Moduli of homomorphisms of sheaves

A Dissertation

Submitted in Partial Fulfilment of the Requirements for

the Degree of Doctor rerum naturalium to the

Department of Mathematics and Computer Science of

Freie Universität Berlin

by

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Berlin, 2019

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Date of defense: 16.09.2019

Danksagung

Für die exzellente Betreuung meiner Doktorarbeit und Begleitung meiner wissenschaftlichen Tätigkeit an der FU-Berlin bedanke ich mich bei Prof. Dr. Victoria Hoskins, für viele Anregungen, die Zweitkorrektur und Anreise aus Oxford bei Frau Prof. Dr. Francis Kirwan. Ein besonderer Dank gilt meinen Mitdoktoranden Alejandra Rincon, Dominic Bunett und Hangxing Lin für den spannenden und kritischen Austausch. Unvergessen bleibt die Freude über Cyril, Sylke und Martin, die sich auf den langen Weg nach Berlin gemacht haben, um bei der Verteidigung meiner Doktorarbeit dabei zu sein. Einen herzlichen Dank meinen Eltern Christine und Günter.

Abstract

Moduli spaces arise in classification problem in algebraic geometry; typically when we try to classify geometric objects we find that they have discrete invariants but these are not sufficient to classify them. Thus we would like to be able to construct moduli spaces whose points correspond to equivalence classes of the objects to be classified (with fixed discrete invariants) and whose geometry reflects the way they can vary in families.

This thesis studies moduli problems for homomorphisms of sheaves over a fixed projective variety X indexed by a quiver; that is, we are looking at representations of a quiver in the category of coherent sheaves over X where the sheaves indexed by the vertices of the quiver are fixed and it is only the homomorphisms between them indexed by the arrows of the quiver which vary. More precisely, we define a the moduli functor for homomorphisms of sheaves over a fixed projective variety X and show that the construction of a moduli space for homomorphisms of sheaves over X indexed by a quiver Q can be reduced to the construction of quotients for actions of the product of the automorphism groups of the sheaves over X labelled by the vertices of Q on affine varieties. Additionally we show that the automorphism groups of the sheaves over X are linear algebraic groups. In the case that these are reductive groups, such quotients can be constructed and studied using Mumford's classical geometric invariant theory (GIT). However in general these automorphism groups are not reductive, so a significant part of this thesis studies ways in which Mumford's GIT can be extended to actions of non-reductive linear algebraic groups on an affine variety, before applying them to representations of quivers in homomorphisms of semisimple sheaves over X.

Abstract

Modulräume entstehen bei der Betrachtung von Klassifikationsproblemen in der algebraischen Geometrie. Bei der Klassifizierung von geometrische Objekten ergibt sich gewöhnlich, dass die diskreten Invarianten nicht ausreichen, um diese geometrischen Objekte zu klassifizieren. Das Ziel ist Modulräume zu konstruieren, dessen Punkte mit den Äquivalenzklassen der zu klassifizierenden Objekte korrespondieren, wobei die diskreten Invarianten fixiert werden, und deren Geometrie das Verhalten von Familien der zu klassifizierenden Objekte reflektiert.

Diese Arbeit betrachtet Modulprobleme für Homomorphismen von Garben über einer projektiven Varietät X indiziert von einem Köcher; dass heißt, wir betrachten Darstellungen eines Köchers in der Kategorie der kohärenten Garben über X wobei wir die Garben fixieren, die von dem Punkten des Köchers indiziert werden, und nur die Homomorphismen, indiziert durch die Pfeile, variieren. Wir definieren Modulfunktoren für Homomorphismen von Garben über einer projektiven Varietät X indiziert von einem Köcher $\boldsymbol{Q}$. Für diese Modulfunktoren zeigen wir, dass die Konstruktion von Modulräumen durch Quotienten für die Gruppenwirkung von Produkten der Automorphismengruppen der Garben über X auf affinen Varietäten, die die Homomorphismen parametrisieren realisiert werden kann.

Darüber hinaus wird gezeigt, dass die Automorphismengruppen von Garben über X lineare algebraische Gruppen sind. Im Falle, dass diese Gruppen reduktiv sind, können diese Quotienten mit Hilfe von Mumfords klassischer geometrischer Invariantentheorie (GIT) studiert und konstruiert werden. Im allgemeinen Fall sind diese Automorphismengruppen nicht reduktiv, so dass ein signifikanter Teil dieser Arbeit Wege studiert, um Mumfords GIT auf Gruppenwirkungen nicht-reduktiver linear algebraischer Gruppen auf affinen Varietäten anzuwenden. Im letzten Teil der Arbeit nehmen wir an das alle Garben halbeinfach sind. Unter dieser Annahme werden die Ergebnisse zur

nicht-reduktiven GIT auf Modulprobleme für Homomorphismen indiziert von einem Köcher $\boldsymbol{Q}$ über einer projektiven Varietät X angewandt.

Contents

Introduction

The aim of this thesis is to study the moduli problem for homomorphisms of sheaves over a projective variety indexed by a quiver. This problem has been studied previously in two special cases. If we fix the projective variety $\operatorname{Spec} k$ then we are considering quiver representations in the category of finite dimensional vector spaces; this moduli problem has been studied by King (see [44]). Specialising in the other direction by fixing the quiver consisting of an arrow connecting two distinct vertices we obtain the moduli problem of sheaf homomorphisms studied by Drézet and Trautmann (see [25]). In both cases the problem is approached from the perspective of geometric invariant theory. The purpose of Mumford's reductive geometric invariant theory (see [30]) is to construct quotients for actions of reductive groups on varieties. For moduli of quiver representations the groups are all reductive and reductive geometric invariant theory can be directly applied. The moduli problem of sheaf homomorphisms involves non-reductive groups. The approach of Drézet and Trautmann is to translate this problem of non-reductive geometric invariant theory into a problem of reductive geometric invariant theory. More recently Bérczi, Doran, Hawes and Kirwan (see [9]) developed non-reductive geometric invariant theory for algebraic actions of linear algebraic groups on projective varieties under certain additional constraints. Our aim is to construct moduli spaces for homomorphisms of sheaves indexed by a quiver via non-reductive geometric invariant theory. The translation of the moduli problem into a geometric invariant theory problem results in algebraic actions of linear algebraic groups on affine varieties which leads us to study non-reductive geometric invariant theory for affine varieties.

The thesis is structured in the following way.

Chapter 1 We define quivers and quiver representations in an abelian category. Afterwards we define several moduli functors for homomorphisms of sheaves and show for these moduli functors that the construction of moduli spaces can be reduced to the construction of an orbit space for

$$H := \bigtimes_{v \in \boldsymbol{V}} \operatorname{Aut}(\mathcal{E}_v)$$

acting on

$$W := \bigoplus_{a \in \boldsymbol{A}} \operatorname{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)}).$$

Chapter 2 We recall the classical results on reductive geometric invariant theory. Additionally we recall the recent result by Bérczi, Doran, Hawes and Kirwan ([9, Theorem 0.1] in non-reductive geometric invariant theory. Roughly speaking they consider linearised actions of a non-reductive group H with unipotent radical U on a projective variety X under the assumption that the group H admits a so-called *graded unipotent subgroup* $\hat{U} = U \rtimes \mathbb{G}_{\mathrm{m}}$ where the multiplicative subgroup $\mathbb{G}_{\mathrm{m}}$ behaves as outlined in Definition 2.5.3. Using the grading subgroup $\mathbb{G}_{\mathrm{m}}$ they define a locus $X^{+}_{\min} \subset X$ which admits a principal U-bundle quotient if every U-stabiliser in $X^{+}_{\min}$ is trivial. Using a suitable linearisation $\mathcal{L} \to X$ of the H-action they obtain a rational map $X \dashrightarrow \mathbb{P} := \mathbb{P}(H^0(X, \mathcal{L})^U)$ which is compatible with the graded unipotent subgroup $\hat{U}$ in the sense that the locus $X^{+}_{\min}$ is contained in the domain of definition of $X \dashrightarrow \mathbb{P}$ in such a way that the morphism $X^{+}_{\min} \to \mathbb{P}^{+}_{\min}$ factorises via the principal U-bundle quotient $X^{+}_{\min} \to X^{+}_{\min}/U$ with $X^{+}_{\min}/U \to \mathbb{P}^{+}_{\min}$ a closed immersion. In the final step reductive geometric invariant theory is applied for the reductive group H/U to obtain the H-quotient for details see Theorem 2.5.8.

Chapter 3 This chapter concerns technical results which we will use later to construct quotients for action of non-reductive groups on affine varieties. Since our approach later on is based on quotienting in stages, we focus on actions of unipotent groups on affine varieties; in particular we focus on principal U-bundle quotients. In the first section we recall some results on actions of unipotent groups on affine varieties such as the Kostant-Rosenlicht Theorem which states that for a given action of a unipotent group U on an affine variety V each U-orbit is closed.

Based on the notion of a graded unipotent group by Bérczi, Doran, Hawes and Kirwan, we consider an action of a unipotent group U on an affine variety V which can be extended by a torus T to a so-called *graded U-action* (see Definition 3.0.5).

The following two sections are more technical and are used two prove the following two results:

1. If the unipotent group acts set-theoretically freely and the action can be extended to a graded unipotent action, then we obtain that $V \to \operatorname{Spec} \mathcal{O}(V)^U$ is a trivial principal U-bundle.
2. Conversely given an action of a semi-direct product $U \rtimes T$ with U unipotent and T a torus on V with $V \to \operatorname{Spec} \mathcal{O}(V)^U$ a trivial principal U-bundle then we show that $V \to \operatorname{Spec} \mathcal{O}(V)^U$ admits a T-invariant section.

Chapter 4 In this chapter we obtain our main results on non-reductive geometric invariant theory for affine varieties. We consider an action of a non-reductive linear algebraic group H on an affine variety X. Our goal is to construct good or, even better, geometric quotients for open subsets of X. We distinguish between two approaches to construct these quotients:

1. The first notion is a classical approach to construct a quotient for a graded H-action on X with respect to a linearisation $\mathfrak{L}$ of the trivial line bundle $X \times \mathbb{A}^1$. To obtain a quotient we use certain H-invariant sections belonging to $\mathcal{O}(X)^H$.
2. The other notion is an approach of using quotienting in stages and a certain embedding inspired by [9].

In both cases we will assume that the action of the unipotent radical $U \subset H$ extends to a graded action $U \rtimes T$ on X for a suitable subgroup of $T \subset H$. For the first approach we use a family of H-invariant sections $(\sigma_i)_{i \in I}$ belonging to $\mathcal{O}(X)^H$ which each admit a trivial principal U-bundle quotient $X_{\sigma_i} \to \operatorname{Spec} \mathcal{O}(X)^U_{\sigma_i}$ (see Definition 4.1.8). We obtain Theorem 4.1.9: Let H be a linear algebraic group with a Levi decomposition $H \cong U \rtimes R$ and X be a graded H-variety. We lift the H-action on X to the trivial line bundle $X \times \mathbb{A}^1$ via a character $\chi : H \to \mathbb{G}_m$ and denote the associated linearisation by $\mathfrak{L}$. Then

1. The open subset $X^{lss(H,\chi)} := X^{ss(R,\mathfrak{L})} \cap \bigcup_{i \in I} X_{\sigma_i}$ admits a good quotient via restricting

$$q_{\mathfrak{L}} : X^{nss(H,\mathfrak{L})} \to X /\!\!/_{\mathfrak{L}} H = \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$$

to $X^{lss(H,\chi)}$, where $X^{nss(H,\mathfrak{L})}$ is the domain of definition of the rational map $q_{\mathfrak{L}} : X^{nss(H,\mathfrak{L})} \dashrightarrow X /\!\!/_{\mathfrak{L}} H$. The image of $X^{lss(H,\chi)}$ under $q_{\mathfrak{L}}$ is open in $X /\!\!/_{\mathfrak{L}} H$ and $q_{\mathfrak{L}}(X^{lss(H,\chi)})$ is a variety.

2. The restriction $X^{ls(H,\chi)} := X^{s(R,\mathfrak{L})} \cap \bigcup_{i \in I} X_{\sigma_i} \to X^{ls(H,\chi)}/H$ is a geometric H-quotient that is open in $X^{lss(H,\chi)} /\!\!/ H$.

3. If U acts set-theoretically free on X, then the loci $X^{lss(H,\chi)}$ and $X^{ls(H,\chi)}$ admit the following Hilbert-Mumford criterion:

$$X^{l(s)s(H,\mathfrak{L})} = \bigcap_{h \in H} h \cdot X^{(s)s(T,\mathfrak{L})}$$

where $T \subset H$ is a maximal torus and $X^{(s)s(T,\mathfrak{L})}$ and $X^{(s)s(R,\mathfrak{L})}$ are the T-(semi)stable and R-(semi)stable loci as introduced in Definition 2.4.1 and Remark 2.4.2.

For the second approach we consider an H-representation W and a family of one-parameter $\lambda_n : \mathbb{G}_\mathrm{m} \to H$ subgroups which grade the unipotent radical of H (see Definition 2.5.3) together with a family of characters $\chi_n : H \to \mathbb{G}_\mathrm{m}$ satisfying $\langle \chi_n, \lambda_n \rangle < 0$ to obtain an analogue of the projective $\hat{U}$-Theorem by Bérczi, Doran, Hawes and Kirwan. In particular we construct a morphism $W \to V$ equivariant relative to $H \to H/U$ with W an $R := H/U$-representation. In the following we assume that the unipotent radical acts set-theoretically freely on the open locus $W^+_{\min}$ defined in Definition 4.2.4 and show that we can choose the H/U-representation V in such a way that for $n >> n_0$ we obtain via restriction a morphism $W^+_{\min} \to V^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ which factorises via the principal U-bundle $W^+_{\min} \to W^+_{\min}/U$ with $W^+_{\min} \to V^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ a closed immersion. Using this procedure we obtain the affine $\hat{U}$-Theorem (Theorem 4.2.5).

Let H be a linear algebraic group with a Levi-decomposition $U \rtimes R$. Suppose that $\lambda_n : \mathbb{G}_\mathrm{m} \to Z(R) \subset H$ is an $\mathbb{N}$-indexed family of central 1-PS adapted to a H-representation W. Suppose for each $n \in \mathbb{N}$, there is a character $\chi_n : H \to \mathbb{G}_{\mathrm{m}}$ such that $\langle \chi_n, \lambda_n \rangle < 0$. Then for n large enough the following statements hold.

1. The locus $W^{+}_{\min}$ admits a geometric $\hat{U}_n$-quotient $q : W^{+}_{\min} \to W^{+}_{\min}/\hat{U}_n$ with $W^{+}_{\min}/\hat{U}_n$ a projective over affine variety and $W^{+}_{\min} = W^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} = W^{s(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$.
2. There exists an affine completion of $W^{+}_{\min} \to W^{+}_{\min}/U$ such that the corresponding relative semistable locus $W^{rss(H,\chi_n)}$ (see Definition 4.2.1) admits a good H-quotient
$$q : W^{rss(H,\chi_n)} \to W^{rss(H,\chi_n)} /\!\!/ H$$
with $W^{rss(H,\chi_n)} /\!\!/ H$ a projective over affine variety.
3. The restriction $q\big|_{W^{rs(H,\chi_n)}} : V^{rs(H,\chi_n)} \to q(W^{s(H,\chi_n)})$ is a geometric H-quotient with $q(W^{rs(H,\chi_n)})$ open in $W^{rss(H,\chi_n)} /\!\!/ H$.

Chapter 5 In this chapter we consider sheaf homomorphisms indexed by a quiver. More concretely, we fix the following data $D = (X, \boldsymbol{Q}, (\mathcal{E}_v)_{v \in \boldsymbol{V}})$, where

1. X is a projective scheme of finite type over $\mathbf{k}$,
2. $\boldsymbol{Q} = (\boldsymbol{V}, \boldsymbol{A}, s, t : \boldsymbol{A} \to \boldsymbol{V})$ is a quiver, and;
3. $(\mathcal{E}_v)_{v \in \boldsymbol{V}}$ is a collection of semisimple coherent sheaves over X.

To this data we associated several moduli functors in Chapter 1 and reduced the construction of a moduli space to the construction of a good quotient for the linear action of

$$H := \bigtimes_{v \in \boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$$

on

$$W := \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$$

given by

$$(h_v)_{v \in \boldsymbol{V}} \cdot (w_a)_{a \in \boldsymbol{A}} := (h_{t(a)} \circ w_a \circ h^{-1}_{s(a)})_{a \in \boldsymbol{A}}.$$

To obtain these good quotients we apply the results obtained in Chapter 4 that is Theorem 4.1.9 and Theorem 4.2.5. In order to apply these results we recall in the first section several descriptions of the automorphism groups involved. In the second and final section we recall the approach by Drézet and Trautmann who considered the case where the quiver is given by $\boldsymbol{Q} = \overset{1}{\circ} \to \overset{2}{\circ}$. Additionally we apply Theorem 4.2.5 to obtain a projective over affine good quotient for an open subset of W, for the data $D = (\mathbb{P}^n, \circ \to \circ, (\mathcal{E}_v)_{v\in\{1,2\}})$ where

$$\mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^n}(c_1)^{\oplus m_1} \oplus \mathcal{O}_{\mathbb{P}^n}(c_2)$$

and

$$\mathcal{E}_2 = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2)^{\oplus m_2}$$

with $c_1 \leq c_2 < d_1 < d_2$ (see Theorem 5.2.3). In order to apply Theorem 4.1.9 we consider a quiver with loops and recall some classical results due to Sylvester. By applying these results we obtain the necessary H-invariant sections which we call *Sylvester sections* and their non-vanishing locus is called the Sylvester locus (see Definition 5.2.9 and 5.2.12) to obtain Theorem 5.2.15. Finally, we consider the case where the automorphism groups are all reductive: in this case we can apply classical reductive geometric invariant theory recalled in Chapter 2 and by using King's Hilbert-Mumford criterion we obtain a description of (semi)stability in terms of one-parameter subgroups.

Chapter 1

The moduli problem for sheaf homomorphisms

We define a moduli problem for sheaf homomorphisms over a projective scheme of finite type over an algebraically closed field $\mathbf{k} = \bar{\mathbf{k}}$ that are indexed by a quiver. More precisely, we consider a fibre of the moduli functor for quiver sheaves with fixed Hilbert polynomials. First we recall quivers and quiver representations in an abelian category. Afterwards, we define a moduli functor for quiver sheaves, which are quiver representations in the abelian category $\mathbf{Coh}(X)$ of coherent sheaves over a projective $\mathbf{k}$-scheme X. For $X = \operatorname{Spec} \mathbf{k}$, this approach yields the classical moduli problem of quiver representations with a fixed dimension vector in the category of finite dimensional $\mathbf{k}$-vector spaces. Finally, we consider the geometric invariant theory approach to construct a (coarse) moduli space associated to this moduli problem.

1.1 Quiver representations in an abelian category

In this section, we recall the definition of quivers and quiver representations in an abelian category.

1.1.1 Quivers

Definition 1.1.1. A quiver $\boldsymbol{Q}$ consists of a quadruple $(\boldsymbol{V}, \boldsymbol{A}, s, t)$ where $\boldsymbol{V}$ is the set of vertices, $\boldsymbol{A}$ is the set of arrows and $s, t : \boldsymbol{A} \to \boldsymbol{V}$ are maps called source and target respectively. We only consider *finite* quivers, i.e. quivers such that the sets $\boldsymbol{V}$ and $\boldsymbol{A}$ are both finite. A *path* in the quiver is a sequence of arrows $p = a_1 a_2 \cdots a_n$ such that $s(a_{i+1}) = t(a_i)$, $i = 1, \ldots, n-1$. For a vertex $v \in \boldsymbol{V}$, we denote the set of arrows with target v by $T(v) := \{a \in \boldsymbol{A} | t(a) = v\}$ and the set of arrows with source v by $S(v) := \{a \in \boldsymbol{A} | s(a) = v\}$. A vertex $v \in V$ is called a *source* if $T(v)$ is empty and a *sink* if $S(v)$ is empty.

Example 1.1.2. Let n be a natural number.

1. An n-Kronecker quiver is a quiver with two vertices and n arrows such that one vertex is a source and the other vertex is a sink.

   $\boldsymbol{K}_1 : \overset{v_1}{\circ} \longrightarrow \overset{v_2}{\circ}$ $\quad\boldsymbol{K}_2 : \overset{v_1}{\circ} \rightrightarrows \overset{v_2}{\circ}$ $\quad\boldsymbol{K}_3 : \overset{v_1}{\circ} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \overset{v_2}{\circ}$ etc.

2. $\boldsymbol{A}_n$-quivers defined recursively. Let $\boldsymbol{A}_0$ be an isolated vertex. Given $\boldsymbol{A}_n$ we obtain $\boldsymbol{A}_{n+1}$ from $\boldsymbol{A}_n$ by adding one vertex v and one arrow from the sink s of $\boldsymbol{A}_n$ to the vertex v.

   $\boldsymbol{A}_0 : \overset{s_0}{\circ}$ $\quad\boldsymbol{A}_1 : \overset{s}{\circ} \longrightarrow \overset{v=s_1}{\circ}$ $\quad\boldsymbol{A}_2 : \overset{s_0}{\circ} \longrightarrow \overset{s_1}{\circ} \longrightarrow \overset{s_2}{\circ}$

   Note that for each $n \in \mathbb{N}$ the quiver $\boldsymbol{A}_n$ has a unique source and sink.

3. A cyclic quiver $\boldsymbol{Z}_n$ is obtained from the quiver $\boldsymbol{A}_n$ by adding one arrow from the sink of $\boldsymbol{A}_n$ to the source of $\boldsymbol{A}_n$.

   $$\begin{tikzcd} \overset{1}{\circ} \arrow[loop right] \end{tikzcd} \qquad \begin{tikzcd} \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[l, bend left] \end{tikzcd} \qquad \begin{tikzcd} \overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r] & \overset{3}{\circ} \arrow[ll, bend left] \end{tikzcd}$$

Definition 1.1.3. Let $\boldsymbol{Q}$ be a quiver. We call $\boldsymbol{Q}$ *acyclic*, if $\boldsymbol{Q}$ does not contain any oriented cycles that is if and only if $\boldsymbol{Q}$ does not contain any of the cyclic quivers $\boldsymbol{Z}_n$ as a subquiver for all $n \in \mathbb{N}$.

Definition 1.1.4. Let $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ be quivers. Then the *product* of $\boldsymbol{Q}_1$ and $\boldsymbol{Q}_2$ is denoted by $\boldsymbol{Q}_1 \times \boldsymbol{Q}_2$ and is given by $\boldsymbol{V}_{\boldsymbol{Q}_1 \times \boldsymbol{Q}_2} := \boldsymbol{V}_{\boldsymbol{Q}_1} \times \boldsymbol{V}_{\boldsymbol{Q}_2}$ and $\boldsymbol{A}_{\boldsymbol{Q}_1 \times \boldsymbol{Q}_2} := \boldsymbol{A}_{\boldsymbol{Q}_1} \times \boldsymbol{V}_{\boldsymbol{Q}_2} \sqcup \boldsymbol{V}_{\boldsymbol{Q}_1} \times \boldsymbol{A}_{\boldsymbol{Q}_2}$ together with the source map

$$\begin{aligned} s : \boldsymbol{A} &\to \boldsymbol{V} \\ x &\mapsto \begin{cases} (s_1(a), v) & \text{if } x = (a, v) \in \boldsymbol{A}_{\boldsymbol{Q}_1} \times \boldsymbol{V}_{\boldsymbol{Q}_2} \\ (v, s_2(a)) & \text{if } x = (v, a) \in \boldsymbol{V}_{\boldsymbol{Q}_1} \times \boldsymbol{A}_{\boldsymbol{Q}_2} \end{cases} \end{aligned}$$

and the target map, which is given by an analogous construction to the source map.

Example 1.1.5. The double loop quiver is given as the product of two loop quivers $\boldsymbol{Z}_1 \times \boldsymbol{Z}_1$:

$$\begin{tikzcd} \overset{1}{\circ} \arrow[loop left] \arrow[loop right] \end{tikzcd} .$$

$\boldsymbol{K}_2 \times \boldsymbol{Z}_1$ is the quiver $\begin{tikzcd} \overset{1}{\circ} \arrow[loop left] \arrow[r, shift left] \arrow[r, shift right] & \overset{2}{\circ} \arrow[loop right] \end{tikzcd}$.

1.1.2 Quiver representations

We want to consider quiver representations in an abelian category such as the category $\mathbf{Coh}(X)$ of coherent sheaves over a projective scheme X of finite type over $\mathbb{C}$ or the category $\mathbf{vect_k}$ of finite dimensional $\mathbb{C}$-vector spaces.

Definition 1.1.6. A representation of a quiver $\boldsymbol{Q}$ in a category A consists of

1. for each vertex $v \in \boldsymbol{V}$, an object M_v of A;

2. for each arrow $a \in \boldsymbol{A}$, a morphism $\varphi_a \in \mathrm{Hom}_{\mathsf{A}}(M_{s(a)}, M_{t(a)})$.

Given two representations $(M_v, \varphi_a)_{v\in\boldsymbol{V}, a\in\boldsymbol{A}}$ and $(N_v, \psi_a)_{v\in\boldsymbol{V}, a\in\boldsymbol{A}}$ of the quiver $\boldsymbol{Q}$ in a category A, a morphism from $(M_v, \varphi_a)_{v\in\boldsymbol{V}, a\in\boldsymbol{A}}$ to $(N_v, \psi_a)_{v\in\boldsymbol{V}, a\in\boldsymbol{A}}$ is given by morphisms $\Phi_v \in \mathrm{Hom}_{\mathsf{A}}(M_v, N_v)$ for each $v \in \boldsymbol{V}$ such that for each arrow $a \in \boldsymbol{A}$ the following diagram commutes

$$\begin{tikzcd} M_{s(a)} \arrow[r, "\Phi_{s(a)}"] \arrow[d, "\varphi_a"] & N_{s(a)} \arrow[d, "\psi_a"] \\ M_{t(a)} \arrow[r, "\Phi_{t(a)}"] & N_{t(a)}. \end{tikzcd}$$

A morphism of quiver representations given by $(\Phi_v : M_v \to N_v)_{v\in\boldsymbol{V}}$ is

1. injective, if for each $v \in V$, Φ_v is injective,

2. surjective, if for each $v \in V$, Φ_v is surjective,

3. and bijective, if for each $v \in V$, Φ_v is bijective.

Let $(M_v, \varphi_a)_{v\in\boldsymbol{V},\, a\in\boldsymbol{A}}$ be a representation; then a subrepresentation of $(M_v, \varphi_a)_{v\in\boldsymbol{V},\, a\in\boldsymbol{A}}$ consists of subobjects $M'_v \subset M_v$ together with the morphisms $\varphi_a|_{M'_{s(a)}}$ such that for each arrow $a \in \boldsymbol{A}$, the restriction $\varphi_a|_{M'_{s(a)}}$ defines a morphism from $M'_{s(a)}$ to $M'_{t(a)}$. The representations of the quiver $\boldsymbol{Q}$ in A form a category which we denote by $\mathrm{Rep}(\boldsymbol{Q}, \mathsf{A})$.

Remark 1.1.7. In categorical terms, the quiver $\boldsymbol{Q}$ corresponds to a small category Q and a representation of the quiver $\boldsymbol{Q}$ is a functor $F : \mathsf{Q} \to \mathsf{A}$. Similarly, under this identification, a morphism is a natural transformation of functors.

1. If A is an abelian (additive) category, then also $\operatorname{Rep}(\boldsymbol{Q}, \mathsf{A})$ is an abelian (additive) category.

2. $\operatorname{Rep}(\boldsymbol{Q}^{op}, \mathsf{A}^{op}) \cong \operatorname{Rep}(\boldsymbol{Q}, \mathsf{A})^{op}$, where $\boldsymbol{Q}^{op}$ denotes the quiver with the same vertices and arrows but the roles of the source and targets maps are interchanged; that is, $s_{\boldsymbol{Q}^{op}} := t_{\boldsymbol{Q}}$ and $t_{\boldsymbol{Q}^{op}} := s_{\boldsymbol{Q}}$.

1.2 The moduli functor

Let $\mathbf{Set}$ denote the category of sets and $\mathbf{Sch}/\mathbf{k}$ the category of schemes of finite type over $\operatorname{Spec} \mathbf{k}$ with $\mathbf{k}$ an algebraically closed field. Since the category $\mathbf{Sch}/\mathbf{k}$ is essentially small, we can consider the category $\operatorname{PSh}(\mathbf{Sch}/\mathbf{k}) := \operatorname{PSh}(\mathbf{Sch}/\mathbf{k}, \mathbf{Set})$ of contravariant functors from $\mathbf{Sch}/\mathbf{k}$ to $\mathbf{Set}$. Equipping the category $\mathbf{Sch}/\mathbf{k}$ with the Zariski topology we can determine, whether a presheaf $\mathsf{F} : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$ is actually a sheaf; that is, for any $T \in \mathbf{Sch}/\mathbf{k}$ and any Zariski-covering $\{U_i \to T\}$; the following diagram is exact

$$\mathsf{F}(T) \longrightarrow \prod_{i \in I} \mathsf{F}(U_i) \underset{\mathrm{pr}_2^*}{\overset{\mathrm{pr}_1^*}{\rightrightarrows}} \prod_{i,j \in I} \mathsf{F}(U_i \times_T U_j).$$

Here we denote by $\mathrm{pr}_1 : U_i \times_T U_j \to U_i$ and $\mathrm{pr}_2 : U_i \times_T U_j \to U_j$ the first and second projection respectively. In other words, F is a Zariski-sheaf if for every $T \in \mathbf{Sch}/\mathbf{k}$ and any Zariski-covering the following condition is satisfied: given a Zariski-covering $\{U_i \to T\}$ and a set of elements $a_i \in \mathsf{F}(U_i)$ such that $\mathrm{pr}_1^* a_i = \mathrm{pr}_2^* a_j \in \mathsf{F}(U_i \times_T U_j)$ for all i and j, then there exists a unique section $a \in \mathsf{F}(T)$ whose pullback to $\mathsf{F}(U_i)$ is a_i for all i.

Definition 1.2.1. A moduli problem is given by

1. sets A_T of families over T and an equivalence relation $\sim_T$ for each $T \in \mathbf{Sch}/\mathbf{k}$,

2. pullback maps $f^* : A_{T'} \to A_T$ for each morphism of schemes $T \to T'$ satisfying the following properties:

(a) for a family $\mathfrak{F}$ over T we have $\mathrm{id}_T^* \mathfrak{F} = \mathfrak{F}$

(b) for a morphism $f : T \to T'$ and equivalent T'-families $\mathfrak{F} \sim_{T'} \mathfrak{F}'$ we obtain equivalent T-families $f^*\mathfrak{F} \sim_T f^*\mathfrak{F}'$.

(c) for morphisms $f : T \to T'$ and $g : T' \to T''$ and a T''-family $\mathfrak{F}$ we have $(g \circ f)^*\mathfrak{F} \sim_T f^* g^* \mathfrak{F}$.

The moduli problem defines a moduli functor

$$\mathsf{M} : (\mathbf{Sch/k})^{op} \to \mathbf{Set}$$
$$T \mapsto A_T/{\sim_T}.$$

Definition 1.2.2. Let $\mathsf{F}, \mathsf{G} \in \mathrm{PSh}(\mathbf{Sch/k})$ be presheaves and G be a group object in $\mathrm{PSh}(\mathbf{Sch/k})$. We call a natural transformation $\sigma : \mathsf{G} \times \mathsf{F} \to \mathsf{F}$ a group action, if for every object $T \in \mathbf{Sch/k}$ we have that $\sigma_T : (\mathsf{G} \times \mathsf{F})(T) \cong \mathsf{G}(T) \times \mathsf{F}(T) \to \mathsf{F}(T)$ is a group action.

Example 1.2.3. Given a group action in the category of presheaves $\sigma : \mathsf{G} \times \mathsf{F} \to \mathsf{F}$, then taking the orbits of $\mathsf{G}(T)$ in $\mathsf{F}(T)$ to be the equivalence classes we obtain a moduli problem in the sense of definition 1.2.1. We denote the associated moduli functor by F/G.

In the following we define several moduli functors for homomorphisms of sheaves indexed by the arrows of a quiver $\boldsymbol{Q}$.

Notation 1.2.4. Let $X \to S$ and $T \to S$ be a S-schemes. We denote the fibre product $X \times_S T$ by X_T :

$$\begin{tikzcd} X_T \arrow[r, "p"] \arrow[d, "q"] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\ T \arrow[r] & S. \end{tikzcd}$$

For a coherent sheaf $\mathcal{E}$ over X we denote the pullback of $\mathcal{E}$ along p by $\mathcal{E}_T$.

Definition 1.2.5. Let $f : X \to S$ be a morphism of finite type of noetherian schemes. A flat family of coherent sheaves on the fibres of f is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ which is flat over S. Recall that this means that for each point $x \in X$ the stalk $\mathcal{F}_x$ is flat over the local ring $\mathcal{O}_{S,f(x)}$.

Definition 1.2.6. Let $\mathbf{k}$ be an algebraically closed field. We call the tuple $(X, \mathcal{O}_X(1))$ a *polarised projective scheme* if

1. X is a projective scheme of finite type over $\mathbf{k}$, and
2. $\mathcal{O}_X(1)$ is a very ample invertible sheaf over X.

Definition 1.2.7. Let $\boldsymbol{Q} = (\boldsymbol{V}, \boldsymbol{A}, s, t)$ be a quiver and X be a connected projective scheme of finite type over $\mathbf{k}$. A family of $\boldsymbol{Q}$-sheaves on X parametrised by a scheme T is a $\boldsymbol{Q}$-sheaf in $\mathbf{Coh}(X \times T)$ which is flat over T.

Definition 1.2.8. Let $\boldsymbol{Q} = (\boldsymbol{V}, \boldsymbol{A}, s, t)$ be a quiver and X be a connected projective scheme of finite type over $\mathbf{k}$. Let T be a scheme and $(\mathcal{E}_v, \varphi_a)$ and $(\mathcal{F}_v, \psi_a)$ be two T-families of $\boldsymbol{Q}$-sheaves. We call $(\mathcal{E}_v, \varphi_a)$ and $(\mathcal{F}_v, \psi_a)$

1. *absolutely equivalent* denoted by $\sim_{T,abs}$, if there exists an isomorphism of $\boldsymbol{Q}$-sheaves $\Phi : (\mathcal{E}_v, \varphi_a) \xrightarrow{\sim} (\mathcal{F}_v, \psi_a)$; that is isomorphisms $\Phi_v : \mathcal{E}_v \to \mathcal{F}_v$ for all $v \in \boldsymbol{V}$ such that for each $a \in \boldsymbol{A}$ the following diagram

$$\begin{array}{ccc} \mathcal{E}_{s(a)} & \xrightarrow{\varphi_a} & \mathcal{E}_{t(a)} \\ \downarrow{\scriptstyle \Phi_{s(a)}} & & \downarrow{\scriptstyle \Phi_{t(a)}} \\ \mathcal{F}_{s(a)} & \xrightarrow{\psi_a} & \mathcal{F}_{t(a)} \end{array}$$

commutes;

2. *relatively equivalent* denoted by $\sim_{T,rel}$, if there exists a line bundle $\mathcal{L}$ over T such that $(\mathcal{E}_v, \varphi_a)$ is absolutely equivalent to $(\mathcal{F}_v \otimes q^*\mathcal{L}, \psi_a \otimes \mathrm{id}_{q^*\mathcal{L}})$.

Definition 1.2.9. Let $\boldsymbol{Q} = (\boldsymbol{V}, \boldsymbol{A}, s, t)$ be a finite quiver and X be a connected projective scheme of finite type over $\mathbf{k}$ with $\mathcal{O}_X(1)$ a very ample invertible sheaf. Fix a polynomial h_v for each vertex $v \in V$. For a coherent sheaf $\mathcal{F}$, we denote the Hilbert polynomial with respect to $\mathcal{O}_X(1)$ by $\chi(\mathcal{F})$. The absolute moduli functor for coherent $\boldsymbol{Q}$-sheaves in $\mathbf{Coh}(X)$ with Hilbert polynomial data

$(h_v)_{v\in V}$ is given by

$$\mathfrak{M}_{X,\boldsymbol{Q},(h_v)} : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$$
$$T \mapsto \mathfrak{M}_{X,\boldsymbol{Q},(h_v)}(T) := \left\{ [(\mathcal{E}_v, \varphi_a)]_{\sim_{T,\text{abs}}} \middle| \begin{array}{l} (\mathcal{E}_v, \varphi_a) \text{ is a } T\text{-family of } \boldsymbol{Q}\text{-sheaves} \\ \forall v \in V \ \forall t \in T(\mathbf{k}) : \chi(\mathcal{E}_v|_{X\times\{t\}}) = h_v \end{array} \right\}$$
$$f \mapsto \mathfrak{M}_{X,\boldsymbol{Q},(h_v)}(f) := (\mathrm{id}_X \times f)^*$$

We define the relative moduli functor by

$$\mathsf{M}_{X,\boldsymbol{Q},(h_v)_{v\in\boldsymbol{V}}} : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$$

by considering the same families up to relative equivalence. In the case that $\boldsymbol{Q}$ is the trivial quiver consisting of a single vertex and no arrows, we denote the absolute and relative moduli functors by $\mathfrak{M}_{X,h_v}$ and M_{X,h_v}. We obtain a forgetful natural transformation π from $\mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in\boldsymbol{V}}}$ to $\prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}$ by forgetting the arrows: for $T \in \mathbf{Sch}/\mathbf{k}$ and $[(\mathcal{E}_v, \varphi_a)] \in \mathfrak{M}_{X,\boldsymbol{Q},(h_v)}(T)$, we associate $([E_v])_{v\in\boldsymbol{V}} \in \prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}(T)$. Recall that $h_{\operatorname{Spec}\mathbf{k}} \to \prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}$ corresponds to an element in $\prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}(\operatorname{Spec}\mathbf{k})$; that is, a tuple of equivalence classes of sheaves $([\mathcal{E}_v])_{v\in\boldsymbol{V}}$ on X with Hilbert polynomial data $(h_v)_{v\in\boldsymbol{V}}$.

In the following we are interested in a fibre of the absolute moduli functor for $\boldsymbol{Q}$-sheaves i.e. we want to consider the fibre product

$$\begin{tikzcd} \mathfrak{M}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in\boldsymbol{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in\boldsymbol{V}}} \arrow[d, "\pi"] \\ h_{\operatorname{Spec}\mathbf{k}} \arrow[r, "([\mathcal{E}_v])_{v\in\boldsymbol{V}}"] & \prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}. \end{tikzcd}$$

To define the relative moduli functor for sheaf homomorphism indexed by $\boldsymbol{Q}$, note that we have an action of the group object $\operatorname{Pic} \in \operatorname{PSh}(\mathbf{Sch}/\mathbf{k})$ on $\mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in\boldsymbol{V}}}$ and $\prod_{v\in\boldsymbol{V}} \mathfrak{M}_{X,h_v}$, where $\mathcal{L} \in \operatorname{Pic}(T)$ acts on $[\mathcal{E}_v, \varphi_a] \in \mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in\boldsymbol{V}}}(T)$ via

$$\mathcal{L} \cdot [\mathcal{E}_v, \varphi_a] = [\mathcal{E}_v \otimes q^*\mathcal{L}, \varphi_a \otimes \mathrm{id}_{q^*\mathcal{L}}],$$

where $q : X \times T \to T$ is the projection to T. The action of Pic on $\prod_{v\in \boldsymbol{V}} \mathfrak{M}_{X,h_v}$ is given by

$$\mathcal{L} \cdot [\mathcal{E}_v] = [\mathcal{E}_v \otimes q^*\mathcal{L}]$$

for $([\mathcal{E}_v])_{v\in V} \in \prod_{v\in \boldsymbol{V}} \mathfrak{M}_{X,h_v}(T)$ and $\mathcal{L} \in \mathrm{Pic}(T)$. The natural transformation π is equivariant with respect to the action of Pic and furthermore we have that

$$\mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}} / \mathrm{Pic} = \mathsf{M}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}}.$$

By an abuse of notation, let

$$\pi : \mathsf{M}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}} = \mathfrak{M}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}} / \mathrm{Pic} \to \Big(\prod_{v\in \boldsymbol{V}} \mathfrak{M}_{X,h_v} \Big) / \mathrm{Pic}$$

be the natural transformation which forgets the arrows. We obtain the analogous fibre product for the relative moduli functor

$$\begin{tikzcd}
\mathsf{M}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in \boldsymbol{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathsf{M}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}} \arrow[d, "\pi"] \\
h_{\mathrm{Spec}\,\mathbf{k}} \arrow[r, "([\mathcal{E}_v])_{v\in \boldsymbol{V}}"] & \Big(\prod_{v\in \boldsymbol{V}} \mathfrak{M}_{X,h_v} \Big) / \mathrm{Pic}\,.
\end{tikzcd}$$

Remark 1.2.10. By [42, 17.4.9.] the sheafification functor on presheaves of sets commutes with finite limits.

For a presheaf $\mathsf{P} : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$ let us denote the sheafification with respect to the Zariski topology by $\widetilde{\mathsf{P}}^{(Zar)}$. By the above remark the sheafification of the relative moduli functor $\mathsf{M}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in \boldsymbol{V}}}$ with respect to the Zariski topology (or any other Grothendieck topology) will fit into the analogue commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathsf{M}}^{(Zar)}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in \boldsymbol{V}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \widetilde{\mathsf{M}}^{(Zar)}_{X,\boldsymbol{Q},(h_v)_{v\in \boldsymbol{V}}} \arrow[d, "\pi"] \\
h_{\mathrm{Spec}\,\mathbf{k}} \arrow[r] & \widetilde{\Big(\prod_{v\in \boldsymbol{V}} \mathsf{M}_{X,h_v} \Big) / \mathrm{Pic}}^{(Zar)}.
\end{tikzcd}$$

Theorem 1.2.11. *([33, III 7.7.6]) Let S be a noetherian scheme and $r : X \to S$ be a proper morphism. Let $\mathcal{F}$ be a coherent sheaf on X which is flat over S. Then there exists a coherent sheaf $\mathcal{Q}$ on S together* *with a functorial $\mathcal{O}_S$-linear isomorphism*

$$\theta : r_*(\mathcal{F} \otimes_{\mathcal{O}_X} r^* -) \to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\mathcal{Q}, -)$$

on the category $\mathbf{QCoh}(S)$*. By its universal property, the pair* $(\mathcal{Q}, \theta)$ *is unique up to a unique isomorphism.*

Then following [52] we have the following result which is a combination of [33, III 7.7.8 and 7.7.9].

Theorem 1.2.12. *Let S be a noetherian scheme and $r : X \to S$ be a projective morphism. Let $\mathcal{E}$ and $\mathcal{F}$ be coherent sheaves on X. Consider the set-valued contravariant functor $\mathrm{Hom}_{\mathcal{E},\mathcal{F}}$ on S-schemes, which associates to any $T \to S$ the set of all $\mathcal{O}_{X_T}$-linear homomorphisms $\mathrm{Hom}_{\mathcal{E},\mathcal{F}}(T) := \mathrm{Hom}_{\mathcal{O}_{X_T}}(\mathcal{E}_T, \mathcal{F}_T)$ where $\mathcal{E}_T$ and $\mathcal{F}_T$ denote the pull-backs of $\mathcal{E}$ and $\mathcal{F}$ under the projection $X_T \to X$. If $\mathcal{F}$ is flat over S, then the functor $\mathrm{Hom}_{\mathcal{E},\mathcal{F}}$ is representable by a linear scheme V over S.*

Remark 1.2.13. Let $\mathcal{E}$ be a locally free sheaf and $\mathcal{F}$ be a reflexive sheaf. Furthermore, let us assume that $S = \mathrm{Spec}\, \mathbf{k}$ where $\mathbf{k}$ is an algebraically closed field. Then by the proof of the above theorem (see [52]), it follows that $V = \mathrm{Spec}\, \mathrm{Sym}^\bullet_{\mathcal{O}_S} \mathcal{Q}$, where $\mathcal{Q} = r_* \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})^\vee$.

Definition 1.2.14. Let $\boldsymbol{Q}$ be a quiver and $(\mathcal{E}_v)_{v \in \boldsymbol{V}}$ be a collection of coherent sheaves over X indexed by $\boldsymbol{V}$. The tautological family $\mathfrak{T} = (\mathcal{F}_v, \mathrm{ev}_a)$ over $\bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$ is given by

1. sheaves $\mathcal{F}_v$ for $v \in \boldsymbol{V}$, which are the pullbacks of $\mathcal{E}_v$ along the projection map

$$X \times_{\mathrm{Spec}\, \mathbf{k}} \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)}) \to X$$

   and

2. morphisms $\mathrm{ev}_{a_0} : \mathcal{F}_{s(a_0)} \to \mathcal{F}_{t(a_0)}$ for $a_0 \in \boldsymbol{A}$, which correspond to inclusion maps

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{s(a_0)}, \mathcal{E}_{t(a_0)}) \to \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)}).$$

Definition 1.2.15. For a moduli functor M, a family $\mathfrak{T}$ over a scheme S has the *local universal* *property,* if for any family $\mathfrak{F}$ over a scheme T and for any $\mathbf{k}$-point $t \in T$ there exists a neighbourhood U of $t \in T$ and a morphism $f : U \to S$ such that $\mathfrak{F}|_U \sim_U f^*\mathfrak{T}$.

Lemma 1.2.16. *Let X be a connected projective scheme of finite type over $S = \operatorname{Spec} \mathbf{k}$. Assume that $\mathcal{E}_v$ is a locally free sheaf over X for each $v \in V$, then the relative moduli functor*

$$\mathsf{M}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in \boldsymbol{V}}} : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$$
$$T \mapsto \{\text{families over } T \text{ up to relative equivalence } \sim_T\}$$

admits a family with the local universal property. Explicitly this is given by $\mathfrak{T}$ the tautological family over $\bigoplus_{a\in \boldsymbol{A}} \operatorname{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$.

Proof. A T-family is given by an equivalence class of $\boldsymbol{Q}$-sheaves. By construction of the moduli functor as a fibre product any representative of the equivalence class corresponds to a $\boldsymbol{Q}$-sheaf $\mathfrak{F} = (\mathcal{F}_v, \varphi_a : \mathcal{F}_{s(a)} \to \mathcal{F}_{t(a)})$ such that $\mathcal{F}_v \cong (\mathrm{id}_X \times f)^*\mathcal{E}_v \otimes q^*\mathcal{L}$ for each $v \in \boldsymbol{V}$ where $\mathrm{id}_X \times f : X \times T \to X \times \operatorname{Spec} \mathbf{k}$. Given a $\mathbf{k}$-point $t \in T$, there exists an affine open neighbourhood $U \subset T$ containing t such that $\mathcal{L}|_U \cong \mathcal{O}_U$ and hence $\mathcal{F}_v|_{X\times U} \cong (\mathrm{id}_X \times f|_U)^*\mathcal{E}_v \otimes q_U^*\mathcal{O}_U \cong (\mathrm{id}_X \times f|_U)^*\mathcal{E}_v$. Therefore, the family $\mathfrak{F}|_U$ restricted to U corresponds to a morphism $g : U \to \bigoplus_{a\in \boldsymbol{A}} \operatorname{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$ and $(\mathrm{id}_X \times g)^*\mathfrak{T} \cong \mathfrak{F}|_U$ which implies that the tautological family has the local universal property. □

Recall the following result, which allows us to construct our potential (coarse) moduli space via methods from (non-reductive) geometric invariant theory. For the convenience of the reader several notions of quotients, and in particular categorical quotients which will be employed in the following proposition, are recalled in Definition 2.3.9.

Proposition 1.2.17. *[51, Proposition 2.13] For a moduli functor M, let $\mathfrak{T}$ be a family with the local universal property over a scheme $S \in \mathbf{Sch}/\mathbf{k}$. Furthermore, suppose that an algebraic group H acts on S such that two $\mathbf{k}$-points p, q of S lie in the same H-orbit if and only if $\mathfrak{T}_p \sim \mathfrak{T}_q$. Then*

1. *any coarse moduli space is a categorical quotient of the H-action on S;*
2. *a categorical quotient of the H-action on S is a coarse moduli space if and only if it is an orbit space.*

1.3 The geometric invariant theory set up for the moduli problem

Analogously to King's GIT-Ansatz (see [44]) for moduli of quiver representations in the category $\mathbf{vect_k}$, we want to consider moduli of homomorphisms of sheaves over X indexed by arrows in $\boldsymbol{Q}$ where we fix a sheaf at each vertex of $\boldsymbol{Q}$. The justification of our GIT-Ansatz for constructing (coarse) moduli spaces is given by Proposition 1.2.17. We already showed that we have a family with the local universal property. It remains to show that this local universal family is compatible with an action of a linear algebraic group for which Proposition 1.2.17 applies. In the following, let X be a projective scheme of finite type over $\mathbb{C}$ and $\mathcal{E}_v \in \mathbf{Coh}(X)$ a collection of coherent sheaves for $v \in \boldsymbol{V}$ and consider the space

$$W := \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$$

together with a linear action $\alpha : H \times W \to W$ of the group

$$H := \bigtimes_{v \in \boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$$

where

$$\alpha\big((h_v)_{v \in \boldsymbol{V}}, (\varphi_a)_{a \in \boldsymbol{A}}\big) := (h_{t(a)} \varphi_a (h_{s(a)})^{-1})_{a \in \boldsymbol{A}}.$$

Proposition 1.3.1. *Let $\mathfrak{T}$ be the family over V with the local universal property. Then two $\mathbf{k}$-points $s, t \in W$ satisfy $\mathfrak{T}_s \cong \mathfrak{T}_t$ if and only if s and t belong to the same H-orbit. Furthermore any $p \in W$ has a nontrivial stabiliser $\mathrm{Stab}_H(p)$ which identifies with the automorphisms of the $\boldsymbol{Q}$-sheaf corresponding to the point p. Since each $\boldsymbol{Q}$-sheaf has at least the isomorphisms $t \cdot \mathrm{id}$ for $t \in \mathbb{G}_{\mathrm{m}}$, we obtain a global stabiliser given by this $\mathbb{G}_{\mathrm{m}}$.*

Proof. Let $(\mathcal{E}_v, \varphi_a)$ be the $\boldsymbol{Q}$-sheaf corresponding to the point $(\varphi_a)_{a \in \boldsymbol{A}} \in W$. By definition an automorphism $\Phi : (\mathcal{E}_v, \varphi_a) \to (\mathcal{E}_v, \varphi_a)$ corresponds to $(\Phi_v)_{v \in \boldsymbol{V}} \in H = \bigtimes_{v \in \boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$ such that for each arrow $a \in \boldsymbol{A}$ the following diagram is commutative

$$\begin{array}{ccc} \mathcal{E}_{s(a)} & \xrightarrow{\Phi_{s(a)}} & \mathcal{E}_{s(a)} \\ \Big\downarrow{\scriptstyle \varphi_a} & & \Big\downarrow{\scriptstyle \varphi_a} \\ \mathcal{E}_{t(a)} & \xrightarrow{\Phi_{t(a)}} & \mathcal{E}_{t(a)}. \end{array}$$

Equivalently, we have for each arrow $a \in \boldsymbol{A}$ the following equation

$$\Phi_{t(a)} \circ \varphi_a \circ \Phi_{s(a)}^{-1} = \varphi_a.$$

So the point $(\varphi_a)_{a\in\boldsymbol{A}} \in W$ gets fixed by $(\Phi_v)_{v\in\boldsymbol{V}} \in H$, hence $\operatorname{Aut}\big((\mathcal{E}_v, \varphi_a)\big) = \operatorname{Stab}_H((\varphi_a)_{a\in\boldsymbol{A}})$.

Given two $\mathbf{k}$-points (φ_a) and (ψ_a) of W then $\mathfrak{T}_{(\varphi_a)} = (\mathcal{E}_v, \varphi_a)$ and $\mathfrak{T}_{(\psi_a)} = (\mathcal{E}_v, \psi_a)$. From the definition of an isomorphism of $\boldsymbol{Q}$-sheaves it follows immediately that the fibres of $\mathfrak{T}$ are isomorphic if and only if (φ_a) and (ψ_a) belong to the same H-orbit. $\square$

Remark 1.3.2. In the case where $\boldsymbol{Q}$ is the $\boldsymbol{A}_1$-quiver the linear action of $H := \operatorname{Aut}(\mathcal{E}) \times \operatorname{Aut}(\mathcal{F})$ on $W := \operatorname{Hom}(\mathcal{E}, \mathcal{F})$ has been partially considered by Drézet and Trautmann (see [25]).

Chapter 2

Background on Geometric Invariant Theory

The construction of moduli spaces often involves the construction of orbit spaces (or quotients) for the action of an affine algebraic group on a suitable variety. Classical Geometric Invariant Theory (GIT) can be used to construct these quotients if the algebraic group is reductive. Since there are moduli problems which involve non-reductive groups, in particular the moduli problem considered in Chapter 1 is such a moduli problem, we will also recall more recent results on non-reductive Geometric Invariant Theory.

In this chapter and the following chapters we assume that $\mathbf{k}$ is an algebraically closed field of characteristic zero.

2.1 Algebraic groups

In this section, we recall the definitions of $\mathbf{k}$-group schemes and algebraic groups.

Definition 2.1.1. A $\mathbf{k}$-*group scheme* is a $\mathbf{k}$-scheme G equipped with morphisms

$$\mu_G : G \times G \to G,\ \iota_G : G \to G \text{ and } e_G : \operatorname{Spec} \mathbf{k} \to G$$

such that for any $\mathbf{k}$-scheme T, the set of T-points $G(T)$ is a group with composition law $\mu(T)$, inverse map $\iota(T)$ and neutral element $T \to \operatorname{Spec} \mathbf{k} \stackrel{e_G}{\to} G$.

Proposition 2.1.2. *Given a* $\mathbf{k}$*-scheme* G *equipped with morphisms*

$$\mu_G : G \times G \to G,\ \iota_G : G \to G \text{ and } e_G : \operatorname{Spec} \mathbf{k} \to G.$$

The following statements are equivalent:

1. *G equipped with above morphisms is a* $\mathbf{k}$*-group scheme.*
2. *The morphisms satisfy the following commutative diagrams*

$$\begin{tikzcd} G \times G \times G \arrow[r, "\mu_G \times \mathrm{id}_G"] \arrow[d, "\mathrm{id}_G \times \mu_G"'] & G \times G \arrow[d, "\mu_G"] \\ G \times G \arrow[r, "\mu_G"'] & G \end{tikzcd} \quad \textit{(}\mu_G \textit{ is associative),}$$

$$\begin{tikzcd} G \cong \operatorname{Spec} \mathbf{k} \times G \arrow[r, "e_G \times \mathrm{id}_G"] \arrow[dr, "\mathrm{id}_G"'] & G \times G \arrow[d, "\mu_G"] & G \cong \operatorname{Spec} \mathbf{k} \times G \arrow[l, "\mathrm{id}_G \times e_G"'] \arrow[dl, "\mathrm{id}_G"] \\ & G & \end{tikzcd}$$

(e_G *is the unit morphism),*

$$\begin{tikzcd} G \cong \operatorname{Spec} \mathbf{k} \times G \arrow[r, "\iota_G \times \mathrm{id}_G"] \arrow[d, "s_G"'] & G \times G \arrow[d, "\mu_G"] & G \cong \operatorname{Spec} \mathbf{k} \times G \arrow[l, "\mathrm{id}_G \times \iota_G"'] \arrow[d, "s_G"] \\ \operatorname{Spec} \mathbf{k} \arrow[r, "e_G"'] & G & \operatorname{Spec} \mathbf{k} \arrow[l, "e_G"] \end{tikzcd} \quad \textit{(}\iota_G \textit{ is the inverse morphism).}$$

Definition 2.1.3. A *morphism of group schemes* is a morphism of schemes $f : G \to H$ where G and H are group schemes and the following diagram commutes

$$\begin{tikzcd} G \times G \arrow[r, "\mu_G"] \arrow[d, "f \times f"'] & G \arrow[d, "f"] \\ H \times H \arrow[r, "\mu_H"'] & H. \end{tikzcd}$$

Definition 2.1.4. An *algebraic group* is a $\mathbf{k}$-group scheme G such that the underlying $\mathbf{k}$-scheme is a variety. A homomorphism of algebraic groups $H_1 \to H_2$ is a morphism of group schemes. If the homomorphism is a closed immersion then we say that H_1 is a closed subgroup of H_2. A *linear algebraic group* is an algebraic group that is a closed subgroup of $\mathrm{GL}(n, \mathbf{k})$, for some $n \geq 0$.

Example 2.1.5. There are two connected linear algebraic groups of dimension one:

1. The additive group $\mathbb{G}_\mathrm{a} = \operatorname{Spec}\mathbf{k}[t]$ with the group composition $\mu : \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$ corresponding to $\mu^\# : \mathbf{k}[t] \to \mathbf{k}[t] \otimes \mathbf{k}[t]$ with $\mu^\#(t) = t \otimes 1 + 1 \otimes t$ and the inversion $\iota : \mathbb{G}_\mathrm{a} \to \mathbb{G}_\mathrm{a}$ given by $\iota^\# : \mathbf{k}[t] \to \mathbf{k}[t]$ where $\iota^\#(t) = -t$. Moreover $e : \operatorname{Spec}\mathbf{k} \to \mathbb{G}_\mathrm{a}$ corresponds to $e^\# : \mathbf{k}[t] \to \mathbf{k}$ where $t \mapsto 0$.

2. The multiplicative group $\mathbb{G}_\mathrm{m} = \operatorname{Spec}\mathbf{k}[t, t^{-1}]$ analogously has $\mu^\# : \mathbf{k}[t, t^{-1}] \to \mathbf{k}[t, t^{-1}] \otimes \mathbf{k}[t, t^{-1}]$ with $\mu^\#(t) = t \cdot t$ and $\iota^\# : \mathbf{k}[t, t^{-1}] \to \mathbf{k}[t, t^{-1}]$ where $\iota^\#(t) = t^{-1}$ and $e^\# : \mathbf{k}[t, t^{-1}] \to \mathbf{k}$ with $e^\#(t) = 1$.

Definition 2.1.6. Let H be an algebraic group. A morphism of group schemes $H \to \mathbb{G}_\mathrm{m}$ is called a *character of H*.

Definition 2.1.7. A linear algebraic group T is called an *algebraic torus* (or simply a *torus*) if it is isomorphic to $\mathbb{G}_m^n$ for some natural number n.

Lemma 2.1.8. *[48, Lemma 8.34,Lemma 8.35] Let P be a property of algebraic groups. We assume the following:*

1. *Every quotient of a group with property P has property P.*

2. *Every extension of groups with property P has property P.*

Let G be an algebraic group and H, N be algebraic subgroups G with N normal. Then

1. *If H and N have property P, then HN also has property P.*

2. *The algebraic group G has at most one maximal normal algebraic subgroup with property P.*

Definition 2.1.9. An algebraic group is called *solvable,* if it admits a subnormal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_t = \{e\}$$

such that each quotient G_i/G_{i+1} is commutative.

Proposition 2.1.10. *[48, Proposition 8.13] Algebraic subgroups, quotients, and extensions of solvable algebraic groups are solvable.*

Definition 2.1.11. An algebraic group G contains a maximal connected solvable normal subgroup, called the *radical* of G. A linear algebraic group G is called *reductive* if its radical is a torus.

2.2 Linear algebraic monoids

Definition 2.2.1. A linear algebraic monoid $(M, \circ)$ is an affine variety M together with an associative composition law $\circ : M \times M \to M$ which is a morphism of varieties and admits a unit element $1 \in M$.

Theorem 2.2.2. *Let* $\mathbf{k}$ *be an algebraically closed field and* M *a linear algebraic monoid over* $\mathbf{k}$.

1. *[18, 2.2 Theorem 1] The group of units* $G(M)$ *is an algebraic group, which is open in* M. *Moreover,* $G(M)$ *consists of nonsingular points of* M.

2. *[55, Theorem 3.15] If* M *is a linear algebraic monoid, then* M *is isomorphic to a closed submonoid of* $\mathrm{Mat}(n \times n, \mathbf{k})$ *for some* $n \in \mathbb{N}$.

Remark 2.2.3. By the above theorem, the group of units $G(M)$ of a linear algebraic monoid is a closed subgroup of some $\mathrm{GL}(n, \mathbf{k})$ [55, 3.25].

Example 2.2.4. Any unital finite dimensional $\mathbf{k}$-algebra A is with respect to multiplication a linear algebraic monoid. In particular, we obtain for a projective $\mathbf{k}$-scheme $(X, \mathcal{O}_X)$ and a coherent $\mathcal{O}_X$-module $\mathcal{F}$ that $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$ is a finite dimensional $\mathbf{k}$-algebra. By forgetting the addition $\mathrm{End}_{\mathcal{O}_X}(\mathcal{F})$ becomes a linear algebraic monoid with unit group $\mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F})$. We conclude that $\mathrm{Aut}_{\mathcal{O}_X}(\mathcal{F})$ is a linear algebraic group.

2.3 Group actions and quotients of group actions

In this section, we define algebraic actions of group schemes on schemes in the category of schemes over $\mathbf{k}$.

Definition 2.3.1. Given a $\mathbf{k}$-group scheme G and a $\mathbf{k}$-scheme X, then we call X a G-scheme, if X is equipped with an algebraic G-action. An *algebraic G-action* on X is given by a morphism $\sigma : G \times X \to X$ which makes the following two diagrams commute:

$$\begin{tikzcd} G \times G \times X \arrow[r, "\mu_G \times \mathrm{id}_X"] \arrow[d, "\mathrm{id}_G \times \sigma"] & G \times X \arrow[d, "\sigma"] \\ G \times X \arrow[r, "\sigma"] & X \end{tikzcd} \qquad \begin{tikzcd} X \cong \operatorname{Spec} \mathbf{k} \times X \arrow[r, "e_G \times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_X"] & G \times X \arrow[d, "\sigma"] \\ & X. \end{tikzcd}$$

Definition 2.3.2. Let G be a $\mathbf{k}$-group scheme and X be a G-scheme given by a G-action σ on X. The action σ is said to be

1. *closed* if for all geometric points x of X, the orbit $G \cdot x$ is closed.
2. *separated* if the image of $\Psi = (\sigma, \mathrm{pr}_2) : G \times X \to X \times X$ is closed.
3. *proper* if Ψ is proper,
4. *set-theoretically free* if $\Psi : \Psi^{-1}(\Delta_X) \to \Delta_X$ is an isomorphism,
5. *free* if Ψ is a closed immersion.

Lemma 2.3.3. *[27, Lemma 8] An action of a linear algebraic group G on X is free if and only if it is set-theoretically free and proper.*

Lemma 2.3.4. *[10, Lemma 4.2 (i)] Let G be a group scheme of finite type over $\mathbf{k}$ and X be a G-scheme of finite type over $\mathbf{k}$. Then G acts set-theoretically free on X if and only if $G(\mathbf{k})$ acts freely on $X(\mathbf{k})$.*

Definition 2.3.5. Given a $\mathbf{k}$-group scheme G and a $\mathbf{k}$-group scheme H equipped with an algebraic G-action $\sigma : G \times H \to H$, we can form the semi-direct product

$$G \ltimes_\sigma H = (G \times H, \circ_\sigma)$$

with the composition law $(g, h) \circ_\sigma (g', h') = (g \cdot g', \sigma(g', h) \cdot h')$.

Definition 2.3.6. Let X be a H-scheme and Y be a G-scheme. We say that a morphism of schemes $\varphi : X \to Y$ is *equivariant relative to* a morphism of group schemes $f : H \to G$, if the following diagram commutes

$$\begin{tikzcd} H \times X \arrow[r, "\sigma_{H,X}"] \arrow[d, "f \times \varphi"] & X \arrow[d, "\varphi"] \\ G \times Y \arrow[r, "\sigma_{G,Y}"] & Y. \end{tikzcd}$$

If φ is equivariant relative to $f : H \to \operatorname{Spec} \mathbf{k}$, where $\operatorname{Spec} \mathbf{k}$ the trivial $\mathbf{k}$-group scheme, then we say that φ is *H-invariant*.

Proposition 2.3.7. *[14, Proposition 2.8.5] Let H be an algebraic group, $G \subseteq H$ be a subgroup scheme and $N \trianglelefteq H$ be a normal subgroup scheme. Consider the semi-direct product $N \rtimes G$, where G acts on N by conjugation.*

1. *The map $f : N \rtimes G \to H$, given by $(x, y) \mapsto xy$ is a homomorphism with kernel $G \cap N$, which is identified with a subgroup scheme of $N \rtimes G$ via $x \mapsto (x^{-1}, x)$.*

2. *The image $G \cdot N$ of f is the smallest subgroup scheme of H containing G and N.*

3. *The natural maps $G/G \cap N \to G \cdot N/N$ and $N/G \cap N \to G \cdot N/G$ are isomorphisms.*

4. *If G is normal in H, then $G \cdot N$ is normal in H as well.*

Proposition 2.3.8. *[15, Proposition 2.1.10] Let G be an algebraic group and X be a G-variety. For $x \in X$, the following statements hold.*

1. *The orbit $G \cdot x$ is a locally closed, smooth subvariety of X.*

2. *The closure $\overline{G \cdot x}$ is the union of $G \cdot x$ and G-orbits of strictly lower dimension.*

3. *Every orbit of minimal dimension is closed. In particular, $\overline{G \cdot x}$ contains a closed orbit.*

Definition 2.3.9. Let H be a linear algebraic group and X be a H-variety. A pair (Y, φ) consisting of a variety Y and a H-invariant morphism $\varphi : X \to Y$ is

1. a *categorical quotient,* if for any other H-invariant morphism $X \xrightarrow{\psi} Z$ to a variety Z there exists a unique morphism $Y \xrightarrow{\overline{\psi}} Z$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "\psi"] \arrow[d, "\varphi"] & Z \\ Y \arrow[ur, "\overline{\psi}"] & \end{tikzcd}$$

   commutes.

2. a *good quotient,* if

(a) the morphism φ is surjective and affine;

(b) for each open subset $U \subseteq Y$ the pull-back map $\varphi^\sharp : \mathcal{O}_Y \to \varphi_*\mathcal{O}_X$ induces an isomorphism of sheaves $\mathcal{O}_Y \cong (\varphi_*\mathcal{O}_X)^H$, where $(\varphi_*\mathcal{O}_X)^H(U) := \mathcal{O}_X(\varphi^{-1}(U))^H$;

and

(c) if W_1, W_2 are disjoint H-invariant closed subsets of X, then $\varphi(W_1)$ and $\varphi(W_2)$ are disjoint closed subsets of Y.

In this case, we write $Y = X/\!\!/H$.

3. a *geometric quotient* is a good quotient (Y, φ) that is also an orbit space; i.e. $\varphi^{-1}(y)$ is a single H-orbit for each $y \in Y$. In this case, we write $Y = X/H$.

4. A *principal H-bundle* is a H-variety X with a H-invariant morphism $\varphi : X \to Y$ such that, for every point $y \in Y$, there is a Zariski-open neighbourhood $U_y \subseteq Y$ of y and a finite étale morphism $\tilde{U}_y \to U_y$, such that there exists a H-equivariant isomorphism $H \times \tilde{U}_y \cong \tilde{U}_y \times_Y X$, where the fibred product $\tilde{U}_y \times_Y X$ has the canonical H-action coming from the H-action on X and $H \times \tilde{U}_y$ has the trivial H-bundle action, induced by left multiplication by H on itself:

$$\begin{aligned} H \times (H \times \tilde{U}_y) &\to H \times \tilde{U}_y \\ (h, (h_0, u)) &\mapsto (h \cdot h_0, u). \end{aligned}$$

Proposition 2.3.10. *[51, Proposition 3.10] Let H be a linear algebraic group and X be a H-variety. Then*

1. *A H-invariant morphism $\varphi : X \to Y$ is a good (respectively, geometric) quotient if and only if there is an open cover $\{U_i\}$ of Y such that each restriction $\varphi|_{\varphi^{-1}(U_i)} : \varphi^{-1}(U_i) \to U_i$ is a good (respectively, geometric) quotient of H acting on $\varphi^{-1}(U_i)$.*

2. *Conversely, if $\varphi : X \to Y$ is a good (respectively, geometric) quotient, then for each open subset $U \subseteq Y$ the restriction $\varphi|_{\varphi^{-1}(U)} : \varphi^{-1}(U) \to U$ is a good (respectively, geometric) quotient of H acting on $\varphi^{-1}(U)$.*

Proposition 2.3.11. *[30, Proposition 0.9] A geometric quotient $X \to X/H$ has the structure of a principal H-bundle if and only if the action of H on X is free.*

Definition 2.3.12. Let X be a H-variety with a good quotient $\varphi : X \to Y$. We define the locus $X^0 \subset X$ to be the set of points $x \in X$ such that the H-orbit of x is closed and of maximal dimension.

Proposition 2.3.13. *Let X be a H-variety and $\varphi : X \to Y$ be a good quotient. The locus X^0 is open in X and the restriction $\varphi : X^0 \to \varphi(X^0)$ is a geometric quotient with $\varphi(X^0)$ open in Y.*

Proof. By definition X^0 consists of closed orbits in X and since φ is a good quotient we know that φ separates disjoint closed H-invariant subsets of X thus $\varphi : X^0 \to \varphi(X^0)$ is a geometric quotient provided $\varphi(X_0) \subset Y$ is open (Prop. 2.3.10). We have to show that $\varphi(X^0)$ is open with $\varphi^{-1}(\varphi(X^0)) = X^0$. If $X^0 = \emptyset$ there is nothing to show, so suppose $X^0 \neq \emptyset$. Let $m = \max\{\dim H \cdot x | x \in X\}$. The set $Z = \{z \in X | \dim H \cdot z < m\}$ is closed (see [37, Proposition 3.21]) and H-stable. Hence $\varphi(Z) \subset Y$ is closed, and $U := Y \setminus \varphi(Z)$ is open. Thus $\varphi^{-1}(U) \subseteq X$ is open and H-stable and consists only of m-dimensional orbits. It remains to show $\varphi^{-1}(U) = X^0$. Assume $x \in \varphi^{-1}(U)$ then $\varphi(x) \notin \varphi(Z)$. Hence any closed orbit that is contained in the closure of $H \cdot x$ has dimension at least m, but $\dim H \cdot x \leq m$, hence by Proposition 2.3.8 $H \cdot x$ itself is this closed orbit, thus $x \in X^0$. Conversely, if $x \in X^0$, then $\varphi(x) \notin \varphi(Z)$, hence $x \in \varphi^{-1}(U)$. We conclude that $\varphi(X^0)$ is open with $\varphi^{-1}(\varphi(X^0)) = X^0$. □

Example 2.3.14. It can happen that the locus X^0 is empty. For instance, consider the action of $\mathbb{G}_m$ on $X = \mathbb{A}^n = \operatorname{Spec} \mathbf{k}[x_1, \ldots, x_n]$ given by scalar multiplication then $\mathbf{k}[x_1, \ldots, x_n]^{\mathbb{G}_m} = \mathbf{k}$ and hence the good quotient is a single point. This is a consequence of the fact that all orbits contain the origin in their closure and get identified by the good quotient.

Theorem 2.3.15. *[58] Let H be a linear algebraic group and X be a H-variety. There exists a H-invariant open subset U of X such that U has a geometric quotient U/H and U/H is a quasi-projective variety.*

The determination/description of such open subsets as described by the above theorem is in general an open problem. Classical geometric invariant theory for linearised actions of reductive groups provides a partial answer to finding open invariant subsets depending only on a so-called linearisation which admit a geometric quotient.

Definition 2.3.16. Let H be a linear algebraic group and X be a H-scheme with associated H-action $\sigma : H \times X \to X$. A *linearisation* of the H-action on X is a line bundle $p : \mathfrak{L} \to X$ over X with an isomorphism of line bundles $p_X^*\mathfrak{L} \cong H \times \mathfrak{L} \cong \sigma^*\mathfrak{L}$ where $p_X : H \times X \to X$ is the projection, such that the induced bundle homomorphism $\Sigma : H \times \mathfrak{L} \to \mathfrak{L}$ defined by

$$\begin{tikzcd}
H \times \mathfrak{L} \arrow[dr, "\cong"] \arrow[drr, bend left, "\Sigma"] \arrow[ddr, bend right, "\mathrm{id}_H \times p"'] & & \\
& \sigma^*\mathfrak{L} \arrow[r] \arrow[d] & \mathfrak{L} \arrow[d, "p"] \\
& H \times X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

induces an action of H on $\mathfrak{L}$; that is, we have a commutative square of bundle homomorphisms

$$\begin{tikzcd}
H \times H \times \mathfrak{L} \arrow[r, "\mathrm{id}_H \times \Sigma"] \arrow[d, "\mu_H \times \mathrm{id}_{\mathfrak{L}}"] & H \times \mathfrak{L} \arrow[d, "\Sigma"] \\
H \times \mathfrak{L} \arrow[r, "\Sigma"] & \mathfrak{L}.
\end{tikzcd}$$

Remark 2.3.17. Let $\mathfrak{L} \to X$ be a linearisation of an action $\sigma : H \times X \to X$. We obtain an H-action on the global sections $f \in H^0(X, \mathfrak{L})$ via

$$h \cdot f(x) = \Sigma\Big(h, f\big(\sigma(h^{-1}, x)\big)\Big).$$

Example 2.3.18. Let H be a connected algebraic group and X be an irreducible H-variety. A linearisation $\mathfrak{L}$ of the trivial line bundle $p : X \times \mathbb{A}^1 \to X$ can be defined via a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$. More precisely, the H-action on $X \times \mathbb{A}^1$ is defined via $h \cdot (x, v) := (h \cdot x, \chi(h)v)$. Conversely, any H-linearisation of the trivial line bundle is equivalent to the choice of a character of H; see [15, Lemma 4.1.7].

Proposition 2.3.19. *[15, p. 18] Let X be a H-variety and $\mathfrak{L}_1, \mathfrak{L}_2$ be H-linearisations over X. The tensor product $\mathfrak{L}_1 \otimes \mathfrak{L}_2$ and the dual line bundles $\mathfrak{L}_1^\vee$ and $\mathfrak{L}_2^\vee$ naturally carry the structure of an H-linearisation.*

Definition 2.3.20. Let H be a linear algebraic group. A *rational H-module* is given by a pair (V, ρ), where V is a vector space and $\rho : H \to \mathrm{GL}(V)$ is a group homomorphism, such that every $v \in V$ is contained in a finite dimensional H-invariant subspace of V on which H acts algebraically. A rational H-module (V, ρ) is called a *(rational) H-representation* if V is a finite dimensional vector space.

Example 2.3.21. Let $X = \operatorname{Spec} A$ be an affine H-variety. The H-action on A given by $h \cdot f(x) := f(h^{-1} \cdot x)$ for each $h \in H$ and $x \in X$ turns A into a rational H-module.

Proposition 2.3.22. *[15, p. 7] Given an algebraic action σ of H on a finite dimensional vector space V. Then the H-action on V corresponds to a rational representation of H if and only if for each $h \in H$ the map $\sigma(h, \cdot) : V \to V$ is linear.*

Definition 2.3.23. An algebraic group U is *unipotent,* if every nonzero (rational) representation of U has a nonzero fixed vector.

From the definition it follows that extensions of unipotent groups and quotients of unipotent groups are unipotent; see [48, p. 135-136].

Definition 2.3.24. Let H be an algebraic group. The *unipotent radical* $R_u(H)$ of H is the maximal normal unipotent subgroup of H.

Definition 2.3.25. Let H be a smooth affine $\mathbf{k}$-group scheme with unipotent radical U. A *Levi-subgroup* of H is a smooth $\mathbf{k}$-subgroup scheme $L \subset H$ such that $L \to H/U$ is an isomorphism.

Remark 2.3.26. According to [19, p. 171] the following two conditions are equivalent for a smooth subgroup L of H:

1. L is a Levi-subgroup.

2. There exists an isomorphism $U \rtimes L \to H$.

Proposition 2.3.27 (Mostow)**.** *([19, Proposition 5.4.1]) Let H be a smooth affine $\mathbf{k}$-group scheme. Then Levi subgroups of H exist and the unipotent radical U of H acts transitively via conjugation on the set of Levi subgroups of H.*

Example 2.3.28. Given a finite dimensional vector space V, we denote by $\mathbb{P}(V)$ the projective space of lines in V. Suppose that V is a rational representation of H, then $\mathbb{P}(V)$ is a H-variety. Moreover the H-action on $\mathbb{P}(V)$ lifts to an action on the tautological line bundle $\mathfrak{L} = \mathcal{O}_{\mathbb{P}(V)}(-1)$.

To see this, recall that the tautological line bundle is the subvariety of $\mathbb{P}(V) \times V$ given by pairs (x, v) such that v lies on the line corresponding to x. The diagonal H-action on $\mathbb{P}(V) \times V$ stabilises $\mathfrak{L}$ and the projection $p : \mathfrak{L} \subset \mathbb{P}(V) \times V \to \mathbb{P}(V)$ is H-equivariant. Moreover, H acts linearly on the fibres of $\mathfrak{L}$. In conclusion we obtain that $\mathfrak{L}$ is a G-linearisation and by Proposition 2.3.19 we obtain a H-linearised dual line bundle $\mathfrak{L}^\vee = \mathcal{O}_{\mathbb{P}(V)}(1)$.

Definition 2.3.29. Let H be a linear algebraic group and X be a H-variety together with a linearisation $\mathfrak{L}$ of the H-action on X. We define the section ring of $\mathfrak{L}$ to be

$$\mathrm{R}(X, \mathfrak{L}) := \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})$$

where the multiplication is induced by the natural maps $H^0(X, \mathfrak{L}^{\otimes r}) \otimes H^0(X, \mathfrak{L}^{\otimes s}) \to H^0(X, \mathfrak{L}^{\otimes r+s})$.

2.4 Reductive geometric invariant theory

Reductive geometric invariant theory is a method for constructing quotients by actions of reductive groups in algebraic geometry. It was developed by Mumford. For Mumford's classical book on reductive geometric invariant theory see [30]. Following Mumford, we recall the definition of the stable and semi-stable locus for a of a linearised action of a reductive group G on a variety X. If $\mathfrak{L}$ is a linearisation of the G-action with respect to an ample line bundle $\mathfrak{L}$ and X is a projective over affine variety (see Definition 2.4.4), then we will state the Hilbert–Mumford criterion, which allows us to determine the (semi)stable locus without having to calculate the ring of invariant sections $\mathrm{R}(X, \mathfrak{L})^G = \bigoplus_{n \geq 0} H^0(X, \mathfrak{L}^{\otimes n})^G$.

Definition 2.4.1. Let X be a G-variety and $\mathfrak{L}$ be a G-linearisation. A point $x \in X$ is called

1. *semistable,* if for some $r > 0$ there exists an invariant $f \in H^0(X, \mathfrak{L}^{\otimes r})^G$ such that X_f is affine and $x \in X_f$.

2. *stable,* if for some $r > 0$ there exists an invariant $f \in H^0(X, \mathfrak{L}^{\otimes r})^G$ such that

   (a) X_f is affine and the G-action on X_f is closed,

   (b) $x \in X_f$ and for all $y \in X_f$ the stabiliser $\mathrm{Stab}_G(y)$ is finite.

3. *unstable* if it is not semistable.

Remark 2.4.2. In the following, we will denote the subvarieties of semistable, stable and unstable points by $X^{ss(G,\mathfrak{L})}$, $X^{s(G,\mathfrak{L})}$ and $X^{us(G,\mathfrak{L})}$.

Theorem 2.4.3. *[30, Theorem 1.10] Let X be a $\mathbf{k}$-scheme and let G be a reductive algebraic group acting on X. Suppose $\mathfrak{L}$ is a G-linearised line bundle over X. Then a categorical quotient (Y, ϕ) of $X^{ss(G,\mathfrak{L})}$ exists. Moreover:*

1. *ϕ is affine and universally submersive,*
2. *there is an ample line bundle $\mathfrak{L}'$ on Y such that $\phi^*(\mathfrak{L}') = \mathfrak{L}^{\otimes n}$ for some $n \in \mathbb{N}_{>0}$; hence Y is a quasi-projective algebraic scheme;*
3. *there is an open subset $\overset{\circ}{Y} \subset Y$ such that $X^{s(G,\mathfrak{L})} = \phi^{-1}(\overset{\circ}{Y})$ and $(\overset{\circ}{Y}, \phi|_{X^{s(G,\mathfrak{L})}})$ is a geometric quotient of $X^{s(G,\mathfrak{L})}$ by G.*

Definition 2.4.4. A variety X is a *projective over affine variety* if one of the following equivalent conditions hold:

1. There exists a closed immersion $X \to \mathbb{P}^n \times \mathbb{A}^m$.
2. The morphism $X \to \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is projective and $\operatorname{Spec} H^0(X, \mathcal{O}_X)$ is a variety.

Seshadri introduced a notion of relative Geometric Invariant Theory (see [61]). Gulbrandsen, Halle and Hulek obtained in this relative setup a Hilbert–Mumford criterion for the loci $Y^{ss(G,\mathcal{L})}$ and $Y^{s(G,\mathcal{L})}$ under the assumptions of the following theorem (for details see [34]).

Theorem 2.4.5. *[30, Theorem 1.10] Let $S = \operatorname{Spec} A$ be an affine scheme of finite type over $\mathbf{k}$, and let $f : Y \to S$ be a projective morphism. Let G be a reductive linear algebraic group over $\mathbf{k}$. Assume that G acts on Y and S such that f is equivariant. Let $\mathcal{L}$ be an ample G-linearised invertible sheaf on Y. Then one can define the set of stable points $Y^{s(G,\mathcal{L})}$ and the set of semistable points $Y^{ss(G,\mathcal{L})}$ in a similar fashion as in the absolute case. These sets are open and invariant. For the semi-stable locus, there exists a universally good quotient*

$$\phi : Y^{ss(G,\mathcal{L})}(\mathcal{L}) \to Z.$$

We shall often refer to Z as the GIT quotient of Y by G. Moreover, there is an open subscheme $Z' \subset Z$ with $Y^{s(G,\mathcal{L})} = \phi^{-1}(Z')$, the restriction is such that

$$Y^{s(G,\mathcal{L})} \to Z'$$

is a universally geometric quotient. Furthermore Z is projective over the quotient $S/\!\!/G = \operatorname{Spec} A^G$.

From the preceding theorem we obtain a categorical quotient for the semistable locus and a geometric quotient for the stable locus. In the following we are going to recall the Hilbert–Mumford criterion to determine these loci for so-called projective over affine varieties.

Definition 2.4.6. Let X be a G-variety. A one-parameter subgroup (1-PS) of G is an injective group homomorphism $\lambda : \mathbb{G}_{\mathrm{m}} \to G$. If $x \in X$ then we obtain a morphism $\sigma_x : \mathbb{G}_{\mathrm{m}} \to X$ given by $t \mapsto \lambda(t) \cdot x$. If the morphism σ_x extends to a $\mathbb{G}_{\mathrm{m}}$-equivariant morphism from $\mathbb{A}^1 \to X$ where $\mathbb{G}_{\mathrm{m}}$ acts on $\mathbb{A}^1$ by $t \cdot x = t^1 x$ then we denote $\sigma_x(0)$ by $\lim_{t\to 0} \lambda(t) \cdot x$. It follows that if $y := \lim_{t\to 0} \lambda(t) \cdot x$ exists, then y is fixed by $\mathbb{G}_{\mathrm{m}}$ under λ. So for any G-linearisation $\mathfrak{L}$ over X the fibre $\mathfrak{L}_y$ is a one dimensional representation of $\mathbb{G}_{\mathrm{m}}$. We let $\mu_{\mathfrak{L}}(\lambda, x)$ denote the weight with which $\mathbb{G}_{\mathrm{m}}$ acts on $\mathfrak{L}_y$ (i.e. the integer corresponding to the character) of this representation.

For an affine G-scheme X there is the following topological criterion by Kempf.

Theorem 2.4.7. *[43, Theorem 1.4] Let G be a connected reductive group over $\mathbf{k}$ and X be an affine G-scheme over $\mathbf{k}$. For a $\mathbf{k}$-point x of X, let S be a closed G-subscheme of X, which meets the closure of the orbit $G \cdot x$. Then there exists a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists and is contained in S.*

Theorem (Hilbert–Mumford criterion) 2.4.8. [30], [44], [34] Assume that

1. X is affine with $\mathfrak{L}$ a G-linearisation of the trivial line bundle $X \times \mathbb{A}^1$,

   or

2. X projective over affine with $\mathfrak{L}$ an ample G-linearisation.

Then $x \in X$ is

1. stable, if for every 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ such that $\lim_{\substack{t\to 0 \\ t\in\mathbb{G}_{\mathrm{m}}}} \lambda(t)x$ exists, we have $\mu_{\mathfrak{L}}(\lambda, x) > 0$.

2. semistable, if for every 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ such that $\lim\limits_{\substack{t\to 0\\ t\in\mathbb{G}_{\mathrm{m}}}} \lambda(t)x$ exists, we have $\mu_{\mathfrak{L}}(\lambda, x) \geq 0$.

3. unstable, if there exists a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to G$ such that $\lim\limits_{\substack{t\to 0\\ t\in\mathbb{G}_{\mathrm{m}}}} \lambda(t)x$ exists, we have $\mu_{\mathfrak{L}}(\lambda, x) < 0$.

Following Białynicki-Birula (see [12, p. 54]), we also call an equality of the following form a Hilbert–Mumford criterion: Let G be a reductive linear algebraic group and $T \subset G$ be a maximal torus of G. Given a linearised action $G \curvearrowright \mathfrak{L} \to X$ on a G-variety X, we also call an equality of the form

$$X^{ss(G,\mathfrak{L})} = \bigcap_{g\in G} gX^{ss(T,\mathfrak{L})}$$

a Hilbert–Mumford criterion. In this sense, the following proposition is a Hilbert–Mumford criterion for an action of a reductive group on a projective variety.

Proposition 2.4.9. *(Hilbert–Mumford criterion for a projective variety) [24, Theorem 9.2 and 9.3] Let G be a reductive group and X be a projective G-variety. Let $\mathfrak{L}$ be an ample G-linearisation which defines a G-equivariant projective embedding $X \subseteq \mathbb{P}^n$ such that the G-linearisation $\mathcal{O}_{\mathbb{P}^n}(1)$ pulls back to a positive tensor power of $\mathfrak{L}$.*

1. *Let T be a maximal torus of G. The loci of semistable and stable points satisfy*

$$X^{ss(G,\mathfrak{L})} = \bigcap_{g\in G} gX^{ss(T,\mathfrak{L})}$$

and

$$X^{s(G,\mathfrak{L})} = \bigcap_{g\in G} gX^{s(T,\mathfrak{L})}.$$

2. *A point $x \in X$ with homogeneous coordinates $[x_0 : \ldots : x_n]$ in some coordinate system on $\mathbb{P}^n$ is semistable (respectively stable) for the action of a maximal torus of G acting diagonally on $\mathbb{P}^n$ with weights $\alpha_0, \ldots, \alpha_n$ if and only if the convex hull $\mathrm{Conv}(\{\alpha_i | x_i \neq 0\})$ contains 0 (respectively contains 0 in its interior).*

Analogously to the projective Hilbert–Mumford criterion the affine Hilbert–Mumford criterion can also be stated in terms of a maximal torus of a reductive group G. Since we will be mostly concerned with affine GIT, we postpone the affine Hilbert–Mumford criterion (see subsection 2.7).

2.5 Non-reductive geometric invariant theory

In this section, we recall some definitions and results on non-reductive GIT. The first paper on GIT for a linear algebraic group H with a non-trivial unipotent radical U by Fauntleroy (see [28]) approaches the construction of GIT-quotients with a two step method. He assumes that the linear algebraic group H acts with respect to a linearisation on a quasi-projective variety. According to Fauntleroy the idea to construct a quotient in stages goes back at least to Nagata, who proposed to construct first a quotient for the action of the unipotent radical $U \triangleleft H$ and then a quotient for the residual action of H/U on the U-quotient. This procedure was slightly modified in the sense that Fauntleroy considered a H/U-action on a projective completion of the U-quotient instead which allowed him to apply results from classical geometric invariant theory.

Later Bérczi, Doran, Hawes and Kirwan (see [9]) also followed the same approach under the assumption that there exists a one parameter subgroup $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ such that the induced linear $\mathbb{G}_{\mathrm{m}}$-action on the Lie algebra of the unipotent radical of H is *graded* in the sense that all $\mathbb{G}_{\mathrm{m}}$-weights on $\mathrm{Lie}(U)$ are either strictly positive or strictly negative. The resulting semi-direct product $U \rtimes \mathbb{G}_{\mathrm{m}}$ is called a 'graded unipotent' group. Following the notation and convention of Bérczi, Doran, Hawes and Kirwan, we denote a graded unipotent group by $\hat{U} = U \rtimes \mathbb{G}_{\mathrm{m}} \subset H$ and assume that all $\mathbb{G}_{\mathrm{m}}$-weights on $\mathrm{Lie}(U)$ are strictly positive. Suppose a graded unipotent group $\hat{U}$ acts via a 'well-adapted' linearisation (see Definition 2.5.7) on a projective variety and the U-stabilisers are all trivial on the open stratum from the Białynicki-Birula-decomposition containing the $\mathbb{G}_{\mathrm{m}}$-stable locus with respect to the well-adapted linearisation. Then they obtain a geometric $\hat{U}$-quotient given by a projective variety which admits a 'Hilbert–Mumford-like' characterisation of the stable locus. This is then further generalised to linear algebraic groups H with unipotent radical U and such that $\hat{U} = U \rtimes \mathbb{G}_{\mathrm{m}}$ is a normal graded unipotent subgroup of H.

2.5.1 Several notions of (semi)stability for linear algebraic groups

In this section, we introduce the notions of stability from Kirwan et al following [7].

Definition 2.5.1. Let H be a linear algebraic group and $\mathfrak{L} \to X$ be a linearised H-variety. The *naively semistable locus* is the open subset

$$X^{nss} := \bigcup_{f \in I^{nss}} X_f$$

of X, where $I^{nss} := \bigcup_{r>0} H^0(X, \mathfrak{L}^{\otimes r})^H$ is the set of invariant sections of positive tensor powers of $\mathfrak{L}$. The *finitely generated semistable locus* is the open subset

$$X^{ss,fg} := \bigcup_{f \in I^{ss,fg}} X_f$$

of X^{nss}, where

$$I^{ss,fg} := \{f \in I^{nss} \mid (S^H)_{(f)} \text{ is finitely generated}\}$$

and $S = \bigoplus_{r \geq 0} H^0(X, \mathfrak{L})$ is the ring of sections.

Definition 2.5.2. Let H be a linear algebraic group with unipotent radical U and X be a H-variety. For a given linearisation $H \curvearrowright \mathcal{L} \to X$ of the H-action on X we define the *stable locus* as the open subset

$$X^{st(H,\mathcal{L})} := \bigcup_{f \in I^{st}} X_f$$

of X, where $I^{st} \subset \bigcup_{r>0} H^0(X, \mathcal{L}^{\otimes r})^H$ is the set of invariant sections of positive tensor powers of $\mathcal{L}$ such that

1. the open set X_f is affine
2. the action of H on X_f is closed with all stabilisers being finite groups; and
3. the restriction of the rational map $q_U : X \dashrightarrow \operatorname{Proj} \bigoplus_{r \geq 0} H^0(X, \mathfrak{L})^U$ to X_f

$$q_U|_{X_f} : X_f \to \operatorname{Spec}\left((\bigoplus_{r \geq 0} H^0(X, \mathfrak{L})^U)_{(f)} \right)$$

is a trivial principal U-bundle for the action of U on X_f.

2.5.2 The $\hat{U}$-Theorem

We recall the $\hat{U}$-Theorem by Bérczi, Doran, Hawes and Kirwan.

Definition 2.5.3. [8] Let U be a unipotent group together with a one-parameter group of automorphisms

$$\begin{aligned} \varphi : \mathbb{G}_{\mathrm{m}} &\to \mathrm{Aut}(U) \\ t &\mapsto \varphi_t : U \to U \end{aligned}$$

such that the weights of the induced $\mathbb{G}_{\mathrm{m}}$-action on the Lie algebra $\mathfrak{u} = \mathrm{Lie}(U)$ of U are all strictly positive (respectively negative). Then we call the semi-direct product $\hat{U} = U \rtimes_{\varphi} \mathbb{G}_{\mathrm{m}}$ given by $U \times \mathbb{G}_{\mathrm{m}}$ with composition law $(u_1, t_1) \cdot (u_2, t_2) = (\varphi_{t_2}^{-1}(u_1)u_2, t_1 t_2)$ a *graded unipotent group*. In the following we will use the convention of [9] that a graded unipotent group has positive grading.

Example 2.5.4. Consider a linear action of $U := \mathbb{G}_{\mathrm{a}}$ on $\mathbb{A}^n$. By Seshadri´s proof of Weitzenbock´s Theorem, the $\mathbb{G}_{\mathrm{a}}$-action extends to an SL_2-action on $\mathbb{A}^n$. Consider the Borel subgroup $B \subset \mathrm{SL}_2$ of upper triangular matrices. Then $B \cong \mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$ and B is a graded extension of $\mathbb{G}_{\mathrm{a}}$. We conclude that any linear $\mathbb{G}_{\mathrm{a}}$-action on $\mathbb{A}^n$ extends to an action of a graded unipotent group $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$.

Proposition 2.5.5. *[9, Lemma 4.2] Let $H := \mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$ be a graded unipotent group and X be an affine H-variety. Suppose that the Levi-factor $\mathbb{G}_{\mathrm{m}}$ of H acts on $\mathrm{Lie}(\mathbb{G}_{\mathrm{a}}) \oplus \mathcal{O}(X)$ such that the weight of $\mathrm{Lie}(\mathbb{G}_{\mathrm{a}})$ is positive and the weights of $\mathcal{O}(X)$ are non-positive. Then every point in x has a limit in X under the action of $t \in \mathbb{G}_{\mathrm{m}}$ as $t \to 0$. If additionally $\mathrm{Stab}_{\mathbb{G}_{\mathrm{a}}}(x) = \{e\}$ for each $x \in X^{\mathbb{G}_{\mathrm{m}}}$, then $X \to \mathrm{Spec}\,\mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}}$ is a trivial principal $\mathbb{G}_{\mathrm{a}}$-bundle.*

Definition 2.5.6. Let $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ be a 1-PS of H. Then $U_\lambda = U \rtimes_{c_\lambda} \mathbb{G}_{\mathrm{m}} \subset H$ where

$$\begin{aligned} c_\lambda : \mathbb{G}_{\mathrm{m}} &\to \mathrm{Aut}(H) \\ h &\mapsto \lambda(t) h \lambda(t)^{-1} \end{aligned}$$

is the inner automorphism of H induced by conjugation and U is the unipotent radical of H. The automorphism c_λ restricts to an automorphism of U, since U is a normal subgroup of H.

To state the $\hat{U}$-Theorem, we consider a linear algebraic group H, with unipotent radical U admitting a one parameter subgroup $\lambda : \mathbb{G}_{\mathrm{m}} \to H$, such that $\hat{U}_\lambda$ is a normal graded unipotent subgroup of H.

If X is a projective variety with a very ample H-linearisation $\mathcal{L}$, we can identify X with a closed H-invariant subvariety of $\mathbb{P}(V)$ where $V := H^0(X, \mathcal{L})^\vee$ and the H-linearisation $\mathcal{L}$ is the pullback of the natural H-linearisation $\mathcal{O}_{\mathbb{P}(V)}(1)$.

Let $\omega_{\min}$ be the minimal weight for $\mathbb{G}_{\mathrm{m}}$ acting on $V := H^0(X, \mathcal{L})^\vee$ and let $V_{\min}$ be the weight space of $\omega_{\min}$ in V. Define

$$\begin{aligned} Z_{\min} :=& X \cap \mathbb{P}(V_{\min}) \\ =& \{x \in X | x \text{ is a } \mathbb{G}_{\mathrm{m}}\text{-fixed point and } \mathbb{G}_{\mathrm{m}} \text{ acts on } \mathcal{L}^\vee\big|_x \text{ with weight } \omega_{\min}\} \end{aligned}$$

and

$$X^0_{\min} := \{x \in X | \lim_{\substack{t \to 0 \\ t \in \mathbb{G}_{\mathrm{m}}}} t \cdot x \in Z_{\min}\}.$$

Definition 2.5.7. Consider a linear action of the graded unipotent group $\hat{U}$ on X with respect to the linearisation $\mathcal{L}$. Let $\chi : \hat{U} \to \mathbb{G}_{\mathrm{m}}$ be a character of $\hat{U}$. We identify such characters χ with integers so that the integer 1 corresponds to the character which fits into the exact sequence $\{1\} \to U \to \hat{U} \to \mathbb{G}_{\mathrm{m}} \to \{1\}$. Suppose that $\omega_{\min} < \omega_{\min+1} < \cdots < \omega_{\max}$ are the weights with which the one-parameter subgroup $\mathbb{G}_{\mathrm{m}}$ of $\hat{U}$ acts on the fibres of the line bundle $\mathcal{O}_{\mathbb{P}((H^0(X,\mathcal{L})^\vee)}(1)$ over points of the connected components of the fixed point set $\mathbb{P}\big(H^0(X, \mathcal{L})^\vee\big)$ for the action of $\mathbb{G}_{\mathrm{m}}$ on $\mathbb{P}\big(H^0(X, \mathcal{L})^\vee\big)$. We will assume that there exist at least two distinct such weights since otherwise the action of U on X is trivial. Let c be a positive integer such that $\frac{\chi}{c} = \omega_{\min} + \epsilon$ where $\epsilon > 0$ is a sufficiently small rational number; we will call rational characters $\frac{\chi}{c}$ with this property *well adapted* and we will call the linearisation well adapted if the trivial character 0 is well adapted. The linearisation of the action of $\hat{U}$ on X with respect to the ample line bundle $\mathcal{L}^{\otimes c}$ can be twisted by the character χ so that the weights ω_j are replaced with $\omega_j c - \chi$; let $\mathcal{L}_\chi^{\otimes c}$ denote this twisted linearisation. Then $\omega_{\min}(\mathcal{L}_\chi^{\otimes c}) = \omega_{\min}(\mathcal{L})c - \chi = -\epsilon c < 0 < \omega_{\min+1}(\mathcal{L}_\chi^{\otimes c})$.

We say that *semistability coincides with stability* for the well adapted $\hat{U}$-linearisation $\mathcal{L} \to X$ if $\mathrm{Stab}_U(x) = \{e\}$ for every $x \in Z_{\min}$.

Theorem 2.5.8. *[9] Let H act on an irreducible projective variety X over $\mathbb{C}$ and let $\hat{U}$ be a graded* *unipotent subgroup of H such that $\mathbb{G}_{\mathrm{m}} = \hat{U}/U$ lies in the center of H/U. Let $\mathcal{L}$ be a well adapted linearisation with respect to a very ample line bundle $\mathfrak{L}$. Suppose that the linear $\hat{U}$-action satisfies the condition semistability coincides with stability. Then*

1. *The $\hat{U}$-invariants are finitely generated, and the inclusion of the $\hat{U}$-invariant algebra induces a projective geometric quotient of an open subvariety $X^{st(\hat{U},\mathcal{L})} = X^{ss,\hat{U}}$ of X*

$$X^{ss,\hat{U}} \to X/\!\!/_{\mathcal{L}}\, \hat{U}.$$

2. *Consequently the H-invariants are finitely generated, and the inclusion of the H-invariant subalgebra induces a projective good quotient of an open subvariety $X^{ss,H}$ of X*

$$X^{ss,H} \to X/\!\!/_{\mathcal{L}}\, H.$$

   where $X/\!\!/_{\mathcal{L}}\, H$ is the GIT-quotient of $X/\!\!/_{\mathcal{L}}\, \hat{U}$ by the induced action of the reductive group $H/\hat{U}$ with respect to the induced linearisation.

3. *The good quotient $X^{ss,H} \to X/\!\!/_{\mathcal{L}}\, H$ restricts to a geometric quotient*

$$X^{s,H} \to X^{s,H}/H \subseteq X/\!\!/_{\mathcal{L}}\, H.$$

   of an open subset $X^{s,H} \subset X^{ss,H}$.

4. *(Non-reductive Hilbert–Mumford Criterion) For $H = \hat{U}$ we have*

$$X^{ss(\hat{U},\mathcal{L})} = \bigcap_{u \in U} uX^{ss(\mathbb{G}_{\mathrm{m}},\mathcal{L})} = X^0_{\min} \setminus UZ_{\min}.$$

In the case that each point $x \in X$ has well behaved positive dimensional stabilisers for the U-action on $X^0_{\min}$ there are variations of the $\hat{U}$-Theorem using a blow-up procedure. For further details we refer the interested reader to [9].

2.6 Linearised actions on affine varieties

Given a linear algebraic group H and an affine H-variety X together with a linearisation $\mathfrak{L}$ of the trivial line bundle $X \times \mathbb{A}^1$. For this choice of linearisation, we obtain an isomorphism of $\mathbb{N}$-graded rings $\mathrm{R}(X, \mathfrak{L}) \cong \mathcal{O}(X)[t]$. It follows that $\mathrm{Proj}\, \mathcal{O}(X)[t] \cong \mathrm{Spec}\, \mathcal{O}(X)$ which induces a rational map

$$X \cong \mathrm{Proj}\, \mathrm{R}(X, \mathfrak{L}) \dashrightarrow \mathrm{Proj}\, \mathrm{R}(X, \mathfrak{L})^H$$

corresponding to the inclusion of graded rings $\mathrm{R}(X, \mathfrak{L})^H \subset \mathrm{R}(X, \mathfrak{L})$. By Example 2.3.18, any linearisation of the trivial line bundle corresponds to the choice of a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$. If the linearisation $\mathfrak{L}$ corresponds to the trivial character χ with $\ker \chi = H$ we obtain $\mathrm{Spec}\, \mathcal{O}(X)^H \cong \mathrm{Proj}\, \mathrm{R}(X, \mathfrak{L})^H$. In this case the rational map $X \dashrightarrow \mathrm{Proj}\, \mathrm{R}(X, \mathfrak{L})^H$ is a morphism of affine schemes $X \to \mathrm{Spec}\, \mathcal{O}(X)^H$ induced by the inclusion $\mathcal{O}(X)^H \subset \mathcal{O}(X)$.
The following result implies that the ring of invariants is in general not finitely generated. Consequently we have to impose additional assumptions on an action of a linear algebraic group on an affine variety in order to obtain a finitely generated ring of invariants.

Popov's theorem 2.6.1. [54] For a linear algebraic group H, the following statements are equivalent:

1. H is reductive.

2. Every rational H-action on a finitely generated **k**-algebra A has a finitely generated subalgebra of invariants A^H.

Definition 2.6.2. Let H be an affine algebraic group and X be an affine H-variety. Suppose the H-action on X is linearised with respect to the trivial line bundle $\mathfrak{L}$.

1. Two points $x, y \in X$ are $(H, \mathfrak{L})$-*separable* if there exists $r \geq 1$ and $f \in H^0(X, \mathfrak{L}^{\otimes r})^H$ such that $f(x) \neq f(y)$.

2. A subset $S \subset \bigcup_{n \geq 1} H^0(X, \mathfrak{L}^{\otimes r})^H$ is said to be $(H, \mathfrak{L})$-*separating* if for any two $(H, \mathfrak{L})$-separable points $x, y \in X$ there exists $g \in S$ with $g(x) \neq g(y)$.

If the linearisation $\mathfrak{L}$ corresponds to the trivial character we drop $\mathfrak{L}$ from the notation and call points H-separable if Condition 1 is satisfied and a subset satisfying Condition 2 a H-separating set.

By Popov's theorem, we cannot expect the ring $\mathcal{O}(X)^H$ to be finitely generated but we have the following result by Derksen and Kemper which tells us that we can find a finite set of H-separating invariants.

Theorem 2.6.3. *[22, Theorem 2.4.8] Let X be an affine H-variety. Then there exists a finite separating set for the induced H-action on $\mathcal{O}(X)$.*

Corollary 2.6.4. *Let X be an affine H-variety and $\mathfrak{L} = X \times \mathbb{A}^1$ be a linearisation of the trivial line bundle over X. Suppose that $H^0(X, \mathcal{O}_X)^H = \mathbf{k}$; then there exists a finite $(H, \mathfrak{L})$-separating set $S \subset \bigcup_{n \geq 1} H^0(X, \mathfrak{L}^{\otimes r})^H$.*

Proof. This follows immediately from Theorem 2.6.3 and Proposition 2.6.5. □

Proposition 2.6.5. *Let X be an affine H-variety and $\chi : H \to \mathbb{G}_{\mathrm{m}}$ be a character of H. We consider the action of H on $X \times \mathbb{A}^1$ via $h \cdot (x, t) := (h \cdot x, \chi^{-1}(h)t)$. Then $X \times \mathbb{A}^1$ is an affine H-variety and provides a linearisation of the H-action on X. Let $\mathcal{O}(X \times \mathbb{A}^1)^H = (\mathcal{O}(X) \otimes_{\mathbf{k}} \mathbf{k}[t])^H$ be the ring of H-invariants for H acting on $X \times \mathbb{A}^1$ graded via setting $\deg t = 1$. Then as a graded ring we have $\mathcal{O}(X \times \mathbb{A}^1)^H \cong \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H$.*

Proof. Recall that $H^0(X, \mathfrak{L}^{\otimes r})^H \cong \{f \in \mathcal{O}(X) | \forall h \in H : h \cdot f = \chi^r(h) f\}$. Via this identification we obtain the following homomorphism given by

$$\varphi : \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H \to \mathcal{O}(X \times \mathbb{A}^1)$$
$$(f_r)_{r \geq 0} \mapsto \sum_{r \geq 0} f_r t^r.$$

Given $f = (f_r)_{r \geq 0}, g = (g_r)_{r \geq 0} \in \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H$ then $\varphi(f) = \varphi(g)$ implies that $f_r = g_r$ for all $r \geq 0$ thus φ is injective. Additionally we have that $h \cdot f_r = \chi^r f_r$ and $h \cdot t^r = \chi^{-r}$ thus $f_r t^r$ is H-invariant. We conclude that φ is compatible with the grading of $\mathcal{O}(X \times \mathbb{A}^1)$ by the degree of t and the image of φ is contained in $\mathcal{O}(X \times \mathbb{A}^1)^H$. It remains to show that φ surjects onto $\mathcal{O}(X \times \mathbb{A}^1)^H$.

Given $g \in \mathcal{O}(X \times \mathbb{A}^1)^H \cong \mathcal{O}(X)[t]^H$ then g decomposes as $g = \sum_{i=0}^{n_g} g_i t^i$. Since g is H-invariant we have that $\forall h \in H$ that $h \cdot g = g$. By comparing the degree i-part of the right hand side and left hand side it follows that $h \cdot g_i = \chi^i(h) g_i$. Thus the element g' defined by $g'_i = g_i$ for $i \leq n_g$ and $g'_i = 0$ for $i > n_g$ is such that $\varphi(g') = g$. We conclude that φ is an isomorphism of graded rings. □

Definition 2.6.6. Let H be a linear algebraic group and G be a subgroup of H.

1. G is an *observable* subgroup of H, if there exists a rational H-module V and $v \in V$ such that $\mathrm{Stab}_H(v) = G$.

2. G is a *Grosshans* subgroup of H, if for every affine H-variety X the ring of G-invariants $\mathcal{O}(X)^G$ is finitely generated.

Theorem (Grosshans Criterion) 2.6.7. [31] Let R be a reductive group over an algebraically closed field $\mathbf{k}$ and G be an observable subgroup of R. Then the following conditions are equivalent:

1. G is a Grosshans subgroup of R.

2. The algebra $\mathcal{O}(R)^G$ is finitely generated, where G acts on R via right translation.

3. There exists a rational R-representation V and some $v \in V$ such that $G = \mathrm{Stab}_R(v)$ and $\dim(\overline{R \cdot v} \setminus R \cdot v) \leq \dim(R \cdot v) - 2$.

Example 2.6.8. The additive group $\mathbb{G}_\mathrm{a}$ is a Grosshans subgroup of $\mathrm{SL}(2;\mathbf{k})$. If furthermore $\mathbf{k}$ has characteristic zero, then by Seshadri's proof of Weitzenböck's Theorem any rational $\mathbb{G}_\mathrm{a}$-representation V is also a rational $\mathrm{SL}(2;\mathbf{k})$-representation and hence $\mathcal{O}(V)^{\mathbb{G}_\mathrm{a}}$ is finitely generated.

2.7 Geometric invariant theory for affine varieties

We recall the GIT-quotient for reductive groups acting algebraically on affine varieties with respect to a linearisation of the trivial line bundle. We also recall the characterisation of stability, as defined by King, for reductive actions on affine spaces in terms of the Hilbert–Mumford criterion as stated by Hoskins (see [36, Proposition 2.7]). For torus actions on affine N-space linearised by a character ϱ, we recall the geometric Hilbert–Mumford criterion which expresses (semi)stability via polyhedral cones which are defined in terms of the weights and one parameter subgroups of the torus action. Via duality for cones we obtain the equivalent dual characterisation of (semi)stability.

Theorem 2.7.1. *[51, Theorem 3.4 and Theorem 3.5] Let G be a reductive algebraic group and X be an affine G-variety over $\mathbf{k}$. Then*

1. *The morphism $X \to \operatorname{Spec} \mathcal{O}(X)^G$ is a good quotient.*
2. *The ring of invariants $\mathcal{O}(X)^G$ is finitely generated and $\operatorname{Spec} \mathcal{O}(X)^G$ is an affine variety.*
3. *For $x, y \in X$ we have $\varphi(x) = \varphi(y)$ if and only if $\overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset$.*
4. *Any G-orbit in X contains a unique closed orbit.*

By applying Proposition 2.6.5 we obtain the following Corollary to the above theorem.

Corollary 2.7.2. *Let G be reductive linear algebraic group and X be an affine G-variety linearised with respect to the trivial line bundle $\mathfrak{L} = X \times \mathbb{A}^1$. The ring of G-invariant sections $\bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^G$ is finitely generated.*

Remark 2.7.3. Given an action of a reductive group G on an affine G-variety X linearised with respect to the trivial line bundle, then the morphism $\operatorname{Proj} \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^G \to \operatorname{Spec} \mathcal{O}(X)^G$ is projective [44, p. 517-518].

Theorem 2.7.4. *[30], [44] Let G be a reductive linear algebraic group and X be an affine G-variety. Let $\mathfrak{L}$ be the linearisation of the trivial line bundle corresponding to the character $\chi : G \to \mathbb{G}_{\mathrm{m}}$ then*

1. *The locus $X^{ss(G,\chi)} = X^{ss(G,\mathfrak{L})}$ admits a good quotient*
$$q_{\mathfrak{L}} : X^{ss(G,\mathfrak{L})} \to \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^G$$
where the $X^{ss(G,\mathfrak{L})}$ is the domain of definition of the rational map
$$q_{\mathfrak{L}} : X \dashrightarrow \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^G.$$
2. *The good quotient restricts to a geometric quotient $X^{s(G,\mathfrak{L})} \to X^{s(G,\mathfrak{L})}/G$ with $X^{s(G,\mathfrak{L})}/G$ open in $\operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^G$.*

Affine Hilbert–Mumford criterion

Notation 2.7.5. Let T be a torus. We denote the group of characters by $\mathbf{X}^*(T) := \mathrm{Hom}_{grps}(T, \mathbb{G}_\mathrm{m})$ and the group of one parameter subgroups by $\mathbf{X}_*(T) := \mathrm{Hom}_{grps}(\mathbb{G}_\mathrm{m}, T)$. By composition we obtain a natural pairing

$$\langle\ ,\ \rangle : \mathbf{X}^*(T) \times \mathbf{X}_*(T) \to \mathrm{Hom}_{grps}(\mathbb{G}_\mathrm{m}, \mathbb{G}_\mathrm{m}) \cong \mathbb{Z}$$
$$(\chi, \lambda) \mapsto \chi \circ \lambda.$$

By tensoring with $- \otimes_\mathbb{Z} \mathbb{R}$, we extend the natural pairing to

$$\langle\ ,\ \rangle : \mathbf{X}^*(T)_\mathbb{R} \times \mathbf{X}_*(T)_\mathbb{R} \to \mathbb{R}.$$

To a character $\chi \in \mathbf{X}^*(T)$ we associate the *halfspace* $H_\chi := \{\lambda \in \mathbf{X}_*(T)_\mathbb{R} | \langle \chi, \lambda \rangle \geq 0\}$.

Proposition 2.7.6. *Let T be a torus and V be a (rational) T-module. Then there exists a weight space decomposition*

$$V = \bigoplus_{\chi \in \mathbf{X}^*(T)} V_\chi$$

where $V_\chi := \{v \in V | \forall t \in T : t \cdot v = \chi(t)v\}$.

Definition 2.7.7. Let V be a (rational) T-module with weight space decomposition $V = \bigoplus_{\chi \in \mathbf{X}^*(T)} V_\chi$.

1. We denote by pr_χ the projection to the weight space V_χ.

2. For $v \in V$, let $\mathrm{wt}_T(v) := \{\chi \in \mathbf{X}^*(T) | \mathrm{pr}_\chi(v) \neq 0\}$ be the *weight set* associated to v and $\sigma_v := \bigcap_{\chi \in \mathrm{wt}_T(v)} H_\chi$ the *weight cone* associated to v.

The Hilbert-Mumford criterion obtain by King in [44] was conveniently restated in terms of torus weights by Hoskins [36] in the following Proposition.

Proposition 2.7.8. *[36, Proposition 2.7] Given a rational G-representation V of a reductive group G with maximal torus T and let $\varrho : G \to \mathbb{G}_\mathrm{m}$ be a character. Then $v \in V$ belongs to $x \in V^{(s)s(G,\varrho)}$ if and only if for each $g \in G$ we have $gx \in V^{(s)s(T,\varrho)}$. Let $v \in V$ then*

1. *$v \in V^{ss(T,\varrho)}$ if and only if is H_ϱ is a supporting halfspace for σ_v that is, $\sigma_v \subset H_\varrho$.*

2. *$v \in V^{s(T,\varrho)}$ if and only if $\sigma_v \setminus \{0\} \subset H_\varrho \setminus (H_\varrho \cap H_{-\varrho}) = \{\lambda \in \mathbf{X}_*(T)_\mathbb{R} | \langle \varrho, \lambda \rangle > 0\}$.*

By duality for polyhedral cones we obtain the following dual statement to the Hilbert–Mumford criterion for rational G-representations which can be more useful then Hoskins's reinstatement in some cases.

Corollary 2.7.9. *The affine Hilbert–Mumford criterion for a linear torus action admits the following dual version:*

1. *$\sigma_v = \bigcap_{\chi \in \mathrm{wt}_T(v)} H_\chi \subset H_\varrho$ if and only if $\varrho \in \mathrm{Cone}(\mathrm{wt}_T(v))$.*

2. *$\sigma_v \setminus \{0\} \subset H_\varrho \setminus (H_\varrho \cap H_{-\varrho})$ if and only if ϱ belongs to the interior of $\mathrm{Cone}(\mathrm{wt}_T(v))$.*

Proof. By Proposition A.1.4 we have that $\mathrm{Cone}(\varrho) = H_\varrho^\vee \subset \sigma_v^\vee = \mathrm{Cone}(\mathrm{wt}_T(v))$ if and only if $\sigma^\vee \subset H_\varrho$. For the second part we use that $\sigma_v \setminus \{0\} \subset H_\varrho \setminus (H_\varrho \cap H_{-\varrho})$ is equivalent to $\sigma_v \subset H_\varrho$ and $\sigma_v \cap H_{-\varrho} = \{0\}$. Now apply conic duality to obtain $\mathrm{Cone}(\mathrm{wt}_T(v), -\varrho) = \mathbf{X}^*(T)_\mathbb{R}$ and $\mathrm{Cone}(\varrho) \subset \mathrm{Cone}(\mathrm{wt}_T(v))$ which is equivalent to ϱ belonging to the interior of $\mathrm{Cone}(\mathrm{wt}_T(v))$. □

Proposition 2.7.10. *(Hilbert-Mumford criterion for an affine G-variety) Let G be a reductive group and X be an affine G-variety. Let $\mathfrak{L}$ be a linearisation of the trivial line bundle over X corresponding to a character $\chi : G \to \mathbb{G}_\mathrm{m}$. Then there exists a G-representation together with a G-equivariant closed immersion $\iota : X \to V$. Then if we linearise the trivial line bundle over V with respect to χ we obtain a G-equivariant commutative square*

$$\begin{tikzcd} X \times \mathbb{A}^1 \arrow[r, "\iota \times \mathrm{id}_{\mathbb{A}^1}"] \arrow[d, "p_X"'] & V \times \mathbb{A}^1 \arrow[d, "p_V"] \\ X \arrow[r, "\iota"'] & V. \end{tikzcd}$$

1. *Let T be a maximal torus of G. The loci of semistable and stable points satisfy*

$$X^{ss(G,\chi)} = \bigcap_{g \in G} g X^{ss(T,\chi)}$$

and

$$X^{s(G,\chi)} = \bigcap_{g \in G} gX^{s(T,\chi)}.$$

2. *Let T be a maximal torus of G. A point $x \in X$ with linear coordinates in V consisting of T-weight vectors $\iota(x) = \sum_{i=1}^{n} v_{\chi_i}$ is χ-semistable (respectively χ-stable) for the restricted action of T if and only if $\mathrm{Cone}(\{\chi_1, \ldots, \chi_n\})$ contains χ (respectively contains χ in its interior).*

Corollary 2.7.11. *Let $f : V \to W$ be T-equivariant with respect to linear T-actions on V and W. Let $v \in V$ and set $w := f(v)$ then the following statements hold.*

1. *We have that $\sigma_v \subset \sigma_w$.*

2. *Suppose that v is a weight vector of weight χ. Then w is a T-fixed point or there exists $n \geq 1$ such that $\chi^n \in \mathrm{wt}_T(w)$.*

3. *Let w be a weight vector of weight χ, then χ is a non-negative combination of weights from $\mathrm{wt}_T(v)$.*

Proof.

1. Given a 1-PS in σ_v then $\lim_{t \to 0} t \cdot v$ exists. By equivariance it follows that also $\lim_{t \to 0} t \cdot w$ exists and hence $\sigma_v \subset \sigma_w$.

2. From $\sigma_v \subset \sigma_w$ and duality it follows that $\mathrm{Cone}(\mathrm{wt}_T(w)) \subseteq \mathrm{Cone}(\mathrm{wt}_T(v)) = \mathbb{R}_{\geq 0}\chi$. We conclude that either $\mathrm{Cone}(\mathrm{wt}_T(w)) = \{0\}$ which implies that w is a T-fixed point or $\mathrm{Cone}(\mathrm{wt}_T(w)) = \mathrm{Cone}(\mathrm{wt}_T(v))$.

3. From the weight space decomposition of V, we can write $v = \sum_{i=1}^{k} v_i$ with $v_i \in V_{\chi_i}$. From $\bigcap_{i=1}^{k} H_{\chi_i} = \sigma_v \subset \sigma_w = H_\chi$, it follows that χ is a non-negative combination of $\chi_1, \ldots, \chi_k$ and a positive combination of a linearly independent subfamily from $\chi_1, \ldots, \chi_k$.

$\square$

Chapter 3

Actions of unipotent groups on affine varieties

Let U be a unipotent linear algebraic group. It is easy to see that U admits only the trivial character $\chi : U \to \mathbb{G}_{\mathrm{m}}$ with $\ker \chi = U$. This implies that for an affine U-variety X, we have a unique linearisation of the trivial line bundle $\mathfrak{L} = X \times \mathbb{A}^1$ and are left to consider the morphism $X \to \operatorname{Spec} \mathcal{O}(X)^U$.

Kostant–Rosenlicht Theorem 3.0.1. [59, Theorem 2] Let X be an affine U-variety. Then all U-orbits in X are closed.

Corollary 3.0.2. *Let X an affine U-variety with a good U-quotient $\varphi : X \to Y$. Then $\varphi : X \to Y$ is a geometric U-quotient.*

Proposition 3.0.3. *Let X be a U-variety. The U-action is free if and only if it is proper.*

Proof. If $\Psi : U \times X \to X \times X$ is proper then $\Psi : \Psi^{-1}(\Delta_X) \to \Delta_X$ is finite so the U-action has finite stabilisers. The stabiliser subgroups are finite and unipotent. It follows that the stabilisers are trivial since we are working in characteristic zero. We conclude that proper implies set-theoretically free. By Lemma 2.3.3 the action is free if and only if it is proper and set-theoretically free. □

If U acts properly on an affine variety, then any geometric quotient is a principal U-bundle. In the following we are considering a sufficient condition to obtain a trivial principal U-bundle for actions on affine varieties (see [9]). This naturally leads us to the notion of graded actions on affine varieties. As a motivation for the consideration of trivial principal U-bundles, we recall the following proposition.

Proposition 3.0.4. *[7, Proposition 2.1.26] Suppose X is an affine U-variety and a locally trivial quotient $X \to X/U$ exists. Then X/U is affine if and only if $X \to X/U$ is a trivial principal U-bundle.*

Let T be a torus and X be an affine variety. To a T-action on X, we can associate the coaction map $\mathcal{O}(X) \to \mathcal{O}(X) \otimes \mathcal{O}(T)$ which induces an $\mathbf{X}^*(T)$-grading on $\mathcal{O}(X)$. Conversely given an $\mathbf{X}^*(T)$-grading on $\mathcal{O}(X)$ we obtain a coaction map which corresponds to a T-action on X. This construction gives a bijective correspondence between

$$\{T - \text{actions on } X\} \leftrightarrow \{\mathbf{X}^*(T) - \text{gradings on } \mathcal{O}(X)\}.$$

Based on the notion 'graded unipotent radical' from [9] we define a graded action.

Definition 3.0.5. Let X be an affine H-variety for a linear algebraic group H with unipotent radical U. We call the action of H on X a *graded action* with respect to a grading subgroup $U_{T_g} = U \rtimes T_g \subset H$ with T_g a torus if the following conditions are fulfilled:

1. The grading monoid $M := \{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w \neq \{0\}\} \subset \mathbf{X}^*(T_g)$ is *positive;* that is, if $x \in M$ and $-x \in M$, then $x = 0$.

2. Additionally we assume for the linear T_g-action on $\mathrm{Lie}(U)$ induced by the semi-direct product structure of $U \rtimes T_g$ that $\{w \in \mathbf{X}^*(T_g) | \, \mathrm{Lie}(U)_w \neq \{0\}\} \cap M = \emptyset$.

We call X a *graded H-variety* and U_{T_g} a *grading subgroup* for the H-action on X, if the H-action on X restricted to U_{T_g} satisfies Conditions 1 and 2.

Proposition 3.0.6. *Let $H = U \rtimes \mathbb{G}_{\mathrm{m}}$ be a linear algebraic group with U unipotent and $\mathbb{G}_{\mathrm{m}}$ the multiplicative group. Let V be an H representation. If $H = \hat{U}$ is a graded extension of a unipotent group, then there exists a character $\chi : \hat{U} \to \mathbb{G}_{\mathrm{m}}$ such that the χ-twisted $\hat{U}$-action on V is a graded action. Conversely given a graded action of $U \rtimes T$ with $T = \mathbb{G}_{\mathrm{m}}$ then $U \rtimes \mathbb{G}_{\mathrm{m}}$ is a graded extension of U.*

Proof. Let $\hat{U}$ is a graded extension of a unipotent group and V a $\hat{U}$-representation. Let χ be the character corresponding to $-\omega_{\min}$ where $\omega_{\min}$ is the minimal weight of $\mathbb{G}_{\mathrm{m}}$ acting on V. By twisting with χ the $\hat{U}$-representation V_χ has a graded action. We postpone the proof of this fact to Proposition 4.2.8. Conversely, let V be a H-representation such that the H-action on V is graded. The grading monoid $\mathcal{O}(V)$ contains zero thus $\mathbb{G}_{\mathrm{m}}$ either acts with positive or negative weights on $\mathrm{Lie}(U)$. By interchanging the roles of t and t^{-1} if necessary we can assume that $\mathbb{G}_{\mathrm{m}}$ acts with positive weights on $\mathrm{Lie}(U)$ then $H = U \rtimes \mathbb{G}_{\mathrm{m}}$ is a graded extension of U as claimed. $\square$

Proposition 3.0.7. *Let H be a linear algebraic group and X be an affine H-variety with a graded H-action. Let $U \rtimes T_g$ be a grading subgroup for the H-action on X and $N \subset U$ be a normal subgroup of $U \rtimes T_g$. Then the action of $U \rtimes T_g/N$ on $\operatorname{Spec} \mathcal{O}(X)^N$ satisfies Condition 1 and 2 of Definition 3.0.5.*

Proof. We have that

$$\left\{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w^N \neq \{0\}\right\} \subset \left\{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w \neq \{0\}\right\}$$

and

$$\mathrm{wt}_{T_g}(\mathrm{Lie}(U/N)) \subset \mathrm{wt}_{T_g}(\mathrm{Lie}(U)).$$

Since $U \rtimes T_g$ is a grading subgroup for the H-action on X we have that

$$\emptyset = \left\{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w \neq \{0\}\right\} \cap \mathrm{wt}_{T_g}(\mathrm{Lie}(U)) \supset \left\{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w^N \neq \{0\}\right\} \cap \mathrm{wt}_{T_g}(\mathrm{Lie}(U/N)).$$

We conclude that the action of $U/N \rtimes T_g$ on $\operatorname{Spec} \mathcal{O}(X)^N$ satisfies Condition 1 and 2 of Definition 3.0.5. $\square$

Proposition 3.0.8. *[41, p. 247] Let T be a torus and X be an affine T-variety. Suppose that the grading monoid of $\mathcal{O}(X)$ is positive. Then the fixed point variety X^T is affine and a good quotient for the T-action on X. As a consequence the only closed T-orbits are given by the T-fixed points.*

Proof. Let $I := \langle \bigoplus_{m \in M \setminus \{0\}} \mathcal{O}(X)_m \rangle$ be the ideal generated by $\bigoplus_{m \in M \setminus \{0\}} \mathcal{O}(X)_m$. It is well known that $\operatorname{Spec} \mathcal{O}(X)/I \cong X^T$ and by positivity of the grading monoid M it follows that

$I = \bigoplus_{m \in M \setminus \{0\}} \mathcal{O}(X)_m$. We conclude that $\mathcal{O}(X)/I \cong \mathcal{O}(X)_0 = \mathcal{O}(X)^T$. From the identification $\mathcal{O}(X)/I \cong \mathcal{O}(X)_0$ we obtain a morphism $\mathcal{O}(X)_0 \to \mathcal{O}(X) \to \mathcal{O}(X)/I = \mathcal{O}(X)_0$ which is the identity map on $\mathcal{O}(X)_0$. Let $X \to X/\!\!/T = \operatorname{Spec} \mathcal{O}(X)^T$ be the affine GIT-quotient. By Theorem 2.7.1 we know that $X \to X/\!\!/T = \operatorname{Spec} \mathcal{O}(X)_0 \cong X^T$ is a good quotient. Each fibre of a good quotient contains a unique closed orbit. It follows that the closed T-orbits in X are given by T-fixed points, since each fibre of the good quotient contains a T-fixed point. □

3.1 Background on $\mathbb{G}_\mathrm{a}$-actions

In this subsection we recall some basic facts on $\mathbb{G}_\mathrm{a}$-actions which will be used in proofs related to graded actions on affine varieties.

Definition 3.1.1. Let A be an affine $\mathbf{k}$-algebra, so then $\operatorname{Spec} A$ is an affine variety.

1. Denote by $\operatorname{LND}(A)$ the set of locally nilpotent derivations of A, where a locally nilpotent derivation is a $\mathbf{k}$-linear map $\delta : A \to A$ such that
   (a) for each $f \in A$ there exists $n_f \in \mathbb{N}$ such that $\delta^{n_f}(f) = 0$, and
   (b) $\forall f, g \in A : \delta(fg) = \delta(f)g + f\delta(g)$.
2. Assume that the affine $\mathbf{k}$-algebra A is graded by a submonoid $M \subseteq \mathbb{Z}^n$, then $\delta \in \operatorname{LND}(A)$ is a *homogeneous* locally nilpotent derivation of the M-graded algebra $A = \bigoplus_{k \in M} A_k$, if there exists $\deg \delta \in \mathbb{Z}^n$ such that for each $k \in M$ we have $\delta|_{A_k} : A_k \to A_{k+\deg \delta}$.
3. We call an element of $s \in A$ a *slice* of the locally nilpotent derivation δ if $\delta(s) = 1$.

Proposition 3.1.2. *[62, Subsection 14.2.1] Let X be an irreducible affine variety over $\mathbf{k}$. We obtain a bijective correspondence between algebraic $\mathbb{G}_\mathrm{a}$-actions on X and locally nilpotent derivations of the ring of regular functions $\mathcal{O}(X)$.*

Example 3.1.3. Let X be an affine $\mathbb{G}_\mathrm{a} \rtimes T$-variety. Under the above correspondence the restriction of the $\mathbb{G}_\mathrm{a} \rtimes T$-action on X to $\mathbb{G}_\mathrm{a}$ corresponds to a homogeneous locally nilpotent derivation $\delta : \mathcal{O}(X) \to \mathcal{O}(X)$ of the M-graded $\mathbf{k}$-algebra $\mathcal{O}(X)$ where the grading monoid $M \subseteq \mathbf{X}^*(T) \cong \mathbb{Z}^n$ is induced by the T-action on X. Here, we have $\deg \delta \in \mathbf{X}^*(T) \cong \mathbb{Z}^{\dim T}$.

Definition 3.1.4. Let A be an affine $\mathbf{k}$-algebra, so then $\operatorname{Spec} A$ is an affine variety. Given $\delta \in \operatorname{LND}(A)$, define $\operatorname{Fix}(\delta) := \{\mathfrak{p} \in \operatorname{Spec}(A) | \delta(A) \subset \mathfrak{p}\}$. Note that $\operatorname{Fix}(\delta)$ is a closed subset of $\operatorname{Spec}(A)$.

Proposition 3.1.5. *[21, Prop 9.7.] Let A be an affine $\mathbf{k}$-algebra and $\delta \in \operatorname{LND}(A)$. Consider the associated $\mathbb{G}_a$-action α_δ on $X := \operatorname{Spec} A$. Then the following holds*

1. *The ring of invariants of α_δ is the subring $\ker(\delta)$ of A.*
2. *The fixed points of α_δ are precisely the closed points of $\operatorname{Fix}(\delta)$.*

Lemma 3.1.6. *[53, Lemme de Taylor] Suppose X is an affine $\mathbb{G}_a$-variety and let $\delta : \mathcal{O}(X) \to \mathcal{O}(X)$ be the associated locally nilpotent derivation. Suppose that δ has a slice, then $X \to \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$ is a trivial principal $\mathbb{G}_a$-bundle.*

3.2 On trivial $\mathbb{G}_a$-quotients of affine varieties

The following lemma is the base case of the inductive proof of Lemma 3.3.1 and will also be reapplied in the proof of the induction step.

Lemma 3.2.1. *Given an affine variety X with a graded action of $\mathbb{G}_a \rtimes T$. Suppose that each T-fixed point $x \in X^T$ has trivial $\mathbb{G}_a$-stabiliser, then the homogeneous locally nilpotent derivation corresponding to the $\mathbb{G}_a$-action admits a section and hence $X \to \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_a}$ is a trivial principal $\mathbb{G}_a$-bundle.*

Remark 3.2.2. For $T = \mathbb{G}_m$ we obtain that Lemma 3.2.1 is equivalent to [9, Lemma 4.2].

Before we proceed with a proof of the above lemma we are going to need the following results:

Proposition 3.2.3. *Consider an algebraic action on an affine variety X by a semi-direct product $H := U \rtimes \mathbb{G}_m$ of an unipotent group with the multiplicative group. Suppose that $x_0 \in X^{\mathbb{G}_m}$ and $x \in X$ such that $\lim_{t\to 0} t \cdot x = x_0$ then we have that $\dim \operatorname{Stab}_U(x) \le \dim \operatorname{Stab}_U(x_0)$.*

Proof. To prove the above statement, we use the fact that $x_0 = \lim_{t\to 0} \lambda(t) \cdot x$ lies in the closure of the H-orbit $H \cdot x$. Recall, that by Proposition 2.3.8, the closure of an orbit consists of the orbit itself and orbits of strictly lower dimension. If x_0 lies in the H-orbit of x then there exists $(u,t) \in H$ such that $x_0 = (u,t)x$ and if $c_{(u,t)} : H \to H$ is the inner automorphism mapping $(v,s) \mapsto (u,t)(v,s)(u,t)^{-1}$

we get an isomorphism from $c_{(u,t)} : \mathrm{Stab}_H(x) \to \mathrm{Stab}_H(x_0)$ with inverse $c_{(u,t)^{-1}}$. Since U is a normal subgroup of H we have that for $(v,1) \in \mathrm{Stab}_U(x)$ it follows that $c_{(u,t)}\big((v,1)\big)$ lies in $\mathrm{Stab}_H(x_0)$ and $c_{(u,t)}\big((v,1)\big) \in U$ which implies that we get a restricted morphism

$$c_{(u,t)} : \mathrm{Stab}_U(x) \to \mathrm{Stab}_U(x_0)$$

with inverse $c_{(u,t)^{-1}}$ and hence $\dim \mathrm{Stab}_U(x) = \dim \mathrm{Stab}_U(x_0)$. Otherwise, the orbit of $H \cdot x_0$ has strictly lower dimension than the orbit of $H \cdot x$ and by the orbit-stabiliser formula (see [37, Proposition 3.20])

$$\dim H = \dim H \cdot y + \dim \mathrm{Stab}_H(y)$$

we obtain the inequality $\dim \mathrm{Stab}_H(x_0) \geq \dim \mathrm{Stab}_H(x) + 1$ but $x_0 = \lim_{t \to 0} t \cdot x$ is a $\mathbb{G}_\mathrm{m}$-fixed point and hence $\dim \mathrm{Stab}_U(x_0) = \dim \mathrm{Stab}_H(x_0) - 1 \geq \dim \mathrm{Stab}_H(x) \geq \dim \mathrm{Stab}_U(x)$. □

Corollary 3.2.4. *Consider a graded action of H on X with grading subgroup $U \rtimes T_g$. Then the following statements are equivalent*

1. $\forall x \in X^T : \mathrm{Stab}_U(x) = \{e\}$
2. $\forall x \in X : \mathrm{Stab}_U(x) = \{e\}$.

Proof. The implication 2. implies 1. is trivial so we will only prove that 1. implies 2. Let $x \in X$ then since $U \rtimes T_g$ is a grading subgroup for the H-action on X it follows that the T_g-orbit $T_g \cdot x$ contains a T_g-fixed point in its closure (see Proposition 3.0.8). By Kempf's theorem (Theorem 2.4.7) there exists a 1-PS $\lambda : \mathbb{G}_\mathrm{m} \to T_g$ such that $x_0 = \lim_{t \to 0} \lambda(t)x$ is a T_g-fixed point. Now apply Proposition 3.2.3 to obtain $0 \leq \dim \mathrm{Stab}_U(x) \leq \dim \mathrm{Stab}_U(x_0) = 0$. We conclude that $\mathrm{Stab}_U(x) = \{e\}$ since $\mathrm{char}\, \mathbf{k} = 0$. □

Proof of Lemma 3.2.1. By Example 3.1.3 an action of $\mathbb{G}_\mathrm{a} \rtimes T_g$ on an affine variety X corresponds to a grading of $\mathcal{O}(X)$ induced by the T_g-action on X together with a homogeneous locally nilpotent derivation $\delta : \mathcal{O}(X) \to \mathcal{O}(X)$. Since the $\mathbb{G}_\mathrm{a} \rtimes T_g$-action is graded, it follows that $\deg \delta \notin M$ where M is the grading monoid for T_g acting on X. To verify that δ admits a slice ($s \in \mathcal{O}(X)$ with $\delta(s) = 1$) it is enough to consider the restriction $\delta|_{\mathcal{O}(X)_{-\deg \delta}} : \mathcal{O}(X)_{-\deg \delta} \to \mathcal{O}(X)_0$. In the following, we want to establish that the restricted linear map $\delta|_{\mathcal{O}(X)_{-\deg \delta}}$ is surjective and hence that δ admits a slice. To do this consider the following T_g-graded subvector space

$$I := \delta(\mathcal{O}(X)_{-\deg\delta}) \oplus \bigoplus_{w\in M\setminus\{0\}} \mathcal{O}(X)_w$$

of $\mathcal{O}(X)$. We claim that I is an ideal in $\mathcal{O}(X)$ and the surjectivity of $\delta|_{\mathcal{O}(X)_{-\deg\delta}}$ translates to the fact that I is not a proper ideal. To show that I is an ideal in $\mathcal{O}(X)$ it is suffices to show for homogeneous elements $a \in \mathcal{O}(X)_k$ and $b \in I_l$ that $ab \in I_{k+l}$, since I is a T_g-graded $\mathbf{k}$-subvector space of $\mathcal{O}(X)$. The only non-trivial case is for $a \in I_0 = \delta(\mathcal{O}(X)_{-\deg\delta})$ and $b \in \mathcal{O}(X)_0$, but then $\delta(b) \in \mathcal{O}(X)_{\deg\delta}$ and $\deg\delta \notin M$ hence $\mathcal{O}(X)_{\deg\delta} = \{0\}$. By definition of I_0 there exists $c \in \mathcal{O}(X)_{-\deg\delta}$ such that $\delta(c) = a$ and

$$\delta(cb) = \delta(c)b + c\delta(b) = ab + c0 = ab$$

which implies $ab \in \delta(\mathcal{O}(X)_{-\deg\delta}) = I_0$. Now that we know that I is an ideal let us assume that $\delta : \mathcal{O}(X)_{-\deg\delta} \to \mathcal{O}(X)_0$ is not surjective which implies that I is a proper ideal of $\mathcal{O}(X)$. Then there exists a maximal ideal $\mathfrak{m}$ containing I. Since $\delta(\mathcal{O}(X)) \subset I \subset \mathfrak{m}$ it follows from the definition of $\mathrm{Fix}(\delta)$ that $\mathfrak{m} \in \mathrm{Fix}(\delta)$. By Proposition 3.1.5 we conclude that the $\mathbb{G}_a$-action α associated to δ has a fixed point which contradicts the assumption on trivial stabilisers (see Corollary 3.2.4). It follows that $\delta : \mathcal{O}(X)_{-\deg\delta} \to \mathcal{O}(X)_0$ is surjective. □

Remark 3.2.5. Consider the following algebraic $\mathbb{G}_a$-action on $\mathbb{A}^5$ given by

$$u \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 + ux_1 \\ x_3 \\ x_4 + ux_3 \\ x_5 + u(1 + x_1x_4^2) + u^2x_1x_3x_4 + \frac{u^3}{6}x_1x_3^2 \end{pmatrix}$$

The locally nilpotent derivation corresponding to the above $\mathbb{G}_a$-action on $\mathbb{A}^5$ is studied in [23] and it is shown that the action is not locally trivial. This gives us an example of an algebraic $\mathbb{G}_a$-action with trivial stabilisers which does not extend to a graded action since the action does not admit a locally trivial quotient.

3.3 On trivial U-quotients of affine varieties

The following result generalises [9, Proposition 4.3] to graded H-actions (see Definition 3.0.5). More precisely, we allow the grading of unipotent radical by a higher dimensional tori.

Lemma 3.3.1. *Let H be a linear algebraic group and X be a graded H-variety with a grading subgroup U_{T_g}. Suppose that each $x \in X^{T_g}$ has $\mathrm{Stab}_U(x) = \{e\}$, then $X \to X/U$ is a trivial principal U-bundle.*

The proof can be reduced to a successive construction of trivial principal $\mathbb{G}_\mathrm{a}$-bundles. Before we can proceed with the proof of Lemma 3.3.1, we will recall some more results.

Definition 3.3.2. Let H be a group and U a subgroup. We call U a *characteristic subgroup* of H, if every automorphism $\alpha : H \to H$ preserves the subgroup U.

Remark 3.3.3. Note that any characteristic subgroup U is also a normal subgroup, as by definition a normal subgroup is a subgroup that is preserved by all inner automorphisms. An inner automorphism is given by

$$\begin{aligned} c_h : H &\to H \\ g &\mapsto hgh^{-1} \end{aligned}$$

where h is some fixed element of H.

Definition 3.3.4. Let H be a linear algebraic group (over $\mathbf{k}$). The *derived subgroup* of H is the intersection of the normal algebraic subgroups N of H such that H/N is commutative. The derived subgroup of H is denoted DH.

Proposition 3.3.5. *[48, Proposition 8.20, Corollary 8.21, Corollary 8.30] The derived subgroup DH of H has the following properties:*

1. *DH is a characteristic subgroup and*

2. *DH is an algebraic subgroup.*

If furthermore H is solvable, then if H is connected (resp. smooth, resp. smooth and connected), then DH is connected (resp. smooth, resp. smooth and connected).

Proposition 3.3.6. *Let $U_T := U \rtimes T$ be a linear algebraic group where U is the unipotent radical and T is a torus. Then we can find a subnormal series of U given by*

$$\{e\} = U_0 \lhd U_1 \lhd \ldots \lhd U_n = U$$

such that the following statements hold.

1. *$U_{i+1}/U_i \cong \mathbb{G}_\mathrm{a}$ for each $i = 0, \ldots, n-1$,*
2. *each U_{i+1} is T-invariant and hence T also acts on U_{i+1}/U_i and*
3. *the weight of T acting on $\mathrm{Lie}(U_{i+1})/\mathrm{Lie}(U_i)$ is one of the weights of T acting on $\mathrm{Lie}(U_{i+1})$.*

Proof. Let us first consider the case where U is abelian. Then we obtain an algebraic isomorphism $\exp : U \to \mathrm{Lie}\, U$. Choose a basis of $\mathrm{Lie}\, U$ consisting of T weight vectors $\xi_1, \ldots, \xi_n \in \mathrm{Lie}\, U$ for the induced action of T on $\mathrm{Lie}(U)$. The subgroups generated by $\xi_1, \ldots \xi_k$ for $k = 1, \ldots n$ fulfil Properties 1, 2 and 3. For the general case, we use the derived series of U where DU is the derived subgroup of U and if the n-th derived group D^nU is already defined for $n \geq 1$, then the $(n+1)$-th derived group is defined by $D^{n+1}H := D(D^nH)$. The derived series for U is the normal series

$$U \rhd DU \rhd D^2U \rhd \ldots$$

Since U is unipotent and hence solvable, it follows that derived series as defined above terminates after finitely many steps with the trivial subgroup $\{e\}$. So let $k := \min\{n \in \mathbb{N} | D^nU = \{e\}\}$ then we obtain

$$\{e\} = D^kU \lhd D^{k-1}U \lhd \ldots \lhd DU \lhd U.$$

Each $D^iU \subsetneq U$ is a *characteristic subgroup* of the smooth group U by Proposition 3.3.5. Since D^iU is preserved by any automorphism of U, it follows that D^iU is preserved by each automorphism in the family $\Phi : T \to \mathrm{Aut}(U)$ associated to the semi-direct product structure. The restriction of Φ to $\mathrm{Aut}(D^iU)$ yields an inclusion of algebraic groups $D^iU \rtimes T \subset U \rtimes T$. Each of the quotients $D^iU/D^{i+1}U$ is an abelian group and since $D^{i+1}U \subsetneq D^iU$ is T-invariant we obtain a semi-direct product $D^iU/D^{i+1}U \rtimes T$. This reduces everything to the abelian case but we already know that we can find a T-invariant subnormal series for $D^iU/D^{i+1}U \rtimes T$ using the projection $\pi_i : D^iU \to D^iU/D^{i+1}U$ in combination with the T-invariant subnormal series $U_{i,0} \triangleleft U_{i,1} \triangleleft \ldots \triangleleft U_{i,n_i}$ for $D^iU/D^{i+1}U$. We consider

$$\pi_{i,j} : D^iU \to D^iU/D^{i+1}U \to (D^iU/D^{i+1}U)/U_{i,j}$$

and let $U_0 := \ker \pi_{k,0}, \ldots, U_{n_k} := \ker \pi_{k,n_k} = \ker \pi_{k-1,0},\ U_{n_k+1} := \ker \pi_{k-1,1}$ and so on. This yields the T-invariant subnormal series for U given by

$$\{e\} = U_0 \triangleleft \ldots \triangleleft U_n = U$$

where $n := n_k + \ldots + n_1$. □

To proceed with the induction step in the proof of Lemma 3.3.1, we use the following results.

Proposition 3.3.7. *[9, Lemma 1.20] Let U be a unipotent linear algebraic group with a normal subgroup N such that the projection $U \to U/N$ splits and let X be an affine H-variety. Suppose X has the structure of the total space of a principal N-bundle, and the quotient X/N is the total space of a principal U/N-bundle, for the canonical action of U/N on X. Then X is a principal U-bundle over X/U.*

Lemma 3.3.8. *[9, Lemma 1.21] Suppose H is a linear algebraic group, N is a normal subgroup of H and X is a H-variety. Suppose all the stabilisers for the restricted action of N on X are finite and this action has a geometric quotient $p : X \to X/N$. Note that H/N acts canonically on X/N. Then given $y \in X/N$, the stabiliser $\mathrm{Stab}_{H/N}(y)$ is finite if and only if $\mathrm{Stab}_H(x)$ is finite for some and hence all $x \in p^{-1}(y)$.*

Proof of Lemma 3.3.1. We proceed via induction on $n := \dim U$. For $n = 1$ we have by Lemma 3.2.1 that $X \to \operatorname{Spec} \mathcal{O}(X)^U$ is a trivial principal U-bundle. To proceed with an inductive proof we use the subnormal series

$$\{e\} = U_0 \triangleleft \ldots \triangleleft U_n = U$$

from Proposition 3.3.6. Suppose that for some $k \geq 1$ that $q_k : X \to \operatorname{Spec} \mathcal{O}(X)^{U_k} =: X'$ is a trivial principal U_k-bundle. We want to show that $q' : X' \to \operatorname{Spec} \mathcal{O}(X')^{U_{k+1}/U_k}$ is a trivial principal $\mathbb{G}_a$-bundle and that the composition $q_{k+1} := q' \circ q_k : X \to \operatorname{Spec} \mathcal{O}(X)^{U_{k+1}}$ is a trivial principal U_{k+1}-bundle. By Lemma 3.2.1 $q' : X' \to \operatorname{Spec} \mathcal{O}(X')^{U_{k+1}/U_k}$ is a trivial principal $\mathbb{G}_a$-bundle if the action $U_{k+1}/U_k \rtimes T_g$ is graded and U_{k+1}/U_k acts set-theoretically free on X'. Since X' is an affine variety it follows by Proposition 3.0.7 that the action of $U_{k+1}/U_k \rtimes T_g$ on X' is graded. To see that U_{k+1}/U_k acts set-theoretically free on X' let $y \in X'$ then there exists $x \in X$ such that $q_k(x) = y$. By Lemma 3.3.8 it follows that $\operatorname{Stab}_{U_{k+1}}(x)$ is trivial if and only if $\operatorname{Stab}_{U_{k+1}/U_k}(y)$ is trivial. By the assumptions of Lemma 3.3.1 it follows that $\{e\} = \operatorname{Stab}_U(x) \supset \operatorname{Stab}_{U_{k+1}}$ and we conclude that $\operatorname{Stab}_{U_{k+1}/U_k}(y)$ is trivial. Since all assumptions from Lemma 3.2.1 are satisfied we obtain that

$$q' : X' \to \operatorname{Spec} \mathcal{O}(X')^{U_{k+1}/U_k} = \operatorname{Spec} \mathcal{O}(X)^{U_{k+1}}$$

is a trivial principal $\mathbb{G}_a$-bundle. It remains to show that $q_{k+1} = q' \circ q_k : X \to \operatorname{Spec} \mathcal{O}(X)^{U_{k+1}}$ is a trivial principal U_{k+1}-bundle. This follows from Proposition 3.3.7 if $U_{k+1} \to U_{k+1}/U_k$ splits. Take $\Xi \in \operatorname{Lie}(U_{j+1}) \setminus \operatorname{Lie}(U_j)$ then the associated subgroup of U_{j+1} is isomorphic to $\mathbb{G}_a$ and splits the above projection map. It follows that the composition $q_{k+1} = q' \circ q_k : X \to \operatorname{Spec} \mathcal{O}(X)^{U_{k+1}}$ is a trivial principal U_{k+1}-bundle. □

Lemma 3.3.9. *Let H be a linear algebraic group and X be an affine H-variety. Suppose that U is the unipotent radical of H and $p : X \to X/U$ is a trivial principal U-bundle, then for T a torus of H there exists a T-equivariant section $s : X/U \to X$.*

To prove the above lemma we will use the following results.

Proposition 3.3.10. *Let U be a unipotent linear algebraic group and N be a closed normal subgroup of U. Then $q : U \to U/N$ is a trivial principal N-bundle.*

Proof. We have that $q : U \to U/N$ is a geometric N-quotient with U/N a linear algebraic group (see [62, p. 19]). By Proposition 2.3.11, a geometric N-quotient has the structure of a principal

N-bundle if and only if the N-action is free. By Definition 2.3.2 the action of N on U is free if

$$\Psi : N \times U \to U \times U$$

is a closed immersion. Using the notation from Chapter 2 we factorise $\Psi = (\sigma, \mathrm{id}) \circ (\mathrm{inc}, \mathrm{id})$ where

$$(\mathrm{inc}, \mathrm{id}) : N \times U \to U \times U$$

is a closed immersion and

$$(\sigma, \mathrm{id}) : U \times U \to U \times U$$

is an isomorphism. We conclude that Ψ is a closed immersion and hence $q : U \to U/N$ is a principal N-bundle. By Proposition 3.0.4 it follows that $q : U \to U/N$ is a trivial principal N-bundle since both U and U/N are affine. □

Corollary 3.3.11. *Let U be a unipotent linear algebraic group with a normal subgroup N. Suppose that $q : X \to X/U$ is a trivial principal U-bundle, then X has the structure of a trivial principal N-bundle.*

Proof. Since X is a trivial principal U-bundle there exists a U-equivariant isomorphism $X \to U \times X/U$. Then $X \to U \times X/U \to U/N \times X/U$ is a trivial principal N-bundle. □

Proof of Lemma 3.3.9. Let H be a linear algebraic group with $T \subset H$ a torus and unipotent radical U. By proceeding analogously to the proof of Propositon 3.3.6 we obtain a subnormal series for U

$$\{e\} = U_0 \trianglelefteq U_1 \trianglelefteq \ldots \trianglelefteq U_n = U$$

such that $U_i \rtimes T$ is a subgroup of $U \rtimes T$ and U_{i+1}/U_i is isomorphic to $\mathbb{G}_\mathrm{a}$. By applying Corollary 3.3.11 to $N = U_{n-1}$ we obtain that $X \to X/U_{n-1}$ is a trivial principal U_{n-1}-bundle. By iteration we obtain that $X \to X/U_i$ is a trivial principal U_i-bundle for each $1 \leq i \leq n$. Furthermore it follows that $X/U_i = \operatorname{Spec} \mathcal{O}(X)^{U_i} \to \operatorname{Spec} \left(\mathcal{O}(X)^{U_i}\right)^{U_{i+1}/U_i} \cong X/U_{i+1}$ is a trivial principal U_{i+1}/U_i-bundle.

To show that $X \to X/U$ admits a T-equivariant section we proceed by induction over $n = \dim U$. For $\dim U = 1$ we have that $U \cong \mathbb{G}_\mathrm{a}$ and the action of $\mathbb{G}_\mathrm{a} \rtimes T$ on X gives us a locally nilpotent derivation $\delta : \mathcal{O}(X) \to \mathcal{O}(X)$ which is homogenous with respect to the $\mathbf{X}^*(T)$-grading of $\mathcal{O}(X)$. Since $X \to X/\mathbb{G}_{\mathrm{a}}$ is a trivial principal U-bundle there exists a slice $\sigma \in \mathcal{O}(X)$. We write

$$\sigma = \sum_{\chi \in \mathbf{X}^*(T)} \sigma_\chi$$

with all but finitely many σ_χ equal to zero. Let $\chi_0 \in \mathbf{X}^*(T)$ be the degree of the homogeneous locally nilpotent derivation δ then σ_{χ_0} is also a slice for δ and $\mathcal{O}(X)$ is T-equivariantly isomorphic to $\mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}} \otimes_{\mathbf{k}} \mathbf{k}[\sigma_{\chi_0}]$. We conclude that the evaluation at $\sigma_{\chi_0} = 0$ gives a morphism of $\mathbf{k}$-algebras $\mathcal{O}(X) \to \mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}}$ which is T-equivariant. Here we consider an element of $\mathcal{O}(X)$ to be a polynomial in s with coefficients in $\mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}}$. By applying the functor Spec to the T-equivariant morphism of $\mathbf{k}$-algebras, we obtain a T-equivariant section

$$s : \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}} \to \operatorname{Spec} \mathcal{O}(X).$$

By the induction hypothesis, we suppose that $X \to X/U_i$ has a T-equivariant section $s_i : X/U_i \to X$ for some $i \geq 1$. Then we obtain a T-equivariant section $s' : (X/U_i)/(U_{i+1}/U_i) \to X/U_i$ since $\dim U_{i+1}/U_i = 1$. We conclude that $s_{i+1} := s_i \circ s'$ is a T-equivariant section of $X \to X/U_{i+1}$. To see this recall that $q_{i+1} : X \to X/U_{i+1}$ factorises via $q_i : X \to X/U_i$ composed with $q' : X/U_i \to (X/U_i)/(U_{i+1}/U_i)$ and $q_{i+1} \circ s_{i+1} = (q' \circ q_i) \circ (s_i \circ s') = q' \circ (q_i \circ s_i) \circ s' = q' \circ \mathrm{id} \circ s' = q' \circ s' = \mathrm{id}$. We conclude that $s_{i+1} : X/U_{i+1} \to X$ is a T-equivariant section of the trivial principal U_{i+1}-bundle $X \to X/U_{i+1}$ thus the trivial principal U-bundle $X \to X/U$ admits a T-equivariant section. $\square$

Chapter 4

Non-reductive geometric invariant theory for affine varieties

In the preprint [9] by Bérczi, Doran, Hawes and Kirwan, the notion of a graded unipotent group $\hat{U}$, was introduced (see Definition 2.5.3). For a linear algebraic group H, they consider the case where the unipotent radical U of H can be graded by a 1-PS of H. They obtain a theorem for suitably linearised H-actions on a projective variety under the assumption that U acts freely on the open stratum for the Białynicki-Birula-decomposition with respect to the flow to 0 for the 1-PS $\mathbb{G}_\mathrm{m}$ that grades U. By using a blow-up procedure they obtain several varations of the previous theorem which have weaker assumptions on the action of unipotent radical U (see Subsection 2.5.2).

In this chapter, we will consider an action of a non-reductive linear algebraic group H on an affine variety X. Our goal is to construct good or, even better, geometric quotients for open subsets of X. We distinguish between two approaches to construct these quotients:

1. The first notion is a classical approach to construct a quotient for a graded H-action on X with respect to a linearisation $\mathfrak{L}$ of the trivial line bundle $X \times \mathbb{A}^1$. To obtain a quotient we use certain H-invariant sections belonging to $\mathcal{O}(X)^H$.

2. The other notion is an approach of using quotienting in stages and a certain embedding inspired by [9].

Let X be an affine H-variety for a linear algebraic group H with unipotent radical U. We recall from Definition 3.0.5 that the action of H on X a *graded action* with respect to a grading subgroup $U_{T_g} = U \rtimes T_g \subset H$ with T_g a torus if the following conditions are fulfilled:

1. The grading monoid $M := \{w \in \mathbf{X}^*(T_g) | \mathcal{O}(X)_w \neq \{0\}\} \subset \mathbf{X}^*(T_g)$ is *positive;* that is, if $x \in M$ and $-x \in M$, then $x = 0$.
2. Additionally we assume for the linear T_g-action on $\mathrm{Lie}(U)$ induced by the semi-direct product structure of $U \rtimes T_g$ that $\{w \in \mathbf{X}^*(T_g) | \mathrm{Lie}(U)_w \neq \{0\}\} \cap M = \emptyset$.

We call X a *graded H-variety* and U_{T_g} a *grading subgroup* for the H-action on X, if the H-action on X restricted to U_{T_g} satisfies Conditions 1 and 2.

From Definition 3.0.5, we obtain Lemma 3.3.1: Suppose, we have a graded action with grading subgroup U_{T_g} on an affine variety X such that for each $x \in X^{T_g}$ we have that $\mathrm{Stab}_U(x) = \{e\}$. Then the morphism $X \to \operatorname{Spec} \mathcal{O}(X)^U$, corresponding to the inclusion $\mathcal{O}(X)^U \to \mathcal{O}(X)$, is a trivial principal U-bundle.

4.1 Classical approach to construct quotients

Given a linear algebraic group H and an affine H-variety X we lift the H-action on X to the trivial line bundle $X \times \mathbb{A}^1$ and denote the linearisation by $\mathfrak{L}$. Any linearisation obtained in this manner corresponds to the choice of a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ (see Example 2.3.18). From the associated $\mathbb{N}$-graded ring of sections

$$\mathrm{R}(X, \mathfrak{L}) := \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})$$

by taking the H-invariant subring, we obtain a rational map

$$q_{\mathfrak{L}} : X \dashrightarrow \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H.$$

Recall that the maximal domain of definition of $q_{\mathfrak{L}}$ is called the naively semistable locus and denoted by $X^{nss(H,\mathfrak{L})}$ (see Definition 2.5.1). If the linearisation $\mathfrak{L}$ corresponds to the trivial character then

$$\operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H \cong \operatorname{Spec} \mathcal{O}(X)^H$$

is an isomorphism and all points are naively semistable.

Definition 4.1.1. Let H be a linear algebraic group and X be an affine H-variety. Let $\mathfrak{L}$ be the linearisation corresponding to the character $\chi : H \to \mathbb{G}_{\mathrm{m}}$. We define the *intrinsic semistable locus* to be

$$X^{iss(H,\chi)} := \bigcup_{f \in I^{iss}} X_f$$

where $f \in I^{iss}$ if $f \in \bigcup_{r \geq 1} H^0(X, \mathfrak{L}^{\otimes r})^H$ and $X_f \to \operatorname{Spec} \mathcal{O}(X_f)^H$ is a good quotient.

Let $T \subset H$ be a maximal torus then the *Hilbert-Mumford (semi)stable locus* with respect to the linearised action $H \curvearrowright \mathfrak{L} \to X$ is given by

$$X^{HM-(s)s(H,\chi)} := \bigcap_{h \in H} hX^{(s)s(T,\chi)}.$$

If additionally

$$X^{iss(H,\chi)} = X^{HM-ss(H,\chi)}$$

then we say that the linearised action $H \curvearrowright \mathfrak{L} \to X$ admits a *Hilbert-Mumford description of the semistable set*.

Remark 4.1.2. It follows from Proposition 2.3.10 that $X^{iss(H,\chi)}$ admits a good H-quotient. Note that the Hilbert-Mumford (semi)stable locus does not depend on the choice of the maximal torus $T \subset H$.

Proposition 4.1.3. *We obtain the following chain of inclusions*

$$X^{iss(H,\chi)} \subseteq X^{nss(H,\chi)} \subseteq X^{HM-ss(H,\chi)}.$$

Proof. If $x \in X^{iss(H,\chi)}$ then there exists $n \geq 1$ and $f \in \mathcal{O}(X)^H_{\chi^n}$ with $f(x) \neq 0$ and $X_f \to \operatorname{Spec} \mathcal{O}(X)^H_f$ a good quotient. By the definition of the naively semistable locus, it follows that $x \in X_f \subset X^{nss(H,\chi)}$. For the second inclusion, let $y \in X^{nss(H,\chi)}$ then there exists $m \geq 1$ and $g \in \mathcal{O}(X)^H_{\chi^m}$ with $g(y) \neq 0$. Let $T \subset H$ be a maximal torus of H. The inclusion $\mathcal{O}(X)^H_{\chi^m} \subset \mathcal{O}(X)^T_{\chi^m}$ yields that $X_g \subset X^{ss(T,\chi)}$. Additionally X_g is H-invariant since $g \in \mathcal{O}(X)^H_{\chi^m}$ thus $X_g \subset X^{HM-ss(H,\chi)}$. $\square$

Proposition 4.1.4. *Let H be a linear algebraic group with unipotent radical U and X be an irreducible* *affine H-variety. Let $\mathfrak{L}$ be a linearisation corresponding to a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$. Given $r \geq 1$ and $f \in H^0(X, \mathfrak{L}^{\otimes r})^H$ with $X_f \to \operatorname{Spec} \mathcal{O}(X_f)^U$ a trivial principal U-bundle; then $X_f \to \operatorname{Spec} \mathcal{O}(X_f)^H$ is a good quotient.*

Proof. If $X_f \to \operatorname{Spec} \mathcal{O}(X_f)^U$ is a trivial principal U-bundle, then $\mathcal{O}(X_f)^U$ is finitely generated since $\mathcal{O}(X_f) \cong \mathcal{O}(X_f)^U \otimes_{\mathbf{k}} \mathcal{O}(U)$ and the finite generation of $\mathcal{O}(X_f)$ and $\mathcal{O}(U)$ implies the finite generation of $\mathcal{O}(X_f)^U$. For the induced action of the reductive group H/U on the affine variety $\operatorname{Spec} \mathcal{O}(X_f)^U$, we obtain a good H/U-quotient given by the affine GIT quotient $\operatorname{Spec} \mathcal{O}(X_f)^U \to \operatorname{Spec}(\mathcal{O}(X_f)^U)^{H/U}$. Since X is irreducible, we have $\mathcal{O}(X_f)^H \cong \left(\mathcal{O}(X_f)^U\right)^{H/U}$. We conclude that $X_f \to \operatorname{Spec} \mathcal{O}(X_f)^H$ is a good H-quotient since it is a composition of a geometric U-quotient with a good H/U-quotient. □

Let $q : Z \to Y$ be a good H-quotient. Recall from Definition 2.3.12 that we defined the open subset $Z^0 \subset Z$, consisting of closed H-orbits of maximal dimension, such that $q|_{Z^0} : Z^0 \to q(X^0)$ is a geometric H-quotient with $q(Z^0)$ open in Y (see Proposition 2.3.13).

Notation 4.1.5. In the following lemma and corollary, we set $Z := X^{iss(H,\chi)}$ and denote the subset $Z^0 = (X^{iss(H,\chi)})^0$ by $X^{\overline{s}(H,\chi)}$.

Lemma 4.1.6. *Let H be a linear algebraic group and let X be a graded H-variety with respect to the grading subgroup U_{T_g} with T_g a torus. Let $\mathfrak{L}$ be a linearisation corresponding to a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ with $T_g \subset \ker(\chi)$. Then for the linearised action $H \curvearrowright \mathfrak{L} \to X$ the following statements hold.*

1. *The ring of semi-invariant sections $\mathrm{R}(X, \mathfrak{L})^H$ is determined by the restricted action of the Levi-factor R; that is, $\mathrm{R}(X, \mathfrak{L})^H = \mathrm{R}(X, \mathfrak{L})^R$. It follows that $\mathrm{R}(X, \mathfrak{L})^H$ is finitely generated.*

2. *(Hilbert-Mumford criterion) The intrinsic semistable locus is the Hilbert-Mumford semistable locus, that is $X^{iss(H,\chi)} = X^{HM-ss(H,\chi)} = X^{ss(R,\chi)}$ and*

$$q_{\mathfrak{L}} : X^{HM-ss(H,\chi)} \to X /\!\!/_{\mathfrak{L}} H := \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$$

is a good quotient and moreover $X /\!\!/_{\mathfrak{L}} H$ is a projective over affine variety.

3. *If we restrict the morphism* $q_{\mathfrak{L}} : X^{HM-ss(H,\chi)} \to X /\!\!/_{\mathfrak{L}} H$ *to the open subset* $X^{\overline{s}(H,\chi)} \subset X^{HM-ss(H,\chi)}$*, then we obtain a geometric quotient*

$$q_{\mathfrak{L}} : X^{\overline{s}(H,\chi)} \to q(X^{\overline{s}(H,\chi)}) \subset X /\!\!/_{\mathfrak{L}} H$$

with $q_{\mathfrak{L}}(X^{\overline{s}(H,\chi)})$ *open in* $X /\!\!/_{\mathfrak{L}} H$.

Proof. It is enough to show that $H^0(X, \mathfrak{L}^{\otimes n})^R \subseteq H^0(X, \mathfrak{L}^{\otimes n})^H$. Let $f \in H^0(X, \mathfrak{L}^{\otimes n})^R$ be a χ^n-semi-invariant (that is, for all $r \in R$ we have $r \cdot f = \chi^n(r)f$); in particular, since $T_g \subset \ker\chi \subset \ker\chi^n$, it follows that f is T_g-invariant. We claim that f is also U-invariant (and thus a H-semi-invariant). To see this consider the grading subgroup $U \rtimes T_g$. By Proposition 3.3.6, there exists a subnormal series of U

$$\{e\} = U_0 \lhd U_1 \lhd \ldots \lhd U_m = U$$

such that $U_{i+1}/U_i \cong \mathbb{G}_\mathrm{a}$ and for each U_i we obtain a restricted action $U_i \rtimes T_g$ on X which is graded. By quotienting in stages, we obtain the ring of U_{i+1}-invariants via

$$\mathcal{O}(X)^{U_{i+1}} = (\mathcal{O}(X)^{U_i})^{U_{i+1}/U_i} = (\mathcal{O}(X)^{U_i})^{\mathbb{G}_\mathrm{a}}.$$

By induction over $\dim U$, we will show that $\mathcal{O}(X)^{T_g} = \mathcal{O}(X)^{U \rtimes T_g} \subset \mathcal{O}(X)^U$. The T_g-action on $\mathcal{O}(X)$ defines a $\mathbf{X}^*(T_g)$-grading of $\mathcal{O}(X)$. For $\dim U = 1$, we have a $\mathbb{G}_\mathrm{a} \rtimes T_g$ action. The $\mathbb{G}_\mathrm{a}$-action on X corresponds to a locally nilpotent derivation $\delta : \mathcal{O}(X) \to \mathcal{O}(X)$ that is homogeneous with respect to the grading induced by T_g. The ring of T_g-invariants is given by the degree zero part of the grading on $\mathcal{O}(X)$ and the ring of $\mathbb{G}_\mathrm{a}$-invariants is the kernel of the associated locally nilpotent derivation δ. The action of $\mathbb{G}_\mathrm{a} \rtimes T_g$ is graded and hence the weight of δ is not contained in the grading monoid M. Therefore

$$\delta : \mathcal{O}(X)_0 \to \mathcal{O}(X)_{\deg\delta}$$

is the zero map since $\deg\delta \notin M$ implies $\mathcal{O}(X)_{\deg\delta} = \{0\}$. Thus $\mathcal{O}(X)^{T_g} = \mathcal{O}(X)_0 = \mathcal{O}(X)_0^{\mathbb{G}_\mathrm{a}} \subset \mathcal{O}(X)^{\mathbb{G}_\mathrm{a}}$ as required. The ring of U_i-invariants $\mathcal{O}(X)^{U_i}$ inherits the $\mathbf{X}^*(T_g)$-grading from $\mathcal{O}(X)$ and by the induction hypothesis we assume that $\mathcal{O}(X)_0 = \mathcal{O}(X)_0^{U_i}$. Now consider the homogeneous locally nilpotent derivation $\delta_{i+1} : \mathcal{O}(X)^{U_i} \to \mathcal{O}(X)^{U_i}$ correspond to the action of $\mathbb{G}_\mathrm{a} = U_{i+1}/U_i$ on

$\mathcal{O}(X)^{U_i}$. The restriction

$$\delta_{i+1} : \mathcal{O}(X)_0^{U_i} \to \mathcal{O}(X)^{U_i}_{\deg \delta_{i+1}}$$

is the zero map since by Proposition 3.0.7 the action of $U_{i+1}/U_i \rtimes T_g$ satisfies Conditions 1 and 2 of a graded action. We conclude that $\mathcal{O}(X)^{T_g} = \mathcal{O}(X)_0 = \mathcal{O}(X)_0^U = \mathcal{O}(X)^{U \rtimes T_g} \subset \mathcal{O}(X)^U$ and obtain

$$\mathrm{R}(X, \mathfrak{L})^H = \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H = \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^R = \mathrm{R}(X, \mathfrak{L})^R.$$

We conclude that $X /\!\!/_{\mathfrak{L}} H = X /\!\!/_{\mathfrak{L}} R$. The remaining statements now follow from reductive Geometric Invariant Theory for affine varieties (see Theorem 2.7.4) and Proposition 2.3.13. □

Corollary 4.1.7. *Let X be a graded H-variety. Then*

1. *The morphism $q : X \to \operatorname{Spec} \mathcal{O}(X)^H$ is a good quotient.*

2. *The ring of invariants $\mathcal{O}(X)^H$ is finitely generated and $\operatorname{Spec} \mathcal{O}(X)^H$ is an affine variety.*

3. *For $x, y \in X$ we have $q(x) = q(y)$ if and only if $\overline{H \cdot x} \cap \overline{H \cdot y} \neq \emptyset$. Consequently, each H-orbit in X contains a unique closed orbit.*

4. *The open subset $X^{\overline{s}(H,0)}$ admits a geometric quotient $q : X^{\overline{s}(H,0)} \to q(X^{\overline{s}(H,0)}) \subset \operatorname{Spec} \mathcal{O}(X)^H$ whose image $q(X^{\overline{s}(H,0)})$ is open in $\operatorname{Spec} \mathcal{O}(X)^H$.*

Proof. Properties 1, 2 and 4 are an immediate consequence of Lemma 4.1.6 and Theorem 2.7.1, where one takes χ to be trivial.

For convenience of the reader we give an elementary proof of property 3: Since q is a good H-quotient, if the closures of the H-orbit of x and y are disjoint then $q(\overline{H \cdot x})$ and $q(\overline{H \cdot x})$ are disjoint which implies that $q(x) \neq q(y)$. Conversely, suppose that $q(x) \neq q(y)$ then $q^{-1}(q(x)) \cap q^{-1}(q(y)) = \emptyset$. For any $z \in X$ we have that $q^{-1}(q(z))$ is closed and by H-invariance of q, it follows that $\overline{H \cdot z} \subset q^{-1}(q(z))$. We conclude that $\overline{H \cdot x} \cap \overline{H \cdot y} \subset q^{-1}(q(x)) \cap q^{-1}(q(y)) = \emptyset$. □

In the following result, we consider a graded H-action on an affine variety X. We use certain H-invariant sections to remove the restriction on the choice of character $\chi : H \to \mathbb{G}_{\mathrm{m}}$.

Definition 4.1.8. Let H be a linear algebraic group with a Levi decomposition $H \cong U \rtimes R$. Suppose that X is an affine H-variety.

1. We define the *strongly stable locus of the unipotent radical subordinate to* H to be the open subset
$$X^{ssu(H,U)} := \bigcup_{\sigma \in I^{ssu(H,U)}} X_\sigma$$
where
$$I^{ssu(H,U)} := \{\sigma \in \mathcal{O}(X)^H \text{ such that } X_\sigma \to \operatorname{Spec} \mathcal{O}(X)^U_\sigma \text{ is a principal } U\text{-bundle}\}.$$

2. For a character $\chi : H \to \mathbb{G}_\mathrm{m}$ and a maximal torus $T \subset H$ we define the local (semi)stable locus for the H-action linearised by χ to be
$$X^{l(s)s(H,\chi)} := X^{ssu(H,U)} \cap X^{HM-(s)s(H,\chi)} = X^{ssu(H,U)} \cap \bigcap_{h \in H} h \cdot X^{(s)s(T,\chi)}.$$

Theorem 4.1.9. *Let H be a linear algebraic group with a Levi decomposition $H \cong U \rtimes R$ and X be a graded H-variety. We lift the H-action on X to the trivial line bundle $X \times \mathbb{A}^1$ via a character $\chi : H \to \mathbb{G}_\mathrm{m}$ and denote the associated linearisation by $\mathfrak{L}$. Then*

1. *The open subset $X^{lss(H,\chi)}$ admits a good quotient via restricting*
$$q_\mathfrak{L} : X^{nss(H,\mathfrak{L})} \to X /\!\!/_\mathfrak{L} H = \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$$
to $X^{lss(H,\chi)}$. The image of $X^{lss(H,\chi)}$ under $q_\mathfrak{L}$ is open in $X /\!\!/_\mathfrak{L} H$ and $q_\mathfrak{L}(X^{lss(H,\chi)})$ is a variety.

2. *The restriction $X^{ls(H,\chi)} \to X^{ls(H,\chi)}/H$ is a geometric H-quotient that is open in $X^{lss(H,\chi)} /\!\!/ H$.*

3. *If U acts set-theoretically free on X, then the loci $X^{lss(H,\chi)}$ and $X^{ls(H,\chi)}$ admit the following Hilbert-Mumford criterion:*
$$X^{l(s)s(H,\mathfrak{L})} = \bigcap_{h \in H} h \cdot X^{(s)s(T,\mathfrak{L})}$$
where $T \subset H$ is a maximal torus.

Corollary 4.1.10. *Let $\sigma_1, \ldots, \sigma_n \in I^{ssu(H,U)}$ then we obtain a good quotient for $X^{HM-ss(H,\chi)} \cap \bigcup_{i=1}^{n} X_{\sigma_i}$ and a geometric quotient for $X^{HM-s(H,\chi)} \cap \bigcup_{i=1}^{n} X_{\sigma_i}$.*

Before we proceed with the proof of the above theorem we will state and prove the following Propositions.

Proposition 4.1.11. *Let H be a linear algebraic group with unipotent radical U and X be an affine variety with a graded H-action. Then $I^{ssu(H,U)} \cup \{0\}$ is an ideal of $\mathcal{O}(X)^H$ and $I^{ssu(H,U)} = \big\{\sigma \in \mathcal{O}(X)^H \setminus \{0\} | \forall x \in X_\sigma : \mathrm{Stab}_U(x) = \{e\}\big\}$.*

Proof. Given $\sigma \in I^{ssu(H,U)}$ and $\tau \in \mathcal{O}(X)^H \setminus \{0\}$ we have to show that $\tau\sigma \in I^{ssu(H,U)}$. If $\sigma = 0$ then there is nothing to show so we assume that $\sigma \neq 0$. By definition of $I^{ssu(H,U)}$ we have that $X_\sigma \to \operatorname{Spec} \mathcal{O}(X)^U_\sigma$ is a principal U-bundle. Since $X_{\tau\sigma}$ is open in X_σ and U-invariant, the restriction of $X_\sigma \to \operatorname{Spec} \mathcal{O}(X)^U_\sigma$ to $X_{\tau\sigma} \to \operatorname{Spec} \mathcal{O}(X)^U_{\sigma\tau}$ is a trivial principal U-bundle. We conclude $\tau\sigma \in I^{ssu(H,U)}$. It remains to show that for $\sigma_1, \sigma_2 \in I^{ssu(H,U)}$ we have $\sigma_1 + \sigma_2 \in I^{ssu(H,U)}$. If $\sigma_1 + \sigma_2 = 0$ there is nothing to show. So let $\sigma_1 + \sigma_2 \neq 0$ then $X_{\sigma_1+\sigma_2} \subset X_{\sigma_1} \cup X_{\sigma_2}$. By definition of $I^{ssu(H,U)}$ we have that $X_{\sigma_i} \to \operatorname{Spec} \mathcal{O}(X)^U_{\sigma_i}$ is a trivial principal U-bundle and so we conclude that each $x \in X_{\sigma_1+\sigma_2}$ has trivial U-stabiliser. Since $\sigma_1 + \sigma_2$ is a H-invariant function it has degree zero for the torus T_g of the grading subgroup $U \rtimes T_g \subset H$ for the H-action on X. It follows that the H-action on $X_{\sigma_1+\sigma_2}$ is graded with trivial U-stabilisers and we conclude that $X_{\sigma_1+\sigma_2} \to \operatorname{Spec} \mathcal{O}(X_{\sigma_1+\sigma_2})^U$ is a trivial principal U-bundle by Lemma 3.3.1. From the equality $\mathcal{O}(X_{\sigma_1+\sigma_2})^U = \mathcal{O}(X)^U_{\sigma_1+\sigma_2}$ we conclude $\sigma_1 + \sigma_2 \in I^{ssu(H,U)}$.

It is obvious that $I^{ssu(H,U)} \subseteq \big\{\sigma \in \mathcal{O}(X)^H \setminus \{0\} | \forall x \in X_\sigma : \mathrm{Stab}_U(x) = \{e\}\big\}$. For the converse inclusion we use the fact that the H-action on the affine open subsets X_σ is graded, because σ is H-invariant and the H-action on X is graded. By assumption each $x \in X_{\sigma_i}$ has trivial U-stabiliser thus Lemma 3.3.1 applies and we obtain that $X_{\sigma_i} \to X_{\sigma_i}/U$ is a trivial principal U-bundle. We conclude that $I^{ssu(H,U)} = \big\{\sigma \in \mathcal{O}(X)^H \setminus \{0\} | \forall x \in X_\sigma : \mathrm{Stab}_U(x) = \{e\}\big\}$. □

Proposition 4.1.12. *Let H be a linear algebraic group with a Levi-decomposition $H \cong U \rtimes R$ and let X be an affine H-variety. Further suppose that $p : X \to X/U$ is a trivial principal U-bundle. If we* *linearise the trivial line bundles over X and X/U with respect to a character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ then*

$$X^{HM-ss(H,\chi)} = p^{-1}\big((X/U)^{ss(H/U,\chi)}\big) \;\; \textit{and} \;\; X^{HM-s(H,\chi)} = p^{-1}\big((X/U)^{s(H/U,\chi)}\big).$$

Proof. By Lemma 3.3.9 there exists for each torus T of H a T-equivariant section $s_T : X/U \to X$. Recall for a reductive group G and an affine G-variety Y that (semi)stability with respect to a character $\chi : G \to \mathbb{G}_{\mathrm{m}}$ can be characterised by

$$Y^{(s)s(G,\chi)} = \bigcap_{g\in G} gY^{(s)s(T',\chi)} = \bigcap_{T \text{a maximal torus of } G} Y^{(s)s(T,\chi)}$$

where T' is any fixed maximal torus of G. We conclude that $x \in (X/U)^{ss(H/U,\chi)}$ if and only if for each maximal torus T of H/U we have that $x \in (X/U)^{ss(T,\chi)}$. To obtain the equality

$$\bigcap_{h\in H} hX^{(s)s(T,\chi)} = p^{-1}\big((X/U)^{(s)s(H/U,\chi)}\big)$$

it suffices to show for $x \in X$ and a fixed maximal torus T that $z := p(x)$ is (semi)stable with respect to T if and only if each $x' \in U \cdot x$ is (semi)stable with respect to T. Let $\lambda : \mathbb{G}_{\mathrm{m}} \to T$ be a 1-PS such that $\lim_{t\to 0} \lambda(t) \cdot x$ exists. Then by T-equivariance of p it follows that also $\lim_{t\to 0} \lambda(t) \cdot z$ exists. Conversely, let $\lambda : \mathbb{G}_{\mathrm{m}} \to T$ be a 1-PS such that $\lim_{t\to 0} \lambda(t) \cdot z$ exists then by T-equivariance of $s_T : X/U \to X$ it follows that $\lim_{t\to 0} \lambda(t) \cdot x'$ exists where $x' := s_T(x)$ with $x' \in U \cdot x$. In both cases we can determine the Hilbert-Mumford weight from the pairing $\langle \chi, \lambda \rangle$. We conclude that $p(x) \in X/U$ is semistable (respectively stable) with respect to T if and only if each $x' \in U \cdot x$ is semistable (respectively stable) with respect to T. $\square$

Proposition 4.1.13. *Let $A = \bigoplus_{d\geq 0} A_d$ be an $\mathbb{N}$-graded algebra and $\sigma \in A_0$ be a homogeneous non-zero divisor of degree zero. We let $B = A_\sigma$ be the localisation of A at σ. Then $B = \bigoplus_{d\geq 0} B_d$ is an $\mathbb{N}$-graded algebra with a natural morphism of graded algebras $\iota : A \to B$. Moreover the induced rational morphism $\operatorname{Proj} B \dashrightarrow \operatorname{Proj} A$ is an open immersion with image $\{\mathfrak{p} \in \operatorname{Proj} A | \sigma \notin \mathfrak{p}\}$.*

Proof. We assume that the reader is familiar with the Proj construction and any facts that we use in the following proof can be found in [32, pp. 366-373]. The domain of definition of

$\operatorname{Proj} B \dashrightarrow \operatorname{Proj} A$ is given by

$$D := \{\mathfrak{p} \in \operatorname{Proj} B | \iota(A_+) \not\subset \mathfrak{p}\}.$$

To see that $D = \operatorname{Proj} B$ we will show that the ideal generated by $\iota(A_+)$ is B_+. Since ι is a homomorphism of $\mathbb{N}$-graded algebras we have that $\iota(A_d) \subset B_d$ and hence $\iota(A_+) \subset B_+$. Conversely, let $f \in B_d$ be a homogeneous element for some $d \geq 1$. For a suitable large $n \in \mathbb{N}$ it follows that $\sigma^n f = \frac{h}{1}$ for some $h \in A_d$. We conclude that $\frac{h}{1} \in \langle \iota(A_+) \rangle$ but then also $\frac{h}{\sigma^n} \in \langle \iota(A_+) \rangle$. By [32, Proposition 2.12] the image $\operatorname{Proj} B \to \operatorname{Proj} A$ is given by

$$\{\mathfrak{p} \in \operatorname{Proj} A | \sigma \notin \mathfrak{p}\}.$$

Let $(f_i)_{i \in I}$ be a family of homogeneous elements in A that generates the irrelevant ideal A_+. Then $\{\mathfrak{p} \in \operatorname{Proj} A | \sigma \notin \mathfrak{p}\} = \bigcup_{i \in I} D_+(\sigma f_i)$ and we have that $\langle \iota(\sigma f_i) | i \in I \rangle = \langle \iota(f_i) | i \in I \rangle = B_+$. Since an open immersion is local on the target, it is enough to show for each $i \in I$ that

$$D_+(\iota(\sigma f_i)) \to D_+(\sigma f_i)$$

is an open immersion. This restricted morphism is given by

$$D_+(\iota(\sigma f_i)) \cong \operatorname{Spec} B_{\left(\iota(\sigma f_i)\right)} \to \operatorname{Spec} A_{(\sigma f_i)} \cong D_+(\sigma f_i)$$

which corresponds to the isomorphism

$$A_{(\sigma f_i)} \to B_{\left(\iota(\sigma f_i)\right)}.$$

We conclude that $\operatorname{Proj} A_\sigma = \operatorname{Proj} B \to \operatorname{Proj} A$ is an open immersion. □

Proof of Theorem 4.1.9. By Proposition 4.1.11 we have that $I^{ssu(H,U)}$ is an ideal in $\mathcal{O}(X)^H$. By Corollary 4.1.7 it follows that $\mathcal{O}(X)^H$ is a finitely generated $\mathbf{k}$-algebra thus the ideal $I^{ssu(H,U)}$ is finitely generated. Let $\sigma_1, \ldots, \sigma_r$ generate the ideal $I^{ssu(H,U)}$ then $X^{ssu(H,U)} = \bigcup_{i=1}^r X_{\sigma_i}$.

We lift the action of $R \cong H/U$ on X_{σ_i}/U to the trivial line bundle $(X_{\sigma_i}/U) \times \mathbb{A}^1$ via the character $\chi : H/U \to \mathbb{G}_{\mathrm{m}}$ and denote the linearisation by $\mathfrak{L}$. Note that characters of H and of H/U are in bijective correspondence since each character of H contains the unipotent radical of H in its kernel. By reductive geometric invariant theory, we obtain a rational morphism $q_i : X_{\sigma_i}/U \dashrightarrow (X_{\sigma_i}/U)/\!\!/_{\mathfrak{L}} (H/U) = \operatorname{Proj} \mathrm{R}(X_{\sigma_i}/U, \mathfrak{L})^{H/U}$ with $q_i : (X_{\sigma_i}/U)^{ss(H/U,\chi)} \to (X_{\sigma_i}/U)/\!\!/_{\mathfrak{L}} (H/U)$ a good quotient. If we take the preimage of $\left(X_{\sigma_i}/U\right)^{ss(H/U,\chi)}$ under the trivial U-quotient $q_U : X_{\sigma_i} \to X_{\sigma_i}/U$ we obtain an open subset of X_{σ_i} that has a good quotient via composition with the good quotient for H/U.

It remains to show that $q_U^{-1}(X_{\sigma_i}^{(s)s(H/U,\chi)}) = X_{\sigma_i} \cap X^{HM-(s)s(H,\chi)}$. By Proposition 4.1.12 we have that

$$q_U^{-1}(X_{\sigma_i}^{(s)s(H/U,\chi)}) = \bigcap_{h \in H} hX_{\sigma_i}^{(s)s(T,\chi)} = X_{\sigma_i}^{HM-(s)s(H,\chi)}.$$

To see that $X_{\sigma_i}^{HM-(s)s(H,\chi)} = X_{\sigma_i} \cap X^{HM-(s)s(H,\chi)}$ we will show that $x \in X_{\sigma_i}$ is T-(semi)stable as a point in X if and only if x is T-(semi)stable as a point in X_{σ_i}. The inclusion $X_{\sigma_i}^{(s)s(T,\chi)} \subset X^{(s)s(T,\chi)}$ is trivial. For the converse inclusion we claim that for a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to T$ and $x \in X_{\sigma_i}$ such that the limit $\lim_{t \to 0} \lambda(t) \cdot x$ exists in X then $\lim_{t \to 0} \lambda(t) \cdot x$ belongs to X_{σ_i}. By definition of the limit the morphism $\mathbb{G}_{\mathrm{m}} \to X$ given by $t \mapsto \lambda(t) \cdot x$ extends to a continuous map $\mathbb{A}^1 \to X$. The composition $\mathbb{A}^1 \to X \xrightarrow{\sigma_i} \mathbb{A}^1$ is constant on $\mathbb{G}_{\mathrm{m}} \subset \mathbb{A}^1$ by the H-invariance of σ_i. It follows by continuity of σ_i that $\sigma_i(\lim_{t \to 0} \lambda(t) \cdot x) = \sigma_i(x) \neq 0$. We conclude that $X_{\sigma_i} \cap X^{HM-(s)s(H,\chi)} = X_{\sigma_i}^{HM-(s)s(H,\chi)}$.

By the following chain of isomorphisms

$$\mathrm{R}(X, \mathfrak{L})_{\sigma_i}^H = \mathrm{R}(X_{\sigma_i}, \mathfrak{L})^H = \left(\mathrm{R}(X_{\sigma_i}, \mathfrak{L}))^U\right)^{H/U} = \mathrm{R}(X_{\sigma_i}/U, \mathfrak{L})^{H/U}$$

we obtain an isomorphism $\operatorname{Proj} \mathrm{R}(X_{\sigma_i}/U, \mathfrak{L})^{H/U} \cong \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})_{\sigma_i}^H$. By Proposition 4.1.13, the morphism $\operatorname{Proj} R(X, \mathfrak{L})_{\sigma_i}^H \to \operatorname{Proj} \mathrm{R}(X, \mathfrak{L}^{\otimes r})^H$ is an open immersion. By patching together these local quotients on the overlaps we obtain a good quotient for

$$X^{lss(H,\chi)} \to Y := \bigcup_{i=1}^{r} \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})_{\sigma_i}^H \subset \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$$

and a geometric quotient for $X^{ls(H,\chi)} \to Z \subset Y$. The quotient $X^{lss(H,\chi)} \to Y$ is obtained by gluing finitely many varieties and by [47, Proposition 4.13] it is a prevariety. It remains to show that Y is separated. The open immersion from $Y \to \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$ is separated. Additionally $\operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H \to \operatorname{Spec} \mathcal{O}(X)^H$ is separated. By Corollary 4.1.7, $\operatorname{Spec} \mathcal{O}(X)^H$ is a variety, thus $\operatorname{Spec} \mathcal{O}(X)^H \to \operatorname{Spec} \mathbf{k}$ is separated. It follows that the morphism from $Y \to \operatorname{Spec} \mathbf{k}$ is separated, since a composition of separated morphisms is separated. We conclude that Y is a variety.

To show that the third statement of Theorem 4.1.9 holds, note that the H-invariant section $\sigma = 1$ belongs to $I^{ssu(H,U)}$. □

Proof of Corollary 4.1.10. Since $\sigma_i \in I^{ssu(H,U)}$, it follows $X_{\sigma_i} \to X_{\sigma_i}/U$ is a trivial principal U-bundle and if $x \in X_{\sigma_i}$ admits a limit under a 1-PS λ, then the limit belongs to X_{σ_i}. We conclude that as in the proof of Theorem 4.1.9 that $X^{HM-ss(H,\chi)} \cap \bigcup_{i=1}^{n} X_{\sigma_i}$ admits a good quotient and $X^{HM-s(H,\chi)} \cap \bigcup_{i=1}^{r} X_{\sigma_i}$ admits a geometric quotient. □

Remark 4.1.14. If the H-action on X is not graded, then $\mathcal{O}(X)^H$ may not be finitely generated. Furthermore we did use the assumption that the H-action on X is graded to show that $I^{ssu(H,U)} \cup \{0\}$ is an ideal. Therefore the covering $X^{ssu(H,U)} = \bigcup_{\sigma \in I^{ssu}} X_\sigma$ need not be finite. We obtain the weaker result that $q_{\mathfrak{L}} : X^{lss(H,\chi)} \to \bigcup_{\sigma \in I^{ssu(H,U)}} \operatorname{Proj} \bigoplus_{r \geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H_\sigma$ is a good quotient with $\bigcup_{\sigma \in I^{ssu(H,U)}} \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H_\sigma$ an open subscheme of $\operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H$. We conclude that $\bigcup_{\sigma \in I^{ssu(H,U)}} \operatorname{Proj} \mathrm{R}(X, \mathfrak{L})^H_\sigma$ is a reduced, separated scheme locally of finite type over $\mathbf{k}$.

"

4.2 Using quotienting in stages and a certain embedding to construct quotients

The main result of this section is Theorem 4.2.5 which is the affine analogue to the projective $\hat{U}$-Theorem by Bérczi, Doran, Hawes and Kirwan (see Theorem 2.5.8).

Definition 4.2.1. Let H be a linear algebraic group with a fixed Levi decomposition $H = U \rtimes R$ and W be a rational representation of H. Additionally, let W be a H/U-representation and $\mathrm{ev} : W \to V$ be a morphism which is equivariant relative to $H \to H/U$. Suppose that $V' \subset V$ is an open H-invariant subset such that $W' \to W'/U$ is a principal U-bundle. If the morphism $\mathrm{ev} : W \to V$

restricted to W' factorises via $W' \to W'/U$ with $W'/U \to V'$ a closed immersion and V' open in V then we call the pair (ev, V) an affine completion of the U-quotient $W' \to W'/U$. Suppose that (ev, V) is an affine completion of the U-quotient $W' \to W'/U$ and linearise the H/U-action on V with respect to a character $\chi : H/U \to \mathbb{G}_{\mathrm{m}}$. We define the *relative (semi)stable loci*

$$W^{rss(H,\chi)} = V^{rss(H,\chi,\mathrm{ev},W')} := \mathrm{ev}^{-1}(\mathrm{ev}(W') \cap V^{ss(H/U,\chi)})$$

and

$$W^{rs(H,\chi)} = V^{rs(H,\chi,\mathrm{ev},W')} := \mathrm{ev}^{-1}(\mathrm{ev}(W') \cap V^{s(H/U,\chi)}).$$

Remark 4.2.2.

1. The approach to construct an H-quotient is to first construct a quotient for the unipotent radical of H and then to embed this quotient space in an affine completion and apply reductive geometric invariant theory on the affine completion. This mimics the approach of Bérczi, Doran, Hawes and Kirwan (see Theorem 2.5.8) of first constructing a (projective) completion of the U-quotient and then applying reductive GIT to the projective completion of the U-quotient. The idea to use an affine completion instead to apply reductive geometric invariant theory was to obtain a Hilbert-Mumford criterion for the stable and semistable locus as claimed by Bérczi, Doran, Hawes and Kirwan. It is unclear whether such a Hilbert-Mumford criterion is true in the affine case.

2. Note that the obvious morphism $X \to \operatorname{Spec} \mathcal{O}(X)^U$ is an affine completion in the category of affine schemes, since $\mathcal{O}(X)^U$ needs not be finitely generated. Additionally, the morphism $X \to \operatorname{Spec} \mathcal{O}(X)^U$ is in general not surjective. In order to mimic the construction from the projective $\hat{U}$-theorem we will use an affine completion given by some $\mathbb{A}^n$. This yields a quotient variety whereas the affine completion $X \to \operatorname{Spec} \mathcal{O}(X)^U$ gives a quotient in the category of schemes.

3. If we assume that $V^{ss(H/U,\chi)} \subset V'$ then it follows that $W^{rss(H,\chi)}$ admits a good quotient and $W^{rs(H,\chi)}$ admits a geometric quotient. To obtain the inclusion $V^{ss(H/U,\chi)} \subset V'$ we introduce the notion of an $\mathbb{N}$-indexed family of multiplicative subgroups of H; for further details, see Definition 4.2.4.

Notation 4.2.3. For an action $\beta : (U \rtimes \mathbb{G}_{\mathrm{m}}) \times W \to W$ such that $\beta|_{\mathbb{G}_{\mathrm{m}}} : \mathbb{G}_{\mathrm{m}} \times W \to W$ acts linearly, we introduce the following notation.

1. The β-weight space decomposition for $\mathbb{G}_{\mathrm{m}}$ acting via β on W is denoted by
$$W = \bigoplus_{i=1}^{r(\beta)} W_{\omega_i(\beta)}$$
where $W_{\omega_i(\beta)} = \{w \in W | \beta(t, w) = t^{\omega_i(\beta)} w\}$ and we order the $\mathbb{G}_{\mathrm{m}}$-weights so
$$\omega_1(\beta) < \ldots < \omega_{r(\beta)}(\beta).$$
2. For any vector subspace $V \subset W$, we write $V^0 := V \setminus \{0\}$. We denote by $W_{\min,\beta} := W^0_{\omega_1(\beta)}$.
3. Let $p_{\min,\beta} : W \to W$ denote the projection onto $W_{\omega_1(\beta)}$ and write $W^+_{\min,\beta} := W \setminus \ker p_{\min,\beta}$.

Definition 4.2.4. (Compare Definition 2.5.7) Let H be a linear algebraic group with unipotent radical U and W be a H-representation. To an $\mathbb{N}$-indexed family of central one-parameter subgroups $\lambda_n : \mathbb{G}_{\mathrm{m}} \to Z(R)$ of the Levi factor R, we associate a family of $\mathbb{G}_{\mathrm{m}}$-extensions of the unipotent racial U of H via

$$\begin{aligned} U \rtimes \mathbb{G}_{\mathrm{m}} &\to H \\ (u; t) &\mapsto u \cdot \lambda_n(t). \end{aligned}$$

We denote the corresponding subgroup of H by $\hat{U}_n$. The family $(\hat{U}_n)_{n\in\mathbb{N}}$ is said to be *adapted to the H-representation W,* if the following assumptions are satisfied, for the actions $\alpha_n : \hat{U}_n \times W \to W$ obtained from restricting the H-action to $\hat{U}_n$.

(A1) For all $n \in \mathbb{N}$ we have
$$\omega_1(\alpha_0) \leq \omega_1(\alpha_n) \quad \text{and} \quad \omega_2(\alpha_n) < \omega_2(\alpha_{n+1})$$
where the weights are assumed to be totally ordered such that
$$\omega_1(\alpha_i) < 0 \leq \omega_2(\alpha_i) < \ldots < \omega_{r_i}(\alpha_i)$$
for each 1-PS λ_i.

(A2) For all $n \in \mathbb{N}$ we assume the equality $W_{\min,\alpha_0} = V_{\min,\alpha_n}$. In the following we drop the dependence on α_n from the notation and write $W_{\min}$ and $W^+_{\min}$.

(A3) The U-stabiliser for each $x \in W_{\min}$ is trivial and $\hat{U}_0$ is a graded extension of the unipotent radical of H.

Theorem 4.2.5. *Let H be a linear algebraic group with a Levi-decomposition $U \rtimes R$. Suppose that $\lambda_n : \mathbb{G}_\mathrm{m} \to Z(R) \subset H$ is an $\mathbb{N}$-indexed family of central 1-PS adapted to a H-representation W. Suppose for each $n \in \mathbb{N}$, there is a character $\chi_n : H \to \mathbb{G}_\mathrm{m}$ such that $\langle \chi_n, \lambda_n \rangle < 0$. Then for n large enough the following statements hold.*

1. *The locus $W^+_{\min}$ admits a geometric $\hat{U}_n$-quotient $q : W^+_{\min} \to W^+_{\min}/\hat{U}_n$ with $W^+_{\min}/\hat{U}_n$ a projective over affine variety and $W^+_{\min} = W^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = W^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$.*

2. *There exists an affine completion of $W^+_{\min} \to W^+_{\min}/U$ such that the corresponding relative semistable locus $W^{rss(H,\chi_n)}$ admits a good H-quotient*

$$q : W^{rss(H,\chi_n)} \to W^{rss(H,\chi_n)}/\!\!/H$$

with $W^{rss(H,\chi_n)}/\!\!/H$ a projective over affine variety.

3. *The restriction $q\big|_{W^{rs(H,\chi_n)}} : V^{rs(H,\chi_n)} \to q(W^{s(H,\chi_n)})$ is a geometric H-quotient with $q(W^{rs(H,\chi_n)})$ open in $W^{rss(H,\chi_n)}/\!\!/H$.*

4.2.1 The proof of Theorem 4.2.5

To prove the affine $\hat{U}$-Theorem we need several preliminary results:

1. We show that the open subset $W' := W^+_{\min}$ admits a principal U-bundle quotient. Furthermore for a suitable $\mathbb{N}$-indexed sequence of characters $\chi_n : H \to \mathbb{G}_\mathrm{m}$ we show that $W^+_{\min} = W^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = W^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$.

2. We show that there exists an affine completion (ev, V) for $W^+_{\min} \to W^+_{\min}/U$. Additionally, for a suitable choice of an affine completion $\mathrm{ev} : W \to V$, the closed immersion $W^+_{\min}/U \to W'$ has the additional property that $W' = W^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = W^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ for sufficiently large n.

Some properties of the locus $W^+_{\min}$

In this subsection we show that $W^+_{\min}$ is H-invariant and admits a principal U-bundle quotient.

Proposition 4.2.6. *Let H be a linear algebraic group with unipotent radical U and $\hat{U} = U \rtimes \mathbb{G}_\mathrm{m}$ be a graded unipotent group such that $\hat{U}/U$ lies in the centre of H/U. For a H-representation W, we obtain that the linear projection $p_{\min} : W \to W$ to the minimal weight space $W_{\omega_{\min}}$ is equivariant relative to $H \to H/U$. Consequently it follows that $W^+_{\min}$ is H-invariant.*

Proof. By assumption $\hat{U}$ is a graded unipotent group and $p_{\min}$ is the projection to the minimal weight space of the grading subgroup $\mathbb{G}_\mathrm{m} \subset \hat{U}$. In particular the grading subgroup $\mathbb{G}_\mathrm{m}$ acts on $\mathrm{Lie}(U)$ with positive weights. Consider $A \in \mathrm{Lie}\, U \setminus \{0\}$ a $\mathbb{G}_\mathrm{m}$-weight vector of weight $\omega > 0$ which determines a locally nilpotent derivation δ_A on W that is also a $\mathbb{G}_\mathrm{m}$-weight vector of weight $\omega > 0$. The corresponding subgroup of U, which is isomorphic to $\mathbb{G}_\mathrm{a}$, acts on W via

$$u \cdot w := \exp(u\delta_A)(w)$$

for $u \in U$ and $w \in W$. Let w be a $\mathbb{G}_\mathrm{m}$-weight vector. We claim for $u \in \mathbb{G}_\mathrm{a}$ that we can write $u \cdot w = w + w'$ with $\mathrm{wt}_{\mathbb{G}_\mathrm{m}}(w) < \min \mathrm{wt}_{\mathbb{G}_\mathrm{m}}(w')$. Note that for $u \in U$

$$u \cdot w = \exp(u\delta_A)w = \sum_{i=0}^{n_w} \frac{u^i \delta_A^i(w)}{i!} = w + \sum_{i=1}^{n_w} \frac{u^i \delta_A^i(w)}{i!} = w + w'$$

where $n_w := \min\{n \in \mathbb{N} | \delta_A^n(w) = 0\}$. Together with $t \cdot (u \cdot w) = (t \cdot u) \cdot w = \big((t^\omega u) \cdot t\big) \cdot w$ we obtain $\mathrm{wt}_{\mathbb{G}_\mathrm{m}}(w) < \mathrm{wt}_{\mathbb{G}_\mathrm{m}}(w) + \omega \leq \min(\mathrm{wt}_{\mathbb{G}_\mathrm{m}}(w'))$. We conclude by linearity of the U-action, for any $w \in W$ and any $\mathbb{G}_\mathrm{m}$-weight vector $A \in \mathrm{Lie}(U)$, that $p_{\min}(\exp(u\delta_A)w) = p_{\min}(w)$. It follows that for all $u \in U$ and $w \in W$ that $p_{\min}(u \cdot w) = p_{\min}(w)$, since $\mathrm{Lie}(U)$ is generated by $\mathbb{G}_\mathrm{m}$-weight vectors.

We claim for the central 1-PS $\mathbb{G}_\mathrm{m}$ of our Levi-factor $R \cong H/U$ that the $\mathbb{G}_\mathrm{m}$-weight spaces get preserved under the restricted R-action. Let $w \in W_\omega \subset W$ be a $\mathbb{G}_\mathrm{m}$-weight vector; then we have for all $t \in \mathbb{G}_\mathrm{m}$ and $r \in R$ that $t \cdot w = t^\omega w$ and

$$r \cdot (t \cdot w) = (rt) \cdot w = (tr) \cdot w = t \cdot (r \cdot w)$$

holds. By linearity of the R-action we conclude that indeed $t \cdot (r \cdot w) = t^\omega (r \cdot w)$.

For $r \in R$ and $w \in W$, we decompose $w = w_{\min} + w'$ with $w_{\min} \in W_{\omega_{\min}}$ and $w' \in \bigoplus_{\omega > \omega_{\min}} W_\omega$ then

$$p_{\min}(r \cdot w) = p_{\min}\big(r \cdot (w_{\min} + w')\big) = p_{\min}\big(r \cdot w_{\min} + r \cdot w'\big) = p_{\min}(r \cdot w_{\min}) + p_{\min}(r \cdot w')$$
$$= r \cdot w_{\min} + 0 = r \cdot p_{\min}(w_{\min}) = r \cdot p_{\min}(w),$$

which proves the claimed equivalence of $p_{\min}$, where the equality between the first and the second line follows as the $\mathbb{G}_{\mathrm{m}}$-weight spaces are R-equivariant. We conclude for all $h \in H$ and $w \in W$ that $p_{\min}(w) \neq 0$ if and only if $p_{\min}(h \cdot w) \neq 0$ which implies that $W_{\min}^+ := W \backslash \ker p_{\min}$ is H-invariant. $\square$

Proposition 4.2.7. *Let W be a H-representation satisfying the assumptions of Definition 4.2.4 and $\chi_n : H \to \mathbb{G}_{\mathrm{m}}$ be a character such that $\langle \chi_n, \lambda_n \rangle < 0$, then the locus $W_{\min}^+$ is equal to the $\mathbb{G}_{\mathrm{m}}$-(semi)stable locus $W^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} = W^{s(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$.*

Proof. The claim follows immediately from the discrete-geometric Hilbert-Mumford criterion (see Corollary 2.7.9) a point $x \in W$ is

1. semistable with respect to ϱ if and only if $\varrho \in \mathrm{Cone}(\mathrm{wt}_T(x))$; and,
2. stable with respect to ϱ if and only if ϱ belongs to the interior of $\mathrm{Cone}(\mathrm{wt}_T(x))$.

Since $T = \mathbb{G}_{\mathrm{m}}$ we have that $\mathrm{Cone}(\mathrm{wt}_T(x))$ is the origin, a half space containing the origin or the space $\mathbf{X}^*(\mathbb{G}_{\mathrm{m}})_{\mathbb{R}}$. By our assumption $\langle \chi_n, \lambda_n \rangle < 0$ thus $w \in W$ is semistable if and only if $p_{\min}(w) \neq 0$. Furthermore if $\chi_n|_{\lambda_n(\mathbb{G}_{\mathrm{m}})}$ belongs to $\mathrm{Cone}(\mathrm{wt}_{\lambda_n(\mathbb{G}_{\mathrm{m}})}(w))$ then the only potential boundary point of $\mathrm{Cone}(\mathrm{wt}_{\lambda_n(\mathbb{G}_{\mathrm{m}})}(w))$ is 0 thus $\chi_n|_{\lambda_n(\mathbb{G}_{\mathrm{m}})}$ belongs to $\mathrm{Cone}(\mathrm{wt}_{\lambda_n(\mathbb{G}_{\mathrm{m}})}(w))$ if and only if $\chi_n|_{\lambda_n(\mathbb{G}_{\mathrm{m}})}$ belongs to the interior of $\mathrm{Cone}(\mathrm{wt}_{\lambda_n(\mathbb{G}_{\mathrm{m}})}(w))$. $\square$

Proposition 4.2.8. *Let $\hat{U}$ be a graded unipotent group and W be a $\hat{U}$-representation with minimal weight space $W_{\omega_{\min}}$. Suppose that $\sigma \in \mathcal{O}(W_{\omega_{\min}}) \subset \mathcal{O}(W)$ is such that each $x \in W_\sigma$ has trivial U-stabiliser. Then $W_\sigma \to \operatorname{Spec} \mathcal{O}(W)^U_\sigma$ is a trivial principal U-bundle. If each $x \in W_{\min}$ has a trivial U-stabiliser, then $W_{\min}^+ \to W_{\min}^+/U$ is a principal U-bundle.*

Proof. To obtain that W_σ admits a trivial principal U-bundle quotient, we invoke Lemma 3.3.1. Thus it is enough to show that the U-action on W extends to a graded $\hat{U}$-action, since U acts set-theoretically free on W_σ. By twisting the linear $\hat{U}$-action with the character corresponding to the weight $-\omega_{\min}$, we obtain for the twisted $\hat{U}$-action that $\omega_{\min} = 0$ and that the function σ is a $\hat{U}$-invariant. In other words the weight space decomposition of W for the twisted $\hat{U}$-action consists of non-negative weights. Thus the $\mathbb{G}_m$-weights of $\mathcal{O}(W)$ are non-positive and by our assumption we have that $\mathbb{G}_m$ acts on $\mathrm{Lie}(U)$ with positive weights. It follows that the twisted $\hat{U}$-action on W is graded in the sense of Definition 3.0.5. The grading of $\mathcal{O}(W_\sigma) \cong \mathcal{O}(W)_\sigma$ for the twisted $\hat{U}$-action is also non-positive since σ is a $\hat{U}$-invariant. Thus the twisted $\hat{U}$-action on W_σ is graded with trivial U-stabilisers. By Lemma 3.3.1, it follows that $W_\sigma \to \operatorname{Spec} \mathcal{O}(W_\sigma)^U$ is a trivial principal U-bundle. If each $x \in W^+_{\min}$ has trivial U-stabiliser, then we can cover $W^+_{\min}$ by finitely many $\hat{U}$-invariant principal open subsets thus $W^+_{\min} \to W^+_{\min}/U$ is a principal U-bundle. □

Embedding the locally trivial U-quotient

Let X be an affine H-variety and $F \subset \mathcal{O}(X)$ be a finite dimensional H-invariant vector subspace. The inclusion $F \subset \mathcal{O}(X)$ extends to a morphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(X)$ which defines a morphism $X \to \operatorname{Spec} \mathrm{Sym}^\bullet(F)$. Moreover, if F contains a generating system of $\mathcal{O}(X)$ as a $\mathbf{k}$-algebra, then the morphism $X \to F^\vee \cong \operatorname{Spec} \mathrm{Sym}^\bullet(F)$ is a H-equivariant closed immersion (see [17, Lemma A.1.9]). In the following definition we will assume that $X = V$ is a rational H-representation and U is the unipotent radical of H.

Definition 4.2.9. Let W' be an open subset of W together with a finite covering by principal open U-invariant subsets $W_{\sigma_i} \subset W$ indexed by $i = 1, \dots, r$ such that each $W_{\sigma_i} \to \operatorname{Spec} \mathcal{O}(W_{\sigma_i})^U$ is a trivial principal U-bundle. We call a finite dimensional vector subspace F of $\mathcal{O}(W)^U$ a *U-separating subspace adapted to the finite covering* $W' = \bigcup_{i=1}^{r} W_{\sigma_i}$ if the following conditions are satisfied:

1. The space F contains regular functions $g_1, \dots, g_l \in \mathcal{O}(W)^U$ such that $\mathcal{O}(W_{\sigma_i})^U \cong \mathcal{O}(W)^U_{\sigma_i}$ is generated $g_1, \dots, g_l$ and $\frac{1}{\sigma_i}$ as a $\mathbf{k}$-algebra for each $i = 1, \dots, r$.

2. F contains a finite H-separating set S for U acting on W such that $\sigma_1, \dots, \sigma_r \in S$ (see Definition 2.6.2).

3. F is H-invariant.

We also consider a H-invariant finite dimensional $\mathbf{k}$-subspace G of $\mathcal{O}(W)$ such that G contains F and a generating system of $\mathcal{O}(W)$ as a $\mathbf{k}$-algebra. Define $Y := G^\vee$ and $V := F^\vee$. Let $\mathrm{ev} : W \to V$ be the morphism corresponding to the ring homomorphism from $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ induced by the inclusion $F \subset \mathcal{O}(W)$.

Let $\mathrm{pr} : Y \to V$ be the restriction map and $W \to Y$ the morphism corresponding to the ring homomorphism $\mathrm{Sym}^\bullet(G) \to \mathcal{O}(W)$ induced by the inclusion $G \subset \mathcal{O}(W)$.

Proposition 4.2.10. *Given $W' = \cup_{i=1}^m W_{\sigma_i} \subset W$ and $F \subset G \subset \mathcal{O}(W)$ as in Definition 4.2.9, consider the morphism $\mathrm{ev} : W \to V$ together with the natural H-action on V induced by the H-action on $F \subset \mathcal{O}(W)^U$. Then the following statements hold.*

1. *The morphism ev is equivariant relative to $\pi : H \to H/U$.*

2. *The following diagram is commutative*

$$\begin{tikzcd} W \arrow[r, hook] \arrow[d, "\pi_U"'] \arrow[dr, "\mathrm{ev}"] & Y \arrow[d, two heads, "\mathrm{pr}"] \\ \mathrm{Spec}\,\mathcal{O}(W)^U \arrow[r] & V. \end{tikzcd}$$

3. *If we restrict the morphism ev to W', then we obtain the following commutative diagram*

$$\begin{tikzcd} W' \arrow[r, "\mathrm{ev}"] \arrow[dr, two heads, "\pi"'] & V' \\ & W'/U \arrow[u, hook, "\iota"] \end{tikzcd}$$

where $V' := \cup_{i=1}^r V_{\sigma_i}$ is an open subset of V and $W'/U \xrightarrow{\iota} V'$ is a closed immersion.

Proof. By definition the evaluation map $\mathrm{ev} : W \to V := \mathrm{Hom}_{lin}(F, \mathbb{A}^1)$ corresponds to the homomorphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ induced by the inclusion of F into $\mathcal{O}(W)$. Analogously, the morphism $V \to Y$ is defined via the homomorphism $\mathrm{Sym}^\bullet(G) \to \mathcal{O}(W)$ induced by the inclusion $G \subset \mathcal{O}(W)$. Since G contains a generating system of the $\mathbf{k}$-algebra $\mathcal{O}(W)$ it follows that $W \to Y$ is a closed immersion. The inclusion $F \subset G$ implies that $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ factorises via $\mathrm{Sym}^\bullet G$. By definition of F, it follows that $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ also factorises via $\mathcal{O}(W)^U$ since $F \subset \mathcal{O}(W)^U$. We obtain a commutative diagram

$$\begin{tikzcd} \mathcal{O}(W) & \mathrm{Sym}^\bullet(G) \arrow[l, two heads] \\ \mathcal{O}(W)^U \arrow[u, hook] & \mathrm{Sym}^\bullet(F). \arrow[l] \arrow[u] \arrow[ul] \end{tikzcd}$$

By applying the functor Spec we obtain Statement 2.

For Statement 1, to see that $\mathrm{ev} : W \to V = F^\vee$ is equivariant relative to $H \to H/U$, we first claim that $\mathrm{ev}(x) = \mathrm{ev}_x : F \to \mathbf{k}$ is the linear map given by $\mathrm{ev}_x(f) = f(x)$ for $x \in V$. Since a morphism of varieties over an algebraically closed field $\mathbf{k}$ is determined on the level of $\mathbf{k}$-points, it is enough to show for a $\mathbf{k}$-point $\mathrm{Spec}\,\mathbf{k} \xrightarrow{x} V$ that $\mathrm{ev}(x) = \mathrm{ev}_x$. Given

$$\mathrm{Spec}\,\mathbf{k} \xrightarrow{x} W \xrightarrow{\mathrm{ev}} F^\vee$$

we obtain the corresponding morphism of $\mathbf{k}$-algebras

$$\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W) \to \mathcal{O}(W)/\mathfrak{m}_x \cong \mathbf{k}.$$

By restricting to F we obtain

$$\begin{aligned} F &\xrightarrow{\mathrm{inc}} \mathcal{O}(W) \to \mathcal{O}(W)/\mathfrak{m}_x \cong \mathbf{k} \\ f &\mapsto \quad f \quad \longmapsto \quad \bar{f} \end{aligned}$$

which corresponds to evaluating f at x as claimed.

Let $L \in V = F^\vee$ (respectively $L \in Y = G^\vee$). We define the H-action on V (respectively Y) by $(h \cdot L)(f) := L(h^{-1} \cdot f)$ where $f : V \to \mathbb{A}^1$ is an element of F (respectively G) and $(h \cdot f)(x) := f(h^{-1}x)$ for $h \in H$ and $x \in V$. Note that the H-action on V (respectively Y) is well-defined since for $h \in H$ and $f \in F$ (respectively $f \in G$) it follows that $h^{-1}f$ belongs F (respectively G) by the H-invariance of F (respectively G). For $L = \mathrm{ev}_x$, we obtain

$$(h \cdot \mathrm{ev}_x)(f) = \mathrm{ev}_x(h^{-1} \cdot f) = (h^{-1} \cdot f)(x) = f(h \cdot x) = \mathrm{ev}_{h \cdot x}(f).$$

For $u \in U$, it follows from the fact that each $f \in F$ is U-invariant that

$$u \cdot \mathrm{ev}_x(f) = \mathrm{ev}_{u \cdot x}(f) = f(ux) = f(x) = \mathrm{ev}_x(f).$$

We conclude that $\mathrm{ev} : W \to V$ is equivariant relative to $\pi : H \to H/U$.

It remains to prove the third Statement. By definition, we have a cover $W' = \bigcup_{i=1}^r W_{\sigma_i}$ and for each $i = 1, \ldots, r$ we have that $\sigma_i \in F \subset \mathrm{Sym}^\bullet(F)$. Consider the open subset $V' := \bigcup_{i=1}^r V_{\sigma_i}$ of V where $V_{\sigma_i} = \mathrm{Spec}(\mathrm{Sym}^\bullet(F))_{\sigma_i}$. We claim that the induced morphism $W'/U \to V'$ is a closed immersion. The property of a morphism to be a closed immersion is local on the target. Therefore, it is enough to show that for each $i = 1, \ldots, r$ the morphism $W_{\sigma_i}/U \to V_{\sigma_i}$ is a closed immersion. By the first property of the definition of F, the associated morphism of $\mathbf{k}$-algebras $(\mathrm{Sym}^\bullet(F))_{\sigma_i} \to \mathcal{O}(W_{\sigma_i})^U$ is surjective. Thus $W_{\sigma_i}/U \to V_{\sigma_i}$ is a closed immersion. It follows that $W'/U \to V'$ is a closed immersion. □

Remark 4.2.11. Let W be a H-representation and $(\hat{U}_n)_{n\in\mathbb{N}}$ be an $\mathbb{N}$-indexed family adapted to the H-representation W (see Definition 4.2.4). In the following we consider a covering of $W^+_{\min}$ induced by a basis $b_1, \ldots, b_r$ of the weight space $W_{\omega_{\min}} = W_{\omega_{\min}(\alpha_n)}$. Let $\sigma_1, \ldots, \sigma_r$ be the dual basis to $b_1, \ldots, b_r$. We fix the covering $W^+_{\min} = \bigcup\limits_{i=1}^{r} W_{\sigma_i}$.

Lemma 4.2.12. *Let H be a linear algebraic group and V be a rational H-representation admitting an $\mathbb{N}$-indexed family $(\hat{U}_n)_{n\in\mathbb{N}}$. Then there exists a U-separating subspace $F \subset \mathcal{O}(W)^U$ adapted to the covering $W^+_{\min} = \bigcup\limits_{i=1}^{r} W_{\sigma_i}$.*

Proof. Given the covering $W^+_{\min} = \bigcup\limits_{i=1}^{r} W_{\sigma_i}$ we will show that there exists a U-separating subspace F adapted to this covering. By Proposition 4.2.8, it follows that $W_{\sigma_i} \to \mathrm{Spec}\, \mathcal{O}(W_{\sigma_i})^U$ is a trivial principal U-bundle and thus $W^+_{\min} \to W^+_{\min}/U$ is a principal U-bundle. By Theorem 2.6.3, there exists a finite separating system of U-invariants $s_1, \ldots, s_k$ for the U-action on W. Since $W_{\sigma_i} \to \mathrm{Spec}\, \mathcal{O}(W_{\sigma_i})^U$, is a trivial principal U-bundle, it follows that $\mathcal{O}(W_{\sigma_i})^U$ is finitely generated. Together with the isomorphism

$$\mathcal{O}(W_{\sigma_i})^U \cong \left(\mathcal{O}(W)^U\right)_{\sigma_i}$$

we deduce that there exist finitely many elements $g_1, \ldots, g_l \in \mathcal{O}(W)^U$ such that each $\mathcal{O}(W_{\sigma_i})^U$ is generated as a $\mathbf{k}$-algebra by $g_1, \ldots, g_l$ and $\frac{1}{\sigma_i}$.

Let F be the $\mathbf{k}$-vector space generated by $h \cdot s_j$ and $h \cdot g_i$ for $h \in H$, $1 \leq i \leq k$ and $1 \leq j \leq l$. By definition of F it follows that F is H-invariant. Furthermore, since H acts rationally on $\mathcal{O}(W)^U$ the vector subspace $F \subset \mathcal{O}(W)^U$ is finite dimensional. Additionally F contains a finite separating system, $\sigma_1, \ldots, \sigma_r$ and the functions $g_1, \ldots, g_l$ thus we conclude that F is a U-separating subspace adapted to the covering $W^+_{\min} = \bigcup_{i=1}^{r} W_{\sigma_i}$ in the sense of Definition 4.2.9. □

Lemma 4.2.13. *Let H be a linear algebraic group and V be a rational H-representation admitting an $\mathbb{N}$-indexed family $(\hat{U}_n)_{n\in\mathbb{N}}$. Then we can choose a U-separating subspace $F \subset \mathcal{O}(W)^U$ adapted to the covering $W^+_{\min} = \bigcup_{i=1}^{r} W_{\sigma_i}$ such that for $n \in \mathbb{N}$ large enough $W^+_{\min}/U$ is closed in $V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} = V^{s(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$ where $\chi_n : H \to \mathbb{G}_{\mathrm{m}}$ is a character satisfying $\langle \chi_n, \lambda_n \rangle < 0$.*

Proof. Let F be the U-separating subspace adapted to the covering $W^+_{\min} = \bigcup_{i=1}^{r} W_{\sigma_i}$ as constructed in the proof of Lemma 4.2.12. We claim that the following statements hold for $n \in \mathbb{N}$ a sufficiently large natural number.

1. A $\lambda_n(\mathbb{G}_{\mathrm{m}})$-weight vector $f \in F$ has a positive weight if and only if $f \mapsto p_f(\underline{\sigma})$ via the homomorphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ where $p_f(\underline{\sigma})$ is a polynomial in $\sigma_1, \ldots, \sigma_r$.

2. $V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} = \bigcup V_f$ where the union is taken over $f \in F$ such that $f \mapsto p_f(\underline{\sigma})$ via the homomorphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(V)$ where $p_f(\underline{\sigma})$ is a non-constant polynomial in $\sigma_1, \ldots, \sigma_r$.

3. The image of the morphism $\mathrm{ev} : W \to V$ is contained in the locus $Y := \{f - p_f(\underline{\sigma}) = 0 | f \in F \text{ with } f \mapsto p_f \text{ under } \mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)\}$.

4. We have $V' \cap Y = V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} \cap Y$.

Suppose Statements 1 to 4 are true. We already know that $W^+_{\min}/U \subset \cup_{i=1}^{r} V_{\sigma_i} = V'$ is closed by Proposition 4.2.10. By Statement 3 we obtain the inclusion $W^+_{\min}/U \subset V' \cap Y$. We conclude that $W^+_{\min}/U$ is closed in $V' \cap Y$. Finally, by Statement 4 $V' \cap Y = V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)} \cap Y$ which is closed in $V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$ thus we conclude that $W^+_{\min}/U$ is closed in $V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$.

It remains to show that Statements 1 to 4 hold. We denote by $D_F := \max\{\deg g | g \in F\}$. By Property (A1) of Definition 4.2.4 there exists a natural number $n \geq 1$ such that $|\omega_1(\alpha_0)| D_F <$

$n \leq \omega_2(\alpha_n)$. For Statement 1 let $f \in F$ be a $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector. The H/U-action on F is the induced action from H/U on $\mathcal{O}(V)^U$ by restricting to F. We claim that any $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector $f \in F$ has a positive $\lambda_n(\mathbb{G}_\mathrm{m})$-weight if and only if it gets mapped to a polynomial of the form $p_f(\underline{\sigma})$ by the homomorphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)^U$. Let $q \in \mathcal{O}(W)^U$ be the image of f then $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} f = \mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q$. In particular it follows that q is a $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector since f is a $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector. Suppose that q is of the form $p_f(\underline{\sigma})$. By remark 4.2.11, we have chosen a basis $b_1, \ldots, b_r$ of the weight space $W_{\omega_{\min}}$ and the corresponding dual basis $\sigma_1, \ldots, \sigma_r$ of the dual $\lambda_n(\mathbb{G}_\mathrm{m})$-representation. From the definition of the $\mathbb{N}$-indexed family of 1-PS it follows for each $n \in \mathbb{N}$ that the $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector b_i has the negative weight $\omega_{\min}(\lambda_n)$, thus the dual vector σ_i is a $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector of positive weight $-\omega_{\min}(\lambda_n)$. Any monomial of $q = p_f$ has the same weight as the $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector q and is given by a product of the form $\prod_{i=1}^{r} \sigma_i^{l_i}$. Thus the weight of q is a non-negative combination of the weights of the $\sigma_1, \ldots, \sigma_r$ and therefore positive as claimed.

Conversely, suppose that the $\lambda_n(\mathbb{G}_\mathrm{m})$-weight of q is positive then we have to show that q is of the form $p_f(\underline{\sigma})$. We extend $b_1, \ldots, b_r$ by $c_1, \ldots, c_s$ to obtain a basis of V consisting of $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vectors. Using the corresponding dual basis $\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s$ we can write q as a finite sum of monomials in $\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_s$. Each of these monomials is a $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector of the same weight as q. Thus we assume without loss of generality that q is a monomial $\lambda_n(\mathbb{G}_\mathrm{m})$-weight vector. We can decompose the monomial q into a product $q = q_+ q_-$ or $q = q_+$ with $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_+ > 0$ and $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_- < 0$. If $q = q_+$ then q is a polynomial in $\sigma_1, \ldots, \sigma_r$ since the $\lambda_n(\mathbb{G}_\mathrm{m})$-weights of V satisfy $\omega_1(\alpha_n) < 0 < \omega_2(\alpha_n) < \ldots < \omega_{\max}(\alpha_n)$ and $\sigma_1, \ldots, \sigma_r$ is a dual basis for the weight space V_{ω_1}. So suppose that $q = q_+ q_-$ then it follows that

$$\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_+ \leq |\omega_1(\alpha_n)| \deg q_+ \leq |\omega_1(\alpha_n)|(D_F - 1) \leq |\omega_1(\alpha_0)|(D_F - 1)$$

and $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_- \leq -\omega_2(\alpha_n)$. From

$$\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q = \mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_+ + \mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q_-$$

it follows that $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})} q \leq |\omega_1(\alpha_n)|(D_F - 1) - \omega_2(\alpha_n) \leq |\omega_1(\alpha_0)|(D_F - 1) - \omega_2(\alpha_n) < 0$. We conclude that q has positive weight if and only if $q = q_+$ but then q is a polynomial in $\sigma_1, \dots, \sigma_r$ thus Statement 1 holds. By Statement 1 and the discrete-geometric Hilbert-Mumford criterion, it follows that $v \in V$ is χ_n-semistable for the 1-PS λ_n, if and only if $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})}(v)$ contains a negative weight. From $V = F^\vee$, it follows that $\mathrm{wt}_{\lambda_n(\mathbb{G}_\mathrm{m})}(v)$ contains a negative weight if and only if there exists a positive $\lambda_n(\mathbb{G}_\mathrm{m})$-weight $f \in F$ such that $f(v) \neq 0$. We conclude as in the proof of Proposition 4.2.7 that $V^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = \bigcup V_f$ where the union is taken over $f \in F$ such that $f = f_+$. We already saw that f_+ gets mapped to a non-constant polynomial of the form $p_f(\underline{\sigma})$ via $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)^U$, thus the claim follows. For Statement 3 note that $\sigma_1, \dots, \sigma_r \in F$ thus $p_f(\underline{\sigma}) \in \mathrm{Sym}^\bullet(F)$. Under the homomorphism $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)$ the elements f and p_f get mapped to p_f thus $f - p_f$ is in the kernel. We conclude that $\mathrm{im}(\mathrm{ev}) \subset Y$. It remains to show that Statement 4 holds. By Statement 2 we have that $V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = \bigcup V_f$ where the union is taken over $f \in F$ such that f gets mapped to a non-constant polynomial of the form $p_f(\underline{\sigma})$ via $\mathrm{Sym}^\bullet(F) \to \mathcal{O}(W)^U$. By definition

$$V' = \bigcup_{i=1}^{r} V_{\sigma_i}.$$

We conclude that $V' \subset V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ which implies $V' \cap Y \subset V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} \cap Y$. For the converse inclusion note that $V_{p_f(\underline{\sigma})} \subset V'$ together with $V_f \cap Y = V_{p_f(\underline{\sigma})} \cap Y$ implies that $V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} \cap Y \subset V' \cap Y$.

□

Proof of Theorem 4.2.5. By Proposition 4.2.8, which we can apply as property (A3) of Definition 4.2.4 holds, we obtain a principal U-bundle

$$W^+_{\min} \to W^+_{\min}/U.$$

Furthermore by Lemma 4.2.12 and Proposition 4.2.10 there exists a morphism $\mathrm{ev} : W \to V$ which is equivariant relative to $H \to H/U$ such that the restriction $\mathrm{ev}|_{W^+_{\min}}$ factorises via a closed immersion ι as follows

$$\begin{tikzcd} W^+_{\min} \arrow[r, "\mathrm{ev}|_{W^+_{\min}}"] \arrow[dr, two heads, "\pi_U"'] & V' \\ & W^+_{\min}/U \arrow[u, hook, "\iota"] \end{tikzcd}$$

with $V' := \bigcup_{i=1}^r V_{\sigma_i}$ open in V. From our $\mathbb{N}$-indexed family $\lambda_n : \mathbb{G}_\mathrm{m} \to H$ and Lemma 4.2.13 we obtain for $n \in \mathbb{N}$ large enough that

$$V' \subset V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} = V^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$$

holds. Additionally by Lemma 4.2.13, we know that $W^+_{\min}/U$ is a closed H-invariant subset of $V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ for n large enough. Consider the following commutative diagram

$$\begin{tikzcd} W^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n))} = W^+_{\min} \arrow[r] \arrow[dr, two heads, "\pi_U"'] & V^{s(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)} \arrow[r, two heads, "\pi_{\mathbb{G}_\mathrm{m}}"] & V /\!\!/_{\chi_n} \lambda_n(\mathbb{G}_\mathrm{m}) \\ & W^+_{\min}/U \arrow[u, hook, "\iota"] \arrow[r, two heads, "\pi_{\mathbb{G}_\mathrm{m}}|_{W^+_{\min}/U}"] & (W^+_{\min}/U)/\lambda_n(\mathbb{G}_\mathrm{m}) \arrow[u, hook] \end{tikzcd}$$

where $V /\!\!/_{\chi_n} \lambda_n(\mathbb{G}_\mathrm{m})$ is a projective over affine variety as it is the reductive GIT quotient of the affine space V with respect the a character χ_n. The morphism $(W^+_{\min}/U)/\lambda_n(\mathbb{G}_\mathrm{m}) \to V'/\lambda_n(\mathbb{G}_\mathrm{m})$ is a closed immersion since ι is a closed immersion. This concludes the proof of the first statement of Theorem 4.2.5.

To obtain the second statement, we apply reductive geometric invariant theory for the $R \cong H/U$-action on V with respect to a character χ_n where n is taken sufficiently large so that $\iota(W^+_{\min}/U)$ is closed in $V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$. In the following we use that the GIT quotient has good functorial properties with respect to closed immersions. The GIT-quotient is given by restricting the following rational map to its domain of definition

$$q_{\mathfrak{L}} : V \dashrightarrow V /\!\!/_{\mathfrak{L}} R := \operatorname{Proj} \mathrm{R}(W, \mathfrak{L})^R$$

where $\mathfrak{L}$ is the linearisation of trivial line bundle over V linearised with respect to the character χ_n. We have the obvious inclusion $V^{ss(R,\chi_n)} \subset V^{ss(\lambda_n(\mathbb{G}_\mathrm{m}),\chi_n)}$ which implies that $(W^+_{\min}/U) \cap V^{ss(R,\chi_n)}$

is a closed R-invariant subset of $V^{ss(R,\chi_n)}$. It remains to show that

$$W^{rss(H,\chi_n)} := \mathrm{ev}^{-1}((W^+_{\min}/U) \cap V^{ss(R,\chi_n)})$$

admits a good H-quotient and that the restriction to

$$W^{rs(H,\chi_n)} := \mathrm{ev}^{-1}((W^+_{\min}/U) \cap V^{s(R,\chi_n)})$$

is a geometric H-quotient. We have that

$$W^{rss(H,\chi_n)} \to (W^+_{\min}/U) \cap V^{ss(R,\chi_n)}$$

is a geometric U-quotient and

$$q_{\mathfrak{L}} : (W^+_{\min}/U) \cap V^{ss(R,\chi_n)} \to q_{\mathfrak{L}}\big((W^+_{\min}/U) \cap V^{ss(R,\chi_n)}\big) \subset V /\!\!/_{\mathfrak{L}} R$$

is a good R-quotient with $q_{\mathfrak{L}}\big((W^+_{\min}/U) \cap V^{ss(R,\chi_n)}\big)$ closed in $V /\!\!/_{\mathfrak{L}} R$. The composition

$$W^{rss(H,\chi_n)} \to (W^+_{\min}/U) \cap V^{ss(R,\chi_n)} \to q_{\mathfrak{L}}\big((W^+_{\min}/U) \cap V^{ss(R,\chi_n)}\big)$$

is a good H-quotient with $q_{\mathfrak{L}}\big((W^+_{\min}/U) \cap V^{ss(R,\chi_n)}\big)$ a projective over affine variety as it is closed in the projective over affine variety $V /\!\!/_{\mathfrak{L}} R$. Analogously

$$W^{rs(H,\chi_n)} \to (W^+_{\min}/U) \cap V^{s(R,\chi_n)}$$

is a geometric U-quotient and

$$q_{\mathfrak{L}} : (W^+_{\min}/U) \cap V^{s(R,\chi_n)} \to q_{\mathfrak{L}}\big((W^+_{\min}/U) \cap V^{s(R,\chi_n)}\big) \subset V /\!\!/_{\mathfrak{L}} R$$

is a geometric R-quotient and thus the composition is a geometric H-quotient which concludes the proof of Theorem 4.2.5. □

4.2.2 Towards potential generalisations of Theorem 4.2.5

In this section, we consider ideas to modify the affine $\hat{U}$-Theorem for the case where non-trivial stabilisers exists for the U-action on $W_{\min}^{+}$. If the U-action on $W_{\min}^{+}$ is sufficiently well behaved it is possible to obtain a variation of the affine $\hat{U}$-Theorem by replacing the $\mathbb{N}$-indexed family of 1-PS by an $\mathbb{N}$-indexed family of subtori of H.

Unlike the projective setting where it is possible to a apply a blow-up procedure which results again in a projective variety with constant dimensional stabilisers for the locus $\hat{X}_{\min}^{0}$ of the blow up $\hat{X}$. If the U-stabilisers for $\hat{X}_{\min}^{0}$ are trivial, then the original $\hat{U}$-Theorem can be applied to $\hat{X}$. For our consideration, if we start with an affine variety a blow-up procedure is also possible but we obtain a projective over affine variety. Thus it is not possible to apply the affine $\hat{U}$-Theorem after a blow up procedure.

To generalise the affine $\hat{U}$-Theorem we could try to allow constant dimensional U-stabilisers for the locus $W_{\min}^{+}$. Another idea is to replace the $\mathbb{N}$-indexed family of central 1-PS with an $\mathbb{N}$-indexed family of central subtori of H.

Constant dimensional stabilisers on $V_{\min}^{+}$

Here we will assume that the dimension of $\mathrm{Stab}_U(x)$ is constant for all $x \in W_{\min}^{+}$. Following Richardson [57], we define an algebraic family of subgroups:

Definition 4.2.14. A *family of algebraic subgroups* over an variety S is a locally closed subvariety F of $G \times S$ such that the projection $p : G \times S \to S$ restricted to F has the following properties:

1. $p|_F$ is a surjective submersion
2. for every $x \in S : (p|_F)^{-1}(x) = G_x \times \{x\}$ where G_x is a subgroup of G.

Example 4.2.15. Consider an action α of $\hat{U}$ on $W = \mathbb{A}^n$ such that for each $x \in W_{\min}^{+}$ we have that $\dim \mathrm{Stab}_U(x) = d > 0$. Then consider the following fibre product

$$\begin{tikzcd} \mathcal{F}_U \arrow[r, "p"] \arrow[d, "q"] & U \times V_{\min}^{+} \arrow[d, "\alpha"] \\ U \times V_{\min}^{+} \arrow[r, "\pi_{V_{\min}^{+}}"] & V_{\min}^{+}. \end{tikzcd}$$

By [57, Theorem 9.3.1 (iv)], we obtain that $\mathcal{F}_U$ is an algebraic family of unipotent subgroups in the sense of Richardson.

Definition 4.2.16. A *complement* of a subgroup H in a group G is a subgroup K of G such that $HK = G$ and $H \cap K = \{1_G\}$. A *complemented group* is a group where every subgroup admits a complement.

Example 4.2.17. An abelian unipotent linear algebraic group U is a complemented group.

Remark 4.2.18. Complements need not exist, and if they do they need not be unique.

Remark 4.2.19. Let $\mathcal{F}$ be a *family of algebraic subgroups* over S a connected variety. Further assume that d is the common dimension of the subgroups G_x for $x \in S$. Then Richardson claims in [57, Lemma 6.2.2] that the map $x \mapsto \mathrm{Lie}(G_x)$ determines a morphism of varieties $S \to \mathrm{Grass}(d, \mathrm{Lie}\, G)$. By [57, Theorem 9.3.1 (iv)] our family of unipotent subgroups is algebraic in the sense of Richardson so it defines a morphism to $\mathrm{Grass}(d, \mathrm{Lie}\, G)$.

From this we obtain the following proposition, which is stated without proof in [9].

Proposition 4.2.20. *Let V be a variety and $V' \subset V$ be an open U-invariant subset such that for each $x \in V'$ we have $\dim \mathrm{Stab}_U(x) = d$. Then we get a U-equivariant morphism*

$$\begin{aligned} V' &\to \mathrm{Grass}(d, \mathrm{Lie}\, U) \\ x &\mapsto \mathrm{Lie}\, \mathrm{Stab}_U(x). \end{aligned}$$

Remark 4.2.21. According to [9], the subset of $\mathrm{Grass}(d, \mathrm{Lie}\, U)$ which admits a fixed complementary subgroup is an affine open subset. The preimage defines an affine open subset of V'. To obtain an affine $\hat{U}$-Theorem in the case that the dimension of $\mathrm{Stab}_U(x)$ is constant for all $x \in V^+_{\min} = V'$ we would need to show for a linear graded $U \rtimes \mathbb{G}_{\mathrm{m}}$-action on V that the affine open subsets V_σ which admit a fixed complementary subgroup is closed under taking the limit $\lim\limits_{t \to 0} t \cdot x$ in the sense that for each $x \in V_\sigma$ the limit exists and stays in V_σ. Or equivalently that the $U \rtimes \mathbb{G}_{\mathrm{m}}$-action on V_σ is graded.

Example 4.2.22. Consider the action of $U = \mathbb{U}_3 \subset \mathrm{GL}(3, \mathbf{k})$, the subgroup of upper triangular matrices with diagonal entries equal to one, on $W = \mathbb{A}^3$ which we will extend to a $\hat{U}$-action via

$$\begin{pmatrix} t^2 & u_1 & u_3 \\ 0 & t & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

For each $x \in W^+_{\min} = V_z$ we have that $\dim \mathrm{Stab}_U(x) = 1$ and the morphism

$$\begin{aligned} W^+_{\min} &\to \mathrm{Grass}(1, \mathrm{Lie}\, U) = \mathbb{P}^2 \\ (x, y, z) &\mapsto \mathrm{Lie}\, \mathrm{Stab}_U(x) = [-y : z : 0]. \end{aligned}$$

Now consider $(0, 0, 1)$ and $(0, 1, 1)$ which belong to the same U-orbit but do not map to the same point in $\mathrm{Grass}(1, \mathrm{Lie}\, U)$. Analogously $(0, 1, 1)$ is in the same $\mathbb{G}_m$-orbit as $(0, 2, 1)$ but their image under the morphism does not coincide. This shows that $W^+_{\min} \to \mathrm{Grass}(1, \mathrm{Lie}\, U)$ is neither U- nor $\mathbb{G}_m$-invariant and in particular not $\hat{U}$-invariant.

Example 4.2.23. Consider the action of $U = \mathbb{U}_3 \subset \mathrm{GL}(3, \mathbf{k})$, the subgroup of upper triangular matrices with diagonal entries equal to one, on $W = \mathbb{A}^3$ which extends to a linear $\hat{U}$-action via

$$\begin{pmatrix} t^2 & u_1 & u_3 \\ 0 & t & u_2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

We want to obtain a quotient for U acting on $W^+_{\min} = V_z$. Note that for each $x \in W^+_{\min}$ we have that $\dim \mathrm{Stab}_U(x) = 1$ so we can not directly apply the $\hat{U}$-Theorem. We consider the following subgroup of $\hat{U'} \subset \hat{U}$ given by matrices of the following form

$$\begin{pmatrix} t^2 & u & v + u^2 \\ 0 & t & 2u \\ 0 & 0 & 1 \end{pmatrix}.$$

This graded unipotent subgroup is a complementary subgroup of $\mathrm{Stab}_U(x)$ for any $x \in W^+_{\min}$ that is $\mathrm{Stab}_U(x) \cap U' = \{e\}$ and $U = \mathrm{Stab}_U(x)U'$. The quotient for the U-action is now given by the quotient for the U'-action on $W^+_{\min}$ which is the trivial principal U'-bundle

$$\begin{aligned} V_z &\to \mathbb{A}^1 \setminus \{0\} \\ (x, y, z) &\mapsto z. \end{aligned}$$

4.3 Examples

In this section we illustrate the results from section 3.1 and 3.2 to construct quotients for affine H-varieties with some simple examples.

Example 4.3.1. In this example we apply Lemma 4.1.6 and Corollary 4.1.7 to construct quotients for graded H-actions on affine varieties.

1. Let $H := \mathbb{G}_a^2 \rtimes \mathbb{G}_{\mathrm{m}}$ acting linearly on $X = \mathbb{A}^3$ via

$$\begin{aligned} \rho : H &\to \mathrm{GL}(3, \mathbf{k}) \\ (u_1, u_2, t) &\mapsto \begin{pmatrix} t & u_1 & u_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

then $\mathbb{A}^3 \to \mathbb{A}^2$ is a good H-quotient with $X^{\overline{s}(H,0)} = \mathbb{A}^3 \setminus \big(V(x_2) \cup V(x_3)\big)$ the stable locus which admits a quasi-affine geometric quotient

$$X^{\overline{s}(H,0)} \to \mathbb{A}^2 \setminus \{0\}.$$

2. Consider $H := \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$ acting linearly on $\mathbb{A}^4$ via

$$\begin{aligned} \rho : H &\to \mathrm{GL}(3, \mathbf{k}) \\ (u_1, u_2, t, s) &\mapsto \begin{pmatrix} t & u_1 & 0 & 0 \\ 0 & s & 0 & 0 \\ 0 & 0 & t & u_2 \\ 0 & 0 & 0 & s \end{pmatrix} \end{aligned}$$

If we linearise the H-action with respect to the character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ given by $\chi(u, t, s) = s$, we obtain a good quotient

$$(\mathbb{A}^4)^{ss} = \mathbb{A}^4_{x_2} \cup \mathbb{A}^4_{x_4} \to \operatorname{Proj} \bigoplus_{r\geq 0} H^0(X, \mathfrak{L}^{\otimes r})^H = \mathbb{P}^1$$

and geometric quotient

$$(\mathbb{A}^4)^{\overline{s}(H,\chi)} = \mathbb{A}^4_{x_2x_4} \to \mathbb{P}^1 \setminus \{0, \infty\}.$$

Example 4.3.2. In [4] A´Campo-Neuen considered $U := \mathbb{G}_a^{12}$ together with a 19-dimensional rational U-module V and showed that the ring of U-invariants $\mathcal{O}(V)^U$ is not finitely generated. Here we extend the linear U-action induced by $U \to \mathrm{GL}(V)$ to graded action of $H := \mathbb{G}_a^{12} \rtimes \mathbb{G}_\mathrm{m}$. For the following choice of coordinates we obtain the locus $V^+_{\min} = \bigcup_{i=1}^{4} V_{x_i}$ but we do not obtain trivial U-stabilisers for all $\mathbf{k}$-points in $V^+_{\min}$ so we can not apply the affine $\hat{U}$-Theorem.

Instead we are lead to consider the strongly stable locus of the unipotent radical subordinate to H given by

$$V^{ssu(H,U)} := V_{x_1x_2} \cup V_{x_1x_3} \cup V_{x_1x_4} \cup V_{x_2x_3x_4}.$$

We do obtain a geometric H-quotient for the open subset given by

$$V^{ssu(H,U)} \cap \bigcap_{u\in U} u \cdot V^{s(\mathbb{G}_\mathrm{m},\chi)} = V^{ssu(H,U)} \setminus UV^{\mathbb{G}_\mathrm{m}}$$

which sits as an open subvariety inside the separated $\mathbf{k}$-scheme $\operatorname{Proj} \mathcal{O}(V)^U$. In the above choice of coordinates the morphism $\hat{U} \to \mathrm{GL}(V)$ is given by matrices of the following form

$$\left(\begin{array}{cccc|ccc|ccc|ccc|cccccc}
1 & 0 & 0 & 0 & & & & & & & & & & & & & & & \\
0 & 1 & 0 & 0 & & & & & & & & & & & & & & & \\
0 & 0 & 1 & 0 & & & & & & & & & & & & & & & \\
0 & 0 & 0 & 1 & & & & & & & & & & & & & & & \\
\hline
u_1 & u_0 & 0 & 0 & t & 0 & 0 & & & & & & & & & & & & \\
u_2 & u_1 & 0 & 0 & 0 & t & 0 & & & & & & & & & & & & \\
0 & u_2 & 0 & 0 & 0 & 0 & t & & & & & & & & & & & & \\
\hline
u_3 & 0 & u_0 & 0 & & & & t & 0 & 0 & & & & & & & & & \\
u_4 & 0 & u_3 & 0 & & & & 0 & t & 0 & & & & & & & & & \\
0 & 0 & u_4 & 0 & & & & 0 & 0 & t & & & & & & & & & \\
\hline
u_5 & 0 & 0 & u_0 & & & & & & & t & 0 & 0 & & & & & & \\
u_6 & 0 & 0 & u_5 & & & & & & & 0 & t & 0 & & & & & & \\
0 & 0 & 0 & u_6 & & & & & & & 0 & 0 & t & & & & & & \\
\hline
u_7 & u_0 & 0 & 0 & & & & & & & & & & t & 0 & 0 & 0 & 0 & 0 \\
u_8 & u_7 & 0 & 0 & & & & & & & & & & 0 & t & 0 & 0 & 0 & 0 \\
u_9 & 0 & u_8 & 0 & & & & & & & & & & 0 & 0 & t & 0 & 0 & 0 \\
u_{10} & 0 & u_9 & 0 & & & & & & & & & & 0 & 0 & 0 & t & 0 & 0 \\
u_{11} & 0 & 0 & u_{10} & & & & & & & & & & 0 & 0 & 0 & 0 & t & 0 \\
0 & 0 & 0 & u_{11} & & & & & & & & & & 0 & 0 & 0 & 0 & 0 & t
\end{array}\right)
\left(\begin{array}{c}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ \hline x_5 \\ x_6 \\ x_7 \\ \hline x_8 \\ x_9 \\ x_{10} \\ \hline x_{11} \\ x_{12} \\ x_{13} \\ \hline x_{14} \\ x_{15} \\ x_{16} \\ x_{17} \\ x_{18} \\ x_{19}
\end{array}\right)$$

Example 4.3.3. A simple example for the affine $\hat{U}$-Theorem (see Theorem 4.2.5). Let $H := \mathbb{G}_a^2 \rtimes \mathbb{G}_m^2$ act linearly on $W := \mathbb{A}^5$ via

$$\big((u_1, u_2), (s,t)\big) \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} s & 0 & 0 & 0 & 0 \\ 0 & s & 0 & 0 & 0 \\ u_1 & 0 & t & 0 & 0 \\ u_2 & u_1 & 0 & t & 0 \\ 0 & u_2 & 0 & 0 & t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}.$$

For $n \in \mathbb{N} \setminus \{0\}$, let $\lambda_n : \mathbb{G}_{\mathrm{m}} \to Z(R) = \mathbb{G}_m^2$ be the adapted $\mathbb{N}$-indexed family of central 1-PS given by $\tau \mapsto (\tau^{-1}, \tau^n)$. Then $\hat{U}_n := U \rtimes_{\lambda_n} \mathbb{G}_{\mathrm{m}}$ is graded unipotent and the induced action is given by

$$\big((u_1,u_2),\tau\big)\cdot\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix} = \begin{pmatrix}\tau^{-1} & 0 & 0 & 0 & 0\\ 0 & \tau^{-1} & 0 & 0 & 0\\ \tau^n u_1 & 0 & \tau^n & 0 & 0\\ \tau^n u_2 & \tau^n u_1 & 0 & \tau^n & 0\\ 0 & \tau^n u_2 & 0 & 0 & \tau^n\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{pmatrix}.$$

To apply the affine $\hat{U}$-Theorem we choose the following $\mathbb{N}$-indexed family of characters

$$\begin{aligned}\chi_n : H &\to \mathbb{G}_{\mathrm{m}}\\ \big((u_1,u_2),(s,t)\big) &\mapsto s^n t.\end{aligned}$$

The λ_n-weight space decomposition is $W = W_{-1} \oplus W_n$ and for $W^+_{\min} = W_{x_1} \cup W_{x_2}$ we have trivial U-stabilisers. Therefore, we get a locally-trivial U-quotient $W^+_{\min} \to W^+_{\min}/U$ which we will identify with $V^+_{\min} \subset V = \mathbb{A}^3$ where $f : W \to V$ is induced by $f_* : \mathbf{k}[y_0,y_1,y_2] \to \mathcal{O}(W)$

$$y_i \mapsto \begin{cases} x_2^2x_3 - x_1x_2x_4 + x_1^2x_5 & \text{if i=0}\\ x_i & \text{if i} > 0\end{cases}$$

The induced H/U-action on V given by $(s,t)\cdot(w_0,w_1,w_2) = (s^2tw_0, sw_1, sw_2)$ turns the morphism f equivariant with respect to $H \to H/U$. The action of λ_n on V has the weights $n-2$ and -1 which are both negative $n \le 1$. It follows that

$$V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}},\chi_n))} = \begin{cases} V\setminus\{0\} & \text{if n=0 or n=1}\\ V_{y_2}\cup V_{y_3} & \text{if n} > 1\end{cases}$$

For $n = 0$ or $n = 1$ the image under f of $\mathbb{G}_{\mathrm{m}}$-stable locus $W^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}}),\chi_n)}$ is not closed in the $\mathbb{G}_{\mathrm{m}}$-stable locus of $V^{ss(\lambda_n(\mathbb{G}_{\mathrm{m}},\chi_n))} = V\setminus\{0\}$. We did not show that F is a fully U-separating subspace in the sense of Definition 4.2.9. Nevertheless we obtain a good quotient for

$$W^{rss(H,\chi_n)} \to q(W^{rss(H,\chi_n)}) \subset V/\!\!/_{\chi_n} \mathbb{G}_m^2$$

if $n \geq 2$. Moreover, since $V^{ss(\mathbb{G}_m^2,\chi_n)} = V^{s(\mathbb{G}_m^2,\chi_n)}$ it follows that the good quotient is even a geometric quotient.

Chapter 5

Towards moduli spaces for sheaf homomorphisms indexed by a quiver

In this chapter we consider sheaf homomorphisms indexed by a quiver. More concretely, we fix the following data $D = (X, \boldsymbol{Q}, (\mathcal{E}_v)_{v\in\boldsymbol{V}})$, where

1. X is a projective scheme of finite type over $\mathbf{k}$,
2. $\boldsymbol{Q} = (\boldsymbol{V}, \boldsymbol{A}, s, t : \boldsymbol{A} \to \boldsymbol{V})$ is a quiver, and;
3. $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ is a collection of semisimple coherent sheaves over X.

To this data we associated several moduli functors in Chapter 1 and reduced the construction of a moduli space to the construction of a good quotient for the linear action of

$$H := \bigtimes_{v\in\boldsymbol{V}} \operatorname{Aut}(\mathcal{E}_v)$$

on

$$W := \bigoplus_{a\in\boldsymbol{A}} \operatorname{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$$

given by

$$(h_v)_{v\in\boldsymbol{V}} \cdot (w_a)_{a\in\boldsymbol{A}} := (h_{t(a)} \circ w_a \circ h_{s(a)}^{-1})_{a\in\boldsymbol{A}}.$$

The construction of good and geometric quotients for algebraic group actions is the aim of reductive and non-reductive geometric invariant theory.

5.1 Description of $\mathrm{Aut}(\mathcal{E})$ for a coherent sheaf

In this section, we recall two descriptions of the automorphism groups for a coherent sheaf over X a projective scheme:

1. the concrete description by Drézet and Trautmann for semisimple locally free sheaves over a complex projective variety. In this case we will only consider $\mathbf{k} = \mathbb{C}$.

2. a description of the automorphism as a combination of a result of Álvarez-Cónsul and King (see [1] and [2]) with a result of Brion (see [13]). Here $\mathbf{k}$ is any algebraically closed field.

The second description of the automorphism groups identifies the Levi factor of such a group with a finite product of general linear groups.

5.1.1 The description by Drézet and Trautmann

From now on, assume that X is a complex projective variety and the sheaf $\mathcal{E}$ is semisimple and locally free, i.e. a finite direct sum of simple subsheaves $\mathcal{E}_i$. Recall that a sheaf $\mathcal{G}$ is simple if its endomorphisms consist solely of homotheties, that is $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G},\mathcal{G}) \cong \mathbb{C}$. For convenience, we collect recurring factors and write $\mathcal{E} = \bigoplus_{i=1}^{r} M_i \otimes \mathcal{E}_i$, where M_i are finite dimensional $\mathbb{C}$-vector spaces. Finally, assume that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_k, \mathcal{E}_l) = 0$ for $k > l$. Following Drézet and Trautmann, we describe $\mathrm{Aut}(\mathcal{E})$ in terms of matrices

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}_{\mathbb{C}}\left(M_i, M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\right)$. Note that under these identifications the maximal normal unipotent subgroup $U_{\mathcal{E}}$ of $\mathrm{Aut}(\mathcal{E})$ is given by matrices of the following form

$$\begin{pmatrix} \mathrm{Id}_{M_1} & 0 & \cdots & 0 \\ u_{2,1} & \mathrm{Id}_{M_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & \mathrm{Id}_{M_r} \end{pmatrix}.$$

with $u_{j,i} \in \mathrm{Hom}_{\mathbb{C}}\left(M_i, M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\right)$. We can also consider the Levi-factor $R_{\mathcal{E}}$ of $\mathrm{Aut}(\mathcal{E})$ which is given by the condition that $u_{j,i} = 0$ for all i, j; that is

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_r \end{pmatrix}.$$

In particular we have that $\mathrm{Aut}(\mathcal{E})/U_{\mathcal{E}} \cong R_{\mathcal{E}}$.

5.1.2 Automorphism groups of quiver representations in the category of finite dimensional vector spaces

The following result of Álvarez-Cónsul and King (see Theorem 5.1.5) allows us to identify n-regular coherent sheaves over a projective $\mathbf{k}$-scheme with Kronecker-modules (representations of the generalised Kronecker quiver $\boldsymbol{K}_m$ in the category $\mathrm{vect}_{\mathbf{k}}$). For an n-regular sheaf $\mathcal{F}$, the associated Kronecker-module $K_{\mathcal{F}}$ provides a description of the automorphism group of $\mathcal{F}$, by Proposition 5.1.7 which is a result of Brion.

In the following let $(X, \mathcal{O}_X(1))$ be a polarised projective scheme.

Definition (Castelnuovo–Mumford regularity) 5.1.1. A coherent sheaf $\mathcal{E}$ over X is n-regular with respect to the polarisation $\mathcal{O}_X(1)$, if $H^i(X, \mathcal{E}(n-i)) = 0$ for all $i > 0$. We call a coherent sheaf $\mathcal{E}$ regular, if $\mathcal{E}$ is 0-regular.

Example 5.1.2. Consider $(\mathbb{P}^{\mathrm{N}}, \mathcal{O}_{\mathbb{P}^{\mathrm{N}}}(1))$; then $\mathcal{O}_{\mathbb{P}^{\mathrm{N}}}(d)$ is n-regular for all $n \geq -d$.

Lemma 5.1.3. *[1, Lemma 3.2] If $\mathcal{E}$ is n-regular, then*

1. *$\mathcal{E}$ is m-regular for all $m \geq n$,*

2. *$H^i(\mathcal{E}(n)) = 0$ for all $i > 0$, hence $\dim H^0(\mathcal{E}(n)) = P(\mathcal{E}, n)$,*

3. *$\mathcal{E}(n)$ is globally generated, meaning that the natural evaluation map $\epsilon_n : H^0(\mathcal{E}(n)) \otimes \mathcal{O}_X(-n) \to \mathcal{E}$ is surjective,*

4. *the multiplication maps $H^0(\mathcal{E}(n)) \otimes H^0(\mathcal{O}_X(m-n)) \to H^0(\mathcal{E}(m))$ are surjective, for all $m \geq n$.*

Remark 5.1.4. Let $\mathcal{T} := \mathcal{O}_X(-n) \oplus \mathcal{O}_X(-m)$ and set $H := H^0(\mathcal{O}_X(m-n))$. Set $d := \dim H$, recall that a d-Kronecker module is a quiver representation of the d-Kronecker quiver $\boldsymbol{K}_d$. A d-Kronecker module can be identified with the data $V \otimes H \stackrel{\alpha}{\to} W$. The Kronecker module $V \otimes H \stackrel{\alpha}{\to} W$ corresponds bijectively to the data required to give $V \oplus W$ the structure of a (right) module for the algebra

$$A = \begin{pmatrix} \mathbf{k} & H \\ 0 & \mathbf{k} \end{pmatrix}$$

where $\mathbf{k}$ is the base field. Since $\mathcal{T}$ is (left) A-module it follows that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{F})$ is a (left) A-module. Thus we obtain a functor

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, -) : \mathrm{mod}_{\mathcal{O}_X} \to \mathrm{mod}_A .$$

Theorem 5.1.5. *[1, Theorem 3.4] Assume that $\mathcal{O}_X(m-n)$ is regular then the functor $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, -)$ is fully faithful on the full subcategory of n-regular sheaves. In other words, if $\mathcal{E}$ is an n-regular sheaf, then the natural evaluation map $\epsilon_{\mathcal{E}} : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{E}) \otimes_A \mathcal{T} \to \mathcal{E}$ is an isomorphism.*

Remark 5.1.6. Let $\mathcal{F}$ be an n-regular sheaf. We denote the Kronecker module associated to $\mathcal{F}$ by $K_{\mathcal{F}} := H^0(X, \mathcal{F}(n)) \otimes H^0(X, \mathcal{O}(m-n)) \stackrel{m_{\mathcal{F}}}{\to} H^0(X, \mathcal{F}(m))$. The following result by Brion states that the automorphism group of a Kronecker module or more generally, the automorphism group of a finite quiver representation in the category of finite dimensional $\mathbf{k}$-vector spaces is a connected linear algebraic group, and moreover a semi-direct of a unipotent group and product of general linear groups.

Proposition 5.1.7. *[13, Prop 2.2.1] Let $\boldsymbol{Q}$ be a finite quiver and M a finite-dimensional representation of $\boldsymbol{Q}$ in the category $\mathrm{vect}_{\mathbf{k}}$, where $\mathbf{k}$ is an algebraically closed field. Then*

1. *The automorphism group* $\mathrm{Aut}_{\boldsymbol{Q}}(M)$ *is an open affine subset of the connected linear algebraic monoid* $\mathrm{End}_{\boldsymbol{Q}}(M)$*. As a consequence,* $\mathrm{Aut}_{\boldsymbol{Q}}(M)$ *is a connected linear algebraic group.*

2. *There exists a decomposition*

$$\mathrm{Aut}_{\boldsymbol{Q}}(M) = U \rtimes \bigtimes_{i=1}^{r} \mathrm{GL}(m_i),$$

where U *is a closed normal unipotent subgroup and* $m_1, \ldots, m_r$ *denote the multiplicities of the indecomposable summands of* M.

Since every coherent sheaf over X is n-regular for $n >> 0$ by Serre's vanishing theorem, we obtain the following result

Corollary 5.1.8. *The automorphism group of every coherent sheaf on a connected projective scheme* X *over an algebraically closed field* $\mathbf{k}$ *is a linear algebraic group, which is a semi-direct product of a unipotent group with a product of general linear groups.*

Proof. Let $\mathcal{F}$ be an n-regular $\mathcal{O}_X$-module. The functor from Theorem 5.1.5 is fully faithful, that means $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{H}) \to \mathrm{Hom}_A\left(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{G}), \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T}, \mathcal{H})\right)$ is bijective for n-regular $\mathcal{O}_X$-modules $\mathcal{G}$ and $\mathcal{H}$. Since functors preserve isomorphisms and $\mathcal{F}$ is n-regular it follows that $\mathrm{Aut}(\mathcal{F}) \cong \mathrm{Aut}_A\left(\mathrm{Hom}(\mathcal{T}, \mathcal{F})\right)$. In any case $A = \mathbf{k}[\boldsymbol{K}_d]$ is the path algebra of the Kronecker quiver with $d := \dim H^0(\mathcal{O}_X(m-n))$ arrows. It is a classical result that we have a correspondence between (left)-modules over $\mathbf{k}[\boldsymbol{Q}]$ the path algebra of $\boldsymbol{Q}$ and $\boldsymbol{Q}$-representations in the category of $\mathbf{k}$-vector spaces. It follows that $\mathrm{Aut}(\mathcal{F}) \cong \mathrm{Aut}_A\left(\mathrm{Hom}(\mathcal{T}, \mathcal{F})\right) \cong \mathrm{Aut}(\mathcal{K}_{\mathcal{F}})$ thus by Proposition 5.1.7 the automorphism group is a semi-direct product of a unipotent group and a product of general linear groups. □

5.1.3 Grading subgroups

Let $\boldsymbol{Q}$ be a quiver and X a projective scheme of finite type over $\mathbf{k}$. We assume that each vertex sheaf $\mathcal{E}_v$ is a direct sum of simple coherent sheaves.

Definition 5.1.9. Let $\mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{E}_i^{n_i}$ be a semisimple coherent sheaf over X with simple summands

$\mathcal{E}_i$. We say that the sheaf $\mathcal{E}$ satisfies the *condition (⊾)*, if for all $1 \leq l < k \leq m$ we have

$$\mathrm{Hom}(\mathcal{E}_k, \mathcal{E}_l) = 0.$$

Example 5.1.10. Let $\mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^N}(-d_i)^{n_i}$ with $d_1 > d_2 > \ldots > d_m$. Then $\mathcal{E}$ satisfies the (⊾)-condition.

Remark 5.1.11. Let $\mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{E}_i^{n_i}$ be a coherent semisimple sheaf satisfying the (⊾)-condition. Then the automorphism group $\mathrm{Aut}(\mathcal{E})$ corresponds to the set of block matrices given by

$$\mathrm{Aut}(\mathcal{E}) \cong \left\{ \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix} \middle| \begin{array}{l} g_i \in \mathrm{Aut}\left(\mathcal{E}_i^{n_i}\right) \cong \mathrm{GL}_{n_i} \\ u_{j,i} \in \mathrm{Hom}\left(\mathcal{E}_i^{n_i}, \mathcal{E}_j^{n_j}\right) \end{array} \right\}.$$

In the following proposition, we will assume that each automorphism group $\mathrm{Aut}(\mathcal{E}_v)$ is a lower triangular block matrix.

Proposition 5.1.12. *Let $\boldsymbol{Q}$ be a quiver and $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ be a collection of coherent semisimple vertex sheaves over X satisfying the ⊾-condition. Let $H = \bigtimes_{v\in\boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$ and $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ be a 1-PS such that for each $v \in \boldsymbol{V}$ the induced 1-PS $\lambda_v : \mathbb{G}_{\mathrm{m}} \to H \to \mathrm{Aut}(\mathcal{E}_v)$ is given by*

$$t \mapsto \begin{pmatrix} t^{l_{v,1}}\,\mathrm{id} & 0 & \cdots & 0 \\ 0 & t^{l_{v,2}}\,\mathrm{id} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & t^{l_{v,r_v}}\,\mathrm{id} \end{pmatrix}.$$

with $l_{v,1} < l_{v,2} < \ldots < l_{v,r_v}$. Then λ grades the unipotent radical U of H.

Proof. The unipotent radical U of H is given by $U = \bigtimes_{v\in\boldsymbol{V}} U_v$ where U_v is the unipotent radical of $\mathrm{Aut}(\mathcal{E}_v)$. It is therefore enough to show that λ_v grades U_v the unipotent radical of $\mathrm{Aut}(\mathcal{E}_v)$. The 1-PS λ_v acts on $\mathrm{Lie}(U_v)$ with weights $l_{v,j} - l_{v,i}$ where $1 \leq i < j \leq r$. We conclude that λ_v grades U_v since $l_{v,1} < l_{v,2} < \ldots < l_{v,r_v}$, thus λ grades U as claimed. □

5.2 Moduli spaces for sheaf homomorphisms indexed by a quiver

As we saw in Chapter 1, the moduli functor $\mathsf{M}_{X,\boldsymbol{Q},[\mathcal{E}_v]_{v\in\boldsymbol{V}}}$ admits a family $\mathfrak{T}$ with the locus universal property over $W = \bigoplus_{a\in\boldsymbol{A}} W_a$ where $W_a := \operatorname{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$. Additionally, the action of $H := \bigtimes_{v\in\boldsymbol{V}} \operatorname{Aut}(\mathcal{E}_v)$ given by

$$(\alpha_v)_{v\in\boldsymbol{V}} \cdot (\varphi_a)_{a\in\boldsymbol{A}} := (\alpha_{t(a)} \circ \varphi_a \circ \alpha_{s(a)}^{-1})_{a\in\boldsymbol{A}}$$

satisfies the condition that two $\mathbf{k}$-points $s, t \in W$ belong to the same H-orbit if and only if $\mathfrak{T}_s \cong \mathfrak{T}_t$. Therefore we are lead to consider the problem of construction a quotient for an open H-invariant subset of W.

In the first subsection, we fix the quiver $\boldsymbol{Q} = \overset{1}{\circ} \to \overset{2}{\circ}$ and apply the affine $\hat{U}$-Theorem and indicate how the projective $\hat{U}$-Theorem can be applied.

The second subsection assumes that the quiver $\boldsymbol{Q}$ contains loops. In this context we apply the results from Section 4.1.

Remark 5.2.1. If we assume that the automorphism groups $\operatorname{Aut}(\mathcal{E}_v)$ of the vertex sheaves $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ are reductive. Then we are in a situation quite similar to King's moduli of finite dimensional algebras (see [44]). Thus we can apply Theorem 2.7.4. For the concrete discription of semistability based on King's Hilbert-Mumford criterion obtained by Drézet and Trautmann we refer the reader to Remark B.3.4.

5.2.1 Several approaches to moduli for sheaf homomorphisms

In this section, we consider the moduli problem for sheaf homomorphisms which is the moduli problem obtained from the moduli problem of sheaf homomorphisms indexed by a quiver by fixing the quiver $\boldsymbol{Q} = \overset{1}{\circ} \to \overset{2}{\circ}$. We fix semisimple locally free vertex sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ over a projective $\mathbf{k}$-scheme X of finite type. In the case of moduli of sheaf homomorphisms we are lead to consider the action

$$H = \mathrm{Aut}(\mathcal{E}_2) \times \mathrm{Aut}(\mathcal{E}_1) \curvearrowright W = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \tag{5.1}$$

$$(h_2, h_1) \cdot \varphi := h_2 \circ \varphi \circ h_1^{-1}. \tag{5.2}$$

To construct a quotient for the above action we consider the following approaches:

1. The approach of Drézet and Trautmann (see Section B),
2. the affine $\hat{U}$-Theorem (see Theorem 4.2.5),
3. the projective $\hat{U}$-theorem by Bérczi, Doran, Hawes and Kirwan (see Theorem 2.5.8 and [9]), by choosing a projective embedding of W.

The approach by Drézet and Trautmann

The approach of Drézet and Trautmann [25] is to transfer the non-reductive GIT-problem to a problem of reductive GIT. From the moduli problem of homomorphism of sheaves we obtain the linear algebraic group H together with a H-representation W. Drézet and Trautmann construct a reductive linear algebraic group G together with a rational G-representation V such that

1. H is an observable subgroup of G.
2. For an admissible character χ of H there exists an associated character χ' of G such that $\chi'|_H$ is a positive power of χ.
3. There exists a morphism $\zeta : W \to V$ which is equivariant relative to the closed immersion $H \to G$.
4. The morphism $\varphi : G \times W \overset{\mathrm{id}_G \times \zeta}{\to} G \times V \to V$ factorises via $G \times^H W$ and the morphism $G \times^H W \to V$ is injective.

In this set-up, the following inclusion

$$\zeta^{-1}(V^{ss(G,\chi')}) \subseteq W^{HM-ss(H,\chi)} = \bigcap_{h \in H} hW^{ss(R,\chi)}$$

is trivial. The converse inclusion is non-trivial. Some sufficient conditions for $\zeta^{-1}(V^{ss(G,\chi')}) \supseteq W^{HM-ss(H,\chi)}$ can be found at [25, Section 7].

Proposition 5.2.2. *[25, Proposition 6.1.1] Keep the above set-up.*

1. *If $\zeta^{-1}(V^{s(G,\chi')}) = W^{HM-s(H,\chi)}$, then there exists a geometric quotient*
$$W^{HM-s(H,\chi)} \to M,$$
with M a quasi-projective variety.

2. *If additionally*
$$\zeta^{-1}(V^{ss}(G,\chi')) = W^{HM-ss(H,\chi)} \text{ and } (\overline{\operatorname{Im}\varphi} \setminus \operatorname{Im}\varphi) \cap V^{ss(G,\chi')} = \emptyset,$$
then there exists a good quotient
$$\pi : W^{HM-ss(H,\chi)} \to P,$$
with P a normal projective variety and M is an open subset of P such that the geometric quotient $W^{HM-s(H,\chi)} \to M$ is the restriction of π to $W^{HM-s(H,\chi)}$.

Applying the affine $\hat{U}$-Theorem

In this subsection, we apply the affine $\hat{U}$-Theorem to the problem of homomorphisms of sheaves.

Theorem 5.2.3. *For the data $D = (\mathbb{P}^n, \circ \to \circ, (\mathcal{E}_v)_{v\in\{1,2\}})$ we can apply the affine $\hat{U}$-Theorem, if*
$$\mathcal{E}_1 = \mathcal{O}_{\mathbb{P}^n}(c_1)^{\oplus m_1} \oplus \mathcal{O}_{\mathbb{P}^n}(c_2)$$
and
$$\mathcal{E}_2 = \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2)^{\oplus m_2}$$
with $c_1 \leq c_2 < d_1 < d_2$. In particular by applying the affine $\hat{U}$-Theorem we can construct a projective over affine good quotient for an open subset of W.

Proof. We consider the action of $H := \mathrm{Aut}(\mathcal{E}_2) \times \mathrm{Aut}(\mathcal{E}_1)$ on $W := \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ given by

$$\begin{aligned} H = \mathrm{Aut}(\mathcal{E}_2) \times \mathrm{Aut}(\mathcal{E}_1) &\curvearrowright W = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) \\ &(h_2, h_1) \cdot \varphi := h_2 \circ \varphi \circ h_1^{-1}. \end{aligned}$$

We distinguish the following cases.

1. $c := c_1 = c_2$.

2. $c_1 < c_2$.

In the first case, we consider the action

$$\mathrm{Aut}(\mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2)^{m_2}) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^n}(c)^{m_1+1}) \curvearrowright \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(c)^{m_1+1}, \mathcal{O}_{\mathbb{P}^n}(d_1) \oplus \mathcal{O}_{\mathbb{P}^n}(d_2)^{m_2}).$$

To apply the affine $\hat{U}$-Theorem we have to construct an $\mathbb{N}$-indexed family of 1-PS in the sense of Definition 4.2.4 together with a suitable sequence of characters. We claim that the following sequence of 1-PS $\left(\lambda_l\right)_{l\in\mathbb{N}}$ is an $\mathbb{N}$-indexed family. For every $l \in \mathbb{N}$ and fixed natural numbers $k_1, k_2, k_3 \in \mathbb{N}$ satisfying the inequality $k_1 < k_3 < k_2$, we define the 1-PS

$$\begin{aligned} \lambda_l : \mathbb{G}_{\mathrm{m}} &\to H = \mathrm{Aut}(\mathcal{E}_2) \times \mathrm{Aut}(\mathcal{E}_1) \\ t &\mapsto \big(\mathrm{diag}(t_1^k, t^{k_2+l}\,\mathrm{Id}), t^{k_3}\,\mathrm{Id}\big) \end{aligned}$$

The 1-PS λ_l acts on W via

$$\lambda_l(t) \cdot \left(\begin{array}{c} C \\ \hline D \end{array}\right) = \left(\begin{array}{c|c} t^{k_1} & 0 \; \cdots \; 0 \\ \hline 0 & \\ \vdots & t^{k_2+l}\,\mathrm{Id} \\ 0 & \end{array}\right) \left(\begin{array}{c} C \\ \hline D \end{array}\right) \left(\; t^{k_3}\,\mathrm{Id} \; \right)^{-1}$$

with weight $\omega_1(\alpha_l) := k_1 - k_3$ on the block C and weight $\omega_2(\alpha_l) := k_2 + l - k_3$ on the block D. We conclude that the underlying vector spaces of the weight spaces given by $D = 0$ respectively $C = 0$ are independent of $l \in \mathbb{N}$. Thus $W^+_{\min} := \{v \in W | C_v \neq 0\}$ is independent of $l \in \mathbb{N}$. By the inequality $k_1 < k_3 < k_2$, it follows that

$$\omega_1 = \omega_1(\alpha_l) < 0 < \omega_2(\alpha_l) < \omega_2(\alpha_{l+1}).$$

We conclude that conditions (A1) and (A2) in Definition 4.2.4 are satisfied. By Proposition 5.1.12, it follows that for each $l \in \mathbb{N}$ that $\lambda_l(\mathbb{G}_\mathrm{m})$ grades the unipotent radical U. A point $v \in V$ belongs to $W_{\min}$ if the blocks $C_v = (x_1 \dots x_{m_1+1})$ and $D_v = (y_{i,j})_{\substack{1\le i\le m_1+1\\ 1\le j\le m_2}}$ satisfy $C_v \neq 0$ and $D_v = 0$. The unipotent radical acts via

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline u_1 & & & \\ \vdots & & \mathrm{Id} & \\ u_{m_2} & & & \end{array}\right) \cdot \left(\begin{array}{ccc} & C_v & \\ \hline 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{array}\right)$$

It follows that $\mathrm{Stab}_U(v) = \{e\}$ for each $v \in W_{\min}$ since $C_v \neq 0$. Thus also condition (A3) holds.

To construct a family of characters adapted to the $\mathbb{N}$-indexed family of 1-PS $(\lambda_l)_{l\in\mathbb{N}}$, let r be a natural number such that

$$r > \frac{k_2}{k_3 - k_1}.$$

We define the associated family of characters

$$\chi_l : H \to \mathbb{G}_\mathrm{m}$$
$$\left(\left(\begin{array}{c|c} A & 0 \\ \hline U & B \end{array}\right), C\right) \mapsto \det(A)^{s_{1,l}} \det(B)^{s_{2,l}} \det(C)^{s_{3,l}}.$$

for integers

$$
\begin{aligned}
s_{1,l} &:= (m_1+1)m_2(l+r) \\
s_{2,l} &:= m_1+1 \\
s_{3,l} &:= -m_2(l+r+1).
\end{aligned}
$$

We claim that the characters χ_l are chosen in such a way that the 1-PS $\Delta : \mathbb{G}_\mathrm{m} \to H$ given by $t \mapsto (t\,\mathrm{id}_{\mathcal{E}_2}, t\,\mathrm{id}_{\mathcal{E}_1})$ satisfies $\langle \chi_l, \Delta\rangle = 0$. We have

$$
\begin{aligned}
\langle \chi_l, \Delta\rangle &= s_{1,l} + m_2 s_{2,l} + (m_1+1)s_{3,l} \\
&= (m_1+1)m_2(l+r) + m_2(m_1+1) - (m_1+1)(l+r+1) = 0
\end{aligned}
$$

thus $\langle \chi_l, \Delta\rangle = 0$ as claimed. It remains to show for each $l \in \mathbb{N}$ that $\langle \chi_l, \lambda_l\rangle < 0$. From the definition of λ_l and χ_l it follows that

$$
\begin{aligned}
\langle \chi_l, \lambda_l\rangle &= k_1 s_{1,l} + m_2(k_2+l)s_{2,l} + (m_1+1)k_3 s_{3,l} \\
&= (m_1+1)m_2\big((l+r)(k_1-k_3) + k_2 + l - k_3\big).
\end{aligned}
$$

Since $(m_1+1)m_2 > 0$, it is enough to show that $(l+r)(k_1-k_3)+k_2+l-k_3 < 0$. From $l(k_1-k_3)+l \leq 0$ and $r(k_1-k_3)+k_2 < \frac{k_2}{k_3-k_1}(k_1-k_3)+k_2 = -k_2+k_2 = 0$ it follows that

$$
(l+r)(k_1-k_3) + k_2 + l - k_3 \leq -k_3 < 0
$$

thus $\langle \chi_l, \lambda_l\rangle < 0$ as claimed. Furthermore each $x \in W^+_{\min}$ has a trivial U-stabiliser where U is the unipotent radical of H. Thus we can apply the affine $\hat{U}$-Theorem.

In the second case given by $c_1 < c_2$, we need to show that there exists an $\mathbb{N}$-indexed family of 1-PS $(\lambda_l : \mathbb{G}_\mathrm{m} \to H)_{l\in\mathbb{N}}$ in the sense of Definition 4.2.4 together with a family of characters $(\chi_l : H \to \mathbb{G}_\mathrm{m})_{l\in\mathbb{N}}$ such that for each $l \in \mathbb{N}$ the inequality $\langle \chi_l, \lambda_l\rangle < 0$ holds. Elements of H are given by a tuple of matrices

$$\left(\left(\begin{array}{c|ccc} g_1 & 0 & \cdots & 0 \\ \hline u_1 & & & \\ \vdots & & g_2 & \\ u_r & & & \end{array}\right), \left(\begin{array}{ccc|c} & & & 0 \\ & g_3 & & \vdots \\ & & & 0 \\ \hline v_1 & \cdots & v_s & g_4 \end{array}\right)\right),$$

where $g_1, g_4 \in \mathbb{G}_{\mathrm{m}},\ g_2 \in \mathrm{GL}(m_2, \mathbf{k}),\ g_3 \in \mathrm{GL}(m_1, \mathbf{k}),\ u_i \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(d_1), \mathcal{O}_{\mathbb{P}^n}(d_2))$ and $v_i \in \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^n}(c_1), \mathcal{O}_{\mathbb{P}^n}(c_2))$. We define for each $l \in \mathbb{N}$ and fixed integers $k_1, \ldots, k_4 \in \mathbb{Z}$ satisfying the inequality $k_3 < k_1 < k_4 < k_2$ the 1-PS

$$\begin{aligned} \lambda_l : \mathbb{G}_{\mathrm{m}} &\to H \\ t &\mapsto \big(\operatorname{diag}(t^{k_{1,l}}, t^{k_{2,l}} \operatorname{Id}), \operatorname{diag}(t^{k_{3,l}} \operatorname{Id}, t^{k_{4,l}})\big) \end{aligned}$$

where $k_{1,l} := k_1 + l, k_{2,l} := k_2 + 2l, k_{3,l} := k_3, k_{4,l} := k_4 + l$. The λ_l action on W has the weights $\omega_1(\alpha_l) < \omega_2(\alpha_l) < \omega_3(\alpha_l) < \omega_4(\alpha_l)$ with

$$\begin{aligned} \omega_1(\alpha_l) &= k_{1,l} - k_{4,l} = k_1 - k_4, \\ \omega_2(\alpha_l) &= \min\{k_{1,l} - k_{3,l}, k_{2,l} - k_{4,l}\} = \min\{k_1 - k_3, k_3 - k_4\} + l, \\ \omega_3(\alpha_l) &= \max\{k_{1,l} - k_{3,l}, k_{2,l} - k_{4,l}\} = \min\{k_1 - k_3, k_3 - k_4\} + l, \\ \omega_4(\alpha_l) &= k_{2,l} - k_{3,l} = k_2 - k_3 + l. \end{aligned}$$

We conclude for each $l \in \mathbb{N}$ that $\omega_1(\alpha_l) < 0 < \omega_2(\alpha_l)$. By Proposition 5.1.12, it follows that λ_0 grades the unipotent radical of H. It remains to show that for each $M \in W_{\min}$ we have that $\mathrm{Stab}_U(M) = \{e\}$. A matrix M belongs to $W_{\min}$ if all entries except the last entry of the first row are zero. To see that $\mathrm{Stab}_U(M) = \{e\}$ consider

$$\left(\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline u_1 & & & \\ \vdots & & \mathrm{Id} & \\ u_r & & & \end{array}\right)\left(\begin{array}{ccc|c} 0 & \cdots & 0 & x \\ \hline 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{array}\right)\left(\begin{array}{ccc|c} & & & 0 \\ & \mathrm{Id} & & \vdots \\ & & & 0 \\ \hline -v_1 & \cdots & -v_s & 1 \end{array}\right) = \left(\begin{array}{ccc|c} -v_1x & \cdots & -v_sx & x \\ \hline -v_1xu_1 & \cdots & -v_sxu_1 & u_1x \\ \vdots & \ddots & \vdots & \vdots \\ -v_1xu_r & \cdots & -v_sxu_r & u_rx \end{array}\right)$$

since $x \neq 0$ it follows that $\mathrm{Stab}_U(M) = \{e\}$. We conclude that $(\lambda_l)_{l\in\mathbb{N}}$ defines an $\mathbb{N}$-indexed family of 1-PS.

To define a suitable sequence of characters, recall that any character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ is of the form

$$\chi : H \to \mathbb{G}_{\mathrm{m}}$$
$$\left(\left(\begin{array}{c|c} A & 0 \\ \hline U & B \end{array}\right), \left(\begin{array}{c|c} C & 0 \\ \hline V & D \end{array}\right)\right) \mapsto \det(A)^{r_1}\det(B)^{r_2}\det(C)^{r_3}\det(D)^{r_4}.$$

for integers $r_1, \ldots, r_4$. We construct a family of characters $\chi_l : H \to \mathbb{G}_{\mathrm{m}}$ by choosing integers $r_{1,l}, r_{2,l}, r_{3,l}, r_{4,l} \in \mathbb{Z}$ such that the following two conditions are satisfied

1. The pairing $\langle \chi_l, \lambda_l \rangle$ corresponds to a negative integer.

2. The 1-PS $\Delta : \mathbb{G}_{\mathrm{m}} \to H$ given by $\Delta(t) = (t\,\mathrm{id}_{\mathcal{E}_2}, t\,\mathrm{id}_{\mathcal{E}_1})$ acts trivially on W, so we assume for each $l \in \mathbb{N}$ that $\langle \chi_l, \Delta \rangle = 0$.

Explicitly, these two conditions translate to

1. $\sum\limits_{i=1}^{4} k_{i,l} r'_{i,l} < 0$

2. $\sum\limits_{i=1}^{4} r'_{i,l} = 0$

where $r'_l = \mathrm{diag}(1, m_1, m_2, 1) r_l$.

To show that such a family of characters exists, we will add the simplifying assumption

$$r'_{4,l} = -r'_{1,l}$$
$$r'_{3,l} = -r'_{2,l}$$

We conclude that condition 2 is satisfied. It remains to show that

$$\sum_{i=1}^{4} k_{i,l} r'_{i,l} < 0$$

We claim that for $r'_{1,l} >> r'_{2,l} > 0$ the conditions can be satisfied. We have

$$\begin{aligned} k_{1,l}r'_{1,l} + k_{2,l}r'_{2,l} + k_{3,l}r'_{3,l} + k_{4,l}r'_{4,l} &= (k_1 + l)r'_{1,l} + (k_2 + 2l)r'_{2,l} - (k_3)r'_{2,l} - (k_4 + l)r'_{1,l} \\ &= (k_1 - k_4)r'_{1,l} + (k_2 - k_3 + 2l)r'_{2,l} \end{aligned}$$

where $(k_1 - k_4)r_{1,l} \leq -r_{1,l} < 0$ and $(k_2 - k_3 + 2l)r'_{2,l} > 0$. Let

$$r'_{1,l} > (k_2 - k_3 + 2l)r'_{2,l}$$

then $\langle \chi_l, \lambda_l \rangle < 0$. We conclude that there exists an $\mathbb{N}$-indexed family of 1-PS together with a family of characters such that conditions of the affine $\hat{U}$-Theorem (Theorem 4.2.5) are satisfied. □

Remark 5.2.4.

1. Keeping the assumptions from Theorem 5.2.3. We could alternatively apply the projective $\hat{U}$-Theorem. To apply the projective $\hat{U}$-Theorem we need to replace W by $\mathbb{P}(\mathbf{k} \times W)$ via the H-equivariant morphism

$$\begin{aligned} V &\to \mathbb{P}(\mathbf{k} \times W) \\ v &\mapsto [1 : v]. \end{aligned}$$

   Finally, it is necessary to find a suitable linearisation of the H-action on $X := \mathbb{P}(\mathbf{k} \times W)$ and check that the unipotent radical of H acts set-theoretically free on the locus $X^0_{\min}$. By applying the projective $\hat{U}$-Theorem we obtain a projective good H-quotient, while the affine $\hat{U}$-Theorem only implies that the H-quotient is projective over affine.

2. Additionally, if we consider the moduli problem of sheaf homomorphisms indexed by a quiver neither the projective nor the affine $\hat{U}$-Theorem can be applied since the condition on trivial stabilisers for the unipotent radical is almost never satisfied. Potentially the variants of the

projective $\hat{U}$ -Theorem, which require a blow-up procedure could be applied in these cases.

5.2.2 Moduli for homomorphisms of sheaves indexed by a quiver with loops

In this section, we consider a quiver $\boldsymbol{Q}$ with at least one loop ℓ. Recall that a loop ℓ is an arrow whose source $s(\ell)$ and target $t(\ell)$ coincide. Let X be a projective scheme of finite type over $\mathbf{k}$. Additionally, we only consider collections of coherent semisimple sheaves $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ over X such that each vertex sheaf satisfies the (◺)-condition in the sense of Definition 5.1.9.

Let $H := \bigtimes_{v\in\boldsymbol{V}} \operatorname{Aut}\left(\mathcal{E}_v\right)$ and $W = \bigoplus_{a\in\boldsymbol{A}} W_a$ where $W_a := \operatorname{Hom}\left(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)}\right)$. The action $H \times W \to W$ given by

$$(\alpha_v)_{v\in\boldsymbol{V}} \cdot (\varphi_a)_{a\in\boldsymbol{A}} := (\alpha_{t(a)} \circ \varphi_a \circ \alpha_{s(a)}^{-1})_{a\in\boldsymbol{A}}$$

turns W into an affine H-variety. Furthermore, let U denote the unipotent radical of H. To apply Theorem 4.1.9 to the affine H-variety W we need at least one H-invariant section $\sigma \in I^{ssu(H,U)} \subset \mathcal{O}(W)^H$. Recall that $\sigma \in \mathcal{O}(W)^H$ belongs to $I^{ssu(H,U)}$ if $W_\sigma \to \operatorname{Spec}\mathcal{O}(W_\sigma)^U$ is a trivial principal U-bundle. In general, it is hard to determine the locus $W^{ssu(H,U)}$; instead we are going to define the Sylvester locus $W_{\mathrm{Syl}} \subset W^{ssu(H,U)}$ and prove that in some cases we obtain that $W^{ssu(H,U)} = W_{\mathrm{Syl}}$. To define the Sylvester locus we will crucially use the fact that our quiver $\boldsymbol{Q}$ has at least one loop. Before we proceed with the construction of the Sylvester locus, we will recall two results due to Sylvester, the so-called Sylvester equation and the resultant of two univariate polynomials p and q.

Definition 5.2.5. Consider $M \in \operatorname{Mat}(n\times n, \mathbf{k})$, $N \in \operatorname{Mat}(m \times m, \mathbf{k})$ and $R \in \operatorname{Mat}(n\times m, \mathbf{k})$ then $X \in \operatorname{Mat}(n \times m, \mathbf{k})$ is a solution of the *Sylvester equation* given by M, N and R if $MX - XN = R$.

Proposition 5.2.6. *[6, Sylvester–Rosenblum Theorem] For M, N, R as above, the Sylvester equation $MX - XN = R$ has a unique solution $X \in \operatorname{Mat}(n\times m, \mathbf{k})$ if and only if the characteristic polynomials of M and N have no common zero.*

Definition 5.2.7. Let B be a $\mathbf{k}$-algebra and let $p, q \in B[t]$ be two polynomials represented by $p = \sum_{i=0}^{n} a_i t^{n-i}$ and $q = \sum_{i=0}^{m} b_i t^{m-i}$ such that the leading coefficients a_n and b_m are both non-zero. The *resultant* of p and q denoted by $\operatorname{res}(p, q)$ is defined as the determinant of $(m + n) \times (m + n)$-

Sylvester-matrix $S_{p,q}$ associated to the polynomials p and q. Where $S_{p,q}$ is given by

$$S_{p,q} := \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & \cdots & 0 \\ a_1 & a_0 & \ddots & \vdots & b_1 & b_0 & \ddots & \vdots \\ \vdots & a_1 & \ddots & 0 & \vdots & b_1 & \ddots & 0 \\ a_{n-1} & \vdots & \ddots & a_0 & b_{m-1} & & \ddots & b_0 \\ a_n & a_{n-1} & & a_1 & b_m & b_{m-1} & & b_1 \\ 0 & a_n & \ddots & \vdots & 0 & b_m & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_{n-1} & \vdots & \ddots & \ddots & b_{m-1} \\ 0 & \cdots & 0 & a_n & 0 & \cdots & 0 & b_m \end{pmatrix}.$$

Lemma 5.2.8. *Assume that $B = \mathbf{k}$. The polynomials $p, q \in \mathbf{k}[t]$ have a common factor if and only if* $\operatorname{res}(p, q) = 0$.

Definition 5.2.9. Let $\boldsymbol{Q}$ be a quiver with loops $\ell_1, \dots, \ell_k$ at the vertices $v_1, \dots, v_k$ and X be a projective scheme of finite type over $\mathbf{k}$. We fix semisimple vertex sheaves $(\mathcal{E}_v)_{v \in \boldsymbol{V}}$ with each $\mathcal{E}_v \in \mathbf{Coh}(X)$. Fix a loop ℓ, as stated in the begin of the section the corresponding vertex sheaf $\mathcal{E}_\ell := \mathcal{E}_{s(\ell)} = \bigoplus_{i=1}^{m_\ell} \mathcal{E}_{\ell,i}^{\oplus n_{\ell,i}}$ satisfies the ($\triangleright$)-condition. Thus any endomorphism $\varphi_\ell \in W_\ell := \operatorname{End}\big(\mathcal{E}_\ell\big)$ corresponds to a lower triangular block matrix of the form

$$\left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline V_{2,1} & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline V_{n_\ell,1} & \cdots & V_{n_\ell,n_\ell-1} & A_{n_\ell} \end{array}\right)$$

where $A_i \in \operatorname{Hom}\big(\mathcal{E}_{\ell,i}^{\oplus n_{\ell,i}}, \mathcal{E}_{\ell,i}^{\oplus n_{\ell,i}}\big) \cong \operatorname{Mat}(n_{\ell,i} \times n_{\ell,i}, \mathbf{k})$ and $V_{j,i} \in \operatorname{Hom}\big(\mathcal{E}_i^{\oplus n_{\ell,i}}, \mathcal{E}_j^{\oplus n_{\ell,j}}\big)$. We define

$$\hat{\sigma}_\ell := \prod_{(i,j) \in J_\ell} \operatorname{res}(\chi_{\ell,i}, \chi_{\ell,j}) : W_\ell \to \mathbb{A}^1$$

where $J_\ell := \{(i,j) | 1 \le i < j \le m_\ell \text{ and } \operatorname{Hom}(\mathcal{E}_{\ell,i}, \mathcal{E}_{\ell,j}) \neq 0\}$ and $\chi_{\ell,i}(t) \in \mathcal{O}\big(\operatorname{End}(\mathcal{E}_{\ell,i}^{\oplus n_{\ell,i}})\big)[t]^{\mathrm{GL}(n_{\ell,i},\mathbf{k})} \subset \mathcal{O}(V)[t]^H$ is the characteristic polynomial. More generally, for given loops $\ell_{i_1}, \dots, \ell_{i_r}$ we define a *pre-Sylvester section*

$$\sigma_{\ell_{i_1},\ldots,\ell_{i_r}} := \prod_{j=1}^{r} \hat{\sigma}_{\ell_{i_j}} \circ \mathrm{pr}_{\ell_{i_j}}$$

where $\mathrm{pr}_{\ell_{i_j}} : W \to W_{\ell_{i_j}}$ is the projection to $W_{\ell_{i_j}} = \mathrm{Hom}\left(\mathcal{E}_{s(\ell_{i_j})}, \mathcal{E}_{t(\ell_{i_j})}\right) = \mathrm{End}\left(\mathcal{E}_{\ell_{i_j}}\right)$.

We call a pre-Sylvester section $\sigma_{\ell_{i_1},\ldots,\ell_{i_r}}$ a *Sylvester section* if the action $H \times W \to W$ restricted to $U \times W_{\sigma_{\ell_{i_1},\ldots,\ell_{i_r}}} \to W_{\sigma_{\ell_{i_1},\ldots,\ell_{i_r}}}$ is set-theoretically free.

Example 5.2.10. Let $X = \mathbb{P}^n$, and $\boldsymbol{Q} = \begin{tikzcd} \overset{1}{\circ} \arrow[loop left, "\ell_1"] \arrow[r, "\alpha"] & \overset{2}{\circ} \arrow[loop right, "\ell_2"] \end{tikzcd}$. We fix the vertex sheaves

$$\mathcal{E}_1 = \mathcal{E}_{\ell_1} := \bigoplus_{i=1}^{m_{\ell_1}} \mathcal{O}_X(c_i)^{n_{\ell_1,i}} \text{ and } \mathcal{E}_2 = \mathcal{E}_{\ell_2} := \bigoplus_{j=1}^{m_{\ell_2}} \mathcal{O}_X(d_j)^{n_{\ell_2,i}}$$

for natural numbers $n_{\ell_1,i}, n_{\ell_2,i}, m_{\ell_1,i}, m_{\ell_2,i}, c_i, d_j \in \mathbb{N} \setminus \{0\}$ such that $c_1 < \ldots < c_n < d_1 < \ldots < d_m$ and $m_{\ell_1}, m_{\ell_2} \geq 2$. There are three pre-Sylvester sections $\sigma_{\ell_1}, \sigma_{\ell_2}$ and σ_{ℓ_1,ℓ_2}. It is easy to see, that σ_{ℓ_1,ℓ_2} is only Sylvester section.

Proposition 5.2.11. *Given the action of*

$$H := \bigtimes_{v \in \boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$$

on

$$W := \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$$

Then any pre-Sylvester section $\sigma_{\ell_{i_1},\ldots,\ell_{i_r}}$ defines a H-invariant regular function on W.

Proof. By definition, we have that $\sigma_{\ell_{i_1},\ldots,\ell_{i_r}} = \prod_{j=1}^{r} \hat{\sigma}_{\ell_{i_j}} \circ \mathrm{pr}_{\ell_{i_j}}$, thus it is enough to show that $\hat{\sigma}_\ell \circ \mathrm{pr}_\ell$ is H-invariant function for $\ell \in \{\ell_{i_1}, \ldots, \ell_{i_r}\}$. The projection $\mathrm{pr}_\ell : W \to W_\ell$ is equivariant relative to $H \to \mathrm{Aut}(\mathcal{E}_\ell)$. It is therefore enough to show that $\hat{\sigma}_\ell$ is an $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant function. By definition $\hat{\sigma}_\ell$ is a product of resultants $\mathrm{res}(\chi_{\ell,i}, \chi_{\ell,j})$. Since a product of $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant functions is a $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant function, it is enough to show that the function $\mathrm{res}(\chi_{\ell,i}, \chi_{\ell,j})$ is $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant. To see that the resultant $\mathrm{res}(\chi_{\ell,i}, \chi_{\ell,j})$ is $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant, we consider the action of $\mathrm{Aut}(\mathcal{E}_\ell)$ on

$\mathrm{End}(\mathcal{E}_\ell)$ by conjugation

$$\left(\begin{array}{c|c|c|c} M_1 & 0 & \cdots & 0 \\ \hline U_{2,1} & M_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & M_n \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline V_{2,1} & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline V_{n,1} & \cdots & V_{n,n-1} & A_n \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} M_1 & 0 & \cdots & 0 \\ \hline U_{2,1} & M_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & M_n \end{array}\right)^{-1}.$$

Note that the diagonal entries are given by $M_iA_iM_i^{-1}$. It follows that the coefficients of the characteristic polynomial $\chi_{X_i} \in \mathcal{O}(W_\ell)[t]$ are $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant. We conclude that the function $\mathrm{res}(\chi_{\ell,i}, \chi_{\ell,j}) \in \mathcal{O}(W_\ell)$ is $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant. It follows that $\hat{\sigma}_\ell = \prod_{1\le i<j\le n_\ell} \mathrm{res}(\chi_{\ell,i}, \chi_{\ell,j})$ is $\mathrm{Aut}(\mathcal{E}_\ell)$-invariant and $\hat{\sigma}_\ell \circ \mathrm{pr}_\ell$ is H-invariant. We conclude that $\sigma_{\ell_{i_1},\ldots,\ell_{i_k}}$ is H-invariant function. □

Definition 5.2.12. We consider a finite quiver $\boldsymbol{Q}$ with loops $\ell_1, \ldots, \ell_k$ and semisimple vertex sheaves $\mathcal{E}_v$ which are coherent over X a projective scheme of finite type over $\mathbf{k}$. We consider the associated H-action on W and define the *Sylvester locus* by

$$W_{\mathrm{Syl}} := \bigcup_{\sigma\in S} W_\sigma$$

where S is the finite set of Sylvester sections.

Proposition 5.2.13. *Let $\boldsymbol{Q}$ be a quiver with loops and $(\mathcal{E}_v)_{v\in \boldsymbol{V}}$ be a collection of vertex sheaves satisfying condition (⊾). Suppose that ℓ is a loop in the quiver $\boldsymbol{Q}$ and let U_ℓ be the unipotent radical of* $\mathrm{Aut}(\mathcal{E}_\ell)$. *Then the action of U_ℓ on W_{σ_ℓ} induced by the inclusion $U_\ell \to \mathrm{Aut}(\mathcal{E}_\ell) \to H$ is set-theoretically free.*

Proof. We show that U_ℓ acts set-theoretically free on W_{σ_ℓ}. The group U_ℓ acts via conjugation on $\mathrm{End}(\mathcal{E}_\ell) \subset W$. We have an H-equivariant projection

$$\mathrm{pr}_\ell : W_{\sigma_\ell} \to \mathrm{End}(\mathcal{E}_\ell)_{\sigma_\ell}.$$

It follows that $\mathrm{Stab}_{U_\ell}(x) \subset \mathrm{Stab}_{U_\ell}(\mathrm{pr}_\ell(x))$; thus it is enough to show that U_ℓ acts set-theoretically free on $\mathrm{End}(\mathcal{E}_\ell)_{\sigma_\ell}$. Note that the action of U_ℓ extends to a graded $U_\ell \rtimes \mathbb{G}_{\mathrm{m}}$-action via

$$((U_{i,j})_{1\leq j<i\leq n}, t) \mapsto \left(\begin{array}{c|c|c|c} t\,\mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & t^2\,\mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & t^n\,\mathrm{id} \end{array}\right).$$

By Corollary 3.2.4, it is enough to show that U_ℓ acts set-theoretically free on the $\mathbb{G}_{\mathrm{m}}$-fixed locus $\mathrm{End}(\mathcal{E}_\ell)_{\sigma_\ell}^{\mathbb{G}_{\mathrm{m}}}$ which is given by matrices of the form $\mathrm{diag}(A_1, \dots, A_n)$. We act by conjugation on such a matrix via

$$\left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & A_n \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right)^{-1}.$$

Let $A \in \mathrm{End}(\mathcal{E}_\ell)_{\sigma_\ell}^{\mathbb{G}_{\mathrm{m}}}$. Note that $U \in \mathrm{Stab}_{U_\ell}(A)$ if $UAU^{-1} = A$ which is equivalent to A satisfies the equation $UA = AU$. In terms of the entries $U_{i,j}$ we obtain the Sylvester equation $U_{i,j}A_j - A_iU_{i,j} = 0$. If $\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i) = 0$ then $U_{i,j} = 0$ and there is nothing to show. Otherwise if $\mathrm{Hom}(\mathcal{E}_j, \mathcal{E}_i) \neq \{0\}$ we have that $U_{i,j} = 0$ is a solution to the Sylvester equation. Since $A \in \mathrm{End}(\mathcal{E}_\ell)_{\sigma_\ell}$ it follows that the eigenvalues of A_i and A_j are distinct, thus $U_{i,j} = 0$ is the unque solution. To see the uniqueness, consider a basis $s_1, \dots, s_N$ of $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j)$. We write $U_{i,j} = \sum_{l=1}^{N} s_l U_{i,j,l}$ and each Sylvester equation $U_{i,j,l}A_j - A_iU_{i,j,l} = 0$ has the unique solution $U_{i,j,l} = 0$, since A_i and A_j have no common eigenvalues, which implies that $U_{i,j} = 0$. We conclude that U_ℓ acts set-theoretically free on W_{σ_ℓ} □

Corollary 5.2.14. *Let Q be the loop quiver and $\mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{E}_i^{n_i}$ be a semisimple sheaf satisfying the condition (⊾). Then $x \in \mathrm{End}(\mathcal{E})$ satisfies $\mathrm{Stab}_U(x) = \{e\}$ if and only if $x \in \mathrm{End}(\mathcal{E})_\sigma$ where $\sigma = \sigma_\ell$ is the unique Sylvester section of the loop quiver.*

Proof. By the condition (⊾), it follows that either $\sigma_\ell = 1$ and for each $1 \leq i < j \leq m$ we have $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j) = 0$ or there exist $1 \leq i_0 < j_0 \leq m$ such that $\mathrm{Hom}(\mathcal{E}_{i_0}, \mathcal{E}_{j_0}) \neq 0$. In the first case the unipotent radical is trivial and there is nothing to show. We assume in the following that the unipotent radical U of $\mathrm{Aut}(\mathcal{E})$ is non-trivial. The $\mathrm{Aut}(\mathcal{E})$-action on $\mathrm{End}(\mathcal{E})$ restricted to U is given by

$$\left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline V_{2,1} & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline V_{n,1} & \cdots & V_{n,n-1} & A_n \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right)^{-1}.$$

We saw in the proof of Proposition 5.2.13, that the U-action on W can be extended to a graded action via a 1-PS $\lambda_g : \mathbb{G}_\mathrm{m} \to \mathrm{Aut}(\mathcal{E})$ on $\mathrm{End}(\mathcal{E})$. For a given $x \in W$, it is enough to consider the associated fixed point $x' := \lim_{t \to 0} \lambda_g(t)x$, since $\dim \mathrm{Stab}_U(x) \leq \dim \mathrm{Stab}_U(x')$ by Proposition 3.2.3. For the λ_g-fixed point x' the entries $V_{i,j}$ are all equal to zero, thus we are considering

$$\left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} A_1 & 0 & \cdots & 0 \\ \hline 0 & A_2 & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & A_n \end{array}\right) \cdot \left(\begin{array}{c|c|c|c} \mathrm{id} & 0 & \cdots & 0 \\ \hline U_{2,1} & \mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline U_{n,1} & \cdots & U_{n,n-1} & \mathrm{id} \end{array}\right)^{-1}.$$

We conclude as in the proof of Proposition 5.2.13, that x' has a trivial U-stabiliser if and only if every Sylvester equation $U_{i,j}A_j - A_iU_{i,j} = 0$ is uniquely solvable by $U_{i,j} = 0$. We conclude that $\mathrm{Stab}_U(x') = \{e\}$ if and only if $x' \in W_\sigma$. By Proposition 5.2.11, σ is an H-invariant function thus if $\sigma(x') \neq 0$ then also $\sigma(x) = \sigma(\lim_{t\to 0} \lambda_g(t)x) = \sigma(x') \neq 0$. We conclude that $\mathrm{Stab}_U(x) = \{e\}$ if and only if $x \in W_\sigma$. □

Theorem 5.2.15. *Let $\boldsymbol{Q}$ be a finite quiver without directed cycles of length at least 2. Fix a collection of semisimple vertex sheaves $(\mathcal{E}_v)_{v \in \boldsymbol{V}}$ satisfying the condition (⊾) and a character $\chi : H \to \mathbb{G}_\mathrm{m}$. Then we obtain a good H-quotient for the open subset $W_{\mathrm{Syl}} \cap W^{HM-ss(H,\chi)}$ and a geometric quotient for $W_{\mathrm{Syl}} \cap W^{HM-s(H,\chi)}$.*

Proof. We want to apply Corollary 4.1.10 to the locus W_{Syl}. Since W admits only finitely many Sylvester sections, it is enough to show that any Sylvester section belongs to $I^{ssu(H,U)}$ (see Definition 4.1.8). We already saw that any Sylvester section is H-invariant. It remains to show for a

Sylvester section σ that $W_\sigma \to W_\sigma/U$ is a trivial principal U-bundle.

We will show in the following lemma that the action of H on W is graded, thus W_σ is a graded H-variety and by the definition of a Sylvestersection it follows that the unipotent radical U acts set-theoretically freely on X_σ. We conclude that Lemma 3.3.1 applies, thus $W_\sigma \to W_\sigma/U$ is a trivial principal U-bundle. $\square$

Lemma 5.2.16. *Let $\boldsymbol{Q}$ be a quiver without directed cycles of length at least 2 and $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ be a collection of semisimple vertex sheaves satisfying condition (⊾). Then the H-action on W is graded.*

Proof. Let $\boldsymbol{Q}'$ be the quiver obtained from $\boldsymbol{Q}$ by removing the set of loops $\{a \in \boldsymbol{A} | s(a) = t(a)\}$. By our assumption $\boldsymbol{Q}$ does not contain a cycle of length at least two, thus since $\boldsymbol{Q}'$ is obtained from $\boldsymbol{Q}$ by removing the cycles of length one, it follows that $\boldsymbol{Q}'$ is a acyclic.

It follows that $\boldsymbol{V}'_{\text{source}} := \boldsymbol{V} \setminus \{v \in \boldsymbol{V} | \exists a \in \boldsymbol{A}' : t(a) = v\}$ is non-empty. Otherwise we could construct a path in $\boldsymbol{Q}'$ of arbitrary length which implies that the quiver $\boldsymbol{Q}'$ has an infinite vertex set or contains a cycle which contradicts our assumptions.

We claim that there exists a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ such that

1. The image of $\lambda : \mathbb{G}_{\mathrm{m}} \to H \to H/U$ lies in the centre of H/U.
2. The action of $\lambda(\mathbb{G}_{\mathrm{m}})$ on the unipotent radical U of H has only positive weights.
3. The action of $\lambda(\mathbb{G}_{\mathrm{m}})$ on W has only non-negative weights.

We prove the existence of such a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ inductively on the number of vertices. Since $\boldsymbol{Q}'$ is acyclic, we need at least two distinct vertices in the quiver to have any arrows. So assume that $\boldsymbol{Q}'$ has exactly two vertices then $\boldsymbol{Q}' = \boldsymbol{K}_n$ is a Kronecker quiver and the weights of each arrow space W_a are the same so we can assume without loss of generality that $\boldsymbol{Q}' = K_1$. The automorphism groups can be assumed to be given in lower triangular block matrix form and hence the unipotent radical is graded, by a 1-PS $\lambda : \mathbb{G}_{\mathrm{m}} \to H$ of the following form

$$\lambda(t) = \left(\left(\begin{array}{c|c|c|c} t^{l_{1,t(a)}}\,\mathrm{id} & 0 & \cdots & 0 \\ \hline 0 & t^{l_{2,t(a)}}\,\mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & t^{l_{k_{t(a)},t(a)}}\,\mathrm{id} \end{array}\right), \left(\begin{array}{c|c|c|c} t^{l_{1,s(a)}}\,\mathrm{id} & 0 & \cdots & 0 \\ \hline 0 & t^{l_{2,s(a)}}\,\mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & t^{l_{k_{s(a)},s(a)}}\,\mathrm{id} \end{array}\right) \right)$$

if the block weights $l_{i,v}$ satisfy $l_{i,v} < l_{i+1,v}$ for $v \in \boldsymbol{V} = \{s(a), t(a)\}$ and $l_{1,t(a)} \geq l_{k_{s(a)},s(a)}$. The weights with which this 1-PS λ acts on W are $l_{i,t(a)} - l_{j,s(a)}$ thus the condition $l_{1,t(a)} \geq l_{k_{s(a)},s(a)}$ implies that all weights are non-negative. We conclude that $\lambda(\mathbb{G}_{\mathrm{m}})$ acts with positive weights on $\mathrm{Lie}(U)$ and with non-positive weights on $\mathcal{O}(V)$.

More generally, let $\boldsymbol{Q}'$ be a finite quiver with at least $n \geq 3$ vertices and $v_0 \in \boldsymbol{V}_{\text{source}}$. We obtain a quiver $\boldsymbol{Q}^\top$ by setting $\boldsymbol{A}^\top := \boldsymbol{A} \setminus \{a \in \boldsymbol{A} | s(a) = v_0\}$ and $\boldsymbol{V}^\top := \{v \in \boldsymbol{V} | \exists a \in \boldsymbol{A}^\top : v = s(a) \text{ or } v = t(a)\}$ and restricting the source and target maps to $\boldsymbol{A}^\top$. By the induction hypothesis, we can grade the unipotent radical for the restricted $H^\top$-action on $W^\top$ such that we act with non-negative weights on $W^\top := \bigoplus_{a \in \boldsymbol{A}^\top} \mathrm{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$. The linear projection $\mathrm{pr} : W \to W^\top$ is equivariant relative to $H \to H^\top$. By replacing λ with λ^{-1} if necessary we can assume without loss of generality that $\lambda(\mathbb{G}_{\mathrm{m}})$ acts with positive weights on $W^\top$. We have that $H = H^\top \times \mathrm{Aut}(\mathcal{E}_{v_0})$ and $W = W^\top \oplus \bigoplus_{a \in \boldsymbol{A} : s(a) = v_0} W_a$. To extend $\lambda : \mathbb{G}_{\mathrm{m}} \to H^\top$ to H, recall that $\lambda : \mathbb{G}_{\mathrm{m}} \to H^\top$ is given by $\lambda(t) = (\lambda_v(t))_{v \in \boldsymbol{V}^\top}$ where each λ_v is of the form

$$\lambda_v(t) = \left(\begin{array}{c|c|c|c} t^{l_{1,v}}\,\mathrm{id} & 0 & \cdots & 0 \\ \hline 0 & t^{l_{2,v}}\,\mathrm{id} & \ddots & \vdots \\ \hline \vdots & \ddots & \ddots & 0 \\ \hline 0 & \cdots & 0 & t^{l_{k_v,v}}\,\mathrm{id} \end{array}\right)$$

with $l_{i,v} < l_{i+1,v}$ and $l_{1,t(a)} \geq l_{k_{s(a)},s(a)}$for all $a \in \boldsymbol{A}^\top$. Finally, we select exponents for v_0 satisfying $l_{1,v_0} < \ldots < l_{k_{v_0},v_0}$ and $\min\{l_{1,t(a)} | a \in \boldsymbol{A} : s(a) = v_0\} > l_{k_{v_0},v_0}$. We claim for this extension $\lambda_{\text{ext}} : \mathbb{G}_{\mathrm{m}} \to H$ that the following conditions are satisfied.

1. The image of $\lambda_{\text{ext}} : \mathbb{G}_{\mathrm{m}} \to H \to H/U$ lies in the centre of H/U.

2. The 1-PS λ_{ext} acts with non-negative weights on W.

3. The 1-PS λ_{ext} acts with positive weights on $\mathrm{Lie}(U)$.

The first statement is obvious. For the second statement note that λ_{ext} acts on the subspace W_a with weights $l_{i,t(a)} - l_{j,s(a)}$ which are non-negative by definition of λ_{ext}. For the final statement note that $U = \bigtimes_{v \in \boldsymbol{V}} U_v$, thus $\mathrm{Lie}(U) = \bigoplus_{v \in \boldsymbol{V}} \mathrm{Lie}(U_v)$. Each U_v is a normal subgroup of H thus the subspaces $\mathrm{Lie}(U_v)$ are λ_{ext}-invariant. The 1-PS λ_{ext} acts on $\mathrm{Lie}(U_v)$ with weights $l_{j,v} - l_{i,v}$ where $i < j$. We have $l_{1,v} < \ldots < l_{k_v,v}$, thus λ_{ext} acts with positive weights on $\mathrm{Lie}(U)$. We conclude that the H-action on W is graded. □

Example 5.2.17. We consider homomorphisms of sheaves indexed by the loop quiver $\boldsymbol{Q} = \mathbf{Z}_1$ consisting of a vertex v and loop ℓ. By Lemma 5.2.16 the action of $H = \mathrm{Aut}(\mathcal{E})$ on $W = \mathrm{End}(\mathcal{E})$ is graded. There exists a unique pre-Sylvester section $\sigma := \sigma_\ell$. By Proposition 5.2.13 this section is a Sylvester section. We conclude that

$$W_{\mathrm{Syl}} = W_\sigma \subset W^{ssu(H,U)}$$

and by Corollary 5.2.14 it follows that $W_{\mathrm{Syl}} = W^{ssu(H,U)}$. We obtain a good quotient for

$$W_\sigma \cap W^{HM-ss(H,\chi)} \to W_\sigma /\!\!/_\chi H,$$

with $W_\sigma /\!\!/_\chi H := \mathrm{Proj}\, \mathrm{R}(W_\sigma, \mathfrak{L})^H$ a projective over affine variety.

We can more generally consider the quiver

$$\boldsymbol{Q} = \begin{tikzcd} \ell_1 \arrow[loop left] & \overset{1}{\circ} \arrow[r, "\alpha"] & \overset{2}{\circ} & \ell_2 \arrow[loop right] \end{tikzcd} .$$

where both vertex sheaves have non-reductive automorphism groups. Again we have a unique Sylvester section $\sigma = \sigma_{\ell_1,\ell_2}$ and obtain a good quotient for

$$W_\sigma \cap W^{HM-ss(H,\chi)} \to W_\sigma /\!\!/_\chi H,$$

with $W_\sigma /\!\!/_\chi H$ a projective over affine variety.

5.2.3 Moduli spaces for sheaf homomorphisms indexed by a quiver via reductive Geometric Invariant Theory

For a projective scheme X of finite type over $\mathbf{k}$, the automorphism groups of coherent sheaves over X will not necessarily be reductive. In particular, for a quiver $\boldsymbol{Q}$ and semisimple vertex sheaves $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$ the automorphism group of a semisimple sheaf $\mathcal{E}_{v_0}$ is reductive then either up to isomorphism $\mathcal{E}_{v_0}$ has exactly one simple factor or every pair $\mathcal{S},\mathcal{T}$ of non-isomorphic simple factors of $\mathcal{E}_{v_0}$ is incomparable; that is, both $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{S},\mathcal{T})$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{T},\mathcal{S})$ are trivial. In this situation we can apply reductive Geometric Invariant Theory for affine spaces as established by King (see [44]). For the quiver $\boldsymbol{Q} = \circ \to \circ$ Drézet and Trautmann already translated the 1-PS Hilbert-Mumford criterion to homomorphisms of sheaves (see Appendix B.3.4 or [25]).

The (left) action of $G = \bigtimes_{v\in\boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v)$ on $W = \bigoplus_{a\in\boldsymbol{A}} \mathrm{Hom}(\mathcal{E}_{s(a)},\mathcal{E}_{t(a)})$ turns W into a rational G-module. Note that $\Delta := \left\{(t\cdot \mathrm{id}_{\mathcal{E}_v})_{v\in\boldsymbol{V}} \middle| t\in\mathbb{G}_{\mathrm{m}}\right\}$ is a subgroup that stabilises all points $w\in W$. Under the assumption that the automorphism groups are reductive, we obtain that

$$G = \bigtimes_{v\in\boldsymbol{V}} \mathrm{Aut}(\mathcal{E}_v) = \bigtimes_{v\in\boldsymbol{V}} (\bigtimes_{i=1}^{k_v} \mathrm{GL}(n_{v,i},\mathbf{k}),$$

where $n_{v,1},\ldots,n_{v,k_v}$ denote the multiplicities of the indecomposable summands of $\mathcal{E}_v$ described at the start of §5.2.3. The characters of $\mathbf{X}^*(G)$ correspond bijectively elements of $\mathbb{Z}^{\sum_{v\in\boldsymbol{V}} k_v}$. Under this $\chi\in\mathbf{X}^*(G)$ correspondences to $\Theta = \big(\Theta(v,i)\big)_{\substack{v\in\boldsymbol{V}\\ i=1,\ldots,k_v}} \in \mathbb{Z}^{\sum\limits_{v\in\boldsymbol{V}} k_v}$ where $\Theta(v,i)$ is the exponent of the factor $\det : \mathrm{GL}(n_{v,i},\mathbf{k}) \to \mathbb{G}_{\mathrm{m}}$ in the character $\chi = \chi_\Theta$.

Let $E := \big(\mathcal{E}_v, \varphi_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)}\big)$ be a representation of the quiver $\boldsymbol{Q}$ in $\mathbf{Coh}(X)$ with fixed semisimple vertex sheaves $(\mathcal{E}_v)_{v\in\boldsymbol{V}}$. More concretely; suppose $\mathcal{E}_v = \bigoplus_{i=1}^{k_v} M_{v,i}\otimes\mathcal{E}_{v,i}$ where $M_{v,i}$ are finite dimensional vector spaces and the sheaves $\mathcal{E}_{v,i}$ are simple.

Definition 5.2.18. $E := \big(\mathcal{E}_v, \varphi_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)}\big)$ be a representation of the quiver $\boldsymbol{Q}$ with fixed vertex sheaves $(\mathcal{E}_v)_{v\in\boldsymbol{V}} = \big(\bigoplus_{i=1}^{k_v} M_{v,i}\otimes\mathcal{E}_{v,i}\big)_{v\in\boldsymbol{V}}$ where $M_{v,i}$ are finite dimensional vector spaces and the sheaves $\mathcal{E}_{v,i}$ are simple. We call a subrepresentation E' of E *admissible,* if E' is defined via a sequence of subvector spaces $\big(M'_{v,i}\subset M_{v,i}\big)_{\substack{v\in\boldsymbol{V}\\ i=1,\ldots,k_v}}$; that is $\mathcal{E}'_v = \bigoplus_{i=1}^{k_v} M'_{v,i}\otimes\mathcal{E}_{v,i}$ and for each arrow $a\in\boldsymbol{A}$ the morphism $\varphi_a : \mathcal{E}_{s(a)} \to \mathcal{E}_{t(a)}$ restricts to $\varphi_a : \mathcal{E}'_{s(a)} \to \mathcal{E}'_{t(a)}$.

Proposition 5.2.19. *Let $E = (\mathcal{E}_v,\varphi_a)$ be a representation of the quiver $\boldsymbol{Q}$ in $\mathbf{Coh}(X)$ with semisimple vertex sheaves $\mathcal{E}_v$ for each $v\in\boldsymbol{V}$. Consider a 1-PS λ of G such that $\lim_{t\to 0}\lambda(t)\cdot E$ exists; this 1-PS induces a finite filtration of E by admissible subrepresentations. Conversely, any finite filtration of E by admissible subrepresentations corresponds to a 1-PS λ for which $\lim_{t\to 0}\lambda(t)E$ exists in W.*

Proof. Let λ be a 1-PS of G. Consider the 1-PS $\lambda_{v,i}$ induced by λ via $\mathbb{G}_{\mathrm{m}} \overset{\lambda}{\to} G \overset{\pi_{v,i}}{\to} \mathrm{GL}(M_{v,i})$.

We obtain that each $M_{v,i}$ decomposes into weight spaces $M_{v,i}^{(n)}$ for $n \in \mathbb{Z}$ and denote by $\mathcal{E}_v^{(m)} := \bigoplus_{i=1}^{k_v} M_{v,i}^{(m)} \otimes \mathcal{E}_{v,i}$ then

$$\mathcal{E}_v^{(\geq m)} := \bigoplus_{i=1}^{k_v} \big(\bigoplus_{n \geq m} M_{v,i}^{(n)} \big) \otimes \mathcal{E}_{v,i} = \bigoplus_{n \geq m} \mathcal{E}_v^{(n)}.$$

Analogously to [44] we can show that $\lim_{t \to 0} \lambda(t) E$ exists if and only if for all $m \in \mathbb{Z}$ and all $a \in \boldsymbol{A}$ we have that

$$\varphi_a(\mathcal{E}_{s(a)}^{(\geq m)})) \subset \mathcal{E}_{t(a)}^{(\geq m)}.$$

To see this let $\lambda_a : \mathbb{G}_{\mathrm{m}} \xrightarrow{\lambda} G \to \mathrm{Aut}(\mathcal{E}_{s(a)}) \times \mathrm{Aut}(\mathcal{E}_{t(a)})$. We define the morphism $\varphi_a^{m,n}$ to be

$$\mathcal{E}_{s(a)}^{(n)} \xrightarrow{\iota} \mathcal{E}_{s(a)} \xrightarrow{\varphi_a} \mathcal{E}_{t(a)} \xrightarrow{\pi} \mathcal{E}_{t(a)}^{(m)} \xrightarrow{\iota} \mathcal{E}_{t(a)}.$$

The action of λ_a on $\varphi_a^{m,n}$ is given by scalar multiplication with t^{m-n}. The limit of $(\varphi_a)_{a \in \boldsymbol{A}}$ with respect to λ exists, if for each arrow $a \in \boldsymbol{A}$ the restricted 1-PS λ_a acts with non-negative weights on φ_a; that is $\varphi_a^{m,n} = 0$ for $m < n$. Therefore, the limit for a 1-PS λ exists if and only if we have a filtration of the representation $E = (\mathcal{E}_v, \varphi_a)$ by admissible subrepresentations

$$\ldots \supset E^{(\geq m)} = (\mathcal{E}_v^{(\geq m)}, \varphi_a) \supset E^{(\geq m+1)} \supset \ldots$$

such that $E^{(\geq M)} = E$ is for $M << 0$ and $E^{\geq L} = 0$ for $L >> 0$. $\square$

To an admissible subrepresentation of E' of E, we associate a dimension vector $\dim E' \in \mathbb{Z}^{\sum_{v \in \boldsymbol{V}} k_v}$ with $(\dim E)_{(v,i)}$ denoting the multiplicity of the i-th indecomposable summand of $\mathcal{E}'_v$. For two elements f and g of $\mathbb{Z}^{\sum_{v \in \boldsymbol{V}} k_v}$ we associate the following pairing

$$\langle f, g \rangle := \sum_{v \in \boldsymbol{V}} \sum_{i=1}^{k_v} f(v,i) \cdot g(v,i).$$

Definition 5.2.20. We call a representation Θ-stable respectively Θ-semistable if for any non-trivial admissible subrepresentation E' of E we have that $\langle \Theta, \dim E' \rangle > 0$ respectively $\langle \Theta, \dim E' \rangle \geq 0$

Proposition 5.2.21. *A representation $E = (\mathcal{E}_v, \varphi_a)$ with semisimple vertex sheaves corresponds to the point $(\varphi_a)_{a \in \boldsymbol{A}} \in W := \bigoplus_{a \in \boldsymbol{A}} \mathrm{Hom}(\mathcal{E}_{s(a)}, \mathcal{E}_{t(a)})$. Then the point $(\varphi_a)_{a \in \boldsymbol{A}}$ is (semi)stable with respect* *to the G-action linearised by χ_Θ on W if and only if the representation E is Θ-(semi)stable.*

Proof. A point $w \in W$ is G-(semi)stable with respect to the χ_Θ-linearised action on W if for any 1-PS λ of G such that $\lim_{t \to 0} \lambda(t)w$ exists, we have $\langle \chi_\Theta, \lambda \rangle > (\geq) 0$. □

Appendix A

Some results related to Geometric Invariant Theory

A.1 Polyhedral cones

Let us fix a pair of dual vector spaces $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ for our applications this will be $\mathbf{X}^*(T)_{\mathbb{R}}$ and $\mathbf{X}_*(T)_{\mathbb{R}}$.

Definition A.1.1. A *convex polyhedral cone* in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \left\{ \sum_{u \in S} \lambda_u u \middle| \lambda_u \geq 0 \right\} \subseteq N_{\mathbb{R}}$$

where $S \subsetneq N_{\mathbb{R}}$ is finite. We say that σ is generated by S. For $S = \emptyset$ we define $\mathrm{Cone}(\emptyset) := \{0\}$.

Remark A.1.2. Since all cones that we want to consider in the following will be convex we will call them polyhedral cones instead of convex polyhedral cones.

In the following, we denote the pairing between $M_{\mathbb{R}}$ and $N_{\mathbb{R}}$ by $\langle\ ,\ \rangle$.

Definition A.1.3. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, its *dual cone* is defined by

$$\sigma^{\vee} := \{m \in M_{\mathbb{R}} | \forall u \in \sigma : \langle m, u \rangle \geq 0\}.$$

For $m \in M_{\mathbb{R}}$ we define the closed halfspace

$$H_m := \{u \in N_{\mathbb{R}} | \langle m, u \rangle \geq 0\} \subset N_r.$$

If $\sigma \subseteq N_{\mathbb{R}}$ is a polyhedral cone with $\sigma \subset H_m$ we call H_m a *supporting halfspace*. Note that H_m is a supporting halfspace of σ if, and only if, $m \in \sigma^{\vee}$.

Duality has the following important properties

Proposition A.1.4. *[20, Proposition 1.2.4 & 1.2.8] Let $\sigma \subseteq N_{\mathbb{R}}$ be a polyhedral cone. Then:*

1. *$\sigma^{\vee}$ is a polyhedral cone in $M_{\mathbb{R}}$ and $(\sigma^{\vee})^{\vee} = \sigma$.*
2. *If $\sigma = H_{m_1} \cap \ldots \cap H_{m_s}$, then $\sigma^{\vee} = \mathrm{Cone}(m_1, \ldots, m_s)$.*
3. *If $\sigma_1 \subset \sigma_2 \subseteq N_{\mathbb{R}}$ are polyhedral cones, then $\sigma_2^{\vee} \subset \sigma_1^{\vee}$.*

Remark A.1.5. Given a polyhedral cone $\sigma \subseteq N_{\mathbb{R}}$, then by the above proposition $\sigma^{\vee}$ is also a polyhedral cone that is there exist $m_1, \ldots, m_s \in M_{\mathbb{R}}$ such that $\sigma^{\vee} = \mathrm{Cone}(m_1, \ldots, m_s)$. It is easy to check that $\sigma = H_{m_1} \cap \ldots \cap H_{m_s}$ so any polyhedral cone is also a finite intersection of halfspaces.

Definition A.1.6. The group of characters of T denoted by $\mathbf{X}^*(T) = \mathrm{Hom}_{\mathbf{Grp}}(T, \mathbb{G}_{\mathrm{m}})$. Dually we have the group of one parameter subgroups $\mathbf{X}_*(T) = \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_{\mathrm{m}}, T)$. We obtain a perfect pairing $\langle \cdot, \cdot \rangle : \mathbf{X}^*(T) \times \mathbf{X}_*(T) \to \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}})$.

A.2 $\mathbb{G}_{\mathrm{a}}$-actions on affine varieties

Let V be an irreducible affine $\mathbb{G}_a$-variety over $\mathbf{k}$ an algebraically closed field of characteristic zero. Algebraic actions of $\mathbb{G}_a$ correspond bijectively to the locally nilpotent derivatives of the ring of regular functions $A := \mathcal{O}(V)$. More precisely, if we have a $\mathbb{G}_{\mathrm{a}}$-action on the affine variety $V = \mathrm{Spec}\, A$ then the action $\alpha : \mathbb{G}_{\mathrm{a}} \times V \to V$ corresponds to a morphism $\alpha_* : A \to A \otimes \mathbb{C}[t] \cong A[t]$. For $f \in A$ we can write $\alpha_* f = \sum_{n \geq 0} D_n(f) t^n$ with almost all $D_n(f) = 0$ and each $D_n : A \to A$

$\mathbf{k}$-linear maps. The fact that α_* is a $\mathbb{G}_a$-action is equivalent to the following properties:

$$\begin{cases} D_0 = id_A, \\ \text{for } f, g \in A : D_n(fg) = \sum_{k+l=n}(D_k f)(D_l g) \\ D_n D_m = \binom{n+m}{n} D_{n+m}. \end{cases}$$

From this it follows that $D := D_1$ is a $\mathbf{k}$-derivation of A and since we are working in characteristic zero we also have that $D_n = \frac{1}{n!}D^n$. So D is a locally nilpotent derivation. Let us denote the set of locally nilpotent $\mathbf{k}$-derivations of the $\mathbf{k}$-algebra A by $\mathrm{LND}(A)$.

Definition A.2.1. Given $D \in \mathrm{LND}(A)$, define $\mathrm{Fix}(D) = \{\mathfrak{p} \in \mathrm{Spec}(A) | D(A) \subset \mathfrak{p}\}$. Note that $\mathrm{Fix}(D)$ is a closed subset of $\mathrm{Spec}(A)$.

Proposition A.2.2. *Let $D \in \mathrm{LND}(A)$ and consider the associated $\mathbb{G}_a$-action α_D on X. Then*

1. *The ring of invariants of α_D is the subring $\ker(D)$ of A.*

2. *The fixed points of α_D are precisely the closed points which belong to $\mathrm{Fix}(D)$.*

Remark A.2.3. In Proposition A.2.2 the first part is well known and the second can be found in [21, Prop 9.7.].

Let us denote by $A^D = \ker D$ then we get a short exact sequence of A^D modules

$$0 \to A^D \to A \to \mathrm{Im}(D) \to 0.$$

Further for $S \subset A$ denote by $\langle S \rangle$ the ideal of A generated by S.

Proposition A.2.4. *We get the following results:*

1. *If $1 \in \langle \mathrm{Im}(D) \rangle$ then the $\mathbb{G}_a$-action is free.*

2. *If $1 \in \langle \mathrm{Im}(D) \cap A^D \rangle$ then the action is locally trivial.*

3. *If $1 \in \mathrm{Im}(D)$ then the action has a slice and $X = \mathrm{Spec}\, A \to \mathrm{Spec}\, A^D$ is a trivial principal $\mathbb{G}_a$-bundle.*

4. *If $A^D \to A$ is faithfully flat we get a trivial principal $\mathbb{G}_{\mathrm{a}}$-bundle $\operatorname{Spec} A \to \operatorname{Spec} A^D$.*

Definition A.2.5. A degree function on a ring B is a map $\deg : B \to \mathbb{N} \cup \{-\infty\}$ satisfying:

1. $\forall x \in B : \deg x = -\infty \iff x = 0$
2. $\forall x, y \in B : \deg(xy) = \deg x + \deg y$
3. $\forall x, y \in B : \deg(x + y) \leq \max(\deg x, \deg y)$.

Definition A.2.6. Let B be a ring. Then each $D \in \operatorname{LND}(B)$ determines a map $\deg_D : B \to \mathbb{N}\cup\{-\infty\}$ defined as follows: $\deg_D(x) = \max\{n \in \mathbb{N} | D^n x \neq 0\}$ for $x \in B\backslash\{0\}$, and $deg_D(0) = -\infty$. Note that $\ker D = \{x \in B | \deg_D(x) \leq 0\}$.

Proposition A.2.7. *[21, Proposition 4.8] Let B be a domain of characteristic zero and $D \in \operatorname{LND}(B)$. Then the map $\deg_D : B \to \mathbb{N} \cup \{-\infty\}$ is a degree function.*

Remark A.2.8. Since $\deg_D$ is a degree function we can associate to the locally nilpotent derivation D a filtration $F_n := \{x \in B | \deg_D(x) \leq n\} = \ker D^{n+1}$.

Corollary A.2.9. *Let X be an affine $\mathbb{G}_{\mathrm{a}}$-variety. The following statements are equivalent:*

1. *The locally nilpotent derivation associated to the $\mathbb{G}_{\mathrm{a}}$-action on X has a slice.*
2. *The deg-function $\deg_D$ associated to the locally nilpotent derivation $D : \mathcal{O}(X) \to \mathcal{O}(X)$ corresponding to the $\mathbb{G}_{\mathrm{a}}$-action on X induces a $\mathbb{N}$-grading of $\mathcal{O}(X)$ and the ideal $D(\mathcal{O}(X)_1) \oplus \bigoplus_{n\geq 1} \mathcal{O}(X)_n$ is the ring $\mathcal{O}(X)$.*
3. *The $\mathbb{G}_{\mathrm{a}}$-action extends to a graded action of $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$ and for each $x \in V$ we have that $\operatorname{Stab}_{\mathbb{G}_{\mathrm{a}}}(x) = \{e\}$.*

Proof. Suppose we have a $\mathbb{G}_{\mathrm{a}}$-action on an affine variety X such that the associated derivation $D : \mathcal{O}(X) \to \mathcal{O}(X)$ has a slice $s \in \mathcal{O}(X)$. Then we obtain $\mathcal{O}(X) = \mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}} \otimes_{\mathbf{k}} \mathbf{k}[s]$ which gives us a grading $\mathcal{O}(X)_n := \mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}} s^n$ for each $n \in \mathbb{N}$ with respect to this grading D is homogeneous of degree -1 and hence the $\mathbb{G}_{\mathrm{a}}$-action extends to a graded action $\mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$. Since $s \in \mathcal{O}(X)$ it follows that $D(\mathcal{O}(X)_1) \oplus \bigoplus_{n\geq 1} \mathcal{O}(X)_n = \mathcal{O}(X)$ so 1 implies 2 and 3. This implies together with Lemma 3.2.1 that a $\mathbb{G}_{\mathrm{a}}$-action on X gives a trivial principal $\mathbb{G}_{\mathrm{a}}$-bundle $X \to \operatorname{Spec} \mathcal{O}(X)^{\mathbb{G}_{\mathrm{a}}}$ if and only if the $\mathbb{G}_a$-action is fixed point free and extends to a graded action $\mathbb{G}_a \rtimes \mathbb{G}_m$. Therefore, the three statements are equivalent. $\square$

Definition A.2.10. Let A be a finitely generated $\mathbf{k}$-algebra (with $\mathbf{k}$ a field of characteristic zero) and $D \in \mathrm{LND}(A)$ be a locally nilpotent derivation on A. We call $s \in A$

1. a *slice* of the locally nilpotent derivative D, if $Ds = 1$.

2. a *local slice* of the locally nilpotent derivative D, if $Ds \neq 0$ and $D^2 s = 0$.

A.3 Białynicki-Birula-decomposition

Let X to be a a non-singular $\mathbb{G}_m$-variety over $\mathbb{C}$. We assume that the $\mathbb{G}_m$-fixed locus $X^{\mathbb{G}_m}$ is non-empty with connected components $F_1, \ldots, F_r$. For $x \in X$ we have the orbit morphism

$$\sigma_x : \mathbb{G}_m \to X$$
$$t \mapsto t \cdot x.$$

If X is complete then σ_x extends to a morphism

$$\overline{\sigma}_x : \mathbb{P}^1 \to X.$$

We denote $\overline{\sigma}_x(0)$ by $\lim_{t\to 0} t \cdot x$ and $\overline{\sigma}_x(\infty)$ by $\lim_{t\to\infty} t \cdot x$. Furthermore we define the following locally-closed subschemes

$$X_i^+ := \{x \in X | \lim_{\substack{t\to 0 \\ t\in\mathbb{G}_m}} t \cdot x \in F_i\}$$

and

$$X_i^- := \{x \in X | \lim_{\substack{t\to \infty \\ t\in\mathbb{G}_m}} t \cdot x \in F_i\}$$

of X which are called plus-cells respectively minus-cells of the Białynicki-Birula-decomposition. Then

$$X = \coprod_{i=0}^{r} X_i^+ = \coprod_{i=0}^{r} X_i^-$$

and there exists a source X_{i_0} and a sink X_{i_1} of the action characterised by the fact that $X_{i_0}^+$ is open in X and $X_{i_0}^- = X_{i_0}$ and analogously $X_{i_1}^-$ is open in X and $X_{i_1}^+ = X_{i_1}$.[11]

For $\mathbb{G}_\mathrm{m}$ acting linearly with respect to $\mathcal{O}_{\mathbb{P}(V)}(1)$ on $\mathbb{P}(V)$ the following proposition characterises the Białynicki-Birula-decomposition.

Proposition A.3.1. *Let V be a representation of $\mathbb{G}_\mathrm{m}$ and let $v \in V$. Denote by $[v]$ its class in $\mathbb{P}(V)$.*

1. *$[v]$ is a fixed point if and only if v is an eigenvector for $\mathbb{G}_\mathrm{m}$.*

2. *If $[v]$ is not fixed under the $\mathbb{G}_\mathrm{m}$-action write*

$$v = \sum_{i=r}^{s} v_i$$

with $v_i \in V_i$ and $r < s$ (the space V_i is the eigenspace for the eigenvalue $i \in \mathbb{Z}$). In this case the morphism $\sigma : \mathbb{G}_\mathrm{m} \to \mathbb{P}(V)$ defined by $\sigma(t) := [t \cdot v]$ extends uniquely to a morphism $\overline{\sigma} : \mathbb{P}^1 \to \mathbb{P}(V)$ with $\overline{\sigma}(0) = [v_r]$ and $\overline{\sigma}(\infty) = [v_s]$.

A.3.1 Categorical Białynicki-Birula-decomposition for $\mathbb{G}_\mathrm{m}$

In [26] a categorical Białynicki-Birula-decomposition is stated in terms of algebraic spaces of finite type over $\mathbf{k}$ and it is shown that the corresponding functors are representable by algebraic spaces. Furthermore if the algebraic space under consideration is a scheme then the functors corresponding to the categorical Białynicki-Birula-decomposition are represented by schemes. Here we will restrict to the case of schemes.

Let Z be a $\mathbb{G}_\mathrm{m}$-scheme of finite type over $\mathbf{k}$. We identify the category $\mathbf{Sch}/\mathbf{k}$ with $\mathbb{G}_\mathrm{m}$-schemes via equipping each scheme with the trivial $\mathbb{G}_\mathrm{m}$-action. For $\mathbb{G}_\mathrm{m}$-schemes X and Y wee denote by $\mathrm{Hom}^{\mathbb{G}_\mathrm{m}}_{\mathbf{Sch}/\mathbf{k}}(X, Y) \subset \mathrm{Hom}_{\mathbf{Sch}/\mathbf{k}}(X, Y)$ the set of $\mathbb{G}_\mathrm{m}$-equivariant morphisms from X to Y. Consider the following functors

$$\mathsf{Z}^0 : (\mathbf{Sch}/\mathbf{k})^{op} \to \mathbf{Set}$$
$$S \mapsto \mathrm{Hom}^{\mathbb{G}_\mathrm{m}}_{\mathbf{Sch}/\mathbf{k}}(S, Z),$$

and

$$\begin{aligned}\mathsf{Z}^+ : (\mathbf{Sch}/\mathbf{k})^{op} &\to \mathbf{Set}\\ S &\mapsto \mathrm{Hom}^{\mathbb{G}_\mathrm{m}}_{\mathbf{Sch}/\mathbf{k}}(\mathbb{A}^1 \times S, Z)\end{aligned}$$

where $\mathbb{G}_\mathrm{m}$ acts on $\mathbb{A}^1$ via homotheties. Furthermore we also consider the opposite action of $\mathbb{G}_\mathrm{m}$ on $\mathbb{A}^1$ and denote the corresponding $\mathbb{G}_\mathrm{m}$-variety by $\mathbb{A}^1_-$. This allows us to define

$$\begin{aligned}\mathsf{Z}^- : (\mathbf{Sch}/\mathbf{k})^{op} &\to \mathbf{Set}\\ S &\mapsto \mathrm{Hom}^{\mathbb{G}_\mathrm{m}}_{\mathbf{Sch}/\mathbf{k}}(\mathbb{A}^1_- \times S, Z).\end{aligned}$$

We obtain natural transformations $q^\pm : \mathsf{Z}^\pm \to \mathsf{Z}^0$ by evaluation at $0 \in \mathbb{A}^1$

Proposition A.3.2. *[26, Proposition 1.2.2] If Z is a scheme of finite type over $\mathbf{k}$ then Z^0 is represented by a scheme Z^0 of finite type over $\mathbf{k}$. Moreover the morphism $Z^0 \to Z$ is a closed embedding.*

Proposition A.3.3. *[26, Theorem 1.4.2, Corollary 1.4.3] If Z is a scheme of finite type over $\mathbf{k}$ then Z^+ is represented by a scheme Z^+ of finite type over $\mathbf{k}$. The morphism $q^+ : Z^+ \to Z$ is affine.*

Lemma A.3.4. *The morphisms $i^\pm : Z^0 \to Z^\pm$ are closed embeddings.*

A.3.2 Categorical Białynicki-Birula-decomposition for certain reductive groups

Following Drinfeld, Jelisiejew and Sienkiewicz define a categorical Białynicki-Birula-decomposition for a reductive group G such that G is the group of units of M a linear algebraic monoid with zero. We identify the category $\mathbf{Sch}/\mathbf{k}$ with G-schemes via equipping each scheme with the trivial G-action.

Consider the following functors

$$\begin{aligned}\mathsf{Z}^0 : (\mathbf{Sch}/\mathbf{k})^{op} &\to \mathbf{Set}\\ S &\mapsto \mathrm{Hom}^{G}_{\mathbf{Sch}/\mathbf{k}}(S, Z),\end{aligned}$$

and

$$\begin{aligned} \mathsf{D}_Z = \mathsf{D}_{Z,M} : (\mathbf{Sch}/\mathbf{k})^{op} &\to \mathbf{Set} \\ S &\mapsto \mathrm{Hom}^{\mathbb{G}_m}_{\mathbf{Sch}/\mathbf{k}}(M \times S, Z) \end{aligned}$$

where G acts on M via the composition law of M :

$$G \times M \to M \times M \xrightarrow{\circ} M.$$

Theorem A.3.5. *[40, Theorem 1] Let X be a G-scheme locally of finite type over $\mathbf{k}$. Then the functor D_X is represented by a scheme X^+ locally of finite type over $\mathbf{k}$. Moreover, the scheme X^+ has a natural M-action and the morphism $X^+ \to X^G$ is affine of finite type and equivariant.*

Appendix B

The work of Drézet and Trautmann

In this section, we follow the approach of Drézet and Trautmann and try to replace the non-reductive groups with reductive groups. Drézet and Trautmann define a new notion of GIT-(semi)stability for sufficiently nice objects in $\mathrm{Rep}(\overset{1}{\circ} \to \overset{2}{\circ}, \mathbf{Coh}(X))$ where X is a projective algebraic variety over the field of complex numbers. More concretely, they assume that the sheaves $\mathcal{E}_1$and $\mathcal{E}_2$ are both semisimple and locally free. In this situation they give an explicit description of the automorphism groups involved [25, 1.1]. To construct quotients for their linearised group actions they replace the action of the (non-reductive) automorphism groups with an action of an reductive group and compare the resulting stability conditions for both linearised actions.

B.1 An explicit description of the automorphism groups

Consider locally free sheaves $\mathcal{E}, \mathcal{F}$ over X that are semisimple, i.e. a direct sum of simple subsheaves $\mathcal{E}_i, \mathcal{F}_j$. Recall that a sheaf $\mathcal{G}$ is simple, if its endomorphism consist solely of homotheties, that is, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{G}) \cong \mathbf{k}$. For convenience, we collect recurring factors and write $\mathcal{E} = \bigoplus_{i=1}^{r} M_i \otimes \mathcal{E}_i$ and $\mathcal{F} = \bigoplus_{j=1}^{s} N_j \otimes \mathcal{F}_j$ where M_i, N_j are finite dimensional $\mathbf{k}$-vector spaces. Finally, assume that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_k, \mathcal{E}_l) = 0$ for $k > l$ and $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_r, \mathcal{F}_s) = 0$ for $r > s$. The groups $\mathrm{Aut}(\mathcal{E})$ and $\mathrm{Aut}(\mathcal{F})$ can be viewed as matrix groups. For instance $\mathrm{Aut}(\mathcal{E})$ identifies with the group of matrices of the following form

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}_{\mathbb{C}}\left(M_i, M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\right)$. Analogously, we can consider $\mathrm{Aut}(\mathcal{F})$ to be a matrix group. Under this identification the maximal normal unipotent subgroup $U_{\mathcal{E}}$ of $\mathrm{Aut}(\mathcal{E})$ is given by elements of the following form

$$\begin{pmatrix} \mathrm{Id}_{M_1} & 0 & \cdots & 0 \\ u_{2,1} & \mathrm{Id}_{M_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & \mathrm{Id}_{M_r} \end{pmatrix}.$$

with $u_{j,i} \in \mathrm{Hom}_{\mathbb{C}}\left(M_i, M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\right)$. We can also consider the reductive part $R_{\mathcal{E}}$ of $\mathrm{Aut}(\mathcal{E})$ which is given by the condition that $u_{j,i} = 0$ for all i, j.

$$\begin{pmatrix} g_1 & 0 & \cdots & 0 \\ 0 & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & g_r \end{pmatrix}.$$

In particular, we have that $\mathrm{Aut}(\mathcal{E})/U_{\mathcal{E}} \cong R_{\mathcal{E}}$. In the following we consider the natural left action of the linear algebraic group $H := \mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ on the affine space $V := \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$. Let us denote the reductive respectively the unipotent part of H by $R := R_{\mathcal{E}} \times R_{\mathcal{F}}$ respectively by $U := U_{\mathcal{E}} \times U_{\mathcal{F}}$. Under this identification we also obtain a block matrix description of $\phi \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ where $\phi_{k,l} : M_l \otimes \mathcal{E}_l \to \mathcal{F}_k \otimes N_k$ identifies with an element of $\mathrm{Hom}(M_l \otimes \mathrm{Hom}(\mathcal{E}_l, \mathcal{F}_k)^{\vee}, N_k) = \mathrm{Hom}(M_l, N_k \otimes \mathrm{Hom}(\mathcal{E}_l, \mathcal{F}_k))$.

B.2 An extension of the action

In this section, we will replace the action $H \times V \to V$ by an action of $G \times W \to W$ that is compatible with the original action in the following sense:

1. H is an observable subgroup of G
2. G has enough characters in the following sense: To any character $\chi : H \to \mathbb{G}_{\mathrm{m}}$ we can associate a character $\psi : G \to \mathbb{G}_{\mathrm{m}}$.
3. The diagram
$$\begin{array}{ccc} H \times V & \xrightarrow{\alpha} & V \\ \downarrow{\scriptstyle \mathrm{id}_H \times \zeta} & & \downarrow{\scriptstyle \zeta} \\ G \times W & \xrightarrow{\beta} & W \end{array}$$
commutes.
4. $\mathrm{Stab}_H(x) = \mathrm{Stab}_G(\zeta(x))$
5. The morphism $\beta \circ (\mathrm{id}_G \times \zeta) : G \times V \to W$ factorizes via $G \times^H V$.

Note that the following construction of Drézet and Trautmann uses the decomposability of the sheaves $\mathcal{E}_1$ and $\mathcal{E}_2$ in an essential way. We use this construction only for the special case $\boldsymbol{Q} = \overset{1}{\circ} \to \overset{2}{\circ}$ and denote $\mathcal{E} := \mathcal{E}_1$ and $\mathcal{F} := \mathcal{E}_2$. Let $\mathcal{E} = \bigoplus_{i=1}^{r} M_i \otimes \mathcal{E}_i$ and $\mathcal{F} = \bigoplus_{j=1}^{s} N_j \otimes \mathcal{F}_j$, where $\mathcal{E}_i$ and $\mathcal{F}_j$ are simple reflexive sheaves of $\mathcal{O}_X$-modules and M_i, N_j are finite dimensional $\mathbb{C}$-vector spaces.

If $\mathrm{Aut}(\mathcal{E})$ respectively $\mathrm{Aut}(\mathcal{F})$ is reductive there is no need to replace the automorphism group. In the following suppose that sheaf $\mathcal{E}$ respectively $\mathcal{F}$ has a non-reductive automorphism group. We try to replace the sheaf with a simpler sheaf admitting a reductive group as a replacement for $\mathrm{Aut}(\mathcal{E})$. Here we have to different choices depending on whether we are working with the source $\mathcal{E}$ or target $\mathcal{F}$ of our unique arrow a. Set

1. $\mathcal{E}^\circ := \mathcal{E}_1 \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E})$,
2. $\mathcal{F}^\bullet := \mathcal{F}_s \otimes \mathrm{Hom}(\mathcal{F}, \mathcal{F}_s)^\vee$ and
3. $W_0 := \mathrm{Hom}(\mathcal{E}^\circ, \mathcal{F}^\bullet)$.

To construct an injective morphism from $V = \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$ to W_0 consider the following set-up. For a given morphism $\phi : \mathcal{E} \to \mathcal{F}$ we construct a morphism $\Phi : \mathcal{E}^\circ \to \mathcal{F}^\bullet$:

$$\mathcal{E}^\circ = \mathcal{E}_1 \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_1, \mathcal{E}) \xrightarrow{\mathrm{ev}_{\mathcal{E}_1}} \mathcal{E} \xrightarrow{\phi} \mathcal{F} \xrightarrow{\kappa^\vee_{\mathcal{F},\mathcal{F}_s}} \mathcal{F}_s \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_s)^\vee = \mathcal{F}^\bullet.$$

All together this determines a closed immersion

$$\gamma : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}^\circ, \mathcal{F}^\bullet)$$
$$\phi \mapsto \gamma(\phi) = \mathrm{ev}_{\mathcal{E}_1} \circ \phi \circ \kappa^\vee_{\mathcal{F},\mathcal{F}_s}.$$

On W_0 we have a natural action of the reductive group $\mathrm{Aut}(\mathcal{E}^\circ) \times \mathrm{Aut}(\mathcal{F}^\bullet)$ but this group does admit enough characters [25, p. 134]. Since we are interested in comparing notions of (semi)stability with respect to a polarization (character) we find the group $\mathrm{Aut}(\mathcal{E}^\circ) \times \mathrm{Aut}(\mathcal{F}^\bullet)$ unsuitable. To circumvent this problem we extend W_0 to W together with a group action of $\bigtimes_{j=1}^r \mathrm{GL}\left(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E})\right) \times \bigtimes_{j=1}^r \mathrm{GL}\left(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_j)^\vee\right)$.

Consider the following composition maps

$$\kappa_i : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}) \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{i-1}, \mathcal{E}_i) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{i-1}, \mathcal{E})$$

given by $\phi \otimes \psi \mapsto \phi \circ \psi$ and

$$\eta_l : \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_{l+1})^\vee \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_l, \mathcal{F}_{l+1}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_l)^\vee$$

mapping $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_l) \xrightarrow{F} k \otimes \mathcal{F}_l \xrightarrow{\psi} \mathcal{F}_{l+1}$ to $F(\psi \circ -) := \eta_l(F \otimes \psi)$.

Setting

$$W_L := \bigoplus_{i=2}^{r} \mathrm{Hom}\left(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}) \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{i-1}, \mathcal{E}_i), \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_{i-1}, \mathcal{E})\right)$$

and

$$W_R := \bigoplus_{l=1}^{s-1} \mathrm{Hom}\left(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_{l+1})^\vee \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_l, \mathcal{F}_{l+1}), \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{F}_l)\right)$$

$$\begin{aligned}\zeta : V &\to W_R \oplus W_0 \oplus W_L\\ \varphi &\mapsto \big((\kappa_2, \ldots, \kappa_r), \gamma(\varphi), (\eta_1, \ldots, \eta_{s-1})\big).\end{aligned}$$

Proposition B.2.1. *The morphism ζ is a closed embedding of affine schemes [25, p.137].*

On $W := W_L \oplus \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}^\circ, \mathcal{F}^\bullet) \oplus W_R$ we have an action of $G := G_L \times G_R$ where

$$G_L := \bigtimes_{j=1}^{r} \mathrm{GL}\,\Big(\bigoplus_{i=1}^{j} M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\Big) = \bigtimes_{j=1}^{r} \mathrm{GL}\,\big(\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E})\big)$$

and

$$G_R := \bigtimes_{l=1}^{s} \mathrm{GL}\,\Big(\bigoplus_{m=1}^{l} N_l \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}_m, \mathcal{F}_l)^\vee\Big).$$

We construct a group homomorphism $\theta : H \to G$. Since $H = H_R \times H_L$ and $G = G_R \times G_L$ we will only construct $\theta_L : H_L \to G_L$ where $\theta = \theta_L \times \theta_R$ and θ_R is defined analogously. Recall that $H_L = \mathrm{Aut}(\mathcal{E})$ where $\mathcal{E} = \bigoplus_{i=1}^{r} M_i \otimes \mathcal{E}_i$ with $\mathcal{E}_i$ a simple locally free sheaf. An element $h \in H_L$ corresponds to a matrix

$$h = \begin{pmatrix} g_1 & 0 & \cdots & 0 \\ u_{2,1} & g_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ u_{r,1} & \cdots & u_{r,r-1} & g_r \end{pmatrix}$$

where $g_i \in \mathrm{GL}(M_i)$ and $u_{j,i} \in \mathrm{Hom}_{\mathbb{C}}\big(M_i, M_j \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)\big)$. For $j \geq i$, let $\bar{g}_j^i := g_j \otimes \mathrm{Id}_{\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_j)}$ and $\bar{u}_{k,l}^i$ be constructed from

$$u_{k,l} : M_l \to M_k \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_l, \mathcal{E}_k).$$

Given $u_{k,l}$ then define $\bar{u}_{k,l}^i$ for $i \leq l \leq k$ by

$$M_l \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_l) \xrightarrow{u_{k,l} \otimes \mathrm{Id}} M_k \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_l, \mathcal{E}_k) \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_l) \xrightarrow{\mathrm{Id} \otimes \kappa} M_k \otimes \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}_i, \mathcal{E}_k)$$

where κ is the composition map. Finally set $\theta_L = (\theta_{L,1}, \ldots, \theta_{L,r})$ where

$$\theta_{L,i}(h) := \begin{pmatrix} \bar{g}_i^i & 0 & \cdots & 0 \\ \bar{u}_{i+1,i}^i & \bar{g}_{i+1}^i & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \bar{u}_{r,i}^i & \cdots & \bar{u}_{r,r-1}^i & \bar{g}_r^i \end{pmatrix}$$

According to Drézet and Trautmann [25, p. 138] it follows that θ_L is a closed embedding of algebraic groups.

Remark B.2.2. Consider the rational H-module W and the rational G-module V then $\zeta : W \to V$ is a closed embedding that is equivariant with respect to the closed embedding of linear algebraic groups $\theta : H \to G$. In other words θ is a closed embedding of algebraic groups and $\zeta : W \to V$ is a closed embedding of affine varieties such that the following diagram commutes

$$\begin{array}{ccc} H \times W & \xrightarrow{\sigma_{H,W}} & W \\ \Big\downarrow{\scriptstyle \theta\times\zeta} & & \Big\downarrow{\scriptstyle \zeta} \\ G \times V & \xrightarrow{\sigma_{G,V}} & V \end{array}$$

In particular we have that $w, w' \in W$ belong to the same H-orbit if and only if $\zeta(w)$ and $\zeta(w')$ belong to the same G-orbit in V [25, Cor 5.3.2]. Furthermore H is the stabilizer of $\zeta(0)$ under the G-action on V [25, Lemma 5.3.1].

B.3 The notion of stability

Recall there is a linear action of $H := \mathrm{Aut}(\mathcal{E}) \times \mathrm{Aut}(\mathcal{F})$ on $W = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})$. In this section, we define following the paper by Drézet and Trautmann [25] the (semi)stable subset $W^{(s)s}(H, \Phi) \subset W$ for the H-action on V with respect to a character χ_Φ of H, where Φ is a proper polarization as defined below.

Definition B.3.1. A *proper polarization* $\Phi = (\lambda_1, \ldots \lambda_r, -\mu_1, \ldots, -\mu_s)$ for the R-action on V is given by a sequence of positive rational numbers λ_i, μ_j such that

$$\sum_{i=1}^{r} \lambda_i \dim M_i = \sum_{j=1}^{s} \mu_j \dim N_j = 1. \tag{B.1}$$

Remark B.3.2. A proper polarization corresponds to a linearisation of the action by a character. Given a proper polarization Φ, let t be the smallest common denominator of the numbers $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s$ then

$$\chi_\Phi(h_\mathcal{E}, h_\mathcal{F}) = \bigtimes_{i=1}^{r} \det(h_{\mathcal{E},i})^{t\lambda_i} \bigtimes_{j=1}^{s} \det(h_{\mathcal{F},j})^{-t\mu_j}$$

is a character of H, where $h_{\mathcal{E},i}$ respectively $h_{\mathcal{F},j}$ are the restricted automorphisms from $M_i \otimes \mathcal{E}_i \to M_i \otimes \mathcal{E}_i$ respectively $N_j \otimes \mathcal{F}_j \to M_j \otimes \mathcal{F}_j$.

Definition B.3.3. Let V be a rational R-module, $\chi : R \to \mathbb{G}_\mathrm{m}$ be a character and $f \in \mathbb{C}[V]$ be a polynomial. We call f χ-invariant, if for every $v \in V$ and $r \in R$ we have $f(r \cdot v) = \chi(r) f(v)$.

A point $v \in V$ is (R, χ)-semistable if there exists an integer $n \geq 1$ and a χ^n-invariant polynomial $f \in \mathbb{C}[V]$ such that $f(v) \neq 0$. The point v is (R, χ)-stable if and only if

1. v is (R, χ)-semistable,
2. $\dim(R \cdot v) = \dim(R/\mathbb{G}_m) = \dim(R) - 1$ and
3. the action of R on $V_f := \{v' \in V | f(v') \neq 0\}$ is closed.

Remark B.3.4.
1. Let $\chi : R \to \mathbb{G}_\mathrm{m}$ be a character associated to a proper polarisation Φ. Then $W^{ss}(R, \chi) = \pi^{-1}(\mathbb{P}(W)^{ss}(\mathcal{O}_{\mathbb{P}(W)}(t))$ for a suitable linearised $\mathcal{O}_{\mathbb{P}(W)}(t)$ see [25, p. 122-123] where $\pi : W \setminus \{0\} \to \mathbb{P}(W)$ is the canonical quotient morphism.
2. For (R, χ)-(semi)stability we have the following criterion by A. King: A point $f \in \mathrm{Hom}(\mathcal{E}, \mathcal{F})$ is χ-(semi)stable if and only if for each family of subvector spaces $((M_i')_{1 \leq i \leq r}, (N_j')_{1 \leq j \leq s})$ which is neither the trivial family $((0), (0))$ nor the maximal family $((M_i), (N_j))$ and which satisfies for each $i, j : f(M_i' \otimes \mathcal{E}_i) \subset N_j' \otimes \mathcal{F}_j$ we have $\sum_{i=1}^{r} t\lambda_i \dim M_i' + \sum_{j=1}^{s} -t\mu_j \dim N_j' (\leq) < 0$.

Definition B.3.5. For the pair (H, χ) a point $v_0 \in V$ is (semi)stable, if every $v \in H \cdot v_0$ is (R, χ)-(semi)stable.

Remark B.3.6. According to [25, p. 120] it follows, that if we want to have stable points that is $V^s(H, \Phi) \neq \varnothing$ then we need for each i that $\lambda_i > 0$ and for each j we have $-\mu_j < 0$.

B.4 Examples

Example B.4.1. [25, §4.2]

Consider $V := \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-2)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2})$. In this case we have three notions of H-(semi)stability, two of these notions correspond to nonsingular polarizations. A polarization for the action of H on V is a triple $\Phi = (\frac{1}{2}, -\mu_1, -\mu_2)$ with μ_1, μ_2 positive rational numbers satisfying $\mu_1 + \mu_2 = 1$. In this example the polarization $\Phi_0 := (\frac{1}{2}, \frac{-3}{4}, \frac{-1}{4})$ corresponds to the character $\chi : H \to \mathbb{G}_m$ given by $\chi(g, (\begin{smallmatrix} \lambda_1 & z \\ 0 & \lambda_2 \end{smallmatrix})) = \det(g)^{-2}\lambda_1^3\lambda_2$. In particular $\mu_2 = 1 - \mu_1$, hence we only depend on μ_1. There is only one singular polarization corresponding to $\mu_1 = \frac{1}{2}$. If $\mu_1 > (<)\frac{1}{2}$ then stability equals semistabiliy. If $\mu_1 > \frac{1}{2}$ then $V^s(H, \Phi)$ has a geometric quotient which is the universal cubic $Z \subset \mathbb{P}^2 \times \mathbb{P}^9$.

Example B.4.2. [39, §2]

$V_d := \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^2}(-d+1)^{\oplus 2}, \mathcal{O}_{\mathbb{P}^2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}^2})$, which is isomorphic to $\mathbb{A}^{d^2+d+6}$. For $d = 3$ this example was obtained already by Drézet and Trautmann. Let X_d be the space of morphisms $\mathcal{O}_{\mathbb{P}^2}(-d+1)^{\oplus 2} \xrightarrow{A} \mathcal{O}_{\mathbb{P}^2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}^2}$ where the matrix A is given by $\begin{pmatrix} z_1 & z_2 \\ q_1 & q_2 \end{pmatrix}$ and has the following properties:

1. the entries z_1 and z_2 are linearly independent, and

2. A has a non-zero 'determinant' $z_1q_2 - z_2q_1$.

The group $H = \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^2}(-d+1)^{\oplus 2}) \times \mathrm{Aut}(\mathcal{O}_{\mathbb{P}^2}(-d+2) \oplus \mathcal{O}_{\mathbb{P}^2})$ acts linearly on X by the rule $(h, g) \cdot A = gAh^{-1}$.

The morphism $X \xrightarrow{\pi} M$, $\begin{pmatrix} z_1 & q_1 \\ z_2 & q_2 \end{pmatrix} \mapsto (\langle z_1q_2 - z_2q_1 \rangle, z_1 \wedge z_2)$ where $z_1 \wedge z_2$ is the common zero of z_1 and z_2.

Any point $x \in X_d$ has $\mathbb{G}_m \cong \mathrm{Stab}_H(x) = \Delta \triangleleft H$. Denote by $PH := H/\Delta$ then $\pi : X \to M$ is a principal PH-bundle and in particular a geometrical quotient for PH acting on X_d.

B.5 Comparison with reductive GIT

In the previous section we have defined the Drézet-Trautmann (semi)stable set $V^{ss}(H,\Phi) \subset V$ for the H-action on V with respect to a character χ_Φ. Recall that there is a commutative diagram

$$\begin{array}{ccc} H \times V & \longrightarrow & V \\ \big\downarrow{\scriptstyle \theta\times\zeta} & & \big\downarrow{\scriptstyle \zeta} \\ G \times W & \longrightarrow & W \end{array}$$

such that G is a reductive group. In this situation we can consider W as an affine completion of $G \times^H V$ and compare $\zeta^{-1}\big(W^{ss}(G,\Psi)\big)$ with $V^{ss}(H,\Phi)$.

Let Φ be the associated polarization for the action of R on V, then there exists a polarization Ψ for the action of G on V that is compatible with $\zeta : W \to V$ and Φ. More precisely,

Definition B.5.1. Let Φ be a proper polarization of the action of H on V, in the sense of Definition B.3.1. Then $\Psi = (\alpha_1, \ldots, \alpha_r, -\beta_1, \ldots, -\beta_s)$ is given as the unique solution to the following two linear equations

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \vdots \\ \lambda_r \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ a_{2,1} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ a_{r,1} & \cdots & a_{r,r-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_r \end{pmatrix}$$

and

$$\begin{pmatrix} \mu_1 \\ \vdots \\ \vdots \\ \mu_s \end{pmatrix} = \begin{pmatrix} 1 & b_{2,1} & \cdots & b_{2,s} \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & b_{s,s-1} \\ 0 & \cdots & 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \vdots \\ \beta_s \end{pmatrix}$$

where $a_{j,i} := \dim \operatorname{Hom}(\mathcal{E}_i, \mathcal{E}_j)$ and $b_{m,l} := \dim \operatorname{Hom}(\mathcal{F}_m, \mathcal{F}_l)^\vee$. Setting $p_i := \dim \operatorname{Hom}(\mathcal{E}_i, \mathcal{E})$ and $q_j := \dim \operatorname{Hom}(\mathcal{F}, \mathcal{F}_j)^\vee$ we have that

$$\sum_{i=1}^{r} \alpha_i p_i = \sum_{j=1}^{s} \beta_j q_j = 1 \tag{B.2}$$

Definition B.5.2. The semi-stable locus $V^{ss}(G,\Psi)$ with respect to the polarization Ψ is more precisely defined by the character χ_Ψ associated to it. Let q the lowest common denominator of $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$, we have $\chi_\Psi(g,h) = \prod_{i=1}^r \det(g_i)^{q\alpha_i} \prod_{j=1}^s \det(h_j)^{q\beta_j}$.

Remark B.5.3. The relation between $\chi_\Phi : H \to \mathbb{G}_\mathrm{m}$ and $\chi_\Psi : G \to \mathbb{G}_\mathrm{m}$ is given by $\chi_\Phi = \chi_\Psi^u|_H$ where we consider H as a subgroup of G and p respectively q are the lowest common denominators with respect to λ_i, μ_j respectively $\alpha_i, \beta|j$ and u is given by $q = up$. [25, p. 142]

Lemma B.5.4. *Consider the polarized action of (H, Φ) on W let Ψ be the polarization of G associated to Φ. Then*

1. $\zeta^{-1}\big(W^{ss}(G,\Psi)\big) \subset V^{ss}(H,\Phi)$

   and

2. $\zeta^{-1}\big(W^{s}(G,\Psi)\big) \subset V^{s}(H,\Phi)$

That is, if $v \in V$ and $\zeta(v)$ is (G,Ψ)-(semi)stable, then v is (H,Φ)-(semi)stable in the sense of Drézet and Trautmann [25, Lemma 5.5.1, Lemma 5.5.3].

Let us denote by $Z := G\zeta(V) \subset W$. Then in the case that $\zeta^{-1}\big(W^{s}(G,\Psi)\big) = V^{s}(H,\Phi)$ we have the following result:

Proposition B.5.5. *If $\zeta^{-1}\big(W^{s}(G,\Psi)\big) = V^{s}(H,\Phi)$, then there exists a geometric quotient $V^s(H,\Phi) \to M^s$ which is a quasi-projective non-singular variety. If in addition $\zeta^{-1}\big(W^{ss}(G,\Psi)\big) = V^{ss}(H,\Phi)$ and $(\bar{Z} \setminus Z) \cap W^{ss}(G,\Psi) = \emptyset$ then there exists a good quotient $V^{ss}(H,\Phi) \xrightarrow{\pi} M$ and $V^s(H,\Phi) \to M^s$ is the restriction of π to $V^s(H,\Phi)$. [25, Prop 6.1.1]*

Considering the previous Lemma and Proposition we are interested in the case that also the opposite inclusions hold $\zeta^{-1}\big(W^{ss}(G,\Psi)\big) \supset V^{ss}(H,\Phi)$ and $\zeta^{-1}\big(W^{s}(G,\Psi)\big) \supset V^{s}(H,\Phi)$.
For any l we consider the composition maps

$$\mathrm{Hom}(\mathcal{E}_1,\mathcal{F}_l)^\vee \otimes \mathrm{Hom}(\mathcal{E}_1,\mathcal{E}_i) \to \mathrm{Hom}(\mathcal{E}_i,\mathcal{F}_l)^\vee$$
$$w \otimes x \mapsto ev_{w\otimes x} : \mathrm{Hom}(\mathcal{E}_i,\mathcal{F}_l) \to \mathbb{C}$$
$$v \mapsto w(v \circ x)$$

which induce the map

$$\delta_l : \bigoplus_{i=2}^{r} M_i \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{F}_l)^\vee \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i) \to \bigoplus_{i=2}^{r} M_i \otimes \mathrm{Hom}(\mathcal{E}_i, \mathcal{F}_l)^\vee.$$

Let $\mathcal{K}$ be the set of proper linear subspaces $K \subset \bigoplus_{i=2}^{r} M_i \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i)$ that are not contained in $\bigoplus_{i=2}^{r} M_i' \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i)$ with $\big(M_i'\big)_{i=2,\dots,r}$ a family of subspaces of $\big(M_i\big)_{i=2,\dots,r}$ such that there exists at least at least one i_0 with $M_{i_0}' \subsetneq M_{i_0}$.

Definition B.5.6. Let

$$c_l(m_2, \dots, m_r) := \sup_{K \in \mathcal{K}} \rho(K)$$

where

$$\rho(K) := \frac{\operatorname{codim} \delta_l(K \otimes \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_i))}{\operatorname{codim} K}$$

Corollary B.5.7. *Assume that $s = 1$ that is $\mathcal{F} = N_1 \otimes \mathcal{F}_1$. Let $\Phi = (\lambda_1, \dots, \lambda_r, \frac{-1}{n_1})$ be a polarisation and let $\Psi = (\alpha_1, \dots, \alpha_r, \frac{-1}{n_1})$ the associated polarisation. Then if each $\alpha_i > 0$ and if $\lambda_2 \geq \frac{\dim \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)}{n_1} c_1(m_2, \dots, m_r)$ then $\zeta^{-1}\big(W^{ss}(G, \Psi)\big) \supset V^{ss}(H, \Phi)$ and $\zeta^{-1}\big(W^{s}(G, \Psi)\big) \supset V^{s}(H, \Phi)$. [25, Cor 7.2.2]*

Example B.5.8. Let $\mathcal{E} := M_1 \otimes \mathcal{O}_{\mathbb{P}^N}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^N}(-d_2)$ and $\mathcal{F} = N_1 \otimes \mathcal{O}_{\mathbb{P}^N}$. Here

$$\delta_l : \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^N}(-d_1), \mathcal{O}_{\mathbb{P}^N})^\vee \otimes \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^N}(-d_1), \mathcal{O}_{\mathbb{P}^N}(-d_2)) \to \mathrm{Hom}(\mathcal{O}_{\mathbb{P}^N}(-d_2), \mathcal{O}_{\mathbb{P}^N})^\vee$$

identifies to

$$\delta_l : S^{d_1}V^\vee \otimes S^{d_1-d_2}V \to S^{d_2}V^\vee.$$

Let K be a subspace of $S^{d_1-d_2}V$ then if $K \neq 0$ we have that δ_l is surjective hence $\rho(K) = 0$. To see that δ_l is surjective take $0 \neq f \in K \subset S^{d_1-d_2}V$ and consider the natural map $\mu : S^{d_1-d_2}V \otimes S^{d_2}V \to S^{d_1}V$ and restrict to $\mu(f \otimes -) : S^{d_2}V \to S^{d_1}V$. Let $\lambda_2 : S^{d_2}V \to \mathbb{C}$ be linear then pick $\lambda_1 : S^{d_1}V \to \mathbb{C}$ to be any linear map λ_1 such that the diagram

$$\begin{tikzcd} S^{d_2}V \arrow[r, "\lambda_2"] \arrow[d, "\mu(f\otimes -)"] & \mathbb{C} \arrow[d, "\mathrm{Id}"] \\ S^{d_1}V \arrow[r, "\lambda_1"] & \mathbb{C} \end{tikzcd}$$

commutes. Since $\rho(K) = 0$ it follows that $c_1(1) = 0$. Let $\Phi = (\lambda_1, \lambda_2, \frac{-1}{n_1})$ be an admissible polarisation with $\lambda_1, \lambda_2 > 0$ then the associated polarisation $\Psi = (\alpha_1, \alpha_2, \frac{-1}{n_1})$ is given by

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \dim \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}.$$

Then we have

$$\begin{aligned} \alpha_1 &= \lambda_1 \\ \alpha_2 &= \lambda_2 - \dim \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)\lambda_1 \\ &= 1 - (\dim \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) + m_1)\lambda_1 \end{aligned}$$

hence for $0 < \lambda_1 < \frac{1}{\dim \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2) + m_1}$ it follows that $\alpha_1, \alpha_2 > 0$. By the above results there exists a good quotient for $W^{ss}(G, \Phi)$ which restricts to a geometric quotient on $W^s(G, \Phi)$.

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Selbständigkeitserklärung

Hiermit versichere ich, Benedikt Vincent Josias Trageser,

1. dass ich alle Hilfsmittel und Hilfen angegeben habe,
2. dass ich auf dieser Grundlage die Arbeit selbständig verfasst habe,
3. dass diese Arbeit nicht in einem früheren Promotionsverfahren eingereicht worden ist.

Unterschrift: