On Local Real Algebraic Geometry and Applications to Kinematics

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Abstract

In contrast to complex varieties, a real algebraic variety X embedded in euclidean space $\mathbb{R}^n$ can still be smooth at a singular point p, in the sense that X is locally an analytic submanifold of $\mathbb{R}^n$. This happens because the analytic Nullstellensatz does not hold for real analytic varieties, i.e. parts of analytic branches of X at p might not be visible in real space.

Identifying non-manifold points of real algebraic sets requires a novel approach and theory from analytic and real algebraic geometry. We will present several results to (dis)prove smoothness and also address the problem to check those criteria algorithmically. Based on those results we have implemented a `Singular`-procedure for algebraic curves.

As an application we will determine all configuration space singularities of several well-known kinematic linkages.

Contents

Chapter 1

Introduction

1.1 Motivation

Real (semi-)algebraic sets, i.e. real solution sets of polynomial (in)equations arise naturally in theoretical kinematics as configuration spaces (workspaces) of linkages. In a basic description a linkage is a set of k rigid bodies, the **links**, which are connected by joints. As a simple example let us consider the planar slider-crank of Figure 1.1 which is well-known from piston designs. It consists of a rotating rod (the crank) of

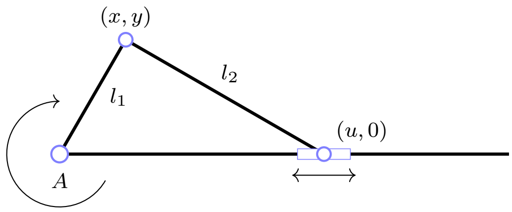


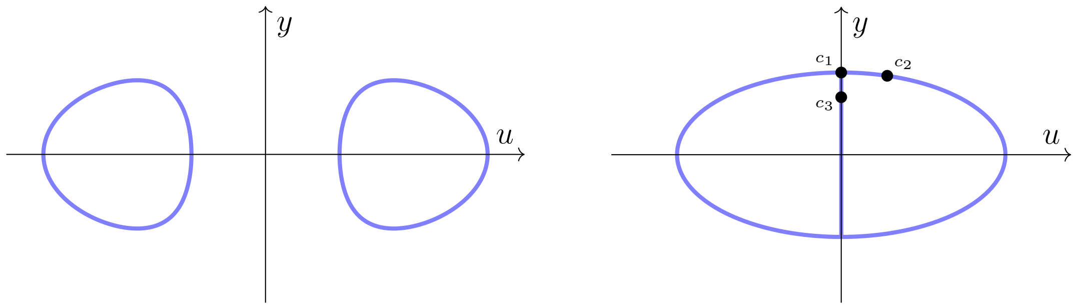
Figure 1.1: Slider-crank linkage.

length l_1 connected to a sliding piston with a rod of length l_2. Blue circles in the figure mean revolute (rotational) joints. The configuration space X of the slider-crank is defined as the set of all possible assembly configurations. It can be represented by the real algebraic set $X = \mathbf{V}_{\mathbb{R}}(I)$, where $I = \langle q_1, q_2 \rangle \leq \mathbb{R}[x, y, u]$ is generated by the polynomials

$$\begin{aligned} q_1 &= x^2 + y^2 - l_1^2, \\ q_2 &= (u - x)^2 + y^2 - l_2^2, \end{aligned} \tag{1.1}$$

which are just the length constraints in euclidean coordinates. Clearly, any complex zero of I does not correspond to an actual configuration of the manipulator. For this reason we need to consider the real algebraic set X if we want to study kinematic properties of the linkage. Features of the complex algebraic set $X_{\mathbb{C}} = \mathbf{V}(I) \subset \mathbb{C}^3$ are usually misleading.

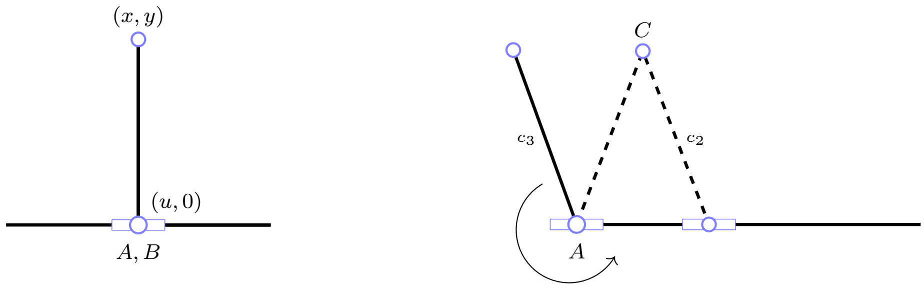
For example, if $l_1 \neq l_2$ then $I \cdot \mathbb{C}[x, y, u]$ is a prime ideal, so $X_{\mathbb{C}}$ must be path-connected, but X is not path-connected (or connected for that matter). In fact, in the euclidean topology X is just the disjoint union of two circles. This can be



(a) Configuration space for $l_1 < l_2$. (b) Configuration space for $l_1 = l_2$.

Figure 1.2: Configuration spaces for the slider-crank.

seen in Figure 1.2a, which shows the projection of X on the yu-plane. The mapping $x \mapsto -x$, $y \mapsto y$, $u \mapsto -u$ gives a symmetry of the configuration space corresponding to the reflection of the slider-crank pose at the vertical axis through the joint A. The disconnectedness of the configuration space means for applications that we need to disassemble the mechanism if we want to switch between configurations in different components.



(a) Singular configuration c_1. (b) Two configurations c_2, c_3 close to c_1.

Figure 1.3: CS-singularity of the slider-crank linkage.

Points $p \in X$, where X is not locally a submanifold of euclidean space are called **configuration space (CS) singularities** of the linkage. In such configurations the linkage will exhibit degenerate kinematic behavior.

Consider the slider-crank linkage for $l_1 = l_2$. Its configuration space X – projected on the yu-plane – is shown in Figure 1.2b with points marking the configurations of Figure 1.3. X decomposes in the Zariski topology in two irreducible components intersecting in the configuration c_1 of Figure 1.3a. In this pose the crank can rotate freely while the sliding joint B is fixed to the origin, see Figure 1.3b. But the slider can also move left or right, which opens the angle at the middle joint C. We have drawn this with dashed lines in Figure 1.3b. The configuration c_1 is clearly critical for the mechanism, since it can get jammed and any wrong torque or force in the joints can damage the linkage.

This motivates the guiding problem for the following thesis: **Identify and analyze all points p of a real algebraic set X, in which X is not locally a submanifold of euclidean space**. We will call such points **non-manifold points** of X.

Note that for complex algebraic sets the problem of identifying non-manifold points reduces to calculating the singular locus of the coordinate ring. But we will see it is more intricate in the real setting.

1.2 Singularities and manifold points

We will use standard terminology of algebraic geometry in this section [24, Chapters 1-2], [33, Chapter I].

Let Y be a complex variety embedded in euclidean space $\mathbb{C}^n$ and $p \in Y$. It is well-known, that Y is locally a smooth submanifold of $\mathbb{R}^{2n}$ at p if and only if the local ring $\mathcal{O}_{p,Y}$ is regular, i.e. p is a nonsingular point of Y. If p is nonsingular then the Jacobian criterion and the implicit function theorem immediately imply that Y is locally a submanifold. On the other hand, the proof that Y is locally not a submanifold at a singular point p is more involved. We will use the shorthand (SNM) for this harder proposition.

One straightforward approach to prove (SNM) for complex algebraic sets is given in [43, Theorem IV.4.3]. Here, the author is analyzing a singular point $p \in Y$ using Cauchy's integral formula to count the number of points in Y for a choice of $\dim(X)$ coordinates fixed, close to p. He concludes that Y will not be the graph of a function in any coordinates and thus is not even a topological submanifold of $\mathbb{R}^{2n}$. This proof relies heavily on the fact that $\mathbb{C}$ is algebraically closed and unsurprisingly the same does not hold for real algebraic sets.

A simple counterexample to (SNM) for real algebraic sets is given in remark 7.8 of [43]: Let $f_1(x,y) = (x^2+y^2)\cdot(y-x^2) \in \mathbb{R}[x,y]$. Then, the set of real zeros $X_1 = \mathbf{V}_{\mathbb{R}}(f_1)$ is just the parabola $y - x^2 = 0$, but the local ring $(\mathbb{R}[x,y]/\langle f\rangle)_{\langle x,y\rangle}$ is certainly not regular. This should not worry us much since clearly $\langle f\rangle \neq \mathbf{I}(X_1) = \langle y - x^2\rangle$ and the $\mathbb{R}$-algebra $\mathbb{R}[x,y]/\langle y-x^2\rangle$ is regular, i.e. all localizations at prime ideals are regular.

One common misconception supported by this example is that the problem with (SNM) for real algebraic sets lies just in the determination of the structure sheaf $\mathcal{O}$ and will disappear if we consider only **real ideals** $I \subset \mathbb{R}[x_1,\dots,x_n]$, with the property that $I = \mathbf{I}(\mathbf{V}_{\mathbb{R}}(I))$.

Unfortunately, this is incorrect which can be seen in the following example, recorded by Milnor [49]: Let $f_2(x,y) = y^3 + 2\,x^2\,y - x^4 \in \mathbb{R}[x,y]$. Then, the local ring $(\mathbb{R}[x,y]/\langle f_2\rangle)_{\langle x,y\rangle}$ is not regular and $\mathbf{I}(\mathbf{V}_{\mathbb{R}}(f_2)) = \langle f_2\rangle$, making $\langle f_2\rangle$ a real ideal. However, $X_2 = \mathbf{V}_{\mathbb{R}}(f_2)$ is still a smooth submanifold of $\mathbb{R}^2$, see Figure 1.4.

The reason for this unexpected behavior is that not all analytic branches of $\mathbf{V}_{\mathbb{C}}(f_2)$ at the origin are visible in the real picture. Let us decompose $\langle f_2\rangle$ in the ring of

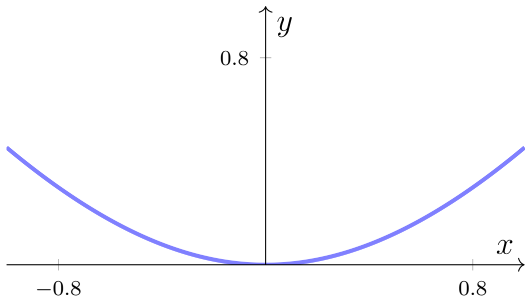


Figure 1.4: $X_2 = \mathbf{V}_{\mathbb{R}}(y^3 + 2\,x^2\,y - x^4)$.

convergent power series $\mathbb{R}\{x, y\}$. The following factorization can be found easily with the quadratic formula:

$$\begin{aligned} f(x,y) &= y^3 + 2\,x^2\,y - x^4 = \\ &= \left(x^2 - y\left(1 + \sqrt{1+y}\right)\right) \cdot \left(x^2 - y\left(1 - \sqrt{1+y}\right)\right). \end{aligned} \tag{1.2}$$

Now $y\left(1 - \sqrt{1+y}\right)$ is negative for y close to 0, hence the real zero set of $g(x,y) = x^2 - y\,\left(1 + \sqrt{1+y}\right) \in \mathbb{R}\{x,y\}$ coincides with the real zero set of f on the domain of g. Since $(\partial_x g, \partial_y g) \neq (0,0)$ at the origin, X_2 is clearly a smooth submanifold there. The analogy to the problem with X_1 is obvious since in both examples something is disappearing in the real picture, making the singular variety a manifold. In both cases the “vanishing ideal" is actually bigger. But in place of Hilbert's Nullstellensatz it is now Rückert's analytic Nullstellensatz [32] which does not hold.

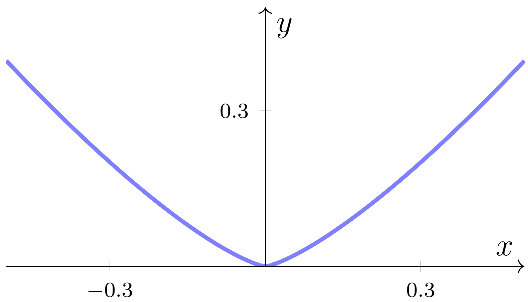


Figure 1.5: $\mathbf{V}_{\mathbb{R}}(y^3 - x^4)$.

We need to mention another phenomenon in the real case. Even if we have a singular irreducible analytic branch, the corresponding real set can still be a C^k-submanifold for k finite. For instance, consider $X_3 = \mathbf{V}_{\mathbb{R}}(y^3 - x^4)$ from Figure 1.5. X_3 is a C^1-submanifold of $\mathbb{R}^2$, but not C^2.

The picture with its visible bend at the origin is misleading in this case, as we can make X_3 looking arbitrarily smooth. The curve $y^3 - x^{4+3\,k} = 0$ gives a singular irreducible analytic branch, which is a C^{k+1}-submanifold of $\mathbb{R}^2$.

In light of the examples X_2, X_3 we see that for real algebraic sets X we need to discern between nonsingular points and points where X is locally a smooth submanifold of $\mathbb{R}^n$, which we call **manifold points**. Any nonsingular point will be a manifold point, but not the other way around.

1.3 Real tangent cones and Nash fibers

Closely related to the problem of identifying non-manifold points, but more general, is the task of calculating the real **geometric tangent (semi-)cone** and the **geometric Nash fiber** at a point p of a real algebraic set X. The tangent (semi-)cone is just the union of lines (rays) which arise as limits of secant lines (rays) with one intersection point fixed at p and the other intersection point approaching p. The geometric Nash fiber [60] is the set of all linear subspaces which arise as limit of tangent spaces at nonsingular points approaching p, where the limit is taken in the Grassmannian.

It is well known that over the complex numbers the geometric tangent cone equals the algebraic tangent cone, which is defined as the zero set of the ideal generated by the initial forms of the polynomials (expanded about p) in the ideal of X. For example, the algebraic tangent cone of $X_2 = \mathbf{V}_{\mathbb{R}}(y^3 + 2\,y\,x^2 - x^4)$ at the origin is given by the zeros of $y^3 + 2\,y\,x^2 = y\,(y^2 + 2\,x^2)$.

In this case the real part of the algebraic tangent cone is just $y = 0$ which is the real geometric tangent cone and also the geometric Nash fiber.

Unfortunately, this fails in general, even for curves. Let $f_4(x, y) = x^2\,y + y^5 - x^6$ and $X_4 = \mathbf{V}_{\mathbb{R}}(f_4)$, see Figure 1.6. From the algebraic tangent cone $x^2\,y = 0$ we expect real vertical and horizontal tangents, however X_3 is again a smooth submanifold of $\mathbb{R}^2$. For higher-dimensional varieties the real tangent is in general not even algebraic anymore but rather semi-algebraic (e.g. $z^2 - x^3\,y^3 = 0$ has the first and third quadrant of the $z = 0$ plane as geometric tangent cone).

The calculation of the real tangent cone in general is a notoriously difficult problem, which has attracted only little attention over the years, so that not many published results exist. Much of what is known only works for curves and is a byproduct of parameterizing the curve with Puiseux series and discarding complex solutions [18].

A standout work regarding Nash fibers is [60], in which they formulated a result for surfaces, close in nature to the theory of Le, Henry, Teissier [35, 45] for complex surfaces (and also generalized for arbitrary varieties). It describes the local geometry of a real surface X at a singular point by the so called **aureole** comprising the geometric tangent cone C and a set of exceptional lines in C such that the Nash fiber of X is the union of the Nash fiber of C and the set of all planes containing one of the exceptional lines. But while the exceptional lines and the Nash fiber for complex surfaces can be algorithmically determined by elimination [59], similar methods fail for real surfaces. To the authors knowledge, there is no symbolic algorithm known to calculate the (Zariski closure of the) real tangent cone or the Nash fiber.

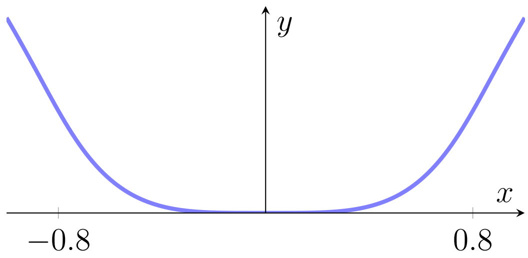
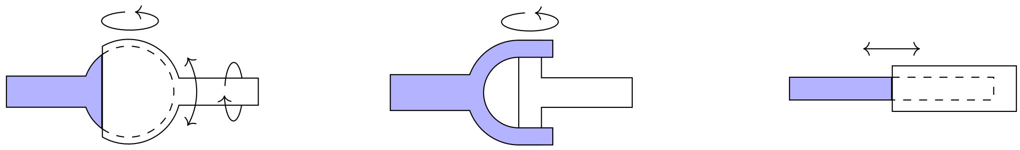


Figure 1.6: The zero set of $-x^2y - y^5 + x^6$



(a) Spherical joint. (b) Revolute joint. (c) Prismatic joint.

Figure 1.7: Three types of joints, corresponding to the lie-subgroups $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(2,\mathbb{R})$ and $\mathbb{R}$.

We should mention one more related approach to the problem of regularity of real algebraic sets. In [25] techniques of geometric measure theory are utilized to analyze real hypersurfaces for which every geometric tangent cone is a hyperplane. Among other things they were able to show that real analytic hypersurfaces which form the boundary of an open convex set are C^1. This cannot be improved upon. X_3 (Figure 1.5) is a convex real algebraic curve that is not C^2 and can easily be modified to a closed curve by taking a different chart in the projective closure.

1.4 Linkages and configuration spaces

We pick up from the end of Section 1.1. The position and orientation of a rigid body in euclidean 3-space is given by an element of the special euclidean group $\mathrm{SE}(3,\mathbb{R}) = \mathrm{SO}(3) \ltimes \mathbb{R}^3$. Thus, the configuration space X of a linkage with k links is the set of all configurations in $(\mathrm{SE}(3,\mathbb{R}))^k$, subject to constraints imposed by the connecting joints. Since $\mathrm{SE}(3,\mathbb{R})$ is a real algebraic set any linkage configuration space will be a real algebraic set as long as the joints constrain the relative movement of the links to algebraic subgroups of $\mathrm{SE}(3,\mathbb{R})$. This is the case for spherical, revolute (rotational) and prismatic joints, corresponding to algebraic (lie)-subgroups $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(\mathbb{R},2)$ and $\mathbb{R}$, Figure 1.7. See Section 6.1 or [68] for details.

The following examples will illustrate several concepts from kinematics. Consider the planar 2R-chain of Figure 1.8. This mechanism consists of two links connected by revolute joints which are indicated by blue circles. The joint coinciding with the origin of the coordinate frame is meant to be fixed to the plane. Any linkage in which

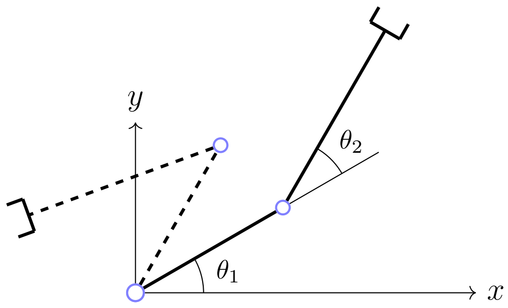


Figure 1.8: Planar 2R kinematic chain

the links are connected in series is called a **serial kinematic chain**. If the serial chain is connected in a loop, we say it is **closed**, otherwise it is **open**. Since the 2R-chain of Figure 1.8 is open, every joint constrains only the next link in the series. Thus, every configuration of this linkage is determined by the two joint coordinates (θ_1, θ_2). This implies that the configuration space is isomorphic (as real algebraic set) to the torus $\mathrm{SO}(2, \mathbb{R}) \times \mathrm{SO}(2, \mathbb{R}) \cong S^1 \times S^1$.

For modern industrial robots as in Figure 1.14, usually 6 (or 7 with redundant systems) revolute joints are connected by links. Then, the configuration space will be $(S^1)^t$, where t is the number of joints. In some instances, e.g. the Stanford arm [55, Example 3.9], revolute joints are replaced with prismatic joints. This gives a configuration space of $(S^1)^r \times \mathbb{R}^{t-r}$, where r is the number of revolute joints.

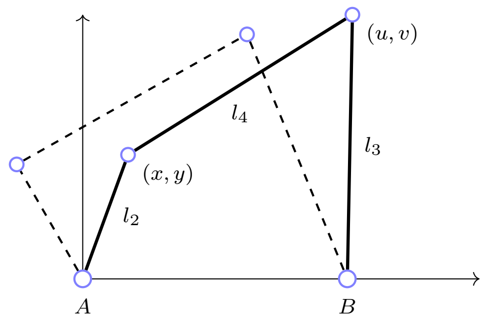


Figure 1.9: Four-bar linkage.

The study of configuration spaces gets more difficult for linkages with closed kinematic chains. One example is the slider-crank linkage from Section 1.1. Consider now the planar mechanism of Figure 1.9 which consists of four rigid rods connected by revolute joints. The four-bar linkage is one of the oldest and most widely used closed chain linkages. Notable first four-bar applications are the pantograph (1603, Figure 1.10) to scale and copy drawings, and Watt's Linkage (1784) to generate approximate straight line motions. Today, the four-bar can be found in numerous areas of engineering (Ackermann steering, windshield wipers, tire suspension, folding chairs).

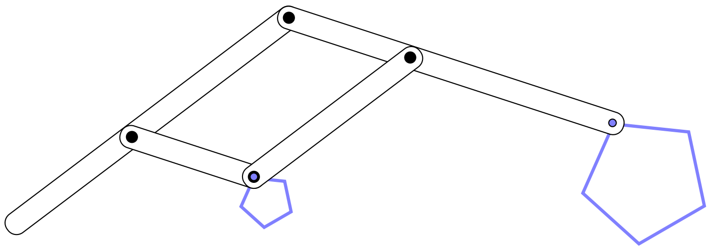


Figure 1.10: Pantograph mechanism.

To derive equations for the configuration space X of the four-bar, we express the length constraints in euclidean coordinates again. Then, we have $X = \mathbf{V}_{\mathbb{R}}(I)$, where the ideal $I = \langle p_1, p_2, p_3 \rangle \leq \mathbb{R}[x, y, u, v]$ is generated by the polynomials

$$\begin{aligned} p_1 &= x^2 + y^2 - l_2^2, \\ p_2 &= (u-2)^2 + v^2 - l_3^2, \\ p_3 &= (u-x)^2 + (v-y)^2 - l_4^2. \end{aligned}$$

The l_i are the parameters of the linkage, which are assumed to be positive real numbers. We fixed $l_1 = \overline{AB} = 2$, as any other length can be treated by scaling the system.

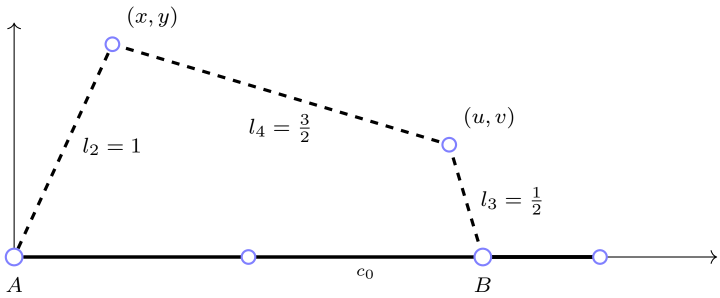
In contrast to serial kinematic chains, there can be singularities in the configuration space of linkages with closed kinematic chains. In Figure 1.3 we have seen a singular configuration of the slider-crank linkage. For the four-bar linkage there are singularities in X if and only if the design parameters (l_1, l_2, l_3, l_4) fulfill the Grashof condition [10]:

$$\pm l_2 \pm l_3 \pm l_4 = l_1 = 2. \tag{1.3}$$

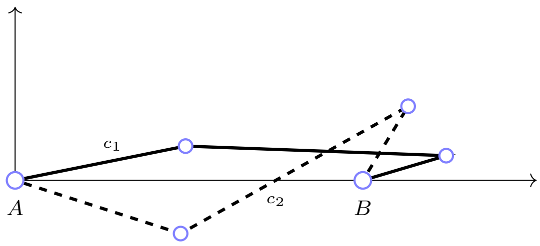
In this case a configuration of the four-bar exists for which all the links are lying on the x-axis.

As an example, we choose $l^* = (l_2, l_3, l_4) = \left(1, \frac{3}{2}, \frac{1}{2}\right)$, see Figure 1.11. Figure 1.11c shows the projection of X on the (y, v)-plane, where the origin is the folded configuration c_0, sketched with solid lines in Figure 1.11a. The two dots in the plot mark the configurations of Figure 1.11b, which are close to the singular configuration but on different analytic branches.

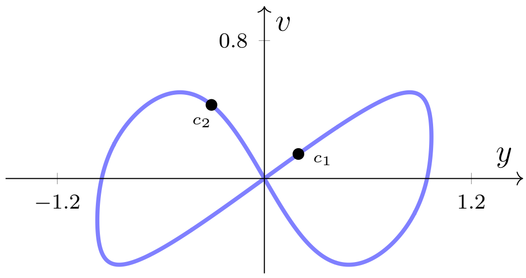
In contrast to the singular configuration space of the crank-slider in Figure 1.2b, I is prime for the chosen values. This makes it difficult to formally prove that the singularity $c_0 = (l_2, 0, l_2 + l_4, 0)$ is not a manifold point of X. But this is necessary to show that c_0 is a configuration space singularity of the four-bar linkage. Remember that the plot in Figure 1.11c is a projection and gives no indication without further analysis.



(a) Four-bar folded configuration.



(b) Two configurations close to singularity.



(c) Configuration space of four-bar.

Figure 1.11: CS-singularity of the four-bar with $l_2 = 1$, $l_3 = \frac{3}{2}$, $l_4 = \frac{1}{2}$.

Fortunately, the configuration space of single-loop planar linkages with only rotational joints is well understood. It is usually known as **polygonal space** in differential topology [21, 40, 41]. Morse theory is one possibility to prove that c_0 is not a manifold point [21, Theorem 2.6]. We quickly sketch this method.

Assume we break up the connection in the four-bar linkage between the joints at B and (u, v), Figure 1.12. Then, the configuration space of the remaining 2R-subchain anchored at A is the torus $T := S^1 \times S^1$. Now we define the regular mapping

$$\begin{aligned} g\colon T &\to \mathbb{R}, \\ (\vec{w}_1, \vec{w}_2) &\mapsto \left| l_2 \cdot \vec{w}_1 + l_4 \cdot \vec{w}_2 - (2, 0)^T \right|^2, \end{aligned}$$

where $g(\vec{w}_1, \vec{w}_2)$ gives the euclidean distance from B to (u, v). One can easily check, that

$$(x, y, u, v) \mapsto \left(\frac{x}{l_2}, \frac{y}{l_2}, \frac{u - x}{l_4}, \frac{v - y}{l_4} \right)$$

induces an isomorphism $\psi\colon X \xrightarrow{\sim} g^{-1}(l_3^2)$ of real algebraic sets.

According to [21, Lemma 1.4] the mapping g restricted to $T \backslash g^{-1}(0)$ has only non-degenerate singularities in the sense of Morse theory [28, Definition 6.1]. Hence g is a Morse function on $T \backslash g^{-1}(0)$.

We recall the parameter values $(l_2, l_3, l_4) = \left(1, \frac{1}{2}, \frac{3}{2}\right)$ of the four-bar from our example. With [21, Lemma 1.4] we check, that g has exactly one singularity $c_0' = \psi(c_0) =$

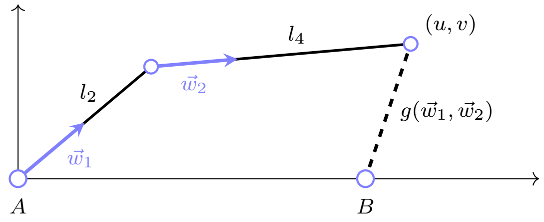
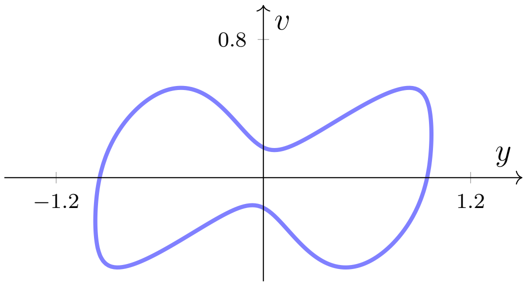


Figure 1.12: 2R-subchain of four-bar linkage.

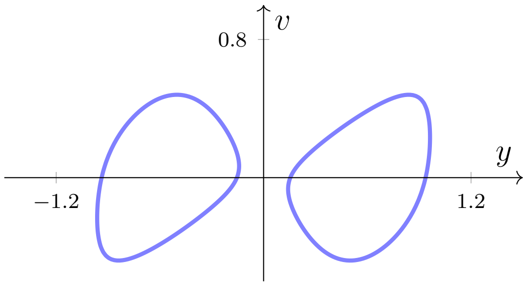
$(1, 0, 1, 0)$ in $g^{-1}(l_3^2)$ and its Morse index is 1. Now we can apply the Morse Theorem, see e.g. [28, Theorem 6.9]. There is a coordinate chart (z_1, z_2) centered at c_0' such that g expressed in this chart is

$$g(z_1, z_2) = l_3^2 + z_1^2 - z_2^2.$$

Thus, $g^{-1}(l_3^2)$ is locally homeomorphic at c_0' to the line intersection $z_1^2 - z_2^2$ in the plane. Therefore, c_0' is not a manifold point of $g^{-1}(l_3^2)$ and c_0 is not a manifold point of X.



(a) Configuration space for $l_3 = 0.52$.



(b) Configuration space for $l_3 = 0.48$.

Figure 1.13: Configuration space of the four-bar linkage with $l_2 = 1$, $l_4 = \frac{3}{2}$ and different values for l_3 close to $\frac{1}{2}$.

We can observe how the topology of X changes with variation of l_3 close to $\frac{1}{2}$ and the parameters $(l_2, l_4) = \left(1, \frac{3}{2}\right)$ fixed, see Figure 1.13. For $l_3 = \frac{1}{2}$ we pass the critical value $\frac{1}{4}$ of the Morse function g. When we examine all four-bars with $(l_2, l_4) = \left(1, \frac{3}{2}\right)$ the configuration space X looks as follows, depending on l_3 [10, 21, 47]:

$$X \cong \begin{cases} S^1 & \text{for } \frac{1}{2} < l_3 < \frac{3}{2} \text{ or } \frac{5}{2} < l_3 < \frac{9}{2}, \\ S^1 \sqcup S^1 & \text{for } 0 < l_3 < \frac{1}{2} \text{ or } \frac{3}{2} < l_3 < \frac{5}{2}, \\ S^1 \vee S^1 & \text{for } l_3 \in \left\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right\}, \\ \{0\} & \text{for } l_3 \in \left\{0, \frac{9}{2}\right\}, \end{cases}$$

where $\cong$ means homeomorphic, $\sqcup$ is the disjoint union and $\vee$ is the wedge sum, i.e. $S^1 \vee S^1$ denotes the figure eight.

This concludes our examples on CS-singularities. We have seen that Morse theory gives a powerful tool for the analysis of polygonal spaces or plane single-loop linkages. It is possible to use these ideas to analyze multi-loop linkages, see e.g. [6] for a strong genericity result about linkage singularities. However, this approach is usually difficult since there can be degenerate singularities in the sense of Morse and we need to account for multiple closure equations. An additional problem are linkages with prismatic joints and revolute spatial joints. Methods derived from the theory of polygonal spaces deal exclusively with graph-embeddings in euclidean space. Hence, the corresponding linkages have only spherical spatial joints or planar revolute joints.

Recently, efforts have been made in the kinematics community to define and categorize kinematic singularities of linkages in a rigorous way [54], [53], [56]. It has been observed [56, Example 6.3.4] that a closed 6R-chain exists with rank drop in the Jacobian of the constraint equations, but smooth configuration space nevertheless. In this example, the singular configuration turned out to be an embedded point of the configuration space, and would vanish upon taking the radical of the ideal of constraint equations.

Although it was not a singularity of the reduced configuration space, this finding renewed the interest in methods to algebraically identify singularities as non-manifold points. It needs to be mentioned that up until now there is no linkage known for which the configuration space has singular manifold points.

Other kinematic singularities To wrap up this section we want to mention two other kinds of linkage singularities. We will not put much focus on them in this thesis but they are still of high interest for any linkage analysis.

Assume we have chosen the **actuated** joints and an **end effector-map** of a linkage with configuration space X. Each actuated joint is presumed to be motor-driven and controlled by the user of the linkage whereas the other joints are passive. This gives a regular **actuation-map**

$$a\colon X \to \mathrm{SO}(3,\mathbb{R})^{k_1} \times \mathrm{SO}(2,\mathbb{R})^{k_2} \times \mathbb{R}^{k_3},$$

where k_1, k_2, k_3 are the numbers of actuated spherical, revolute and prismatic joints respectively. a maps a configuration to the parameters of the actuated joints in this configuration.

On the other hand, the regular end effector-map

$$e\colon X \to \mathrm{SE}(3,\mathbb{R})$$

gives a mapping between the configurations of the linkage and the position and orientation of a tool-point connected to the linkage (e.g. a welding laser on an industrial robot).

An **actuator singularity** is a singularity of the actuation-map restricted to the configuration space without CS-singularities. In such a point the configuration space

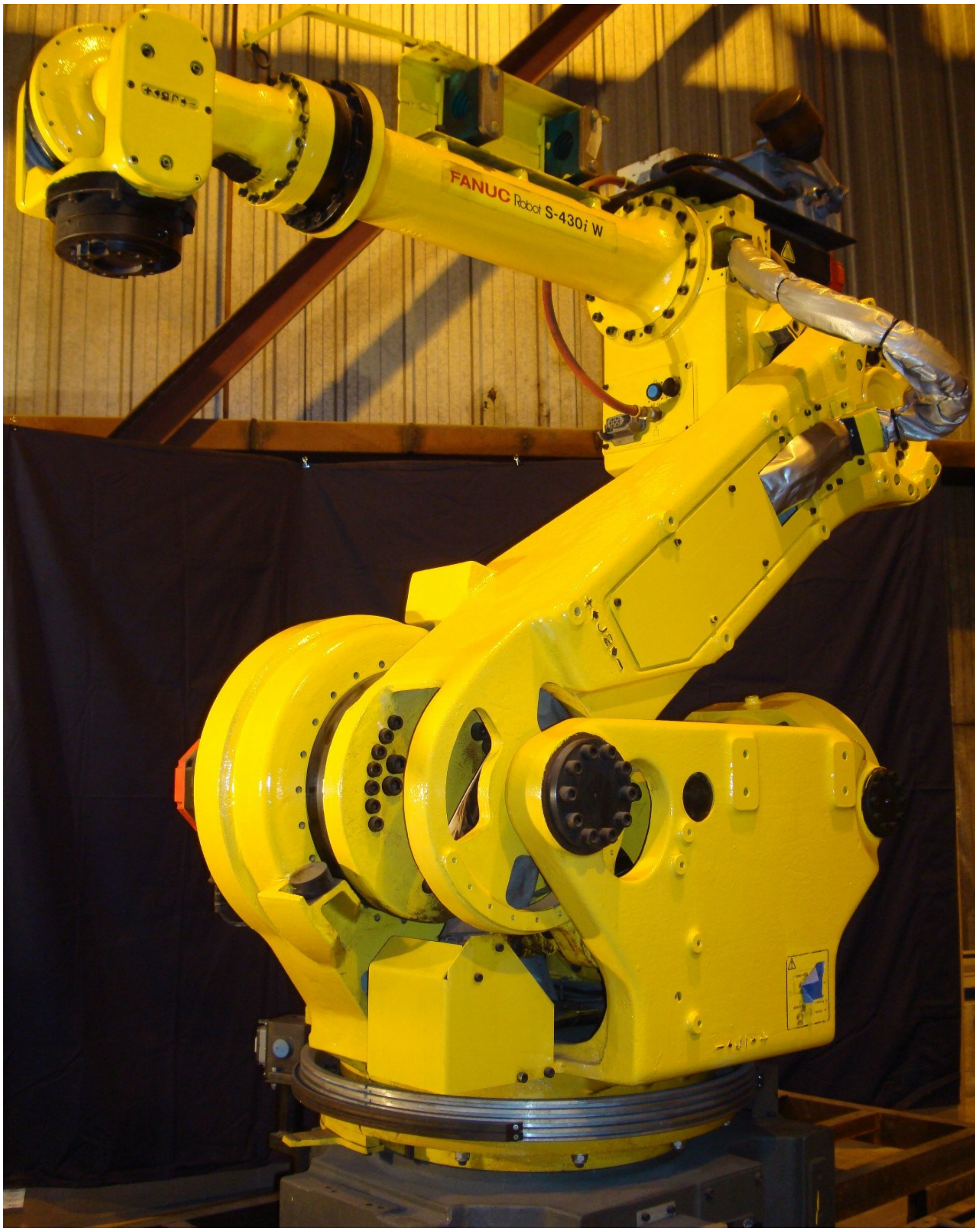


Figure 1.14: A Fanuc industrial robot with 6 revolute joints (6R chain).[1]

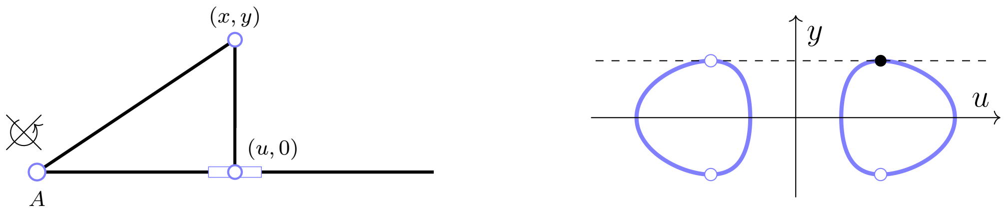


Figure 1.15: Actuator singularities.

cannot be parameterized by the coordinates of the actuated joints. As an example, we look at the slider-crank from Figure 1.1 but with $l_1 > l_2$. We assume the revolute joint in A to be actuated. Then, we get the actuation map

$$\begin{aligned} a\colon X = \mathbf{V}_{\mathbb{R}}(q_1, q_1) &\to \mathrm{SO}(2, \mathbb{R}), \\ (x, y, u) &\mapsto (x, y) \in S^1 \cong \mathrm{SO}(2, \mathbb{R}). \end{aligned}$$

Recall, that X is a C^ω-manifold. Clearly, a will be singular at $c \in X$ if and only if both the coordinate functions x and y are singular at c. Thus, we examine the Jacobian

$$D(q_1, q_1) = \begin{pmatrix} 2x & 2y & 0 \\ -2(u-x) & 2y & 2(u-x) \end{pmatrix}$$

with the 2-minors

$$M_x = \begin{vmatrix} 2y & 0 \\ 2y & 2(u-x) \end{vmatrix}, \quad M_y = \begin{vmatrix} 2x & 0 \\ -2(u-x) & 2(u-x) \end{vmatrix}, \quad M_u = \begin{vmatrix} 2x & 2y \\ -2(u-x) & 2y \end{vmatrix}.$$

A necessary condition for x and y to be singular is that both M_x and M_y vanish, therefore we need to calculate $S_{x,y} = \mathbf{V}_{\mathbb{R}}(q_1, q_2, M_x, M_y)$. One verifies immediately $S_{x,y} = \mathbf{V}_{\mathbb{R}}(q_1, q_2, u - x)$, hence

$$S_{x,y} = \left\{ \left(\sqrt{l_1^2 - l_2^2}, \pm l_2, \sqrt{l_1^2 - l_2^2} \right), \left(-\sqrt{l_1^2 - l_2^2}, \pm l_2, -\sqrt{l_1^2 - l_2^2} \right) \right\}.$$

Since u is a coordinate chart for X at this points, we can check that all points in $S_{x,y}$ are indeed singularities for x and y both. So, $S_{x,y}$ is the complete set of actuator singularities.

The points of $S_{x,y}$ are depicted in Figure 1.15, where the configuration on the left is marked black in the plot of the configuration space. In this configuration the revolute joint cannot rotate left, so a will not be a chart for X. However, the sliding joint at $(u, 0)$ can still move in both directions and its coordinate u will parameterize the configuration space. This distinguishes this kind of singularity from a CS-singularity where no combination of joints will parameterize the configuration space.

[1]Picture by courtesy of Kitmondo [64].

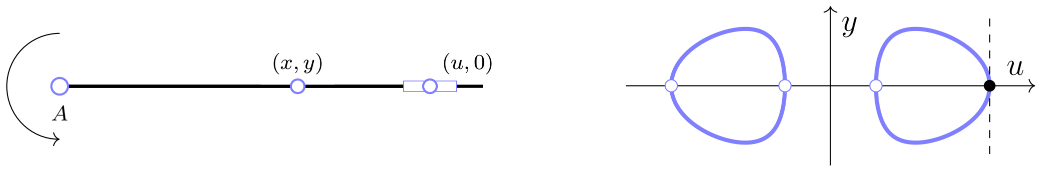


Figure 1.16: End effector singularities.

Please note that $S_{x,y}$ is empty for $l_1 < l_2$. This means there are no actuator singularities in this case provided we choose again the rotational joint at A to be actuated.

An **end effector singularity** is a singularity of the end effector map restricted to the configuration space minus CS-singularities. Consider the slider-crank once more, with the sliding joint as end effector. In this case we get the end effector-map

$$\begin{aligned} f\colon X = \mathbf{V}_{\mathbb{R}}(q_1, q_1) &\to \mathrm{SE}(3, \mathbb{R}), \\ (x, y, u) &\mapsto u \in \mathbb{R} \subset \mathrm{SE}(3, \mathbb{R}). \end{aligned}$$

Arguing as before the set of end-effector singularities will be

$$S_u = \mathbf{V}_{\mathbb{R}}(q_1, q_2, M_u) = \left\{ (\pm l_1, 0, \pm(l_1 + l_2)), (\pm l_1, 0, \pm(l_1 - l_2)) \right\}.$$

All points of S_u are depicted in Figure 1.16. Observe how u remains stationary under actuation of the crank in the selected configuration. End effector singularities are generally problematic in serial chains like industrial robots, as specific velocities of the end effector cannot be achieved. This is often critical for applications where the end effector needs to trace a curve with fixed orientation, such as welding tasks.

1.5 Aim and structure of the thesis

In the following thesis we address the problem of identifying non-manifold points of real algebraic sets. In particular, we present constructive results which can be checked by means of computational algebra. Thereafter, we apply the developed theory to the task of detecting CS-singularities of linkages. Our main results are:

(1) An equivalent algebraic condition for manifold points on the completion of the local ring, Proposition 3.4. Using this, we can prove that being a manifold point is an intrinsic property of real algebraic varieties, Corollary 3.6.

(2) An algebraic criterion to decide for any point p of a one-dimensional real algebraic set C whether p is a manifold point of C, Theorem 4.4.

(3) We formulate and prove a symbolic algorithm to check the criterion of (2), Proposition 5.3. Additionally, we provide an implementation of this algorithm in the computer algebra system (CAS) `Singular` [16], Listing A.1.

(4) For a plane real algebraic curve C we give sufficient conditions on the algebraic tangent cone at a point $p \in C$ for p to be a manifold point or isolated, Theorem 4.11.

(5) A criterion for manifold points of normal real algebraic varieties, derived from Efroymson's criterion on local reality [19], Corollary 3.9. Together with Theorem 3.12 this criterion helps to identify non-manifold points of arbitrary real algebraic sets, see the remark after Theorem 3.12.

(6) The identification of all CS-singularities of the following linkages:

 (i) The complete class of four-bars, Section 6.2.

 (ii) The complete class of five-bars, Section 6.3.

 (iii) The delta-robot, Section 6.4.

 For all linkages and singularities we will prove that the configuration space is locally not a manifold. This will demonstrate several of the presented techniques to identify non-manifold points of real algebraic sets. The CS-singularities of four-bars and five-bars are known [21, Theorem 1.6], [41, Theorem 2.6]. However, CS-singularities of the delta-robot have not been determined before.

Item (6) suggests solutions for several issues raised in [63, p. 227]. We make a note, that Conjecture 1 in [63] is known to be true [21, Theorem 1.6]. Item (3) supplements results in [14, pp. 307–309].

Our contribution will also be an exhaustive compilation of facts and results regarding base changes of real affine algebras, which are very useful for expanding results of symbolic computations over $\mathbb{Q}$.

The thesis will be structured as follows: In Chapter 2, we review some well-known facts from commutative algebra, real algebra and differential geometry.

In Chapter 3 we analyze the completion $\mathcal{F}$ of the local ring at a singular point of a real algebraic variety to derive (1) and (5) of our main results. In addition, we will examine the normalization of $\mathcal{F}$ which will be necessary for our results on algebraic curves and to apply Efroymson's criterion to non-normal algebraic varieties.

In Chapter 4 we investigate real algebraic curves with the proposed theory. This chapter includes (2) and (4) of our main results.

In Chapter 5 we develop the algorithm of (3) and discuss several problems which arise when certain results of symbolic computations over $\mathbb{Q}$ are extended to the real numbers.

Finally, in Chapter 6 we will give a short introduction to the representation of configuration spaces as algebraic sets. Subsequently, we carry out the analysis of (6).

In the appendix you will find the `Singular` code of all our library implementations. Moreover, we provide a link to helpful online resources, including downloads for all code listings. Those files should also be attached to the digital version of this thesis.

Chapter 2

Preliminaries

To prepare for the next chapters we review some terminology and theory from commutative algebra, real algebra and differential geometry. Proofs and/or references for all results will be included, since some of them are hard to find.

In the following exposition, all rings will be commutative, noetherian and with multiplicative identity. All modules will be noetherian and unital. The field $\mathbb{K}$ will always be of characteristic zero, i.e. $\mathbb{Q} \subset \mathbb{K}$, and we will use the abbreviation $\mathbf{x} = x_1, \ldots, x_n$, so that $\mathbb{K}[x_1, \ldots, x_n] = \mathbb{K}[\mathbf{x}]$. In some places we also work with multi-index notation: For $J \in \mathbb{N}^n$ we write $\mathbf{x}^J$ for $x_1^{J_1} \cdots x_n^{J_n}$.

2.1 Algebraic sets

Let $A := \mathbb{K}[\mathbf{x}]$, $I \leq \mathbb{K}[\mathbf{x}]$ and $\overline{\mathbb{K}}$ be the algebraic closure of $\mathbb{K}$.

Definition 2.1. For subsets $S \subset A$, $U \subset \overline{\mathbb{K}}^n$ and extension fields $\mathbb{K} \subset \mathbb{F}$ we set

$$\begin{aligned} \mathbf{V}(S) &:= \{\, x \in \overline{\mathbb{K}}^n \mid f(x) = 0, \text{ for all } f \in S \,\}, \\ \mathbf{V}_{\mathbb{F}}(S) &:= \{\, x \in \mathbb{F}^n \mid f(x) = 0, \text{ for all } f \in S \,\}, \\ \mathbf{I}(U) &:= \{\, f \in A \mid f(x) = 0, \text{ for all } x \in U \,\}. \end{aligned}$$

Any subset $\mathbf{V}(I) \subset \overline{\mathbb{K}}^n$, $I \leq \mathbb{K}[\mathbf{x}]$ is called **algebraic set**. If $\mathbb{K} \subset \mathbb{R}$, a subset $\mathbf{V}_{\mathbb{R}}(I)$ is called **real algebraic set**. The **Zariski topology** on $\mathbb{R}^n$ is the topology with closed sets $\mathbf{V}_{\mathbb{R}}(S)$, for $S \subset \mathbb{R}[\mathbf{x}]$.

Remark. We usually have $\mathbb{K} \subset \mathbb{C}$. In this case we could just replace $\overline{\mathbb{K}}^n$ with $\mathbb{C}^n$ in the definition of $\mathbf{V}(S)$ and any "algebraic statement" involving $\mathbf{V}(S)$ will be true if and only if it is true for $\overline{\mathbb{K}}^n$. This is usually called Lefschetz principle or transfer principle for algebraically closed fields. See e.g. [4, Theorem 1.26].

The analogue in real algebra is the Tarski-Seidenberg principle [4, Theorem 2.80] and allows us to use $\mathbb{R}$ and $\mathbb{R}_{\text{alg}} = \overline{\mathbb{Q}} \cap \mathbb{R}$ interchangeably.

2.2 Formal completion

To study properties in algebraic geometry which are local with respect to the euclidean topology we need the notion of completion of a topological ring.

Let A be a ring and $\mathfrak{a} \leq A$ an ideal. The **$\mathfrak{a}$-adic topology** is defined by taking powers of $\mathfrak{a}$ as a fundamental system of neighborhoods of $0 \in A$, see [3, Chapter 10]. The sequence of rings $\left(A/\mathfrak{a}^k\right)_{k\geq 1}$ with natural homomorphism $A/\mathfrak{a}^{k+1} \to A/\mathfrak{a}^k$ forms an inverse system. Then, the **$\mathfrak{a}$-adic completion** A^* of A is given by the inverse limit

$$A^* = \varprojlim_{k\geq 1} A/\mathfrak{a}^k$$

and the natural homomorphism $\phi\colon A \to A^*$. For any A-module M, the **$\mathfrak{a}$-adic** completion M^* of M is defined as

$$M^* = \varprojlim_{k\geq 1} M/\mathfrak{a}^k M.$$

A^* can be identified naturally with the set of Cauchy sequences in A with respect to the $\mathfrak{a}$-adic topology, modulo the equivalence relation

$$(a_k)_{k\geq 1} \sim (b_k)_{k\geq 1} \Leftrightarrow \lim_{k\to\infty} (a_k - b_k) = 0.$$

Hence A^* is the usual completion of A as a topological space. For local rings $(B, \mathfrak{m})$ the **completion of B** will always mean the $\mathfrak{m}$-adic completion.

We need a technical result regarding completion of factor rings.

Proposition 2.1. *Let A be a noetherian ring and $I \subset \mathfrak{m} \subset A$ for ideals $I, \mathfrak{m} \leq A$. Furthermore, let $\psi\colon A \to A^*$ be the $\mathfrak{m}$-adic completion and*

$$\Psi\colon A/I \to A^*/(I \cdot A^*)$$

induced by ψ, where $I \cdot A^*$ denotes the ideal of A^* generated by $\psi(I)$. Then Ψ is the $\mathfrak{m}'$-adic completion of A/I, where $\mathfrak{m}'$ is the canonical image of $\mathfrak{m}$ in A/I.

Proof. Let $\xi\colon A/I \to Q$ be the $\mathfrak{m}'$-adic completion of A/I. Since $I^* \cong I \cdot A^*$ according to [3, Proposition 10.15], it is easy to show that $Q \cong A^*/(I \cdot A^*)$. The tricky part is to verify that $\Psi\colon A/I \to A^*/(I \cdot A^*)$ is natural, i.e. equivalent to ξ.

First, we look at the $\mathfrak{m}$-adic completion $\varphi\colon I \to I^*$ as A-module. We have the following diagram where all arrows are A-module homomorphism:

$$\begin{tikzcd} I \arrow[r, "\varphi"] \arrow[dr] & I^* \arrow[r, "\gamma"] & A^* \\ & A^* \otimes_A I \arrow[u, "\eta"] \arrow[ur, "\tau"'] & \end{tikzcd} \tag{2.1}$$

Here γ is induced by $I \hookrightarrow A$, τ is induced by the bilinear map $(a, i) \mapsto a \cdot \psi(i)$ and η is the composed homomorphism

$$A^* \otimes_A I \to A^* \otimes_A I^* \to A^* \otimes_{A^*} I^* = I^*.$$

Now we can use

$$I^* = \varprojlim_{k \geq 1} I/\mathfrak{m}^k I, \qquad A^* = \varprojlim_{k \geq 1} A/\mathfrak{m}^k.$$

Thus, η is given on simple tensors by

$$\eta\left((a_k)_k \otimes_A i\right) = (a_k \cdot i)_k \in \varprojlim I/\mathfrak{m}^k I, \tag{2.2}$$

With equation (2.2) we can check that diagram (2.1) commutes. It is well-known that η is an isomorphism [3, Proposition 10.13]. Then, since τ is a monomorphism, we conclude that $\operatorname{Im}\tau = I \cdot A^*$ is isomorphic to I^* with the embedding $\iota\colon I \cdot A^* \to A^*$ equivalent to γ.

Next, we consider $\xi\colon A/I \to Q$ again. In the following diagram we have drawn ξ together with the $\mathfrak{m}$-adic completion of the short exact sequence $0 \to I \to A \to A/I \to 0$.

$$\begin{tikzcd}
0 \arrow[r] & I \arrow[r] \arrow[d, "\varphi"] & A \arrow[r, "\pi"] \arrow[d, "\psi"] & A/I \arrow[r] \arrow[d, "\xi"] \arrow[dd, bend left, "\Psi"] & 0, \\
0 \arrow[r] & I^* \arrow[r, "\gamma"] \arrow[d, "\cong"] & A^* \arrow[r, "p"] \arrow[dr, "\pi^*"] & Q \arrow[r] & 0 \\
& I \cdot A^* \arrow[ur, "\iota"] & & A^*/(I \cdot A^*) \arrow[u, dashed, "\exists\nu"] &
\end{tikzcd} \tag{2.3}$$

First, we note that the upper squares commute and since completion is exact [3, Proposition 10.12], we know that the second row is exact, too.

We have seen in the first part of the proof that the lower left triangle commutes. Now π^* is the cokernel of ι and since $p \circ \iota = 0$, we get from the universal property of cokernels a homomorphism ν such that the lower right triangle of diagram (2.3) commutes. We also check easily by diagram chasing that ν is an isomorphism. Because Ψ is induced by ψ, we have $\Psi \circ \pi = \pi^* \circ \psi$ and it follows

$$\nu \circ \Psi \circ \pi = \nu \circ \pi^* \circ \psi = p \circ \psi = \xi \circ \pi.$$

Since π is an epimorphism, we can cancel it on the right and have $\nu \circ \Psi = \xi$. $\square$

2.3 Faithfully flat ring extensions

Ring extensions which are faithfully flat behave especially nice. We will meet them for ground field extensions in polynomial rings and completions of local rings, see Example 2.1.

Let A be a ring and M an A-module. M is called **faithfully flat** if, any sequence

$$\cdots \longrightarrow N \longrightarrow N' \longrightarrow N'' \longrightarrow \cdots$$

of A-modules is exact if and only if

$$\cdots \longrightarrow N \otimes_A M \longrightarrow N' \otimes_A M \longrightarrow N'' \otimes_A M \longrightarrow \cdots$$

is exact. According to [38, 48] M is faithfully flat if and only if $\cdot \otimes_A M$ is an exact and faithful functor, if and only if M is A-flat and for any nonzero A-module N we have $N \otimes_A M \neq 0$.

The next Proposition collects some well-known facts about faithfully flat ring extensions. We will use some notation from [3]. For a ring homomorphism $\psi \colon A \to B$ and an ideal $I \leq A$, I^e denotes the **extension** of I, i.e. the ideal of B generated by $\psi(I)$. Also, for an ideal $J \leq B$, $J^c = \phi^{-1}(J)$ is the **contraction** of J. We will also write $I \cdot B$ or IB for the extension I^e.

Proposition 2.2. *Let $\psi \colon A \to B$ be a faithfully flat ring homomorphism, i.e. B is faithfully flat as A-module. Then*

(i) $(I_1 \cap I_2)^e = I_1^e \cap I_2^e$*, for ideals* $I_1, I_2 \leq A$.

(ii) $I^{ec} = I$*, for* $I \leq A$*. In particular,* ψ *is a monomorphism.*

(iii) The induced set-map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ *is surjective.*

(iv) The going-down property holds for $A \subset B$*: For any* $\mathfrak{p}, \mathfrak{p}' \in \mathrm{Spec}(A)$*, with* $\mathfrak{p} \subset \mathfrak{p}'$ *and for any* $\mathfrak{P}' \in \mathrm{Spec}(B)$ *lying over* $\mathfrak{p}'$*, there exists* $\mathfrak{P}$ *lying over* $\mathfrak{p}$ *such that* $\mathfrak{P} \subset \mathfrak{P}'$.

Remarks.

(1) For property (i) we only need flatness.

(2) Property (iii) and ψ flat ensures that ψ is faithfully flat [48, 4.D].

Proof of Proposition 2.2. All results can be found in [48]. (i) is (3.H), (ii) and (iii) are in (4.C) and (iv) is (5.D Theorem 4). □

For local homomorphism, flatness means already faithful flatness:

Proposition 2.3. *Let A, B be local rings with maximal ideals $\mathfrak{a}$, $\mathfrak{b}$ and $\psi \colon A \to B$ a local homomorphism, i.e. $\psi(\mathfrak{a}) \subset \mathfrak{b}$. Then, any finitely generated B-module M is faithfully flat over A if and only if M is flat over A. In particular, B is flat over A if and only if B is faithfully flat over A.*

Proof. See Matsumura [48, 4.A Corollary]. □

One very useful feature of faithfully flat ring extensions of noetherian rings is that the height of extended ideals remains unchanged. Since for faithfully flat ring extensions the going-down property holds and the induced map on the ring spectra is surjective, this follows from [48, 13.B]. Since the arguments are very instructional we will also include a proof here with all the details worked out.

Proposition 2.4. *Let $A \to B$ be a faithful flat ring extension and $I \leq A$ an ideal. Then*

$$\operatorname{ht} I^e = \operatorname{ht} I.$$

Proof. First, we show the statement for prime ideals. Let $\mathfrak{p} \leq A$ be prime with $\operatorname{ht} \mathfrak{p} = k$ and $\mathfrak{P}$ a prime ideal of minimal height containing $\mathfrak{p}^e$. We need to show that $\operatorname{ht} \mathfrak{P} = k$.

As initial step we prove $\mathfrak{P}^c = \mathfrak{p}$. If $\mathfrak{p} = \mathfrak{p}^{ec} \subsetneq \mathfrak{P}^c$, then according to the going-down property there exists $\mathfrak{P}'$ lying over $\mathfrak{p}$ with $\mathfrak{P}' \subsetneq \mathfrak{P}$. Because $\mathfrak{p}^e \subset \mathfrak{P}'$, we have a contradiction to the fact that $\mathfrak{P}$ has minimal height among the primes containing $\mathfrak{p}^e$. Hence, it must be $\mathfrak{P}^c = \mathfrak{p}$.

Next, since $\operatorname{ht} \mathfrak{p} = k$, we have a chain of distinct prime ideals

$$\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_k = \mathfrak{p}$$

and with the going-down property of Proposition 2.2 (iv) we get a chain

$$\mathfrak{P}_0 \subset \mathfrak{P}_1 \subset \ldots \subset \mathfrak{P}_k = \mathfrak{P}.$$

of prime ideals in B. Note that those primes must be distinct since their intersections with A are distinct. It follows $\operatorname{ht} \mathfrak{P} \geq k$.

The other inequality is slightly more technical. With a converse to the general principal ideal theorem [20, Corollary 10.5], we find k elements $a_1, \ldots, a_k \in \mathfrak{p}$ such that $\mathfrak{p}$ is a minimal prime associated to $\langle a_1, \ldots, a_k \rangle$. This means $\sqrt{\langle a_1, \ldots, a_k \rangle A_\mathfrak{p}} = \mathfrak{p} A_\mathfrak{p}$ [3, Proposition 4.9] and because $A_\mathfrak{p}$ is noetherian, a power of $\mathfrak{p} A_\mathfrak{p}$ must be contained in $\langle a_1, \ldots, a_k \rangle$ according to [3, Proposition 7.14].

Now we can choose $f \in \mathbb{N}$ with $\mathfrak{p}^f A_\mathfrak{p} \subset \langle a_1, \ldots a_k \rangle A_\mathfrak{p}$. As $\mathfrak{P}$ is a minimal prime associated to $\mathfrak{p}^e$, we find with the same reasoning an integer $g \in \mathbb{N}$ with $\mathfrak{P}^g B_\mathfrak{P} \subset \mathfrak{p}^e B_\mathfrak{P} = \mathfrak{p} B_\mathfrak{P}$. Thus, we get

$$\mathfrak{P}^{fg} B_\mathfrak{P} = (\mathfrak{P}^g B_\mathfrak{P})^f \subset (\mathfrak{p} B_\mathfrak{P})^f = \mathfrak{p}^f B_\mathfrak{P} \subset \langle a_1, \ldots, a_k \rangle B_\mathfrak{P}. \tag{2.4}$$

According to [3, Corollary 7.16] this means that $\langle a_1, \ldots, a_k \rangle$ is a $\mathfrak{P} B_\mathfrak{P}$-primary ideal in the local ring $B_\mathfrak{P}$. But in any noetherian local ring $(S, \mathfrak{m})$ the minimal numbers of generators of an $\mathfrak{m}$-primary ideal equals the dimension of S [3, Theorem 11.14], so we have $k \geq \dim B_\mathfrak{P} = \operatorname{ht} \mathfrak{P}$ and consequently $\operatorname{ht} \mathfrak{P} = k$.

Now we consider an arbitrary ideal $I \leq A$. Let $\mathfrak{P}$ be a prime of B containing I^e with $\operatorname{ht}\mathfrak{P} = \operatorname{ht} I^e$ and let $\mathfrak{p} = \mathfrak{P} \cap A$. Then $\mathfrak{p}^e \subset \mathfrak{P}$ and $I = I^{ec} \subset \mathfrak{P}^c = \mathfrak{p}$. Therefore

$$\operatorname{ht} I^e = \operatorname{ht}\mathfrak{P} \geq \operatorname{ht}\mathfrak{p}^e = \operatorname{ht}\mathfrak{p} \geq \operatorname{ht} I.$$

For the other inequality let $\mathfrak{p}$ be a prime ideal containing I with $\operatorname{ht} I = \operatorname{ht}\mathfrak{p}$. Then it is

$$\operatorname{ht} I = \operatorname{ht}\mathfrak{p} = \operatorname{ht}\mathfrak{p}^e \geq \operatorname{ht} I^e. \qquad \square$$

We will use the following faithfully flat ring extensions.

Example 2.1.

(a) Let $\mathbb{K} \subset \mathbb{K}'$ be any field extension. Then, $\mathbb{K}[\mathbf{x}] \to \mathbb{K}' \otimes_{\mathbb{K}} \mathbb{K}[\mathbf{x}] = \mathbb{K}'[\mathbf{x}]$ is faithfully flat. In particular, $\big(I \cdot \mathbb{K}'[\mathbf{x}]\big) \cap \mathbb{K}[\mathbf{x}] = I$, for all $I \leq \mathbb{K}[\mathbf{x}]$.

(b) Let $\psi\colon A \to A^*$ be the $\mathfrak{a}$-adic completion of a local ring A with maximal ideal $\mathfrak{a}$. Then, ψ is faithfully flat.

Proof.

(a) Let $A = \mathbb{K}[\mathbf{x}]$. For any A-module M we have

$$(\mathbb{K}' \otimes_{\mathbb{K}} A) \otimes_A M \cong \mathbb{K}' \otimes_{\mathbb{K}} (A \otimes_A M) \cong \mathbb{K}' \otimes_{\mathbb{K}} M.$$

Since every field extension is faithfully flat, $\mathbb{K}' \otimes_{\mathbb{K}} A$ is faithfully flat over A.

(b) It is well-known that A^* is a flat A-algebra [3, Proposition 10.14] and $\mathfrak{a}^* = \mathfrak{a}\, A^*$ is the maximal ideal of A^* [3, Proposition 10.16]. Consequently, ψ must be faithfully flat due to Proposition 2.3. $\square$

2.4 Power series rings

In this section let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ and $A = \mathbb{K}[\mathbf{x}]$ with maximal ideal $\mathfrak{m} := \langle \mathbf{x} \rangle$. We have the following commutative diagram

$$\begin{tikzcd} A \arrow[r, "\iota"] \arrow[d] & \mathbb{K}\{\mathbf{x}\} \arrow[r, "\gamma"] & \mathbb{K}[[\mathbf{x}]] \\ A_{\mathfrak{m}} \arrow[ur, "\varphi"'] & & \end{tikzcd} \tag{2.5}$$

where $\mathbb{K}\{\bar{x}\}$ is the ring of convergent power series and $\mathbb{K}[[\mathbf{x}]]$ the ring of formal power series. Furthermore, ι and γ are the natural embeddings and φ is given by the universal property of localization since elements of $A \backslash \mathfrak{m}$ are units in $\mathbb{K}\{\mathbf{x}\}$ [22, Korollar 6.2]. As ι is a monomorphism and $\mathbb{K}\{\mathbf{x}\}$ is a domain φ is a monomorphism, too.

We will need the following facts about γ and φ:

Proposition 2.5.

(i) $\gamma \circ \iota \colon A \to \mathbb{K}[[\mathbf{x}]]$ *is the* $\mathfrak{m}$*-adic completion of* A*.*

(ii) $\gamma \circ \varphi \colon A_\mathfrak{m} \to \mathbb{K}[[\mathbf{x}]]$ *is the* $\mathfrak{m}A_\mathfrak{m}$*-adic completion of* $A_\mathfrak{m}$*.*

(iii) $\gamma \colon \mathbb{K}\{\mathbf{x}\} \to \mathbb{K}[[\bar{x}]]$ *is the* $\mathfrak{m}\,\mathbb{K}\{\mathbf{x}\}$*-adic completion of* $\mathbb{K}\{\mathbf{x}\}$*.*

Remark. Since A is noetherian any $\mathfrak{a}$-adic completion of A is noetherian as well, for ideals $\mathfrak{a} \leq A$ [3, Theorem 10.26]. Consequently, $\mathbb{K}[[\mathbf{x}]]$ is noetherian. We will see that γ is faithfully flat. Then, $\mathbb{K}\{\mathbf{x}\}$ is also noetherian, a fact which is usually proven with induction and the Weierstrass preparation theorem.

Proof.

(i) For $k \geq 1$ define homomorphism $\gamma_k \colon \mathbb{K}[[\mathbf{x}]] \to A/\mathfrak{m}^k$ by

$$\gamma_k\left(\sum_{I \in \mathbb{N}^n} a_I \mathbf{x}^I\right) = \sum_{|I|<k} a_I \mathbf{x}^I + \mathfrak{m}^k.$$

Clearly, $\pi_k \circ \gamma_{k+1} = \gamma_k$ for the natural homomorphism $\pi_k \colon A/\mathfrak{m}^{k+1} \to A/\mathfrak{m}^k$. Therefore we can use the universal property of the inverse limit and get a homomorphism $\Gamma \colon \mathbb{K}[[\mathbf{x}]] \to \varprojlim_{k\geq 1} A/\mathfrak{m}^k$. With the representation

$$\varprojlim_{k\geq 1} A/\mathfrak{m}^k = \left\{ (b_k)_{k\geq 1} \in \prod A/\mathfrak{m}^k \;\middle|\; b_{k+1} \equiv b_k \mod \mathfrak{m}^k \right\}$$

we can quickly check, that Γ is an isomorphism and $\Gamma \circ \gamma \circ \iota \colon A \to \varprojlim_{k\geq 1} A/\mathfrak{m}^k$ is the canonical map to a constant sequence, where γ, ι are as in (2.5).

(ii) Since diagram (2.5) commutes and $A_\mathfrak{m}/(\mathfrak{m}A_\mathfrak{m})^k = A_\mathfrak{m}/\mathfrak{m}^k A_\mathfrak{m} \cong A/\mathfrak{m}^k$ we can use the same proof as in (i).

(iii) As in (ii) all follows from $\mathbb{K}\{\mathbf{x}\}/(\mathfrak{m}\,\mathbb{K}\{\mathbf{x}\})^k = \mathbb{K}\{\mathbf{x}\}/(\mathfrak{m}^k\,\mathbb{K}\{\mathbf{x}\}) \cong A/\mathfrak{m}^k$. $\square$

Corollary 2.6 (Flatness)**.** *The monomorphism* ι *is flat. The monomorphism* γ *and* φ *are both faithfully flat.*

Remarks.

(1) Since (faithfully) flatness is a transitive property [48, 3.B, 4.B] $\gamma \circ \iota$ is flat and $\gamma \circ \varphi$ is faithfully flat.

(2) ι is not faithfully flat for $n \geq 1$: Choose an ideal $J \leq A$ with $J \backslash \mathfrak{m} \neq \emptyset$. Then, every element $r \in J\backslash\mathfrak{m}$ is a unit in $\mathbb{K}\{\mathbf{x}\}$ hence $J \cdot \mathbb{K}\{\mathbf{x}\} \cap A = \mathbb{K}\{\mathbf{x}\} \cap A = A$. This contradicts Proposition 2.2 (ii).

(3) We will see that Corollary 2.6 leads to the fact that $\mathbb{K}[[\mathbf{x}]]/\big(I\,\mathbb{K}[[\mathbf{x}]]\big)$ is faithfully flat over $\mathbb{K}[\mathbf{x}]_{\mathfrak{m}}/(I\,\mathbb{K}[\mathbf{x}]_{\mathfrak{m}})$. That means many insights about the local analytic structure of an algebraic set can be gained from the local algebraic structure. This was exploited for the first time by J. P. Serre in his GAGA paper [69] relating many properties in the algebraic and analytic categories.

Proof of Corollary 2.6. γ and $\gamma \circ \varphi$ are faithfully flat because of Proposition 2.5 and Example 2.1. Note that A is not a local ring therefore we cannot use the argument of Example 2.1 for $\gamma \circ \iota$. Now, φ is faithfully flat as well because faithful flatness has the descent property [48, 4.B].

It remains to show that $\mathbb{K}\{\mathbf{x}\}$ is a flat A-module. Since flatness can be checked locally at maximal ideals [48, 3.J] it is enough to prove that $\mathbb{K}\{\mathbf{x}\}_{\mathfrak{m}\mathbb{K}\{\mathbf{x}\}} = \mathbb{K}\{\mathbf{x}\}$ is flat over $A_{\mathfrak{m}}$. But we have already seen that φ is flat. ☐

Following the terminology of [67] we call a quotient $\mathbb{K}\{\mathbf{x}\}/J$ of $\mathbb{K}\{\mathbf{x}\}$ an **analytic ring** and a quotient $\mathbb{K}[[\mathbf{x}]]/J$ of $\mathbb{K}[[\mathbf{x}]]$ a **formal ring**.

Our next result will be needed in Section 3.4, when we study the normalization of completions of local rings.

Proposition 2.7. *Let $I \leq \mathbb{R}[\mathbf{x}]$ be any ideal and $\mathfrak{n} \leq \mathbb{R}[\mathbf{x}]/I$, maximal. Suppose also that $\mathfrak{n}\big(\mathbb{C}[\mathbf{x}]/I\,\mathbb{C}[\mathbf{x}]\big) = \mathfrak{m}' \cap \mathfrak{m}$ is not maximal in $\mathbb{C}[\mathbf{x}]/I\,\mathbb{C}[\mathbf{x}]$. Then*

$$\big(\mathbb{R}[\mathbf{x}]/I\big)^*_{\mathfrak{n}} \cong \big(\mathbb{C}[\mathbf{x}]/I\,\mathbb{C}[\mathbf{x}]\big)^*_{\mathfrak{m}'} \cong \big(\mathbb{C}[\mathbf{x}]/I\,\mathbb{C}[\mathbf{x}]\big)^*_{\mathfrak{m}}$$

where $*$ stands for completion with respect to the maximal ideal.

Proof. Let $A = \mathbb{R}[\mathbf{x}]/I$ and $B = \mathbb{C}[\mathbf{x}]/I\,\mathbb{C}[\mathbf{x}]$. For every $k \geq 1$, we have the following commutative diagram:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, hook] & A_{\mathfrak{n}} \arrow[r] \arrow[d, hook] & A_{\mathfrak{n}}/\mathfrak{n}^k A_{\mathfrak{n}} \arrow[r] \arrow[d, hook, "\psi"] & A_{\mathfrak{n}}/\mathfrak{n}A_{\mathfrak{n}} \arrow[r, "\cong"] \arrow[d, "\cong"] & \mathbb{C} \arrow[d, equal] \\
B \arrow[r] & B_{\mathfrak{m}} \arrow[r] & B_{\mathfrak{m}}/\mathfrak{m}^k B_{\mathfrak{m}} \arrow[r] & B_{\mathfrak{m}}/\mathfrak{m}B_{\mathfrak{m}} \arrow[r, "\cong"] & \mathbb{C}
\end{tikzcd}$$

To prove the proposition it is enough to show that ψ is an epimorphism. First, we check that $\mathfrak{n}B \equiv \mathfrak{m} \bmod \mathfrak{m}^k$ in B. So let $a \in \mathfrak{m}$. Because $\mathfrak{m}'$ is maximal in B and $\mathfrak{m}^k \not\subset \mathfrak{m}'$, we have

$$\mathfrak{m}' + \mathfrak{m}^k = (1).$$

Hence we can choose $e \in \mathfrak{m}'$, $f \in \mathfrak{m}^k$, with $e + f = 1$. Then

$$a - a\,e = a(1-e) = a\,f \in \mathfrak{m}^k,$$

but $ae \in \mathfrak{m}' \cap \mathfrak{m} - = \mathfrak{n}B$.

Now let $\frac{a}{s} \in \mathfrak{m}B_{\mathfrak{m}}$. We have seen that $a = b + d$, with $d \in \mathfrak{m}^k$ and $b \in \mathfrak{n}B$. Thus $\frac{a}{s} = \frac{b}{s} + \frac{d}{s} \equiv \frac{b}{s} \bmod \mathfrak{m}^k B_{\mathfrak{m}}$. This means that the maximal ideal of $A_{\mathfrak{n}}/\mathfrak{n}^k A_{\mathfrak{n}}$ generates the maximal ideal in $B_{\mathfrak{m}}/\mathfrak{m}^k B_{\mathfrak{m}}$. Moreover, we can easily check that $B_{\mathfrak{m}}$ is a finite $A_{\mathfrak{n}}$ module generated by $\{1, i\}$. Then, according to Lemma 2.8 ψ is an epimorphism. □

Lemma 2.8. *Let $(A, \mathfrak{m}_A) \subset (B, \mathfrak{m}_B)$ be a finite extension of local rings. Assume additionally $\mathfrak{m}_A B = \mathfrak{m}_B$ and $B/\mathfrak{m}_B = A/\mathfrak{m}_A$. Then*

$$A = B.$$

Proof. Let $b \in B$ be arbitrary. Since $A/\mathfrak{m}_A = B/\mathfrak{m}_B$ there exists $a \in A$ with $b - a \in \mathfrak{m}_B = \mathfrak{m}_A B$. It follows

$$B = A + \mathfrak{m}_A B.$$

Because m_A is the Jacobson-radical of A and B is a finite A-module, the statement of the lemma follows from Nakayama's Lemma. □

The last proposition in this section characterizes regular formal rings and is very useful. See Lemma II.1.9 from [67] for a reference.

Proposition 2.9. *Let $A = \mathbb{K}[[\mathbf{x}]]/I$ be a formal ring with $\dim A = d$. If A is regular then*

$$A \cong \mathbb{K}[[t_1, \ldots, t_d]].$$

2.5 Nagata's Jacobian criterion

Nagata's Jacobian Criterion will be an important tool for us. We introduce the terminology following the exposition of [67, Section II.4]. Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{C}$ again and $A = \mathbb{K}\{\mathbf{x}\}$ or $\mathbb{K}[[\mathbf{x}]]$. For $f_1, \ldots, f_s \in A$ and indices $1 \leq i_1 < \ldots < i_s \leq n$ we define:

$$\frac{D(f_1, \ldots, f_s)}{D(x_{i_1}, \ldots, x_{i_s})} = \det \left[\frac{\partial f_k}{\partial x_{i_l}} \right]_{k,l} \tag{2.6}$$

where $\frac{\partial f_k}{\partial x_i}$ is defined in the usual way [67, Section I.2.7]. Any power series in the form (2.6) is called a **Jacobian** of order s. For an ideal $L \leq A$ we denote with $J_s(L)$ the ideal of A generated by L and all Jacobians of order s for $f_1, \ldots, f_s \in L$.

Lemma 2.10. *Let $L = \langle g_1, \ldots, g_k \rangle \leq A$ with $k \geq s$. Then $J_s(L)$ is generated by the elements in*

$$\{g_1, \ldots, g_k\} \cup \left\{ \frac{D(g_{j_1}, \ldots, g_{j_s})}{D(x_{i_1}, \ldots, x_{i_s})} \;\middle|\; \begin{array}{l} 1 \leq j_1 < \ldots, < j_s \leq k \\ 1 \leq i_1 < \ldots < i_s \leq n \end{array} \right\} \tag{2.7}$$

Proof. Let $S \subset A$ be the finite set (2.7). Then, clearly $\langle S \rangle \subset J_s(L)$. For the other direction note that for indices $1 \le i_1 < \ldots < i_s \le n$ the map

$$\begin{aligned} D\colon A^s &\to A \\ (f_1, \ldots, f_s) &\mapsto \frac{D(f_1, \ldots, f_s)}{D(x_{i_1}, \ldots, x_{i_s})} \end{aligned}$$

is an alternating multilinear map over $\mathbb{K}$. In addition, let $c_1, \ldots, c_k \in A$. Then, we derive with the Leibniz formula for power series:

$$\begin{aligned} \frac{D(c_1 \cdot g_{j_1}, \ldots, c_s \cdot g_{j_s})}{D(x_{i_1}, \ldots, x_{i_s})} &= \det\big[\nabla(c_1 \cdot g_{j_1}), \ldots, \nabla(c_s \cdot g_{j_s})\big] \\ &= \det\big[(\nabla c_1) \cdot g_{j_1} + c_1 \cdot (\nabla g_{j_1}), \ldots, (\nabla c_s) \cdot g_{j_s} + c_s \cdot (\nabla g_{j_s})\big] \\ &= c_1 \cdots c_s \frac{D(g_{j_1}, \ldots, g_{j_s})}{D(x_{i_1}, \ldots, x_{i_s})} + \sum_{I \in \{0,1\}^s \setminus \{0\}^s} \det\big[\Lambda^1_{I_1}, \ldots, \Lambda^s_{I_s}\big], \end{aligned} \tag{2.8}$$

where $\nabla f = (\partial_{i_1} f, \ldots, \partial_{i_s} f)^T$ for $f \in A$ and

$$\Lambda^r_t = \begin{cases} (\nabla c_r) \cdot g_{j_r}, & t = 1, \\ c_r \cdot (\nabla g_{j_r}), & t = 0. \end{cases}$$

Since every determinant in the sum contains at least one column of type $(\nabla c_r) \cdot g_{j_r}$, where g_{j_r} can be pulled out, it is clear that the element (2.8) is contained in the ideal generated by S. $\square$

Theorem 2.11 (Nagata's Jacobian criterion [67, Proposition II.4.3])**.** *Let $\mathfrak{p}$ be a prime ideal of A and $L \le A$ an arbitrary ideal with $L \subset \mathfrak{p}$. The following assertions are equivalent:*

(a) The local ring $A_\mathfrak{p}/LA_\mathfrak{p}$ is regular of dimension $\mathrm{ht}(\mathfrak{p}) - s$.

(b) $\mathfrak{p} \not\supset J_s(L)$ *and* $\mathrm{ht}(LA_\mathfrak{p}) \le s$.

2.6 Base change

In this section let $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{K}'$ be a field extension. Then, we get a faithful flat ring extensions $\mathbb{K}[\mathbf{x}] \to \mathbb{K}'[\mathbf{x}]$, see Example 2.1. For $I \le \mathbb{K}[\mathbf{x}]$ we will prove some important facts about extended ideals $I_{\mathbb{K}'} := I^e = I \cdot \mathbb{K}'[\mathbf{x}]$.

Proposition 2.12. *Let $I \le \mathbb{K}[\mathbf{x}]$, $A = \mathbb{K}[\mathbf{x}]/I$ and $A' = \mathbb{K}'[\mathbf{x}]/I^e = \mathbb{K}' \otimes_\mathbb{K} A$. Then*

(i) $I^e \cap \mathbb{K}[\mathbf{x}] = I$.

(ii) $\operatorname{ht} I^e = \operatorname{ht} I$, $\dim A' = \dim A$.

(iii) $\sqrt{I^e} = \sqrt{I} \cdot \mathbb{K}'[\mathbf{x}]$.

(iv) Let $\mathfrak{p} \leq A$ *be prime. Then,* $A_{\mathfrak{p}}$ *is regular if and only if* $A'_{\mathfrak{P}}$ *is regular for one and then all associated primes* $\mathfrak{P}$ *of* $\mathfrak{p}A'$.

Remark. Since we require $\operatorname{char}(\mathbb{K}) = 0$, $\mathbb{K}$ is a perfect field and therefore $\mathbb{K}'$ separable over $\mathbb{K}$. This means (note that $\mathbb{K} \subset \mathbb{K}'$ does not need to be algebraic) that every finitely generated subextension is separably generated over $\mathbb{K}$, cf. [20, A1.2]. Whereas (i) and (ii) would work for any field extension (iii) and (iv) are in general wrong if $\mathbb{K} \subset \mathbb{K}'$ is not separable.

Proof. (i) and $\operatorname{ht} I^e = \operatorname{ht} I$ follow because $\mathbb{K}[\mathbf{x}] \subset \mathbb{K}'[\mathbf{x}]$ is a faithfully flat ring extension, see Proposition 2.2 and Proposition 2.4. Since $\operatorname{ht} I = \dim \mathbb{K}[\mathbf{x}] - \dim A$ we then have $\dim A = \dim A'$, see [20, Corollary 13.4]. (iii) is a consequence of the fact, that any reduced $\mathbb{K}$-algebra is geometrically reduced [13, Lemma 10.42.6 and Lemma 10.44.6]. Now we will show (iv) with the general Jacobian criterion in the form of [31, Thm. 5.7.1].

First, choose any associated prime $\mathfrak{P}$ of $\mathfrak{p}A'$ and let $\hat{\mathfrak{p}}$, $\hat{\mathfrak{P}}$ denote the preimages of $\mathfrak{p}$ and $\mathfrak{P}$ in $\mathbb{K}[\mathbf{x}]$ and $\mathbb{K}'[\mathbf{x}]$ respectively. Next, write K for the quotient field of $\mathbb{K}[\mathbf{x}]/\hat{\mathfrak{p}}$ and K' for the quotient field of $\mathbb{K}'[\mathbf{x}]/\hat{\mathfrak{P}}$. Since $\hat{\mathfrak{P}} \cap \mathbb{K}[\mathbf{x}] = \hat{\mathfrak{p}}$ [77, VII Theorem 36], K is clearly a subfield of K'. For any K-vector space V we then have $\dim_K V = \dim_{K'} K' \otimes_K V$ since the tensor product commutes with direct sums. Consequently

$$\operatorname{rk} \left[\frac{\partial f_i}{\partial x_j} \bmod \hat{\mathfrak{p}} \right]_{i,j} = \operatorname{rk} \left[\frac{\partial f_i}{\partial x_j} \bmod \hat{\mathfrak{P}} \right]_{i,j} =: h,$$

where we have chosen $f_1, \dots, f_n \in \mathbb{K}[\mathbf{x}]$ such that $\langle f_1, \dots, f_n \rangle = I$.

Now assume $A_{\mathfrak{p}}$ is a regular local ring and choose an associated prime $\mathfrak{q}$ of I with $\mathfrak{q} \subset \hat{\mathfrak{p}}$ (note that there should be only one prime with this property, otherwise $A_{\mathfrak{p}}$ wouldn't be regular). Then, we conclude $\operatorname{ht} \mathfrak{q} = h$ with the general Jacobian criterion. Now any associated prime of $\mathfrak{q}^e$ has height h as well [77, VII Theorem 36] and one of them is contained in $\hat{\mathfrak{P}}$. But then $A'_{\mathfrak{P}}$ is regular according to the general Jacobian criterion.

Suppose to the contrary that $A'_{\mathfrak{P}}$ is regular. Then, there exists an associated prime $\mathfrak{Q}$ of I^e with $\mathfrak{Q} \subset \hat{\mathfrak{P}}$ and $\operatorname{ht} \mathfrak{Q} = h$. Now, since $\mathfrak{Q}$ is associated to I^e, it is associated to $\mathfrak{r}^e$ for a primary ideal $\mathfrak{r} \in \mathbb{K}[\mathbf{x}]$ which is part of a primary decomposition of I (use $(J_1 \cap J_2)\,\mathbb{K}'[\mathbf{x}] = J_1\mathbb{K}'[\mathbf{x}] \cap J_2\mathbb{K}'[\mathbf{x}]$ for ideals $J_1, J_2 \leq \mathbb{K}[\mathbf{x}]$ and $\mathfrak{Q} = (I^e : \langle b \rangle)$, for some $b \in \mathbb{K}'[\mathbf{x}]$). So $\mathfrak{q} := \sqrt{\mathfrak{r}}$ is a prime ideal associated to I and

$$\mathfrak{r} = \mathfrak{r}^e \cap \mathbb{K}[\mathbf{x}] \subset \mathfrak{Q} \cap \mathbb{K}[\mathbf{x}] \subset \hat{\mathfrak{P}} \cap \mathbb{K}[\mathbf{x}] = \hat{\mathfrak{p}}.$$

But then $\mathfrak{q} = \sqrt{\mathfrak{r}} \subset \hat{\mathfrak{p}}$. Also, $h = \operatorname{ht} \mathfrak{Q} \geq \operatorname{ht} \mathfrak{q}$. Consequently, $A_{\mathfrak{p}}$ is regular according to the general Jacobian criterion. $\square$

The final lemma of this section is a technical result and will be needed in Section 5.2:

Lemma 2.13. *Let $\mathbb{K} \subset \mathbb{K}'$ be any field extension and $I \leq \mathbb{K}[\mathbf{x}]$ an equidimensional ideal. If*

$$I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$$

is a minimal primary decomposition and $(\mathfrak{q}_i)_{\mathbb{K}'} = \cap_j \mathfrak{q}_{i,j}$ are minimal primary decompositions in $\mathbb{K}'[\mathbf{x}]$. Then

$$I_{\mathbb{K}'} = \bigcap_{i,j} \mathfrak{q}_{i,j}$$

is a minimal primary decomposition in $\mathbb{K}'[\mathbf{x}]$.

Proof. First, we show that $\sqrt{\mathfrak{q}_{i,j}} \neq \sqrt{\mathfrak{q}_{i',j'}}$ for $i \neq i'$. If the contrary is true then $\sqrt{\mathfrak{q}_i} = \mathbb{K}[\mathbf{x}] \cap \sqrt{\mathfrak{q}_{i,j}} = \mathbb{K}[\mathbf{x}] \cap \sqrt{\mathfrak{q}_{i',j'}} = \sqrt{\mathfrak{q}_{i'}}$ according to [77, Theorem 11.36] which is a contradiction. Now suppose $\cap\, \mathfrak{q}_{i,j} \subset \mathfrak{q}_{i',j'}$. But then $\cap \sqrt{\mathfrak{q}_{i,j}} \subset \sqrt{\mathfrak{q}_{i',j'}}$ and consequently $\sqrt{\mathfrak{q}_{i,j}} \subset \sqrt{\mathfrak{q}_{i',j'}}$, for some $i \neq i'$. This is a contradiction with the argument above because I is equidimensional. $\square$

2.7 Real algebra

We review some facts from real algebra. Most of them can be found in [44] or [7].

Definition 2.2. Let B be any commutative ring and $I \leq B$ an ideal. B is called (formally) **real** if any equation

$$b_1^2 + \ldots + b_k^2 = 0, \quad k \geq 1,$$

implies $b_1 = \ldots = b_k = 0$. I is called **real** if B/I is real. Also, we define the **real radical**

$$\sqrt[r]{I} = \left\{ x \in B \;\middle|\; x^{2r} + b_1^2 + \ldots + b_k^2 \in I, \text{ for } r, k \geq 0,\, b_i \in B \right\},$$

which is either the smallest real ideal containing I or B if there are no real ideals between I and B, see [7, Proposition 4.1.7]. Thus, I is real if and only if $\sqrt[r]{I} = I$.

Example 2.2.

(i) $\mathbb{C}$ is clearly not real since $1^2 + i^2 = 0$, but $\mathbb{Q}$ and $\mathbb{R}$ are. Likewise, any domain B is real if and only if its field of fractions is real. Fields are real if and only if they can be ordered [7, Theorem 1.1.8].

A field k is called **real closed** if it is real and $k[x]/(x^2+1)$ is algebraically closed. Clearly, $\mathbb{R}$ is real closed but not $\mathbb{Q}$.

(ii) Consider the ideal $I = \langle x^2 + y^2 \rangle \leq \mathbb{R}[x, y]$. Then, I is not real since $x, y \notin I$. We see easily from the definition that $x, y \in \sqrt[r]{I}$. But according to the real Nullstellensatz (Proposition 2.14 below) it must be $1 \notin \sqrt[r]{I}$. Hence, $\sqrt[r]{I} = \langle x, y \rangle$.

(iii) Consider $I = \langle x^3 - 5\, y^3 \rangle \leq \mathbb{Q}[x, y]$. Then, I is prime in $\mathbb{Q}[x, y]$. In addition, there exist points $p \in \mathbf{V}_{\mathbb{R}}(I)$ such that $\mathbb{R}[x, y]/I_{\mathbb{R}}$ localized at the maximal ideal corresponding to p is regular. Then, I must be real (in $\mathbb{Q}[x, y]$) according to the simple point criterion, see Proposition 2.15 with the first remark. $I_{\mathbb{R}}$ is not real however since $\sqrt[r]{I_{\mathbb{R}}} = \mathbf{I}_{\mathbb{R}}(\mathbf{V}_{\mathbb{R}}(I_{\mathbb{R}})) = \langle x - \sqrt[3]{5}\, y \rangle$.

(iv) $f(x, y) = y^3 + 2\, x^2 y - x^4$ is an irreducible polynomial in $\mathbb{R}[\mathbf{x}]$ and for any $x_0 \neq 0$, there exists a real solution $y_0 \in \mathbb{R}$ of $f(x_0, y) = 0$ since this is a polynomial of degree 3. Also, the local ring at (x_0, y_0) is regular with the Jacobian criterion. Hence, $I = \langle f \rangle$ is a real ideal of $\mathbb{R}[\mathbf{x}]$ according to the simple point criterion.

The analogue to Hilbert's Nullstellensatz in real algebraic geometry is the

Proposition 2.14 (Risler's Real Nullstellensatz [44]). *Let $I \leq \mathbb{R}[\mathbf{x}]$ be any ideal. Then*

$$\mathbf{I}(\mathbf{V}_{\mathbb{R}}(I)) = \sqrt[r]{I}.$$

Proposition 2.15 (Simple Point Criterion [44]). *Let $I \leq \mathbb{R}[\mathbf{x}]$. Then, I is real if and only if I is radical and for every associated prime $\mathfrak{p}$ of I there exists $p \in \mathbf{V}_{\mathbb{R}}(\mathfrak{p})$ such that $\mathbb{R}[\mathbf{x}]/I$ localized at $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ is regular.*

Remarks.

(1) We can easily modify the proof of Proposition 2.15 in [44] to show the following generalization for $I \leq \mathbb{K}[\mathbf{x}]$ with $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{R}$: Assume I is radical and for every associated prime $\mathfrak{p}$ of I there exists $p \in \mathbf{V}_{\mathbb{R}}(\mathfrak{p})$ such that $\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}$ localized at the maximal ideal $\langle x_1 - p_1, \ldots, x_n - p_n \rangle$ is regular. Then, I is real in $\mathbb{K}[\mathbf{x}]$.

(2) In contrast to the usual radical of ideals of affine rings we have to be careful with the real radical and field extensions: Let $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{K}'$ be a field extension. Then, for ideals $J \leq \mathbb{K}[\mathbf{x}]$ it is in general $\sqrt[r]{J_{\mathbb{K}'}} \neq \sqrt[r]{J} \cdot \mathbb{K}'[\mathbf{x}]$. Consider for instance $I = \langle x^3 - 5\, y^3 \rangle$ from Example 2.2 (iii) which is real as ideal in $\mathbb{Q}[x]$ but not in $\mathbb{R}[x]$.

(3) There are algorithms to compute the real radical of an ideal $J \leq \mathbb{Q}[\mathbf{x}]$ (e.g. `realrad` in `Singular` [16, 71, 72]). But to the authors knowledge all existing implementations only compute over $\mathbb{Q}$ (and will not calculate $\sqrt[r]{J \cdot \mathbb{R}[\mathbf{x}]}$), since it is arduous to encode an ordering in a simple field extension of $\mathbb{Q}$, see [57] and Section 7.2. For example, in `Singular` we have `realrad(x^3 - 5y^3) = x^3 - 5y^3`.

Lemma 2.16. *Let* $\mathfrak{a} \le \mathbb{R}[\mathbf{x}]$ *be a real prime. Then,* $\mathfrak{a}_{\mathbb{C}} = \mathfrak{a}\,\mathbb{C}[\mathbf{x}]$ *is prime.*

Proof. Let $A = \mathbb{R}[\mathbf{x}]/\mathfrak{a}$ and $K = Q(A)$ the real quotient field of A. Now suppose $\mathfrak{a}\,\mathbb{C}[\mathbf{x}] = \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}$ is not prime. Then $\mathbb{C} \otimes_{\mathbb{R}} A$ is not a domain and neither is $\mathbb{C} \otimes_{\mathbb{R}} K$. But

$$\mathbb{C} \otimes_{\mathbb{R}} K = \mathbb{R}[x]/(x^2+1) \otimes_{\mathbb{R}} K \cong K[x]/(x^2+1).$$

We see that $K[x]/(x^2+1)$ is not a domain. Then, there exists $a \in K$ with $a^2+1=0$. This is a contradiction to the fact that K is a real field. □

2.8 Manifold points

In this section let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ again. We consider functions $f\colon U \to \mathbb{K}$ on an open subset $U \subset \mathbb{K}^n$. In order to shorten the exposition, we consider the domain of f over the fields $\mathbb{R}$ and $\mathbb{C}$ simultaneously which leads to a small degree of ambiguity when we talk about questions of differentiability. We will use constructions like the Jacobian matrix which will work in both settings but mention if some properties are named differently for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.

f is called **analytic** at $p \in \mathbb{K}^n$ (or **holomorphic** for $\mathbb{K} = \mathbb{C}$), if

$$f(z) = \sum_{i_1,\ldots,i_n} c_{i_1,\ldots,i_n} (z_1 - p_1)^{i_1} \ldots (z_n - p_n)^{i_n},$$

in a neighborhood of $p = (p_1, \ldots, p_n)$.

Since we only need submanifolds of euclidean space, we will use the following definition which complies with definitions given in [9] (or [23] for the case $\mathbb{K} = \mathbb{C}$).

A d-dimensional smooth (analytic, complex) **submanifold** of $\mathbb{K}^n$ is a set $X \subset \mathbb{K}^n$ such that for every p in X there is an open neighborhood $U \subset \mathbb{K}^n$ and a C^∞-diffeomorphism (C^ω, biholomorphism) $\phi\colon U \to V$ to an open set $V \subset \mathbb{K}^n$, with

$$X \cap U = \{x \in U \mid \phi_{d+1}(x) = \ldots = \phi_n(x) = 0\}. \tag{2.9}$$

We need another definition in order to formulate the next result. Let $\rho\colon \mathbb{K}^n \to \mathbb{K}^d$ be the projection on the first d coordinates and $X \subset \mathbb{K}^n$ any nonempty set. If for an open neighborhood U of $p \in X$ there is an analytic (smooth, holomorphic) mapping $\psi\colon \rho(U) \to \mathbb{K}^{n-d}$ such that

$$X \cap U = \{\, (y, \psi(y)) \mid y \in \rho(U) \,\}, \tag{2.10}$$

we say that X is **locally a graph** of an analytic (smooth, holomorphic) mapping at p. It is not clear a priori if this gives a local definition, so we need to verify it:

Lemma 2.17. *Let* $p \in X$ *and* $U \subset \mathbb{K}^n$ *be open with* $p \in U$ *such that* (2.10) *is true with analytic (smooth, holomorphic) mapping* $\psi\colon \rho(U) \to \mathbb{K}^{n-d}$. *For any open subset* $U' \subset U$ *with* $p \in U'$ *there is an open neighborhood* $V \subset U'$ *of* p *such that*

$$X \cap V = \{\, (y, \psi(y)) \mid y \in \rho(V) \,\}.$$

Proof. We set

$$W_{\varepsilon_1,\varepsilon_2} = \left\{ p + (x_1, \dots, x_n) \in \mathbb{K}^n \;\middle|\; |x_1|, \dots, |x_d| \le \varepsilon_1,\ |x_{d+1}|, \dots, |x_n| \le \varepsilon_2 \right\} \quad (2.11)$$

Since U' is open, there exists $\varepsilon > 0$ with $W_{\varepsilon,\varepsilon} \subset U'$. Then, as ψ is continuous, we can choose $\delta > 0$ with $\delta < \varepsilon$ and

$$\{ (y, \psi(y)) \mid y \in \rho(W_{\delta,\varepsilon}) \} \subset W_{\delta,\varepsilon} \subset U'.$$

One checks easily now that $X \cap W_{\delta,\varepsilon} = \{ (y, \psi(y)) \mid y \in \rho(W_{\delta,\varepsilon}) \}$. □

Proposition 2.18. *Let $X \subset \mathbb{K}^n$ be any nonempty set and $p \in X$. The following conditions are equivalent:*

(a) There is an euclidean neighborhood U of p such that $X \cap U$ is an analytic (smooth, holomorphic) submanifold of $\mathbb{K}^n$.

(b) There exists a permutation $\pi\colon \mathbb{K}^n \to \mathbb{K}^n$ of coordinates such that

$$\pi(X) = \{ \pi(x) \mid x \in X \}$$

is locally the graph of an analytic (smooth, holomorphic) mapping at $\pi(p)$.

(c) For a generic choice of $A \in \mathrm{GL}(n, \mathbb{K})$, $A(X)$ is locally the graph of an analytic (smooth, holomorphic) mapping at Ap.

Definition 2.3. A point p of X for which the conditions of Proposition 2.18 hold is called a (smooth, analytic, holomorphic) **manifold point** of X.

Proof. First, we show that (a) implies (b) and (c). Suppose (a) is true. Then, we can choose $U, V \subset \mathbb{K}^n$ open with $p \in U$ and a diffeomorphism (biholomorphism) $\phi\colon U \to V$ with

$$X \cap U = \{ x \in U \mid \phi_{d+1}(x) = \ldots = \phi_n(x) = 0 \}.$$

Note that for any $U' \subset U$ open, $\phi' = \phi|_{U'}\colon U' \to \phi(U')$ is a diffeomorphism (biholomorphism) between open sets and

$$X \cap U' = \{x \in U' \mid \phi'_{d+1}(x) = \ldots = \phi'_n(x) = 0 \}. \quad (2.12)$$

Now let $A \in \mathrm{GL}(n, \mathbb{K})$ and $\rho'\colon \mathbb{K}^n \to \mathbb{K}^{n-d}$ be the projection onto the last $n - d$ coordinates. We set

$$\delta = (\rho' \circ \phi) \circ A^{-1}\colon A(U) \to \mathbb{K}^{n-d}.$$

Then, the Jacobian $D\delta$ evaluated at Ap consists of the last $n-d$ rows of $D\phi(p) \cdot A^{-1}$, where $D\phi(p)$ is a regular matrix as ϕ is a diffeomorphism (biholomorphism).

According to Lemma 2.19 and Lemma 2.20 A can be chosen generically for (b) or as permutation matrix for (c) such that the last $n-d$ columns of $D\delta(Ap)$ form a regular matrix. Next, we set $Ap = (r, s)$, $r \in \mathbb{K}^d$, $s \in \mathbb{K}^{n-d}$ and apply the (analytic) implicit function theorem [76, 1.37] (see [23, 7.6], [22, A.3] for (real) analytic versions). We can choose open sets $V \subset \mathbb{K}^d$, $W \subset \mathbb{K}^{n-d}$ and a smooth (analytic, holomorphic) map $\psi\colon V \to W$ such that $r \in V$, $s \in W$, $V \times W \subset A(U)$ and for all $(a, b) \in V \times W$

$$\delta(a, b) = 0 \Leftrightarrow b = \psi(a).$$

With this we derive

$$\begin{aligned}\{\, (y, \psi(y)) \mid y \in V \} &= \{\, x \in V \times W \mid \delta(x) = 0 \,\} \\ &= A \{\, x' \in A^{-1}(V \times W) \mid \delta(Ax') = 0 \,\} \\ &= A \{\, x' \in A^{-1}(V \times W) \mid \phi_{d+1}(x') = \ldots = \phi_n(x') = 0 \,\}.\end{aligned}$$

Because of (2.12), we also have

$$\{\, x' \in A^{-1}(V \times W) \mid \phi_{d+1}(x') = \ldots = \phi_n(x') = 0 \,\} = X \cap A^{-1}(V \times W).$$

But then $\{\, (y, \psi(y)) \mid y \in V \} = A(X) \cap (V \times W)$ and $A(X)$ is locally the graph of a smooth (analytic, holomorphic) mapping.

Suppose to the contrary that $A(X)$ is locally the graph of a smooth (analytic, holomorphic) mapping at Ap, for a regular matrix $A \in \mathrm{GL}(n, \mathbb{K})$. Then, there is $U \subset \mathbb{K}^n$ open and a smooth (analytic, holomorphic) mapping $\psi\colon \rho(U) \to \mathbb{K}^n$ such that $Ap \in U$ and

$$U \cap A(X) = \{\, (y, \psi(y)) \mid y \in \rho(U) \,\}.$$

We will show that $A^{-1}(U) \cap X$ is a submanifold of $\mathbb{K}^n$. So let $q \in A^{-1}(U) \cap X$ and set $q' = Aq \in U$. We define

$$\begin{aligned}&\phi\colon U \to \mathbb{K}^n, \\ &(r, s) \mapsto (r, s - \psi(r)).\end{aligned}$$

ϕ is a smooth (analytic, holomorphic) map and $D\phi(q')$ is the identity matrix hence regular. Thus, we can apply the (analytic) inverse mapping theorem, see [76, 1.30] or [23, 7.5] in the analytic setting. There are open neighborhoods U', V' of q' and $\phi(q')$ such that $U' \subset U$ and $\phi|_{U'}\colon U' \to V'$ is an (analytic) diffeomorphism (biholomorphism). By shrinking U' if necessary, we can make sure that

$$\{\, (y, \psi(y)) \mid y \in \rho(U') \,\} = A(X) \cap U'.$$

Now let $U'' = A^{-1}(U') \ni q$ and $\phi' = (\phi \circ A)|_{U''}$. Since $\phi|_{U'}$ is an (analytic) diffeomorphism (biholomorphism), the same is true for ϕ' and

$$X \cap U'' = A^{-1}(A(X) \cap U') = A^{-1} \{\, (y, \psi(y)) \mid y \in \rho(U') \,\}$$

$$
\begin{aligned}
&= A^{-1}\{\, x \in U' \mid \phi_{d+1}(x) = \ldots = \phi_n(x) = 0 \,\} \\
&= \{\, y \in U'' \mid \phi'_{d+1}(y) = \ldots = \phi'_n(y) = 0 \,\},
\end{aligned}
$$

which we wanted to show. □

Lemma 2.19. *Let $C \in \mathbb{K}^{r\times n}$, where $r \le n$. Assume C has full rank. For a generic choice of $A \in \mathrm{GL}(n,\mathbb{K})$, the last r columns of CA^{-1} are linearly independent.*

Proof. Let $p \in \mathbb{K}[\{x_{ij}, 1 \le i,j \le n\}]$ be the determinant of the last r columns of CB, where B is the $n \times n$ matrix with entries x_{ij}. Since $p(T) \neq 0$ for some permutation matrix T (Lemma 2.20), p must be nonzero. Now

$$\bigl\{B \in \mathrm{GL}(n,\mathbb{K}) \bigm| p(B) \neq 0\bigr\}$$

is a nonempty Zariski open subset of $\mathrm{GL}(n,\mathbb{K})$. Because $i\colon B \mapsto B^{-1}$ is an isomorphism of the quasi-affine variety $\mathrm{GL}(n,\mathbb{K})$ onto itself,

$$\left\{A \in \mathrm{GL}(n,\mathbb{K}) \mid \bigl[CA^{-1}\bigr]_{\substack{i,j \\ n-r<j\le n}} \text{ regular}\right\} = i\left(\bigl\{B \in \mathrm{GL}(n,\mathbb{K}) \bigm| p(B) \neq 0\,\bigr\}\right)$$

must be Zariski open in $\mathrm{GL}(n,\mathbb{K})$, too. □

Lemma 2.20. *Let $v_1,\ldots,v_r$ be linearly independent in $\mathbb{K}^n$ with standard coordinate system $(x_1,\ldots,x_n)$. There exists a choice of projection $p\colon \mathbb{K}^n \to \mathbb{K}^r$ on coordinates $x_{i_1},\ldots,x_{i_r}$ such that $p(v_1),\ldots,p(v_r)$ are linearly independent.*

Proof. Write the vectors $v_1,\ldots,v_r$ in a matrix C. Since the row rank of C equals r, we find r linearly independent rows. Let p be the projection on the coordinates indexed by the row numbers. Then

$$\dim\langle p(v_1),\ldots,p(v_r)\rangle_{\mathbb{K}} = r.$$ □

The following proposition shows that for real algebraic sets C^∞-manifold points and C^ω-manifold points are the same. This will be important for us since we can work with power series to parameterize smooth real algebraic sets.

Proposition 2.21. *Let $\mathbb{K} = \mathbb{R}$ and $X \subset \mathbb{R}^n$ be a real algebraic set. Any $p \in X$ is an analytic manifold point of X if and only if it is a smooth manifold point of X.*

Proof. Let $p \in X$ be a smooth manifold point of X. Without loss of generality we can choose an open neighborhood $U \subset \mathbb{R}^n$ of p and a smooth function $\psi\colon \rho(U) \to \mathbb{R}^{n-d}$ with

$$X \cap U = \{\, (y, \psi(y)) \mid y \in \rho(U) \,\},$$

where ρ is the projection on the first d coordinates. According to Lemma 2.17, we can assume that U is semi-algebraic [7, Section 2.1] by replacing U with $W_{\varepsilon_1,\varepsilon_2}$ from (2.11) if necessary. Now we have

$$\operatorname{Graph}\psi_i = \bigl\{(y,\psi_i(y)) \in \mathbb{R}^{d+1} \mid y \in \rho(U)\bigr\} = \rho'(X \cap U), \quad i = 1,\ldots,n-d,$$

where ρ' is the projection on the d first and the $d+i$-th coordinate. Therefore, Graph ψ_i is semi-algebraic for all i since projections of semi-algebraic sets are semi-algebraic [7, Theorem 2.2.1]. Then, ψ_i is a semi-algebraic C^∞-function, which is a Nash-function and in particular analytic [7, Corollary 8.1.6, 8.1.7 and Proposition 8.1.8]. $\square$

The next proposition is an easy consequence of [43, Definition II.7.3 and Theorem IV.4.1]. We will not use this result but it is of interest in this context.

Proposition 2.22. *Let $\mathbb{K}=\mathbb{C}$ and $X\subset\mathbb{C}^n$ be a complex algebraic set, with $p\in X$. p is a real smooth manifold point of X considered as subset of $\mathbb{R}^{2n}$ if and only if p is a holomorphic manifold point of $X\subset\mathbb{C}^n$.*

2.9 Normalization of local rings

We will need some technical results about the normalization of local rings, i.e. the integral closure in its total ring of fractions. The example to keep in mind is $A=\mathbb{R}[\mathbf{x}]/I$ with I radical and $I\subset\mathfrak{m}=\langle x\rangle$.

Proposition 2.23. *Let A be a reduced noetherian ring and B the integral closure of A in its total ring of fractions. Assume that B is noetherian, $\mathfrak{m}\leq A$ a maximal ideal and $S=A\backslash\mathfrak{m}$. $A\hookrightarrow B$ induces the following commutative diagram:*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A_{\mathfrak{m}} & \longrightarrow & S^{-1}B \end{array}$$

$S^{-1}B$ is the integral closure of $A_{\mathfrak{m}}$ in its total ring of fractions. For the unique decomposition

$$\sqrt{\mathfrak{m}\,S^{-1}B}=\mathfrak{n}'_1\cap\ldots\cap\mathfrak{n}'_k,$$

there is a unique decomposition

$$\sqrt{\mathfrak{m}B}=\mathfrak{n}_1\cap\ldots\cap\mathfrak{n}_k.$$

with $\mathfrak{n}_i$ maximal in B and $S^{-1}\mathfrak{n}_i=\mathfrak{n}'_i$. Moreover, $B_{\mathfrak{n}_i}\cong(S^{-1}B)_{\mathfrak{n}'_i}$.

Proof. The existence of homomorphism such that the diagram is commutative is clear. Furthermore, Lemma 2.24 will show that $S^{-1}B$ is the normalization of $A_{\mathfrak{m}}$.

Now let $\hat{\mathfrak{m}}=\mathfrak{m}B$ be the extension of $\mathfrak{m}$ in B. Then, clearly $S^{-1}\hat{\mathfrak{m}}=\mathfrak{m}\cdot S^{-1}B$, hence

$$\sqrt{S^{-1}\hat{\mathfrak{m}}}=\sqrt{\mathfrak{m}\cdot S^{-1}B}. \tag{2.13}$$

Since formation of fractions commutes with taking of radicals we have also

$$\sqrt{S^{-1}\hat{\mathfrak{m}}} = S^{-1}\sqrt{\hat{\mathfrak{m}}}. \tag{2.14}$$

On the other hand, as B is noetherian, we have

$$\sqrt{\hat{\mathfrak{m}}} = \mathfrak{n}_1 \cap \ldots \cap \mathfrak{n}_k.$$

with $\mathfrak{n}_i$ prime for $i = 1, \ldots, k$. Since $\mathfrak{m} \subset \mathfrak{n}_i \cap A$ and $\mathfrak{m}$ is maximal, $\mathfrak{n}_i \cap A = \mathfrak{m}$ for all i and consequently $\mathfrak{n}_i$ must be maximal because B is integral over A [3, Corollary 5.8]. As formation of fractions commutes with intersections we get

$$S^{-1}\sqrt{\hat{\mathfrak{m}}} = S^{-1}\mathfrak{n}_1 \cap \ldots \cap S^{-1}\mathfrak{n}_k. \tag{2.15}$$

Clearly, $S^{-1}\mathfrak{n}_i$ is either maximal or the whole ring $S^{-1}B$, depending on whether $S \cap \mathfrak{n}_i = \emptyset$. But since $S \cap \mathfrak{n}_i \subset (S \cap A) \cap \mathfrak{n}_i = S \cap \mathfrak{m} = \emptyset$, all $S^{-1}\mathfrak{n}_i$ must be maximal. One can check as well that no $S^{-1}\mathfrak{n}_i$ is contained in any other $S^{-1}\mathfrak{n}_j$, since the same is true for the $\mathfrak{n}_i$. Putting (2.13), (2.14) and the previous arguments together, we see that (2.15) is the unique decomposition of $\sqrt{\mathfrak{m} \cdot S^{-1}B}$ into maximal ideals.

It remains to show that $B_{\mathfrak{n}_i} \cong (S^{-1}B)_{\mathfrak{n}_i'}$ for all i. Because $\mathfrak{n}_i \cap S = \emptyset$, this follows from Lemma 2.25. $\square$

Lemma 2.24. *Let A be a reduced noetherian ring with*

$$(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_k$$

the unique decomposition into primes. Set $B = B_1 \times \cdots \times B_k$, where B_i is the integral closure of $A/\mathfrak{p}_i$. Then, B is the integral closure of A.

Moreover, let $\mathfrak{m} \leq A$ be a maximal ideal and $S = A \backslash \mathfrak{m}$. Suppose the $\mathfrak{p}_i$ are ordered such that $\mathfrak{p}_i \subset \mathfrak{m}$, $i = 1 \ldots, r$ and $\mathfrak{p}_i \not\subset \mathfrak{m}$, $i = r+1, \ldots, n$. Set $S_i := \big(A/\mathfrak{p}_i\big)\backslash\big(\mathfrak{m}/\mathfrak{p}_i\big)$, $i = 1, \ldots, r$. Then

$$S^{-1}B \cong S_1^{-1}B_1 \times \cdots \times S_r^{-1}B_r$$

is the integral closure of $A_{\mathfrak{m}}$ in its total ring of fractions.

Proof. We consider the following commutative diagram

$$\begin{tikzcd} A \arrow[r, hook, "\gamma"] \arrow[d] & B_1 \times \cdots \times B_k \arrow[r] \arrow[d, "\phi_i"] & S^{-1}B \arrow[d] \\ A/\mathfrak{p}_i \arrow[r, hook] & B_i \arrow[r, "\iota_i"] & S_i^{-1}B_i \end{tikzcd} \tag{2.16}$$

It is well-known that $B = B_1 \times \cdots \times B_k$ is the integral closure of A in its total ring of fractions [13, Lemma 29.52.3]. We define the homomorphism

$$\eta' \colon B \to S_1^{-1}B_1 \times \cdots \times S_r^{-1}B_r$$

induced by the homomorphism $\iota_i \circ \phi_i$, for $i = 1, \dots, r$. Now for any $b = (b_1, \dots, b_k) \in \gamma(A \backslash \mathfrak{m})$ we have $b_i = \phi_i(b) \in S_i$, for $i = 1, \dots, r$ and consequently $(b_1, \dots, b_r) = \eta'(b)$ is a unit in $S_1^{-1}B_1 \times \cdots S_r^{-1}B_r$, so η' induces a homomorphism

$$\eta \colon S^{-1}B \to S_1^{-1}B_1 \times \cdots \times S_r^{-1}B_r.$$

We will show that η is actually an isomorphism. First, let

$$\left(\frac{b_1}{q_1}, \dots, \frac{b_r}{q_r}\right) \in S_1^{-1}B_1 \times \cdots \times S_r^{-1}B_r$$

with $b_i \in B_i$ and $q_i \in S_i$. Then

$$\left(\frac{b_1}{q_1}, \dots, \frac{b_r}{q_r}\right) = \eta'(b_1, \dots, b_r, 0, \dots, 0) \cdot \eta'(q_1, \dots, q_r, 0, \dots, 0)^{-1}.$$

Therefore, η is an epimorphism. Now let $b = (b_1, \dots, b_n) \in B$ and $(s_1, \dots, s_r) \in S_1 \times \cdots \times S_r$ such that $b_i\, s_i = 0$, for all $i = 1, \dots, r$. We choose $s_1', \dots, s_n' \in S$, with $\phi_i(\gamma(s_i')) = s_i$, for $i = 1, \dots, r$ and $\phi_i(\gamma(s_i)) = 0$, for $i = r+1, \dots, n$ which is possible according to our assumption. Now let $s = \gamma(s_1' \cdots , s_n') \in S$. Then

$$\phi_i(s\, b) = \phi_i(\gamma(s_i'))\, b_i = 0, \quad i = 1, \dots, n.$$

So, $s\, b = 0$ and η is a monomorphism.

It remains to show that $\bigtimes_{i=1}^{r} S_i^{-1}B_i$ is the normalization of $A_\mathfrak{m}$. However, this is clear since $0 = \mathfrak{p}_1 A_\mathfrak{m} \cap \ldots \cap \mathfrak{p}_r A_\mathfrak{m}$ is the unique decomposition of the zero ideal in $A_\mathfrak{m}$ and $S_i^{-1}B_i$ is the integral closure of $A_\mathfrak{m}/\big(\mathfrak{p}_i\, A_\mathfrak{m}\big) = \big(A/\mathfrak{p}_i\big)_\mathfrak{m}$ in $S_i^{-1}Q\big((A/\mathfrak{p}_i)_\mathfrak{m}\big) = Q(A/\mathfrak{p}_i)$ [3, Proposition 5.12]. $\square$

Lemma 2.25. *Let A be a ring and $\mathfrak{p} \leq A$ a prime ideal. Also, let $S \subset A$ be a multiplicative set with $S \cap \mathfrak{p} = \emptyset$. Then, $S^{-1}\mathfrak{p} = \mathfrak{p}\, S^{-1}A$ is a prime ideal and*

$$(S^{-1}A)_{S^{-1}\mathfrak{p}} \cong A_\mathfrak{p}.$$

Proof. It is a characteristic feature of rings of fractions that $S^{-1}\mathfrak{n}$ is prime [3, Proposition 3.11].

We get a homomorphism $\varphi \colon A_\mathfrak{p} \to (S^{-1}A)_{S^{-1}\mathfrak{p}}$ from the universal property of localization. One can check easily that φ is an isomorphism. $\square$

2.10 Analytic varieties and set germs

We give a brief overview of analytic varieties to introduce the notion of vanishing ideal of an analytic set germ and to formulate the analytic Nullstellensätze. Assume again $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Definition 2.4.

(i) Let $U \subset \mathbb{K}^n$ be open, and $g_1, \ldots, g_k$ be analytic functions on U. Then, we write $\mathbf{V}_U^a(g_1, \ldots g_k)$ for the set of common zeros of the g_i:

$$\mathbf{V}_U^a(g_1, \ldots g_k) = \{x \in U \mid g_1(x) = \ldots = g_k(x) = 0\}.$$

(ii) A subset $V \subset U$ of an open set $U \subset \mathbb{K}^n$ is called **analytic variety in U** if V is closed in U and for each $p \in V$, there is a $\mathbb{K}^n$-neighborhood U' of p and $g_1, \ldots g_k$ analytic functions on U' such that

$$U' \cap V = \mathbf{V}_{U'}^a(g_1, \ldots, g_k).$$

(iii) Let $p \in \mathbb{K}^n$ and $\mathcal{Y}_p$ be the set of all subsets $Y \subset \mathbb{K}^n$ such that $p \in Y$ and Y is an analytic variety in an open neighborhood of p. We now define the usual equivalence relation on $\mathcal{Y}_p$: $Y \sim Y'$ if there exists an open neighborhood W of p with $Y \cap W = Y' \cap W$. We call an equivalence class G of $\mathcal{Y}_p/\sim$ an **analytical set germ at p** and write $G = (Y, p)$ for a representative $Y \in \mathcal{Y}_p$.

(iv) Let $p \in \mathbb{K}^n$. We will write $\mathcal{O}_n(p)$ for the ring of analytic function germs [23, Section 3.4] at p. Clearly $\mathcal{O}_n(p) \cong \mathcal{O}_n(0) = \mathbb{K}\{\mathbf{x}\}$. Let $G = (Y, p)$ be an analytical set germ. Then we define $\mathbf{I}^a(G)$ as the ideal of $\mathcal{O}_n(p)$ with a representative $g\colon U \to \mathbb{K}$, for which $g|_{U \cap Y} = 0$. $\mathbf{I}^a(G)$ is called the **vanishing ideal** of the set germ G and $\mathcal{O}(G) = \mathcal{O}_n(p)/\mathbf{I}^a(G)$ the **analytic coordinate ring** of G.

(v) For any ideal $I \leq \mathcal{O}_n(p)$ we have $I = \langle g_1, \ldots, g_k \rangle$, since $\mathcal{O}_n(p)$ is noetherian. Hence I gives rise to a set germ $\mathbf{V}^a(I) = (Y, p)$, where $Y = \mathbf{V}_U^a(g_1, \ldots, g_k)$ for a common domain U of $g_1, \ldots, g_k$.

To simplify the theory we will fix p to the origin from now on. The general case can be retrieved by translation. We collect some basic facts which will help us later.

Proposition 2.26. *Let $G = (X, 0)$ be an analytic set germ. Then, we have*

(a) $G = (\{0\}, 0)$ *if and only if* $\dim \mathcal{O}(G) = 0$ *if and only if* $\mathcal{O}(G) \cong \mathbb{K}$.

(b) $G = (B_\varepsilon, 0)$ *for any* $\varepsilon > 0$, *if and only if* $\dim \mathcal{O}(G) = n$ *if and only if* $\mathcal{O}(G) \cong \mathbb{K}\{\mathbf{x}\}$.

Proof.

(a) First, assume $G = (\{0\}, 0)$. In this case it is $\mathbf{I}^a(G) = \langle \mathbf{x} \rangle$. Therefore

$$\mathcal{O}(G) \cong \mathbb{K}\{\mathbf{x}\}/\langle \mathbf{x} \rangle \cong \mathbb{K}$$

and we have $\dim \mathcal{O}(G) = 0$. Now let $\dim \mathcal{O}(G) = 0$. Because $\mathcal{O}(G)$ is noetherian, local and zero-dimensional, it is well-known that $\langle \mathbf{x} \rangle^k = 0$ in $\mathcal{O}(G)$ for some $k \geq 1$. This means that $\langle \mathbf{x} \rangle^k \subset \mathbf{I}^a(G)$. But $\mathbf{I}^a(G)$ is radical, hence $\langle \mathbf{x} \rangle = \mathbf{I}^a(G)$ and G must be the isolated origin.

(b) First, let $G = (B_\varepsilon, 0)$. Then, every function germ in $\mathbf{I}^a(G)$ has a representative which vanishes identically on B_ε and must be the zero accordingly. So $\mathbf{I}^a(G) = (0)$ and $\mathcal{O}(G) = \mathbb{K}\{\mathbf{x}\}$ which means $\dim \mathcal{O}(G) = n$.

Now let $\dim \mathcal{O}(G) = n$. With [67, Proposition 2.4] this means $\operatorname{ht} \mathbf{I}^a(G) = n - \dim \mathcal{O}(G) = 0$, so $\mathbf{I}^a(G) = 0$. But $X \cap U = \mathbf{V}^a_U(g_1, \dots g_k)$ for an open neighborhood U of 0 and g_i analytical on U for $i = 1, \dots, k$. As $g_i \in \mathbf{I}^a(G) = (0)$, we have $X \cap U = U$. Therefore, $(X, 0) = (B_\varepsilon, 0)$. □

For any ideal $I \leq \mathbb{K}\{\mathbf{x}\}$ we clearly have $I \subset \mathbf{I}^a(\mathbf{V}^a(I))$ but this is a proper inclusion in general:

Example 2.3. Let $\mathbb{K} = \mathbb{R}$ and recall from the introduction $I = \langle x^2 - y^3 + 2\, yx^2 - x^4 \rangle \leq \mathbb{R}\{\mathbf{x}\}$. Then

$$\mathbf{I}^a(\mathbf{V}^a(I)) = \langle x^2 - y(1 + \sqrt{1+y}) \rangle \subsetneq \langle y^3 + 2\, yx^2 - x^4 \rangle.$$

The following theorem due to W. Rückert for $\mathbb{K} = \mathbb{C}$ and J. Risler for $\mathbb{K} = \mathbb{R}$ resolves this issue:

Theorem 2.27 (Analytic Nullstellensätze)**.** *Let $I \leq \mathbb{K}\{\mathbf{x}\}$. Then*

(i) If $\mathbb{K} = \mathbb{C}$, then $\mathbf{I}^a(\mathbf{V}^a(I)) = \sqrt{I}$.

(ii) If $\mathbb{K} = \mathbb{R}$, then $\mathbf{I}^a(\mathbf{V}^a(I)) = \sqrt[r]{I}$.

Proof. See [32, Theorem 2.20, Theorem 3.7] for (i) and [65, Théorème 4.1] for (ii). □

2.11 Pseudo Gröbner bases and Gröbner covers

For the analysis of linkage configuration spaces in Chapter 6, we will need some facts about Gröbner bases of parametric ideals.

Let $\mathbb{K}$ be a field containing $\mathbb{Q}$ and $R = \mathbb{K}[t_1, \dots, t_s]$. We work with the polynomial ring $R[\mathbf{x}]$ and consider $\mathbf{t} = (t_1, \dots, t_s)$ as parameters. Fix a global monomial ordering $<$ on the set of monomials in $\mathbf{x} = (x_1, \dots, x_n)$. For $f, g \in R[\mathbf{x}]$ with leading terms $\mathrm{LT}(f) = a\,\mathbf{x}^\alpha$, $\mathrm{LT}(g) = b\,\mathbf{x}^\beta$, $\alpha, \beta \in \mathbb{N}^n$ let

$$\operatorname{spoly}(f, g) = b\mathbf{x}^{\gamma-\alpha} f - a\mathbf{x}^{\gamma-\beta} g, \tag{2.17}$$

the **s-polynomial** of f, g, where $\gamma_r = \max(\alpha_r, \beta_r)$ for $r = 1, \dots, n$.

Definition 2.5. Let $I \leq R[\mathbf{x}]$ be an ideal. A set $\mathcal{G} = \{g_1, \dots, g_d\} \subset I$ is called a **pseudo Gröbner basis** of I, if

(1) $\mathcal{G}$ generates I as $R[\mathbf{x}]$-ideal.

(2) $\operatorname{red}\big(\operatorname{spoly}(g_i, g_j) \mid \mathcal{G}\big) = 0$, for all $1 \leq i, j \leq s$.

See [31, Exercise 2.3.7]. Here $\operatorname{red}(f \mid \mathcal{G})$ denotes the **reduction** or **pseudo normal form** of f with respect to $\mathcal{G}$ as in [31, Algorithm 1.6.10]. This algorithm can be seen as division with remainder in $R[\mathbf{x}]$ and works by repeatedly forming $f_{k+1} = \operatorname{spoly}(f_k, g)$, with $f_0 = f$ and $g \in \mathcal{G}$ such that the leading monomial $\operatorname{LM}(g)$ of g divides $\operatorname{LM}(f_k)$, see [12, Section 2.3] for details. Note that we need to use the symmetric form (2.17) of the s-polynomial in the calculation of the pseudo normal form because we have to avoid dividing by elements of R [31, p. 49].

In the CAS `Singular` [16] we can calculate pseudo normal forms and pseudo Gröbner bases by setting the options `intStrategy` and `contentSB` [31, Exercise 2.3.10].

Proposition 2.28. *Let $\mathcal{G} = \{g_1, \ldots, g_d\}$ be a pseudo Gröbner basis of an ideal $I \leq R[\mathbf{x}]$ and let $t_0 \in \mathbb{K}^s$ such that the leading coefficients $[\operatorname{LC}(g_i)](t_0) \neq 0$, for $i = 1, \ldots, d$. Then, $\mathcal{G}|_{\mathbf{t}=t_0}$ is a Gröbner basis of $I|_{\mathbf{t}=t_0} = \{f|_{\mathbf{t}=t} \mid f \in I\} \leq \mathbb{K}[\mathbf{x}]$.*

Proof. See [31, Exercise 2.3.8]. □

Example 2.4. $\mathcal{G} = \{-a\,y, x^2 - a\}$ is a pseudo Gröbner basis of

$$I \leq \langle x^2 - a, xy\rangle \leq \big(\mathbb{C}[a]\big)[x, y],$$

with respect to the lexicographic ordering on the set of monomials in (x, y). Note that G is a Gröbner basis for any specialization $a \in \mathbb{C}\backslash\{0\}$. But $\mathcal{G}|_{a=0} = \{x^2\}$ is not a Gröbner basis of $I|_{a=0} = \langle x^2, x\,y\rangle$.

Definition 2.6. A **Gröbner cover** [51] or **comprehensive Gröbner system** [42] of $I \leq R[\mathbf{x}]$ is a sequence of pairs $(S_i, B_i)_{i=1,\ldots,k}$, where the S_i are disjoint locally closed segments of the parameter space, i.e.

$$S_i = \mathbf{V}(J_{i,1})\backslash\mathbf{V}(J_{i,2}), \quad \text{for } J_{i,1}, J_{i,2} \leq \mathbb{K}[\mathbf{t}],$$

and for any $t_0 \in S_i$ the specialized set $B_i|_{\mathbf{t}=t^*} \subset R[\mathbf{x}]$ is a reduced Gröbner basis of $I|_{\mathbf{t}=t^*}$. Usually, we also require that $\bigcup_i S_i = O$, where O is a specified Zariski-open subset of the parameter space $\mathbb{K}^s$.

In `Singular` we can calculate Gröbner covers with the `grobcov` algorithm [50] which builds on the Kapur-Sun-Wang algorithm [42]. Very useful is the `grobcov` option `nonnull` to restrict the parameter space for the algorithm to a preselected Zariski open subset of $\mathbb{K}^s$.

To finish this section we present some techniques to calculate Gröbner bases over simple algebraic field extensions of $\mathbb{Q}$. Let $\mathbb{K} = \mathbb{Q}[\alpha]$, for $\alpha \in \mathbb{C}$ algebraic over $\mathbb{Q}$ with minimal polynomial $p_\alpha \in \mathbb{Q}[\lambda]$. We fix any ideal $I \leq \mathbb{K}[\mathbf{x}]$ and set

$$I_\lambda := \big\{\, p(\mathbf{x}, \lambda) \in \mathbb{Q}[\mathbf{x}, \lambda] \;\big|\; p(\mathbf{x}, \alpha) \in I \,\big\},$$

which is an ideal of $\mathbb{Q}[\mathbf{x},\lambda]$. If $I = \langle p_1,\ldots,p_k\rangle \leq \mathbb{K}[\mathbf{x}]$, then one can easily check that $I_y = \langle p'_1,\ldots,p'_k,p_\alpha\rangle \leq \mathbb{Q}[\mathbf{x},\lambda]$, where p'_i is p_i with α substituted by λ. The epimorphism $\varphi\colon \mathbb{Q}[\mathbf{x},\lambda] \to \mathbb{K}[\mathbf{x}]/I$ given by $p(\mathbf{x},\lambda) \mapsto p(\mathbf{x},\alpha)$ has kernel I_λ according to the definition. This gives

$$\mathbb{Q}[\mathbf{x},\lambda]/I_\lambda \cong \mathbb{K}[\mathbf{x}]/I. \tag{2.18}$$

No we choose a product ordering $(<,<_\lambda)$ [31, Example 1.2.8] on the set of monomials in $(\mathbf{x},\lambda)$, where $<$ is a degree ordering and $<_\lambda$ is the unique global ordering on the univariate monomials in λ.

Proposition 2.29. *With the notations above let $\mathcal{G}$ be the reduced Gröbner basis of I_λ with respect to the ordering $(<,<_\lambda)$. Then*

(i) $p_\alpha \in \mathcal{G}$ *and* $\big(\mathcal{G}\backslash\{p_\alpha\}\big)\big|_{\lambda=\alpha}$ *is the reduced Gröbner basis of I over $\mathbb{K} = \mathbb{Q}[\alpha]$ with respect to $<$.*

(ii) $\dim \mathbb{Q}[\mathbf{x},\lambda]/I_\lambda = \dim \mathbb{K}[\mathbf{x}]/I$.

(iii) If $\dim \mathbb{Q}[\mathbf{x},\lambda]/I_\lambda = 0$, *we have* $\dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{x},\lambda]/I_\lambda = \deg p_\alpha \cdot \dim_{\mathbb{K}} \mathbb{K}[\mathbf{x}]/I$.

Proof. (i) is a well-known result from the theory of Gröbner bases [8, 58]. (ii) follows clearly from (2.18). Now consider the following chain of $\mathbb{Q}$-algebras:

$$\mathbb{Q} \subset \mathbb{K} \subset \mathbb{K}[\mathbf{x}]/I \cong \mathbb{Q}[\mathbf{x},\lambda]/I_\lambda.$$

For $r = \deg p_\alpha$, the set $\{1,\alpha,\ldots,\alpha^{r-1}\}$ is a $\mathbb{Q}$-base of $\mathbb{K}$. Choose any $\mathbb{K}$-base $\{v_1,\ldots,v_s\}$ of $\mathbb{K}[\mathbf{x}]/I$. Then, one easily checks that

$$\{v_i, \alpha\, v_i, \ldots, \alpha^{r-1}\, v_i \mid i = 1,\ldots,s\}$$

is a $\mathbb{Q}$-base of $\mathbb{Q}[\mathbf{x},\lambda]/I_\lambda$. This proves (iii). $\square$

Chapter 3

Real analytic and formal rings

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and $I \leq \mathbb{K}[\mathbf{x}]$, with $I \subset \langle \mathbf{x} \rangle$. Geometric properties of the set germ $G = (\mathbf{V}_{\mathbb{K}}(I), 0)$ can be studied via the analytic local ring $\mathcal{F} = \mathbb{K}[[\mathbf{x}]]/I\,\mathbb{K}[[\mathbf{x}]$ which is just the formal completion of the local ring $\mathbb{K}[\mathbf{x}]_{\langle \mathbf{x} \rangle}/I\,\mathbb{K}[\mathbf{x}]_{\langle \mathbf{x} \rangle}$. This is because $\mathcal{F}$ is closely connected to the analytic coordinate ring $\mathcal{O}(G)$ by the analytic Nullstellensätze and M. Artin's approximation theorem.

In this chapter we will study the relationship of $\mathcal{F}$ and $\mathcal{O}(G)$ and work out the differences for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. Since the origin is a manifold point of $\mathbf{V}_{\mathbb{K}}(I)$ if and only if $\mathcal{O}(G)$ is regular (Proposition 3.4), this will explain the contrasting behavior of embeddings of real and complex algebraic sets at singular points.

Additionally, we review some results about the normalization of $\mathcal{F}$. This will be useful in Chapter 4, when we examine real algebraic curves.

3.1 Structure

Let $I \leq \mathbb{K}[\mathbf{x}]$, $\mathfrak{m} := \langle \mathbf{x} \rangle$ and $I \subset \mathfrak{m}$. We set $A = \mathbb{K}[\mathbf{x}]/I$ and define the following local rings. This notation will be used throughout this thesis.

Definition 3.1.

(i) $\mathcal{R} = \mathbb{K}[\mathbf{x}]_{\mathfrak{m}}/\big(I \cdot \mathbb{K}[\mathbf{x}]_{\mathfrak{m}}\big) = A_{\mathfrak{m}}$, with maximal ideal $\mathfrak{m}\,\mathcal{R}$.

(ii) $\mathcal{O} = \mathbb{K}\{\mathbf{x}\}/\big(I \cdot \mathbb{K}\{\mathbf{x}\}\big)$, with maximal ideal $\mathfrak{m}\,\mathcal{O}$.

(iii) $\mathcal{F} = \mathbb{K}[[\mathbf{x}]]/\big(I \cdot \mathbb{K}[[\mathbf{x}]]\big)$, with maximal ideal $\mathfrak{m}\,\mathcal{F}$.

The natural homomorphism $\varphi\colon \mathbb{K}[\mathbf{x}]_{\mathfrak{m}} \to \mathbb{K}\{\mathbf{x}\}$ and $\gamma\colon \mathbb{K}\{\mathbf{x}\} \to \mathbb{K}[[\mathbf{x}]]$ induce homomorphism $\hat{\varphi}\colon \mathcal{R} \to \mathcal{O}$ and $\hat{\gamma}\colon \mathcal{O} \to \mathcal{F}$ such that the following diagram commutes:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & \mathcal{O} \arrow[r, "\hat{\gamma}"] & \mathcal{F} \\ \mathcal{R} \arrow[ur, "\hat{\varphi}"'] & & \end{tikzcd} \tag{3.1}$$

We have the following result regarding flatness:

Corollary 3.1 (Faithfully Flatness)**.** *The homomorphism $\hat{\varphi}$ and $\hat{\gamma}$ are faithfully flat. In particular, they are injective.*

Proof. Because of Proposition 2.1 and Proposition 2.5, $\hat{\gamma}$ and $\hat{\gamma} \circ \hat{\varphi}$ are completions of local rings and consequently faithfully flat. With the descent property [48, 2.4.B] $\hat{\varphi}$ is faithfully flat as well. □

Corollary 3.2. *Let J be an ideal of $\mathcal{R}$ and K be an ideal of $\mathcal{O}$, then*

(a) $\big(J \cdot \mathcal{O}\big) \cap \mathcal{R} = J, \quad \big(K \cdot \mathcal{F}\big) \cap \mathcal{O} = K.$

(b) $\operatorname{ht}\big(J \cdot \mathcal{O}\big) = \operatorname{ht} J, \quad \operatorname{ht}\big(K \cdot \mathcal{F}\big) = \operatorname{ht} K.$

3.2 Regularity

Let $I \leq \mathbb{K}[\mathbf{x}]$ again with $I \subset \mathfrak{m} := \langle \mathbf{x} \rangle$. We set $X = \mathbf{V}_{\mathbb{K}}(I)$ and define the analytic set germ $G = (X, 0)$, see Section 2.10. The next proposition gives an expression for $\mathcal{O}(G)$ and is a direct consequence of the analytic Nullstellensätze.

Proposition 3.3.

(i) If $\mathbb{K} = \mathbb{C}$, then $\mathcal{O}(G) = \mathcal{O}/\sqrt{(0)}$.

(ii) If $\mathbb{K} = \mathbb{R}$, then $\mathcal{O}(G) = \mathcal{O}/\sqrt[r]{(0)}$.

The analytic coordinate ring determines if the origin is a manifold point of X:

Proposition 3.4. *The origin is a manifold point of $X = \mathbf{V}_{\mathbb{K}}(I)$ if and only if $\mathcal{O}(G)$ is regular.*

Proof. First, let the origin be a manifold point of X (of dimension d). According to Proposition 2.18 we find w.l.o.g. an analytic parametrization of X:

$$\begin{aligned} \Psi \colon U &\mapsto \mathbb{K}^n, \\ (x_1, \dots, x_d) &\to (x_1, \dots, x_d, \psi_1(x_1, \dots, x_d), \dots, \psi_{n-d}(x_1, \dots, x_d)), \end{aligned}$$

where U is an euclidean neighborhood of the origin in $\mathbb{K}^d$ and $\Psi(0) = 0$. We set

$$L := \langle x_{d+1} - \psi_1(x_1, \dots, x_d), \dots, x_n - \psi_{n-d}(x_1, \dots x_d) \rangle \leq \mathbb{K}\{\mathbf{x}\}$$

and claim that

$$L = \mathbf{I}^a(G) = \left\{ f \in \mathbb{K}\{\mathbf{x}\} \;\middle|\; \begin{array}{l} \exists\, W \ni 0 \text{ euclidean neighborhood with} \\ f \text{ converges on } W \text{ and } f|_{W \cap X} = 0 \end{array} \right\}.$$

Clearly, we have $L \subset \mathbf{I}^a(G)$, so let $a \in \mathbf{I}^a(G)$. Since $\Psi(0) = 0$, we can compose $a \circ \Psi$ and get a converging power series

$$a(x_1, \ldots x_d, \psi_1(x_1, \ldots, x_d), \ldots \psi_{n-d}(x_1, \ldots x_d)) = 0, \tag{3.2}$$

which follows because $a \circ \Psi$ is identically zero close to the origin. Now we set $\varphi_i := x_{d+i} - \psi_i(x_1, \ldots x_d) \in \mathbb{K}\{\mathbf{x}\}$, for $i = 1, \ldots, n-d$ and have that φ_i is of x_{d+i}-order 1. Then, according to the Weierstrass division theorem [67, Proposition 3.2] we have a representation

$$a = q_1 \cdot \varphi_1 + r,$$

with $q_1 \in \mathbb{K}\{\mathbf{x}\}$ and $r \in \mathbb{K}\{x_1, \ldots, x_d, x_{d+2}, \ldots, x_{n-1}\}$. If we iterate this process with r instead of a, we get a decomposition

$$a = q_1 \cdot \varphi_1 + \ldots + q_{n-d} \cdot \varphi_{n-d} + r,$$

with $r \in \mathbb{K}\{x_1, \ldots, x_d\}$. Because of (3.2) and

$$\begin{aligned} &\varphi_i(x_1, \ldots x_d, \psi_1(x_1, \ldots, x_d), \ldots, \psi_{n-d}(x_1, \ldots, x_d)) \\ &= \psi_i(x_1, \ldots, x_d) - \psi_i(x_1, \ldots x_d) = 0, \end{aligned}$$

we see that $r(x_1, \ldots, x_d) = 0$, so $r = 0$ and therefore $a \in L$.

Now we have to check, that $\mathbb{K}\{\mathbf{x}\}/L$ is a regular local ring. We will use Nagata's Jacobian criterion, Theorem 2.11. It is enough to show that $\mathfrak{m}\mathbb{K}\{\mathbf{x}\} \not\supset J_{n-d}(L)$ and $\mathrm{ht}(L) \leq n-d$, where $J_{n-d}(L)$ is the Jacobian ideal of L of order $n-d$ (2.6). Since L is generated by $n-d$ elements $\mathrm{ht}(L) \leq n-d$ follows easily from Krull's height theorem [3, Corollary 11.16]. Furthermore, the Jacobian

$$\frac{D(\varphi_1, \ldots, \varphi_{n-d})}{D(x_{d+1}, \ldots, x_n)} = \det \left[\frac{\partial \varphi_i}{\partial x_j}\right]_{\substack{i=1,\ldots,n-d \\ j=d+1,\ldots,n}} = 1,$$

hence $J_{n-d}(L) = \mathbb{K}\{\mathbf{x}\} \not\supset \mathfrak{m}\mathbb{K}\{\mathbf{x}\}$. This means that $\mathbb{K}\{\mathbf{x}\}/L$ is a regular local ring and $\dim \mathbb{K}\{\mathbf{x}\}/L = d$.

Suppose to the contrary that $\mathcal{O}(G)$ is regular with $\dim \mathcal{O}(G) = d$. According to Nagata's Jacobian criterion $\mathfrak{m}\mathbb{K}\{\mathbf{x}\} \not\supset J_{n-d}\big(\mathbf{I}^a(G)\big)$. For that reason there are $g_1, \ldots, g_{n-d} \in \mathbf{I}^a(G)$ such that without loss of generality

$$\frac{D(g_1, \ldots, g_{n-d})}{D(x_1, \ldots, x_{n-d})} \notin \mathfrak{m}\mathbb{K}\{\mathbf{x}\}.$$

This means that the submatrix comprising the first $(n-d)$ columns of the Jacobian matrix of $(g_1, \ldots, g_{n-d})$ evaluated at the origin has full rank. Now let U be an euclidean environment of the origin in $\mathbb{K}^n$ such that U is contained in the region of convergence of all g_i. We set

$$X' := \{\, x \in U \mid g_i(x) = 0, \text{ for all } i \,\}.$$

According to the analytic implicit function theorem [22, Anhang 3] the origin is a manifold point of X' and we only have to show, that X' agrees with X on a neighborhood of the origin. This follows easily if we can prove

$$L \coloneqq \langle g_1, \dots, g_{n-d} \rangle = \mathbf{I}^a(G).$$

Since $g_1, \dots, g_{n-d} \in \mathbf{I}^a(G)$ we clearly have $L \subset \mathbf{I}^a(G)$. On the other hand, because $\mathfrak{m}\mathbb{K}\{\mathbf{x}\} \not\supset J_{n-d}(L)$ and $\operatorname{ht}(L) \leq n-d$ we can apply Nagata's Jacobian criterion again to see that $\mathbb{K}\{\mathbf{x}\}/L$ is a regular local ring of dimension d. Then $\mathbb{K}\{\mathbf{x}\}/L$ is also an integral domain [3, Lemma 11.23], so L is a prime ideal. Because $\mathbb{K}\{\mathbf{x}\}$ is local and Cohen-Macaulay (or use [67, Proposition 2.4]) we have $\operatorname{ht}(L) = \dim \mathbb{K}\{\mathbf{x}\} - \dim \mathbb{K}\{\mathbf{x}\}/L = n-d$. But since $\mathcal{O}(G)$ is regular, $\mathbf{I}^a(G)$ must be prime as well with $\operatorname{ht}\big(\mathbf{I}^a(G)\big) = n-d$. As $L \subset \mathbf{I}^a(G)$, we have $L = \mathbf{I}^a(G)$. This completes the proof. □

3.3 Comparative results

With the help of M. Artin's approximation theorem, Nagata's Jacobian theorem and flatness of the local rings, we can compare properties of $\mathcal{R}$, $\mathcal{O}$ and $\mathcal{F}$. The majority of the following results are due to Nagata and Zariski and most proofs can be found in [67] with only minor modifications necessary.

Proposition 3.5.

(a) $\mathcal{R}$ *reduced* $\Leftrightarrow$ $\mathcal{O}$ *reduced* $\Leftrightarrow$ $\mathcal{F}$ *reduced.*

(b) $\mathcal{R}$ *normal domain* $\Leftrightarrow$ $\mathcal{O}$ *normal domain* $\Leftrightarrow$ $\mathcal{F}$ *normal domain.*

(c) $\mathcal{R}$ *regular* $\Leftrightarrow$ $\mathcal{O}$ *regular* $\Leftrightarrow$ $\mathcal{F}$ *regular.*

Moreover for $\mathbb{K} = \mathbb{R}$ *we have*

(d) $\mathcal{O}$ *real* $\Leftrightarrow$ $\mathcal{F}$ *real.*

(e) $\mathcal{O}/\sqrt[r]{(0)}$ *is regular* $\Leftrightarrow$ $\mathcal{F}/\sqrt[r]{(0)}$ *is regular.*

(f) If $\mathcal{F}, \mathcal{O}$ *regular, then* $\mathcal{F}, \mathcal{O}$ *real.*

Remark. Clearly, the first equivalence in (b) does not stay true if we remove the normality condition. A basic example is the simple node $y^2 - x^2(x+1) = (y + x\sqrt{x+1})(y - x\sqrt{x+1})$. On the other hand, it is a result of Nagata [67, Corollary 5.4.4] that $\mathcal{O}$ is a domain if and only if the same is true for $\mathcal{F}$.

Proof of Proposition 3.5. The proofs for (a),(b),(c),(d) can be found in [67, Chapter V,VI]. In our definition a real ring does not need to be a domain, but the proof of Proposition V.4.9 in [67] can be used for (d) with only evident modifications.

To prove (e) we set $I' = I \cdot \mathbb{R}\{\mathbf{x}\}$ and $I'' = I \cdot \mathbb{R}[[\mathbf{x}]]$. It is enough to show that $\sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]] = \sqrt[r]{I''}$. Then, the statement follows from [67, Proposition V.4.5]. $\sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]] \subset \sqrt[r]{I''}$ is clear, so we proceed to show that $\sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]] \supset \sqrt[r]{I''}$ by using the argument in the proof of [67, Theorem V.4.2]. Let $f \in \sqrt[r]{I''}$, which means

$$f^{2s} + p_1^2 + \ldots + p_k^2 \in I'',$$

for elements $p_1, \ldots, p_k \in \mathbb{R}[[\mathbf{x}]]$. According to M. Artin's approximation theorem in the form of [67, Proposition V.4.1] we find elements $\hat{f}, \hat{p}_1, \ldots, \hat{p}_k \in \mathbb{R}\{\mathbf{x}\}$, for every integer $\alpha \geq 1$ such that

$$\hat{f}^{2s} + \hat{p}_1^2 + \ldots \hat{p}_k^2 \in I',$$

and $f = \hat{f} \bmod \mathfrak{m}^\alpha \, \mathbb{R}[[\mathbf{x}]]$ (recall that $\mathfrak{f}$ is the maximal ideal of $\mathcal{F}$). Then, since $\hat{f} \in \sqrt[r]{I'}$ we have for every $\alpha \geq 1$:

$$f \in \sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]] + \mathfrak{m}^\alpha \, \mathbb{R}[[\mathbf{x}]].$$

It follows

$$f \in \bigcap_\alpha \left(\sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]] + \mathfrak{m}^\alpha \, \mathbb{K}[[\mathbf{x}]] \right) = \sqrt[r]{I'} \cdot \mathbb{R}[[\mathbf{x}]]$$

since any ideal of $\mathbb{R}[[\mathbf{x}]]$ is closed in the $\mathfrak{m}\mathbb{R}[[\mathbf{x}]]$-adic topology.

Lastly, we show (f). Suppose $\mathcal{F}$ is a regular local ring. Because $\mathcal{F}/\mathfrak{m}\mathbb{R}[[\mathbf{x}]] \cong \mathcal{R}/\mathfrak{m}\mathcal{R} \cong \mathbb{R}$ is real, $\mathcal{F}$ is regular local ring with real residue field must be real according to [44, Proposition 2.7]. □

With Proposition 3.5 we see why there is no need in complex algebraic geometry to consider the completion of $\mathcal{R}$ to answer questions about the regularity of $\mathcal{O}(G)$. If $\mathbb{K} = \mathbb{C}$ we have $\mathcal{O}(G) = \mathcal{O}/\sqrt{(0)} = \mathcal{O}$ as long as $\mathcal{R}$ is reduced. Then, $\mathcal{O}(G)$ is regular if and only if $\mathcal{R}$ is regular.

For $\mathbb{K} = \mathbb{R}$ it is not enough for I to be real to imply the realness of $I \cdot \mathbb{R}\{\mathbf{x}\}$, see Example 2.3. Hence $\mathcal{O}(G) \ncong \mathcal{O}$ and the non-regularity of $\mathcal{R}$ does not imply the non-regularity of $\mathcal{O}(G)$. On the other hand, if $\mathcal{R}$ is regular then $\mathcal{O}$ is regular and real, hence $\mathcal{O}(G) = \mathcal{O}$ is regular.

From Proposition 3.3, Proposition 3.4 and Proposition 3.5 we get immediately the following corollaries:

Corollary 3.6. *Let $\mathbb{K} = \mathbb{R}$. Then the origin is a manifold point of $\mathbf{V}_\mathbb{R}(I)$ if and only if $\mathcal{F}/\sqrt[r]{(0)}$ is regular.*

Corollary 3.7. *Let $\mathbb{K} = \mathbb{R}$ and $\mathcal{O}$ or $\mathcal{F}$ be real. Then the origin is nonsingular if and only if it is a manifold point.*

In the next chapter we analyze thoroughly how to check that $\mathcal{F}$ is real for $\dim I = 1$. But there is also a useful criterion for many higher-dimensional algebraic sets.

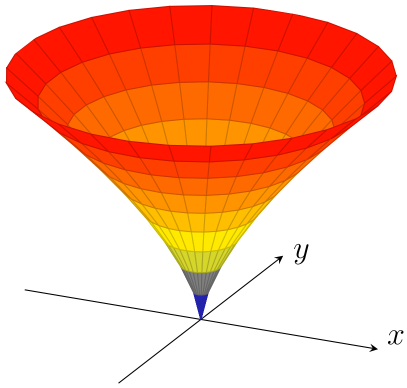


Figure 3.1: $\mathbf{V}_{\mathbb{R}}(x^2+y^2-z^3)$.

Theorem 3.8 (G. Efroymson [19])**.** *Let $\mathbb{K}=\mathbb{R}$, $I \leq \mathbb{R}[\mathbf{x}]$ a real prime with $I \subset \langle \mathbf{x} \rangle$ and $\mathcal{R}$ integrally closed. $\mathcal{F}$ is real if and only if the origin is contained in the euclidean closure of the nonsingular points of $\mathbf{V}_{\mathbb{R}}(I)$.*

Corollary 3.9. *Let X be a normal, irreducible real algebraic variety embedded in euclidean space. Any singular point $p \in X$ is either a non-manifold point of X or not a limit point of the nonsingular locus of X.*

Corollary 3.10. *Let X be a normal, irreducible real algebraic variety embedded in euclidean space. Any isolated singularity of X is either a non-manifold point of X or isolated in X.*

Example 3.1.

(i) Let $I = \langle x^2+y^2-z^3 \rangle \leq \mathbb{R}[x,y,z]$ and $A = \mathbb{R}[x,y,z]/I$. $\mathbf{V}_{\mathbb{R}}(I)$ is plotted in Figure 3.1. I is prime and the singular locus of $\mathbf{V}(I)$ is just the origin. Then I must be real according to the simple point criterion, Proposition 2.15. We also know that A is a Cohen-Macaulay ring, since $\dim A = 2 = 3-1$ and $\mathbf{V}(I)$ is a hypersurface [20, Proposition 18.13]. Then, A is normal with Serre's criterion [48, Theorem 39] or use [20, Theorem 18.15].

Since the origin is not isolated in $\mathbf{V}_{\mathbb{R}}(I)$, it must be a non-manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ according to Corollary 3.10.

(ii) Let $I = \langle x^2+y^2+z^2-x^3 \rangle \leq \mathbb{R}[x,y,z,w]$. The projection of $\mathbf{V}_{\mathbb{R}}(I)$ on (x,y,z) is plotted in Figure 3.2. I is a real prime and $\mathbf{V}(I)$ is a normal variety according to [20, Theorem 18.15]. However, any point on the line $x=y=z=0$ is a manifold point of $X = \mathbf{V}_{\mathbb{R}}(I)$.

It is clear in this example, that the origin is not a limit point of the nonsingular points in X. We can confirm this algebraically if we are able to calculate a desingularization $\pi\colon \tilde{Y} \to Y = \mathbf{V}(I)$. According to [19, Theorem 4.1] the origin

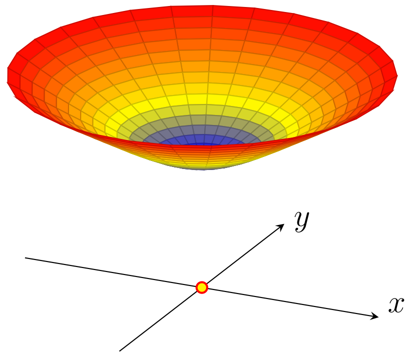


Figure 3.2: $\mathbf{V}_{\mathbb{R}}(x^2 + y^2 + z^2 - z^3)$

is a limit point of real nonsingular points if and only if there is a real point in the fiber of the origin under π.

Let us check this in our example. If we blow-up affine 4-space at the variety $\mathbf{V}(x, y, z)$, we get the corresponding ring homomorphism

$$\psi\colon \mathbb{R}[x, y, z, w] \to \mathbb{R}[x, y, z, w, \hat{x}, \hat{y}, \hat{z}]/\langle \hat{x}y - \hat{y}x, \hat{x}z - \hat{z}x, \hat{y}z - \hat{z}y\rangle.$$

Since Y is a hypersurface, we can easily determine the strict transform of Y on the chart $\hat{x} = 1$. The corresponding ring homomorphism is given by

$$\varphi_x\colon \mathbb{R}[x, y, z, w]/\langle x^2 + y^2 + z^2 - x^3\rangle \to \mathbb{R}[x, \hat{y}, \hat{z}, w]/\langle 1 - x + \hat{y}^2 + \hat{z}^2\rangle,$$

induced by $(x, y, z, w) \mapsto (x, \hat{y}x, \hat{z}x, w)$. Now $\mathbb{R}[x, \hat{y}, \hat{z}, w]/\langle 1 - x + \hat{y}^2 + \hat{z}^2\rangle$ is a regular ring with the Jacobian criterion and we check easily hence the strict transform of Y on the charts $\hat{x} = 1$ is nonsingular. In the same way we can check that the strict transform of Y on the charts $\hat{y} = 1$ and $\hat{z} = 1$ are nonsingular, too. Hence $\varphi_x, \varphi_y, \varphi_z$ correspond to a desingularization of Y.

We have $\varphi_x(\langle x, y, z, w\rangle) = \langle x, w, 1 + \hat{y}^2 + \hat{z}^2\rangle$ in $\mathbb{R}[x, \hat{y}, \hat{z}, w]/\langle 1 - x + \hat{y}^2 + \hat{z}^2\rangle$. Because $1 + \hat{y}^2 + \hat{z}^2$ has no real solutions, there are no real points on the chart $\hat{x} = 1$ in the fiber of the origin. Analogously, we check that there are no real points on the other charts in the fiber of the origin. Thus, according to [19, Theorem 4.1] the origin is not a limit point of nonsingular real points in $Y = \mathbf{V}(I)$.

3.4 Normalization and analytic branches

From now on we fix $\mathbb{K} = \mathbb{R}$. The construction in this section and Theorem 3.12 work for $\mathbb{K} = \mathbb{C}$ as well, but we will observe some special behavior in the real case. Let $I \leq \mathbb{R}[\mathbf{x}]$ again with $I \subset \mathfrak{m} = \langle \mathbf{x}\rangle$. We also assume that I is radical.

In order to decompose the extended ideal $I \cdot \mathbb{R}[[\mathbf{x}]]$ we examine the normalization of $\mathcal{F}$, which has a close connection to the normalization of $\mathcal{R}$. We need some preparation to formulate this result. Since I is radical, $\mathcal{R} = \big(\mathbb{R}[\mathbf{x}]/I\big)_{\mathfrak{m}}$ is reduced and we have the following minimal primary decomposition of the zero ideal in $\mathcal{R}$:

$$(0) = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s, \tag{3.3}$$

where $\mathfrak{p}_i \leq \mathcal{R}$ prime, for $i = 1, \ldots, s$. From now on we will use the notations

$$\mathcal{R}_i := \mathcal{R}/\mathfrak{p}_i, \qquad \mathcal{F}_i := \mathcal{F}/\mathfrak{p}_i\mathcal{F}.$$

Moreover, for any reduced ring A we will write $\overline{A}$ for the integral closure of A in its total ring of fractions. The following lemma collects some facts from Section 2.9 about the normalization of $\mathcal{R}$.

Lemma 3.11. $\overline{\mathcal{R}} = \overline{\mathcal{R}_1} \times \ldots \times \overline{\mathcal{R}_s}$ *is a product of semi-local normal domains. Additionally, we have*

$$\sqrt{\mathfrak{m}\overline{\mathcal{R}}} = (\mathfrak{n}_{1,1} \cap \ldots \cap \mathfrak{n}_{1,k_1}) \cap \ldots \cap (\mathfrak{n}_{s,1} \cap \ldots \cap \mathfrak{n}_{s,k_s})$$

where the $\mathfrak{n}_{i,j}$ are the maximal ideals of $\overline{\mathcal{R}}$ in the form

$$\mathfrak{n}_{i,j} = \overline{\mathcal{R}_1} \times \ldots \times \overline{\mathcal{R}_{i-1}} \times \mathfrak{n}'_{i,j} \times \overline{\mathcal{R}_{i+1}} \times \ldots \times \overline{\mathcal{R}_s},$$

and $\mathfrak{n}'_{i,j}$ is one of the k_i maximal ideals of $\overline{\mathcal{R}_i}$. We also have the following minimal primary decomposition $\sqrt{\mathfrak{m}\overline{\mathcal{R}_i}} = \mathfrak{n}'_{i,1} \cap \ldots \cap \mathfrak{n}'_{i,k_i}$ and

$$\overline{\mathcal{R}}_{\mathfrak{n}_{i,j}} \cong (\overline{\mathcal{R}_i})_{\mathfrak{n}'_{i,j}}.$$

Now we want to compare the normalization of $\mathcal{F}$ and the completion of $\overline{\mathcal{R}}$, so we need to find an expression for $\big(\overline{\mathcal{R}}_{\mathfrak{n}}\big)^*$, for $\mathfrak{n} \leq \overline{\mathcal{R}}$ maximal. Since $\overline{\mathcal{R}}_{\mathfrak{n}} = (\overline{\mathcal{R}_i})_{\mathfrak{n}'}$ for some i and $\mathfrak{n}' \leq \mathcal{R}_i$ maximal, we can assume that $\mathcal{R}$ is a domain. The following exposition and Theorem 3.12 are taken from [67, Section VI.4]. We will verify, that the construction of the completion of $\overline{\mathcal{R}}_{\mathfrak{n}}$ in [67] coincides with the completion of $\overline{\mathcal{R}}_{\mathfrak{n}}$ as topological local ring.

Because the monomorphism

$$\mathbb{R} = \mathcal{R}/\mathfrak{m}\mathcal{R} \subset \overline{\mathcal{R}}/\mathfrak{n}$$

is an algebraic field extension it needs to be $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}, \mathbb{R}$. We distinguish between the following three cases:

(a) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{R}$. Since $\overline{\mathcal{R}}$ is finitely generated over $\mathcal{R}$ [67, Proposition III.2.3], we can extend a surjection $\mathbb{R}[\mathbf{x}]_{\mathfrak{m}} \to \mathcal{R}$ to a surjection $\mathbb{R}[\mathbf{x}, \mathbf{y}]_{\langle \mathbf{x}, \mathbf{y} \rangle} \to \overline{\mathcal{R}}_{\mathfrak{n}}$, for new variables $\mathbf{y} = (y_1, \ldots, y_m)$. Hence $\overline{\mathcal{R}}_{\mathfrak{n}} \cong \mathbb{R}[\mathbf{x}, \mathbf{y}]_{\langle \mathbf{x}, \mathbf{y} \rangle}/J$ and the completion $\mathbb{R}[[\mathbf{x}, \mathbf{y}]]/\big(J \cdot \mathbb{R}[[\mathbf{x}, \mathbf{y}]]\big)$ from [67] coincides with $\big(\overline{\mathcal{R}}_{\mathfrak{n}}\big)^*$.

(b) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}$ and $\sqrt{-1} \in Q(\mathcal{R})$. Since $\overline{\mathcal{R}}$ is integrally closed, $\mathbb{C} \subset \overline{\mathcal{R}}$. Then we get a surjection $\mathbb{C}[\mathbf{x},\mathbf{y}]_{\langle \mathbf{x},\mathbf{y}\rangle} \to \overline{\mathcal{R}}_{\mathfrak{n}}$. Hence $\overline{\mathcal{R}}_{\mathfrak{n}} \cong \mathbb{C}[\mathbf{x},\mathbf{y}]_{\langle \mathbf{x},\mathbf{y}\rangle}/J$ and the formal completion $\mathbb{C}[[\mathbf{x},\mathbf{y}]]/\big(J\cdot\mathbb{C}[[\mathbf{x},\mathbf{y}]]\big)$ from [67] coincides with the completion $(\overline{\mathcal{R}}_{\mathfrak{n}})^*$ of $\overline{\mathcal{R}}_{\mathfrak{n}}$ as topological local ring.

(c) $\overline{\mathcal{R}}/\mathfrak{n} = \mathbb{C}$ and $\sqrt{-1} \notin Q(\mathcal{R})$. Now we need to adjoin $\sqrt{-1}$ to $\overline{\mathcal{R}}$ and we get $\mathfrak{n}\overline{\mathcal{R}}[\sqrt{-1}] = \mathfrak{n}_1 \cap \mathfrak{n}_2\ \mathfrak{n}'$, for two maximal ideals $\mathfrak{n}_i$ in $\overline{\mathcal{R}}[\sqrt{-1}]$. Now as in (b) $\mathbb{C}[\mathbf{x},\mathbf{y}]_{\langle \mathbf{x},\mathbf{y}\rangle}/J_i \cong \overline{\mathcal{R}}[\sqrt{-1}]_{\mathfrak{n}_i}$, $i = 1, 2$. With Proposition 2.7 one checks, that the completion

$$\mathbb{C}[[\mathbf{x},\mathbf{y}]]/\big(J_1\cdot\mathbb{C}[[\mathbf{x},\mathbf{y}]]\big) \cong \mathbb{C}[[\mathbf{x},\mathbf{y}]]/\big(J_2\cdot\mathbb{C}[[\mathbf{x},\mathbf{y}]]\big) \cong \overline{\mathcal{R}}[\sqrt{-1}]^*_{\mathfrak{n}_1} \cong \overline{\mathcal{R}}[\sqrt{-1}]^*_{\mathfrak{n}_2}$$

from [67] coincides with the completion $(\overline{\mathcal{R}}_{\mathfrak{n}})^*$ of $\overline{\mathcal{R}}_{\mathfrak{n}}$ as topological local ring.

Now for $i = 1, \dots, s$ we set $\mathcal{F}_i := \mathcal{F}/(\mathfrak{p}_i \cdot \mathcal{F}) = \mathbb{R}[[\mathbf{x}]]/(\mathfrak{p}' \cdot \mathbb{R}[[\mathbf{x}]])$.

Theorem 3.12 (Ruiz, Zariski [67, Proposition VI.4.4]). *With the notations above and from Lemma 3.11, we have for any $i = 1, \dots, s$:*

$$\overline{\mathcal{F}_i} \cong [(\overline{\mathcal{R}}_i)_{\mathfrak{n}_{i,1}}]^* \times \ldots \times [(\overline{\mathcal{R}}_i)_{\mathfrak{n}_{i,k_i}}]^*.$$

Moreover, there is a bijective correspondence $\mathfrak{n}_{i,j} \mapsto \mathfrak{q}_{i,j}$ between the maximal ideals of $\overline{\mathcal{R}_i}$ and the minimal primes $\mathfrak{q}_{i,j}$ of $\mathcal{F}_i$ such, that $[(\overline{\mathcal{R}}_i)_{\mathfrak{n}_{i,j}}]^* \cong \overline{\mathcal{F}_i/\mathfrak{q}_{i,j}}$ and those isomorphisms naturally extend $\mathcal{R}_i \to \mathcal{F}_i$. Additionally

$$\overline{\mathcal{F}} \cong \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

Remark. The importance of Theorem 3.12 for us lies in the fact that $\mathcal{F}$ is real if and only if $\overline{\mathcal{F}}$ is real, so we can check realness on completions of local rings of normal varieties. This is very useful if we want to apply Efroymson's criterion (Theorem 3.8) for example.

Example 3.2.

(i) Let $A = \mathbb{R}[x,y]/\langle y^2 - x^2 - x^3\rangle$ be the coordinate ring of the simple node, see Figure 3.3. Then $s = \frac{y}{x} \in Q(A)$ is integral over A and $B = A[s] \cong \mathbb{R}[x,y,s]/\langle s^2 - x - 1, sx - y\rangle$ is normal, since any localization at a maximal ideal is regular. Thus, $\overline{A} = B$ and according to Proposition 2.23

$$\overline{\mathcal{R}} = \overline{A_{\langle x,y\rangle}} \cong S^{-1}\big(\mathbb{R}[x,y,s]/\langle s^2 - x - 1, sx - y\rangle\big),$$

where $S = \mathbb{R}[x,y]\backslash\langle x,y\rangle$.

We see that there are two maximal ideals $\mathfrak{c}_1 = \langle s-1, x, y\rangle$ and $\mathfrak{c}_2 = \langle s+1, x, y\rangle$ lying over $\langle x,y\rangle$ in B. For $i = 1, 2$ we also know that $(B_{\mathfrak{c}_i})^*$ is regular

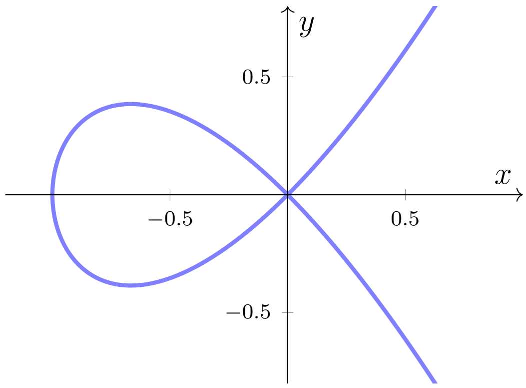


Figure 3.3: $\mathbf{V}_{\mathbb{R}}(y^2 - x^2 - x^3)$.

[3, Proposition 11.24] with residue field $B_{\mathfrak{c}_i}/\mathfrak{c}_i B_{\mathfrak{c}_i} \cong \mathbb{R}$ and $\dim (B_{\mathfrak{c}_i})^* = 1$. Then $(B_{\mathfrak{c}_i})^* \cong \mathbb{R}[[t]]$ according to Proposition 2.9 (or use the Cohen structure theorem). Consequently

$$\left(\overline{\mathcal{R}}_{S^{-1}\mathfrak{c}_1}\right)^* \times \left(\overline{\mathcal{R}}_{S^{-1}\mathfrak{n}_2}\right)^* \cong (B_{\mathfrak{c}_1})^* \times (B_{\mathfrak{c}_2})^* \cong \mathbb{R}[[t]] \times \mathbb{R}[[t]]. \tag{3.4}$$

On the other hand

$$\mathcal{F} = \mathbb{R}[[x,y]]/\langle (y - x\sqrt{1+x}) \cdot (y + x\sqrt{1+x})\rangle.$$

$\mathbb{R}[[x,y]]/\langle y \pm x\sqrt{1+x}\rangle$ is regular according to Nagata's Jacobian theorem hence normal. Thus, this leads to

$$\begin{aligned}\overline{\mathcal{F}} &\cong \overline{\mathbb{R}[[x,y]]/\langle y - x\sqrt{1+x}\rangle} \times \overline{\mathbb{R}[[x,y]]/\langle y + x\sqrt{1+x}\rangle} \\ &\cong \mathbb{R}[[x,y]]/\langle \ldots \rangle \times \mathbb{R}[[x,y]]/\langle \ldots \rangle \\ &\cong \mathbb{R}[[t]] \times \mathbb{R}[[t]] \quad \text{(by Proposition 2.9)},\end{aligned}$$

which coincides with (3.4) just as predicted by Theorem 3.12.

(ii) Let $\mathcal{R} = \left(\mathbb{R}[x,y]/\langle y^2 + x^2\rangle\right)_{\langle x,y\rangle}$ and $S = \mathbb{R}[x,y] \setminus \langle x,y\rangle$. Then, we have

$$\overline{\mathcal{R}} \cong S^{-1}\left(\mathbb{R}[x,y,s]/\langle s^2+1, y - xs\rangle\right).$$

Thus, $\overline{\mathcal{R}}$ is a local ring with maximal ideal $\mathfrak{n} = S^{-1}\langle s^2+1, x, y\rangle$. Clearly, $\overline{\mathcal{R}}/\mathfrak{n} \cong \mathbb{C}$ but also $s = \sqrt{-1} \in \overline{\mathcal{R}}$. This is case (b) in the list above Theorem 3.12. We see $\overline{\mathcal{R}}_{\mathfrak{n}} \cong (\mathbb{C}[x,y]/\langle y \pm ix\rangle)_{\langle x,y\rangle}$ and it follows $\left(\overline{\mathcal{R}}_{\mathfrak{n}}\right)^* \cong \mathbb{C}[[t]]$.

Secondly, it is $\mathcal{F} = \mathbb{R}[[x,y]]/\langle x^2+y^2\rangle$. $\langle x^2+y^2\rangle$ is prime in $\mathbb{R}[[\mathbf{x}]]$ since the initial polynomial of x^2+y^2 is prime in $\mathbb{R}[x,y]$. According to [67, Proposition III.3.2] $\overline{\mathcal{F}}$ must be $\mathbb{R}[[t]]$ or $\mathbb{C}[[t]]$ in this case. But $\overline{\mathcal{F}}$ is not real since $\mathcal{F}$ is not real. Therefore, $\overline{\mathcal{F}} = \mathbb{C}[[t]]$.

(iii) Let $\mathcal{R} = \left(\mathbb{R}[x,y]/\langle y^3 + 2\,y\,x^2 - x^4\rangle\right)_{\langle x,y\rangle}$ be the local ring of the curve from Figure 1.4 of the introduction. As in (i) we deduce

$$\overline{\mathcal{R}} \cong S^{-1}\left(\mathbb{R}[x,y,s]/\langle s^3 + 2s - x, y - xs\rangle\right),$$

where $S = \mathbb{R}[x,y]\backslash\langle x,y\rangle$. We get two maximal ideals $\mathfrak{n}_1 = S^{-1}\langle s,x,y\rangle$ and $\mathfrak{n}_2 = S^{-1}\langle s^2+2,x,y\rangle$ in $\overline{\mathcal{R}}$. Arguing as in (i) it is

$$(\overline{\mathcal{R}}_{\mathfrak{n}_1})^* \cong \mathbb{R}[[t]].$$

For $\mathfrak{n}_2$ we decompose $\langle s^2+2,x,y\rangle = \langle s-i\sqrt{2},x,y\rangle \cap \langle s+i\sqrt{2},x,y\rangle$ in $\mathbb{C}[x,y,s]$. With Proposition 2.7 and Proposition 2.9 this implies

$$\begin{aligned}(\overline{\mathcal{R}}_{\mathfrak{n}_2})^* &= \left[\left(\mathbb{R}[x,y,s]/\langle s^3+2s-x, y-xs\rangle\right)_{\langle s^2+2,x,y\rangle}\right]^* \\ &\cong \left[\left(\mathbb{C}[x,y,s]/\langle s^3+2s-x, y-xs\rangle\right)_{\langle s-i\sqrt{2},x,y\rangle}\right]^* \\ &\cong \mathbb{C}[[t]].\end{aligned}$$

Note that we have $\sqrt{-1} \notin \overline{\mathcal{R}}$ for this example. This is easy to see: For $A = \mathbb{R}[x,y]/\langle y^3 + 2\,y\,x^2 - x^4\rangle$ we have that

$$\overline{A}[\sqrt{-1}] \cong \mathbb{C}[x,y,s]/\langle s^3+2s-x, y-xs\rangle$$

is a domain, hence $\sqrt{-1} \notin Q\left(\overline{A}\right) = Q(A) = Q(\mathcal{R}) = Q(\overline{\mathcal{R}})$. Accordingly, for $\mathfrak{n}_2$ we were in case (c) of the construction before Theorem 3.12.

Now we consider

$$\mathcal{F} = \mathbb{R}[[x,y]]/\langle -(x^2 - y(1+\sqrt{1+y}))\cdot(x^2 - y(1-\sqrt{1+y}))\rangle.$$

Then, we get

$$\overline{\mathcal{F}} = \overline{\mathbb{R}[[x,y]]/\langle x^2 - y(1+\sqrt{1+y})\rangle} \times \overline{\mathbb{R}[x,y]/\langle x^2 - y(1-\sqrt{1+y})\rangle}.$$

$R_1 := \mathbb{R}[[x,y]]/\langle x^2 - y(1+\sqrt{1+y})\rangle$ is regular according to Nagata's Jacobian theorem, since the gradient of $x^2 - y\left(1+\sqrt{1+y}\right)$ doesn't vanish at the origin. Thus $R_1 \cong \mathbb{R}[[t]]$ is normal.

For $R_2 := \mathbb{R}[[x,y]]/\left\langle x^2 - y\left(1-\sqrt{1+y}\right)\right\rangle$ we note that $\left\langle x^2 - y\left(1-\sqrt{1+y}\right)\right\rangle$ is a prime ideal in $\mathbb{R}[[x,y]]$ since the initial polynomial of

$$x^2 - y\left(1-\sqrt{1+y}\right) = x^2 - y\left(1 - \left(1 + \frac{y}{2} - \frac{y^2}{8} + \ldots\right)\right) = x^2 + \frac{y^2}{2} + \ldots$$

is irreducible in $\mathbb{R}[x,y]$. Therefore, $\overline{R_2}$ must be $\mathbb{C}[[t]]$ or $\mathbb{R}[[t]]$ according to [67, Proposition III.3.2]. But we will see that R_2 is not real. Therefore, it must be $\overline{R_2} \cong \mathbb{C}[[t]]$ and

$$\overline{\mathcal{F}} \cong \overline{R_1} \times \overline{R_2} = \mathbb{R}[[t]] \times \mathbb{C}[[t]],$$

as predicted by Theorem 3.12.

It remains to show that R_2 is not real, i.e. $J = \left\langle x^2 - y\left(1-\sqrt{1+y}\right)\right\rangle$ is not a real ideal of $\mathbb{R}[[x,y]]$. For this it is enough to confirm that $J \cap \mathbb{R}\{x,y\}$ is not real. But $\mathbf{V}^a(J \cap \mathbb{R}\{x,y\})$ is the set germ of the isolated origin, hence

$$\mathbf{I}^a(\mathbf{V}^a(J \cap \mathbb{R}\{x,y\})) = \langle x,y\rangle \neq J \cap \mathbb{R}\{x,y\}.$$

Thus, $J \cap \mathbb{R}\{x,y\}$ is not real according to the real analytic Nullstellensatz.

(iv) Let $A = \mathbb{R}[x,y]/\langle y^2 - x^3\rangle$ be the cuspidal cubic from Figure 3.4. Then

$$\overline{\mathcal{R}} \cong S^{-1}\left(\mathbb{R}[x,y,s]/\langle s^2 - x, y - xs\rangle\right).$$

We can check that $\overline{\mathcal{R}}$ is a local ring with unique maximal ideal $\mathfrak{n} = S^{-1}\langle s,x,y\rangle$ and $(\overline{\mathcal{R}}_{\mathfrak{n}})^*$ is regular of dimension one. Thus, with Proposition 2.9

$$(\overline{\mathcal{R}}_{\mathfrak{n}})^* \cong \mathbb{R}[[t]].$$

On the other hand, we have $\mathcal{F} = \mathbb{R}[[x,y]]/\langle y^2 - x^3\rangle$. The ideal $\langle y^2 - x^3\rangle$ is prime in $\mathbb{R}[[x,y]]$ since $y^2 - x^3$ is no product of power series in $\langle x,y\rangle\,\mathbb{R}[[x,y]]$. According to [67, Proposition III.3.2] $\overline{\mathcal{F}}$ must be $\mathbb{R}[[t]]$ or $\mathbb{C}[[t]]$, but we claim that $\mathcal{F}$ is real, hence $\overline{\mathcal{F}}$ is $\mathbb{R}[[t]]$ and coincides with $(\overline{\mathcal{R}}_{\mathfrak{n}})^*$.

We will prove the claim now. Suppose $\mathcal{F}$ is not real, then $\langle x^2 - y^3\rangle \neq \sqrt[r]{\langle x^2 - y^3\rangle}$ in $\mathbb{R}[[x,y]]$ and because $\sqrt[r]{\langle x^2 - y^3\rangle}$ is the intersection of all real primes in $\mathbb{R}[[x,y]]$ containing $\langle x^2 - y^3\rangle$ we have

$$\sqrt[r]{\langle x^2 - y^3\rangle} = \langle x,y\rangle$$

in $\mathbb{R}[[x,y]]$. Then, for the set germ $G = \left(\mathbf{V}_{\mathbb{R}}(y^2 - x^3), 0\right)$ we conclude with the analytic Nullstellensatz

$$\mathcal{O}(G) = \mathbb{R}[[x,y]]/\mathbf{I}^a(G) = \mathbb{R}[[x,y]]/\sqrt[r]{\langle x^2 - y^3\rangle} = \mathbb{R}[[x,y]]/\langle x,y\rangle \cong \mathbb{R}.$$

But in this case G is the isolated origin with Proposition 2.26. This is a contradiction since $x^2 - y^3$ has zeros arbitrarily close to the origin.

Proof of Theorem 3.12. The only thing missing from the proof in [67] is to take into account non-domains $\mathcal{R}$, so we need to check

$$\overline{\mathcal{F}} = \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

According to Chevalley's theorem [67, Proposition VI.2.1] we have a minimal primary decomposition

$$\mathfrak{p}_i \cdot \mathcal{F} = \mathfrak{q}'_{i,1} \cap \ldots \cap \mathfrak{q}'_{i,k_i},$$

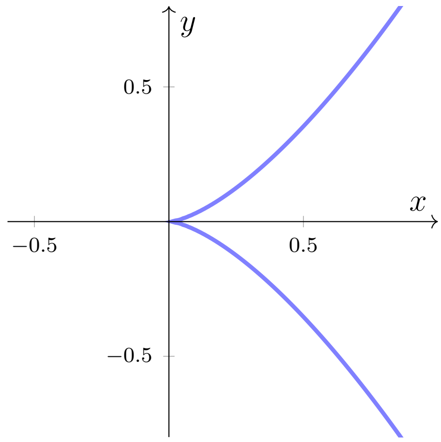


Figure 3.4: $\mathbf{V}_{\mathbb{R}}(y^2 - x^3)$.

with $\mathfrak{q}'_{i,j}$ prime of height $\operatorname{ht} \mathfrak{p}_i =: d_i$ and $\mathfrak{q}_{i,j} = \mathfrak{q}'_{i,j} \cdot \mathcal{F}_i$. It remains to show that

$$(0) = (\mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_s) \cdot \mathcal{F} = \mathfrak{q}'_{1,1} \cap \ldots \cap \mathfrak{q}'_{1,k_1} \cap \ldots \cap \mathfrak{q}'_{s,1} \cap \ldots \cap \mathfrak{q}'_{s,k_s}$$

is a minimal primary decomposition of (0) in $\mathcal{F}$, because then

$$\overline{\mathcal{F}} = \bigtimes_{i,j} \overline{\mathcal{F}/\mathfrak{q}'_{i,j}} = \overline{\mathcal{F}_1} \times \ldots \times \overline{\mathcal{F}_s}.$$

First, suppose without loss of generality $\mathfrak{q}'_{1,1} \supset \bigcap_{i,j \neq 1,1} \mathfrak{q}'_{i,j}$. Then, because $\mathfrak{q}'_{1,1}$ is prime there exists $\mathfrak{q}'_{i,j} \subset \mathfrak{q}'_{1,1}$, where clearly $i \neq 1$. If we can show that $\mathfrak{q}'_{i,j} \cap \mathcal{R} = \mathfrak{p}_i$, we are done since (3.3) is a minimal decomposition.

Assume $\mathfrak{p}_i \subsetneq \mathfrak{a} := \mathfrak{q}'_{i,j} \cap \mathcal{R}$. Since $\mathfrak{a}$ is prime, it follows $\operatorname{ht} \mathfrak{a} > d_i = \operatorname{ht} \mathfrak{p}_i$. Consequently, according to Chevalley's theorem every associated prime of $\mathfrak{a} \cdot \mathcal{F}$ is of height greater than d_i. Because $\mathfrak{a}\mathcal{F} \subset \mathfrak{q}'_{i,j}$ and $\operatorname{ht} \mathfrak{q}'_{i,j} = d_i$, this is a contradiction. □

We put on record a corollary of Theorem 3.12 which will be needed in the next chapter when we analyze algebraic curves.

From now on we will ignore the partition $\overline{\mathcal{R}} = \overline{\mathcal{R}_1} \times \cdots \times \overline{\mathcal{R}_s}$ when $\mathcal{R}$ is not a domain and just write $\mathfrak{n}_1, \ldots, \mathfrak{n}_k$ for the maximal ideals of $\overline{\mathcal{R}}$ with corresponding minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_k$ of $\mathcal{F}$.

Corollary 3.13. *Let $I \leq \mathbb{R}[\mathbf{x}]$ be radical with $\dim I = 1$ and $I \leq \mathfrak{m} = \langle \mathbf{x} \rangle$. For any maximal ideal $\mathfrak{n}_i$ of $\overline{\mathcal{R}}$ with corresponding minimal prime $\mathfrak{q}_i$ of $\mathcal{F}$ we have*

$$\mathcal{F}/\mathfrak{q}_i \text{ real} \Leftrightarrow \mathfrak{n}_i \text{ real.} \tag{3.5}$$

Proof. Clearly $\overline{\mathcal{F}/\mathfrak{q}_i}$ is real if and only if $\mathcal{F}/\mathfrak{q}_i$ is real, since they are both contained in the quotient field of $\mathcal{F}/\mathfrak{q}_i$. So we need to show that $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ is real if and only if $\mathfrak{n}_i$ is real.

First, let $\mathfrak{n}_i$ not be real. Then, $\overline{\mathcal{R}}/\mathfrak{n}_i \cong \mathbb{C}$ and one can see from the cases (b) and (c) before Theorem 3.12 that $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ will not be real (since $\mathbb{C} \subset (\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$).

For the other direction let $\mathfrak{n}_i$ be real. Then, $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ will be the completion of the local ring $\overline{\mathcal{R}}_{\mathfrak{n}_i}$. Since $\overline{\mathcal{R}}$ is normal of dimension one, $\overline{\mathcal{R}}_{\mathfrak{n}_i}$ must be regular according to Serre's regularity criterion R_1 [48, Theorem 39]. Thus, $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ is regular as well [3, Proposition 11.24] with residue field $\overline{\mathcal{R}}/\mathfrak{n}_i = \mathbb{R}$. Therefore, $(\overline{\mathcal{R}}_{\mathfrak{n}_i})^*$ has to be real because of [44, Proposition 2.7]. This proves (3.5). □

Chapter 4

Real algebraic curves

Now we apply the theory of the last chapter to singularities of real algebraic curves.

Let $I \leq \mathbb{R}[\mathbf{x}]$ radical with $\dim I = 1$ and $I \subset \mathfrak{m} = \langle \mathbf{x} \rangle$. Then, the structure of $\mathcal{F}/\sqrt[r]{(0)}$ will be especially nice, since the real radical of an associated prime $\mathfrak{q}$ of $I \cdot \mathbb{R}[[\mathbf{x}]]$ will be either $\mathfrak{q}$ itself or the maximal ideal of $\mathbb{R}[[\mathbf{x}]]$. With this fact and Corollary 3.13 we will prove a criterion which decides whether the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ and can easily be tested by way of computational algebra.

4.1 Curve criterion

Lemma 4.1. *Let $\mathfrak{q} \leq \mathbb{R}[[\mathbf{x}]]$ be prime with $\operatorname{ht} \mathfrak{q} = n - 1$. Then*

$$\sqrt[r]{\mathfrak{q}} = \begin{cases} \mathfrak{q}, & \mathfrak{q} \text{ real}, \\ \mathfrak{m}\,\mathbb{R}[[\mathbf{x}]], & \mathfrak{q} \text{ not real}. \end{cases}$$

Proof. Any ideal is real if and only if it is radical and all associated primes are real [19]. So $\sqrt[r]{\mathfrak{q}}$ is the intersection of all real primes containing $\mathfrak{q}$. We only have to show that $\mathfrak{m}\,\mathbb{R}[[\mathbf{x}]]$ is real. But this is clear since $\mathbb{R}[[\mathbf{x}]]/\big(\mathfrak{m}\,\mathbb{R}[[\mathbf{x}]]\big) \cong \mathbb{R}$ is real. □

For the curve criterion we will need to assess whether certain isolated primary components of $\mathfrak{m}\,\overline{\mathcal{R}}$ are prime, so we require some concepts:

Lemma 4.2. *Let A be a noetherian ring with ideal $J \leq A$. If $\mathfrak{p}$ is an associated prime of J, we have $J \cdot A_{\mathfrak{p}} = \mathfrak{p} \cdot A_{\mathfrak{p}}$ if and only if $\mathfrak{p}$ appears in one and then every minimal primary decomposition of J.*

Definition 4.1. If $J \cdot A_{\mathfrak{p}} = \mathfrak{p} \cdot A_{\mathfrak{p}}$ as in Lemma 4.2, we say J is **$\mathfrak{p}$-reduced**.

Proof of Lemma 4.2. First, note that any prime appearing in a minimal primary decomposition is isolated, so it is independent of the decomposition. Now let $J \cdot A_{\mathfrak{p}} = \mathfrak{p} \cdot A_{\mathfrak{p}}$ and choose a minimal primary decomposition $J = \bigcap_{i=1}^{s} \mathfrak{s}_i$. Since $\mathfrak{p}$ is an associated prime of J we have w.l.o.g. $\sqrt{\mathfrak{s}_1} = \mathfrak{p}$ and according to [3, Proposition 4.9]

$$\mathfrak{p} \cdot A_{\mathfrak{p}} = J \cdot A_{\mathfrak{p}} = \bigcap_{\sqrt{s_i} \subset \mathfrak{p}} \mathfrak{s}_i \cdot A_{\mathfrak{p}} \subset \mathfrak{s}_1 A_{\mathfrak{p}}.$$

It follows $\mathfrak{p} \cdot A_{\mathfrak{p}} = \mathfrak{s}_1 \cdot A_{\mathfrak{p}}$. But $\mathfrak{s}_1$ is a $\mathfrak{p}$-primary ideal. So $\mathfrak{p} = \mathfrak{s}_1$ according to [3, Proposition 4.9].

Conversely, if $\mathfrak{p}$ appears in a minimal primary decomposition of J, it is an isolated prime and $\mathfrak{p} \cdot A_{\mathfrak{p}} = J \cdot A_{\mathfrak{p}}$ follows with [3, Proposition 4.9] again. □

Definition 4.2. Let $a \in \mathbf{V}(J)$, for a zero-dimensional ideal $J \leq \mathbb{C}[\mathbf{x}]$. The **multiplicity** of a in $\mathbf{V}(J)$ is defined as

$$\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]/J)_{\mathfrak{n}},$$

where $\mathfrak{n} = \langle x_1 - a_1, \ldots, x_n - a_n \rangle$.

Let $J \leq \mathbb{C}[\mathbf{x}]$ zero-dimensional and $\mathfrak{n} \leq \mathbb{C}[\mathbf{x}]$ maximal with $J \subset \mathfrak{n}$. Clearly, J is $\mathfrak{n}$-reduced if and only if $\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]/J)_{\mathfrak{n}} = 1$ or $(\mathbb{C}[\mathbf{x}]/J)_{\mathfrak{n}}$ is regular. The following lemma generalizes this fact.

Lemma 4.3. *Let $\mathbb{K}$ be any field with $\mathbb{Q} \subset \mathbb{K} \subset \mathbb{C}$. And $J \leq \mathbb{K}[\mathbf{x}]$ be a zero-dimensional ideal with minimal primary decomposition. In addition, let*

$$J = \mathfrak{a}_1 \cap \ldots \cap \mathfrak{a}_k$$

and associated maximal ideals $\mathfrak{n}_i = \sqrt{\mathfrak{a}_i}$. Then

(i) The decomposition is unique.

(ii) $\mathfrak{a}_i = \mathfrak{n}_i$, i.e. J is $\mathfrak{n}_i$-reduced, if and only if for all $a \in \mathbf{V}(\mathfrak{a}_i)$ the multiplicity of a in $\mathbf{V}(J_{\mathbb{C}})$ is one.

(iii) If $\mathfrak{a}_i \neq \mathfrak{n}_i$, then the multiplicity of all $a \in \mathbf{V}(\mathfrak{a}_i)$ in $\mathbf{V}(J_{\mathbb{C}})$ is greater one.

Proof.

(i) Any associated prime of J is maximal, hence an isolated prime of J. But then the decomposition is unique according to [3, Theorem 4.10].

(ii) Suppose that J is $\mathfrak{n}_1$-reduced. We have

$$J = \mathfrak{n}_1 \cap \mathfrak{a}_2 \ldots \cap \mathfrak{a}_k.$$

Now let $(\mathfrak{n}_1)_{\mathbb{C}} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_r$ be the unique minimal primary decomposition in $\mathbb{C}[\mathbf{x}]$ with $\mathfrak{c}_i$ maximal for all i. According to Lemma 2.13 we find a minimal primary decomposition

$$J_{\mathbb{C}} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_r \cap \ldots$$

of $J_{\mathbb{C}}$. Hence, on account of Lemma 4.2 $J_{\mathbb{C}}\, \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i} = \mathfrak{c}_i\, \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}$, for $i = 1, \ldots, r$. Therefore

$$(\mathbb{C}[\mathbf{x}]/J_{\mathbb{C}})_{\mathfrak{c}_i} = \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}/\big(J_{\mathbb{C}}\, \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}\big) = \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}/\big(\mathfrak{c}_i\, \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}\big) \cong \mathbb{C},$$

and $\dim_{\mathbb{C}}(\mathbb{C}[\mathbf{x}]/J_{\mathbb{C}})_{\mathfrak{c}_i} = 1$, for $i = 1, \ldots, r$.

Suppose to the contrary that $\dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_{\mathfrak{c}}/\big(J\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}\big) = 1$, for $\mathfrak{c} \leq \mathbb{C}[\mathbf{x}]$ maximal, with $(\mathfrak{a}_1)_{\mathbb{C}} \subset \mathfrak{c}$. Then, clearly $\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}/\big(J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}\big)$ is a field and $\mathfrak{c}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}$ is the zero ideal in $\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}/\big(J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}\big)$. Hence, $J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}} = \mathfrak{c}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}$ and since $J_{\mathbb{C}} \subset (\mathfrak{a}_1)_{\mathbb{C}}$ we also have $(\mathfrak{a}_1)_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}} = \mathfrak{n}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}}$. According to Lemma 4.2 this means

$$(\mathfrak{a}_1)_{\mathbb{C}} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_r$$

for $\mathfrak{c}_i \leq \mathbb{C}[\mathbf{x}]$ maximal. But then $(\mathfrak{a}_1)_{\mathbb{C}}$ is radical and

$$\mathfrak{a}_1 = (\mathfrak{a}_1)_{\mathbb{C}} \cap \mathbb{K}[\mathbf{x}] = \sqrt{(\mathfrak{a}_1)_{\mathbb{C}} \cap \mathbb{K}[\mathbf{x}]} = \sqrt{\mathfrak{a}_1} = \mathfrak{n}_1.$$

Thus, J is $\mathfrak{n}_1$-reduced.

(iii) We show the statement by contraposition. Let $\sqrt{(\mathfrak{a}_1)_{\mathbb{C}}} = \mathfrak{c}_1 \cap \ldots \cap \mathfrak{c}_r$ and suppose $\dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_1}/\big(J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}_1}\big) = 1$. Then, clearly $\mathbb{C}[\mathbf{x}]_{\mathfrak{c}_1}/\big(J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}_1}\big) \cong \mathbb{C}$ is a regular local ring. We can apply Proposition 2.12 (iv) now and have that $(\mathbb{C}[\mathbf{x}]/J_{\mathbb{C}})_{\mathfrak{c}_i}$ is regular for $i = 1, \ldots, r$. From this we deduce that $(\mathbb{C}[\mathbf{x}]/J_{\mathbb{C}})_{\mathfrak{c}_i} \cong \mathbb{C}$, see e.g. [4, Proposition 4.94]. Thus, $\dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}/\big(J_{\mathbb{C}}\,\mathbb{C}[\mathbf{x}]_{\mathfrak{c}_i}\big) = 1$, for all i. But then $\mathfrak{a}_1 = \mathfrak{n}_1$ with (ii). □

We can now formulate the main result of this section. Recall the notation $\mathcal{R} = (\mathbb{R}[\mathbf{x}]/I)_{\langle \mathbf{x} \rangle}$ with normalization $\overline{\mathcal{R}}$.

Theorem 4.4. *Let $I \leq \mathbb{R}[\mathbf{x}]$ be radical with $\dim I = 1$ and $I \subset \mathfrak{m} := \langle \mathbf{x} \rangle$. The origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ if and only if one of the following two conditions is true*

(a) There is exactly one real maximal ideal $\mathfrak{n} \leq \overline{\mathcal{R}}$ and $\mathfrak{m}\overline{\mathcal{R}}$ is $\mathfrak{n}$-reduced.

(b) None of the maximal ideals $\mathfrak{n} \leq \overline{\mathcal{R}}$ is real. This is the case if and only if the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(I)$.

Remark. The normalization $f \colon Y \to \mathbf{V}(I)$ of an algebraic curve $\mathbf{V}(I)$ is a desingularization. Hence, there is an illustrative description of Theorem 4.4: The origin is a non-isolated manifold point of a real algebraic curve $\mathbf{V}_{\mathbb{R}}(I)$ if and only if there is exactly one real root in the fiber $f^{-1}(0)$ under the desingularization and this is a simple root.

Proof. First, let $(0) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ be a minimal primary decomposition in $\mathcal{F} = \mathbb{R}[[\mathbf{x}]]/\big(I \cdot \mathbb{R}[[\mathbf{x}]]\big)$. Since $\mathcal{R}$ is reduced $\mathcal{F}$ is also reduced according to Proposition 3.5. Therefore all the $\mathfrak{q}_i$ are prime. Using Theorem 3.12 and Lemma 3.11 we know that there are exactly r maximal ideals $\mathfrak{n}_1, \ldots, \mathfrak{n}_r$ in $\overline{\mathcal{R}}$, which are all lying over $\mathfrak{m}\mathcal{R}$ and

$$\overline{\mathcal{F}/\mathfrak{q}_i} \cong (\overline{\mathcal{R}}_{\mathfrak{n}_i})^*, \quad i = 1, \ldots, r.$$

Next, we consider
$$\sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} = \sqrt[r]{\mathfrak{q}'_1} \cap \ldots \cap \sqrt[r]{\mathfrak{q}'_r}, \tag{4.1}$$
where $\mathfrak{q}'_i$ is the preimage of $\mathfrak{q}_i$ in $\mathbb{R}[[\mathbf{x}]]$. Clearly, $\mathfrak{q}_i$ is real if and only if $\mathfrak{q}'_i$ is real. Thus, applying Corollary 3.13 we get that $\mathfrak{q}'_i$ is real if and only if $\mathfrak{n}_i$ is real.

Now we can show the subsidiary statement in (b). None of the $\mathfrak{n}_i$ is real if and only if none of the $\mathfrak{q}'_i$ is real and by Lemma 4.1 this is the case if and only if $\sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} = \mathfrak{m}\,\mathbb{R}[[\mathbf{x}]]$. Let $G = (\mathbf{V}_{\mathbb{R}}(I), 0)$ bet the set germ of $\mathbf{V}_{\mathbb{R}}(I)$ at the origin. We have seen in the proof of Proposition 3.5 (e) that $\sqrt[r]{I \cdot \mathbb{R}\{\mathbf{x}\}} = \sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} \cap \mathbb{R}\{\mathbf{x}\}$. On the other hand, according to Proposition 2.26 G is the isolated origin if and only if $\mathbf{I}^a(G) = \mathfrak{m}\,\mathbb{R}\{\mathbf{x}\}$. Hence, we know that G is the isolated origin if and only if
$$\mathfrak{m}\,\mathbb{R}\{\mathbf{x}\} = \mathbf{I}^a(G) = \sqrt[r]{I \cdot \mathbb{R}\{\mathbf{x}\}} = \sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} \cap \mathbb{R}\{\mathbf{x}\}. \tag{4.2}$$
It is clear that (4.2) is true if and only if $\sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} = \mathfrak{m}\,\mathbb{R}[[\mathbf{x}]]$ and we know that this equality in turn is equivalent to the fact that none of the $\mathfrak{n}_i$ is real.

Now suppose that two of the $\mathfrak{n}_i$ are real. In this case $\mathcal{F}/\sqrt[r]{(0)}$ would not be a domain and therefore not regular. This means the origin cannot be a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ according to Corollary 3.6.

Finally, we examine the case that exactly one $\mathfrak{n}_i$ is real. Without loss of generality let $\mathfrak{n}_1$ be real. Then, $\sqrt[r]{I \cdot \mathbb{R}[[\mathbf{x}]]} = \mathfrak{q}'_1$ and $\mathcal{F}/\sqrt{(0)} = \mathcal{F}/\mathfrak{q}_1$. According to Theorem 3.12 we have the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathcal{R}}_{\mathfrak{n}_1} \arrow[rr, "\psi"] & & \left(\overline{\mathcal{R}}_{\mathfrak{n}_1}\right)^* \arrow[d, leftrightarrow, "\eta"', "\cong"] \\
\overline{\mathcal{R}} \arrow[u, "l"] & & \left(\overline{\mathcal{F}/\mathfrak{q}_1}\right) \\
\mathcal{R} \arrow[u] \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}/\mathfrak{q}_1 \arrow[u, "\iota"]
\end{tikzcd} \tag{4.3}$$

At first, let us assume that $\mathfrak{m}\,\overline{\mathcal{R}}$ is $\mathfrak{n}_1$-reduced, i.e. $\mathfrak{m}\,\overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{n}_1\,\overline{\mathcal{R}}_{\mathfrak{n}_1}$. We write $\mathfrak{o}$ for the maximal ideal of $\mathcal{F}/\mathfrak{q}_1$ and proceed in two steps.

(1) $\iota(\mathfrak{o})$ generates the maximal ideal of $\overline{\mathcal{F}/\mathfrak{q}_1}$: Since ψ is the $\mathfrak{n}_1\overline{\mathcal{R}}_{\mathfrak{n}_1}$-adic completion of $\overline{\mathcal{R}}_{n_1}$ we know that $\psi\big(\mathfrak{n}_1\,\overline{\mathcal{R}}_{\mathfrak{n}_1}\big) = \psi\big(\mathfrak{m}\,\overline{\mathcal{R}}_{\mathfrak{n}_1}\big)$ generates the maximal ideal of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$. But $\langle \psi\big(\mathfrak{m} \cdot \overline{\mathcal{R}}_{\mathfrak{n}_1}\big)\rangle = \langle \eta(\iota(\mathfrak{o}))\rangle$ and because η is an isomorphism we conclude that $\iota(\mathfrak{o})$ generates the maximal ideal of $\overline{\mathcal{F}/\mathfrak{q}_1}$.

(2) $\mathcal{F}/\mathfrak{q}_1$ is regular: Since $\overline{\mathcal{R}}/\mathfrak{n}_1 \cong \mathbb{R}$ the residue field of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$ must be $\mathbb{R}$, hence the same is true for the residue field of $\overline{\mathcal{F}/\mathfrak{q}_1}$. Also, we know that $\overline{\mathcal{F}/\mathfrak{q}_1}$ is finite over $\mathcal{F}/\mathfrak{q}_1$ [67, Proposition III.2.3]. Now with step (1) we are exactly in the situation of Lemma 2.8 with $A = \mathcal{F}/\mathfrak{q}_1$ and $B = \overline{\mathcal{F}/\mathfrak{q}_1}$. It follows $\overline{\mathcal{F}/\mathfrak{q}_1} = \mathcal{F}/\mathfrak{q}_1$. Thus, $\mathcal{F}/\mathfrak{q}_1$ is a normal local ring of dimension 1. With Serre's regularity criterion R_1 we see

that $\mathcal{F}/\mathfrak{q}_1 = \mathcal{F}/\sqrt[r]{(0)}$ is regular and on account of Corollary 3.6 the origin must be a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$.

Suppose to the contrary that $\mathcal{F}/\mathfrak{q}_1$ is regular. Then, it is a Cohen-Macaulay domain. It fulfills S_2 and R_1 and is therefore normal by Serre's normality criterion [48, Theorem 39]. Consequently, $\mathcal{F}/\mathfrak{q}_1 = \overline{\mathcal{F}/\mathfrak{q}_1} \cong (\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$.

We set $\mathfrak{b} = \mathfrak{m}\overline{\mathcal{R}}_{\mathfrak{n}_1}$. Because the diagram (4.3) commutes and ι is an isomorphism, we have that $\psi(\mathfrak{b})$ generates the maximal ideal $\mathfrak{a}$ of $(\overline{\mathcal{R}}_{\mathfrak{n}_1})^*$. But ψ is faithfully flat (Example 2.1b). Hence

$$\mathfrak{m}\overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{b} = \langle \psi(\mathfrak{b}) \rangle \cap \overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{a} \cap \overline{\mathcal{R}}_{\mathfrak{n}_1} = \mathfrak{n}_1 \overline{\mathcal{R}}_{\mathfrak{n}_1}.$$

So, $\mathfrak{m}\overline{\mathcal{R}}$ is $\mathfrak{n}_1$-reduced. This completes the proof. □

Example 4.1. Let us review the curves from Example 3.2 with Theorem 4.4.

(i) For the simple node $A = \mathbb{R}[x, y]/\langle y^2 - x^2 - x^3 \rangle$ we have seen in Example 3.2

$$\overline{\mathcal{R}} = \overline{A_{\langle x,y \rangle}} \cong S^{-1}\big(\mathbb{R}[x, y, s]/\langle s^2 - x - 1, sx - y \rangle\big),$$

where $S = \mathbb{R}[x, y] \backslash \langle x, y \rangle$. Now $\mathfrak{c}_1 = \langle s - 1, x, y \rangle$ and $\mathfrak{c}_2 = \langle s + 1, x, y \rangle$ are the maximal ideals of $\overline{A} = \mathbb{R}[x, y, s]/\langle s^2 - x - 1, sx - y \rangle$ lying over $\langle x, y \rangle$ and both are real because of the real Nullstellensatz. But then $\overline{\mathcal{R}}$ has the real maximal ideals $S^{-1}\mathfrak{c}_1$ and $S^{-1}\mathfrak{c}_2$ according to Lemma 4.5 below.

Now Theorem 4.4 ensures that the origin is not a manifold point of the real algebraic curve $\mathbf{V}_{\mathbb{R}}(y^2 - x^2 - x^3)$.

(ii) Let $A = \mathbb{R}[x, y]/\langle y^2 + x^2 \rangle$. We have seen

$$\overline{\mathcal{R}} \cong S^{-1}\big(\mathbb{R}[x, y, s]/\langle s^2 + 1, y - xs \rangle\big).$$

Thus, $\overline{\mathcal{R}}$ is a local ring with maximal ideal $\mathfrak{n} = S^{-1}\langle s^2 + 1, x, y \rangle$. Since $\langle s^2 + 1, x, y \rangle$ is not a real ideal of $\overline{A} = (\mathbb{R}[x, y, s]/\langle s^2 + 1, y - xs \rangle)$, $\mathfrak{n}$ cannot be a real ideal of $\overline{\mathcal{R}}$. Hence, all maximal ideals of $\overline{\mathcal{R}}$ are not real.

Theorem 4.4 ensures that the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(x^2 + y^2)$.

(iii) Let $A = \mathbb{R}[x, y]/\langle y^3 + 2\,y\,x^2 - x^4 \rangle$ be the curve from Figure 1.4 again. We have seen

$$\overline{\mathcal{R}} \cong S^{-1}\big(\mathbb{R}[x, y, s]/\langle s^3 + 2s - x, y - xs \rangle\big).$$

Then, $\overline{\mathcal{R}}$ has the maximal ideals $\mathfrak{n}_1 = S^{-1}\langle s, x, y \rangle$ and $\mathfrak{n}_2 = S^{-1}\langle s^2 + 2, x, y \rangle$. As in (i) and (ii) we can show that $\mathfrak{n}_1$ is real and $\mathfrak{n}_2$ is not real. We also know that $\langle x, y \rangle\,\overline{\mathcal{R}}$ is $\mathfrak{n}_1$-reduced since

$$\langle x, y \rangle\,\overline{\mathcal{R}} = \mathfrak{n}_1 \cap \mathfrak{n}_2,$$

is a minimal primary decomposition according to Lemma 4.5.

Now Theorem 4.4 ensures that the origin is a manifold point of the real algebraic curve $\mathbf{V}_{\mathbb{R}}(y^3 + 2\,y\,x^2 - x^4)$.

(iv) Let $A = \mathbb{R}[x,y]/\langle y^2 - x^3\rangle$ be the cuspidal cubic. We have seen

$$\overline{\mathcal{R}} \cong S^{-1}\big(\mathbb{R}[x,y,s]/\langle s^2 - x, y - xs\rangle\big)\,.$$

Then

$$\langle x,y\rangle\,\overline{\mathcal{R}} = S^{-1}\langle s^2,x,y\rangle$$

and since localization commutes with formation of radicals:

$$\sqrt{S^{-1}\langle s^2,x,y\rangle} = S^{-1}\left(\sqrt{\langle s^2,x,y\rangle}\right) = S^{-1}\langle s,x,y\rangle.$$

This means that $\overline{\mathcal{R}}$ is a local ring and has the unique real maximal ideal $\mathfrak{n} = S^{-1}\langle s,x,y\rangle$. But $\langle x,y\rangle\,\overline{\mathcal{R}}$ is not $\mathfrak{n}$-reduced.

Theorem 4.4 ensures that the origin is not a manifold point of the real algebraic curve $\mathbf{V}_{\mathbb{R}}(y^2 - x^3)$.

Lemma 4.5. *Let $A \to B$ be a finite extension of noetherian rings and $\mathfrak{m} \leq A$ a maximal ideal. We set $S = A\backslash\mathfrak{m}$ and $\mathfrak{m}_B = \mathfrak{m}B$. Let*

$$\mathfrak{m}_B = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$$

be a minimal primary decomposition with $\mathfrak{n}_i \coloneqq \sqrt{\mathfrak{q}_i}$ maximal. Then

$$S^{-1}\,\mathfrak{m}_B = S^{-1}\mathfrak{q}_1 \cap \ldots \cap S^{-1}\mathfrak{q}_k \tag{4.4}$$

is a minimal primary decomposition of $S^{-1}\,\mathfrak{m}_B$ in $S^{-1}B$ with $\sqrt{S^{-1}\mathfrak{q}_i} = S^{-1}\mathfrak{n}_i$ and $S^{-1}\mathfrak{n}_1,\ldots,S^{-1}\mathfrak{n}_k$ are all the maximal ideals of the semi-local ring $S^{-1}B$.

In addition, for $i = 1,\ldots,k$ we have that $S^{-1}\mathfrak{m}_B$ is $S^{-1}\mathfrak{n}_i$-reduced if and only if $\mathfrak{m}_B$ is $\mathfrak{n}_i$-reduced and $S^{-1}\mathfrak{n}_i$ is real if and only if $\mathfrak{n}_i$ is real.

Remark. Both the minimal primary decompositions of $\mathfrak{m}_B$ and $S^{-1}\mathfrak{m}_B$ are unique since all associated primes are maximal, hence isolated [3, Corollary 4.11].

Proof of Lemma 4.5. For $i = 1,\ldots,k$ we have $\mathfrak{n}_i \cap S = \mathfrak{n}_i \cap A \cap S = \mathfrak{m} \cap S = \emptyset$. Thus, (4.4) must be a minimal primary decomposition of $S^{-1}\mathfrak{m}_B$, [3, Proposition 4.9]. Furthermore, since radicals commute with localization we have $S^{-1}\mathfrak{n}_i = \sqrt{S^{-1}\mathfrak{q}_i}$.

Now let $\mathfrak{a}$ be a maximal ideal of $S^{-1}B$. Then, there exists a prime ideal $\mathfrak{q} \leq B$, with $S \cap \mathfrak{q} = \emptyset$ and $S^{-1}\mathfrak{q} = \mathfrak{a}$. It is $\mathfrak{q} \cap A \subset \mathfrak{m}$ and with the going-up theorem [3, Theorem 5.11] we find $\mathfrak{q}' \leq B$ prime, with $\mathfrak{q}' \cap A = \mathfrak{m}$ and $\mathfrak{q} \subset \mathfrak{q}'$. Hence, $\sqrt{\mathfrak{m}_B} = \bigcap_i \mathfrak{n}_i \subset \mathfrak{q}'$ and $\mathfrak{q}'$ must be one of the $\mathfrak{n}_i$. Assume $\mathfrak{q}' = \mathfrak{n}_1$. Then, $\mathfrak{q} \subset \mathfrak{n}_1$ and consequently

$$\mathfrak{a} = S^{-1}\mathfrak{q} \subset S^{-1}\mathfrak{n}_1.$$

But $\mathfrak{a}$ is maximal, so $\mathfrak{a} = S^{-1}\mathfrak{n}_1$.

Next, we show that $S^{-1}\mathfrak{m}_B$ is $S^{-1}\mathfrak{n}_i$-reduced if and only if $\mathfrak{m}_B$ is $\mathfrak{n}_i$-reduced. First, let $S^{-1}\mathfrak{m}_B$ be $S^{-1}\mathfrak{n}_i$ reduced. Then, we have $S^{-1}\mathfrak{q}_i = S^{-1}\mathfrak{n}_i$. According to [3, Proposition 3.11, Proposition 4.8] it must be $\mathfrak{q}_i = \mathfrak{n}_i$ and $\mathfrak{m}_B$ is $\mathfrak{n}_i$-reduced. The other direction is easy.

Finally, we prove that $S^{-1}\mathfrak{n}_i$ is real if and only if $\mathfrak{n}_i$ is real. Assume first that $S^{-1}\mathfrak{n}_i$ is real and let $a_1^2+\ldots+a_r^2 \in \mathfrak{n}_i$, for $a_i \in B$. Since $S^{-1}\mathfrak{n}_i$ is real we have $\frac{a_j}{1} \in S^{-1}\mathfrak{n}_i$ and consequently $s_j\, a_j \in \mathfrak{n}_i$, for $s_j \in S$ and $j = 1, \ldots, r$. If $s_j \in \mathfrak{n}_i$ then $s_j \in \mathfrak{n}_i \cap A = \mathfrak{m}$ which is a contradiction, hence $a_j \in \mathfrak{n}_i$ for all j. Thus, $\mathfrak{n}_i$ is real. Conversely, let $\mathfrak{n}_i$ be real and

$$\frac{a_1^2}{s_1^2} + \ldots + \frac{a_r^2}{s_r^2} = \frac{a_1^2\, t_1^2 + \ldots + a_r^2\, t_r^2}{s^2} \in S^{-1}\mathfrak{n}_i,$$

for $a_1, \ldots, a_r \in B$, $s, s_1, \ldots, s_r, t_1, \ldots, t_r \in S$. Then, $s'(a_1^2\, t_1^2 + \ldots + a_r^2\, t_r^2) \in \mathfrak{n}_i$, for some $s' \in S$. Thus, $s'^2(a_1^2\, t_1^2 + \ldots + a_r^2\, t_r^2) \in \mathfrak{n}_i$. Since $\mathfrak{n}_i$ is real, it must be $s'\, a_j\, t_j \in \mathfrak{n}_i$ for all j and therefore

$$\frac{a_j}{s_j} = \frac{s'\, a_j\, t_j}{s'\, s_j\, t_j} \in S^{-1}\mathfrak{n}_i, \quad j = 1, \ldots, r.$$

This means $S^{-1}\mathfrak{n}_i$ is real which completes the proof. □

4.2 Plane curves

Let $X = \mathbf{V}_{\mathbb{R}}(f)$, where $f \in \mathbb{R}[x, y]$. In this section we analyze the algebraic tangent cone of X in regard to the question whether the origin is a manifold point of X. We will need some terminology. Let

$$g = \sum_{k=0}^{\infty} \sum_{i+j=k} a_{i,j}\, x^i y^j = \sum_{k=0}^{\infty} g^{(k)} \in \mathbb{C}[[x, y]],$$

with **homogenous parts** $g^{(k)}$, for $k \in \mathbb{N}$. The **initial polynomial** of g is $g^{(l)}$ with l minimal such that $g^{(l)} \neq 0$. Also, we will write $\overline{g}$ for the **conjugated** power series $\sum_{i,j} \overline{a_{i,j}}\, x^i y^j$. Then $g \cdot \overline{g} \in \mathbb{R}[[x, y]]$. Finally, recall that a **Weierstrass polynomial** [22] or **distinguished polynomial** [67] is a monic element $g \in \mathbb{C}\{x\}[y]$, with

$$g(x, y) = y^k + a_1\, y^{k-1} + \ldots + a_0,$$

where $a_i \in \mathbb{C}\{x\}$ is in the maximal ideal of $\mathbb{C}\{x\}$, i.e. $a_i(0) = 0$. A Weierstrass polynomial g is irreducible as a polynomial in $\mathbb{C}\{x\}[y]$ if and only if g is irreducible in $\mathbb{C}\{x, y\}$ [22, Lemma 6.11].

Lemma 4.6. *Let $f \in \mathbb{R}\{x, y\}$ be irreducible. Then either f is irreducible in $\mathbb{C}\{x, y\}$ or $f = \varepsilon\, g\, \overline{g}$ for a unit ε in $\mathbb{R}\{x, y\}$ and g irreducible in $\mathbb{C}\{x, y\}$.*

Proof. Since $\mathbb{C}\{x,y\}$ is factorial [22, Theorem 6.11], we can decompose f in $\mathbb{C}\{x,y\}$:

$$f = g_1^{k_1} \cdots g_r^{k_r},$$

with $r, k_i \geq 1$ and $g_i \in \mathbb{C}\{x,y\}$ irreducible. Let $J = \langle f \rangle_{\mathbb{C}\{x,y\}}$. For $h := g_1 \cdots g_r$ we have $h^n \in J$ if $n \geq \max\{k_i \mid i = 1, \dots, r\}$. Since f is irreducible, J is a radical ideal by [67, 5.3d], so $h \in J$ and consequently $k_i = 1$ for all i. Now we have the following minimal decomposition in $\mathbb{C}\{x,y\}$:

$$\langle f \rangle = \langle g_1 \rangle \cap \ldots \cap \langle g_r \rangle, \tag{4.5}$$

with $\langle g_i \rangle$ prime for all i. On the other hand, by [67, Theorem II.5.4] we have

$$\langle f \rangle = \mathfrak{p} \cap \overline{\mathfrak{p}},$$

where $\mathfrak{p} \leq \mathbb{C}\{x,y\}$ is prime and $\overline{\mathfrak{p}} = \left\{ \overline{f} \mid f \in \mathfrak{p} \right\}$ is prime as well. We need to distinguish between two cases.

Let $\mathfrak{p} = \overline{\mathfrak{p}}$. Then, because the decomposition (4.5) is unique, we have $r = 1$ and $\langle f \rangle$ is prime. This means that f is irreducible in $\mathbb{C}\{x,y\}$.

Now suppose $\mathfrak{p} \neq \overline{\mathfrak{p}}$. Then because the decomposition (4.5) is unique, we have $r = 2$ and $\langle g_2 \rangle = \overline{\langle g_1 \rangle} = \langle \overline{g_1} \rangle$. Therefore, $g_2 = \varepsilon\, \overline{g_1}$, for $\varepsilon \in \mathbb{C}\{x,y\}$. But g_2 and $\overline{g_1}$ are irreducible, hence ε must be a unit. Now $f = \varepsilon\, g_1\, \overline{g_1}$ and because $g_1\, \overline{g_1} \in \mathbb{R}\{x,y\}$ we have $\varepsilon \in \mathbb{R}\{x,y\}$, too. This completes the proof. ☐

Lemma 4.7. *Let $\alpha, \beta \in \mathbb{C}$, with $(\alpha, \beta) \neq (0,0)$ and*

$$(\alpha x - \beta y)^l = \alpha^l x^l + \ldots + \beta^l y^l \in \mathbb{R}[x,y].$$

Then, there are $a, b \in \mathbb{R}$ with $(\alpha x - \beta y)^l = \pm(ax - by)^l$.

Proof. Since $(\alpha x - \beta y)^l = (\overline{\alpha} x - \overline{\beta} y)^l$ and $\mathbb{C}[x,y]$ is factorial, there is $\gamma \in \mathbb{C} \backslash \{0\}$, with $\gamma \alpha x - \gamma \beta y = \overline{\alpha} x - \overline{\beta} y$. This means $\gamma\, \alpha = \overline{\alpha}$ and $\gamma\, \beta = \overline{\beta}$. Now we easily check that

$$\arg(\alpha) = \arg(\beta) \quad \text{or} \quad \arg(\alpha) = \arg(\beta) + \pi. \tag{4.6}$$

We set $a = |\alpha|$ and $b = \frac{|\alpha|}{\alpha}\, \beta$. Clearly $a \in \mathbb{R}$ and because of (4.6) we have $b \in \mathbb{R}$, too. Moreover, since $\alpha^l \in \mathbb{R}$:

$$(ax - by)^l = \frac{|\alpha|^l}{\alpha^l}\, (\alpha x - \beta y)^l = \pm(\alpha x - \beta y)^l. \qquad ☐$$

For a real convergent power series $f \in \mathbb{R}\{x,y\}$ we wish to categorize the zeros of the homogeneous initial polynomial $f^{(l)}$ in $\mathbb{P}_1(\mathbb{C})$, hence we define:

Definition 4.3.

$$\mathbb{P}_1(\mathbb{C})_{\mathrm{re}} = \{\, [a:b] \in \mathbb{P}_1(\mathbb{C}) \mid a, b \in \mathbb{R} \,\}, \quad \mathbb{P}_1(\mathbb{C})_{\mathrm{im}} = \mathbb{P}_1(\mathbb{C}) \backslash \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}.$$

For example, the homogeneous polynomial $g(x, y) = (x^2 - y^2) \cdot (x^2 + y^2) \cdot y$ has the zeros $[1 : 1], [1 : -1], [1 : 0] \in \mathbb{P}_1(\mathbb{C})_{\text{re}}$ and $[1 : i], [1 : -i] \in \mathbb{P}_1(\mathbb{C})_{\text{im}}$.

The following Proposition is an easy consequence of [22, Satz 8.1]:

Proposition 4.8. *Let $f \in \mathbb{R}\{x, y\}$ be irreducible. Then, the initial polynomial of f has the form*

$$f^{(l)} = \begin{cases} \pm(ax - by)^l, \ a, b \in \mathbb{R}, \ (a, b) \neq (0, 0), \\ \pm(cx - dy)^{\frac{l}{2}} (\bar{c}x - \bar{d}y)^{\frac{l}{2}}, \ [c : d] \in \mathbb{P}_1(\mathbb{C})_{\text{im}}, \end{cases} \tag{4.7}$$

where the second case is only possible for l even.

Remark. We will use Lemma 4.7 for the proof. Note that the initial polynomial of f can be $f^{(l)} = \pm(ax - by)^l, \ a, b \in \mathbb{R}$, even if $f = \varepsilon g \overline{g}$ for g in $\mathbb{C}\{x, y\}$. Consider for example

$$f(x, y) = y^2 - y^3 + x^4 = \left(x^2 - i \cdot y\sqrt{1 - y}\right) \cdot \left(y^2 + i \cdot y\sqrt{1 - y}\right)$$

in $\mathbb{C}\{x, y\}$. See Figure 4.1 for an illustration of $\mathbf{V}_{\mathbb{R}}(f)$.

Proof. Let $f \in \mathbb{R}\{x, y\}$ be irreducible. By Lemma 4.6 f is either irreducible in $\mathbb{C}\{x, y\}$ or $f = \varepsilon g \overline{g}$ with ε a unit in $\mathbb{R}\{x, y\}$ and $g \in \mathbb{C}\{x, y\}$ irreducible. First, assume f is irreducible in $\mathbb{C}\{x, y\}$. Since f is not divisible by y we can use the Weierstrass preparation theorem [22, 6.7] to find a representation

$$f = \nu h,$$

where ν is a unit in $\mathbb{C}\{x, y\}$ and $h \in \mathbb{C}\{x\}[y]$ is a Weierstrass polynomial. Since f is irreducible in $\mathbb{C}\{x, y\}$, h is irreducible as well. Now we can apply [22, Satz 8.1] to see that the initial polynomial $h^{(l)}$ of h has the form

$$h^{(l)} = (\alpha x - \beta y)^l, \ (\alpha, \beta) \in \mathbb{C}^2 \backslash \{(0, 0)\}.$$

Because ν is a unit we have $f^{(l)} = \delta \cdot (\alpha x - \beta y)^l = (\gamma \alpha x - \gamma \beta y)^l$ for any lth root γ of $\delta \in \mathbb{C}$. As $f \in \mathbb{R}\{x, y\}$, Lemma 4.7 confirms that $f^{(l)} = \pm(ax - by)^l$, for $a, b \in \mathbb{R}$.

Now assume that $f = \varepsilon g \overline{g}$. Since y does not divide g we can use the Weierstrass preparation theorem again and find a factorization $g = \nu h$, with an irreducible Weierstrass polynomial $h \in \mathbb{C}\{x\}[y]$ and a unit $\nu \in \mathbb{C}\{x, y\}$. Then

$$f = \varepsilon \cdot \nu \overline{\nu} \cdot h \overline{h}.$$

Clearly, $\varepsilon \cdot \nu \overline{\nu}$ is a unit in $\mathbb{R}\{x, y\}$ and with [22, Satz 8.1] the initial polynomial of h is $h^{(m)} = (cx - dy)^m$, for $[c : d] \in \mathbb{P}(\mathbb{C})$. Thus, the initial polynomial of f is:

$$f^{(2m)} = u \cdot (cx - dy)^m \cdot (\bar{c}x - \bar{d}y)^m,$$

for $u \in \mathbb{R}$. If $[c:d] \in \mathbb{P}_1(\mathbb{C})_{\mathrm{im}}$, we can pull $|u|$ inside the brackets and $f^{(2m)}$ has exactly the second form in (4.7). So let $[c:d] \in \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}$, then $\alpha c, \alpha d \in \mathbb{R}$, for some $\alpha \in \mathbb{C}$. This means

$$\bar{c} = \frac{\overline{\alpha \cdot c}}{\overline{\alpha}} = c \cdot \frac{\alpha}{\overline{\alpha}}, \qquad \bar{d} = \frac{\overline{\alpha \cdot d}}{\overline{\alpha}} = d \cdot \frac{\alpha}{\overline{\alpha}}.$$

And consequently

$$f^{(2m)} = u \cdot (cx - dy)^{2m} \cdot \frac{\alpha^m}{\overline{\alpha}^m} = (c'x - d'y)^{2m}, \quad \text{for } c', d' \in \mathbb{C}.$$

Now we apply Lemma 4.7 again and $f^{(2m)}$ has the first form of (4.7). $\square$

Proposition 4.9. *Let $f \in \mathbb{R}\{x,y\}$ be irreducible in $\mathbb{C}\{x,y\}$. Then $\langle f \rangle$ is a real ideal of $\mathbb{R}\{x,y\}$.*

Remark. This is not true for $f \in \mathbb{R}\{x_1,\ldots,x_n\}$ and $n \geq 3$. Consider $f(x,y,z) = x^2 + y^2 + z^2$. Then, f is irreducible in $\mathbb{C}\{x,y,z\}$ since $\mathbf{V}(f)$ is a normal variety, but $\langle f \rangle$ is clearly not real.

Proof. Suppose that f is irreducible. By [67, Proposition IV.4.4] it is enough to show that $R = \mathbb{R}\{x,y\}/\langle f \rangle$ is 2-real. This means that $\alpha^2 + \beta^2 = 0$ implies $\alpha = \beta = 0$, for all $\alpha, \beta \in R$. So let $a, b, c \in \mathbb{R}\{x,y\}$ with

$$a^2 + b^2 = c\, f.$$

But then $(a+ib)(a-ib) = c\,f$ in $\mathbb{C}\{x,y\}$. Since f is irreducible in $\mathbb{C}\{x,y\}$, we have w.l.o.g. that f divides $(a+ib)$ in $\mathbb{C}\{x,y\}$. Thus, $a + ib \in \langle f \rangle_{\mathbb{C}\{x,y\}}$ and

$$(a - ib) = \overline{a+ib} \in \langle \overline{f} \rangle_{\mathbb{C}\{x,y\}} = \langle f \rangle_{\mathbb{C}\{x,y\}}.$$

Hence, $a, b \in \langle f \rangle_{\mathbb{C}\{x,y\}} \cap \mathbb{R}\{x,y\} = \langle f \rangle$, since $\mathbb{R}\{x,y\} \hookrightarrow \mathbb{C}\{x,y\}$ is faithfully flat according to [67, 5.3 (b)]. $\square$

Lemma 4.10. *Let $f \in \mathbb{R}\{x,y\}$ be irreducible with initial polynomial*

$$f^{(l)} = \pm(cx - dy)^{\frac{l}{2}}(\bar{c}x - \bar{d}y)^{\frac{l}{2}}, \quad [c:d] \in \mathbb{P}_1(\mathbb{C})_{\mathrm{im}}. \tag{4.8}$$

Then $\langle f \rangle$ is not a real ideal of $\mathbb{R}\{x,y\}$.

Proof. Assume $\langle f \rangle$ is real and f has its initial polynomial in the form above. Let $G = \big(\mathbf{V}_{\mathbb{R}}(f), 0\big)$ be the set germ of $\mathbf{V}_{\mathbb{R}}(f)$ at the origin. According to the real analytic Nullstellensatz it must be $\mathbf{I}^a(G) = \sqrt[r]{\langle f \rangle} = \langle f \rangle$. Now

$$\dim \mathcal{O}(G) = \dim \mathbb{R}\{x,y\}/\mathbf{I}^a(G) = 2 - \operatorname{ht} I = 1 \neq 0$$

and according to Proposition 2.26 G is not the isolated origin.

We find a sequence of points $w^{(k)} \in \mathbb{R}^2 \setminus \{0\}$ converging to the origin such that $f(w^{(k)}) = 0$. The sequence

$$v^{(k)} := w^{(k)} / |w^{(k)}|$$

lies in the compact unit sphere in $\mathbb{R}^2$. Therefore, there exists a convergent subsequence of $v^{(k)}$ and we can assume without loss of generality that $v^{(k)}$ converges in $\mathbb{R}^2$. We set $v := \lim_{k\to\infty} v^{(k)}$. Then

$$\begin{aligned} 0 = f(w^{(k)}) = f(v^{(k)} \cdot |w^{(k)}|) &= f^{(l)}(v^{(k)}) \cdot |w^{(k)}|^l + \sum_{i=1}^{\infty} f^{(l+i)}(v^{(k)}) \cdot |w^{(k)}|^{l+i} \\ &= |w^{(k)}|^l \cdot \left(f^{(l)}\left(v^{(k)}\right) + \sum_{i\geq 1} f^{(l+i)}(v^{(k)}) |w^{(k)}|^i \right). \end{aligned}$$

So we conclude:

$$0 = f^{(l)}(v^{(k)}) + \sum_{i\geq 1} f^{(l+i)}(v^{(k)})\, |w^{(k)}|^i. \tag{4.9}$$

We now claim that $\sum_{i\geq 1} f^{(l+i)}(v^{(k)})\, |w^{(k)}|^i$ converges to zero for $k \to \infty$. First, write f with multi-indices in the form

$$f(u) = \sum_{\nu\in\mathbb{N}^2} a_\nu\, u^\nu.$$

Since f is a convergent power series we find $q \in \mathbb{R}$, $q > 0$ such that $f(q, q)$ converges. This means there is a constant $c \in \mathbb{R}$ with $|a_\nu\, q^\nu| = |a_\nu|\, q^{|\nu|} \leq c$, for all $\nu \in \mathbb{N}^2$. Additionally, we have $|u^\nu| \leq 1$, for $u \in \mathbb{R}^2$ on the unit sphere and $\nu \in \mathbb{N}^2$. Thus

$$|f^{(r)}(u)| \leq \sum_{|\nu|=r} |a_\nu\, u^\nu| \leq \sum_{|\nu|=r} \frac{c}{q^{|\nu|}} = \frac{(r+1)\, c}{q^r},$$

for all $r \geq l$. So we get

$$\begin{aligned} \left| \sum_{i\geq 1} f^{(l+i)}(v^{(k)})\, |w^{(k)}|^i \right| &= |w^{(k)}| \cdot \left| \sum_{i=0}^{\infty} f^{(l+i+1)}(v^{(k)})\, |w^{(k)}|^i \right| \\ &\leq |w^{(k)}| \cdot \sum_{i=0}^{\infty} \frac{(l+i+2)\, c}{q^{l+i+1}}\, |w^{(k)}|^i, \end{aligned}$$

where the last sum converges by the root criterion if $|w^{(k)}| < q$. This clearly proves the claim $\sum_{i\geq 1} f^{(l+i)}(v^{(k)})\, |w^{(k)}|^i \to 0$ for $k \to \infty$. Hence, from (4.9) we get

$$0 = \lim_{k\to\infty} f^{(l)}(v^{(k)}) = f^{(l)}(v).$$

But then $\mathbf{V}(f^{(l)}) \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}} \neq \emptyset$, which contradicts (4.8). This proves the lemma. $\square$

The following theorem lists all the information which can be deduced from the algebraic tangent cone.

Theorem 4.11 (Plane Curve Smoothness Criterion). *Let $f \in \mathbb{R}[x,y]$ with homogeneous initial polynomial $f^{(l)}$ and let $\mathcal{N}$ be the set of zeros of $f^{(l)}$ in $\mathbb{P}_1(\mathbb{C})$. Also, let $\mathcal{N}_0 \subset \mathcal{N}$ be the set of zeros of odd multiplicity and let $\mathcal{N}_1 \subset \mathcal{N}_0$ be the set of zeros of multiplicity* 1.

(i) If $\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}} = \emptyset$, then the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(f)$.

(ii) If $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| \geq 1$, the origin is not an isolated point of $\mathbf{V}_{\mathbb{R}}(f)$.

(iii) If $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| > 1$, the origin is not a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

(iv) If $|\mathcal{N}_1 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = |\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = 1$, the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

Remark.

(i) If $\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}} \neq \emptyset$ but $|\mathcal{N}_1 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| < |\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}|$ in general only few information can be recovered from the tangent cone alone, since a double root in $\mathbb{P}_1(\mathbb{C})_{\mathrm{re}}$ of the tangent cone can originate from a real branch or two conjugated complex branches. See Example 4.2 and Figure 4.1 with the following curves: $h_4(x,y) = y^2 - x^5$, $h_5(x,y) = y^2 - y^3 + x^4$ and $h_6(x,y) = x^3 + x^4 + xy^4 + y^6$. h_4 is irreducible in $\mathbb{C}\{x,y\}$ [22], h_5 has two conjugated complex branches and h_6 has one nonsingular real branch and two conjugated complex branches.

(ii) If $\deg f^{(l)}$ is odd, then $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}|$ must be one or higher. Therefore, the origin cannot be isolated in $\mathbf{V}_{\mathbb{R}}(f)$ according to Theorem 4.11.

Proof. Let $f \in \mathbb{R}[x,y]$, with $I = \langle f \rangle \subset \langle x,y \rangle$. We decompose

$$f = f_1 \cdots f_k \tag{4.10}$$

in $\mathbb{R}\{x,y\}$ into irreducible factors. By Proposition 4.8 we have for the initial polynomial of f:

$$h(x,y) = \pm \prod_{i=1}^{s} (a_i\, x - b_i\, y)^{l_i} \cdot \prod_{j=1}^{r} (c_j\, x - d_j\, y)^{l_j} (\overline{c_j}\, x - \bar{d}_j\, y)^{l_j}, \tag{4.11}$$

where $a_i, b_i \in \mathbb{R}$, $(a_i, b_i) \neq (0,0)$, for all i and $[c_j : d_j] \in \mathbb{P}_1(\mathbb{C})_{\mathrm{im}}$ for all j. Moreover, we have ordered $f_1, \ldots, f_k$ such that $\pm(a_i\, x - b_i\, y)^{l_i}$ is the initial polynomial of f_i, $i = 1, \ldots, s$ and $\pm(c_j\, x - d_j\, y)^{l_j}(\overline{c_j}\, x - \bar{d}_j\, y)^{l_j}$ is the initial polynomial of f_{s+j}, $j = 1, \ldots, r$. Finally, we deduce from (4.10):

$$\hat{I} := \sqrt[r]{I \cdot \mathbb{R}\{x,y\}} = \sqrt[r]{\langle f_1 \rangle} \cap \ldots \cap \sqrt[r]{\langle f_k \rangle}. \tag{4.12}$$

With this preparation we can show the parts (i)–(iv). Let $G = \big(\mathbf{V}_{\mathbb{R}}(f), 0\big)$ be the set germ of $\mathbf{V}_{\mathbb{R}}(f)$ at the origin.

(i) If $\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}} = \emptyset$, then $s = 0$ in (4.11). By Lemma 4.10 and Lemma 4.1 we have $\hat{I} = \langle x, y \rangle \, \mathbb{R}\{x, y\}$, hence $\mathcal{O}(G) = \mathbb{R}\{x, y\}/\hat{I} \cong \mathbb{R}$ and the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(f)$ according to Proposition 2.26.

(ii) Assume $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| \geq 1$. Then, there exists $i_0 \in \{1, \ldots, s\}$ in (4.11) such that l_{i_0} is odd. According to Lemma 4.6 f_{i_0} must be irreducible in $\mathbb{C}\{x, y\}$, hence $\langle f_{i_0} \rangle$ is real by Proposition 4.9, i.e. $\sqrt[r]{\langle f_{i_0} \rangle} = \langle f_{i_0} \rangle$. Then $\hat{I} \neq \langle x, y \rangle \, \mathbb{R}\{x, y\}$ and the origin is not isolated in $\mathbf{V}_{\mathbb{R}}(f)$.

(iii) If $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| > 1$, then we have $s \geq 2$ in (4.11) and w.l.o.g. l_1, l_2 odd. This means that f_1, f_2 are irreducible in $\mathbb{C}\{x, y\}$ and therefore with (4.12) $\hat{I} = \langle f_1 \rangle \cap \langle f_2 \rangle \cap J$ for $J \leq \mathbb{R}\{x, y\}$ with $I \subsetneq J$. One checks easily that $\mathcal{O}(G) = \mathbb{R}\{x, y\}/\hat{I}$ is not a domain in this case. This means $\mathcal{O}(G)$ is not regular and the origin cannot be a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

(iv) Let $|\mathcal{N}_1 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = |\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = 1$. Then, we have $s = 1$, $l_1 = 1$ in (4.11). We deduce from (4.12), Lemma 4.10 and Lemma 4.1 that $\hat{I} = \langle f_1 \rangle$.

But f_1 has initial polynomial $(a_1 x - b_1 y)$ and with Nagata's Jacobian criterion we see that $\mathcal{O}(G)$ is a regular ring. Consequently, the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$. $\square$

Example 4.2. All curves are plotted in Figure 4.1.

(i) $h_1(x, y) = x^2 + y^2 - y^3$. Then, $\mathcal{N} = \{[1 : i], [1 : -i]\}$. So $\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}} = \emptyset$ and the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(f)$.

(ii) $h_2(x, y) = x^3 + y^3 - y^4$. Then, $\mathcal{N} = \mathcal{N}_1 = \left\{[1 : -1], [1 : e^{\pi/3}], [1 : e^{-\pi/3}]\right\}$. So $|\mathcal{N} \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = |\mathcal{N}_1 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = 1$ and the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

(iii) $h_3(x, y) = 4xy + 4x^2 + y^3$. Then, $\mathcal{N}_0 = \{[0 : 1], [1 : -1]\}$. So $|\mathcal{N}_0 \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}| = 2$ and the origin is not a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$.

(iv) $h_4(x, y) = y^2 - y^3 + x^4$. Then, $\mathcal{N} = \{[1 : 0]\}$, $\mathcal{N}_0 = \emptyset$. The origin is isolated in $\mathbf{V}_{\mathbb{R}}(f)$ but Theorem 4.11 cannot be applied, see the remark after the Theorem.

(v) $h_5(x, y) = x^2 - y^5$. Then, $\mathcal{N} = \{[0 : 1]\}$ and $\mathcal{N}_0 = \emptyset$. The origin is not a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$ but Theorem 4.11 cannot be applied. See Example 5.1 and the remark after Theorem 4.11.

(vi) $h_6(x, y) = y^3 - y^4 + yx^4 - x^6$. Then, $\mathcal{N} = \mathcal{N}_0 = \{[1 : 0]\}$. So the origin is not isolated in $\mathbf{V}_{\mathbb{R}}(f)$. It is also a manifold point. See Example 5.1 and the remark after Theorem 4.11.

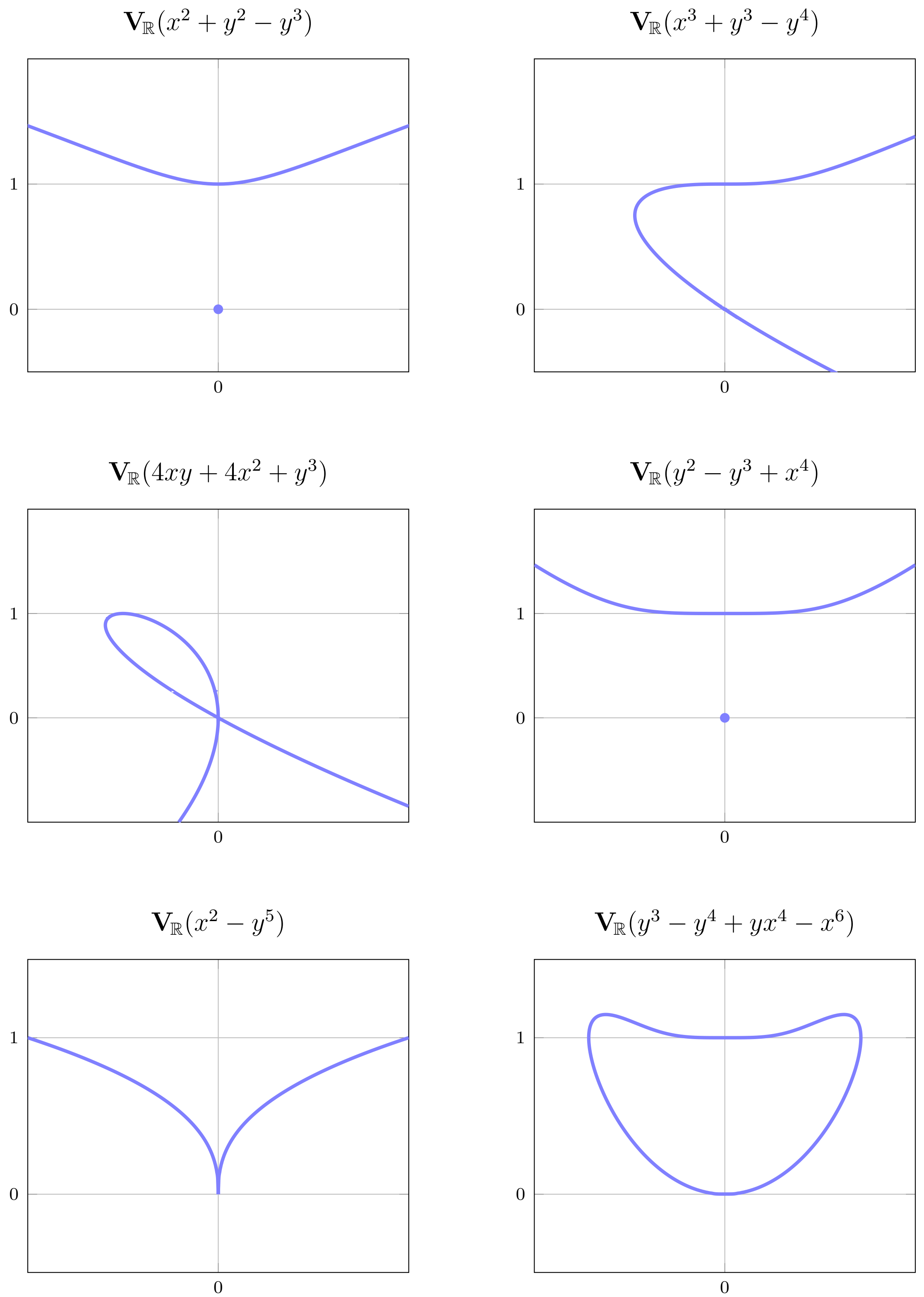


Figure 4.1: Real plane algebraic curves with different algebraic tangent cones.

Chapter 5

Algorithms

We will prepare some of the results of the previous chapters for the application in computational algebra. In particular, we formulate algorithms to check the conditions of Theorem 4.4 and to test whether $I \cdot \mathbb{R}[[\mathbf{x}]]$ is real for a one-dimensional ideal $I \leq \mathbb{R}[\mathbf{x}]$.

All algorithms are implemented in the CAS `Singular`. See Appendix A.1 for the complete source code and a quick discussion of implementation details.

5.1 Realness for zero-dimensional ideals

There are algorithms to check if an ideal $I \leq \mathbb{Q}[\mathbf{x}]$ is real [72]. However, this is not enough in general to show that $I \cdot \mathbb{R}[\mathbf{x}]$ is real as we have seen in Example 2.2 (iii). But we have the following result for zero-dimensional ideals:

Lemma 5.1.

(i) Let $I \leq \mathbb{R}[\mathbf{x}]$ be maximal. Then, I is real if and only if $\mathbf{V}(I) \subset \mathbb{R}^n$. In this case $\mathbf{V}(I)$ is a single real point.

(ii) Let $I \leq \mathbb{R}[\mathbf{x}]$ be radical and $\dim I = 0$*. Then, I is real if and only if $\mathbf{V}(I) \subset \mathbb{R}^n$.*

Proof.

(i) This is [72, Lemma B.15].

(ii) Since I is radical, we have a decomposition into prime ideals

$$I = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_k.$$

As $\dim I = 0$, it must be $\operatorname{ht} I = n$ and consequently $\operatorname{ht} \mathfrak{p}_i = n$ for all i. This means that all $\mathfrak{p}_i$ are maximal ideals. If I is real, then all $\mathfrak{p}_i$ are real because of [19, Lemma 2.2]. Hence, $\mathbf{V}(I) \subset \mathbb{R}^n$ follows from (i). Suppose to the contrary that

$$\mathbf{V}(I) = \bigcup_{i=1}^{k} \mathbf{V}(\mathfrak{p}_i) \subset \mathbb{R}^n.$$

Then, $\mathbf{V}(\mathfrak{p}_i) \subset \mathbb{R}^n$ for all i and $\mathfrak{p}_i$ is real because of (i). Therefore, I is real. $\square$

Let $I \leq \mathbb{Q}[\mathbf{x}]$ be a radical ideal with $\dim I = 0$. To decide whether $I \cdot \mathbb{R}[\mathbf{x}]$ is real, we need to check $|\mathbf{V}_{\mathbb{R}}(I)| = |\mathbf{V}(I)|$ according to Lemma 5.1. The calculation of the number of real solutions of a zero-dimensional ideal I is a classical problem. We will work with the multivariate Tarski-query $\mathrm{TaQ}(1, I)$ [4, Algorithm 12.7] (or Sturm-query in an earlier edition of [4]) which is implemented in `Singular` [73]. The algorithm works by computing the signature of Hermite's quadratic form $\mathrm{Her}(I, 1)$. According to [4, Theorem 4.100] this is just $|\mathbf{V}_{\mathbb{R}}(I)|$.

After the calculation of $|\mathbf{V}_{\mathbb{R}}(I)|$, this number needs to be compared with the count of all solutions in $\mathbf{V}(I)$. Since I is radical we can count the number of zeros with multiplicities by computing the degree of $\mathbf{V}(I)$ or which is the same in this case $\dim_{\mathbb{C}} \mathbb{C}[\mathbf{x}]/(I \cdot \mathbb{C}[\mathbf{x}]) = \dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{x}]/I$. Then, we get the following algorithm:

Algorithm 5.1 Check for $I \leq \mathbb{Q}[\mathbf{x}]$ with $\dim I = 0$, whether $I \cdot \mathbb{R}[\mathbf{x}]$ is real.

Input: $I \leq \mathbb{Q}[\mathbf{x}]$, with $\dim I = 0$.
Output: **true** if I is real, else **false**.

if $I = \sqrt{I}$ **then**
  $k_1 := \dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{x}]/I$
  $k_2 := \mathrm{TaQ}(1, I) = |\mathbf{V}(I) \cap \mathbb{R}^n|$
  if $k_1 = k_2$ **then**
    return true
  else
    return false
  end if
else
  return false
end if

We quickly mention another approach for Algorithm 5.1 which is sometimes faster because it is not necessary to determine $\mathrm{TaQ}(1, I)$. If we calculate a reduced Gröbner basis $\mathcal{B}$ of I with respect to the lexicographic ordering and $x_1 > x_2 > \ldots > x_n$, the last entry of $\mathcal{B}$ will always be a single-variate polynomial in x_n [12, Chapter 3]. We are done if $\mathcal{B}$ looks like

$$\mathcal{B} = \{\, x_1 - g_1(x_n), \ldots, x_{n-1} - g_{n-1}(x_n), g_n(x_n) \,\}. \tag{5.1}$$

In this case I is real if and only if g_n has only real roots which can be checked by the Sturm's sequence [4, Section 2.2.2]. According to the Shape lemma [26, Proposition 1.6] $\mathcal{B}$ is guaranteed to be in the form (5.1) if all the zeros of I have distinct x_n-coordinates. Hence if $\mathcal{B}$ is not in shape (5.1) we proceed to calculate Gröbner bases with respect to lexicographic orderings where we exchange x_n with another variable in the chain $x_1 > \ldots > x_n$. If the algorithm does not terminate after n steps, because the zeros of I do not have distinct values in any coordinate, we can make a generic coordinate change to get $\mathcal{B}$ in the desired shape [31, Proposition 4.2.2]. See also the discussion in [72, Kapitel 4.3] for some considerations about the impact of a generic coordinate change on the speed of the Gröbner base calculations.

5.2 Algorithmic curve criterion

In this section let $I \leq \mathbb{Q}[\mathbf{x}]$ be radical with $\dim I = 1$. For the next algorithm we will require the integral closure of $\mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}$. Hence, we need to check that we can utilize the normalization calculated over $\mathbb{Q}$:

Proposition 5.2. *Let $\psi\colon A \hookrightarrow B$ be the normalization of a finitely generated, reduced $\mathbb{Q}$-algebra A and $\mathbb{Q} \subset \mathbb{K}$ be any field extension. The induced monomorphism*

$$\psi_{\mathbb{K}}\colon \mathbb{K} \otimes_{\mathbb{Q}} A \hookrightarrow \mathbb{K} \otimes_{\mathbb{Q}} B$$

is the normalization of $\mathbb{K} \otimes_{\mathbb{Q}} A$.

Proof. Since $\mathbb{Q} \subset \mathbb{K}$ is a separable field extension $\mathbb{K}$ is geometrically normal over $\mathbb{Q}$ [13, Lemma 10.160.3]. This implies in particular that $\mathbb{K} \otimes_{\mathbb{Q}} B$ is normal, i.e. integrally closed in its total ring of fractions [13, Lemma 10.160.4]. Also, any simple tensor $a \otimes f \in \mathbb{K} \otimes_{\mathbb{Q}} B$ is clearly integral over $\mathbb{K} \otimes_{\mathbb{Q}} A$ since f is integral over A. Then, $\mathbb{K} \otimes_{\mathbb{Q}} B$ is integral over $\mathbb{K} \otimes_{\mathbb{Q}} A$.

The tricky part is to show that we can embed $\mathbb{K} \otimes_{\mathbb{Q}} B \subset Q(\mathbb{K} \otimes_Q A)$. First, we note that $A \hookrightarrow B$ is a finite monomorphism, according to a result of E. Noether [20, Corollary 13.13] or [31, Theorem 3.5.10]. Thus, B is generated by finitely many elements $\frac{a_1}{s_1}, \dots, \frac{a_k}{s_k} \in B \subset Q(A)$, with $a_i \in A$ and s_i a non-zero-divisor. Now we set $s := \prod_{i=1}^k s_i$, then

$$A[s^{-1}] = B[s^{-1}]$$

as subrings of $Q(A)$. We claim that $\hat{s} = 1 \otimes s$ is a non-zero-divisor in $\mathbb{K} \otimes_{\mathbb{Q}} A$ and $\mathbb{K} \otimes_{\mathbb{Q}} B$. But this is clear because $\mathbb{K}$ is flat over $\mathbb{Q}$, hence the monomorphism $A \to A$ given by multiplication with s induces a monomorphism $\mathbb{K} \otimes_{\mathbb{Q}} A \to \mathbb{K} \otimes_{\mathbb{Q}} A$ given by multiplication with $\hat{s}$. Then, we have the following commutative diagram:

$$\begin{tikzcd}
& \mathbb{K} \otimes_{\mathbb{Q}} A \arrow[r] \arrow[dl] \arrow[d] & \mathbb{K} \otimes_{\mathbb{Q}} B \arrow[d] \arrow[dr] & \\
(\mathbb{K} \otimes_{\mathbb{Q}} A)[\hat{s}^{-1}] \arrow[r, "\cong"] & \mathbb{K} \otimes_{\mathbb{Q}} \left(A[s^{-1}]\right) \arrow[r, "\cong"] & \mathbb{K} \otimes_{\mathbb{Q}} \left(B[s^{-1}]\right) \arrow[r, "\cong"] & (\mathbb{K} \otimes_{\mathbb{Q}} B)[\hat{s}^{-1}]
\end{tikzcd}$$

We see that $\mathbb{K} \otimes_{\mathbb{Q}} A \to \mathbb{K} \otimes_{\mathbb{Q}} B$ induces an isomorphism

$$Q(\mathbb{K} \otimes_{\mathbb{Q}} A) = Q((\mathbb{K} \otimes_{\mathbb{Q}} A)[\hat{s}^{-1}]) \cong Q((\mathbb{K} \otimes_{\mathbb{Q}} B)[\hat{s}^{-1}]) = Q(\mathbb{K} \otimes_{\mathbb{Q}} B).$$

This concludes the proof. $\square$

The following proposition gives a different formulation of Theorem 4.4 and proves the correctness of Algorithms 5.2 and 5.3.

Proposition 5.3. *Let* $I \leq \mathbb{Q}[\mathbf{x}]$ *be radical with* $\dim I = 1$ *and* $I \subset \mathfrak{m} := \langle \mathbf{x} \rangle$. *Also, let* $\psi\colon \mathbb{Q}[\mathbf{x}]/I \to \mathbb{Q}[\mathbf{y}]/J$ *be the normalization of* $\mathbb{Q}[\mathbf{x}]/I$. *We set*

$$W = \pi^{-1}(\langle \psi(\mathfrak{m}) \rangle) \leq \mathbb{Q}[\mathbf{y}],$$

where π *is the projection* $\mathbb{Q}[\mathbf{y}] \to \mathbb{Q}[\mathbf{y}]/J$. *Then*

(i) $I \cdot \mathbb{R}[[\mathbf{x}]]$ *is real if and only if* $\sqrt{W_{\mathbb{R}}}$ *is real.*

(ii) The origin is isolated in $\mathbf{V}_{\mathbb{R}}(I)$ *if and only if* $\mathbf{V}_{\mathbb{R}}(W) = \emptyset$.

(iii) The origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ *if and only if* $|\mathbf{V}_{\mathbb{R}}(W)| = 1$ *and there is an isolated primary component* $\mathfrak{q}$ *of* W *in* $\mathbb{Q}[\mathbf{y}]$ *such that* $|\mathbf{V}_{\mathbb{R}}(\mathfrak{q})| = 1$ *and* $\sqrt{\mathfrak{q}} = \mathfrak{q}$, *i.e.* $\mathfrak{q}$ *prime.*

Remark. With the Tarski-query we are unable to count the number of real zeros with multiplicities. Hence, if $|\mathbf{V}_{\mathbb{R}}(W)| = 1$ we need to decide whether this is a simple zero. This makes Algorithm 5.3 somewhat more complicated.

Proof of Proposition 5.3. First, let

$$\psi_{\mathbb{R}}\colon \mathbb{R}[\mathbf{x}]/I_{\mathbb{R}} \to \mathbb{R}[\mathbf{y}]/J_{\mathbb{R}}$$

be the homomorphism induced by ψ. According to Proposition 5.2 $\psi_{\mathbb{R}}$ is the normalization of $A := \mathbb{R}[\mathbf{x}]/I_{\mathbb{R}}$. We will write $B = \mathbb{R}[\mathbf{y}]/J_{\mathbb{R}}$, $S := A \backslash \mathfrak{m}$ and recall the notation $\mathcal{R} = A_{\mathfrak{m}}$, $\mathcal{F} = \mathbb{R}[[\mathbf{x}]]/\big(I \cdot \mathbb{R}[[\mathbf{x}]]\big)$. In addition, let

$$\mathcal{W} = \langle \psi_{\mathbb{R}}(\mathfrak{m}) \rangle = W_{\mathbb{R}}/J_{\mathbb{R}} \leq B. \tag{5.2}$$

Since $\psi_{\mathbb{R}}$ is an integral homomorphism we have a minimal decomposition

$$\sqrt{\mathcal{W}} = \mathfrak{n}_1 \cap \ldots \cap \mathfrak{n}_k, \tag{5.3}$$

where $\mathfrak{n}_i \leq B$ is maximal for all i. We proceed to prove (i)–(iii).

(i) According to Lemma 4.5

$$\sqrt{S^{-1}\mathcal{W}} = S^{-1}\mathfrak{n}_1 \cap \ldots \cap S^{-1}\mathfrak{n}_k$$

is a minimal primary decomposition of $\sqrt{S^{-1}\mathcal{W}}$ in $S^{-1}B = \overline{\mathcal{R}}$. As stated in Corollary 3.13 $\mathcal{F}$ is real if and only if $S^{-1}\mathfrak{n}_i$ is real for all i. But due to Lemma 4.5 $S^{-1}\mathfrak{n}_i$ is real if and only if $\mathfrak{n}_i$ is real. With (5.2),(5.3) we also have that all $\mathfrak{n}_i$ are real if and only if $\sqrt{\mathcal{W}}$ is real. To summarize: $\mathcal{F}$ is real if and only if $\sqrt{W}$ is real. Since $\sqrt{\mathcal{W}}$ is real if and only if $\sqrt{W_{\mathbb{R}}}$ is real, this proves (i).

Algorithm 5.2 Check if $I \cdot \mathbb{R}[[\mathbf{x}]]$ is real.

Input: $I \leq \mathbb{Q}[\mathbf{x}]$ with $\dim I = 1$ and $I \leq \langle \bar{x} \rangle = \mathfrak{m}$.
Output: **true** if $I \cdot \mathbb{R}[[\mathbf{x}]]$ real, else **false**.

if $I = \sqrt{I}$ **then**
  Calculate normalization $\psi\colon \mathbb{Q}[\mathbf{x}]/I \to \mathbb{Q}[\mathbf{y}]/J$.
  $W := \pi^{-1}\big(\langle \psi(\mathfrak{m}) \rangle\big)$ with $\mathfrak{m} = \langle \mathbf{x} \rangle$, $\pi\colon \mathbb{Q}[\mathbf{y}] \to \mathbb{Q}[\mathbf{y}]/J$.
  if $\sqrt{W}$ is real **then**
    return true
  else
    return false
  end if
else
  return false
end if

(ii) According to Lemma 5.1 there are no associated real primes of $\mathcal{W}$ if and only if $\mathbf{V}_{\mathbb{R}}(W) = \emptyset$. We know from Lemma 4.5 that any associated prime ideal $\mathfrak{n}_i$ of $\mathcal{W}$ is real if and only if $S^{-1}\mathfrak{n}_i$ is real. This means $\mathbf{V}_{\mathbb{R}}(W) = \emptyset$ if and only if any associated prime $S^{-1}\mathfrak{n}_i$ of $S^{-1}\mathcal{W}$ is not real. Now (ii) follows from Theorem 4.4.

(iii) We have a minimal primary decomposition of W in $\mathbb{Q}[\mathbf{y}]$:

$$W = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k.$$

Note that this decomposition is unique because every component is isolated. Now for $i = 1, \ldots, k$ let $(\mathfrak{q}_i)_{\mathbb{R}} = \cap \mathfrak{q}_{i,j}$ be a minimal primary decomposition of $(\mathfrak{q}_i)_{\mathbb{R}}$ in $\mathbb{R}[\mathbf{y}]$ with $\mathfrak{n}_{i,j} := \sqrt{\mathfrak{q}_{i,j}}$ maximal in $\mathbb{R}[\mathbf{y}]$. Then, according to Lemma 2.13 we have the following minimal primary decomposition in $\mathbb{R}[\mathbf{y}]$:

$$W_{\mathbb{R}} = \bigcap_{i,j} \mathfrak{q}_{i,j}. \tag{5.4}$$

Now suppose $|\mathbf{V}_{\mathbb{R}}(W)| = |\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_1)| = 1$ and $\sqrt{\mathfrak{q}_1} = \mathfrak{q}_1$. Thus, $(\mathfrak{q}_1)_{\mathbb{R}}$ is radical and consequently $\mathfrak{q}_{1,j} = \mathfrak{n}_{1,j}$, for all j. Since $|\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_1)| = 1$ there exists j_0, such that $|\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_{1,j_0})| = 1$. But all $\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_{i,j})$ are disjoint and $|\mathbf{V}_{\mathbb{R}}(W_{\mathbb{R}})| = 1$, hence $\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_{i,j}) = \emptyset$, for $(i,j) \neq (1, j_0)$. Then, according to Lemma 5.1 $\mathfrak{n}_{1,j_0}$ is the only real prime associated to $W_{\mathbb{R}}$ and $W_{\mathbb{R}}$ is $\mathfrak{n}_{1,j_0}$-reduced. With Lemma 4.5 we see that there is exactly one real prime $\mathfrak{n}$ associated to $S^{-1}\mathcal{W}$ and $S^{-1}\mathcal{W}$ is $\mathfrak{n}$-reduced. Hence, with Theorem 4.4 we deduce that the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$.

Suppose to the contrary that the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$. According to Theorem 4.4 there is exactly one associated real prime ideal $\mathfrak{n}$ of $S^{-1}\mathcal{W}$ and

$S^{-1}\mathcal{W}$ is $\mathfrak{n}$-reduced. Thus, on account of Lemma 4.5 there is exactly one $\mathfrak{q}_{i,j}$, say $\mathfrak{q}_{1,1}$ in (5.4) with $\mathfrak{q}_{1,1} = \mathfrak{n}_{1,1}$ real. This already shows $|\mathbf{V}_{\mathbb{R}}(W)| = |\mathbf{V}_{\mathbb{R}}(W_{\mathbb{R}})| = 1$ and $|\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_1)| = 1$. We still need to check that $\mathfrak{q}_1 = \sqrt{\mathfrak{q}_1}$. Suppose $\mathfrak{q}_1 \neq \sqrt{\mathfrak{q}_1}$. Then, with Lemma 4.3

$$\dim_{\mathbb{C}}\bigl(\mathbb{C}[\mathbf{y}]/(\mathfrak{q}_1)_{\mathbb{C}}\bigr)_{\mathfrak{c}} > 1, \tag{5.5}$$

for all $\mathfrak{c} \leq \mathbb{C}[\mathbf{y}]$ maximal with $(\mathfrak{q}_1)_{\mathbb{C}} \subset \mathfrak{c}$. But $\mathfrak{c}' := (\mathfrak{n}_{1,1})_{\mathbb{C}}$ contains $(\mathfrak{q}_1)_{\mathbb{C}}$ and is maximal because $\mathfrak{n}_{1,1}$ is real and maximal (Lemma 2.16). According to Lemma 4.3 we now have

$$\dim_{\mathbb{C}}\bigl(\mathbb{C}[\mathbf{y}]/(\mathfrak{q}_1)_{\mathbb{C}}\bigr)_{\mathfrak{c}'} = 1,$$

since $(\mathfrak{q}_1)_{\mathbb{R}}$ is $\mathfrak{n}_{1,1}$-reduced. This is a contradiction to (5.5), so $\mathfrak{q}_1 = \sqrt{\mathfrak{q}_1}$. □

Algorithm 5.3 Check if the origin is a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$.

Input: $I \leq \mathbb{Q}[\mathbf{x}]$ with $\dim I = 1$ and $I \leq \langle \bar{x} \rangle = \mathfrak{m}$.
Output: 1 if the origin is manifold point of X, 2 if the origin is isolated, 0 else.

$I = \sqrt{I}$.
Calculate normalization $\psi\colon \mathbb{Q}[\mathbf{x}]/I \to \mathbb{Q}[\mathbf{y}]/J$.
$W := \pi^{-1}\bigl(\langle \psi(\mathfrak{m}) \rangle\bigr)$, with $\mathfrak{m} = \langle \mathbf{x} \rangle$, $\pi\colon \mathbb{Q}[\mathbf{y}] \to \mathbb{Q}[\mathbf{y}]/J$.
$k := |\mathbf{V}_{\mathbb{R}}(W)|$.
if $k = 0$ **then**
    return 2.
else if $k = 1$ **then**
    Calculate minimal primary decomposition $W = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_k$.
    for all $\mathfrak{q}_i$ **do**
        if $|\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_i)| = 1$ **then**
            if $\mathfrak{q}_i = \sqrt{\mathfrak{q}_i}$ **then**
                return 1.
            else
                return 0.
            end if
        end if
    end for
else
    return 0.
end if

Example 5.1. Let us review some curves from the introduction and Example 4.2 with Proposition 5.3 and Algorithm 5.3. For this we can work with any CAS with a normalization algorithm for polynomial rings. We will use `Singular` with the libraries `normal.lib` and `primdec.lib` [15,16,29,30,62] for normalization and primary decomposition.

(i) $f_4(x,y) = x^2y + y^5 - x^6$. This is the curve from Figure 1.6. Consider the following execution in `Singular`:

```
LIB "normal.lib";
ideal I = x^2*y + y^5 - x^6;
def nor = normal(I);
def S = nor[1][1];
setring S;
ideal W = norid + ideal(x,y);
primdecGTZ(W);
```

Listing 5.1: Manifold test for $x^2y + y^5 - x^6$.

This gives the output

```
[1]:
    [1]:
        _[1]=T(2)
        _[2]=T(1)^2+1
        _[3]=y
        _[4]=x
    [2]:
        -- same
```

```
[2]:
    [1]:
        _[1]=T(2)-1
        _[2]=T(1)
        _[3]=y
        _[3]=x
     [2]:
        -- same
```

For $A = \mathbb{Q}[x,y]/\langle f_4 \rangle$ `Singular` calculates $\overline{A} = \mathbb{Q}[x,y,T_1,T_2]/J$ with the normalization map ψ induced by $x \mapsto x$, $y \mapsto y$. In `Singular` J is referenced by the handle `norid`. W from Proposition 5.3 is the preimage of $\psi(\langle x,y \rangle)$ in $\mathbb{Q}[x,y,T_1,T_2]$, so

$$W = J + \langle x,y \rangle.$$

From the output above we deduce the minimal primary decomposition

$$W = \langle T_2, T_1^2+1, y, x \rangle \cap \langle T_2 - 1, T_1, y, x \rangle =: \mathfrak{q}_1 \cap \mathfrak{q}_2,$$

We immediately see that $|\mathbf{V}_{\mathbb{R}}(W)| = |\mathbf{V}_{\mathbb{R}}(\mathfrak{q}_2)| = 1$ and $\mathfrak{q}_2$ is maximal. Hence, the origin must be a manifold point of $\mathbf{V}_{\mathbb{R}}(f)$ according to Proposition 5.3.

(ii) $f_4(x,y) = y^2 - y^3 + x^4$. This is item (iv) of Example 4.2. Listing 5.1 with the second line replaced by `ideal I = y^2 - y^3 + x^4` gives the following output:

```
[1]:
    [1]:
        _[1]=T(1)^2+1
        _[2]=y
        _[3]=x
    [2]:
        -- same
```

As in (i) we deduce $W = \langle T_1^2 + 1, y, x \rangle \leq \mathbb{Q}[x,y,T_1]$. Now $\mathbf{V}_{\mathbb{R}}(W) = \emptyset$. Thus the origin is an isolated point of $\mathbf{V}_{\mathbb{R}}(f_4)$.

(iii) $f_5(x, y) = x^2 - y^5$. This is item (v) of Example 4.2. Listing 5.1 with the second line replaced by `ideal I = x^2 - y^5` gives the following output:

```
[1]:
   [1]:
      _[1]=T(1)^2
      _[2]=y
      _[3]=x
   [2]:
      _[1]=T(1)
      _[2]=y
      _[3]=x
```

As before we deduce $W = \langle T_1^2, y, x\rangle \leq \mathbb{Q}[x, y, T_1]$. W is a primary ideal with $|\mathbf{V}_\mathbb{R}(W)| = 1$, but $\sqrt{W} \neq W$. Thus, the origin is not a manifold point of $\mathbf{V}_\mathbb{R}(f_5)$.

(iv) $f_6(x, y) = y^3 - y^4 + yx^4 - x^6$. This is item (iv) of Example 4.2. Listing 5.1 with the second line replaced by `ideal I = y^3 - y^4 + y*x^4 - x^6` gives the following output:

```
[1]:
   [1]:
      _[1]=T(2)^3-T(2)^2-2*T(2)-1
      _[2]=y
      _[3]=x
      _[4]=-T(2)^2+T(1)+T(2)+1
   [2]:
      -- same
```

As before we deduce $W = \langle T_2^3 - T_2^2 - 2T_2 - 1, y, x, -T_2^2 + T_1 + T_2 + 1\rangle \leq \mathbb{Q}[x, y, T_1, T_2]$. We set $q(T_2) = T_2^3 - T_2^2 - 2T_2 - 1$, which has discriminant

$$\Delta = -36 - 4 + 4 + 32 - 27 = -31 < 0.$$

Hence the cubic q has one real and two non-real complex conjugate roots. This means $|\mathbf{V}_\mathbb{R}(W)| = 1$. Since W is also prime, the origin must be a manifold point of $\mathbf{V}_\mathbb{R}(f_6)$.

Chapter 6

Kinematics

In this chapter we are going to apply the theory from Chapters 3 and 4 to the configuration space of some well-known linkages. For this it will be necessary to calculate Gröbner bases of ideals in $\mathbb{Q}[\mathbf{x}]$. We will mainly rely on the CAS `Singular` [16] for our computations. All employed `Singular` code should be attached to the digital version of this thesis and can be accessed by the attachment dialogue of your PDF-reader. The filename of each code file is stated in the caption the corresponding listing. In Section A.3 you will also find a link to an online repository, where the entire `Singular` code can be downloaded.

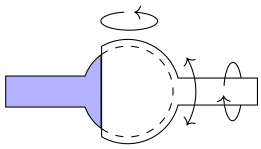
6.1 Representation of configuration spaces

Before we begin to investigate CS-singularities we want to show that any linkage configuration space can be represented by a real algebraic set. This will formalize the notion of a linkage. The following account is close in nature to the approach of [17] and [52].

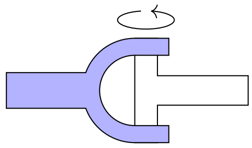
We start with a linkage L consisting of k rigid bodies which are called links. Any joint connecting two links is in general a **lower-order pair** [68,75]. These are surfaces in $\mathbb{R}^3$ invariant under the action of non-finite lie-subgroups of $\mathrm{SE}(3,\mathbb{R})$. A pair of those surfaces can be put together and move without losing surface contact. For example, a sphere is invariant under the action of $\mathrm{SO}(3,\mathbb{R})$ and would give a spherical joint in a linkage, similar to the shoulder joint in the human body, see Figure 6.1a. For a second example take an infinite hollow beam with non-circular cross-section. It is invariant under the action of a group of linear motions $\mathbb{R} \lesssim \mathrm{SE}(3,\mathbb{R})$. This corresponds to a prismatic joint, see Figure 6.1c.

In this thesis we only consider revolute, prismatic and spherical joints, corresponding to the lie-subgroups $\mathrm{SO}(2,\mathbb{R})$, $\mathbb{R}$ and $\mathrm{SO}(3,\mathbb{R})$. In Figure 6.1 – which is a copy of Figure 1.7 – we have illustrated these joints and their allowed motions.

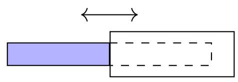
There are three more lower-order pairs: Planar, cylindrical and helical joints. However, helical joints correspond to screw motion subgroups of $\mathrm{SE}(3,\mathbb{R})$ which are not algebraic [68] and planar and cylindrical joints are very rare in applications.



(a) Spherical joint.



(b) Revolute joint.



(c) Prismatic joint.

Figure 6.1: Three types of joints, corresponding to the lie-subgroups $\mathrm{SO}(3,\mathbb{R})$, $\mathrm{SO}(2,\mathbb{R})$ and $\mathbb{R}$.

Now we return to the configuration space of the linkage L. Assume two links of L are connected by a joint f with associated lie-subgroup H. Then, all configurations $(g_1, g_2) \in \mathrm{SE}(3,\mathbb{R}) \times \mathrm{SE}(3,\mathbb{R})$ of the two links obey $g_1\, h = g_2$, $h \in h_1 H h_2$, where $h_1, h_2 \in \mathrm{SE}(3,\mathbb{R})$ are fixed rigid motions depending on the joint position and orientation in relation to the connected links. This means, the constraint by this joint can be expressed as

$$g_1^{-1} \cdot g_2 \in h_1 H h_2.$$

Since $(g_1, g_2) \mapsto g_1^{-1} \cdot g_2$ is a regular mapping, the set

$$X_f = \{(g_1, \ldots, g_k) \in \mathrm{SE}(3,\mathbb{R})^k \mid g_1^{-1} \cdot g_2 \in h_1 H h_2\,\}$$

is algebraic as long as H is an algebraic subgroup of $\mathrm{SE}(3,\mathbb{R})$. As the configuration space X of L is the intersection of the X_f for all joints f in L, it must be a real algebraic set.

In the kinematics community many approaches exist to derive algebraic or analytic equations for X, depending on specific parametrizations of $\mathrm{SE}(3,\mathbb{R})$. We can divide them roughly in two groups. One basic idea is to work with study-coordinates in a double-quaternion representation [36, 75]. A second method is to use the lie-algebra $\mathfrak{se}(3,\mathbb{R})$ to parameterize $\mathrm{SE}(3,\mathbb{R})$ working with the exponential mapping and canonical coordinates of the second kind [52, 55]. Since we are only interested in geometric properties of the configuration space we will mostly use an ad hoc derivation like we have seen in equation (1.1) of Section 1.1.

Another concept to formalize configuration spaces is the embedding of graphs in euclidean space [6, 21, 39] which also leads to the notion of **frameworks** [1]. But in general this does not take into account prismatic joints and non-plane rotational joints.

6.2 The four-bar linkage

We proceed to analyze the configuration space of the four-bar linkage discussed in the introduction. A sketch of the four-bar in two different configurations can be seen in Figure 6.2 which is a copy of Figure 1.9.

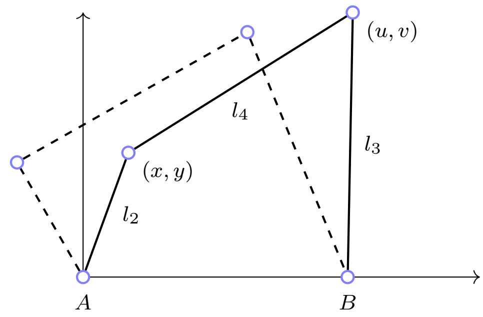


Figure 6.2: The four-bar linkage.

A fairly recent result from differential topology [21, Theorem 1.6] implies that all singularities (if there are any) of a planar closed kinematic chain with revolute joints are non-manifold points. In the introduction we included a short proof of this result (using Morse theory) for a special four-bar linkage. Now we want to give a different proof for the whole class of four-bars using only computational algebraic methods. This will demonstrate how Theorem 4.4 can be used.

We recall from Section 1.4 that the configuration space of the four-bar linkage can be represented by the real algebraic set $X = \mathbf{V}_{\mathbb{R}}(I)$, where $I = \langle p_1, p_2, p_3 \rangle \leq \mathbb{R}[x, y, u, v]$ is generated by the polynomials

$$\begin{aligned} p_1 &= x^2 + y^2 - l_2^2, \\ p_2 &= (u-2)^2 + v^2 - l_3^2, \\ p_3 &= (u-x)^2 + (v-y)^2 - l_4^2, \end{aligned}$$

and l_2, l_3, l_4 are the parameters of the four-bar which are assumed to be positive real numbers. Note the (completely arbitrary) choice $l_1 = 2$ for the length of the ground bar.

Dimension of I We assume $l_2 \neq 2$, $l_4 \neq 2$, since the complementary case can be analyzed in the same way. All following polynomial calculation will be carried out with the open-source CAS `Singular` [16].

First, we calculate a pseudo Gröbner basis B of I in $\mathbb{Q}(l_2, l_3, l_4)[x, y, u, v]$ (Definition 2.5) with respect to the product ordering `(dp(2),dp(2))` on $(v, y), (u, x)$ [31, Example 1.2.8]. The upper section of Listing 6.1 gives $B = \{g_1, \ldots, g_6\}$, where the leading terms of the g_i are:

$$\begin{aligned} \mathrm{LT}(g_1) &= -16\, u^2\, x, & \mathrm{LT}(g_4) &= y^2, \\ \mathrm{LT}(g_2) &= (-2\, l_2^2 + 8)\, v\, u, & \mathrm{LT}(g_5) &= 2\, v\, y, \\ \mathrm{LT}(g_3) &= -2\, v\, x^2, & \mathrm{LT}(g_6) &= v^2. \end{aligned}$$

```
ring r=(0,l2,l3,l4),(v,y,u,x),(dp(2),dp(2));

poly p1 = x^2 + y^2 - l2^2;
poly p2 = (u - 2)^2 + v^2 - l3^2;
poly p3 = (u - x)^2 + (v - y)^2 - l4^2;

ideal I = p1,p2,p3;
option(intStrategy);option(contentSB); //no divison normal form for parameters
std(I);
"---------------------------------";
ideal Is = subst(I,l3,l2 + l4 - 2); //grashof condition*/

ring r2=(0,l2,l4),(v,y,u,x),(dp(2),dp(2)); //no l3
ideal I = imap(r,Is);
ideal S = std(minor(jacob(I),3) + I);
list S_coef;
int num_coef = size(S);

//Leading Coefficients
for (int i=1; i <= size(S); i=i+1) {S_coef[i] = leadcoef(S[i]);}

ring s=0,(l2,l4),dp; //only parameters
list S_coef = imap(r2,S_coef);
for (i=1; i <= num_coef; i=i+1) { newline + "Factorize coefficient Nr." +
    ↪ string(i); factorize(simplify(S_coef[i],1)); }
```

Listing 6.1: Dimension and singular locus of $\mathbb{R}[x, y, u, v]/I$. #fourbar_dim

According to Proposition 2.28 B is a Gröbner basis of I as long as $l_2 \neq \pm 2$ which we required in the beginning. Thus, we get for the dimension of I:

$$\dim I = \dim\langle \mathrm{LM}(G)\rangle = \dim \langle u^2\, x, v\, u, v\, x^2, y^2, v\, y,\ v^2\rangle, \tag{6.1}$$

where $\mathrm{LM}(G)$ denotes the set of leading monomials of the polynomials in G. With a simple combinatorial argument [12, Proposition 9.1.3] we see that the dimension of the right ideal in (6.1) is 1 and consequently $\dim I = 1$.

Since I can be generated by the three elements p_1, p_2, p_3, $A := \mathbb{R}[x, y, u, v]/I$ must be equidimensional Cohen-Macaulay according to [20, Proposition 18.13] and the unmixedness theorem [20, Corollary 18.14].

Singular locus As stated in [63] there exist singular points in X if and only if the Grashof condition

$$\pm l_2 \pm l_3 \pm l_4 = 2$$

is fulfilled. We restrict our investigation to the case $l_2 - l_3 + l_4 = 2$ which means $l_3 = l_2 + l_4 - 2 > 0$. The other cases can be handled in a similar way. Since

$\dim I = 1$ equidimensional we can apply the Jacobian criterion in the form of [31, Theorem 5.7.1]. Then, the singular locus of $A = \mathbb{R}[x, y, u, v]/I$ is the set of all prime ideals in A containing $S \leq \mathbb{R}[x, y, u, v]$ generated by I and all the 3-minors of the Jacobian of (p_1, p_2, p_3). Executing the lower section of Listing 6.1 we get $S = \langle s_1, s_2, s_3, s_4 \rangle$ generated by the polynomials

$$\begin{aligned} s_1 &= q_1(l_2, l_4)\, x + c_1(l_2, l_4) \\ s_2 &= q_2(l_2, l_4)\, u + r_2(l_2, l_4)\, x + c_2(l_2, l_4) \\ s_3 &= q_3(l_2, l_4)\, y \\ s_4 &= q_4(l_2, l_4)\, v + f(l_2, l_4, x, y, u). \end{aligned}$$

We need to examine the coefficients $q_i \in \mathbb{Q}[l_2, l_5]$ of the leading monomials of all s_i to make sure that $\{s_1, s_2, s_3, s_4\}$ is a Gröbner basis of S. Our calculation in `Singular` shows that after simplification over $\mathbb{Q}$:

$$\begin{aligned} q_1(l_2, l_4) &= l_4^2 \cdot (l_2 + l_4 - 2)^2 \cdot (l_4 - 2) \cdot (l_2 + l_4) \cdot (l_2 + 2)^3 \cdot l_2 \cdot (3\, l_2 - 8), \\ q_2(l_2, l_4) &= l_4^2 \cdot (l_2 + 2\, l_4 - 2) \cdot (l_2 + l_4 - 2)^2 \cdot (l_2^2 - 4)^2, \\ q_3(l_2, l_4) &= l_4^2 \cdot (l_2 + l_4 - 2)^2 \cdot (l_2 + 2), \\ q_4(l_2, l_4) &= l_2^2 \cdot (l_2^2 - 4). \end{aligned}$$

Taking into account our assumptions

$$l_4, l_2 > 0, \quad l_2 + l_4 - 2 = l_3 > 0, \quad l_2 \neq 2,\ l_3 \neq 2$$

and in addition $l_2 \neq \frac{8}{3}$ (which we also need to check separately), we see that none of the q_i will vanish and $\{s_1, s_2, s_3, s_4\}$ forms a Gröbner basis of S for all valid parameters l_2, l_4. Then, clearly $\dim S = 0$ and since A is Cohen-Macaulay we can infer from [20, Theorem 18.15] that I must be a radical ideal.

Now we set $p = (l_2, 0, l_2 + l_4, 0) \in \mathbb{R}^4$. One confirms quickly by substitution that p fulfills the linear polynomials in S. So p is the only singularity of $\mathbf{V}(I)$.

Manifold Points To check with Theorem 4.4 whether p is a non-manifold point, we need to calculate the integral closure C of $A_{\mathfrak{n}}$, where $\mathfrak{n} = \langle x - l_2, y, u - l_2 - l_4, v \rangle$. One way to do this is the normalization algorithm of `Singular` [29,30] which we have used in Example 5.1 before. However, it has proven difficult to check the validness of the Gröbner base calculations in each step for the considered values of l_2, l_4. We could still analyze the situation for generic values of l_2, l_4 but we seek a statement for all admissible parameter values.

Instead, we will determine the strict transform $\pi\colon \tilde{Y} \to Y$ with respect to the blow up of affine 4-space at p, where $Y = \mathbf{V}(I)$. Since $\tilde{Y}$ will be nonsingular after one blow up, it must be the normalization of Y, see e.g. [13, Lemma 29.42.7].

So we first move the singularity $p = (l_2, 0, l_2 + l_4, 0)$ to the origin and consider $I_{\mathrm{bl}} = \langle p_1', p_2', p_3', b_1, b_2, b_3, b_4\rangle \leq \mathbb{R}[x, y, u, v, \hat{x}, \hat{y}, \hat{u}, \hat{v}]$ generated by

$$\begin{aligned}
p_1' &= p_1(x + l_2, y, u + l_2 + l_4, v) = x^2 + y^2 + 2l_2 x,\\
p_2' &= p_2(x + l_2, y, u + l_2 + l_4, v) = u^2 + v^2 + (2l_2 + 2l_4 - 4)u,\\
p_3' &= p_3(x + l_2, y, u + l_2 + l_4, v) = x^2 + y^2 - 2xu + u^2 - 2yv + v^2 - 2l_4 x + 2l_4 u,
\end{aligned}$$

and the homogeneous polynomials

$$\begin{aligned}
b_1 &= x\,\hat{y} - y\,\hat{x}, & b_4 &= y\,\hat{u} - u\,\hat{y},\\
b_2 &= x\,\hat{u} - u\,\hat{x}, & b_5 &= y\,\hat{v} - v\,\hat{y},\\
b_3 &= x\,\hat{v} - v\,\hat{x}, & b_6 &= u\,\hat{v} - v\,\hat{u}.
\end{aligned}$$

Now we go to the chart $\hat{y} = 1$ and get the isomorphic system

$$\begin{aligned}
p_1'' &= y \cdot (y\,\hat{x}^2 + y + (2l_2)\hat{x}),\\
p_2'' &= y \cdot (\hat{u}^2 + y\hat{v}^2 + (2l_2 + 2l_4 - 4))\hat{u},\\
p_3'' &= y \cdot (y\,\hat{x}^2 - 2y\hat{x}\hat{u} + y\hat{u}^2 + y\hat{v}^2 - 2y\hat{v} + y + (-2l_4)\hat{x} + (2l_4)\hat{u}).
\end{aligned}$$

We set $I_y := \langle p_1''/y, p_2''/y, p_2''/y\rangle \leq \mathbb{R}[\hat{x}, y, \hat{u}, \hat{v}]$. To get the equations of the strict transform on this chart, we need to factor out the exceptional divisor, so we have to calculate the saturation

$$J := (I_y : \langle y\rangle^\infty).$$

This can be easily achieved with the command `sat` in `Singular`. But again it is difficult to check whether the Gröbner basis calculations are correct for all assumed parameter values. Therefore, we compute the saturation manually using the algorithm of [31, 1.8.9]. The following steps can be carried out in `Singular` with Listing 6.2.

First, we determine $I_y \cap \langle y\rangle$ which we get by eliminating t of

$$I_y\,t + \langle (1 - t)y\rangle.$$

Then, we divide any generator of $I_y \cap \langle y\rangle$ by y. After checking that none of the coefficients of the leading monomials will be zero after valid parameter substitution, we can divide the polynomials each by its leading coefficient in $\mathbb{Q}[l_2, l_4]$ and get the Gröbner basis $B = \langle f_1, \ldots, f_7\rangle$ of $J = (I_y : \langle y\rangle)$ with

$$\begin{aligned}
f_1 &= \hat{u}^2\hat{x}^2 - \frac{2l_2 + 2l_4}{l_2 + 2}\hat{u}\,\hat{x}^3 + \frac{(l_2 + l_4)^2}{(l_2 + 2)^2}\hat{x}^4 + \frac{l_2^2 - 4l_2 + 4}{(l_2 + 2)^2}\hat{u}^2 + \frac{2\eta_1(l_2, l_4)}{(l_2 + 2)^2}\hat{u}\,\hat{x} + \frac{\eta_2(l_2, l4)}{(l_2 + 2)^2}\hat{x}^2,\\
f_2 &= y\hat{x}^2 + y + (2\,l_2)\hat{x},\\
f_3 &= y\hat{u}^2 - y\hat{u}\hat{x} + \frac{l_2^2 + 4l_2 + 4}{4}\hat{u}^2\hat{x} + \frac{\eta_2(l_2, l_4)}{2}\hat{u}\hat{x}^2 + \frac{(l_2 + l_4)^2}{4}\hat{x}^3,\\
f_4 &= \hat{v}\hat{x} + \frac{l_2 + 2}{2l_2}\hat{u}\hat{x}^2 - \frac{l_2 + l_4}{2l_2}\hat{x}^3 + \frac{2 - l_2}{2l_2}\hat{u} - \frac{l_2 + l_4}{2l_2}\hat{x},
\end{aligned}$$

```
LIB "poly.lib";
ring r=(0,l2,l4),(x,y,u,v),dp;

number l3 = l2 + l4 - 2;
poly p1 = x^2 + y^2 - l2^2;
poly p2 = (u - 2)^2 + v^2 - l3^2;
poly p3 = (u - x)^2 + (v - y)^2 - l4^2;

option(contentSB);option(intStrategy);
ideal I = p1,p2,p3;
map phi = r, x + l2, y, u + l2 + l4,v; I = phi(I);

//Blow up
ring s = (0,l2,l4),(x,y,u,v,xs,ys,us,vs),dp;
ideal I= imap(r,I);
ideal Ib= substitute(I,x,xs*y,u,us*y,v,vs*y);
ideal Iy = Ib/y;

//Saturation manually
ring s2 = (0,l2,l4),(t,vs,y,us,xs),(dp(1),dp(2),dp(2));
ideal Iy = imap(s,Iy);
ideal H = t*Iy + ideal((1-t)*y);
H = std(H); //groebner base for all valid parameters
ideal H2 = H[1..7];
H2 = H2/y; //repeat gives same ideal
ideal J = simplify(H2,1+2);
ideal fiber_points = J,y;
"-----points-in-fiber------";
std(fiber_points);

"-----blow-up-nonsingular--";
ring s3 = (0,l2,l4),(vs,xs,ys,us),dp;
ideal J = imap(s2,H2);
ideal sing = J + minor(jacob(J),3);
sing = std(sing);
size(sing);deg(sing[1]); //sing is constant in l2,l4
number c = leadcoef(sing[1]);

//c nonnull for l2,l4 valid?
ring rpar = 0,(l2,l4),dp;
poly c = imap(s3,c);
list fac = factorize(c);fac;
"fac[1][4] = 3*l2*(l2 + l4 - 2) + 2*l4 > 0";
```

Listing 6.2: Normalization via blow up. #fourbar_norm

$$f_5 = \hat{v}\hat{u} + \frac{3\eta_1(l_2,l_4)+4l_4}{(l_2-2)^2}\hat{v}\hat{x} + \frac{(l_2+2)^2}{2l_2(l_2-2)}\hat{u}^2\hat{x}^3 + \frac{\eta_2(l_2,l_4)}{l_2(l_2-2)}\hat{u}\hat{x}^4 + \frac{(l_2+l_4)^2}{2l_2(l_2-2)}\hat{x}^5 + \frac{3l_2^2+4}{2l_2(l_2-2)}\hat{u}^2\hat{x}$$
$$+ \frac{\eta_3(l_2,l4)-4l_2^2(l_2+l_4)}{l_2(l_2-2)^2}\hat{u}\hat{x}^2 + \frac{5l_2\eta_4(l_2,l_4)-2l_2l_4-2l4^2}{2l_2(l_2-2)^2}\hat{x}^3 + \frac{2\,(l_2+l_4)^2-4l_2-4l_4}{(l_2-2)^2}\hat{x},$$
$$f_6 = \hat{v}y + y\hat{u}\hat{x} + \frac{2-2l_2-l_4}{2l_2}y\hat{x}^2 + \frac{2-2l_2-l_4}{2l_2}y + (l_2-2)\hat{u} - (l_2-2)\hat{x},$$
$$f_7 = \hat{v}^2 + \frac{4-2l_2-2l_4}{l_2-2}\hat{v} + \hat{u}^2 + \frac{4-2l_2-2l_4}{l_2-2}\hat{u}\hat{x} + \frac{(l_2+l_4)(l_2+l_4-2)}{l_2(l_2-2)}\hat{x}^2 + \frac{(l_2+l_4)(l_2+l_4-2)}{l_2(l_2-2)},$$

where

$$\eta_1(l_2,l_3) = -l_2^2 - l_2l_4 + 2l_2 - 2l_4, \qquad \eta_2(l_2,l_3) = -l_2^2 - l_2l_4 - 2l_2 - 2l_4,$$
$$\eta_3(l_2,l_3) = 3\,l_2^2 - l_2l_4 + 2l_2 + 2l_4, \qquad \eta_4(l_2,l_3) = l_2^2 + l_4^2 + 2\,l_2l_4 - 2l_2 - 2l_4.$$

If we try to repeat this process we get the same ideal $J = (J : \langle y\rangle) = (I_y : \langle y\rangle^2)$, hence $J = (I_y : \langle y\rangle^\infty)$ is the ideal of the strict transform $\tilde{Y}$ of $Y = \mathbf{V}(I)$ on the chart $\hat{y} = 1$. Further, since the 3-minors of the Jacobian of $(f_1, \ldots, f_7)$ generate $\mathbb{R}[\hat{x}, y, \hat{u}, \hat{v}]$, $\tilde{Y}$ must be nonsingular.

Now we need to identify all points q in the fiber of the origin under π. So we calculate a Gröbner basis $B' = \{g_1, g_2, g_3, g_4\}$ of $J + \langle y\rangle$, where

$$g_1 = (2\,l_2)\,\hat{x},$$
$$g_2 = (l_2-2)\,\hat{u} + (-l_2+2)\,\hat{x},$$
$$g_3 = y,$$
$$g_4 = \hat{v}^2 + \frac{-2l_2-2l_4+4}{l_2-2}\,\hat{v} + \hat{u}^2 + \frac{-2l_2-2l_4+4}{l_2-2}\,\hat{u}\hat{x}$$
$$+ \frac{l_2^2+2l_2l_4-2l_2+l_4^2-2l_4}{l_2^2-2l_2}\hat{x}^2 + \frac{l_2^2+2l_2l_4-2l_2+l_4^2-2l_4}{l_2^2-2l_2}.$$

Then, we substitute $\hat{x} = 0$, $\hat{u} = 0$ from g_1, g_2 into g_4 and multiply with $(l_2^2 - 2\,l_2)$. We get

$$g_4'(\hat{v}) = (l_2^2 - 2\,l_2)\,\hat{v}^2 - l_2\,(2\,l_2 - 2\,l_4 + 4)\hat{v} + (l_2^2 + 2l_2l_4 - 2l_2 + l_4^2 - 2l_4).$$

g_4' is a quadratic equation in $\hat{v}$ with discriminant

$$8\,l_2\,l_4\,(l_2 + l_4 - 2) = 8\,l_2\,l_3\,l_4 > 0.$$

Consequently, there are two real points lying over the origin in the chart $\hat{y} = 1$. It follows with Theorem 4.4 that $p = (l_2, 0, l_2 + l_4, 0)$ is not a manifold point of X.

6.3 The five-bar linkage

The configuration space of the five-bar linkage will be two-dimensional, so we cannot use Theorem 4.4 again. However, one tool for higher-dimensional varieties is Efroymson's criterion, as we have mentioned in Section 3.3. We recall:

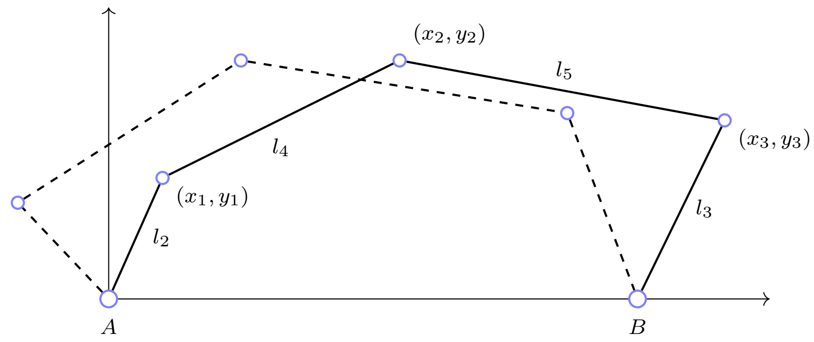


Figure 6.3: The five-bar linkage.

Theorem 3.8 (Efroymson [19]). *Let $I \leq \mathbb{R}[\mathbf{x}]$ be a real prime and $(\mathbb{R}[\mathbf{x}]/I)_{\langle \mathbf{x} \rangle}$ integrally closed. $I \cdot \mathbb{R}[[\mathbf{x}]]$ is real, if and only if the origin is contained in the euclidean closure of the nonsingular points of $\mathbf{V}_{\mathbb{R}}(I)$.*

If $X = \mathbf{V}_{\mathbb{R}}(I)$ is a normal real algebraic variety, this criterion can often be used to show that a singular point $p \in X$ is a non-manifold point, since it is enough to prove that $I_p \cdot \mathbb{R}[[\mathbf{x}]]$ is real, where I_p is the translated ideal such that p becomes the new origin.

The restriction of Theorem 3.8 to irreducible algebraic sets is usually not a problem since normal algebraic sets ordinarily have disjoint irreducible components, see Hartshorne's connectedness theorem [20, Theorem 18.12]. If X is not normal we have the option to calculate a normalization $f \colon Z \to \mathbf{V}(I)$ and proceed with Theorem 3.12 and Efroymson's criterion. For example, to prove that the origin is not a manifold point it would be enough to find two real points in the fiber of the origin under f which are not isolated in the nonsingular real locus of Z.

The best case for us is if X is a normal algebraic set with isolated singularities. Fortunately, this often occurs for linkage configuration spaces of dimensions greater than one. In this case Efroymson's criterion with Corollary 3.7 is very effective.

We want to demonstrate this argument on the configuration space X of the five-bar linkage from Figure 6.3. To derive equations for X we express the length constraints in euclidean coordinates. Then, we have $X = \mathbf{V}_{\mathbb{R}}(I)$, where $I = \langle p_1, \ldots, p_4 \rangle \leq \mathbb{R}\left[\{x_i, y_i \mid i = 1, \ldots, 3\}\right]$ with the polynomials

$$\begin{aligned}
p_1 &= x_1^2 + y_1^2 - l_2^2, \\
p_2 &= (x_3 - 2)^2 + y_3^2 - l_3^2, \\
p_3 &= (x_1 - x_2)^2 + (y_1 - y_2)^2 - l_4^2, \\
p_4 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 - l_5^2.
\end{aligned}$$

As in the case of the four-bar we assume that l_2, l_3, l_4, l_5 are positive real numbers and fix $l_1 = 2$ for the length of the ground bar. The last choice is arbitrary however.

```
ring r=(0,l2,l3,l4,l5),(x1,y1,x2,y2,x3,y3),dp;

ideal I = x1^2 + y1^2 - l2^2,(x3 - 2)^2 + y3^2 - l3^2, (x1 - x2)^2 + (y1 - y2)^2 -
    ↪ l4^2, (x2 - x3)^2 + (y2 - y3)^2 - l5^2;
option(contentSB);
option(intStrategy);

std(I);dim(std(I));
```

Listing 6.3: Dimension of five-bar configuration space. #fivebar_dim

Dimension of I In Listing 6.3 we use the `Singular` options `contentSB` and `intStrategy` to calculate a pseudo Gröbner basis B of I in $\mathbb{R}(l_2, l_3, l_4, l_5)[\{x_i, y_i\}]$, see Section 2.11. Then, we get

$$B = \{q_1, \ldots, q_{10}\}.$$

The coefficients of the leading monomials of the q_i do not depend on (l_2, l_3, l_4, l_5), hence B is a Gröbner basis of I for all parameter values. Thus, we see $\dim I = \dim\langle \mathrm{LM}(B)\rangle = 2$, for all $l_2, l_3, l_4, l_5 \in \mathbb{R}$.

Singular locus First, we note that $A = \mathbb{R}[\{x_i, y_i\}]/I$ is Cohen-Macaulay since $\dim I = 2$ and $I = \langle p_1, \ldots, p_4\rangle$ can be generated by 4 elements [20, Proposition 18.13]. In particular, this means that I must be equidimensional by the unmixedness theorem [20, Corollary 18.14].

Now we can apply the Jacobian criterion [31, Corollary 5.7.5]. The singular locus of A is given by all prime ideals containing the sum of I and J, where J is the ideal generated by all 4-minors of the Jacobian of (p_1, p_2, p_3, p_4). Since the calculation of a pseudo Gröbner basis of $I+J$ is unfeasible, we will first decompose the parameter-free ideal J with the factorizing Gröbner base algorithm, see Listing 6.4. `facstd(J)` in `Singular` gives seven ideals $J_1, \ldots, J_7$ such that $\sqrt{J} = \sqrt{J_1 \cap \ldots \cap J_m}$. Then

$$\begin{aligned}\sqrt{I+J} &= \sqrt{\sqrt{I} + \sqrt{J}} = \sqrt{\sqrt{I} + \left(\sqrt{J_1} \cap \ldots \cap \sqrt{J_7}\right)} \\ &= \sqrt{I+J_1} \cap \sqrt{I+J_2} \cap \ldots \cap \sqrt{I+J_7},\end{aligned} \tag{6.2}$$

where the last equation follows because intersection is distributive over addition inside a radical. We set $K_r = J_r + I$, for $r = 1, \ldots, 7$.

Before we proceed to investigate the singular locus of A we fix one of the parameters. According to [63] a necessary requirement for singularities in X is the Grashof-condition:

$$\pm l_2 \pm l_3 \pm l_4 \pm l_5 = 2.$$

We will only consider $l_2 = 2+l_3-l_4-l_5$ since all other cases can be treated analogously.

```
LIB "grobcov.lib";
ring r=(0,l3,l4,l5),(x1,y1,x2,y2,x3,y3),dp;

number l2 = 2 + l3 - l4 - l5;
ideal I = x1^2 + y1^2 - l2^2,(x3 - 2)^2 + y3^2 - l3^2, (x1 - x2)^2 + (y1 - y2)^2 -
    ↪ l4^2, (x2 - x3)^2 + (y2 - y3)^2 - l5^2;
ideal J = minor(jacob(I),4);
list L = facstd(J);

list sing; list GL;
for (int k=1; k<=size(L); k=k+1){
    //J_k = insert(sing,I+L[k]);
    //Calculate Groebner Cover of J_k
    GL = insert(GL,grobcov(I +
        ↪ L[k],("nonnull",ideal(l3*l4*l5*(2+l3-l4-l5))),("ext",1)));
}

int j;
for (k=1; k<=size(L); k++) {
    "K" + string(k);
    "------------";
    for (j=1; j<=size(GL[k]); j++) {
        "Segment " + string(j);
        "Dimension " + string(dim(std(GL[k][j][1])));
        "";
    }
}
```

Listing 6.4: Gröbner covers of K_r. #fivebar_cov

Now we continue with our analysis of K_r, for $k = 1, \dots, 7$. Unlike our previous approach with pseudo Gröbner bases, this time we work with the grobcov algorithm to calculate a Gröbner cover of all K_r [42,50,51]. Recall from Definition 2.6 that a Gröbner cover of K_r is a sequence of pairs (S_i, B_i), where the S_i are disjoint locally closed segments of the parameter space, i.e. $S_i = \mathbf{V}(T_{i,1}) \setminus \mathbf{V}(T_{i,2})$, for $T_{i,1}, T_{i,2} \leq \mathbb{Q}[l_3, l_4, l_5]$, and $B_i \subset \big(\mathbb{Q}[l_3, l_4, l_5]\big)[\{x_i, y_i\}]$ is a Gröbner basis of K_r for any specialization $B_i(a)$, $a \in S_i$. Furthermore, the union of all S_i is the whole parameter space or a predefined locally closed subset thereof.

Due to our assumption that all l_i are positive real numbers, we can restrict the parameter space in the grobcov algorithm to $\mathbb{R}^3 \setminus \mathbf{V}_{\mathbb{R}}\big(l_3\, l_4\, l_5\, (2+l_3-l_4-l_5)\big)$. Listing 6.4 shows that

$$K_r = (1), \text{ for } r \neq 5 \text{ and } \dim K_5 = 0, \tag{6.3}$$

for all parameters $(l_3\, l_4\, l_5) \in \mathbb{R}^3 \setminus \mathbf{V}_{\mathbb{R}}(l_3\, l_4\, l_5\, (2 + l_3 - l_4 - l_5))$ and $l_2 = 2 + l_3 - l_4 - l_5$. Please note that Listing 6.4 takes about one minute to complete on a system with an Intel Core i5-5300 CPU.

Manifold points From the previous discussion we can deduce the following facts for all $l_3, l_4, l_5 > 0$ and $l_2 = 2 + l_3 - l_4 - l_5$:

(a) The singular locus of $A = \mathbb{R}[\{x_i, y_i\}]/I$ is zero-dimensional on account of (6.2),(6.3).

(b) I is equidimensional and radical [20, Corollary 18.14, Theorem 18.15].

(c) A is a normal ring [20, Theorem 18.15].

(d) The components of $\mathbf{V}(I)$ are disjoint since the singular intersection of components would have codimension ≤ 1 according to Hartshorne's connectedness theorem [20, Theorem 18.13]. But this contradicts (a).

Now we conclude with (a),(d): For any point $p \in X$ and corresponding maximal ideal $\mathfrak{m}(p) = \langle x_1 - p_1, \ldots, x_n - p_n \rangle$, the local ring $A_{\mathfrak{m}(p)}$ is isomorphic to $\left(\mathbb{R}[\{x_i, y_i\}]/\tilde{I}\right)_{\mathfrak{m}(p)}$, for a prime ideal $\tilde{I}$ such that $\mathbb{R}[\{x_i, y_i\}]/\tilde{I}$ is normal and $p \in \mathbf{V}\big(\tilde{I}\big)$. According to the simple point criterion (Proposition 2.15), $\tilde{I}$ is real if and only if there exists a nonsingular point in $\mathbf{V}_{\mathbb{R}}\big(\tilde{I}\big)$. Thus, with Efroymson's criterion we recognize that $I_p \cdot \mathbb{R}[[\mathbf{x}]] = \tilde{I}_p \cdot \mathbb{R}[[\mathbf{x}]]$ is real if and only if p is not isolated in the set of nonsingular points of X. But the singular locus is zero-dimensional. This means that any singularity of X is either isolated in X or a non-manifold point.

It remains to show that the singularities of X are not isolated in X. This can be easily done with geometric arguments. However, for sake of brevity we will only analyze one of the segments of K_5.

After the execution of Listing 6.4, we can run `GL[5][1][2]` and `GL[5][1][3]`, resulting in:

```
> GL[5][1][2];
_[1]=y3                     _[4]=x2+(-l3+l5-2)
_[2]=x3+(-l3-2)             _[5]=y1
_[3]=y2                     _[6]=x1+(-l3+l4+l5-2)

> GL[5][1][3];
[1]:
   [1]:
      _[1]=0
   [2]:
      [1]:                  [6]:                  [11]:
         _[1]=(l3-l4-l5+2)     _[1]=(l3-l5)          _[1]=(l4-2)
      [2]:                  [7]:                  [12]:
         _[1]=(l3-l4-l5)       _[1]=(l5-2)           _[1]=(l4)
      [3]:                  [8]:                  [13]:
         _[1]=(l4+l5-2)        _[1]=(l5)             _[1]=(l3+2)
      [4]:                  [9]:                  [14]:
         _[1]=(l4+l5)          _[1]=(l3-l4+2)        _[1]=(l3)
      [5]:                  [10]:
         _[1]=(l3-l5+2)        _[1]=(l3-l4)
```

The list `GL[5][1][2]` contains the Gröbner basis for the segment of the parameter space which is given as the complement of

$$\bigcup_{i=1,\ldots,14} \mathbf{V}(f_i),$$

where the f_i are the 14 polynomials in `GL[5][1][3]`. We see that for positive real parameters l_2, l_3, l_4, l_5 with $l_2 = 2 + l_3 - l_4 - l_5$ and

$$l_2, l_4, l_5 \neq 2, \qquad l_4, l_5 \neq l_3, \qquad l_4 + l_5 \neq 2, \qquad l_5, l_4 \neq l_3 + 2, \tag{6.4}$$

there exists exactly one singularity in X:

$$p_0 = (x_1, y_1, x_2, y_2, x_3, y_3) = (2 + l_3 - l_4 - l_5, 0, 2 + l_3 - l_5, 0, 2 + l_3, 0).$$

p_0 corresponds to the folded configuration of Figure 6.4.

Suppose we break the connection between the joint at (x_3, y_3) and B in Figure 6.4. Then, the joint at (x_3, y_3) – connected to the joint at (x_1, y_1) by a 2R-chain – can reach at least any point in the annulus $D_1 = \operatorname{ann}\big((0, l_2), |l_5 - l_3|, l_5 + l_3\big)$. Thus, the intersection of D_1 with the disc of radius l_3 around B is nonempty and has the limit point $(2 + l_3, 0)$. Hence, p_0 is not isolated in X and must be a CS-singularity of the five-bar linkage.

For parameters other than in (6.4), one needs to check the remaining segments of I_5 which can be done in a similar way.

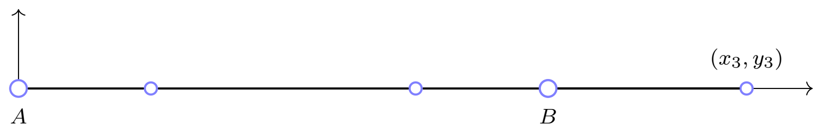


Figure 6.4: Folded configuration of the five-bar linkage.

6.4 The delta robot

The delta robot is a parallel linkage that consists of three identical limbs which carry a platform serving as Cartesian positioning device, see Figures 6.5, 6.7. It was developed 1985 by a research team under the supervision of Reymond Clavel who described it first in his Ph.D. thesis [11].

Applications of the delta robot include pick and place tasks like packaging and pre-assembly work. Recently, delta robots can also be found in many 3D-printers to carry the extruder. Unlike depicted in Figure 6.5 the actual platform is mounted upside down in almost all applications, see Figure 6.7.

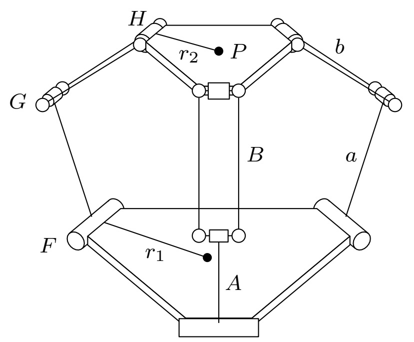


Figure 6.5: Illustration of the delta robot.

Figure 6.6: One arm of the delta robot. Front and side view in different configurations.

In the original design and most currently manufactured delta robots (Fanuc M1, ABB IRB 360, Yaskawa MPP3H) each limb comprises a solid upper arm A of length a and a parallelogram-linkage B with sides of length b, Figure 6.6. B is attached to the upper arm with a revolute joint G. This allows the tip H of the parallelogram to travel on a spherical surface around G.

Finally, the complete limb is connected to the base and the moving platform with revolute joints F and H. Both the joint-connections to the base and to the moving platform of each limb are placed at the vertices of equilateral triangles with apothems r_2 and r_1, respectively.

In applications the base joints F will be actuated (motor-driven). The composition with the parallelogram-linkages forces the platform to always maintain the same orientation. This means that the platform will perform translational movements under actuation of the base joints.

We follow the discussion in [74, Section 3.6] to give equations for the configuration space of the delta robot. Depending on the parameters a, b and $d \coloneqq r_1 - r_2$ we set:

$$
\begin{aligned}
Y_{a,b,d} &= \mathbf{V}(\{s_i, c_j, l_k \mid i,j = 1,2,3 \text{ and } k = 1,\dots,6\}) \subset \mathbb{C}^{15},\\
X_{a,b,d} &= \mathbf{V}_{\mathbb{R}}(\{s_i, c_j, l_k \mid i,j = 1,2,3 \text{ and } k = 1,\dots,6\}) = Y_{a,b,d} \cap \mathbb{R}^{15},
\end{aligned}
$$

Figure 6.7: A delta robot model from Tosy Robotics.[1]

where the polynomials s_i, c_j, l_k are defined as follows:

$$\begin{aligned} s_1 &:= x_1^2 + y_1^2 + z_1^2 - b^2, & c_1 &:= ca_1^2 + sa_1^2 - a^2, \\ s_2 &:= x_2^2 + y_2^2 + z_2^2 - b^2, & c_2 &:= ca_2^2 + sa_2^2 - a^2, \\ s_3 &:= x_3^2 + y_3^2 + z_3^2 - b^2, & c_3 &:= ca_3^2 + sa_3^2 - a^2, \end{aligned} \tag{6.5}$$

and

$$(l_1, l_2, l_3)^T := v_1 - A\, v_2, \qquad (l_4, l_5, l_6)^T := v_1 - A^{-1}\, v_3, \tag{6.6}$$

with

$$v_i := \begin{pmatrix} d + ca_i + x_i \\ y_i \\ z_i + sa_i \end{pmatrix}, \qquad A := \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{SO}(3, \mathbb{R}).$$

The polynomials s_i,c_j,l_k depend on the 15 variables

$$\mathbf{w} = (x_1, y_1, z_1, x_2, y_2, z_2, x_3, y_3, z_3, ca_1, sa_1, ca_2, sa_2, ca_3, sa_3).$$

Therefore

$$I_{a,b,d} := \left\langle \left\{ s_i, c_j, l_k \;\middle|\; i,j = 1,2,3 \text{ and } k = 1,\dots,6 \right\} \right\rangle \leq \left(\mathbb{Q}\left[\sqrt{3}\right] \right) [\mathbf{w}].$$

Values for (x_i, y_i, z_i) and (ca_i, sa_i) in $X_{a,b,d}$ are just coordinates of the direction vectors $\overrightarrow{FG}$ and $\overrightarrow{GH}$ corresponding to the ith limb in the coordinate system A^i, $i = 0, 1, 2$. See [74] for details.

[1] Picture licensed under GFDL [66].

Table 6.1: Singularities of the delta robot.[2]

Variable	q_1	q_2	q_3	q_4
x_1	$-\frac{db}{\sqrt{a^2+3\,d^2}}$	$\frac{db}{\sqrt{a^2+3\,d^2}}$	$\frac{db}{\sqrt{a^2+3\,d^2}}$	$-\frac{db}{\sqrt{a^2+3\,d^2}}$
y_1	$-\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$-\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$
z_1	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$
x_2	$-\frac{db}{\sqrt{a^2+3\,d^2}}$	$\frac{db}{\sqrt{a^2+3\,d^2}}$	$\frac{db}{\sqrt{a^2+3\,d^2}}$	$-\frac{db}{\sqrt{a^2+3\,d^2}}$
y_2	$\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$-\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$-\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{3}\,db}{\sqrt{a^2+3\,d^2}}$
z_2	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$
x_3	$\frac{2db}{\sqrt{a^2+3d^2}}$	$-\frac{2db}{\sqrt{a^2+3d^2}}$	$\frac{2bd(b\tau_1-2a^2+b^2+3d^2)}{2b\tau_2-\tau_1(a^2+b^2)}$	$-\frac{2bd(b\tau_1+2a^2-b^2-3d^2)}{2b\tau_2+\tau_1(a^2+b^2)}$
y_3	0	0	0	0
z_3	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$\frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}}$	$-\frac{b\sqrt{a^2-d^2}(2b\tau_1-\tau_4)}{-2b\tau_2+\tau_1(a^2+b^2)}$	$\frac{b\sqrt{a^2-d^2}(2b\tau_1+\tau_4)}{2b\tau_2+\tau_1(a^2+b^2)}$
ca_1	$-d$	$-d$	$-d$	$-d$
sa_1	$\sqrt{a^2-d^2}$	$-\sqrt{a^2-d^2}$	$-\sqrt{a^2-d^2}$	$\sqrt{a^2-d^2}$
ca_2	$-d$	$-d$	$-d$	$-d$
sa_2	$\sqrt{a^2-d^2}$	$-\sqrt{a^2-d^2}$	$-\sqrt{a^2-d^2}$	$\sqrt{a^2-d^2}$
ca_3	$-d$	$-d$	$\frac{6bd(a^2-d^2)(b-\tau_1)}{-2b\tau_1\tau_2+\tau_1^2(a^2+b^2)}-d$	$\frac{6bd(a^2-d^2)(b+\tau_1)}{2b\tau_1\tau_2+\tau_1^2(a^2+b^2)}-d$
sa_3	$\sqrt{a^2-d^2}$	$-\sqrt{a^2-d^2}$	$\frac{6bd^2\sqrt{a^2-d^2}(\tau_1+2b)}{-2b\tau_1\tau_2+\tau_1^2(a^2+b^2)}-\tau_3$	$\frac{6bd^2\sqrt{a^2-d^2}(\tau_1-2b)}{2b\tau_1\tau_2+\tau_1^2(a^2+b^2)}+\tau_3$

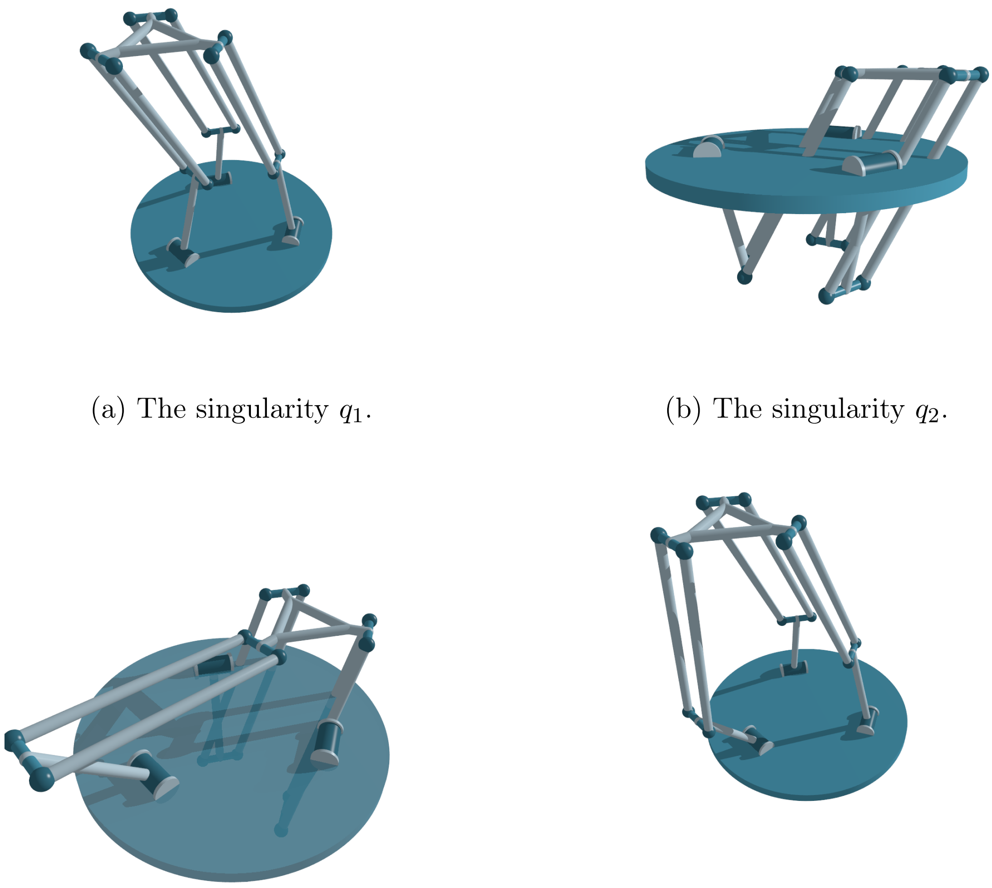
Note that with the presented equations we disregard the four-bar structure of the parallelogram-linkages. Hence, in the following analysis we will find only "inherent" configuration space singularities of the delta robot and no CS-singularities potentially induced by the four-bars in the limbs of the mechanism. See Section 6.2 for CS-singularities of the four-bar linkage.

Now let all the polynomials s_i, c_j, l_k be collected in a polynomial map $F\colon \mathbb{R}^{15} \to \mathbb{R}^{12}$. Then, we can formulate our main results:

Theorem 6.1. *Let $a, b, d \in \mathbb{R}\backslash\{0\}$, with $a^2 > d^2$. The dihedral group D_3 acts on $X_{a,b,d}$ which restricts to a group action on*

$$S_{a,b,c} := \{\, p \in X_{a,b,d} \mid \operatorname{rk} DF(p) < 12 \,\}.$$

[2]with abbreviations $\tau_1 = \sqrt{a^2 + 3d^2}$, $\tau_2 = a^2 - 3d^2$, $\tau_3 = \sqrt{a^2 - d^2}$ and $\tau_4 = a^2 + b^2 - 6d^2$.



(a) The singularity q_1. (b) The singularity q_2.

(c) The singularity q_3. (d) The singularity q_4.

Figure 6.8: Singular poses of the delta robot, $d > 0$.

Moreover, there exist 24 *points* $p_i \in S_{a,b,d}$ *and* D_3 *acts freely on* $P_{a,b,d} := \{ p_i \mid i = 1, \ldots, 24\}$. *Representatives of the four orbits in* $P_{a,b,d}/D_3$ *are given in Table 6.1 and illustrated in Figure 6.8.*

Theorem 6.2. *Let* O *be the Zariski open set of Definition 6.1 and* $(a, b, d) \in O$. *Then, we have* $\dim Y_{a,b,d} = 3$ *and* $S^{\mathbb{C}}_{a,b,d} = \{z \in Y_{a,b,d} \mid \operatorname{rk} DF(z) < 12\}$ *is zero-dimensional with* $|S^{\mathbb{C}}_{a,b,d}| \leq 24$.

Corollary 6.3 (Delta robot)**.** *Let* $(a, b, d) \in O$ *and* $a^2 > d^2$. *Then,* $Y_{a,b,d}$ *is a normal complex algebraic set of dimension* 3 *and there are* 24 *isolated singularities of* $Y_{a,b,d}$ *which are all real and non-manifold points of* $X_{a,b,d}$.

In particular, $S_{a,b,d} = P_{a,b,d}$ *is the whole set of CS-singularities of the delta robot.*

Remarks.

(1) We need to allow negative values for $d = r_1 - r_2$. But note that $r_1 > r_2$ in all commercially available delta robots. On the other hand, the parameters a, b represent physical lengths which cannot be negative. However, since a, b appear only squared in the equations (6.5), we permit negative values for all parameters a, b, d.

(2) Since O is Zariski-open, O is dense in $\mathbb{R}^3$. But the choice of O is for convenience in the proofs of Theorem 6.2 and Corollary 6.3. It is conjectured, that Theorem 6.2 and Corollary 6.3 hold for $a, b, d \in \mathbb{R}\backslash\{0\}$, $a^2 > d^2$. See also Section 7.2.

The special case $d = 0$ is of interest but not included here to shorten the analysis. See [74, p. 283] for some comments on this type of delta robot.

(3) We will see in the proof of Corollary 6.3 that $I_{a,b,d} \cdot \mathbb{R}[\mathbf{w}]$ is radical for $(a, b, d) \in O$ and any associated prime $\mathfrak{p}_i$ with $\mathbf{V}_{\mathbb{R}}(\mathfrak{p}_i) \neq \emptyset$ is real. Hence $X_{a,b,d}$ is a normal real algebraic set [7] of dimension 3. It is conjectured that $I_{a,b,d} \cdot \mathbb{R}[\mathbf{w}]$ is prime. An option to prove this might be the technique of [20, Section 18.3].

Definition 6.1. Let O be the Zariski open set $\mathbb{R}^3\backslash\mathbf{V}(f)$ with $f = \prod_{i=1}^{18} f_i$, where

$$
\begin{aligned}
f_1 &= a\,b\,d, & f_{10} &= a - b - d,\\
f_2 &= a^2 + 3\,d^2 - b\,a - 3\,b\,d, & f_{11} &= a + b + d,\\
f_3 &= a^2 + 3\,d^2 + b\,a + 3\,b\,d, & f_{12} &= a + b - d,\\
f_4 &= a^2 + 3\,d^2 + b\,a - 3\,b\,d, & f_{13} &= a - b + d,\\
f_5 &= a^2 + 3\,d^2 - b\,a + 3\,b\,d, & f_{14} &= a - d,\\
f_6 &= a^2 - 4\,b^2 + 3\,d^2, & f_{15} &= a + d,\\
f_7 &= a^2 + 3\,d^2, & f_{16} &= 3\,a^2 - 6\,a\,d - 4\,b^2 + 3\,d^2,\\
f_8 &= 3\,a^2 + 6\,a\,d - 4\,b^2 + 3\,d^2, & f_{17} &= a^6 - 2\,a^4\,b^2 + 3\,a^4\,d^2 + a^2\,b^4\\
f_9 &= a^2 - b^2 + 3\,d^2, & &\quad + 30\,a^2\,b^2\,d^2 + 3\,b^4\,d^2 - 36\,b^2\,d^4,\\
 & & f_{18} &= 9\,d^4 - 6\,a^2 d^2 - 3\,d^2 b^2 - b^2 a^2 + a^4.
\end{aligned}
$$

6.4.1 The proof of Theorem 6.1

We define the following faithful representation $\Psi\colon D_3 = \langle r, m\rangle \hookrightarrow \mathrm{GL}(15, \mathbb{R})$:

$$
r \mapsto \begin{pmatrix} \mathbf{0} & E & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{0} & E & & \mathbf{0} & \\ E & \mathbf{0} & \mathbf{0} & & & \\ & & & \mathbf{0} & e & \mathbf{0} \\ & \mathbf{0} & & \mathbf{0} & \mathbf{0} & e \\ & & & e & \mathbf{0} & \mathbf{0} \end{pmatrix}, \quad m \mapsto \begin{pmatrix} S & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & S & \mathbf{0} & & \mathbf{0} & \\ \mathbf{0} & \mathbf{0} & S & & & \\ & & & s & \mathbf{0} & \mathbf{0} \\ & \mathbf{0} & & \mathbf{0} & s & \mathbf{0} \\ & & & \mathbf{0} & \mathbf{0} & s \end{pmatrix},
$$

where

$$s := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

and E, e are the identity matrices in $\mathbb{R}^{3\times 3}$ and $\mathbb{R}^{2\times 2}$, respectively. Let Φ denote the induced action on $R = \mathbb{R}[\mathbf{w}]$, i.e. $\Phi(d)(f) = f(\Psi(d)\,\mathbf{w})$, for $d \in D_3$. We will show that $\Phi(d)(I_{a,b,d}) = I_{a,b,d}$. Then, D_3 acts on $X_{a,b,d} = \mathbf{V}_{\mathbb{R}}(I_{a,b,d})$ via Ψ.

First, recall that $I_{a,b,d} = \langle\{c_i, s_j, l_k\}\rangle$ and write π for the permutation $(123) \in S_3$. We have

$$\Phi(r)(s_i) = s_{\pi(i)},\ \Phi(r)(c_i) = c_{\pi(i)},\ \Phi(m)(s_i) = s_i,\ \Phi(m)(c_i) = c_i.$$

Now we consider the action of D_3 on $R^3 = \mathbb{R}[\mathbf{w}]\times\mathbb{R}[\mathbf{w}]\times\mathbb{R}[\mathbf{w}]$ componentwise. Then, we see $\Phi(r)v_i = v_{\pi(i)}$. Hence, we derive with $A^2 = A^{-1}$:

$$\begin{aligned} \Phi(r)(v_1 - A\,v_2) &= v_2 - A\,v_3 = A^{-1}(Av_2 - A^{-1}v_3) \\ &= -A^{-1}(v_1 - Av_2 - (v_1 - A^{-1}v_3)), \\ \Phi(r)(v_1 - A^{-1}\,v_3) &= v_2 - A^{-1}\,v_1 = -A^{-1}(v_1 - Av_2), \end{aligned}$$

where $(l_1, l_2, l_3)^T = v_1 - A\,v_2$ and $(l_4, l_5, l_6)^T = v_1 - A^{-1}\,v_3$ as in (6.6). Also

$$\begin{aligned} &\Phi(m)(l_i) = l_i,\ \text{for } i = 1, 2, 4, 5, \\ &\Phi(m)(l_3) = -l_3, \quad \Phi(s)(l_6) = -l_6. \end{aligned}$$

Consequently, $\Phi(d)(I_{a,b,d}) = I_{a,b,d}$ for all $d \in D_3$. Thus, we have an action of D_3 on $\mathbf{V}_{\mathbb{R}}(I_{a,b,d}) = X_{a,b,d}$.

Next, let $J \leq \mathbb{R}[\mathbf{w}]$ be the ideal of the principal minors of DF. We will check that D_3 acts on J via Φ. Choose any $d \in D_3$. With the chain rule we derive:

$$D\big(\Phi(d)(F)\big) = D\big(F(\Psi(d)\,\mathbf{w})\big) = DF(\Psi(d)\,\mathbf{w}) \cdot \Psi(d) = \Phi(d)\big(DF(\mathbf{w})\big) \cdot \Psi(d), \quad (6.7)$$

where we consider $F \in R^{12}$ and write $\Phi(d)$ for the componentwise actions on R^{12} and $R^{12\times 15}$.

For a tuple $T = (i_1, \ldots i_{12}) \in \mathbb{N}^{12}$ with $1 \leq i_1 \leq i_2 \leq \ldots i_{12} \leq 15$ and a matrix $M \in M^{12\times 15}$ we denote with $M_{(T)}$ the matrix comprising the T columns of M. Then, (6.7) shows $D\big(\Phi(d)(F)\big) \cdot \Psi(d)^{-1}_{(T)} = \Phi(d)\big(DF_{(T)}\big)$. Thus, because the action of $\Phi(d)$ on R respects the ring structure

$$\Phi(d)\ \det DF_{(T)} = \det\left(\Phi(d)DF_{(T)}\right) = \det\big(D(\Phi(d)F) \cdot \Psi(d)^{-1}_{(T)}\big) \qquad (6.8)$$

Since $\Phi(d)$ acts linearly on the $\mathbb{R}$-vector space generated by the polynomials s_i, c_j, l_k, we have $\Phi(d)(F) = A_d \cdot F$, for some $A_d \in \mathbb{R}^{12\times 12}$, hence

$$D\big(\Phi(d)(F)\big) = A_d \cdot DF. \qquad (6.9)$$

With (6.8) and (6.9) we get

$$\Phi(d) \det DF_{(T)} = \det A_d \cdot \det(DF \cdot \Psi(d)^{-1}_{(T)}).$$

Now we can use the Cauchy-Binet Formula and have

$$\Phi(d) \det DF_{(T)} = \sum_{\substack{L=(j_1,\ldots,j_{12})\in\mathbb{N}^{12} \\ 1\leq j_1<\ldots<j_{12}\leq 15}} \det DF_{(L)} \cdot \det \Psi(d)^{-1}_{(T)^{(L)}} \cdot \det A_d,$$

where $\Psi(d)^{-1}_{(T)^{(L)}}$ denotes the matrix comprising the L rows of $\Psi(d)^{-1}_{(T)}$. This shows $\Phi(d)(J) \subset J$. Therefore

$$\Psi(d)(S_{a,b,d}) = \Psi(d)\big(\mathbf{V}_{\mathbb{R}}(I_{a,b,d} + J)\big) \subset \mathbf{V}_{\mathbb{R}}(I_{a,b,d} + J) = S_{a,b,d}$$

and the action of D_3 on $X_{a,b,d}$ restricts to an action on $S_{a,b,d}$.

Finally, one checks easily that $\Psi(D_3)$ acts freely on the orbits generated by the four points q_1, q_2, q_3, q_4 of Table 6.1. To complete the proof it suffices to verify the following two assertions: The points q_1, q_2, q_3, q_4 are well defined and real for all $a, b, d \in \mathbb{R}\backslash\{0\}$ with $a^2 > d^2$. Furthermore, they fulfill all polynomials in $I_{a,b,d} + J$.

We can check the second statement easily with any CAS. See e.g. `minorcheck.py` (or `minorcheck_parallel.py` for a parallelized version) attached to the digital version of this thesis or from the online repository, Section A.3. So it remains to show the first statement. For q_1 and q_2 this is clear. Now we will investigate the denominators of the coordinates of q_3 and q_4.

Let $u := 2\,b\,\tau_2 + \tau_1\,(a^2 + b^2) \neq 0$. If we consider u as quadratic equation in b, its discriminant divided by 4 is

$$\tau_2^2 - s_1^2 a^2 = a^4 - 6\,d^2\,a^2 + 9d^4 - a^2\,(a^2 + 3\,d^2) = -9\,a^2\,d^2 + 9\,d^{\,4} < 0,$$

since $d \neq 0$ and $a^2 > d^2$. Thus, there exists no real zero for u. The same statement follows analogously for the other denominators. □

6.4.2 The proof of Theorem 6.2

First, we show $\dim Y_{a,b,d} = 3$ for $(a, b, d) \in \mathbb{R}^3$. We define

$$I^{s_3}_{a,b,d} \coloneqq \langle \{s_i, c_j, l_k \mid i, j = 1, \ldots, 3, k = 1, \ldots, 6\} \cup \{s_3^2 - 3\}\rangle \leq \mathbb{Q}[\mathbf{w}, s_3],$$

where we replace all occurrences of $\sqrt{3}$ in s_i, c_j, l_k with s_3. We also recall

$$I_{a,b,d} \coloneqq \langle \{s_i, c_j, l_k \mid i, j = 1, \ldots, 3, k = 1, \ldots, 6\}\rangle \leq (\mathbb{Q}[\sqrt{3}])[\mathbf{w}].$$

With Listing 6.5 (see Appendix A.1, for a listing of `delta.lib`) we calculate Gröbner cover of $I^{s_3}_{a,b,d}$ with the `grobcov` algorithm, Section 2.11. Since we have only one

```
LIB "delta.lib";
LIB "grobcov.lib";

def r = DeltaGetCsIdeal();
setring r;

list L = grobcov(I,("ext",1),("can",0)); //Calculate Groebner Cover
"Number of Segments:";
size(L); //--> gives 1

"Dimension of Configuration Space for all (a,b,d):";
attrib(L[1][2],"isSB",1);
dim(L[1][2]); // --> gives 3
```

Listing 6.5: Dimension of delta robot configuration space. #delta_dim

segment, this gives a reduced Gröbner basis G of $I^{s_3}_{a,b,d}$ for all $a, b, d \in \mathbb{R}$. Now we conclude with Proposition 2.29

$$\dim I^{s_3}_{a,b,d} = \dim I_{a,b,d} = \dim\langle \mathrm{LM}(G)\rangle = 3.$$

Therefore, it is

$$\dim Y_{a,b,d} = 3, \quad \text{for all } (a, b, d) \in \mathbb{R}^3.$$

Next, let $J \leq (\mathbb{Q}[\sqrt{3}])[\mathbf{w}]$ be the ideal of the 12-minors of DF. Note that J does not depend on (a, b, d). Since it will simplify the following computations significantly, we will determine the associated minimal primes of the ideal J beforehand. For this we use the algorithm by Gianni, Trager and Zacharias [27,31] implemented in `Singular`. Listing 6.6 gives

$$\sqrt{J} = J_1 \cap \ldots \cap J_{10},$$

with

$$\begin{aligned}
J_1 &:= \langle ca_3, sa_3\rangle,\\
J_2 &:= \langle sa_3\, ca_3 - ca_3\, ca_3, sa_2\, ca_2 - ca_2\, ca_2, 3\, ca_2\, ca_3 - \sqrt{3}\, ca_3\, ca_2 + \sqrt{3}\, ca_2\, ca_3 + 3\, ca_2\, ca_3,\\
&\qquad 3\, sa_2\, ca_3 + \sqrt{3}\, sa_2\, ca_3 + 2\sqrt{3}\, ca_3\, ca_2, 3\, ca_2\, ca_3 + 2\sqrt{3}\, ca_3\, ca_2 + \sqrt{3}\, ca_2\, ca_3,\\
&\qquad 3\, sa_3\, ca_2 - \sqrt{3}\, sa_3\, ca_2 - 2\sqrt{3}\, ca_2\, ca_3, ca_3\, ca_2 + 2\, ca_2\, ca_3 + \sqrt{3}\, ca_3\, ca_2\rangle,\\
J_3 &:= \langle sa_3\, x_3 - z_3\, ca_3, sa_2\, x_2 - z_2\, ca_2, sa_1\, x_1 - z_1\, ca_1, 3\, z_3\, y_1\, y_2 + 3\, z_2\, y_1\, y_3 + 3\, z_1\, y_2\, y_3\\
&\qquad - \sqrt{3}\, z_3\, y_2\, x_1 + \sqrt{3}\, z_2\, y_3\, x_1 + \sqrt{3}\, z_3\, y_1\, x_2 - \sqrt{3}\, z_1\, y_3\, x_2 + 3\, z_3\, x_1\, x_2\\
&\qquad - \sqrt{3}\, z_2\, y_1\, x_3 + \sqrt{3}\, z_1\, y_2\, x_3 + 3\, z_2\, x_1\, x_3 + 3\, z_1\, x_2\, x_3\rangle,\\
J_4 &:= \langle ca_2, sa_2\rangle,
\end{aligned}$$

```
LIB "delta.lib";
LIB "rwlist.lib";
LIB "primdec.lib";

def r = DeltaGetCsIdeal();

//ring with algebraic field extensions and no parameters for decomposition of
    ↪ ideal of minors
ring s = (0,s3),(z1,z2,z3,sa1,sa2,sa3,y1,y2,y3,x1,x2,x3,ca1,ca2,ca3),dp(15);
minpoly = s3^2 - 3;

ideal I = imap(r,I);
ideal J = minor(jacob(I),12);

list L = minAssGTZ(J); //Calculate Associated primes
print(L);
writelist("comps",L);
```

Listing 6.6: Calculation of associated primes. #assoc_primes

$$
\begin{aligned}
J_5 &:= \langle sa_2\, x_2 - z_2\, ca_2, sa_1\, x_1 - z_1\, ca_1, 3\, y_1\, y_2 - \sqrt{3}\, y_2\, x_1 + \sqrt{3}\, y_1\, x_2 + 3\, x_1\, x_2,\\
&\qquad 3\, sa_1\, y_2 + \sqrt{3}\, sa_1\, x_2 + 2\sqrt{3}\, z_2\, ca_1, 3\, z_1\, y_2 + 2\sqrt{3}\, z_2\, x_1 + \sqrt{3}\, z_1\, x_2,\\
&\qquad 3\, sa_2\, y_1 - \sqrt{3}\, sa_2\, x_1 - 2\sqrt{3}\, z_1\, ca_2, z_2\, y_1 + 2\, z_1\, y_2 + \sqrt{3}\, z_2\, x_1\rangle,\\
J_6 &:= \langle x_2, y_2, z_2\rangle,\\
J_7 &:= \langle sa_3\, x_3 - z_3\, ca_3, sa_1\, x_1 - z_1\, ca_1, 3\, y_1\, y_3 + \sqrt{3}\, y_3\, x_1 - \sqrt{3}\, y_1\, x_3 + 3\, x_1\, x_3,\\
&\qquad 3\, sa_1\, y_3 - \sqrt{3}\, sa_1\, x_3 - 2\sqrt{3}\, z_3\, ca_1, 3\, z_1\, y_3 - 2\sqrt{3}\, z_3\, x_1 - \sqrt{3}\, z_1\, x_3,\\
&\qquad 3\, sa_3\, y_1 + \sqrt{3}\, sa_3\, x_1 + 2\sqrt{3}\, z_1\, ca_3, z_3\, y_1 + 2\, z_1\, y_3 - \sqrt{3}\, z_3\, x_1\rangle,\\
J_8 &:= \langle ca_1, sa_1\rangle,\\
J_9 &:= \langle x_1, y_1, z_1\rangle,\\
J_{10} &:= \langle x_3, y_3, z_3\rangle.
\end{aligned}
$$

Listing 6.6 will also store equations for J_i in the text file `comps` for the subsequent computations. This file will be generated with the library `rwlist.lib`, see Appendix A.2.

We set $K_i := I_{a,b,d} + J_i$. Then

$$K := \sqrt{J + I_{a,b,d}} = K_1 \cap \ldots \cap K_{10}.$$

Now we have to analyze $K_1, \ldots, K_{10}$. We start with the easiest components: If $i \in \{1, 4, 6, 8, 9, 10\}$ then J_i is either $\langle ca_j, sa_j\rangle$ or $\langle x_j, y_j, z_j\rangle$ for some $j \in \{1, 2, 3\}$. But $I_{a,b,d}$ contains $ca_j^2 + sa_j^2 - a^2$ and $x_j^2 + y_j^2 + z_j^2 - b^2$. Thus

$$K_i = (1), \quad \text{for } i = 1, 4, 6, 8, 9, 10.$$

To examine the remaining components K_2, K_3, K_5 and K_7 we use the `grobcov` algorithm [42, 51] again. This needs some preparation since the following Gröbner cover computations are harder than before. First, we will scale the system to remove one of the parameters: Let $d \neq 0 \in \mathbb{R}$. Since the generating polynomials of J_i are homogeneous and independent of (a, b, d), one can verify easily that

$$\begin{aligned} \Psi\colon \mathbb{C}[\mathbf{w}] &\to \mathbb{C}[\mathbf{w}] \\ (x_i, y_i, z_i, ca_i, sa_i) &\mapsto (d \cdot x_i, d \cdot y_i, d \cdot z_i, d \cdot ca_i, d \cdot sa_i) \end{aligned}$$

induces an isomorphism of $\mathbf{V}(I_{(a,b,d)} + J_i)$ and $\mathbf{V}\big(I_{\left(\frac{a}{b},\frac{b}{d},1\right)} + J_i\big)$. Thus, from now on we consider the scaled system $I_{a,b,1}$ with the parameters

$$(a, b) \in \mathbb{R}^2, \quad a \neq 0,\ b \neq 0.$$

If we need to exclude an algebraic set $\mathbf{V}_{\mathbb{R}}(f) \subset \mathbb{R}^2$, $f \in \mathbb{R}[a, b]$ from the parameters (a, b) we have to exclude $\mathbf{V}_{\mathbb{R}}(f^h)$ from the original parameters (a, b, d), where

$$f^h(a, b, d) = f\left(\frac{a}{d}, \frac{b}{d}\right) \cdot d^k, \quad k = \deg f,$$

is the homogenization of f. This is clear, since

$$\left(\frac{a}{d}, \frac{b}{d}\right) \notin \mathbf{V}_{\mathbb{R}}(f) \Leftrightarrow (a, b, d) \notin \mathbf{V}_{\mathbb{R}}(f^h),\ d \neq 0.$$

As a second measure we exclude some hyperplanes from the parameter space in the execution of the `grobcov` algorithm. This will simplify the calculations further. See the remarks below for our strategy in choosing the excluded algebraic set.

Finally, there is one remaining obstacle. The `grobcov` implementation does not work with simple algebraic field extensions in `Singular`. Therefore, we proceed as in the calculation of $\dim I_{a,b,d}$. By Proposition 2.29 we can consider

$$K_i^{s_3} = I_{a,b,d}^{s_3} + J_i^{s_3} \leq \mathbb{Q}[\mathbf{w}, s_3], \quad i = 1, \ldots, 10,$$

in place of K_i, where $i = 1, \ldots, 10$ and

$$J_i^{s_3} = \left\{ f(\mathbf{w}, s_3) \in \mathbb{Q}[\mathbf{x}, s_3] \mid f\big(\mathbf{w}, \sqrt{3}\big) \in J_i \right\}$$

is generated by $s_3^2 - 3$ and the generators of J_i with $\sqrt{3}$ replaced by s_3.

With the preceding deliberations we are able to calculate Gröbner covers for K_2 and K_7. When we execute Listing 6.7 – which takes about 2 minutes on a Intel Core i5-5300 CPU – we get the following output:

```
LIB "delta.lib";
LIB "rwlist.lib";
LIB "grobcov.lib";

def r = DeltaGetCsIdeal();
setring r;

//set d=1
I = subst(I,d,1);

//different ordering no d
ring s = (0,a,b),(x1,y1,z1,sa1,ca1,x2,y2,z2,sa2,ca2,x3,y3,z3,sa3,ca3,s3),
    ↪ (dp(15),dp(1));
ideal I = imap(r,I);
list L = readlist("comps");
ideal K2 = L[2],I; //K2

//Calculate Groebner Cover*/
list F2 = grobcov(K2,("nonnull",ideal(a*b*(a^2+3)*(3*a^2+6*a-4*b^2+3)*
    ↪ (3*a^2-6*a-4*b^2+3)*(a^6-2*a^4*b^2+3*a^4+a^2*b^4+
    ↪ 30*a^2*b^2+3*b^4-36*b^2))), ("ext",1));
//Groebner Cover Add ("comment",3) for commented output */
//Calculation about 30 sec*/

ideal K7 = L[7],I; //K7
list F7 = grobcov(K7,("nonnull",ideal(a*b*(a^2 +
    ↪ 3)*(3*a^2+6*a-4*b^2+3)*(3*a^2-6*a-4*b^2+3)*
    ↪ (a^6-2*a^4*b^2+3*a^4+a^2*b^4+30*a^2*b^2+3*b^4-36*b^2))),("ext",1));
//Calculation about 1min;

//Analyze Components
// #Solutions in K2
"Component K2"; "Segment definition:";
F2[1][3];
attrib(F2[1][2],"isSB",1); //is standard basis for our parameter values*/
"--------------------------------";
"Dimension: " + string(dim(F2[1][2])); //--> output 0
"dim_Q Q[x]/K2: " + string(size(kbase(F2[1][2]))) + newline; //--> output 16

// #Solutions in K7
"--------------------------------";
"Component K7"; "Segment definition:";
F7[1][3]; //same as F2[1][3]
attrib(F7[1][2],"isSB",1); //is standard basis for our parameter values
"--------------------------------";
"Dimension: " + string(dim(F7[1][2])); //--> output 0
"dim_Q Q[x]/K7: " + string(size(kbase(F7[1][2]))); //--> output 16
```

Listing 6.7: Components K_2 and K_7. #component27

```
Component K2
Segment definition: //see below
--------------------------------
Dimension: 0
dim_Q Q[w,s3]/K2: 16


Component K7
Segment definition: //see below
--------------------------------
Dimension: 0
dim_Q Q[w,s3]/K7: 16
```

Because of the restriction of the parameter space we get only one segment of the Gröbner cover for K_2 and K_7. The part of the output that we omitted above is both times:

```
Segment definition:
   [1]:
      _[1]=0
[2]:
   [1]:
      _[1]=(a^6-2*a^4*b^2+3*a^4+a^2*b^4+30*a^2*b^2+3*b^4-36*b^2)
   [2]:
      _[1]=(a^2-4*b^2+3)
   [3]:                           [7]:
      _[1]=(3*a^2-6*a-4*b^2+3)       _[1]=(a^2+3)
   [4]:                           [8]:
      _[1]=(3*a^2+6*a-4*b^2+3)       _[1]=(a+1)
   [5]:                           [9]:
      _[1]=(a^2-b^2+3)               _[1]=(a-1)
   [6]:                           [10]:
      _[1]=(b)                       _[1]=(a)
--------------------------------
```

Now let O be the algebraic set from Definition 6.1 and assume $(a, b, d) \in O$. The output from Listing 6.7 shows $\dim K_2 = \dim K_7 = 0$ and

$$|\mathbf{V}(K_2)| \leq 8, \quad |\mathbf{V}(K_3)| \leq 8. \tag{6.10}$$

The inequation (6.10) holds because the number of roots in $\mathbf{V}(K_i)$ is bounded by the $\mathbb{Q}[\sqrt{3}]$-dimension of $\left(\mathbb{Q}[\sqrt{3}]\right)[\mathbf{w}]/K_i$ [4, Theorem 4.85]. Thus, we need to divide $\dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{w}, s_3]/K_i^{s_3} = 16$ from the output by 2, to take in account the simple field extension $\mathbb{Q}[\sqrt{3}]$, see Proposition 2.29(c). Note also that

$$\dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{w}, s_3]/K_i^{s_3} = \dim_{\mathbb{Q}} \mathbb{Q}[\mathbf{w}, s_3]/\mathrm{LM}(K_i^{s_3})$$

according to [31, Corollary 7.5.6]. Hence, this dimension can be calculated independent of (a, b, d).

Lastly, we analyze the most difficult components K_3 and K_5. Listings 6.8 and 6.9 show that

$$K_3 = (1), \quad \dim K_5 = 0, \quad |\mathbf{V}(K_5)| \leq 8,$$

for all parameter values $(a, b, d) \in O$. Please be aware that Listing 6.8 needs 1 GB of RAM and takes about 40 minutes to complete on an Intel Core i5-5300 CPU.

```
LIB "delta.lib";
LIB "rwlist.lib";
LIB "grobcov.lib";

def r = DeltaGetCsIdeal();
setring r;

//set d=1
I = subst(I,d,1);

//different ordering no d
ring s = (0,a,b),(x3,y3,z3,sa3,ca3,x2,y2,z2,sa2,ca2,x1,y1,z1,sa1,ca1,s3),
    ↪ (dp(15),dp(1));
ideal I = imap(r,I);

list L = readlist("comps");
ideal K3 = L[3],I; //K3

//Calculate Groebner Cover
list F3 = grobcov(K3,("nonnull",ideal(a*b*(a^2-1)*(a-b-1)*(a-b+1)*
    ↪ (a+b-1)*(a+b+1)*(a^2+3)*(a^2+a*b+3*b+3)*(a^2+a*b-3*b+3)*
    ↪ (a^2-a*b+3*b+3)*(a^2 - a*b - 3*b + 3))),("ext",1)); //Add ("comment",3) for
    ↪ commented output
//Calculation about 0:30h on Intel Xeon CPU E5-2650 cpu 120 process needs 1.9gb ram
//Calculation about 0:35h on i5-5300U cpu needs 1gb ram

//Analyze Component
"Component K3";
F3[1][2]; //--> gives (1)
"exceptional set:";
F3[1][3]; //--> same as for K2,K7
```

Listing 6.8: Component K_3. #component3

```
LIB "delta.lib";
LIB "rwlist.lib";
LIB "grobcov.lib";

def r = DeltaGetCsIdeal();
setring r;

//set d=1
I = subst(I,d,1);

//different ordering no d
ring s = (0,a,b),(x3,y3,z3,sa3,ca3,x2,y2,z2,sa2,ca2,x1,y1,z1,sa1,ca1,s3),
    ↪ (dp(15),dp(1));
ideal I = imap(r,I);

list L = readlist("comps");
ideal K5 = L[5],I; //K5

//Calculate Groebner Cover
list F5 = grobcov(K5,("nonnull",ideal(a*b*(a-b+1)*(a^2+a*b+3*b+3)*
    ↪ (a-b-1)*(a+1)*(a^2+3)*(3*a^2-6*a-4*b^2+3)*(3*a^2+6*a-4*b^2+3)*
    ↪ (a^6-3*a^4+99*a^2-81)*
    ↪ (a^6-2*a^4*b^2+3*a^4+a^2*b^4+30*a^2*b^2+3*b^4-36*b^2))),("ext",1));
    ↪ //Groebner Cover Add ("comment",3) for commented output
//Calculation about 40 sec on i5 cpu 120 process needs 120 mb ram

//Analyze Component
"Component K5";
"exceptional set:";
F5[1][3]; //--> output same as K2,K7
attrib(F5[1][2],"isSB",1); //is standard basis for our parameter values*/
"---------------------------------";
"Dimension: " + string(dim(F5[1][2])); //--> output 0
"dim_Q Q[x]/K5: " + string(size(kbase(F5[1][2]))) + newline; //--> output 16
```

Listing 6.9: Component K_5. #component5

This concludes our proof. We have seen that $\dim Y_{a,b,d} = 3$, $\dim K = 0$, and $|\mathbf{V}(K)| \leq 3 \cdot 8 = 24$. $\square$

Remark. The calculations of Gröbner covers of large systems are very hard in general. In the hope it will be helpful for others, we would like to provide some clues about the usage of the `grobcov` algorithm. Besides the hints in [42, 51], the following measures proved to be useful for us:

(i) It is important to choose the *right* sequence of variables in the ring definition. Different monomial orderings often give unfeasible systems. For the calculation of the Gröbner covers of K_3 and K_5 we experimented with 8 different orderings parallel on a workstation. Only 2 finished computations in a time of 24 hours.

(ii) One should use as less parameters as possible. The calculations for K_2, K_3, K_5 and K_7 were unfeasible with three parameter (a, b, d), although the systems are isomorphic to their counterparts with two parameters.

(iii) With the option `("comment",3)` of `grobcov` the user gets detailed information on the progress of the algorithm. If the computation seems unfeasible, one can try to exclude algebraic sets from the parameter space for which the algorithm performs poorly. Use the option `nonnull` for that.

Obvious candidates for hypersurfaces to be excluded are given by the prime factors of the leading coefficients of the Gröbner bases in the steps of the Kapur-Sun-Wang algorithm [42] which powers `grobcov`. These are indicated in the commented output by the last polynomials of `GrHi`.

We proceeded like this for the calculations of the Gröbner covers of K_2,K_3,K_5 and K_7.

6.4.3 The proof of Corollary 6.3

Let $(a, b, d) \in O$. Then, according to Theorem 6.2 we have $\dim I_{a,b,d} = 3$, and $\dim K = 0$, for the ideal K generated by $I_{a,b,d}$ and the 12-minors of DF. We also know with [20, Proposition 18.13], that the coordinate ring $A = \mathbb{R}[\mathbf{w}]/(I_{a,b,d} \cdot \mathbb{R}[\mathbf{w}])$ is Cohen-Macaulay, since $I_{a,b,d}$ is generated by 12 elements. We can now argue just as we did for the five-bar linkage but we will repeat the discussion for completeness sake. By Theorem 18.15 of [20] we conclude:

(a) $I_{a,b,d}$ is equidimensional and radical.

(b) The singular locus of $Y_{a,b,d}$ is zero-dimensional.

(c) A is a normal ring.

(d) All the components of $Y_{a,b,d}$ are disjoint since the singular intersection of components would have codimension lesser equal 1 according to Hartshorne's Connectedness Theorem [20, Theorem 18.12].

With (a)–(d) above we can apply Corollary 3.10 and have the result that any point in $P_{a,b,d}$ is either a non-manifold point or isolated in $X_{a,b,d}$.

In subsection 6.4.4 we will construct analytic curves γ_i in $X_{a,b,d}$, with $\gamma_i(0) = q_i$, for q_1, q_2, q_3, q_4 and show that none of the points in $P_{a,b,d}$ are isolated in $X_{a,b,d}$. Consequently, they are CS-singularities of the delta robot. $\square$

6.4.4 On the real tangent cones in q_1,q_2,q_3,q_4

Let $a, b, d \in O$ again. In this section we are going to construct analytic curves $\gamma_i\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^{15}$, with $\varepsilon > 0$, $\gamma(-\varepsilon, \varepsilon) \subset X_{a,b,d}$ and $\gamma(0) = q_i$, for q_1, q_2, q_3, q_4 from Table 6.1. To complete the proof of Corollary 6.3 it is not necessary to construct more than one curve for each q_i. Nevertheless, we will do so for four curves $\gamma_i^{(j)}$, $j = 1, \dots, 4$ and also calculate

$$\lambda_i^{(j)} \coloneqq \dot{\gamma}_i^{(j)}(0),$$

which will be linear independent for $j = 1, \dots, 4$ and contained in the real geometric tangent cone, see [60, Lemma 3]. This gives an alternative proof that $X_{a,b,d}$ is not locally a manifold at q_i and provides insight in the kinematic properties of the delta robot at its CS-singularities. In addition, this proves that $X_{a,b,d}$ is not even a topological manifold at q_i.

Animations of the delta robot tracking some of the constructed paths can be obtained from our online repository, see Appendix A.3.

To keep the notation simple, we consider the isomorphic real algebraic set

$$\tilde{X}_{a,b,d} \coloneqq \mathbf{V}_{\mathbb{R}}(\{s_i, c_j, \tilde{l}_k \mid i, j = 1, 2, 3 \text{ and } k = 1, \dots, 6\}),$$

where

$$\begin{pmatrix} \tilde{l}_1 \\ \tilde{l}_2 \\ \tilde{l}_3 \end{pmatrix} = \begin{pmatrix} d + ca_1 + x_1 \\ y_1 \\ sa_1 + z_1 \end{pmatrix} - A \begin{pmatrix} d + ca_2 \\ 0 \\ sa_2 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix},$$

$$\begin{pmatrix} \tilde{l}_4 \\ \tilde{l}_5 \\ \tilde{l}_6 \end{pmatrix} = \begin{pmatrix} d + ca_1 + x_1 \\ y_1 \\ sa_1 + z_1 \end{pmatrix} - A^{-1} \begin{pmatrix} d + ca_3 \\ 0 \\ sa_3 \end{pmatrix} - \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}.$$

With this formulation (x_i, y_i, z_i) for $i = 1, 2, 3$ are now coordinates for the direction vector of the parallelogram linkages in the same reference coordinate system (x, y, z). We also define:

$$m_1(\psi) \coloneqq A \begin{pmatrix} d + a \cdot \cos(\psi) \\ 0 \\ a \cdot \sin(\psi) \end{pmatrix}, \quad m_2(\psi) \coloneqq A^{-1} \begin{pmatrix} d + a \cdot \cos(\psi) \\ 0 \\ a \cdot \sin(\psi) \end{pmatrix},$$

Table 6.2: Geometric expressions.

Function	Term
$p(q_4)$	$\left(\frac{2bd}{\sqrt{a^2+3d^2}}, 0, \sqrt{a^2-d^2}+\frac{b\sqrt{a^2-d^2}}{a^2+3d^2}\right)^T$
$m_1(t)$	$\left(-\frac{d}{2}-\frac{a\cos(t)}{2}, \sqrt{3}\cdot\frac{d+a\cos(t)}{2}, a\sin(t)\right)^T$
$m_2(t)$	$\left(-\frac{d}{2}-\frac{a\cos(t)}{2}, -\sqrt{3}\cdot\frac{d+a\cos(t)}{2}, a\sin(t)\right)^T$
$m_3(t)$	$(d+a\cos(t), 0, a\sin(t))^T$
$M_1(t)$	$\left(-\frac{d}{2}-\frac{a\cos(t)}{2}, 0, a\sin(t)\right)^T$
$r_1(t)$	$\sqrt{-\frac{3\,a^2\cos^2(t)}{4}-\frac{3\,ad\cos(t)}{2}-\frac{3d^2}{4}+b^2}$

$$m_3(\psi) := \begin{pmatrix} d + a\cdot\cos(\psi) \\ 0 \\ a\cdot\sin(\psi) \end{pmatrix}, \qquad m(\psi) := \begin{pmatrix} a\cos(\psi) \\ a\sin(\psi) \end{pmatrix},$$

and write $S_i(\psi)$ for the sphere with radius b and center $m_i(\psi)$ in the (x, y, z)-coordinate system. Then, we have the following simple characterization of points in the transformed configuration space $\tilde{X}_{a,b,d}$:

Lemma 6.4. *Let $p = (x, y, z) \in \mathbb{R}^3$ and $\psi_i \in \mathbb{R}$, for $i = 1, 2, 3$, then*

$$Q(p, \psi_1, \psi_2, \psi_3) := \big(p - m_1(\psi_1), p - m_2(\psi_2), p - m_3(\psi_3), m(\psi_1), m(\psi_2), m(\psi_3)\big) \in \mathbb{R}^{15}$$

is a point in $\tilde{X}_{a,b,d}$ if and only if $|p - m_i(\psi_i)|^2 = b^2$, for $i = 1, 2, 3$, if and only if $p \in S_1(\psi_1) \cap S_2(\psi_2) \cap S_3(\psi_3)$.

Lemma 6.4 can be proven easily by verifying that $Q(p, \psi_1, \psi_2, \psi_3)$ fulfills all polynomials $c_i, s_j, \tilde{l}_k$ if and only if $|p - m_i(\psi_i)|^2 = b^2$, for $i = 1, 2, 3$. Furthermore, it is clear that every point $q \in \tilde{X}_{a,b,d}$ can be represented uniquely in the form $q = Q(p, \psi_1, \psi_2, \psi_3)$, for a $p \in \mathbb{R}^3$ and $\psi_i \in \mathbb{R}$, $i = 1, 2, 3$. From now on we will write $p(q)$ and $\psi_i(q)$ to reference those coordinates.

By slight abuse of notation we denote with $q_1, \ldots, q_4$ also the corresponding points in the transformed configuration space $\tilde{X}_{a,b,d}$. Only paths for q_1 and q_4 will be constructed, since it is going to become clear, how to construct matching paths for q_2 and q_3.

Singularities q_4 (and q_3)

We fix $p_0 := p(q_4)$ and $\varphi_i = \psi_i(q_4)$ for $i = 1, 2, 3$. Then, we have $\varphi := \varphi_1 = \varphi_2$ and $m_3(\varphi_3)$, $m_1(\varphi_1) = m_2(\varphi_2)$ and p_0 all lie in the xz-plane. According to Lemma 6.4 it

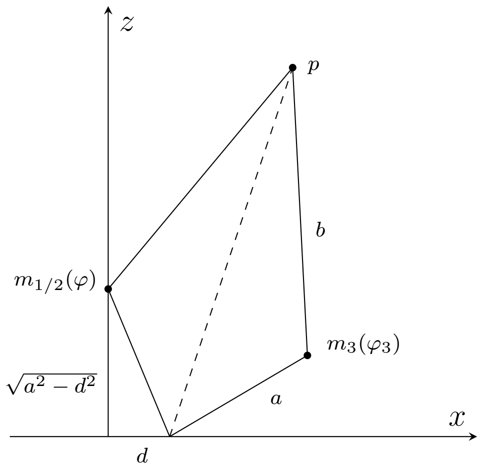


Figure 6.9: Reflection of $m_3(\varphi_3)$.

must be

$$p_0 \in S_1(\varphi_1) \cap S_2(\varphi_2) \cap S_3(\varphi_3).$$

First Path $\gamma_4^{(1)}$: It is $S_1(\varphi) = S_2(\varphi)$ and for $t \neq \varphi$ close to φ,

$$K_1(t) \coloneqq S_1(t) \cap S_2(t)$$

is a circle in the xz-plane. We will denote the center of $K_1(t)$ with $M_1(t)$ and its radius with $r_1(t)$. Both $M_1(t)$ and $r_1(t)$ clearly admit analytic continuations for $t = \varphi$. You can find expressions for these terms in Table 6.2.

As one can check quickly $m_3(\varphi_3)$ is the reflection of $m_1(\varphi) = m_2(\varphi)$ in the xz-plane across the axis through $(d, 0, 0)$ and p, see Figure 6.9. This means that the intersection of $K_1(\varphi)$ with $S_3(\varphi_3)$ is transversal as long as

$$p_0 - m_{1/2}(\varphi) = \begin{pmatrix} \frac{2bd}{\sqrt{a^2+3d^2}} & 0 & \frac{b\sqrt{a^2-d^2}}{\sqrt{a^2+3d^2}} \end{pmatrix}^T$$

is not perpendicular to

$$p_0 - \begin{pmatrix} d & 0 & 0 \end{pmatrix}^T = \begin{pmatrix} \frac{2bd}{\sqrt{a^2+3d^2}} - d & 0 & \sqrt{a^2-d^2} + \frac{b\sqrt{a^2-d^2}}{\sqrt{a^2+3d^2}} \end{pmatrix}^T.$$

The opposite condition is equivalent to

$$b\,(a^2+3d^2) + \sqrt{a^2+3d^2}\,(a^2-3d^2) = 0.$$

and this equation implies

$$0 = (a^2 - 3\,d^2)^2 - b^2\,(a^2 + 3\,d^2) = 9\,d^4 - a^4 - 6\,a^2 d^2 - b^2 a^2 + 3\,d^2 b^2. \tag{6.11}$$

But the right side of (6.11) is f_{18} from Definition 6.1. This means equation (6.11) is not true for $(a, b, d) \in O$.

Accordingly, the intersection $K_1(\varphi) \cap S_3(\varphi_3)$ is transversal. Due to the analytic implicit function theorem we find an analytic path $\delta_1\colon (\varphi - \varepsilon, \varphi + \varepsilon) \to \mathbb{R}^3$, with

$$\delta_1(t) \in K(t) \cap S_3(\varphi_3) \subset S_1(t) \cap S_2(t) \cap S_3(\varphi_3) \tag{6.12}$$

and $\delta_1(\varphi) = p_0$. Now we set $\gamma_4^{(1)}(t) := Q(\delta_1(t), t, t, \varphi^*)$. According to Lemma 6.4 and (6.12) we have $\gamma_1(t) \in \tilde{X}_{a,b,d}$ for all t. We immediately check $\gamma_1(\varphi) = q_4$ and

$$\lambda_4^{(1)} = \dot{\gamma}_4^{(1)}(\varphi) = \begin{pmatrix} * \\ \vdots \\ * \\ -\sqrt{(a^2 - d^2)} \\ -d \\ -\sqrt{(a^2 - d^2)} \\ -d \\ 0 \\ 0 \end{pmatrix}.$$

Remark. The implicit function theorem gives an expression for $\delta_1'(\varphi)$. Hence all coordinates of $\lambda_4^{(1)}$ can be calculated. We refrain from doing so, to keep the discussion shorter. The projection of $\gamma_1(\varphi)$ on the actuated coordinates (c_i, s_i), $i = 1, 2, 3$ is important information for the delta robot controller.

Second Path $\boldsymbol{\gamma_4^{(2)}}$: Using the implicit function theorem again we find an analytic path $\delta_2\colon (\varphi_3 - \varepsilon, \varphi_3 + \varepsilon) \to \mathbb{R}^3$ with

$$\delta_2(t) \subset S_1(\varphi) \cap S_2(\varphi) \cap S_3(t) \cap \{(x, 0, z) \in \mathbb{R}^3\}, \tag{6.13}$$

and $\delta_2(\varphi_3) = p_0$. Hence $\gamma_1^{(2)}(t) := Q(\delta_2(t), \varphi, \varphi, t) \in \tilde{X}_{a,b,d}$ for all t and $\gamma_2(\varphi_3) = q_4$. We determine:

$$\lambda_4^{(2)} = \dot{\gamma}_4^{(2)}(\varphi_3) = \begin{pmatrix} * \\ \vdots \\ * \\ 0 \\ 0 \\ 0 \\ 0 \\ -\frac{6bd^2\sqrt{a^2-d^2}(q-2b)}{2bq(a^2-3d^2)+q^2(a^2+b^2)} + \sqrt{a^2-d^2} \\ \frac{6bd(a^2-d^2)(b+q)}{2bq(a^2-3d^2)+q^2(a^2+b^2)} - d \end{pmatrix}.$$

Note that the last two entries of $\lambda_1^{(2)}$ are well defined and not both zero.

Third Path $\boldsymbol{\gamma_4^{(3)}}$: Since equation (6.11) is not true for $(a,b,d) \in O$, the intersection $S_1(\varphi) \cap S_3(\varphi_3) = S_2(\varphi) \cap S_3(\varphi_3)$ must be a circle and can be parameterized around p_0, i.e. there exists an analytic path $\delta_3 \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$, with

$$\delta_3(t) \subset S_1(\varphi) \cap S_2(\varphi) \cap S_3(\varphi_3),$$

and $\delta_3(0) = p_0$. Again $\gamma_4^{(3)}(t) := Q(\delta_3(t), \varphi, \varphi, \varphi_3) \in \tilde{X}_{a,b,d}$ for all t and $\gamma_4^{(3)}(0) = q_4$. We get

$$\lambda_4^{(3)} = \dot{\gamma}_4^{(3)}(0) = (*, \dots, *, 0, 0, 0, 0, 0, 0)^T .$$

Fourth Path $\boldsymbol{\gamma_4^{(4)}}$: We attempt to find a path $\gamma_4^{(4)}$ with $\gamma_4^{(4)}(t) \in S_1(t) \cap S_2(\varphi) \cap S_3(\varphi_3)$, for t close to φ. Clearly, $K_2(t) = S_1(t) \cap S_2(\varphi)$ is a circle for $t \neq \varphi$ close to φ. We write $M_2(t)$ for the center of $K_2(t)$ and $r_2(t)$ for its radius. Moreover, let $n(t)$ be the normal

$$\frac{m_1(t) - m_2(\varphi)}{|m_1(t) - m_2(\varphi)|}$$

of the circle plane. Some elementary arguments (see e.g. the proof of Lemma 6.5) confirm that $M_2(t)$, $r_2(t)$ and $n(t)$ admit an analytic continuation at $t = \varphi$, with

$$n(\varphi) = M'(\varphi) = m_1'(\varphi), \quad r_2'(\varphi) = 0.$$

Now we show that $p_0 \in K_2(\varphi) \cap S_3(\varphi_3)$, where $K_2(\varphi)$ is the circle belonging to the analytic continuations of M_2, r_2 and n. Since $p_0 \in S_1(\varphi) \cap S_3(\varphi)$ and $K_2(\varphi) \subset S_1(\varphi)$ it suffices to show, that p_0 and $K_2(\varphi)$ lie in the same plane, i.e. $p_0 - m_1(\varphi) \perp n(\varphi) = m_1'(\varphi)$. But with $\varphi = \pi - \arctan\left(\frac{\sqrt{a^2-d^2}}{d}\right)$ we see

$$m_1'(\varphi) = \begin{pmatrix} \frac{\sqrt{a^2-d^2}}{2} \\ -\frac{\sqrt{3}}{2}\sqrt{a^2-d^2} \\ d \end{pmatrix}.$$

Thus we calculate $(p_0 - m_1(\varphi)) \cdot m_1'(\varphi) = 0$.

$K_2(t) \cap S_3(\varphi_3)$ is a transversal intersection for $t = \varphi$ as we have seen in our discussion for $\gamma_4^{(1)}$ and we just showed $p_0 \in K_2(\varphi) \cap S_3(\varphi_3)$. Consequently, there must be an analytic path $\delta_4 \colon (\varphi - \varepsilon, \varphi + \varepsilon) \to \mathbb{R}^3$, with $\delta_4(\varphi) = p_0$. Again we have $\gamma_4^{(4)}(t) := Q(\delta_4(t), t, \varphi, \varphi_3) \in \tilde{X}_{a,b,d}$ for all t and $\gamma_4^{(4)}(\varphi) = q_4$. We get

$$\lambda_4^{(4)} = \dot{\gamma}_4^{(4)}(\varphi) = \left(*, \dots, *, -\sqrt{a^2-d^2}, -d, 0, 0, 0, 0\right)^T .$$

One checks immediately that $\lambda_4^{(1)}, \lambda_4^{(2)}, \lambda_4^{(3)}, \lambda_4^{(4)}$ are linear independent. In particular, $X_{a,b,d}$ is not locally a topological manifold at q_4.

Singularities q_1 (and q_2)

We fix again $p_0 \coloneqq p(q_1)$ and $\varphi_i = \psi_i(q_1)$ for $i = 1, 2, 3$. Then

$$\varphi \coloneqq \varphi_1 = \varphi_2 = \varphi_3 = \arctan\left(-\frac{\sqrt{a^2 - d^2}}{d}\right) + \pi$$

and

$$m_1(\varphi_1) = m_2(\varphi_2) = m_3(\varphi_3) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{a^2 - d^2}. \end{pmatrix}.$$

First and second path $\boldsymbol{\gamma_1^{(1)}}$, $\boldsymbol{\gamma_1^{(2)}}$: For any $p \in S_1(\varphi) = S_2(\varphi) = S_3(\varphi)$ we clearly have $Q(p, \varphi_1, \varphi_2, \varphi_3) \in \tilde{X}_{a,b,d}$. Hence, we find paths $\gamma_1^{(1)}$, $\gamma_1^{(2)} \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^{15}$ such that $\gamma_1^{(1)}(t), \gamma_1^{(2)}(t) \in \tilde{X}_{a,b,d}$ for all t, $\gamma_1^{(1)}(0) = \gamma_1^{(2)}(0) = q_1$ and

$$\lambda_1^{(1)} = \dot{\gamma}_1^{(1)}(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \lambda_1^{(2)} = \dot{\gamma}_1^{(2)}(0) = \begin{pmatrix} -\sqrt{a^2 - d^2} \\ 0 \\ 2\,d \\ -\sqrt{a^2 - d^2} \\ 0 \\ 2\,d \\ -\sqrt{a^2 - d^2} \\ 0 \\ 2\,d \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Third Path $\boldsymbol{\gamma_1^{(3)}}$: As in our discussion of $\gamma_4^{(1)}$, $K_1(t) = S_1(t) \cap S_2(t)$, is a circle in the xz-plane with center $M_1(t)$ and radius $r_1(t)$ from Table 6.2, provided $t \neq \varphi$ close to φ. We also denote with K_3 the circle given by the intersection of $S_3(\varphi_3)$ with the xz-plane. Since

$$M_1'(\varphi_1) \cdot (p_0 - M_1(\varphi)) = \begin{pmatrix} \frac{\sqrt{a^2-d^2}}{2} \\ 0 \\ -d \end{pmatrix} \cdot \begin{pmatrix} \frac{2db}{\sqrt{a^2+3d^2}} \\ 0 \\ \frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}} \end{pmatrix} = 0$$

we find with Lemma 6.5 an analytic path $\delta \colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ such that $\delta(0) = p_0$ and $\delta(t) \in K_1(t) \cap K_3$ for all t. This means that

$$\gamma_1^{(3)}(t) \coloneqq Q(\delta(t), \varphi + t, \varphi + t, 0) \in \tilde{X}_{a,b,d}, \text{ for all } t.$$

We differentiate

$$\lambda_1^{(3)} = \dot{\gamma}_1^{(3)}(0) = \begin{pmatrix} * \\ \vdots \\ * \\ -\sqrt{(a^2-d^2)} \\ -d \\ -\sqrt{(a^2-d^2)} \\ -d \\ 0 \\ 0 \end{pmatrix}.$$

Fourth Path $\gamma_1^{(4)}$: Lemma 6.6 states that we are able to find an analytic path $\delta\colon (\varphi-\varepsilon, \varphi+\varepsilon) \to \mathbb{R}^3$, with $\delta(\varphi) = p_0$ and $\delta(t) \in S_1(t) \cap S_2(\varphi) \cap S_3(\varphi)$ for all t, provided that $p_0 - m_1(\varphi) \perp m_1'(\varphi)$. But

$$m_1'(\varphi) \cdot (p_0 - m_1(\varphi)) = \begin{pmatrix} \frac{\sqrt{a^2-d^2}}{2} \\ \frac{\sqrt{3}}{2}\sqrt{a^2-d^2} \\ -d \end{pmatrix} \cdot \begin{pmatrix} \frac{2db}{\sqrt{a^2+3d^2}} \\ 0 \\ \frac{\sqrt{a^2-d^2}\,b}{\sqrt{a^2+3\,d^2}} \end{pmatrix} = 0.$$

So we set $\gamma_1^{(4)}(t) := Q(\delta(t), t, 0, 0)$. Then, $\gamma_1^{(4)}(t) \in \tilde{X}_{a,b,d}$ for all t and $\gamma_1^{(4)}(\varphi) = q_1$. We compute

$$\lambda_1^{(4)} = \dot{\gamma}_1^{(4)}(\varphi) = \begin{pmatrix} * \\ \vdots \\ * \\ -\sqrt{(a^2-d^2)} \\ -d \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

It is clear again that $\lambda_1^{(1)}, \lambda_1^{(2)}, \lambda_1^{(3)}, \lambda_1^{(4)}$ are linear independent. In particular, $X_{a,b,d}$ is not locally a topological manifold at q_1.

Lemma 6.5. *Let $r, p_x, p_y\colon \mathbb{R} \to \mathbb{R}$ be analytic on a neighborhood of the origin and*

$$r(0) := r_0 > 0, \quad r'(0) = 0 \quad (p_x(0), p_y(0)) = (0,0), \quad (p_x'(0), p_y'(0)) \neq (0,0).$$

We consider the following intersection in $\mathbb{R}^2$ with coordinates (x, y):

$$\begin{aligned} x^2 + y^2 - r_0^2 &= 0, \\ (x - p_x(t))^2 + (y - p_y(t))^2 - r(t)^2 &= 0. \end{aligned} \tag{6.14}$$

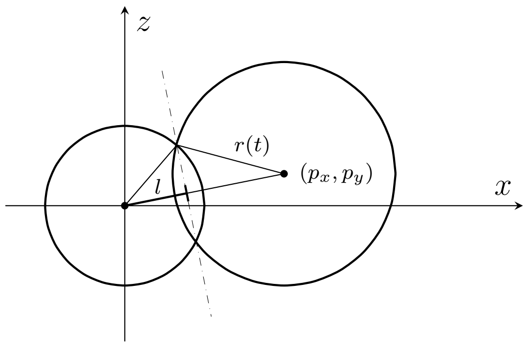


Figure 6.10: Illustration of Lemma 6.5.

There exist two analytic paths $\gamma_{1/2}(t) = (x(t), y(t))$, $|t| < \varepsilon$, *fulfilling* (6.14), *with*

$$\gamma_{1/2}(0) = \pm \frac{r_0}{\sqrt{p_x'(0)^2 + p_y'(0)^2}} \begin{pmatrix} -p_y'(0) \\ p_x'(0) \end{pmatrix} =: b_\pm.$$

Proof. First, choose a coordinate system such that we can assume $p_x'(0) \neq 0$ and $p_y'(0) \neq 0$. Then for $t \in \mathbb{R}\backslash\{0\}$ we set

$$d(t) \coloneqq \sqrt{p_x(t)^2 + p_y(t)^2}, \quad l(t) \coloneqq \frac{{r_0}^2 + p_x(t)^2 + p_y(t)^2 - r(t)^2}{2\, d(t)},$$

and

$$\begin{aligned} q_\pm(t) \coloneqq& \frac{l(t)}{d(t)} \cdot \begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} \pm \frac{\sqrt{r_0^2 - l(t)^2}}{d(t)} \cdot \begin{pmatrix} -p_y(t) \\ p_x(t) \end{pmatrix} \\ =& -\frac{1}{2} \left(\frac{r_0^2 - r(t)^2}{d(t)^2} + 1 \right) \begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} \pm \frac{\sqrt{r_0^2 - l(t)^2}}{d(t)} \cdot \begin{pmatrix} -p_y(t) \\ p_x(t) \end{pmatrix}. \end{aligned}$$

One easily verifies that $q_\pm(t)$ fulfills the system (6.14). To prove the statement of the lemma, we will need to show that $q_\pm(t)$ can be continued analytically around 0, that $q_\pm(t) \in \mathbb{R}$, for t small enough and that either $\lim_{t\to 0} q_\pm(t) = b_\pm$ or $\lim_{t\to 0} q_\pm \to b_\mp$.

Since p_x, p_y and r are analytic around 0 we can extend them to holomorphic functions on a small neighborhood of 0 in $\mathbb{C}$. Then, for every $z \in \mathbb{C}$ where the following expressions are defined we set

$$\begin{aligned} f_1(z) &\coloneqq \frac{r_0^2 - r(z)^2}{p_x(z)^2 + p_y(z)^2}. \\ f_2(z) &\coloneqq l(z)^2. \\ f_3(z) &\coloneqq \frac{p_x(z)^2}{p_x(z)^2 + p_y(z)^2}. \end{aligned}$$

$$f_4(z) := \frac{p_y(z)^2}{p_x(z)^2 + p_y(z)^2}.$$

As the origin cannot be a limit point for the zeros of $p_x(z)^2 + p_y(z)^2$, we find $\varepsilon > 0$ such that $f_1, \dots f_4$, are defined and analytic on $B_\varepsilon \backslash \{0\}$. We claim that $f_1, \dots, f_4$ admit analytic continuations on B_ε with

$$f_1(0) =: b \in \mathbb{R}, \quad f_2(0) = 0, \quad f_3(0), f_4(0) > 0. \tag{6.15}$$

Then, with the main branch of logarithm we can define analytic functions

$$\begin{aligned} g_2(z) &:= \sqrt{r_0 - f_2(z)}, \\ g_3(z) &:= \sqrt{f_3(z)}, \\ g_4(z) &:= \sqrt{f_4(z)}. \end{aligned}$$

Now we can assume, that $g_3(t) = \frac{p_x(t)}{\sqrt{p_x(t)+p_y(t)^2}}$ and $g_4(t) = \frac{-p_y(t)}{\sqrt{p_x(t)+p_y(t)^2}}$, for $t \in \mathbb{R}$, otherwise multiply by -1. Hence

$$q_\pm(t) = -\frac{1}{2}(f_1(t) + 1) \begin{pmatrix} p_x(t) \\ p_y(t) \end{pmatrix} \pm g_2(t) \begin{pmatrix} g_4(t) \\ g_3(t) \end{pmatrix}$$

is an analytic function for t small enough. As $f_2(z) \to 0$ for $z \to 0$ and $r_0 > 0$, we have $g_2(t) \in \mathbb{R}$ and $q_\pm(t) \in \mathbb{R}$ for t small enough. Moreover

$$\lim_{t \to 0} q_\pm(t) = -\frac{1}{2}(b + 1) \begin{pmatrix} 0 \\ 0 \end{pmatrix} \pm g_2(0) \begin{pmatrix} g_4(0) \\ g_3(0) \end{pmatrix}.$$

We will see with (6.16), that $g_3(0) = \frac{p'_x(0)}{\sqrt{p'_x(0)^2+p'_y(0)^2}}$, $g_4(0) = \frac{-p'_y(0)}{\sqrt{p'_x(0)^2+p'_y(0)^2}}$, therefore

$$\lim_{t \to 0} q_\pm(t) = \pm r_0 \cdot \frac{1}{\sqrt{p'_x(0)^2 + p'_y(0)^2}} \begin{pmatrix} -p'_y(0) \\ p'_x(0) \end{pmatrix}.$$

It remains to prove the claim that $f_1, \dots, f_4$ admit analytic continuations on B_ε with (6.15). We only do this for f_3 since the statements for f_1, f_2, f_4 follow in a similar way. First, we calculate:

$$\begin{aligned} (p_y^2 + p_x^2)''(0) &= 2\,(p'_x(0)^2 + p'_y(0)^2) + 2p_x(0)\,p''_x(0)^2 + 2p_y(0)\,p''_y(0)^2 \\ &= 2\,(p'_x(0)^2 + p'_y(0)^2) > 0, \\ (p_x^2)''(0) &= 2\,p'_x(0)^2 + 2\,p_x(0)\,p''_x(0) = 2\,p'_x(0)^2 > 0. \end{aligned}$$

Then, we conclude

$$\begin{aligned}\lim_{z\to 0}\frac{p_x^2(z)}{p_x^2(z)+p_y^2(z)} &= \lim_{z\to 0}\frac{p_x^2(z)/z^2}{(p_x^2(z)+p_y^2(z))/z^2} = \lim_{z\to 0}\frac{(p_x^2)''(z)}{(p_y^2+p_x^2)''(z)} \\ &= \frac{2\,p_x'(0)^2}{2\,(p_x'(0)^2+p_y'(0)^2)} = \frac{p_x'(0)^2}{p_x'(0)^2+p_y'(0)^2} > 0.\end{aligned} \tag{6.16}$$

This means that f_3 is holomorphic on $B_\varepsilon\backslash\{0\}$ and its domain can be extended to the origin such that f_3 is continuous, but then f_3 can be continued analytically on B_ε with $f_3(0) > 0$. □

Lemma 6.6. *Consider the intersection of spheres in* $\mathbb{R}^3$

$$\begin{aligned} x^2+y^2+z^2-r_0 &= 0, \\ (x-p_x(t))^2+(y-p_y(t))^2+(z-p_z(t))^2-r(t) &= 0, \end{aligned} \tag{6.17}$$

where p_x, p_y, p_z, r_1 *are analytic real functions,* $r'(0) = 0$*,* $r(0) = r_0$ *and*

$$\begin{pmatrix} p_x(0) \\ p_y(0) \\ p_z(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} p_x'(0) \\ p_y'(0) \\ p_z'(0) \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

For every $p \in B_{r_0}(0)$ *with* $p \perp (p_x'(0), p_y'(0), p_z'(0))^T$ *there exists an analytic path* $\delta\colon (-\varepsilon, \varepsilon) \to \mathbb{R}^3$ *with* $\delta(0) = p$ *fulfilling* (6.17)*.*

Proof. Let $p \in B_{r_0}$ with $p \perp (p_x'(0), p_y'(0), p_z'(0))^T$ and let E be the plane through the origin spanned by the position vector of p and $(p_x'(0), p_y'(0), p_z'(0))^T$. We choose a two-dimensional coordinate system in E and can now apply Lemma 6.5 to the circles given by the intersection of E with the spheres of (6.17). □

Chapter 7

Conclusion

In this chapter we will give a short summary and conclusion of the work done in this thesis. Additionally, we will point out several possibilities to continue the research in the future.

7.1 Summary and conclusion

We focused on the following decision problem (D): Let p be a singularity of a real algebraic set X embedded in euclidean space $\mathbb{R}^n$. Decide whether p is a non-manifold point of X. A non-manifold point is in this context a point $x \in X$, where the algebraic set is not locally a submanifold of $\mathbb{R}^n$, see Definition 2.3. This research question was motivated primarily by the kinematic study of linkages, as the configuration space of a linkage can be represented by a real algebraic set, see Section 6.1. Any non-manifold point p of the configuration space of a linkage is called a configuration space singularity (CS-singularity) and the linkage will exhibit degenerate kinematic behavior in p. See Section 1.4 for more details on singularities of linkages.

With Proposition 3.4 we proved the following fact: Any point $p \in X$ of a real algebraic set X is a manifold point of X if and only if the coordinate ring of the set germ (X, p) is regular. Subsequently with Proposition 3.5 and the analytic Nullstellensätze we could explain why a real singularity of the complex variety $\mathbf{V}(I)$, $I \leq \mathbb{R}[\mathbf{x}]$, will in general not be a non-manifold point of $\mathbf{V}_{\mathbb{R}}(I)$. In addition, we saw that whether a point in $\mathbf{V}_{\mathbb{R}}(I)$ is a manifold point is an intrinsic property of the real algebraic set $\mathbf{V}_{\mathbb{R}}(I)$, see Section 3.3.

Thereafter, Corollary 3.7 followed immediately: Let $p \in \mathbf{V}_{\mathbb{R}}(I)$ and $I_p \cdot \mathbb{R}[[\mathbf{x}]]$ be real for the translated ideal I_p. Then, p will be a manifold point of $\mathbf{V}_{\mathbb{R}}(I)$ if and only if p is nonsingular. This means we can solve (D) for $p \in \mathbf{V}_{\mathbb{R}}(I)$ provided $I_p \cdot \mathbb{R}[[\mathbf{x}]]$ is real.

At this point we could utilize a criterion by G. Efroymson [19] and got the following result: Real singular points of normal varieties $\mathbf{V}(I)$ are either non-manifold points of $\mathbf{V}_{\mathbb{R}}(I)$ or not a limit point of the nonsingular real locus, Corollary 3.9. This implies e.g. the fact that for normal real varieties X with isolated singularity $p \in X$, p will either be a non-manifold point or isolated in X (also known as "hermit point").

The restriction to normal varieties is superficial in a way. Since normalization and completion of a local ring commute in the sense of Theorem 3.12, we can apply Efroymson's criterion and Corollary 3.7 to arbitrary varieties. This decides (D) in many cases, see the remark after Theorem 3.12.

For real algebraic curves we could prove the criterion of Theorem 4.4 which solves (D) completely. It states the following: Let $I \leq \mathbb{R}[\mathbf{x}]$ be radical with $\dim I = 1$ and $f\colon Y \to Z$ be the normalization of the affine scheme $Z = \operatorname{Spec}(\mathbb{R}[\mathbf{x}]/I)$. Then, $\mathbf{V}_{\mathbb{R}}(I)$ is locally a one-dimensional manifold at $p \in \mathbf{V}_{\mathbb{R}}(I)$ if and only if there is one real point in the fiber $f^{-1}(p)$ and this is a simple root.

Finally, we gave a criterion for real plane algebraic curves X, which connects the algebraic tangent cone C with local geometric properties of X at the origin, Theorem 4.11. Among other things it states – as expected from Theorem 4.4 – that X is locally a manifold at the origin if $C \subset \mathbb{P}_1(\mathbb{C})$ has exactly one root in $\mathbb{P}_1(\mathbb{C})_{\mathrm{re}} = \{[a : b] \in \mathbb{P}(\mathbb{C}) \mid a, b \in \mathbb{R}\}$ and this is a simple root. Theorem 4.11 also explains why not every root in $C \cap \mathbb{P}_1(\mathbb{C})_{\mathrm{re}}$ gives rise to a tangent of X (but any simple root of C in $\mathbb{P}_1(\mathbb{C})_{\mathrm{re}}$ does). Theorem 4.11 is quite useful to create examples with different kinds of local real behavior.

Apart from the theoretical results above, one of our goals was to work out and implement an algorithm to decide (D) for real curves over $\mathbb{Q}$. We were able to do this with Proposition 5.3 using the normalization algorithm from [30] and the multivariate Tarski-query [4], both implemented in `Singular`. See Appendix A for the complete source code reference as well as implementation details and some examples.

Along the way we identified several possible pitfalls of computational aspects of local real algebraic geometry. Like the circumstance that for real ideals $I \leq \mathbb{Q}[\mathbf{x}]$ the extension $I \cdot \mathbb{R}[\mathbf{x}]$ is not real in general. This is contrary to the related concept of radical ideal or reducedness, see Example 2.2.

In addition, we proved a number of statements which are not referenced in most books on commutative algebra and can be useful for designing algorithms in real algebraic geometry. For example, the fact that any primary zero-dimensional non-radical ideal of $\mathbb{Q}[\mathbf{x}]$ has no nonsingular solutions in $\mathbb{C}$, Lemma 4.3. Or that the homomorphism $\mathbb{K} \otimes_{\mathbb{Q}} A \to \mathbb{K} \otimes_{\mathbb{Q}} B$ induced by the normalization $A \to B$ of an affine ring A over $\mathbb{Q}$ is the normalization of $\mathbb{K} \otimes_{\mathbb{Q}} A$ for any field extension $\mathbb{Q} \subset \mathbb{K}$, Proposition 5.2.

To conclude this section, we want to give a quick personal assessment of the implication of our results for the research in local real algebraic geometry and theoretical kinematics.

In contrast to topics like global topology and global optimization, local geometric properties at singular points of real algebraic sets have not attracted much attention since the death of Gus Efroymson (1983), who was one of the leading contributors in this domain. This is partly due to the fact that results will not be very elegant in general and many established tools and techniques from algebraic geometry need to be adjusted – if they work at all. See the introductions of [7, 60] for more on this subject.

We hope that this thesis could provide a convenient entryway to (computational) aspects of local real algebraic geometry. It gives motivating results for the general theory and provides clarification on some crucial points which create confusion for researchers, that are new to the field. Furthermore, it highlights some interesting open problems and applications, see also Section 7.2.

In regard to contributions to theoretical kinematics, we can say that this thesis addressed for the first time the subject that reduced configuration-spaces of linkages could have singular manifold points. However, it has been observed before that linkages exist with fat points in the (non-reduced) configuration space [46] and there exists some conclusive topological theory for planar linkages and spatial frameworks [6, 21].

As part of this discussion we provided for the Delta platform the first rigorous proof that all singularities are non-manifold points. In addition, we initiated the search for a linkage with a singular point manifold point in its reduced configuration space.

To put these results in perspective however, it needs to be said that practical kinematicians and roboticists care more about finding a connected smooth subset of the configuration space big enough to execute all tasks of the robot. Any singularity can often be excluded from the safe configuration space without deliberation whether it could be a manifold point. We should also point out that even though a singular manifold point of the configuration space does not lead to kinematic deviations, it will likely have an impact on the dynamics of the linkage. This leads to the notion of shakiness, see [56, pp. 189-197].

Nevertheless, we are confident that we could dispel some misunderstandings concerning CS-singularities and provide some inspirational impulses for future research.

7.2 Future work

Much of the theory in this thesis can be developed further. We will list a selection of additional research topics here, divided in the areas of (computational) real algebraic geometry and kinematics.

Real algebraic geometry

(i) Theorem 4.4 indicates an algorithm to determine the geometric tangent (half)-cone of a real algebraic curve X. For simple real roots q in the fiber $f^{-1}(0)$ under the normalization $f\colon Y \to X$, we get a tangent line by projecting the tangent space T_qY to X and we can ignore all non-real points in $f^{-1}(0)$. It remains to find the tangent (half)-lines induced by real roots in $f^{-1}(0)$ of higher multiplicity.

On the computational side we would need to decompose $f^{-1}(0)$ over $\mathbb{R}$, so we require an algorithm to factorize a $\mathbb{Q}$-polynomial in the algebraic closure $\overline{\mathbb{Q}}$. One approach by G. Lecerf [31, Algorithm B.7.8] is implemented in `Singular`, see `absFactorize` [62].

(ii) As the problem of identifying non-manifold points was motivated by applications in kinematics, it is quite natural to ask for a version of Algorithm 5.3 for simple algebraic field extension $\mathbb{Q}(\alpha)$ of $\mathbb{Q}$ with $\alpha \in \overline{\mathbb{Q}}$.

The employed normalization algorithm [29] works in $\mathbb{Q}(\alpha)$ without problems, but it remains to find a way to count the real roots of maximal ideals M in $\mathbb{Q}(\alpha)[\mathbf{x}]$. To work with the Tarski-query $\mathrm{TaQ}(1, M)$ we need to count sign variations in a sequence of numbers in $\mathbb{Q}(\alpha)$ which is part of the calculation of the signature of the Hermite quadratic form [4, Algorithm 8.18]. For this we have to encode an ordering in $\mathbb{Q}(\alpha)$, i.e. we need an algorithm which decides if $q(\alpha) > 0$, for $q \in \mathbb{Q}[x]$. This is trivial e.g. for square roots α, but harder for real zeros α of an arbitrary irreducible polynomial $\mu \in \mathbb{Q}[\mathbf{x}]$. In particular, there is the problem of "choosing" a real root of μ. See [4, Section 10.2] for isolating lists and [4, Algorithm 10.6] for a sign determination algorithm. Unfortunately, none of those algorithms are implemented in `Singular` yet.

(iii) Algorithm 5.1 checks for a zero-dimensional ideal $M \leq \mathbb{Q}[\mathbf{x}]$ if M is real in $\mathbb{R}[\mathbf{x}]$. By analyzing the ideas in [5, 57] we should be able to extend this to arbitrary ideals $I \leq \mathbb{Q}[\mathbf{x}]$ (or $I \leq \mathbb{Q}(\alpha)[x]$, see (ii)).

More general, one could implement an algorithm to calculate the real radical over $\mathbb{R}_{\mathrm{alg}}$. A constructive proof of existence can be found in [5]. With the Tarski-Seidenberg principle this will be the real radical over $\mathbb{R}$. The implementation of the algorithm over $\mathbb{Q}$ by S. Spang [71,72] could provide a convenient starting point for this.

(iv) Finally, it would be desirable to generalize Theorem 4.4 and Algorithm 5.3 to higher-dimensional real algebraic varieties $X = \mathbf{V}_{\mathbb{R}}(I)$. The simpler problem to decide if $I \cdot \mathbb{R}[[\mathbf{x}]]$ is real can be done algorithmically if a desingularization $f\colon Z \to Y = \mathbf{V}(I)$ is known, see Example 3.1. Unfortunately, if $I \cdot \mathbb{R}[[\mathbf{x}]]$ is not real only little can be deduced from the fiber $f^{-1}(0)$ alone since it can happen that there are no real points in $f^{-1}(0)$. Consider e.g. the example $I = \langle x^2 + y^2 + z^2 - z^3, w^2 - u^2 - u^3 - x \rangle \leq \mathbb{R}[x, y, z, w, u]$. $\mathbf{V}_{\mathbb{R}}(I)$ contains the curve $\mathbf{V}_{\mathbb{R}}(w^2 - u^2 - u^3, x, y, z)$ as connected component in the euclidean topology, hence the origin is a non-manifold point of $\mathbf{V}_{\mathbb{R}}(I)$.

In the `Singular`-file `#fw_example` – which is attached to the digital version of this thesis or can be downloaded from our online repository A.3 – we calculate a desingularization of the variety $\mathbf{V}(I)$ and show that there are no real points in the fiber of the origin.

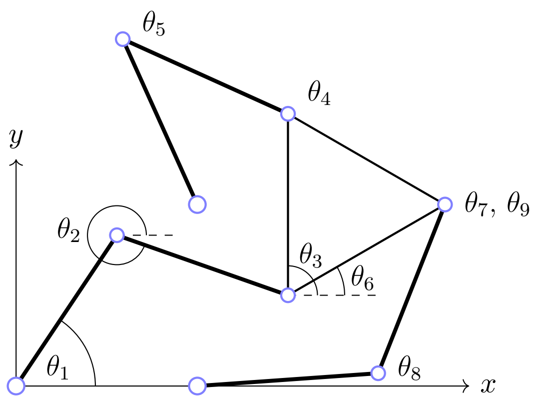


Figure 7.1: A planar 3RRR-mechanism.

Kinematics

(i) Despite considerable effort a linkage which has a configuration space with singular manifold points could not be found. A priori, it is not clear if such a linkage exists. Any example would be very interesting to analyze in regard to its dynamic properties in the singular configurations.

(ii) Another linkage which can be analyzed in a similar fashion to the five-bar linkage and the delta-platform is the 3RRR-linkage of Figure 7.1. Its configuration space X and singularities have been analyzed thoroughly in [6, 63, 70]. For generic parameters of the linkage all singularities are known to be non-manifold points [6].

As in the case of the delta robot, X is a three-dimensional normal real algebraic set with zero-dimensional singular locus. Therefore, any singularity of X will be either a non-manifold point or isolated in X. This could help to examine the non-generic cases in [6].

(iii) Another remaining task is to investigate the configuration space $X_{a,b,d}$ of the delta robot for parameters a, b, d not contained in O, where O is the Zariski open set from Definition 6.1. It is conjectured that Theorem 6.2 and Corollary 6.3 hold for all $a, b, d \in \mathbb{R}\backslash\{0\}$, $a^2 > d^2 > 0$ but we restricted the parameter space in the proofs to O for two reasons:

In the special parameter case $b = \frac{3\,d^2 - a^2}{a^2 + 3\,d^2}$ we were not able to show that q_1 and q_2 of Table 6.1 are not isolated in $X_{a,b,d}$ because we use $b \neq \frac{3\,d^2 - a^2}{a^2 + 3\,d^2}$ in Section 6.4.4 to apply the implicit function theorem. To decide if q_1 and q_2 are not isolated in the configuration space, one idea is to investigate the blow ups at q_1 and q_3 in dependence of the parameters $a, b, d > 0$, with $b = \frac{3\,d^2 - a^2}{a^2 + 3\,d^2}$.

For the proof of Theorem 6.2 we excluded a hyperplane $\mathbf{V}(f_1 \cdots f_k)$ from the parameter space to enable the calculation of Gröbner covers of the components

K_2, K_3, K_5, K_5. To analyze these components for the parameter cases $(a, b, d) \in \mathbf{V}(f_j)$ we could manually perform the steps of the Kapur-Sun-Wang algorithm [42, 4.1] called by `grobcov`. Then, in contrast to the implementation in `Singular` via `grobcov` we are free to use any technique for Gröbner basis calculations including modular methods [2, 61] which often have a higher probability to succeed and can effectively be parallelized [34, 37].

Appendix A

Singular libraries and online resources

In this appendix we provide the full source code for our implementations of Algorithms 5.3 and 5.2 as libraries of the CAS `Singular` [16]. In Section A.2 you will also find the `Singular`-libraries we use in Section 6.4 for the proof of Theorem 6.2. A hyperlink to some helpful online resources, including all `Singular` code, is available in Section A.3.

Every code listing is also attached to the digital version of this thesis with the `embedfile`-package. You should be able to access it using the attachments dialogue of your PDF-viewer.

A.1 The library `curvetest.lib`

Listed below is the source code of the `Singular`-library `curvetest.lib`, which implements Algorithm 5.2 and Algorithm 5.3. After the listing you will find a short description and some implementation details.

#curvetest.lib.

```
////////////////////////////////////////////////////////////////////////////
version="version curvetest.lib 0.1 Feb_2019 ";
category="real algebra";
info="
LIBRARY: curvetest.lib manifold and local reality test of real algebraic curve
AUTHOR : Marc Diesse

OVERVIEW:
   Algorithm to test if a plane algebraic curve has a non-manifold point at the origin.
   Non-manifold point means locally not a smooth (C^infinity) submanifold of euclidean space.
   Second procedure tests if the extensions of the ideal to the ring of formal power series is real.


PROCEDURES:
 curvetest(k); Test if V_R(k) has a manifold point at the origin.
 isrealpow(k); Test if k R[[x]] is a real ideal.
";

LIB "primdec.lib";
LIB "rootsmr.lib";
```

```
LIB "normal.lib";
LIB "sing.lib";
//Manifold test
proc curvetest(ideal k, list #)
"USAGE: curvetest(k); a one-dimensional ideal k
RETURN: 1: if V_R(k) has not a manifold point at the origin
      2: if the origin is isolated in V_R(k)
      0: if the origin is not a manifold point of V_R(k)


EXAMPLE: example curvetest; shows an example"


{
int ver = 0;
if (size(#) >= 1) {
    if (typeof(#[1]) == "string") {
        if (#[1] == "verbose") {ver = 1;} //verbose mode
    }
}


dbprint(ver,newline + "Curve Test");
dbprint(ver,"-------------");
int is_real = 1;
int smooth = 0;
int nonsmoothcomp = 0;
int i,j;
int nreal = 0;
int real_index;
list rings;

def r = basering;
ring rn;
setring r;


//Radical
dbprint(ver,"Calculating radical");
ideal K = radical(k);

//STB
dbprint(ver,"Calculating groebner base");
if (attrib(k,"isSB") != 1) {K = std(K);}

//preliminary checks
if (dim(K) != 1) {ERROR("You can only test affine curves");}

//origin in curve
if (maxdeg1(reduce(K,maxideal(1))) != -1) {ERROR("The origin is not on curve");}

//remaining checks
if (char(r) != 0 || size(parstr(r))!=0 || attrib(r,"global") != 1) {ERROR("Algorithm only works in
     ↪ zero characteristic for nonparametric rings with global term ordering");}

//maximal ideal
ideal M = maxideal(1);

//Normalization
dbprint(ver,"Calculating equidimensional decomposition");
list l = equidim(K);
K = l[size(l)]; //equidim component of same dimension
dbprint(ver,"Calculating normalization");
```

```
def nor = normal(K,"noDeco"); //we use the normalization without equidimensional decomp
int normaldec = size(nor[1]);
dbprint(ver,string(normaldec) + " Components in the normalization to check");


//if K has more components (zerodivisor found)
dbprint(ver,"Calculating the number of all real roots over the origin");
for (i=1; i<=normaldec; i=i+1) {
   rings[i] = nor[1][i];
   rn = rings[i];
   setring rn;
   ideal N = imap(r,M) + norid;
   nreal = nreal + nrRootsDeterm(N);
   if (nreal >= 1) {real_index = i;}
   setring r;
}
dbprint(ver,string(nreal) + " real roots found");

//zero
if (nreal >= 2) {
   dbprint(ver,newline);
   dbprint(ver,"--------------------------------------------");
   dbprint(ver,"the origin is not a manifold point of V_R(k)");
   dbprint(ver,"--------------------------------------------");
   return(0);
}
else {
   if (nreal == 0) {
      dbprint(ver,newline);
      dbprint(ver,"--------------------------------");
      dbprint(ver,"the origin is isolated in V_R(k)");
      dbprint(ver,"--------------------------------");
      return(2);
   }
   else {
   rn = rings[real_index];
   setring rn;
   list comp;
   comp = primdecGTZ(N);
   dbprint(ver,"Checking Decomposition of origin ideal in Normalization Component with real root");
   dbprint(ver,"--------------------------------");
   dbprint(ver,comp);
   dbprint(ver,"--------------------------------");
   for (j=1; j <= size(comp); j=j+1){ //Loop all Components
         if(nrRootsDeterm(comp[j][2]) > 0) {
            if(size(reduce(comp[j][2],std(comp[j][1]))) == 0) {
               dbprint(ver,"Component " + string(j) + " has real point of multiplicity 1");
               dbprint(ver,newline);
               dbprint(ver,"----------------------------------------");
               dbprint(ver,"the origin is a manifold point of V_R(k)");
               dbprint(ver,"----------------------------------------");
               return(1);
            }
            else {
               dbprint(ver,"Component " + string(j) + " has real point of multiplicity greater
                   ↪ 1");
               dbprint(ver,newline);
               dbprint(ver,"--------------------------------------------");
               dbprint(ver,"the origin is not a manifold point of V_R(k)");
               dbprint(ver,"--------------------------------------------");
               return(0);

            }
```

```
            }
        }
        }
    }

    ERROR("Something went wrong");
    }
    example
    { "EXAMPLE:"; echo = 2;
    ring r = 0,(x,y),dp;
    ideal I1 = y^2 - x^2 - x^3;
    ideal I2 = y^3 + y*x^2 - x^4;
    ideal I3 = y^2 - x^3;
    ideal I4 = x^2 - y^11;
    ideal I5 = x^2 - 2*y^2 + x^3;
    ideal I6 = x^2 + y^2 - x^3;
    ideal I7 = y^3 - x^11;
    curvetest(I1);
    curvetest(I2);
    curvetest(I3);
    curvetest(I4);
    curvetest(I5);
    curvetest(I6);
    curvetest(I7);

    ring r2 = 0,(x,y,z),dp;
    ideal I = (y+z)^3 + 5*(y+z)*x^2 - x^4, z-x;
    curvetest(I);
    }


    proc isrealpow(ideal k)
    "USAGE: isreal(k); k a one-dimensional ideal
    RETURN: 1: if k R[[x]] is real
         0: if k_R[[x]] is not real"


    {
    def r = basering;
    int i;
    ring rn;
    setring r;
    ideal M = maxideal(1);

    //preliminary checks
    if (dim(std(k)) != 1) {ERROR("You can only test affine curves");}

    //characteristic
    if (char(r) != 0) {ERROR("Algorithm only works for zero characteristic");}

    //k R[[x]] = R[[x]]?
    if (maxdeg1(reduce(k,maxideal(1))) != -1) {return(1);}

    //remaining checks
    if (size(parstr(r))!=0 || attrib(r,"global") != 1) {ERROR("Algorithm only works for nonparametric
         ↪ rings with global term ordering");}

    //Decomp
    list comp = primdecGTZ(k);
    for (i=1; i<=size(comp); i=i+1) {
        if (size(reduce(comp[i][2],M)) == 0) {
            if (size(reduce(comp[i][2],std(comp[i][1]))) != 0) { return(0);} //not radical => not real
```

```
        if (dim(std(comp[i][2])) == 1) {
            def nor = normal(comp[i][2],"noDeco");
            rn = nor[1][1];
            setring rn;
            ideal N = imap(r,M) + norid;
            N = std(radical(N));
            if (nrRootsDeterm(N) != size(kbase(N))) { return(0);}
            setring r;
        }
    }
}
return(1);
}
example
{ "EXAMPLE:"; echo = 2;
ring r = 0,(x,y),dp;
ideal I1 = y^2 - x^2 - x^3;
ideal I2 = y^3 + y*x^2 - x^4;
ideal I3 = y^2 - x^3;
ideal I4 = x^2 - y^11;
ideal I5 = x^2 - 2*y^2 + x^3;
ideal I6 = x^2 + y^2 - x^3;
ideal I7 = y^3 - x^11;
isrealpow(I1);
isrealpow(I2);
isrealpow(I3);
isrealpow(I4);
isrealpow(I5);
isrealpow(I6);
isrealpow(I7);

ring r2 = 0,(x,y,z),dp;
ideal I = (y+z)^3 + 5*(y+z)*x^2 - x^4, z-x;
isrealpow(I);
}
```

Listing A.1: The library `curvetest.lib`.

User guide and examples The preceding `Singular`-library provides the procedures `curvetest` and `isrealpow` which can be used in the following way:

```
curvetest(I,["verbose"]);
isrealpow(I);
```

Here, I needs to be a one-dimensional ideal of a polynomial ring $\mathbb{Q}[\mathbf{x}]$ with global term ordering and $I \subset \langle \mathbf{x} \rangle$. Both procedures throw errors if those conditions are not met. `curvetest` and `isrealpow` return the following values:

$$\texttt{curvetest(I)} = \begin{cases} 0 & \text{The origin is not a manifold point of } \mathbf{V}_{\mathbb{R}}(I). \\ 1 & \text{The origin is a manifold point of } \mathbf{V}_{\mathbb{R}}(I). \\ 2 & \text{The origin is isolated in } \mathbf{V}_{\mathbb{R}}(I). \end{cases}$$

$$\texttt{isrealpow(I)} = \begin{cases} 0 & I \cdot \mathbb{R}[[\mathbf{x}]] \text{ is not real.} \\ 1 & I \cdot \mathbb{R}[[\mathbf{x}]] \text{ is real.} \end{cases}$$

The optional argument `"verbose"` of `curvetest` gives commented output for any stage of the algorithm. For some examples we show the output of the `example` command.

```
> example curvetest;
// proc curvetest from lib
    ↪ curvetest.lib
EXAMPLE:
ring r = 0,(x,y),dp;
ideal I1 = y^2-x^2-x^3;
ideal I2 = y^3+y*x^2-x^4;
ideal I3 = y^2-x^3;
ideal I4 = x^2-y^11;
ideal I5 = x^2-2*y^2+x^3;
ideal I6 = x^2+y^2-x^3;
ideal I7 = y^3-x^11;
curvetest(I1);
0
curvetest(I2);
1
curvetest(I3);
0
curvetest(I4);
0
curvetest(I5);
0
curvetest(I6);
2
curvetest(I7);
0

ring r2 = 0,(x,y,z),dp;
ideal I =
(y+z)^3+5*(y+z)*x^2-x^4,
z-x;
curvetest(I);
1
```

```
> example isrealpow;
// proc isrealpow from lib
    ↪ curvetest.lib
EXAMPLE:
ring r = 0,(x,y),dp;
ideal I1 = y^2-x^2-x^3;
ideal I2 = y^3+y*x^2-x^4;
ideal I3 = y^2-x^3;
ideal I4 = x^2-y^11;
ideal I5 = x^2-2*y^2+x^3;
ideal I6 = x^2+y^2-x^3;
ideal I7 = y^3-x^11;
isrealpow(I1);
1
isrealpow(I2);
0
isrealpow(I3);
1
isrealpow(I4);
1
isrealpow(I5);
1
isrealpow(I6);
0
isrealpow(I7);
1

ring r2 = 0,(x,y,z),dp;
ideal I =
(y+z)^3+5*(y+z)*x^2-x^4,
z-x;
isrealpow(I);
0
```

Implementation Details Since any zero-dimensional associated prime of I will clearly not have any impact in Algorithm 5.3 we first calculate a equidimensional decomposition of I and continue with the equidimensional locus I_1 of I, see line 79 of Listing A.1. Then we can execute the `normal` procedure with the option `"noDeco"` (line 83) which avoids any preliminary decomposition of I. If the normalization algorithm discovers any zero-divisor in its execution [30], it will calculate several polynomial rings R_i and ideals $N_i \leq R_i$, $i = 1, \ldots, k$ such that

$$R_1/N_1 \times \ldots \times R_k/N_k$$

is the normalization of $\mathbb{Q}[\mathbf{x}]/I$. Thus, we need to loop over the rings R_i/N_i (line 90) in our implementations of Algorithm 5.3.

For Algorithm 5.2 we chose a different approach and calculate a primary decomposition of I beforehand (line 203). Then, we determine a normalization for every one-dimensional prime component separably, which will return only one ring each. In this way we can check that I is radical without further calculations (line 206).

A.2 The libraries `delta.lib` and `rwlist.lib`

Listed below is the source code for the `Singular`-libraries used in Section 6.4 for the proof of Theorem 6.2.

The library `rwlist.lib` provides procedures to write lists of ideals to textfiles and access them in a different `Singular`-session. It will be used to store the results of primary decompositions.

`delta.lib` implements the procedures `DeltaGetCsIdeal` and `DeltaSingularities` to return equations of the delta robot configuration space (6.5) and the singularities from Table 6.1.

#delta.lib

```
////////////////////////////////////////////////////////////////////////////
version="version delta.lib 0.1 Dez_2019 ";
category="applications";
info="
LIBRARY: delta.lib ideals for delta calculations
AUTHOR : Marc Diesse

OVERVIEW:
  Procedures to write lists of ideals to files


PROCEDURES:
 DeltaGetCsIdeal();  get Ideal of Configuration Space
 getFitting();  get Fitting Ideal
";

LIB "inout.lib";

//////////////////////////////////////////////////////////7
proc DeltaGetCsIdeal()
{
ring r = (0,a,b,d),(x1,y1,z1,x2,y2,z2,x3,y3,z3,ca1,sa1,ca2,sa2,ca3,sa3,s3),(dp(15),dp(1));

poly k1 = x1^2 + y1^2 + z1^2 - b^2;
poly k2 = x2^2 + y2^2 + z2^2 - b^2;
poly k3 = x3^2 + y3^2 + z3^2 - b^2;
poly k2d1 = ca1^2 + sa1^2 - a^2;
poly k2d2 = ca2^2 + sa2^2 - a^2;
poly k2d3 = ca3^2 + sa3^2 - a^2;
poly wurzel = s3^2 - 3;

matrix A[3][3] = -1/2,-s3/2,0,s3/2,-1/2,0,0,0,1;
matrix Ai[3][3] = -1/2,s3/2,0,-s3/2,-1/2,0,0,0,1;
```

```
matrix kl1[3][1] = d + ca1 + x1,y1,z1+sa1;
matrix kl2[3][1] = d + ca2 + x2,y2,z2+sa2;
matrix kl3[3][1] = d + ca3 + x3,y3,z3+sa3;

matrix gl1 = kl1 - A*kl2;
matrix gl2 = kl1 - Ai*kl3;

ideal I = wurzel,ideal(gl1),ideal(gl2),k1,k2,k3,k2d1,k2d2,k2d3;
export(I);
return(r);
}


proc DeltaSingularities()
{
def r1 = DeltaGetCsIdeal();
setring r1;
matrix M = jacob(I);
matrix Msub[12][15] = M[2..13,1..15];
ideal J = minor(Msub,12);
ideal B = I,J;

ring r2 = 0,(a,b,d,x1,y1,z1,x2,y2,z2,x3,y3,z3,ca1,sa1,ca2,sa2,ca3,sa3,s1,s2,s3),dp;
ideal B = imap(r1,B);
ideal J = imap(r1,J);

poly w3 = s3^2 - 3;
poly w2 = s2^2 - (a^2 + 3*d^2);
poly w1 = s1^2 - (a^2 - d^2);

ideal sing1 = s2*x1 - (-d*b),s2*y1 - (-s3*d*b), s2*z1 - (s1*b),s2*x2 - (-d*b),s2*y2 - (s3*d*b), s2*z2 -
     ↪ (s1*b), s2*x3 - (2*d*b),y3, s2*z3 - (s1*b), ca1 - (-d), sa1 - s1, ca2 - (-d), sa2 - s1, ca3 -
     ↪ (-d), sa3 - s1,w1,w2,w3;

//option(contentSB);
//option(intStrategy);

export(J);
export(B);
export(sing1);
return(r2);
}
```

Listing A.2: The library delta.lib.

#rwlist.lib

```
///////////////////////////////////////////////////////////////////////////////
version="version rwlist.lib 0.1 Dez_2019 ";
category="I/O";
info="
LIBRARY: rwlist.lib read and write Lists
AUTHOR : Marc Diesse

OVERVIEW:
  Procedures to write lists of ideals to files


PROCEDURES:
 readlist(s);  Read file s
 writelist(s,L); Write list L to file s
";
```

```
LIB "inout.lib";

///////////////////////////////////////////////////////7
proc writelist(s,L)
{
 int i,j;
 link l=":w " + s;
 write(l,"ideal(" + string(L[1]) + "),");
 l = ":a " + s;
    for (i=2; i <= size(L); i++) {
           if (i < size(L)) {
           write(l,"ideal(" + string(L[i]) + "),");
           }
           else
           {
           write(l,"ideal(" + string(L[i]) + ")");
           }
           //write(l,L[i]);
           //write(l,"),");
    }
 close(l);
}


proc readlist(s)
{
 int i,j;
 link l=":r " + s;
 string inp = read(l);
 close(l);
 string listri = "list L = " + inp + ";";
 execute(listri);
 return(L);
}
```

Listing A.3: The library `rwlist.lib`.

A.3 Online repository and animations

You can download the source code of the libraries and all `Singular` scripts at the following URL: `http://pc.cd/WA97`.

In this repository you will also find several animations of the crank-slider, the four-bar, the five-bar and the delta robot in its singular configurations. All animations are realized with the `glowscript`-library and can be rendered live in most current web browsers with JavaScript enabled. You can either download all animation files or run the hosted version.[1]

[1] `https://filedn.com/lps1cXVaYw95l2veqL7W55J/RobotAnimation/index.html`.

Zusammenfassung (Abstract dt.)

Anders als komplexe Varietäten kann eine reelle algebraische Varietät X, eingebettet in $\mathbb{R}^n$, an einem singulären Punkt $p \in X$ glatt sein in dem Sinne, dass X lokal eine analytische Untermannigfaltigkeit von $\mathbb{R}^n$ ist. Das liegt daran, dass der analytische Nullstellensatz nicht für reelle analytische Varietäten gilt, d.h. Teile von analytischen Zweigen von X an p können im reellen nicht sichtbar sein.

Die Identifizierung von Nicht-Mannigfaltigkeitspunkten reeller algebraischer Mengen erfordert einen neuartigen Ansatz und Theorie aus analytischer und reeller algebraischer Geometrie. In dieser Arbeit werden wir mehrere Ergebnisse vorstellen um reelle Glattheit nachzuweisen oder zu widerlegen. Darüber hinaus behandeln wir das Problem, diese Kriterien algorithmisch zu überprüfen. Basierend auf diesen Resultaten wurde eine `Singular`-Prozedur für algebraische Kurven implementiert.

Als Anwendung konnten alle Konfigurationsraum-Singularitäten mehrerer bekannter kinematischer Getriebe bestimmt werden.

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Selbstständigkeitserklärung

Name: **Diesse, Marc**

Ich erkläre gegenüber der Freien Universität Berlin, dass ich die vorliegende Dissertation selbstständig und ohne Benutzung anderer als der angegebenen Quellen und Hilfsmittel angefertigt habe. Die vorliegende Arbeit ist frei von Plagiaten. Alle Ausführungen, die wörtlich oder inhaltlich aus anderen Schriften entnommen sind, habe ich als solche kenntlich gemacht. Diese Dissertation wurde in gleicher oder ähnlicher Form noch in keinem früheren Promotionsverfahren eingereicht.

Datum: **4. Mai, 2020**

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