi = \sum_n W_n(\varphi_n)$:

$$\|\varphi\|^2 = \sum_n n! \sum_{\kappa \in \mathbb{Z}_d^n} |\hat{\varphi}_n(\kappa)|^2.$$

Lemma 5.2 *In Fourier variables the operators $\mathcal{L}_0, \mathcal{G}_+, \mathcal{G}_-$ are given by*

$$\mathcal{F}(\mathcal{L}_0\varphi)_n(\kappa_{1:n}) = -(|2\pi k_1|^2 + \dots + |2\pi k_n|^2)\hat{\varphi}_n(\kappa_{1:n}),$$

$$\mathcal{F}(\mathcal{G}_+\varphi)_n(\kappa_{1:n}) = -(n-1) \sum_{i=1}^d \Gamma_{i_1 i_2}^i 2\pi i (k_1 + k_2) \hat{\varphi}_n((i(k_1 + k_2)), \kappa_{3:n+1}),$$

$$\mathcal{F}(\mathcal{G}_-\varphi)_n(\kappa_{1:n}) = -2\pi i k_1 n(n+1) \sum_{j_1, j_2=1}^d \Gamma_{j_1 j_2}^{i_1} \sum_{p+q=k_1} \hat{\varphi}_n((j_1 p), (j_2 q), \kappa_{2:n+1}),$$

respectively.

Proof The proof is more or less the same as for $d = 1$. □

In other words, \mathcal{G}_+ and \mathcal{G}_- are finite linear combinations of some mild variations of the operators that we considered in $d = 1$. In particular they satisfy the same estimates and we obtain the existence and uniqueness of solutions to the martingale problem for $\mathcal{L} = \mathcal{L}_0 + \mathcal{G}_+ + \mathcal{G}_-$ as before, and also for the cylinder function martingale problem.

5.2 Fractional Burgers equation

In the paper [27] the authors not only study the stochastic Burgers equation, but also the fractional generalization

$$\partial_t u = -A^\theta u + \partial_x u^2 + A^{\theta/2} \xi,$$

for $\theta > 1/2$ and $A = -\Delta$. They define and construct stationary energy solutions for all $\theta > 1/2$, and they prove uniqueness in distribution for $\theta > 5/4$. Here we briefly sketch how to adapt our arguments to obtain the uniqueness for $\theta > 3/4$, also in the non-stationary case as long as the initial condition is absolutely continuous with density in $L^2(\mu)$. Unfortunately we cannot treat the limiting case $\theta = 3/4$ which would formally be scale-invariant and which plays an important role in the work [29].

In Section 4 of [27] it is shown that, just as for $\theta = 1$, the white noise is an invariant measure for μ . By adapting the arguments of Lemma 3.7 in [34] we see that the (formal) generator of u is given by

$$\mathcal{L} = \mathcal{L}_\theta + \mathcal{G},$$

where

$$\mathcal{F}(\mathcal{L}_\theta \varphi)_n(k_{1:n}) = -(|2\pi k_1|^{2\theta} + \dots + |2\pi k_n|^{2\theta}) \hat{\varphi}_n(k_{1:n}).$$

Up to multiples of \mathcal{N} we can estimate $(-\mathcal{L}_\theta)$ by $(-\mathcal{L}_0)^\theta$ and vice versa, so we would expect that $(-\mathcal{L}_\theta)^{-1}$ gains regularity of order $(-\mathcal{L}_0)^{-\theta}$. We saw in Lemma 2.8 that \mathcal{G} loses $(-\mathcal{L}_0)^{3/4}$ regularity, and therefore it is canonical to assume $\theta > 3/4$. To construct controlled functions we only need to slightly adapt Lemma 2.14 and to replace $(-\mathcal{L}_0)^{-1}$ by $(-\mathcal{L}_\theta)^{-1}$. For simplicity we restrict our attention to $\theta \leq 1$ because this allows us to estimate

$$(|k_1|^{2\theta} + \dots + |k_n|^{2\theta})^{-1} \leq (k_1^2 + \dots + k_n^2)^{-\theta}, \quad \text{i.e. } \|(-\mathcal{L}_\theta)^{-1} \varphi\| \leq \|(-\mathcal{L}_0)^{-\theta} \varphi\|. \tag{41}$$

Lemma 5.3 *Let $\theta \in (3/4, 1]$, let w be a weight, let $\gamma \in (1/4, 1/2]$, and let $L \geq 1$. For $N_n = L(1+n)^{3/(4\theta-3)}$ we have*

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_\theta)^{-1} \mathcal{G}^\gamma \varphi\| \lesssim |w| L^{3/2-2\theta} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\|, \tag{42}$$

where the implicit constant on the right hand side is independent of w . Therefore, the construction of controlled functions $\varphi = \mathcal{K}\varphi^\sharp = (-\mathcal{L}_\theta)^{-1} \mathcal{G}_+^\gamma \varphi + \varphi^\sharp$ for given φ^\sharp works as in Lemma 2.14.

Proof We treat \mathcal{G}_+^γ and \mathcal{G}_-^γ separately. We use (41) and that $1 - 2\gamma \geq 0$ to estimate the \mathcal{G}_+^γ term as in the proof of Lemma 2.14:

$$\begin{aligned} & \sum_{k_{1:n}} |\mathcal{F}((-\mathcal{L}_0)^\gamma (-\mathcal{L}_\theta)^{-1} \mathcal{G}_+^\gamma \varphi)_n(k_{1:n})|^2 \\ & \lesssim n \sum_{\ell_{1:n-1}, p} (\mathbb{1}_{|p| \geq N_n/2} + \mathbb{1}_{|\ell_{1:n-1}|_\infty \geq N_n/2}) \frac{(\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma}}{(|p|^2 + |\ell_1|^2 + \dots + |\ell_{n-1}|^2)^{2\theta-1}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \end{aligned}$$

$$\begin{aligned} &\lesssim n \sum_{\ell_{1:n-1}} \left(N_n^{3-4\theta} + \frac{\mathbb{1}_{|\ell_{1:n-1}|_\infty \geq N_n}}{(\ell_1^2 + \dots + \ell_{n-1}^2)^{2\theta-3/2}} \right) (\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ &\lesssim n \sum_{\ell_{1:n-1}} N_n^{3-4\theta} (\ell_1^2 + \dots + \ell_{n-1}^2)^{2\gamma} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2, \end{aligned}$$

where the third step follows from Lemma A.1 (and here we need $\theta < 3/4$).

For the \mathcal{G}_- term we have by the same arguments as in Lemma 2.14 and using (41) and that $\theta > 3/4$

$$\begin{aligned} &\sum_{k_{1:n}} |\mathcal{F}((-\mathcal{L}_0)^\gamma (-\mathcal{L}_\theta)^{-1} \mathcal{G}_-^\gamma \varphi)_n(k_{1:n})|^2 \\ &\lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 (k_1^2)^{3/2}}{(k_1^2 + \dots + k_n^2)^{2\theta}} \sum_{p+q=k_1} (p^2 + q^2)^{2\gamma} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ &\leq N_n^{3-4\theta} n^4 \sum_{\ell_{1:n+1}} (\ell_1^2 + \dots + \ell_{n+1}^2)^{2\gamma} |\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2, \end{aligned}$$

and from now on the proof is the same as for Lemma 2.14. □

Proposition 2.18 remains essentially unchanged in the fractional setting, because for $\varphi = \mathcal{K}\varphi^\sharp$ we have $\mathcal{L}\varphi = \mathcal{G}^<\varphi + \mathcal{L}_\theta\varphi^\sharp$. The only difference is that, since we still want to measure regularity in terms of $(-\mathcal{L}_0)$, we have $\|\mathcal{L}_\theta\varphi^\sharp\| \lesssim \|\mathcal{N}^{1-\theta}(-\mathcal{L}_0)\varphi^\sharp\|$ by Hölder’s inequality. Also the proof of Lemma 2.19 carries over to our setting. And also the analysis of the backward equation is more or less the same as before. The main difference is that now we only have a priori estimates in $w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-\theta/2}\Gamma L^2$ and no longer in $w(\mathcal{N})^{-1}(-\mathcal{L}_0)^{-1/2}\Gamma L^2$. But for the controlled analysis it is only important to have an a priori estimate in $(-\mathcal{L}_0)^{-1/4-\delta}\Gamma L^2$, because that is what we need to control the contribution from $\mathcal{G}^<$. So since $\theta/2 > 3/8 > 1/4$ the same arguments work, and then we obtain the existence and uniqueness of solutions to the backward equation and to the martingale problem by the same arguments as for $\theta = 1$, and also the cylinder function martingale problem has unique solutions.

5.3 Burgers equation on the real line

The stochastic Burgers equation on $\mathbb{R}_+ \times \mathbb{R}$ has essentially the same structure as the equation on $\mathbb{R}_+ \times \mathbb{T}$. The only difference is that now we have to work with Fourier integrals instead of Fourier sums, which might lead to divergences at $k \simeq 0$. But since most of our estimates boil down to an application of Lemma A.1, and this lemma remains true if the sum in k is replaced by an integral, most of our estimates still work on the full space. In fact *all* estimates in Sect. 2 remain true, but some of them are not so useful any more because we no longer have $\|\varphi\| \lesssim \|(-\mathcal{L}_0)^\gamma \varphi\|$ for $\gamma > 0$ and $\int \varphi d\mu = 0$. But we can strengthen the results as follows (with the difference to the previous results marked in blue):

- In Lemma 2.14 we can use the cutoff $\mathbb{1}_{|k_{1:n}|_\infty > N_n}$ to estimate

$$\begin{aligned} & \|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \varphi\| \\ & \lesssim \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \varphi\| \lesssim |w| L^{-1/2} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma \varphi\| \\ & \leq |w| L^{-1/2} \|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma \varphi\| \end{aligned}$$

and thus

$$\begin{aligned} & \|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma \mathcal{K} \varphi^\sharp\| + L^{1/2} \|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma (\mathcal{K} \varphi^\sharp - \varphi^\sharp)\| \\ & \lesssim \|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma \varphi^\sharp\|. \end{aligned}$$

Similarly, we get in Lemma A.2 the better bound

$$\|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \varphi\| \lesssim |w| \|w(\mathcal{N})(1 + \mathcal{N})^{3/2} (-\mathcal{L}_0)^{\gamma-1/4} \varphi\|.$$

- In the proof Proposition 2.18 we used the bound $(k_1^2 + \dots + k_n^2)^{2\gamma} \mathbb{1}_{|k_{1:n}|_\infty \leq N_n} \leq n^{2\gamma} N_n^{4\gamma}$, and of course this works also with $(1 + k_1^2 + \dots + k_n^2)^{2\gamma}$, so that we get the slightly stronger result

$$\|w(\mathcal{N})(1 - \mathcal{L}_0)^\gamma \mathcal{G}^\gamma \varphi\| \lesssim \|w(\mathcal{N})(1 + \mathcal{N})^{9/2+7\gamma} (-\mathcal{L}_0)^{1/4+\delta} \varphi^\sharp\|.$$

- The definition of the domain in Lemma 2.19 is problematic now, because it does not even guarantee that $\mathcal{D}(\mathcal{L}) \subset \Gamma L^2$. So instead we set

$$\mathcal{D}_w(\mathcal{L}) := \{\mathcal{K} \varphi^\sharp : \varphi^\sharp \in w(\mathcal{N})^{-1} (-\mathcal{L}_0)^{-1} \Gamma L^2 \cap w(\mathcal{N})^{-1} (1 + \mathcal{N})^{-9/2} (1 - \mathcal{L}_0)^{-1/2} \Gamma L^2\},$$

and then we get from the stronger version of Lemma 2.14 the better estimate

$$\begin{aligned} & \|w(\mathcal{N})(1 - \mathcal{L}_0)^{1/2} (\varphi^M - \psi)\| \lesssim M^{-1/2} \|w(\mathcal{N})(1 - \mathcal{L}_0)^{1/2} \psi\|, \\ & \|w(\mathcal{N})(1 - \mathcal{L}_0)^{1/2} \varphi^M\| \lesssim \|w(\mathcal{N})(1 - \mathcal{L}_0)^{1/2} \psi\|. \end{aligned}$$

- The analysis in Sect. 3.1 does not change, and Lemma 3.1 together with Corollary 3.2 give as an a priori bound on $\|(1 + \mathcal{N})^\alpha (1 - \mathcal{L}_0)^{1/2} \varphi^m\|$ and $\|(1 + \mathcal{N})^\alpha \partial_t \varphi^m\|$ in terms of φ_0^m .
- In the controlled analysis of Sect. 3.2 we can strengthen the bound from Lemma 3.4 to control $\|(1 + \mathcal{N})^\alpha (1 - \mathcal{L}_0)^{1/2} \varphi^{m,\sharp}\|$ in terms of $\varphi_0^{m,\sharp}$, and this is sufficient to control $(1 - \mathcal{L}_0)^\gamma \mathcal{G}^{m,\prec} \varphi^m$. Throughout, we replace all bounds for terms of the form $(-\mathcal{L}_0)^\gamma (\cdot)$ by corresponding bounds for $(1 - \mathcal{L}_0)^\gamma (\cdot)$. Here we need the strengthened version of Lemma A.2 mentioned in the first bullet point, and we also use that $\|(1 + \mathcal{N})^\alpha (1 - \mathcal{L}_0)^\beta S_t \psi\| \lesssim (t^{-\beta} \vee 1) \|(1 + \mathcal{N})^\alpha \psi\|$.
- The existence proof for strong solutions to the backward equation was based on the fact that, on the torus, bounded sets in $(1 + \mathcal{N})^{-\kappa} (1 - \mathcal{L}_0)^{-\gamma} \Gamma L^2$ are relatively compact in $(1 + \mathcal{N})^{-\kappa'} (1 - \mathcal{L}_0)^{-\gamma'} \Gamma L^2$ if $\kappa' < \kappa$ and $\gamma' < \gamma$. But on \mathbb{R} this is false, for example the Sobolev space $H^1(\mathbb{R})$ is not compactly embedded in

$L^2(\mathbb{R})$. On the other hand, bounded sequences in any separable Hilbert space have weakly convergent subsequences, and in Lemma A.4 we prove a version of the Arzelà–Ascoli theorem for the weak topology. So we let

$$\mathcal{U}_\alpha := \bigcup_{\gamma \in (3/8, 5/8)} \mathcal{K}(1 + \mathcal{N})^{-p(\alpha, \gamma)}(1 - \mathcal{L}_0)^{-1-\gamma} \Gamma L^2 \subseteq \Gamma L^2,$$

and we replace the compactness argument in the proof of Theorem 3.6 by a weak compactness argument. By the Fatou property of the norm under weak convergence, we deduce that any weak limit point φ^\sharp of $(\varphi^{m, \sharp})_m$ is in $C(\mathbb{R}_+, (1 + \mathcal{N})^{-\alpha+\delta}(-\mathcal{L}_0)^{-1} \Gamma L^2)$. Moreover, the weak convergence is sufficient to identify $\varphi(t) - \varphi(0) = \int_0^t (\mathcal{L}_0 \varphi^\sharp(s) + \mathcal{G}^\prec \mathcal{K} \varphi^\sharp(s)) ds$, where $\varphi = \mathcal{K} \varphi^\sharp$. After that, the arguments are the same as on \mathbb{T} .

- Existence and uniqueness for the martingale problem are shown in exactly the same way as on the torus, the only difference is that we have to use the stronger version of Proposition 2.18 to approximate cylinder functions by functions in $\mathcal{D}(\mathcal{L})$.
- The cylinder function martingale problem is more complicated: In the proof of Theorem 4.15 we used that $\|(-\mathcal{L}_0)^{-1/2} \mathcal{G} \varphi\| \lesssim \|(-\mathcal{L}_0)^{1/2} \varphi\|$, which is no longer true on the full space. But we can decompose $\mathcal{G} = \mathcal{G}_- + \mathcal{G}_+$ and estimate the contribution from \mathcal{G}_- by directly using Lemma 2.8 for $\gamma = 3/4$, without applying the Itô trick (it follows from Young’s inequality for products that $\mathcal{D}(\mathcal{L}) \subset (1 + \mathcal{N})^{-1}(-\mathcal{L}_0)^{-3/4} \Gamma L^2$). And for \mathcal{G}_+ we can use the Itô trick together with the bound $\|(-\mathcal{L}_0)^{-1/2} \mathcal{G}_+ \varphi\| \lesssim \|(-\mathcal{L}_0)^{1/4} \varphi\| \lesssim \|(1 - \mathcal{L}_0)^{1/2} \varphi\|$, where the right hand side is under control.

In that way all results from Sects. 2.3–4 apart from Sect. 4.3 carry over to Burgers equation on \mathbb{R} . Of course, the exponential ergodicity of Sect. 4.3 does not hold on the full space, because \mathcal{L}_0 no longer has a spectral gap.

But we can still prove a qualitative ergodicity result. By (the full space version of) Lemma 2.22 we know that

$$\langle \varphi, \mathcal{L} \varphi \rangle = -\|(-\mathcal{L}_0)^{1/2} \varphi\|^2,$$

for all $\varphi \in \mathcal{D}(\mathcal{L})$. By (the full space version of) Proposition 4.11, the Hille–Yosida generator $\hat{\mathcal{L}}$ of the semigroup (T_t) is the closure of \mathcal{L} . So for all $\varphi \in \mathcal{D}(\hat{\mathcal{L}})$ there exists a sequence $(\varphi^M) \subset \mathcal{D}(\mathcal{L})$ such that $\varphi^M \rightarrow \varphi$ and $\mathcal{L} \varphi^M \rightarrow \hat{\mathcal{L}} \varphi$ in ΓL^2 . Then Fatou’s lemma gives

$$\langle \varphi, -\hat{\mathcal{L}} \varphi \rangle = \lim_{M \rightarrow \infty} \langle \varphi^M, -\hat{\mathcal{L}} \varphi^M \rangle \geq \|(-\mathcal{L}_0)^{1/2} \varphi\|^2.$$

So if $\varphi \in \mathcal{D}(\hat{\mathcal{L}})$ is such that μ -almost surely $\hat{\mathcal{L}} \varphi = 0$, then

$$0 = \langle \varphi, -\hat{\mathcal{L}} \varphi \rangle \geq \|(-\mathcal{L}_0)^{1/2} \varphi\|^2,$$

and therefore $(-\mathcal{L}_0)^{1/2} \varphi = 0$. With the Fourier representation of \mathcal{L}_0 this easily implies that $\varphi - \int \varphi d\mu = 0$, i.e. the only functions $\varphi \in \mathcal{D}(\hat{\mathcal{L}})$ with $\hat{\mathcal{L}} \varphi = 0$ are constants. If

$\varphi \in \Gamma L^2$ is such that $T_t \varphi = \varphi$, then $\varphi \in \mathcal{D}(\hat{L})$ and $\hat{L}\varphi = 0$, and therefore the only invariant functions for T_t are constants. This proves ergodicity by general principles, see [16, Theorem 3.2.4].

As far as we are aware, the ergodicity of the stochastic Burgers equation on \mathbb{R} is a new result. For more regular noise, the ergodicity on \mathbb{R} was recently shown by Bakhtin and Li [5] and by Dunlap et al. [11]. Both of these works prove a *one-force-one-solution principle*, which is stronger than ergodicity.

Acknowledgements We would like to thank the Isaac Newton Institute for Mathematical Sciences for support and hospitality during the programme SRQ: Scaling limits, Rough paths, Quantum field theory when part of the work on this paper was undertaken. We are grateful to the anonymous referees for their very detailed reports, which helped to greatly clarify the presentation. Moreover, the argument in Lemma 3.3 was kindly pointed out to us by one of the referees. The main part of the work was done while N.P. was employed at Max-Planck-Institute for Mathematics in the Sciences, Leipzig, and at Humboldt-Universität zu Berlin.

Funding Open Access funding provided by Projekt DEAL.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix A: Auxiliary results

The following simple estimate is used many times, so we formulate it as a lemma.

Lemma A.1 *Let $C \geq 0$, $a > 1/2$, and $k \in \mathbb{Z}$ be such that $k^2 + C > 0$. Then*

$$\sum_{p+q=k} \left(\frac{1}{p^2 + q^2 + C} \right)^a = \sum_p \left(\frac{1}{p^2 + (k - p)^2 + C} \right)^a \lesssim \left(\frac{1}{k^2 + C} \right)^{a-\frac{1}{2}}.$$

Proof Since $p^2 + (k - p)^2 \simeq p^2 + k^2$, we have

$$\begin{aligned} \sum_p \left(\frac{1}{p^2 + (k - p)^2 + C} \right)^a &\lesssim \int_0^\infty \left(\frac{1}{y^2 + k^2 + C} \right)^a dy \\ &= (k^2 + C)^{-a} \int_0^\infty \left(\frac{1}{\left(\frac{y}{\sqrt{k^2 + C}}\right)^2 + 1} \right)^a dy \\ &= (k^2 + C)^{-a+\frac{1}{2}} \int_0^\infty \left(\frac{1}{y^2 + 1} \right)^a dy, \end{aligned}$$

and since $2a > 1$ the integral on the right hand side is finite and our claim follows. \square

Lemma A.2 *In the context of Lemma 2.14 let now $\gamma \in (1/2, 3/4)$. Then we have*

$$\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}^\gamma \varphi\| \lesssim |w| \|w(\mathcal{N}) \mathcal{N}^{3/2} (-\mathcal{L}_0)^{\gamma-1/4} \varphi\|.$$

Proof In Step 1 of the proof of Lemma 2.14 we derived, for all $\gamma \in \mathbb{R}$, the estimate

$$\begin{aligned} \|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}_+^\gamma \varphi\|^2 &\lesssim \sum_{n \geq 2} n! w(n)^2 n \sum_{\ell_{1:n-1}, p} \mathbb{1}_{|\ell_{1:n-1}|_\infty \vee |p| \geq N_n} \\ &\quad \times \frac{\ell_1^2 + \dots + \ell_{n-1}^2}{(p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2. \end{aligned}$$

If $2 - 2\gamma > 1/2$ (which is equivalent to $\gamma < 3/4$), it follows from Lemma A.1 that

$$\begin{aligned} &\sum_{n \geq 2} n! w(n)^2 n \sum_{\ell_{1:n-1}, p} \mathbb{1}_{|\ell_{1:n-1}|_\infty \vee |p| \geq N_n} \frac{\ell_1^2 + \dots + \ell_{n-1}^2}{(p^2 + \ell_1^2 + \dots + \ell_{n-1}^2)^{2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ &\lesssim \sum_{n \geq 2} n! w(n)^2 n \sum_{\ell_{1:n-1}} \frac{\ell_1^2 + \dots + \ell_{n-1}^2}{(\ell_1^2 + \dots + \ell_{n-1}^2)^{3/2-2\gamma}} |\hat{\varphi}_{n-1}(\ell_{1:n-1})|^2 \\ &= \sum_{n \geq 1} n!(n+1)w(n+1)^2(n+1) \sum_{\ell_{1:n}} (\ell_1^2 + \dots + \ell_n^2)^{2\gamma-1/2} |\hat{\varphi}_n(\ell_{1:n})|^2 \\ &\leq |w|^2 \|w(\mathcal{N} + 1) \mathcal{N}(-\mathcal{L}_0)^{\gamma-1/4} \varphi\|^2. \end{aligned}$$

For $(-\mathcal{L}_0)^{-1} \mathcal{G}_-^\gamma \varphi$ we argue similarly as in Step 2 of the proof of Lemma 2.14: We apply (10) with $\beta = \gamma - 1/4 > 1/4$ (here we need $\gamma > 1/2$) to estimate

$$\begin{aligned} &\sum_{k_{1:n}} |\mathcal{F}((-\mathcal{L}_0)^{\gamma-1} \mathcal{G}_-^\gamma \varphi)_n(k_{1:n})|^2 \\ &\lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 k_1^2}{(k_1^2 + \dots + k_n^2)^{2-2\gamma}} \left| \sum_{p+q=k_1} \hat{\varphi}_{n+1}(p, q, k_{2:n}) \right|^2 \\ &\lesssim \sum_{k_{1:n}} \frac{\mathbb{1}_{|k_{1:n}|_\infty \geq N_n} n^4 k_1^2 (k_1^2)^{1-2\gamma}}{(k_1^2 + \dots + k_n^2)^{2-2\gamma}} \sum_{p+q=k_1} (p^2 + q^2)^{2\gamma-1/2} |\hat{\varphi}_{n+1}(p, q, k_{2:n})|^2 \\ &\leq n^4 \sum_{\ell_{1:n+1}} (\ell_1^2 + \dots + \ell_{n+1}^2)^{2\gamma-1/2} |\hat{\varphi}_{n+1}(\ell_{1:n+1})|^2, \end{aligned}$$

which leads to $\|w(\mathcal{N})(-\mathcal{L}_0)^\gamma (-\mathcal{L}_0)^{-1} \mathcal{G}_-^\gamma \varphi\| \lesssim |w| \|w(\mathcal{N})(1+\mathcal{N})^{3/2} (-\mathcal{L}_0)^{\gamma-1/4} \varphi\|$. □

Lemma A.3 *Let $\varphi \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L})) \cap C^1(\mathbb{R}_+, \Gamma L^2)$ and let u be an incompressible solution of the martingale problem for \mathcal{L} . Then*

$$\varphi(t, u_t) - \varphi(0, u_0) - \int_0^t (\partial_s + \mathcal{L})\varphi(s, u_s) ds, \quad t \geq 0,$$

is a martingale.

Proof We discretize time: Set $t_k = kt/n$ and

$$\begin{aligned} \varphi(t, u_t) - \varphi(0, u_0) &= \sum_{k=0}^{n-1} [\varphi(t_{k+1}, u_{t_{k+1}}) - \varphi(t_k, u_{t_{k+1}}) + \varphi(t_k, u_{t_{k+1}}) - \varphi(t_k, u_{t_k})] \\ &= \sum_{k=0}^{n-1} \left[\int_{t_k}^{t_{k+1}} \partial_s \varphi(s, u_{t_{k+1}}) ds + \int_{t_k}^{t_{k+1}} \mathcal{L}\varphi(t_k, u_s) ds + M_{t_{k+1}}^{\varphi(t_k)} - M_{t_k}^{\varphi(t_k)} \right]. \end{aligned}$$

Now for $[s]^n = \min\{t_k : t_k \geq s\}$ (which depends on n because the t_k depend on n)

$$\begin{aligned} &\mathbb{E} \left[\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \partial_s \varphi(s, u_{t_{k+1}}) ds - \int_0^t \partial_s \varphi(s, u_s) ds \right] \\ &\leq \int_0^t \mathbb{E}[|\partial_s \varphi(s, u_{[s]^n}) - \partial_s \varphi(s, u_s)|] ds, \end{aligned}$$

and

$$\mathbb{E}[|\partial_s \varphi(s, u_{[s]^n}) - \partial_s \varphi(s, u_s)|] \leq \mathbb{E}[|\partial_s \varphi(s, u_{[s]^n})|] + \mathbb{E}[|\partial_s \varphi(s, u_s)|] \lesssim \|\partial_s \varphi(s)\|$$

is bounded in $[0, t]$. Moreover, by approximating $\partial_s \varphi(s)$ in ΓL^2 with continuous functions, we get $\lim_{n \rightarrow \infty} \mathbb{E}[|\partial_s \varphi(s, u_{[s]^n}) - \partial_s \varphi(s, u_s)|] = 0$ for all s , and therefore by dominated convergence

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \partial_s \varphi(s, u_{t_{k+1}}) ds - \int_0^t \partial_s \varphi(s, u_s) ds \right] = 0.$$

Since $\varphi \in C(\mathbb{R}_+, \mathcal{D}(\mathcal{L}))$ we know that $\mathcal{L}\varphi \in C(\mathbb{R}_+, \Gamma L^2)$ and thus, using once more the incompressibility,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \mathcal{L}\varphi(t_k, u_s) ds - \int_0^t \mathcal{L}\varphi(s, u_s) ds \right] = 0.$$

The convergence of the Lebesgue integrals is in L^1 , and therefore the martingale property is inherited in the limit:

$$\begin{aligned} 0 &= \lim_{n \rightarrow \infty} \mathbb{E} \left[\varphi(t, u_t) - \varphi(0, u_0) - \sum_{k=0}^{n-1} \left[\int_{t_k}^{t_{k+1}} \partial_s \varphi(s, u_{t_{k+1}}) ds + \int_{t_k}^{t_{k+1}} \mathcal{L}\varphi(t_k, u_s) ds \right] \right] \\ &= \mathbb{E} \left[\varphi(t, u_t) - \varphi(0, u_0) - \int_0^t [\partial_s \varphi(s, u_s) + \mathcal{L}\varphi(s, u_s)] ds \right], \end{aligned}$$

and similarly for the conditional expectations. □

Lemma A.4 (“Arzelà–Ascoli theorem for the weak topology”) *Let H be a separable Hilbert space, let $T > 0$, and let $\mathcal{X} \subset C([0, T], H)$ be a family of functions which is*

- (i) *uniformly bounded, i.e. $\sup_{t \in [0, T]} \sup_{f \in \mathcal{X}} \|f(t)\| < \infty$, and*
 (ii) *uniformly equicontinuous, i.e. $\lim_{\varepsilon \rightarrow 0} \sup_{f \in \mathcal{X}} \sup_{|t-s| \leq \varepsilon} \|f(t) - f(s)\| = 0$.*

Then for any sequence $(f_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ there exists a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ and an $f \in C([0, T], H)$, such that for all $h \in H$:

$$\lim_{k \rightarrow \infty} \sup_{t \in [0, T]} |\langle f(t) - f_{n_k}(t), h \rangle| = 0. \quad (43)$$

Proof Let $(t_m)_{m \in \mathbb{N}}$ be a dense subset of $[0, T]$. By uniform boundedness together with a diagonal sequence argument we can find a subsequence $(f_{n_k})_{k \in \mathbb{N}}$ of $(f_n)_{n \in \mathbb{N}} \subset \mathcal{X}$ such that for all m the sequence $f_{n_k}(t_m)$ converges weakly to some $f(t_m) \in H$. Moreover,

$$\|f(t_m) - f(t_{m'})\| \leq \liminf_{k \rightarrow \infty} \|f_{n_k}(t_m) - f_{n_k}(t_{m'})\| \leq \sup_{g \in \mathcal{X}} \sup_{|t-s| \leq |t_m - t_{m'}|} \|g(t) - g(s)\|,$$

so the uniform equicontinuity of \mathcal{X} implies that f is uniformly continuous on the dense set $(t_m) \subset [0, T]$. Therefore, it has a unique continuous extension to all of $[0, T]$, which we still denote by f . Now we apply the same arguments as in the standard proof of the Arzelà–Ascoli theorem, based on the equicontinuity of \mathcal{X} , to see that the convergence in (43) holds. \square

References

1. Assing, S.: A pregenerator for Burgers equation forced by conservative noise. *Commun. Math. Phys.* **225**(3), 611–632 (2002)
2. Bruned, Y., Chandra, A., Chevyrev, I., Hairer, M.: Renormalising SPDEs in regularity structures. To appear in *J. Europ. Math. Soc.* [arXiv:1711.10239](https://arxiv.org/abs/1711.10239) (2020)
3. Bahouri, H., Chemin, J.-Y., Danchin, R.: *Fourier Analysis and Nonlinear Partial Differential Equations*. Springer, Berlin (2011)
4. Bruned, Y., Hairer, M., Zambotti, L.: Algebraic renormalisation of regularity structures. *Invent. Math.* **215**(3), 1039–1156 (2019)
5. Bakhtin, Y., Li, L.: Thermodynamic limit for directed polymers and stationary solutions of the Burgers equation. *Commun. Pure Appl. Math.* **72**(3), 536–619 (2019)
6. Cannizzaro, G., Chouk, K.: Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.* **46**(3), 1710–1763 (2018)
7. Chandra, A., Hairer, M.: An analytic BPHZ theorem for regularity structures. [arXiv:1612.08138](https://arxiv.org/abs/1612.08138) (2016)
8. Corwin, I.: The Kardar–Parisi–Zhang equation and universality class. *Random Matrices Theory Appl.* **1**(1), 1130001 (2012)
9. Delarue, F., Diel, R.: Rough paths and 1D SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Relat. Fields* **165**(1–2), 1–63 (2016)
10. Diehl, J., Gubinelli, M., Perkowski, N.: The Kardar–Parisi–Zhang equation as scaling limit of weakly asymmetric interacting Brownian motions. *Commun. Math. Phys.* **354**(2), 549–589 (2017)
11. Dunlap, A., Graham, C., Ryzhik, L.: Stationary solutions to the stochastic Burgers equation on the line. [arXiv:1910.07464](https://arxiv.org/abs/1910.07464) (2019)
12. Da Prato, G., Debussche, A.: Two-dimensional Navier–Stokes equations driven by a space-time white noise. *J. Funct. Anal.* **196**(1), 180–210 (2002)

13. Da Prato, G., Debussche, A.: Strong solutions to the stochastic quantization equations. *Ann. Probab.* **31**(4), 1900–1916 (2003)
14. Da Prato, G., Flandoli, F., Priola, E., Röckner, M.: Strong uniqueness for stochastic evolution equations in Hilbert spaces perturbed by a bounded measurable drift. *Ann. Probab.* **41**(5), 3306–3344 (2013)
15. Da Prato, G., Flandoli, F., Röckner, M., Veretennikov, A.Y.: Strong uniqueness for SDEs in Hilbert spaces with nonregular drift. *Ann. Probab.* **44**(3), 1985–2023 (2016)
16. Da Prato, G., Zabczyk, J.: Ergodicity for infinite-dimensional systems. In: London Mathematical Society Lecture Note Series, vol. 229. Cambridge University Press, Cambridge (1996)
17. Da Prato, G., Zabczyk, J.: Stochastic Equations in Infinite Dimensions Encyclopedia, volume 152 of of Mathematics and its Applications, 2nd edn. Cambridge University Press, Cambridge (2014)
18. Ethier, S.N., Kurtz, T.G.: Markov Processes: Characterization and Convergence. Wiley, Hoboken (1986)
19. Franco, T., Gonçalves, P., Simon, M.: Crossover to the stochastic Burgers equation for the WASEP with a slow bond. *Commun. Math. Phys.* **346**(3), 801–838 (2016)
20. Friz, P.K., Hairer, M.: A course on rough paths. In: Universitext. Springer, Cham. With an introduction to regularity structures (2014)
21. Funaki, T., Hoshino, M.: A coupled KPZ equation, its two types of approximations and existence of global solutions. *J. Funct. Anal.* **273**(3), 1165–1204 (2017)
22. Flandoli, F., Luo, D.: Kolmogorov equations associated to the stochastic two dimensional Euler equations. *SIAM J. Math. Anal.* **51**(3), 1761–1791 (2019)
23. Flandoli, F., Luo, D.: Convergence of transport noise to Ornstein–Uhlenbeck for 2D Euler equations under the enstrophy measure. To appear in *Ann. Probab.* [arXiv:1806.09332](https://arxiv.org/abs/1806.09332) (2020)
24. Flandoli, F., Russo, F., Wolf, J.: Some SDEs with distributional drift I. General calculus. *Osaka J. Math.* **40**(2), 493–542 (2003)
25. Flandoli, F., Russo, F., Wolf, J.: Some SDEs with distributional drift II Lyons–Zheng structure, Itô’s formula and semimartingale characterization. *Random Oper. Stoch. Equ.* **12**(2), 145–184 (2004)
26. Gubinelli, M., Imkeller, P., Perkowski, N.: Paracontrolled distributions and singular PDEs. In: Forum of Mathematics, Pi, vol. 3, e6 (2015)
27. Gubinelli, M., Jara, M.: Regularization by noise and stochastic Burgers equations. *Stoch. Partial Differ. Equ. Anal. Comput.* **1**(2), 325–350 (2013)
28. Gonçalves, P., Jara, M.: Nonlinear fluctuations of weakly asymmetric interacting particle systems. *Arch. Ration. Mech. Anal.* **212**(2), 597–644 (2014)
29. Gonçalves, P., Jara, M.: Density fluctuations for exclusion processes with long jumps. *Probab. Theory Relat. Fields* **170**(1–2), 311–362 (2018)
30. Gonçalves, P., Jara, M., Sethuraman, S.: A stochastic Burgers equation from a class of microscopic interactions. *Ann. Probab.* **43**(1), 286–338 (2015)
31. Gubinelli, M., Perkowski, N.: Lectures on singular stochastic PDEs. *Ensaios Mat.* **29** (2015)
32. Gubinelli, M., Perkowski, N.: The Hairer–Quastel universality result at stationarity. *RIMS Kôkyûroku Bessatsu* **B59** (2016)
33. Gubinelli, M., Perkowski, N.: KPZ reloaded. *Commun. Math. Phys.* **349**(1), 165–269 (2017)
34. Gubinelli, M., Perkowski, N.: Energy solutions of KPZ are unique. *J. Am. Math. Soc.* **31**(2), 427–471 (2018)
35. Gubinelli, M., Perkowski, N.: Probabilistic approach to the stochastic Burgers equation. In: Stochastic Partial Differential Equations and Related Fields, pp. 512–527. In Honor of Michael Röckner (2018)
36. Gonçalves, P., Perkowski, N., Simon, M.: Derivation of the stochastic Burgers equation with Dirichlet boundary conditions from the WASEP. [arXiv:1710.11011](https://arxiv.org/abs/1710.11011) (2017)
37. Gubinelli, M.: Controlling rough paths. *J. Funct. Anal.* **216**(1), 86–140 (2004)
38. Gubinelli, M.: A panorama of singular SPDEs. *Proc. Int. Cong. Math.* **2**, 2277–2304 (2018)
39. Guionnet, A., Zegarliński, B.: Lectures on logarithmic Sobolev inequalities. In: Séminaire de Probabilités, XXXVI, volume 1801 of Lecture Notes in Mathematics, pp. 1–134. Springer, Berlin (2003)
40. Hairer, M.: A theory of regularity structures. *Invent. Math.* **198**(2), 269–504 (2014)
41. Hairer, M., Mattingly, J.: The strong Feller property for singular stochastic PDEs. *Ann. Inst. Henri Poincaré Probab. Stat.* **54**(3), 1314–1340 (2018)
42. Janson, S.: Gaussian Hilbert spaces. In: Cambridge Tracts in Mathematics, vol. 129. Cambridge University Press, Cambridge (1997)

43. Komorowski, T., Landim, C., Olla, S.: Fluctuations in Markov Processes, Volume 345 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer, Heidelberg. Time symmetry and martingale approximation (2012)
44. Kupiainen, A., Marozzi, M.: Renormalization of generalized KPZ equation. *J. Stat. Phys.* **166**(3–4), 876–902 (2017)
45. Lyons, T.J., Caruana, M., Lévy, T.: Differential equations driven by rough paths. In: *Lecture Notes in Mathematics*, vol. 1908. Springer, Berlin (2007)
46. Liu, W., Röckner, M.: Stochastic partial differential equations: an introduction. In: *Universitext*. Springer, Cham (2015)
47. Lyons, T.J.: Differential equations driven by rough signals. *Rev. Mat. Iberoam.* **14**(2), 215–310 (1998)
48. Mitoma, I.: Tightness of probabilities on $C([0, 1]; S')$ and $D([0, 1]; S')$. *Ann. Probab.* **11**(4), 989–999 (1983)
49. Nualart, D.: The Malliavin calculus and related topics. In: *Probability and its Applications* (New York), 2nd edn. Springer, Berlin (2006)
50. Quastel, J., Spohn, H.: The one-dimensional KPZ equation and its universality class. *J. Stat. Phys.* **160**(4), 965–984 (2015)
51. Quastel, J.: Introduction to KPZ. In: *Current Developments in Mathematics, 2011*, pp. 125–194. International Press, Somerville (2012)
52. Röckner, M., Zhu, R., Zhu, X.: Restricted Markov uniqueness for the stochastic quantization of $P(\Phi)_2$ and its applications. *J. Funct. Anal.* **272**(10), 4263–4303 (2017)
53. Stannat, W.: A new a priori estimate for the Kolmogorov operator of a 2D-stochastic Navier–Stokes equation. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* **10**(4), 483–497 (2007)
54. Walsh, J.B.: An introduction to stochastic partial differential equations. In: *École d’été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Mathematics*, pp. 265–439. Springer, Berlin (1986)
55. Yang, K.: The KPZ equation, non-equilibrium energy solutions, and weak universality for long-range interactions. [arXiv:1810.02836](https://arxiv.org/abs/1810.02836) (2018)
56. Zhu, R., Zhuǐ, X.: Dirichlet form associated with the ϕ_3^4 model. *Electron. J. Probab.* 23 Paper No. 78, 31 (2018)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.