Freie Universität Berlin

Fachbereich Mathematik und Informatik

The topology of conjugate Berkovich spaces

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Dissertation

zur Erlangung des akademischen Grades eines
Doktors der Naturwissenschaften (Dr. rer. nat.)

2020

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Tag der Disputation: 10.06.2020

Abstract

We want to understand how the topology of Berkovich spaces varies when we conjugate the non-archimedean base field. After a short introduction with a discussion of the original problem solved by Serre [Ser64] in the complex setting, we explain some background material about non-archimedean geometry and non-archimedean analytifications. We are able to construct examples of non-homeomorphic conjugate Berkovich spaces by controlling the homotopy type of the Berkovich analytification via its skeleton and its tropicalization. In the appendix we include some useful programs written in SAGE that compute the examples of the last section.

Zussamenfassung

Wir wollen verstehen, wie sich die Topologie von Berkovich-Räumen verändert, wenn wir der nichtarchimedische Basiskörper konjugieren. Nach einer kurzen Einführung mit einer Diskussion des ursprünglichen Problems, das von Serre [Ser64] in der komplexen Umgebung gelöst wurde, erläutern wir etwas Hintergrundmaterial über nichtarchimedische Geometrie und nichtarchimedische Analysis. Wir konstruieren Beispiele für nicht-homöomorph konjugierte Berkovich-Räume, indem wir den Homotopietyp der Berkovich-Analytifizierung über ihr Berkovich-Skelett und ihre Tropikalisierung kontrollieren. Im Anhang gibt es in SAGE geschriebene Programme, die die Beispiele des letzten Abschnitts berechnen.

Between 2016 and 2020, (at least) 12,685 people died in the Mediterranean trying to reach Europe according to the International Organization for Migrants of the United Nations. This is a low estimate, since many deaths are not even reported nor heard by anyone who was not on the sinking boat. Since World War II there have never been so many unburied bodies in Europe.

This thesis, done in this time period, is dedicated to the memory of all migrants who died on their way to Europe, so much at origin as during voyage or after arrival.

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1 Introduction

Given a projective algebraic variety X over a number field K, and choosing an embedding $\phi : K \hookrightarrow \mathbb{C}$, we can consider its base change $X_\phi := X \times_{K,\phi} \mathbb{C}$ so that it becomes a complex algebraic variety. Now we can follow GAGA [Ser56] and consider its complex analytification X_ϕ^{an}, whose underlying topological space is just the set of complex points $X_\phi(\mathbb{C})$ together with the euclidean topology. Note that considering just the underlying topological space is equivalent to forgetting the analytic structure of X_ϕ^{an}: for example, two elliptic curves with different j-invariants are different as analytic spaces, but they both have the same underlying topological space (homeomorphic to a doughnut).

Some of the topological invariants of $X_\phi(\mathbb{C})$ are independent of the chosen embedding ϕ: for example, the comparison between étale cohomology and singular cohomology due to Artin [SGA4, Exp. XI] implies that the Betti numbers of $X_\phi(\mathbb{C})$ are independent of ϕ.

If X is a smooth projective curve of genus g, we know that $X_\phi(\mathbb{C})$ is a Riemann surface with g holes. Hence, the variation of ϕ doesn't change the topology of the analytification, since choosing a different ϕ doesn't change the genus of the curve.

In arbitrary dimension, we can also say something about the topological fundamental group of $X_\phi(\mathbb{C})$. We first fix a base point $x \in X(\mathbb{C})$, and denote also by x its pullback to $X_\phi(\mathbb{C})$. Since the work of Grothendieck and others (see for example [SGA1], or [Sza09, Theorem 5.7.4] for an easier introduction), we know that the profinite completion of $\pi_1^{\text{top}}(X_\phi(\mathbb{C}), x)$ is isomorphic to the étale fundamental group $\pi_1^{\text{ét}}(X_\phi, x)$, and this group is independent of the chosen ϕ (see for example [Esn17, Proposition 6.1]). Indeed, all base changes of X to an algebraic closure of the base field K are isomorphic as K-schemes; in particular, the geometric étale fundamental group $\pi_1^{\text{ét}}(X_\phi, x)$ is independent of the chosen ϕ up to isomorphism. Hence, the profinite completion of $\pi_1^{\text{top}}(X_\phi(\mathbb{C}), x)$ is independent of the chosen embedding ϕ.

With this kind of examples in mind, it was a plausible question back in the 60's whether the topology of $X_\phi(\mathbb{C})$ is independent of the chosen ϕ. In other words, the scheme X_ϕ is equipped with a structure morphism $X_\phi \to \operatorname{Spec}(\mathbb{C})$ given by the Cartesian square[1]

$$\begin{array}{ccc} X_\phi & \longrightarrow & \operatorname{Spec}(\mathbb{C}) \\ \downarrow & & \downarrow \phi \\ X & \longrightarrow & \operatorname{Spec}(K), \end{array}$$

and the question is whether the topology of $X_\phi(\mathbb{C})$ is intrinsic to the abstract scheme X_ϕ, or if it depends on the structure morphism induced by the embedding $\phi : K \hookrightarrow \mathbb{C}$.

Serre answered this question [Ser64]: he constructs a variety X over a number field K and embeddings $\phi, \psi : K \hookrightarrow \mathbb{C}$ such that the topological fundamental group $\pi_1^{\text{top}}(X_\phi(\mathbb{C}), x)$ is different from $\pi_1^{\text{top}}(X_\psi(\mathbb{C}), x)$. Let's say a word on Serre's example:

- Construction of K. The number field K is the Hilbert class field of a certain imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, where p is a prime number congruent to -1 modulo 4 and the class number of $\mathbb{Q}(\sqrt{-p})$ satisfies an extra condition that we

[1] As usual, we abuse notation and write ϕ instead of $\operatorname{Spec}(\phi)$.

don't explain here; for example, we can take K to be the Hilbert class field of $\mathbb{Q}(\sqrt{-23})$, which happens to be $K = \mathbb{Q}(\sqrt{-23})[x]/(x^3 - x - 1)$.

- Construction of the embeddings $\phi, \psi : K \hookrightarrow \mathbb{C}$. Due to the theory of complex multiplication of elliptic curves, there exists an elliptic curve E defined over K whose ring of endomorphisms is precisely the ring of integers of the imaginary quadratic field. In this situation, there exist embeddings $\phi, \psi : K \hookrightarrow \mathbb{C}$ such that $\pi_1^{\text{top}}(E_\phi(\mathbb{C}), e)$ is a free $\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}$ -module of rank 1, but $\pi_1^{\text{top}}(E_\psi(\mathbb{C}), e)$ is not free as an $\mathcal{O}_{\mathbb{Q}(\sqrt{-p})}$ -module.

- Construction of X defined over K. Let A be the abelian variety given by $A = E^{\frac{p-1}{2}}$, and let Y be the hypersurface inside $\mathbb{P}_K^{p-1}$ given by the equation $\sum_{i=1}^{p} x_i^p = 0$; in our above example, we would have $A = E^{11}$ and Y the 21-dimensional hypersurface given by $\sum_{i=1}^{23} x_i^{23} = 0$. Let G be the cyclic group of order p, which acts both on Y by permutation of the coordinates and on A by a different action that we don't describe here. Then, Serre defines X as the quotient of $Y \times A$ by G, where G acts on $Y \times A$ via $g(y, a) = (g^{-1}y, ga)$. Now we base change X via ϕ and ψ and we get X_ϕ and X_ψ such that $X_\phi(\mathbb{C})$ and $X_\psi(\mathbb{C})$ have different topological fundamental groups.

Serre calls the varieties X_ϕ and X_ψ *conjugate*, as we can go from X_ϕ to X_ψ via a conjugation of the complex numbers $\sigma \in \mathrm{Aut}_K(\mathbb{C})$ (indeed, σ is an extension of a particular element in the absolute Galois group $\mathrm{Gal}(\overline{K}, K)$). In other words, if $\phi = \sigma \circ \psi$, then we have a commutative diagram

$$\begin{array}{ccccc}
X_\phi & & \longrightarrow & & X_\psi \\
\downarrow & & & & \downarrow \\
\mathrm{Spec}(\mathbb{C}) & & \xrightarrow{\sigma} & & \mathrm{Spec}(\mathbb{C}) \\
& \searrow^{\phi} & & {}^{\psi}\swarrow & \\
& & \mathrm{Spec}(K) & &
\end{array}$$

which is Cartesian, i.e. $X_\phi = X_\psi \times_{\mathbb{C},\sigma} \mathbb{C}$. Note that X_ϕ and X_ψ are isomorphic as K-schemes, but not as $\mathbb{C}$-schemes.

We have seen our first example of non-homeomorphic conjugate varieties, where the topological fundamental group of the analytification of one variety is different than the one of its conjugate's. In particular, since the profinite completion of the topological fundamental group is independent of the embedding ϕ, we see that the difference between $\pi_1^{\text{top}}(X_\phi(\mathbb{C}), x)$ and its profinite completion can be far from trivial. Actually, we don't know whether there exists a complex projective variety V such that $\pi_1^{\text{ét}}(V, x)$ is trivial but $\pi_1^{\text{top}}(V(\mathbb{C}), x) \neq 1$ (c.f. [Sza09, Remark 5.7.5]). Another example in this direction is given by Toledo [Tol93], who constructs a smooth projective variety V over $\mathbb{C}$ such that the profinite completion map $\pi_1^{\text{top}}(V(\mathbb{C}), x) \to \pi_1^{\text{ét}}(V, x)$ contains a free kernel of infinite rank.

Since Serre's example, there have been in the literature more examples of non-homeomorphic conjugate (complex) varieties, where different topological invariants have been studied (see for example [Abe74; ACC07; BCG15; MS10; Shi09; Shi18; Shi19]).

The aim of this PhD is to analyze the analog question in the non-archimedean world: are there non-homeomorphic conjugate non-archimedean varieties? What phenomena do occur in this world?

2 Non-archimedean geometry and Berkovich spaces

In this section, we recall the basic notions of non-archimedean geometry. In particular, we recall some facts from non-archimedean fields, rigid geometry and Berkovich spaces.

2.1 Non-archimedean fields

The first example of a non-archimedean field appeared in the work of Hensel, who introduced in 1893 the p-adic numbers as an analog of the local power series expansion around a point in complex analysis (c.f. [Ull95]) in order to analyze discriminants of field extensions. For example, given a non-zero natural number $n \in \mathbb{N}$ and a prime number p, we can write n as a finite sum $n = \sum_{\nu \geq r}^{R} a_\nu p^\nu$ in a unique way, with $\in \mathbb{N}$, $a_v \in \{0, 1, \dots, p-1\}$ and $a_r \neq 0$. Generalizing this idea, we can consider formal infinite series

$$\sum_{\nu \geq r} a_\nu p^\nu,$$

with r now in $\mathbb{Z}$, $a_v \in \{0, 1, \dots, p-1\}$ and $a_r \neq 0$. The set of all such series, together with the usual addition (with carrying) and multiplication, forms the *field of p-adic numbers*, denoted $\mathbb{Q}_p$.

Example 2.1. Let's do a some p-adic arithmetic. For $p = 2$, we can write -1 as $\sum_{\nu \geq 0} 1 \cdot 2^\nu$, because if we add 1 to the above series, then all the coefficients of the power series expansion will become zero[2]. Then, -2 can be written as $2(-1) = \sum_{\nu \geq 1} 1 \cdot 2^\nu$. We can also write $1/3$ as

$$1 + \sum_{\nu \geq 0} 2^{2\nu+1} = 1 + 2^1 + 2^3 + 2^5 + \dots$$

Indeed,

$$\begin{aligned} 3 \cdot \left(1 + \textstyle\sum_{\nu \geq 0} 2^{2\nu+1}\right) &= (1+2) \cdot \left(1 + \textstyle\sum_{\nu \geq 0} 2^{2\nu+1}\right) \\ &= 1 + \textstyle\sum_{\nu \geq 0} 2^{2\nu+1} + 2 + \textstyle\sum_{\nu \geq 0} 2^{2\nu+2} \\ &= 1 + 2 + \textstyle\sum_{\nu \geq 1} 2^\nu \\ &= 1. \end{aligned}$$

The importance of p-adic numbers in arithmetic geometry can be seen for example in the Hasse principle: Hasse [Has24] proved that a quadric defined over $\mathbb{Z}$ has a rational point if and only if it has a real point and a $\mathbb{Q}_p$-point for every prime p. This is very convenient, because the existence of a real point can be deduced from continuity arguments, and the existence of a p-adic point comes from solving congruences modulo p^n, which have finitely many possibilities.

There is a more systematic way to describe the p-adic numbers, namely in terms of valuations. For instance, consider in $\mathbb{Q}\backslash\{0\}$ the p-adic valuation given by $v_p(p^r a/b) := r$, where $r \in \mathbb{Z}$ and p divides neither a nor b. We extend the valuation to the whole $\mathbb{Q}$ by declaring $v_p(0) := +\infty$. Then we can define a p-adic norm in $\mathbb{Q}$ via $|q|_p := p^{-v_p(q)}$, where $|0|_p := p^{-\infty} = 0$.

The p-adic norm defines a metric on $\mathbb{Q}$. If we take the completion of $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$, then we obtain again the p-adic numbers $\mathbb{Q}_p$.

Remark 2.2. The norm $|\cdot|_p$ of the valued field $(\mathbb{Q}_p, |\cdot|_p)$ satisfies the following properties:

- It is *non-archimedean*, i.e. $|1+a|_p \leq \max\{1, |a|_p\}$.
- It is *positive definite*, i.e. $|a|_p = 0$ if and only if $a = 0$ (if there was an element $c \neq 0$ such that $|c|_p = 0$, then $|\cdot|$ would be a seminorm).

[2] The correct way to think about this is as a collection of congruences modulo 2^ν, and the series is then congruent to 1 modulo 2^ν, so adding 1 we obtain a collection of congruences which are 0 modulo 2^ν for all ν, hence 0. For general p, -1 can be written in $\mathbb{Q}_p$ as $\sum_{\nu \geq 0}(p-1)p^\nu$.

- It is *multiplicative*, i.e. $|ab|_p = |a|_p |b|_p$.

These properties motivate the general definition of (real) valued field, which was given by Kürschák in[3] 1913 (c.f. [Kür13]):

Definition 2.3. A *(real) valued field* K is a field K together with a map $|\cdot|_K : K \to \mathbb{R}_{\geq 0}$ such that:

- It satisfies the *triangle inequality*, i.e. for every element $a \in K$, $|1+a|_K \leq 1+|a|_K$.
- It is *positive definite*, i.e. $|a|_K = 0$ if and only if $a = 0$.
- It is *multiplicative*, i.e. $|ab|_K = |a|_K \, |b|_K$.

If, in addition to the triangle inequality, we have the stronger condition that for every element $a \in K$,

$$|1 + a|_K \leq \max\{1, |a|_K\},$$

then we say that $(K, |\cdot|_K)$ is a *non-archimedean* valued field. If this last property is not satisfied, then we say that $(K, |\cdot|_K)$ is an *archimedean* valued field.

Example 2.4. The rational numbers $\mathbb{Q}$ together with the euclidean norm $|q|_\infty := \sqrt{q^2}$ is an archimedean valued field. Indeed, $|1 + 1|_\infty = 2 > \max\{1, 1\} = 1$.

However, the rational numbers $\mathbb{Q}$ together with the p-adic norm $|\cdot|_p$ is a non-archimedean valued field, as we saw before.

Remark 2.5. Given any field F, we can define the *trivial norm* $|\cdot|_0$ given by $|a|_0 := 1$ for every non-zero $a \in F$, and $|0|_0 := 0$. Note that $(F, |\cdot|_0)$ is a complete non-archimedean valued field. In this case, we say that the field F is equipped with the *trivial valuation.*

Remark 2.6. Given a valued field $(K, |\cdot|_K)$, then the map $|\cdot|_K^t$ defines a different norm for every $t \in (0, 1)$. For $t > 1$, the triangle inequality might fail: for example, if we start with $(\mathbb{Q}, |\cdot|_\infty)$, then

$$4 = |1 + 1|_\infty^2 > |1|_\infty^2 + |1|_\infty^2 = 2.$$

However, if the norm $|\cdot|_K$ is non-archimedean, then $|\cdot|_K^t$ will also define a norm for $t > 1$. Note that choosing a different t is equivalent to choosing a different real number when defining the p-adic norm $|x|_p := p^{-v_p(x)}$. Indeed, if $t > 1$, then $t^{-v_p(x)} = \left(p^{\log_p(t)}\right)^{-v_p(x)} = |x|_p^{\log_p(t)}$.

The following example shows how the notion of a valued field unifies phenomena from formal power series, holomorphic functions and p-adic numbers.

Example 2.7.
1. Let $F((T))$ be the field of Puiseux series, where F is any field (for example the complex numbers $\mathbb{C}$), T is the variable, and any non-zero element a in $F((T))$ is of the form
$$a = \sum_{i=i_0}^{\infty} c_i T^i,$$
where $i_0 \in \mathbb{Z}$, $c_i \in F$ and $c_{i_0} \neq 0$. This field is the field of fractions of the ring of formal power series $F[[T]]$. Now, let $|\cdot|_T$ be the norm defined by $|a|_T := e^{-i_0}$ for $a \neq 0$, and $|0|_T := 0$. Then, $(F((T)), |\cdot|_T)$ is a non-archimedean valued field.

[3] Kürschák [Kür13] also asks that the valuation is non-trivial, i.e. that there exists an element of the field with valuation different from 0 or 1. However, we will not include this assumption in our definition. Moreover, we include in our definition the notions of archimedean and non-archimedean field, which don't appear in Kürschák's article.

2. Let $a \in \mathbb{C}$ be a complex number, and $\mathcal{O}_a$ the ring of functions which are holomorphic on a neighborhood of a. The elements f of $\mathcal{O}_a$ that are non-invertible are precisely those such that $f(a) = 0$. For $f \neq 0$, let $v_a(f)$ denote the order of the zero of f at a, where obviously $v_a(f) = 0$ if $f(a) \neq 0$. Let M_a be the field of fractions of $\mathcal{O}_a$, which is precisely the field of meromorphic functions which are holomorphic on a punctured disc around a (i.e. the functions are holomorphic on $B(a,r) \setminus \{a\}$ for r small enough), and extend v_a so that it gives you the order of the pole of f at a in case $f \in M_a \setminus \mathcal{O}_a$. In this situation, we can define the norm $|f|_a := e^{-v_a(f)}$ for $f \neq 0$, and $|0|_a := 0$. Then, the field M_a together with the norm $|\cdot|_a$ is a non-archimedean valued field.

3. In general, if A is a discrete valuation ring (with non-trivial maximal ideal $\mathfrak{m}$), we can consider for every non-zero $a \in A$ the valuation $v_{\mathfrak{m}}(a) := \max\{n | a \in \mathfrak{m}^n A\}$. We can extend this valuation to the fraction field F of A, and we can define the norm $|f|_{\mathfrak{m}} := e^{-v_{\mathfrak{m}}(f)}$ for non-zero f, and $|0|_{\mathfrak{m}} := 0$. Then, $(F, |\cdot|_{\mathfrak{m}})$ is a non-archimedean valued field.

Remark 2.8. Archimedes of Syracuse lived in the third century B.C., so one could ask why these fields are called *non-archimedean* fields. Archimedean fields $(K, |\cdot|_K)$ satisfy the so-called *archimedean property*: given two non-zero elements $a, b \in K$ such that $|a|_K \leq |b|_K$, there exists a natural number n such that $|na|_K > |b|_K$. For example, the field of real numbers together with the euclidean norm $(\mathbb{R}, \|\cdot\|)$ satisfies the archimedean property.

However, if the norm $|\cdot|_K$ of our valued field K satisfies the condition

$$|a + b|_K \leq \max\{|a|_K, |b|_K\},$$

then the field doesn't satisfy the archimedean property. Indeed, assume $|a|_K \leq |b|_K$. Then, for all non-zero integers n we have that $|na|_K = |a + \ldots + a|_K \leq |a|_K \leq |b|_K$.

The archimedean property, which appears as Assumption 4 in *On the Sphere and the Cylinder*, by Archimedes (c.f. [Hea97, page 4, Assumption 4]), is actually attributed to Eudoxus of Cnidus, and can be found in the Book 5 of the Elements of Euclid, Definition 4. It seems that the naming of this property as *archimedean property* or *archimedean axiom* goes back to Otto Stolz (c.f. [Sto82, page 75]).

Definition 2.9. Given a non-archimedean valued field $(K, |\cdot|_K)$, we can define its *ring of integers* $\mathcal{O}_K$, given by elements of K whose norm is less or equal to 1. This ring is a local ring, whose *maximal ideal* $\mathfrak{m}_K$ is given by the elements with norm strictly smaller than 1.

Remark 2.10. If our non-archimedean field $(K, |\cdot|_K)$ is trivially valued, then the above definition still makes sense. However, here $\mathcal{O}_K = K$ and $\mathfrak{m}_K = \{0\}$, so we don't gain any extra structure.

Remark 2.11. Note that if the valued field $(K, |\cdot|_K)$ is archimedean, then the set of elements with norm smaller or equal than 1 doesn't form a ring. Indeed, if a non-zero element $a \in K$ has norm smaller or equal to 1, then the archimedean property tells us that there exists a natural number n such that $|na|_K > |1|_K = 1$, and therefore the sum of n copies of a doesn't belong to the set of elements of norm less or equal to 1.

Example 2.12. All the above examples of non-archimedean fields have a noetherian ring of integers. However, this is not always the case. For example, one can extend uniquely the p-adic norm from $\mathbb{Q}_p$ to a fixed algebraic closure $\overline{\mathbb{Q}_p}$. Then, the local ring $\mathcal{O}_{\overline{\mathbb{Q}_p}}$ is not noetherian.

Another example that might be more explicit is the field of Hahn series. Let F be any field, and consider the field $F[[T^{\mathbb{Q}}]]$ given by elements a that are formal power series of the form

$$a = \sum_{\nu \in \mathbb{Q}} c_\nu T^\nu,$$

where $c_\nu \in F$ and $\mathrm{supp}(f) := \{\nu \in \mathbb{Q} : c_\nu \neq 0\}$ is a well-ordered[4] subset of $\mathbb{Q}$. Then, for every non-zero element a we can define the valuation $v_T(a) := \min\{\nu \in \mathbb{Q} : c_\nu \neq 0\}$. As usual, this extends to a norm on the field via $|a|_T := e^{-v_T(a)}$ for non-zero elements, and $|0|_T := 0$. This is a non-archimedean valued field, and its ring of integers $\mathcal{O}_{F[[T^{\mathbb{Q}}]]}$ is not noetherian: indeed, we have an infinite chain of strictly increasing ideals, namely

$$I_n := \{a \in F[[T^{\mathbb{Q}}]] : |a|_T < 1 - 1/n\},$$

and we see that the element $T^{\frac{2n-3}{2n-1}}$ is in $I_n \setminus I_{n-1}$, for $n > 2$.

Remark 2.13. Let $(K, |\cdot|_K)$ be a non-archimedean field. Then $\mathcal{O}_K$ is noetherian if and only if the (additive) value group $\log(|K^*|_K) \subset \mathbb{R}$ is discrete.

Given a valued field $(K, |\cdot|_K)$, we can define a metric in the field given by the distance function

$$d_{(K,|\cdot|_K)}(x, y) := |x - y|_K.$$

Note that the induced topology makes the addition and the multiplication continuous maps

$$K \times K \to K,$$

where $K \times K$ carries the product topology. Hence, in order to give a basis of this topology, it is enough to give a basis around 0, and then we can carry this basis via addition. For example, for $r > 0$, the collection of balls

$$B(0, r) := \{x \in K : \ d_{(K,|\cdot|_K}(x, 0) < r\} = \{x \in K : \ |x|_K < r\}$$

forms a neighborhood basis of the topology around 0.

Remark 2.14. Given a field K with two different norms $|\cdot|_K$ and $|\cdot|'_K$, we say that they are *equivalent* if they induce the same topology.

Two norms are equivalent if and only if $|\cdot|_K^t = |\cdot|'_K$ for some $t > 0$. Indeed, if the latter is true, then they induce the same topology, because to give a ball of radius $r > 0$ around 0 with respect to $|\cdot|_K$ is the same as giving a ball of radius r^t around 0 with respect to $|\cdot|'_K$.

Reciprocally, two norms $|\cdot|_K$ and $|\cdot|'_K$ induce the same topology if and only if for every $x \in K$,

$$|x|_K < 1 \Leftrightarrow |x|'_K < 1,$$

which by the multiplicative property is equivalent to

$$|x|_K > 1 \Leftrightarrow |x|'_K > 1.$$

The trivial norm is only equivalent to itself, so we can assume that our norms are not and take $x \in K$ such that $|x|_K > 1$ (when $|\cdot|_K$ is not trivial, there exists a non-zero element x with norm different than 1, so either x or x^{-1} has norm strictly greater than 1). Let $t := \log|x|'_K / \log|x|_K$, which is greater than zero because both $|x|_K$ and $|x|'_K$ are greater than one. If we prove that this element t is independent of the chosen x such that $|x|_K > 1$, then we are done, because we deduce that $t \log|x|_K - \log|x|'_K = 0$, which implies that $|x|_K^t = |x|'_K$.

Assume that t is not constant, and consider $y \in K$ such that $|y|_K > 1$ and

$$\frac{\log|x|'_K}{\log|x|_K} < \frac{\log|y|'_K}{\log|y|_K},$$

[4] A *well-ordered* set S is a set S together with a total ordering such that every non-empty subset of S has a minimum element. For example, the set $\{-1/n : n \in \mathbb{Z}_{>0}\}$ is well ordered, while $\{1/n : n \in \mathbb{Z}_{>0}\}$ is not.

and take $m, n \in \mathbb{N}$ such that

$$\frac{\log |x|'_K}{\log |x|_K} < \frac{m}{n} < \frac{\log |y|'_K}{\log |y|_K}.$$

By rearranging and taking exponentials, we get that $|x^n/y^m|'_K < 1$ and $|x^n/y^m|_K > 1$, which is a contradiction. Hence t must be constant and we are done.

Example 2.15 (Ostrowski's Theorem). The topology of the rational numbers with the euclidean norm $(\mathbb{Q}, |\cdot|_\infty)$is the usual topology on $\mathbb{Q}$. On the other hand, the topology induced by the p-adic norm $|\cdot|_p$ is a bit complicated (for example, all balls are simultaneously open and closed) and we will look into it in the next section. Before moving to the next section, let's look again to the norms in the rational numbers $\mathbb{Q}$. Ostrowski [Ost16] classified in 1916 all the non-trivial norms of $\mathbb{Q}$, which are equivalent to:

- The Euclidean norm $|\cdot|_\infty$, or
- the p-adic norm $|\cdot|_p$ for some prime number p.

Note that when $t \to 0$, both norms $|\cdot|_\infty^t$ and $|\cdot|_p^t$ tend to the trivial norm $|\cdot|_0$. Hence, we could represent informally all the norms of the rational numbers as a broom (c.f. Figure 1), where

- the broomstick is the set of (equivalent) norms of the form $|\cdot|_\infty^t$, with $t \in (0,1]$,
- the are infinite broom hairs (one for every prime p), where each hair is the set of (equivalent) norms of the form $|\cdot|_p^t$, with $t \in (0,\infty)$,
- the broom hairs are attached to the broomstick in a point representing the trivial norm $|\cdot|_0$.

Figure 1: The norms of the rational numbers.

This description will be made precise when we introduce Berkovich spaces, c.f. Example 2.44.

2.2 The topology of non-archimedean fields

The topology of non-archimedean fields might seem at first confusing, but it is important for us to understand it because we want to do analysis over these fields. Let's fix a non-archimedean valued field $(K, |\cdot|_K)$. Note that the non-archimedean property

$$|1 + a|_K \leq \max\{1, |a|_K\},$$

where $a \in K$, implies that

$$|a + b|_K \leq \max\{|a|_K, |b|_K\},$$

just by multiplying by $|b|_K$ both sides and using the multiplicative property. We denote by $B(a, r)$ the open ball with center a and radius $r > 0$, and $B^+(a, r)$ the closed ball with same center and radius, i.e.

$$B(a,r) := \{a' \in K : \ |a - a'|_K < r\},$$
$$B^+(a,r) := \{a' \in K : \ |a - a'|_K \leq r\}.$$

Note that every point b inside a ball $B(a, r)$ is also a center of the ball, i.e.

$$B(a, r) = B(b, r).$$

Indeed, if $c \in B(a, r)$, then we have the inequalities

$$|a - b|_K < r$$
$$|a - c|_K < r,$$

which implies that $|b - c|_K < r$ (just use the non-archimedean property in the sum $|b - a + a - c|_K$), and therefore $c \in B(b, r)$. The other content is exactly the same. Moreover, if we start with closed balls, the same argument applies (we just need to substitute the strict inequalities by non-strict inequalities). Hence, we have proved the following:

Lemma 2.16. *In a non-archimedean field $(K, |\cdot|_K)$, all the points of an open (resp. closed) ball are also centers of the ball.*

The intersection of two balls might be confusing. Balls over a non-archimedean field behave like mercury droplets: they are disjoint until they touch each other, and then they become one. In other words: the intersection of two different balls $B(a_1, r_1)$ and $B(a_2, r_2)$ is either empty or one of the balls. Indeed, without loss of generality, assume that $r_1 \leq r_2$ and that the intersection is not empty, i.e. there exists c in both balls. Then, by the previous lemma, the balls are equal to $B(c, r_1)$ and $B(c, r_2)$ respectively, and we conclude that the first is inside the second one. Hence, we have proved the following:

Lemma 2.17. *In a non-archimedean field $(K, |\cdot|_K)$, two balls are either disjoint or one is contained inside the other.*

Now, the open ball is obviously open, and the closed ball closed. But we also have that the open ball is *closed* and the closed ball is *open*. Indeed, $B(0, r)$ is closed if and only if $K \setminus B(0, r)$ is open. If we take $b \notin B(0, r)$, then $|b|_K \geq r$, and to conclude the statement we just need to show that there is an open ball around b disjoint to $B(0, r)$. We claim that the ball $B(b, r/2)$ is such a ball. Indeed, $b \notin B(0, r)$ and similarly $0 \notin B(b, r/2)$, so by our previous lemma we conclude that the intersection is empty, and therefore the open ball $B(0, r)$ is also *closed*.

For the closed ball $B^+(0,r)$, we pick a point b inside the ball, and we claim that $B(b,r)$ is inside $B^+(0,r)$. Indeed, $B^+(0,r) = B^+(b,r) \supset B(b,r)$. Hence, the closed ball $B^+(0,r)$ is also *open*. By translating (addition is a continuous map) we have proved the following:

Lemma 2.18. *In a non-archimedean field $(K, |\cdot|_K)$, balls are simultaneously open and closed.*

As a direct consequence, we obtain:

Corollary 2.19. *Non-archimedean fields $(K, |\cdot|_K)$ are totally disconnected, i.e. the only connected components are points.*

Proof. It is enough to prove that given two different points a and b, we can find two disjoint open sets U and V such that $K = U \cup V$ and $a \in U$, $b \in V$. Let $d := |a-b|_K$, which is greater than zero. Then taking $U := B(a,d)$ is both closed and open, so its complement $V := K \setminus B(a,d)$ is also open, and contains b by definition of d. □

Remark 2.20. It is worth to compare the Euclidean and the p-adic topologies of the rational numbers $\mathbb{Q}$. The rational numbers are totally disconnected with respect to both topologies: our previous corollary proves the claim for the p-adic topology $(\mathbb{Q}, |\cdot|_p)$; for the euclidean topology $(\mathbb{Q}, |\cdot|_\infty)$, we can take two different rational numbers, say a and b such that $a < b$, and then there exists an irrational number r between them. Now, the intervals[5] $U := (-\infty, r) \ni a$ and $V := (r, \infty) \ni b$ are both open (and closed) in $\mathbb{Q}$, and they cover the whole rational numbers. In particular, $(\mathbb{Q}, |\cdot|_\infty)$ is also totally disconnected.

However, we can consider the *completion* of $\mathbb{Q}$ with respect the two topologies. Recall that the completion is the set of equivalence classes of Cauchy sequences, which are defined with respect to a norm and therefore depend on the chosen topology. For the p-adic topology, we get the p-adic numbers $\mathbb{Q}_p$, and this field is non-archimedean, so it is totally disconnected. But when we complete with respect to the euclidean topology, we get the real numbers $\mathbb{R}$, which is a connected topological space (even contractible!).

Somehow, the totally disconnected nature of $(\mathbb{Q}, |\cdot|_\infty)$ is because the field is not complete, and as soon as we complete the field, then the topological space $(\mathbb{R}, |\cdot|_\infty)$ becomes connected.

Remark 2.21. We usually imagine the rational numbers with the euclidean topology as a line with some points missing, and those points appear when we complete the field with respect to the euclidean norm $|\cdot|_\infty$. How should we imagine $(\mathbb{Q}, |\cdot|_p)$? For simplicity, let's assume that $p = 3$. Then, 3-adic numbers should be thought as the Sierpiński triangle (see Figure 2).

Recall that points on the Sierpiński triangle are given by a convergent sum

$$\sum_{\nu=0}^{\infty} \frac{1}{2^\nu} e_{a_\nu},$$

where $a_\nu \in \{0,1,2\}$ and e_{a_ν} is given by

$$e_{a_\nu} := \begin{cases} (0,0) & \text{if } a_\nu = 0, \\ (1,0) & \text{if } a_\nu = 1, \\ (0,1) & \text{if } a_\nu = 2. \end{cases}$$

Then, for any element $a \in \mathbb{Z}_3$ written as $\sum_\nu a_\nu 3^\nu$, we assign the element $\sum_{\nu=0}^{\infty} \frac{1}{2^\nu} e_{a_\nu}$ of the Sierpiński triangle. Note however that this is not a bijection: for example the

[5] We don't really need the irrational number r to define these intervals, and we can also avoid *choosing* an arbitrary irrational number r, but for this remark this is not a big problem. The key point here is that the rational numbers $\mathbb{Q}$ don't satisfy the *least upper bound property*.

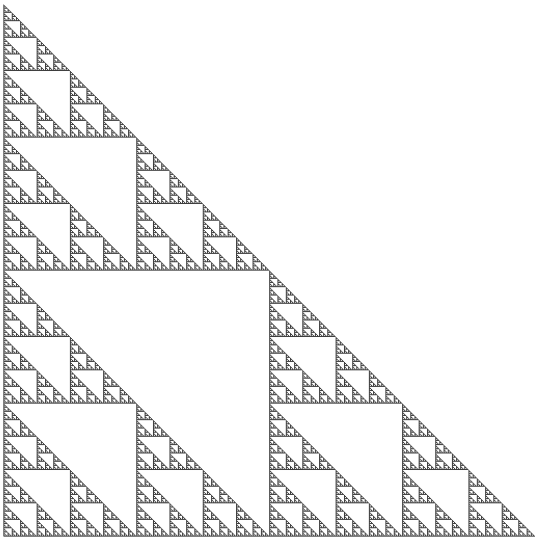


Figure 2: Representation of $\mathbb{Z}_3$ (Wikipedia).

points $-3 = \sum_{\nu\geq 1} 2 \cdot 3^\nu$ and 2 have the same image in this representation, which is the point with coordinates $(0,1)$.

But we can still grasp the fractal nature of $\mathbb{Z}_3$. Indeed, zooming at the biggest lower left copy of the triangle corresponds to looking at the maximal ideal $3\mathbb{Z}_3$, which is the closed ball $B^+(0,1/3)$; the triangle on the right is the conjugate class $1+3\mathbb{Z}_3$, which is the ball $B^+(1,1/3)$; and the upper triangle is $2+3\mathbb{Z}_3$, which is the ball $B^+(2,1/3)$. A little triangle on the lower left corner of the Sierpiński triangle (say the n-th zoom) corresponds to the ideal $3^n\mathbb{Z}_3$: in other words, this is just the ball $B^+(0,1/3^n)$.

The right vertex of the triangle, the point $(0,2)$, corresponds to the element $-1/2 = \sum_\nu 1\cdot 3^\nu \in \mathbb{Z}_3$, and the upper vertex, the point $(2,0)$, to the element $-1 = \sum_\nu 2\cdot 3^\nu \in \mathbb{Z}_3$.

However, the 3-adic metric induced in this representation of $\mathbb{Z}_3$ is a bit more tricky to describe, and shouldn't be confused by the euclidean metric induced from the plane. Indeed, two points of the triangle are very close if we need to make a lot of zooms in order to separate them: for example, the numbers 0 and 3, which correspond to the points $(0,0)$ and $(1/2,0)$, are closer than the the numbers 1 and 39, which correspond to the points $(1,0)$ and $(7/8,0)$. Indeed, if we do a single zoom, we separate 1 and 39, but we need to make two zooms if we want to separate 0 and 3. The number of zooms that are necessary to separate a and b is equal to $1 + v_3(a-b)$.

2.3 Analysis over the rational numbers

Before explaining the analysis over a non-archimedean field, let's recall two different ways of doing analysis over the rational numbers with the euclidean topology $(\mathbb{Q}, |\cdot|_\infty)$.

To do analysis means to give a sheaf of *analytic* functions $\mathcal{O}_\mathbb{Q}$, so that the pair $((\mathbb{Q},|\cdot|_\infty),\mathcal{O}_\mathbb{Q})$ forms a locally ringed space. In real analysis, a function f is said to be analytic on an open set U if for any $x_0 \in U$, f can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n$$

on a neighborhood around x_0, where the coefficients a_n are real numbers. Hence, if we define the sheaf $\mathcal{N}$ of *naive analytic functions of* $\mathbb{Q}$ as the functions that can be written locally as a power series expansion, i.e.

$$\mathcal{N}(U) = \left\{ f : U \to \mathbb{R} \;\middle|\; \begin{array}{l} \forall x_0 \in U,\ \exists \text{ an open neighborhood } U \supset V \ni x_0 \text{ such} \\ \text{that } f|_V(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n \end{array} \right\}$$

where $U \subset \mathbb{Q}$ is an arbitrary open subset, $a_n \in \mathbb{R}$ and the power series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ converges on V, we see that the function defined as

$$f(x) := \begin{cases} 1 & \text{if } |x|_\infty < \sqrt{2}, \\ 0 & \text{else.} \end{cases}$$

is naive-analytic. This means that there are too many analytic functions when we follow the naive approach. In particular, we have no analytic continuation: we would like that an analytic function which is zero on a ball, is zero everywhere (so that we can extend analytic functions defined in a disc to a bigger open set, just as we do with the Riemann zeta function), but this is not the case in this naive approach. The reason behind this is that the only connected open set in $(\mathbb{Q}, |\cdot|_\infty)$ is the empty set, as we saw before, and analytic functions are usually extended on connected sets.

We present here two ways of fixing this, so that we can do analysis over $(\mathbb{Q}, |\cdot|_\infty)$ in such a way that we have analytic continuation.

The first approach consists in completing the field $\mathbb{Q}$ with respect to the euclidean norm $|\cdot|_\infty$, so that we obtain the real numbers $\mathbb{R}$. The real numbers $\mathbb{R}$ form a very nice topological space which is contractible, and here we can do analysis in the usual way. Hence, we could stay in the reals, or if we want to come back to the rational numbers, we could define a sheaf of analytic functions where the sections on an open subset $U \subset$ are exactly the restrictions of the analytic functions defined on $\mathrm{int}(\overline{U})$, where $\overline{U}$ is the euclidean closure of U in the real numbers $\mathbb{R}$, and $\mathrm{int}(C)$ denotes the interior of the closed set C.

The second approach is to stay in the rational numbers and restricting the amount of open sets that we take into account by defining a Grothendieck topology. Recall that a Grothendieck topology is defined on a category C with fiber product as follows:

Definition 2.22 ([SGA4, Exposé II])**.** A *Grothendieck topology* on a category C is given by, for any object U in C, a set of coverings $\mathrm{Cov}(U)$, where

- A covering of U, denoted $\{U_i \to U\}_{i \in I}$, is a collection of morphisms $f_i : U_i \to U$ indexed by a set I.
- We have *stability under base change*, i.e. for every object U and V of C, every covering $\{U_i \to U\}_{i \in I}$ in $\mathrm{Cov}(U)$, and every morphism $V \to U$, the pullback $\{U_i \times_U V \to V\}$ is a covering of V.
- Coverings have a *local character*, i.e. let $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(U)$ be a covering, and consider a collection of morphisms $\{U'_j \to U\}_{j \in J}$. Assume that for every object V and every morphism $V \to U_i$, the collection of morphisms $\{U'_j \times_U V \to V\}_{j \in J}$ coming from the Cartesian diagrams

$$\begin{array}{ccc} U'_j \times_U V & \longrightarrow & V \\ \big\downarrow & & \big\downarrow \\ & & U_i \\ & & \big\downarrow \\ U'_j & \longrightarrow & U \end{array}$$

is a covering $\mathrm{Cov}(V)$. Then, the collection $\{U'_j \to U\}_{j \in J}$ is a covering in $\mathrm{Cov}(U)$.

- For every isomorphism $V \to U$ in C, $\{V \to U\}$ belongs to $\mathrm{Cov}(U)$.

The reader who is not familiar with this definition shouldn't worry, because we are going to define a very concrete Grothendieck topology on the rational numbers. The category $AOS_{\mathbb{Q}}$ that we will use is the category of *admissible open sets of* $\mathbb{Q}$, whose objects are the open intervals of the form

$$\{x \in \mathbb{Q} : q_1 < x < q_2\},$$

where q_1 and q_2 are rational numbers or (plus or minus) infinity, and the morphisms of this category are the inclusions. Note that $\mathbb{Q} = \{x \in \mathbb{Q} : -\infty < x < \infty\}$ is an admissible open set of $\mathbb{Q}$, and that the intersection of two admissible subsets U, V is again admissible and equal to the fiber product $U \times_{\mathbb{Q}} V$.

Now, for every admissible open set U, a collection $\{U_i \to U\}_{i\in I}$ is in $\mathrm{Cov}(U)$ if $U = \bigcup_{i\in I} U_i$ and the index set I is finite. In other words, we only allow finite collections of admissible open subsets of U.

Now, the Grothendieck topology on $\mathbb{Q}$ given by $AOS_{\mathbb{Q}}$ and the finite coverings satisfy the following axioms:

- The empty set is in $AOS_{\mathbb{Q}}$. If U and V are in $AOS_{\mathbb{Q}}$, then $U \cap V$ is also in $AOS_{\mathbb{Q}}$.
- Stability under base change: if $\{U_1, \dots, U_n\}$ is in $\mathrm{Cov}(U)$, and V is an admissible open subset contained in U, then $\{U_1 \cap V, \dots, U_n \cap V\}$ is in $\mathrm{Cov}(V)$.
- Local character: if $\{U_1, \dots, U_n\}$ is in $\mathrm{Cov}(U)$, and for every $i = 1, \dots, n$, $\{U_{i,j} \to U_i\}_{j\in J_i}$ is in $\mathrm{Cov}(U_i)$, then $\{U_{i,j}\}_{i\in\{1,\dots,n\},\, j\in\bigcup_i J_i}$ is in $\mathrm{Cov}(U)$.
- The identity $\mathrm{id} : U \to U$ is a covering, i.e. $\{U\} \in \mathrm{Cov}(U)$.

Remark 2.23. Even if $(\mathbb{Q}, |\cdot|_\infty)$ is totally disconnected as a topological space, it becomes connected with respect to the Grothendieck topology defined above: indeed, if two non-empty admissible open sets U and V cover $\mathbb{Q}$, then their intersection can't be empty.

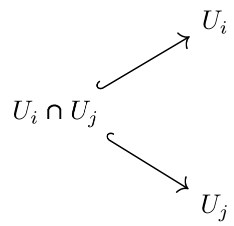
Now, in order to do analysis, we define the structure presheaf $\mathcal{O}_{\mathbb{Q},Groth}$ with respect to the Grothendieck topology defined above as follows: for each admissible open subset U, $\mathcal{O}_{\mathbb{Q},Groth}(U)$ is the ring of functions $f : \mathbb{Q} \to \mathbb{R}$ such that for every point $x_0 \in U$, f can be written as a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

convergent on a neighborhood around x_0, where the coefficients a_n are real numbers. Now $\mathcal{O}_{\mathbb{Q},Groth}$ becomes a sheaf with respect to the Grothendieck topology defined above, in the sense that if a section is locally zero, then it must be globally zero, and if we have a section defined on an admissible covering such that it agrees on the intersections, then we can glue it to a global section. In other words, if $\{U_i \to U\}_{i=1,\dots,n}$ is an admissible covering of an admissible open set U, then the sequence

$$0 \to \mathcal{O}_{\mathbb{Q},Groth}(U) \to \prod_{i=1}^{n} \mathcal{O}_{\mathbb{Q},Groth}(U_i) \to \prod_{i,j=1}^{n} \mathcal{O}_{\mathbb{Q},Groth}(U_i \cap U_j)$$

is exact, where the second arrow is the restriction map coming from $U_i \hookrightarrow U$, and the third arrow is the difference of the two arrows coming from the restriction maps



Remark 2.24. The structure sheaf $\mathcal{O}_{\mathbb{Q},Groth}$ with respect to the Grothendieck topology defined above allows us to do analysis over $(\mathbb{Q}, |\cdot|_\infty)$. In particular, we have analytic continuation. The fact behind this is that now, for every analytic function $f \in \mathcal{O}_{\mathbb{Q},Groth}(U)$, where U is an admissible open subset. we can find an admissible covering $U = U_1 \cup \ldots, U_n$ such that f_{U_i} is given by a power series, and since U is connected with respect to the Grothendieck topology, the smaller subsets U_i overlap and transmit their local properties to the whole U.

2.4 Analysis and geometry over a non-archimedean field

As said by Ducros [Duc07],

> "*Le rôle majeur joué en théorie des nombres et en géométrie arithmétique par les corps p-adiques a incité à développer sur ces derniers, autant que faire se pouvait, une théorie analogue à celle des espaces analytiques complexes.*"

2.4.1 Rigid geometry

We saw before that non-archimedean fields $(K, |\cdot|_K)$ are totally disconnected, so we encounter similar problems as with the rational numbers.

However, if we complete K with respect to $|\cdot|_K$, it will remain totally disconnected. Hence the first approach that we did before doesn't seem to work.

For this section, we will assume that K is complete with respect to a non-trivial norm $|\cdot|_K$, and for simplicity we will assume that it is algebraically closed (we will drop some of these assumptions in the next section). We will follow the presentation by Kato [And03, Appendix A].

Tate's approach to non-archimedean analysis is to define a suitable Grothendieck topology that makes balls connected, and this is what we are going to do now. Since K is algebraically closed, we can identify the closed ball $B^+(0,1) = \mathcal{O}_K \subset K$ with the set of maximal ideals of the ring

$$K\{t\} := \left\{ \sum_{n\geq 0} a_n t^n \in K[[t]] : |a_n|_K \to 0 \right\},$$

which is now called *Tate algebra*. Indeed, since K is algebraically closed, any maximal ideal in $K\{t\}$ is generated by an element of the form $t - a$, with $a \in \mathcal{O}_K$: if $a \neq 0$, then the Taylor expansion

$$\frac{1}{t-a} = -\frac{1}{a} \sum_{\nu\geq 0} \left(\frac{t}{a}\right)^\nu$$

converges if and only if $1/|a|_K^\nu \to 0$, which occurs if and only if $|a|_K > 1$. Hence, $t - a$ generates a maximal ideal if and only if $|a|_K \leq 1$, i.e. if and only if $a \in B^+(0,1)$.

The Tate algebra $K\{t\}$, together with the Gauss norm $\|f\| := \max_n |a_n|$, forms a Banach space. Note that this algebra is precisely the subring of formal power series convergent on $B^+(0,1)$. In particular, $f(x)$ lies in K for every $x \in B^+(0,1)$. The maximum modulus principle holds:

$$\|f\| = \sup_{x\in B^+(0,1)} |f(x)|_K = \max_{x\in B^+(0,1)} |f(x)|_K.$$

The Tate algebra $K\{t\}$, together with the Gauss norm $\|f\| := \max_n |a_n|_K$, forms a K-Banach algebra[6].

[6]Recall that a K-Banach space is a K-vector space V together with a norm $\|\cdot\|_V : V \to \mathbb{R}_{\geq 0}$ such that $\|v\| = 0$ if and only if $v = 0$, $\|v+v'\|_V \leq \max\{\|v\|_V, \|v'\|_V\}$, $\|cv\|_V = |c|_K\|v\|_V$ and V is complete for the metric given by $\|v - v'\|_V$. Moreover, if V is also an algebra, then we say that the K-Banach space $(V, \|\cdot\|_V)$ is a K-Banach algebra if $\|v_1 \cdot_V v_2\|_V = \|v_1\|_V\|v_2\|_V$ holds.

Remark 2.25 ([Con08, Example 1.2.2]). If K is not algebraically closed, then $\mathrm{Spm}(K\{t\})$ has many more points than $B^+(0,1)$, and this is actually the big difference between rigid analytic spaces and the classical notion of K-analytic manifolds. We will not talk about this classical approach here, but this is for example the point of view developed by Serre in his Harvard lectures (c.f. [Ser65]).

The definition for higher dimensional Tate algebras $K\{t_1,\dots,t_n\}$ is similar, and they correspond to higher dimensional balls $B^+(0,1)^n$. We denote the set of maximal ideals of a ring R by $\mathrm{Spm}(R)$, the so-called *maximal spectrum.* The bijection $B^+(0,1) = \mathrm{Spm}(K\{t\})$ induces naturally the non-archimedean topology on $\mathrm{Spm}(K\{t\})$.

Remark 2.26. Note that the Tate algebra $K\{t\}$ plays the role of $K[t]$ in algebraic geometry, and the closed disk $B^+(0,1)$ (or more precisely, $\mathrm{Spm}(K\{t\})$ plays the role of the affine line $\mathrm{Spec}(K[t])$. Indeed, if we forget the norm $|\cdot|_K$, we know that we can endow any field with the trivial norm $|\cdot|_0$, and the Tate algebra of $(K,|\cdot|_0)$ equals $K[t]$.

Remark 2.27. If K is not algebraically closed, we know that for any maximal ideal $x \in \mathrm{Spm}(K\{t\})$, the quotient $K\{t_1,\dots,t_n\}/x$ is a finite extension of K. Since K is complete, this implies that we can extend uniquely $|\cdot|_K$ (c.f. [Ser79, Chapter 2, Proposition 3]). However, if x is just a prime ideal, there is no canonical way to extend $|\cdot|_K$ to $K\{t_1,\dots,t_n\}/x$. This is one of the reasons why we restrict to the maximal spectrum.

Definition 2.28. In general, a *strictly affinoid algebra*[7] A over K is a K-algebra isomorphic (as a K-algebra) to the quotient $K\{t_1,\dots,t_n\}/I$ for some ideal I. Since all ideals in $K\{t_1,\dots,t_n\}$ are closed (because the Tate algebra is noetherian, c.f. [FP04, Theorem 3.2.1]), every isomorphism $\alpha : A \to K\{t_1,\dots,t_n\}/I$ defines a residue norm $|\cdot|_\alpha$. Recall that the residue norm of $\overline{f} \in K\{t_1,\dots,t_n\}/I$ is defined to be the infimum of all the representatives, i.e.

$$\|\overline{f}\|' := \inf_{f \mod I = \overline{f}} \|f\|.$$

However, even if the $|\cdot|_\alpha$ are different, they are all equivalent, in the sense that they define the same topology. Note that $(A,|\cdot|_\alpha)$ is a K-Banach algebra. The corresponding maximal spectrum $\mathrm{Spm}(A)$ is called an *strictly affinoid domain.* Strictly affinoid domains play the role of affine schemes in algebraic geometry.

Now, we can define the Grothendieck topology that we will use. We will follow here the approach of Kato [And03, Appendix A].

First, we fix a strictly affinoid domain $\mathrm{Spm}(A)$. The category that will play the role of admissible open sets is the category of *rational subdomains*, which are subspaces of the form

$$R = \{x \in \mathrm{Spm}(A) : \ |f_i(x)|_K \le |f_0(x)|_K,\ i = 1,\dots,n\},$$

where the f_i's are elements in A without common zeros on $\mathrm{Spm}(A)$. Note that R is the strictly affinoid domain corresponding to the strictly affinoid algebra[8]

$$A_R := A\{s_1,\dots,s_n\}/(f_1 - s_1 f_0,\dots,f_n - s_n f_0),$$

i.e. $\mathrm{Spm}(A_R) = R$. Rational subdomains satisfy the following universal property: for any morphism of strictly affinoid domains $\phi : \mathrm{Spm}(B) \to \mathrm{Spm}(A)$ such that

[7] In rigid analytic literature, this is called just affinoid algebra. We use here the terminology from Berkovich spaces.

[8] The norm on the A-algebra $A\{s\}$ is defined as $\|\sum a_\nu s^\nu\| := \max\{\|a_\nu\|_A\}$, and the norm of the quotient is the usual residue norm.

$\phi(\mathrm{Spm}(B)) \subset R$, there exists a unique K-homomorphism $A_R \to B$ such that

$$\begin{array}{ccc} \mathrm{Spm}(B) & \longrightarrow & \mathrm{Spm}(A) \\ & \searrow & \uparrow \\ & & R \end{array}$$

commutes. One can prove that the intersection of two rational subdomains is again a rational subdomain, and that if R_2 is a rational subdomain in R_1, and R_1 a rational subdomain of $\mathrm{Spm}(A)$, then R_2 is a rational subdomain of $\mathrm{Spm}(A)$ (c.f. [FP04, Lemma 4.1.3]).

Hence, rational subdomains satisfy the properties of the Grothendieck topology on a strictly affinoid domain $\mathrm{Spm}(A)$ where:

- Admissible sets are rational subdomains $\mathrm{Spm}(A_R)$ of $\mathrm{Spm}(A)$.
- For a rational subdomain $R \subset \mathrm{Spm}(A)$, an admissible covering is a finite covering $R_1, \ldots, R_k$ of rational subdomains such that $R = \bigcup_{i=1}^k R_i$.

Now we fix a strictly affinoid domain $X := \mathrm{Spm}(A)$, and we define the structure presheaf $\mathcal{O}_X$ on the admissible open sets (i.e. rational subdomains of $\mathrm{Spm}(A)$) as $\mathcal{O}_X(\mathrm{Spm}(A_R)) := A_R$. The key fact is that the structure presheaf $\mathcal{O}_X$ is indeed a sheaf with respect to the Grothendieck topology:

Theorem 2.29 (Tate's acyclicity theorem, [FP04, Theorem 4.2.2])**.** *Let $X := \mathrm{Spm}(A)$ be a strictly affinoid domain, and $R_1, \ldots, R_n$ rational subdomains covering X. Then, the sequence*

$$0 \to \mathcal{O}_X(X) \to \prod_{i=1}^n \mathcal{O}_X(R_i) \to \prod_{i,j=1}^n \mathcal{O}_X(R_i \cap R_j)$$

is exact, where the last arrow is the difference of the two restriction morphisms.

Remark 2.30. In the theorem we start with a finite covering of a strictly affinoid domain X by rational subdomains. We can always do this thanks to the Gerritzen-Grauert theorem, c.f. [FP04, Theorem 4.10.4].

Note also that thanks to Tate's acyclicity theorem, together with the Gerritzen-Grauert theorem, we can extend the definition of the structure sheaf to strictly affinoid subdomains: indeed, if $U \subset X$ is a strictly affinoid subdomain, then we define $\mathcal{O}_X(U) := \ker\left(\prod_{i=1}^n \mathcal{O}_X(R_i) \to \prod_{i,j=1}^n \mathcal{O}_X(R_i \cap R_j)\right)$, where $\{R_i \to U\}$ is an admissible cover of U (i.e. a finite cover of rational subdomains). Note that $\mathcal{O}_X(U) = \mathcal{O}_U(U)$.

Remark 2.31. This approach follows the philosophy of Grothendieck, as explained by Berkovich [Ber90, Preface]: "*According to A. Grothendieck one really does not need a space to do geometry, all one needs is a category of sheaves on this would-be space*". This is precisely what we do here, and what is done in étale cohomology, crystalline cohomology and many other algebraic geometric contexts.

If A is a strictly affinoid algebra over K, then the strictly affinoid domain $X = \mathrm{Spm}(A)$, together with the Grothendieck topology defined above and the structure sheaf $\mathcal{O}_X$, is called a *strictly affinoid space* $(X, \mathcal{O}_X)$ over K. We can now globalize this definition:

Definition 2.32. A *rigid analytic space* $(X, \mathcal{O}_X)$ over K is a locally ringed space, with respect to a given Grothendieck topology, locally isomorphic to a strictly affinoid. In other words, there exists a (possibly infinite) covering $\{X_i \to X\}_{i \in I}$, where each $(X_i, \mathcal{O}_X|_{X_i})$ is (isomorphic to) a strictly affinoid space over K.

Example 2.33. • Any strictly affinoid space is a rigid analytic space.

- Any open subset U of a strictly affinoid domain $X = \mathrm{Spm}(A)$ is a rigid analytic space. Note that here we don't ask the open subset to be admissible, nor strictly affinoid. For example (c.f. [Nic08, Examples 4 and 5]), let $X = \mathrm{Spm}(K\{t\}) = B^+(0,1)$ be the closed ball, where K is still algebraically closed, and let $B(0,1)$ be the open ball
$$B(0,1) = \{z \in X : \ |t(z)|_K < 1\}.$$
Then, $B(0,1)$ is not a strictly affinoid domain (since the function $|t(\cdot)|_K$ doesn't attain a maximum, and therefore contradicts the maximum modulus principle, c.f. [BGR84, p. 6.2.1.4]), but we can cover it with infinitely many strictly affinoid domains
$$B(0,1) = \bigcup_{r\in(0,1)\cap|K^*|_K} B^+(0,r),$$
where each ball $B^+(0,r) := \mathrm{Spm}\left(K\{t,s\}/(t-rs)\right)$ is a strictly affinoid domain. Note that we can write $B^+(0,1)$ as the disjoint union of the two open subsets
$$B^+(0,1) = B(0,1) \sqcup \{z \in B^+(0,1) : \ |t(z)|_K = 1\}.$$
This is why we don't want to include $B(0,1)$ as an admissible open set in the definition of the Grothendieck topology above (otherwise the closed ball wouldn't be connected with respect to the Grothendieck topology).

- We can glue two disks in order to get the rigid analytic projective space, c.f. [And09, Appendix A, Example 2.6].

- We can also define the rigid analytification of the affine space by gluing infinitely many closed balls of increasing radius, c.f. [Con08, Example 2.4.3]. This procedure is the analog of covering a non-archimedean field $(K, |\cdot|_K)$ with non-trivial valuation with the disks $\pi^{1/n}\mathcal{O}_K$, where π is any non-zero element of the maximal ideal.

Remark 2.34 ([Nic08, Section 4.6]). In general, given any scheme X of finite type over K, we can endow the set of closed points[9] $X(K)$ with the structure of a rigid analytic space $(X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}})$ over K. This *rigid analytification* functor preserves a lot of properties: it commutes with fiber products, preserves immersions, and when X is proper over K, it has the classical GAGA properties between proper K-schemes of finite type and their rigid analytification, i.e. we have an equivalence between coherent $\mathcal{O}_X$-modules and coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-modules, cohomology groups agree, and if Y is another proper K-scheme of finite type, then all rigid analytic morphisms $X^{\mathrm{rig}} \to Y^{\mathrm{rig}}$ are algebraic.

Remark 2.35. As an example of a result in this setting, we have the uniformization of elliptic curves with multiplicative reduction (done by Tate), and even uniformization of higher genus curves with totally degenerate reduction, also known as Mumford curves, which was obtained by Mumford [Mum72].

Hence, we have a successful non-archimedean analytification procedure, but the topology of X^{rig} is not satisfactory: everything works when we restrict to the Grothendieck topology, so we don't have for example a topological fundamental group or similar features. Hence, in order to solve the original problem of analyzing non-archimedean analogues of non-homeomorphic conjugate complex varieties, we need a different approach.

In the next section, we introduce another way to construct non-archimedean analytifications that yields nice topological spaces (they will be Hausdorff, locally path-connected, locally contractible, etc.), due to Berkovich. This will allow us to solve our original problem.

[9] Since K is algebraically closed, the set of closed points coincides with $X(K)$.

2.4.2 Berkovich spaces

Recall that when we try to do analysis over $\mathbb{Q}$, there are two approaches: the first one is to add missing points by completing the field with respect to the euclidean norm (so that we obtain the real numbers), and the second one consists in restricting the amount of open sets that we are looking at by defining a suitable Grothendieck topology.

In the previous section, we studied how to define suitable strictly affinoid algebras (that are the algebras of analytic functions on a disk over a non-archimedean field) and how to define a suitable Grothendieck topology over non-archimedean fields so that we can do analysis and obtain an analytification functor from the category of schemes of finite type over a non-archimedean field K to the category of rigid analytic spaces. This approach corresponds to the second approach when we studied analysis over the rational numbers.

There is, however, another approach to do non-archimedean analysis that considers more points in the analytification, so that we obtain a topological space which is locally connected (actually locally contractible), Hausdorff, and that has many other nice properties. This approach is analog to the one that adds more points to the rationals (so that we obtain the real numbers) in order to do analysis, and was introduced by Berkovich [Ber90]. As he says in his preface:

> "(...) *And so they [rigid analytic spaces] are called spaces only conditionally. Several years ago I found that p-adic analytic spaces really exist. They are quite elegant objects possessing many topological properties of complex analytic spaces that are sufficient, for example, for applying to them the homotopy and singular homology notions in the usual sense.*"

Note that if we start with a complete non-archimedean valued field $(K, |\cdot|_K)$, we can't add more points by completing. So how did Berkovich add more points? We will analyze here the case of the closed ball corresponding to the Tate algebra $(K\{t\}, \|\cdot\|)$, assuming for simplicity that K is algebraically closed.

In rigid analytic geometry, we consider the maximal spectrum $\mathrm{Spm}(K\{t\})$. Note that every maximal ideal $x \in \mathrm{Spm}(K\{t\})$ defines a multiplicative seminorm $|\cdot|_x$ on $K\{t\}$ via $|f|_x := |\overline{f}|_K$, where $\overline{f}$ is the image of f under the projection $K\{t\} \twoheadrightarrow K\{t\}/x \simeq K$. It is clearly not a norm, since every element f inside the maximal ideal x has norm equal to zero, and it is immediate to see that it is multiplicative. If x is generated by $t'b$, we also denote $|f(b)| := |f|_x$. Note that $|\cdot|_x$ is a *bounded seminorm* on $(K\{t\}, \|\cdot\|)$, where a bounded seminorm $|\cdot|$ on a Banach K-algebra $(A, \|\cdot\|)$ is a seminorm $|\cdot|$ on A such that there exists a constant $C > 0$ satisfying the inequality

$$|f| \le C\|f\| \quad \text{for all } f \in A.$$

Indeed, if $f = \sum_{\nu\ge 0} a_\nu t^\nu$, and x is generated by $t - b$, where $|b|_K \le 1$, then we have that

$$|f|_x = |\overline{f}|_K = \left|\sum_{\nu\ge 0} a_\nu b^\nu\right|_K \le \max_\nu\{|a_\nu|_K \cdot |b|_K^\nu\} \le \max_\nu\{|a_\nu|_K\} = \|f\|.$$

Hence, every point $b \in B^+(0,1)$ induces the bounded multiplicative seminorm $|\cdot|_{(t-b)}$ on $(K\{t\}, \|\cdot\|)$. However, there are many more bounded multiplicative seminorms on $(K\{t\}, \|\cdot\|)$: for instance, any closed ball $B^+(c,r)$ with center $c \in B^+(0,1)$ and radius $r \le 1$ defines the seminorm $|\cdot|_{B^+(c,r)}$ given by

$$|f|_{B^+(c,r)} = \left|\sum_{\nu\ge 0} a_\nu t^\nu\right|_{B^+(c,r)} := \max_{b\in B^+(c,r)}\{|f(b)|\},$$

which is clearly a seminorm bounded by $\|\cdot\| = |\cdot|_{B^+(0,1)}$. Indeed,

$$\begin{aligned} |f|_{B^+(c,r)} &:= \max_{b\in B^+(c,r)}\{|f(b)|\} \\ &\leq \max_{b\in B^+(c,r)}\left\{\max_\nu\{|a_\nu b^\nu|_K\}\right\} \\ &\leq \max_{b\in B^+(c,r)}\left\{\max_\nu\{|a_\nu|_K\}\right\} \\ &= \max_\nu\{|a_\nu|_K\} \\ &= \|f\|. \end{aligned}$$

In the next lemma we show that it is multiplicative.

Lemma 2.36. *The seminorm $|\cdot|_{B^+(c,r)}$ defined on $K\{t\}$ is multiplicative.*

Proof. We have to show that $|fg|_{B^+(c,r)} = |f|_{B^+(c,r)}|g|_{B^+(c,r)}$.

We first assume that $r = 1$. If the norm $|\cdot|_K$ is trivial, then the result clearly holds, so let's assume it isn't, and let $k := \mathcal{O}_K/\mathfrak{m}_K$ be the residue field. Now, let $\pi : B^+(0,1) = \mathcal{O}_K\{t\} \to \mathcal{O}_K\{t\}/\mathfrak{m}_K\{t\} \simeq k[t]$ be the projection. Note that $|f|_{B^+(0,1)} < 1$ if and only if $\pi(f) = 0$. Hence, if both $f, g \in B^+(0,1)$ satisfy $|f|_{B^+(0,1)} = |g|_{B^+(0,1)} = 1$, we have that $\pi(f), \pi(g) \neq 0$, and therefore $0 \neq \pi(f)\pi(g) = \pi(fg)$, so $|fg|_{B^+(0,1)} = 1$.

For general $f, g \in K\{t\}$, choose $c, d \in K$ so that[10] $f = cf'$ and $g = dg'$, with $|f'|_{B^+(0,1)} = |g'|_{B^+(0,1)} = 1$. In other words, $|f|_{B^+(0,1)} = |c|_K$ and $|g|_{B^+(0,1)} = |d|_K$. Then,

$$|fg|_{B^+(0,1)} = |cdf'g'|_{B^+(0,1)} = |c|_K|d|_K|f'g'|_{B^+(0,1)} = |f|_{B^+(0,1)}|g|_{B^+(0,1)}.$$

For a general disk $B^+(c,r)$, we can write any element $f \in K\{t\}$ as a power expansion with center c, i.e. $f = \sum_{\nu\geq 0} a'_\nu(t-c)^\nu$. Then, if $f \in K\{t\}$ converges in the disk $B^+(c,r)$ (which is equivalent to $|a'_\nu|_K r^\nu \to 0$), we have that

$$|f|_{B^+(c,r)} = \max_\nu\{|a'_\nu|_K \cdot r^\nu\},$$

and by using the non-archimedean property we can check that the seminorm is multiplicative on the elements converging in the disk.

Note that for any $f \in K\{t\}$, we can find $c \in K$ such that $f = cf'$ with f' convergent on the disk $B^+(c,r)$, and we conclude that the seminorm is multiplicative. □

Hence, in addition to the points of the rigid analytic closed points (which can be viewed as bounded multiplicative seminorms on $(K\{t\}, \|\cdot\|)$), we can consider the set of all bounded multiplicative seminorms of $(K\{t\}, \|\cdot\|)$ to define the Berkovich closed disk. For a general Banach algebra, this leads us to Berkovich's definition of spectrum.

Definition 2.37. Let $(A, \|\cdot\|)$ be a Banach algebra, i.e. a ring A together with a (not necessarily multiplicative, possibly trivial) norm $\|\cdot\| : A \to \mathbb{R}_{\geq 0}$ such that A is complete with respect to $\|\cdot\|$. The *Berkovich spectrum* $M(A)$ is the set of all bounded multiplicative seminorms on A.

Remark 2.38. Note that the norm of a Banach algebra $(A, \|\cdot\|)$ might not be multiplicative. Hence the existence of an element inside $M(A)$ is not trivial (but it is true, c.f. [Ber90, Theorem 1.2.1]).

Remark 2.39. Note that we don't need our Banach algebra to be defined over any field. In particular, since $(\mathbb{Z}, |\cdot|_\infty)$ is a Banach algebra, then we can consider its Berkovich spectrum $M(\mathbb{Z})$.

[10] Such c and d exist. Here we construct c: due to the maximum modulus principle, there exist $b_0 \in B^+(0,1)$ and $\nu_0 \in \mathbb{N}$ satisfying

$$|a_{\nu_0} b_0^{\nu_0}|_K = \max_{b\in B^+(0,1)}\left\{\max_\nu\{|a_\nu b^\nu|_K\}\right\}.$$

Hence, $c := a_{\nu_0} b_0^{\nu_0}$ satisfies $|c|_K = |f|_{B^+(0,1)}$.

Remark 2.40. We can endow $M(A)$ with a natural topology. Note that every (bounded multiplicative) seminorm on A can be seen as a section of $A \times \mathbb{R}_{\geq 0}$: in other words, if we fix a multiplicative bounded seminorm $|\cdot|_x$, then we get the pairs of the form $\{(a, |a|_x) : a \in A\}$. Hence, $M(A)$ is contained in $\mathbb{R}^A$, so the topology of $\mathbb{R}^A$ induces a topology on $M(A)$. This topology coincides with the weakest topology so that for every $a \in A$, the function

$$a : M(A) \longrightarrow \mathbb{R}_{\geq 0}$$

$$|\cdot|_x \longmapsto |a|_x$$

is continuous. An equivalent way of defining this topology is as the topology generated by open sets of the form

$$\{|\cdot|_x \in M(A) : |a|_x < r\}, \quad \{|\cdot|_x \in M(A) : |a|_x > r\},$$

for some $a \in A$ and $r \in \mathbb{R}_{\geq 0}$.

Example 2.41. Let $\mathcal{B}^+(0,1)$ denote the *Berkovich unit closed ball* $M(K\{t\})$. Observe that the notation is different from the unit closed ball inside K, denoted $B^+(0,1)$. How does $\mathcal{B}^+(0,1) := M(K\{t\})$ look like? We have seen already that the (multiplicative bounded) seminorms of the form $|\cdot|_{(t-b)}$, for $b \in B^+(0,1) \subset K$, belong to $\mathcal{B}^+(0,1)$. Hence we have an injection

$$B^+(0,1) \hookrightarrow \mathcal{B}^+(0,1)$$

$$b \longmapsto |\cdot|_{(t-b)}.$$

Note also that for any $|\cdot|_x \in \mathcal{B}^+(0,1)$, we have (c.f. [BR10, Lemma 1.1]):

- For every $f \in K\{t\}$, $|f|_x \leq \|f\|$, where $\|\cdot\|$ is the Gauss norm defined above.
- For all $f, g \in K\{t\}$, $|f + g|_x \leq \max\{|f|_x, |g|_x\}$, with equality if $|f|_x \neq |g|_x$.
- For all $c \in K$, $|c|_x = |c|_K$.

Berkovich's classification theorem (c.f. [Ber90, Example 1.4.4]) says that every bounded seminorm on $K\{t\}$ arises as the (norm corresponding to a) limit of a decreasing sequence of disks. In other words, for $|\cdot|_x \in \mathcal{B}^+(0,1)$, we can find a nested sequence of disks $\{B^+(a_i, r_i)\}_{i\geq 1}$ inside $B^+(0,1)$, where $B^+(a_i, r_i) \supset B^+(a_{i+1}, r_{i+1})$, such that

$$|\cdot|_x = \lim_{i\to\infty} |\cdot|_{B^+(a_i,r_i)}.$$

Moreover, Berkovich classifies the points $|\cdot|_x$ as follows. We say that $|\cdot|_x$:

- ● Is of type I, if $\bigcap_i B^+(a_i, r_i) = \{b\}$. In this case, $|\cdot|_x = |\cdot|_{(t-b)}$.
- ▲ Is of type II, if $\bigcap_i B^+(a_i, r_i) = B^+(b, s)$, for some $b \in B^+(0,1)$ and s *rational*[11] with respect to $|K^*|_K$. In this case, $|\cdot|_x = |\cdot|_{B^+(b,s)}$.
- ◆ Is of type III, if $\bigcap_i B^+(a_i, r_i) = B^+(b, s)$, for some $b \in B^+(0,1)$ and s *irrational*[12] with respect to $|K^*|_K$. In this case, we also have $|\cdot|_x = |\cdot|_{B^+(b,s)}$.

[11] In other words, $s \in |K^*|_K \cap (0,1]$.
[12] In other words, $s \notin |K^*|_K$.

- Is of type IV, if $\bigcap_i B^+(a_i, r_i) = \varnothing$. The existence of these points is related with the concept of spherically completeness. Most of the fields that we will use here (as $\mathbb{C}\{t\}$ or $\mathbb{C}_p$) are not spherically complete, and therefore type 4 points will exist!

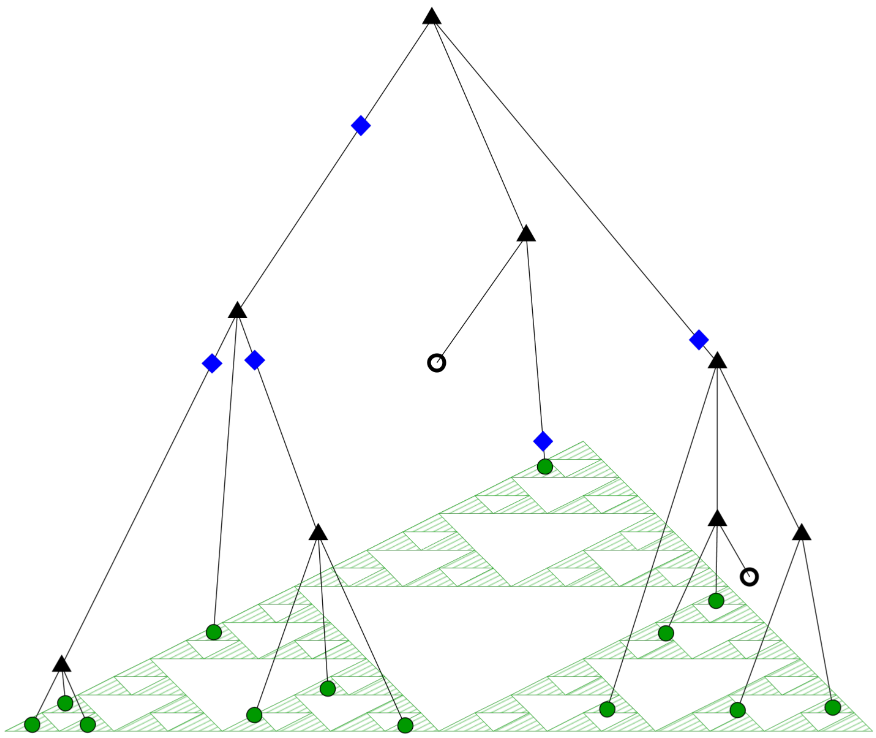


Figure 3: Berkovich closed disk with 13 type I points, 8 type II points, 5 type III points and 2 type IV points marked. Note that the set of type I points look like Figure 2.

One way to represent $\mathcal{B}^+(0,1)$ is as an upside down tree, see Figure 4. The top point would be the *Gauss point* corresponding to the Gauss norm $\|\cdot\| = |\cdot|_{B^+(0,1)}$, which is a type II point. Every type II point $|\cdot|_{B^+(b,s)}$ has many branches: a single branch going up (in case $s < 1$, which corresponds to increasing radius s), and infinitely many branches going down, in one to one correspondence with the open disks $B(b,s) \subset B^+(b,s)$, which are in one to one correspondence with the elements of the residue field $\widetilde{K} := \mathcal{O}_K/\mathfrak{m}$. These branches end up in the points of type I and IV. Points of type III lie between points of type II.

Now, for any $b \in B^+(0,1)$, if we interpret the type I point $|\cdot|_{(t-b)}$ as the seminorm $|\cdot|_{B^+(b,0)}$ corresponding to the degenerate ball $B^+(b,0)$, we have a map

$$\gamma : [0,1] \longrightarrow \mathcal{B}^+(0,1)$$

$$s \longmapsto |\cdot|_{B^+(b,s)}.$$

Geometrically, this corresponds to taking the valuation corresponding to a disk with center b and radius growing until it's the whole disk. One can show that this map is continuous (the intuitive idea behind this is that two norms $|\cdot|_x = |\cdot|_{B^+(a_1,r_1)}$ and

$|\cdot|_y = |\cdot|_{B^+(a_2,r_2)}$ are close to each other if there exists a ball $B^+(a', r')$ containing both $B^+(a_1, r_1)$ and $B^+(a_2, r_2)$ such that $|r' - r_1|$ and $|r' - r_2|$ are small. See also Remark 2.42 below).

In general, if $|\cdot|_x = \lim_i |\cdot|_{B^+(a_i,r_i)}$, then the map $\gamma : [\lim r_i, 1] \to \mathcal{B}^+(0,1)$, defined in the obvious way, is continuous and gives a path between $|\cdot|_x$ and the Gauss point. In particular, $\mathcal{B}^+(0,1)$ is path-connected! See also [Ber90, Theorem 3.2.1] for a generalization of this fact.

Remark 2.42 ([BR10, §1.4]). The closed ball $\mathcal{B}^+(0,1)$ has the structure of a tree. One way to see it is as follows: first, we define a metric d on $\mathcal{B}^+(0,1)$ via

$$d(|\cdot|_x, |\cdot|_y) := 2\,\mathrm{diam}(|\cdot|_x \vee |\cdot|_y) - \mathrm{diam}(|\cdot|_x) - \mathrm{diam}(|\cdot|_y),$$

where

- The points $|\cdot|_x$ and $|\cdot|_y$ are given by $|\cdot|_x = \lim_i |\cdot|_{B^+(a_i,r_i)}$ and $|\cdot|_y = \lim_j |\cdot|_{B^+(a_j,r_j)}$;
- The point $|\cdot|_x \vee |\cdot|_y$ is the least upper bound inside $\mathcal{B}^+(0,1)$ given by[13] the partial order defined as

$$|\cdot|_x \le |\cdot|_y \text{ if and only if } |f|_x \le |f|_y \text{ for all } f \in K\{T\};$$

- The function diam is defined as

$$\mathrm{diam}\left(\lim_i |\cdot|_{B^+(a_i,r_i)}\right) := \lim_i r_i.$$

Then, $M(K\{T\})$ viewed as a set, together with the distance d, forms a metric space $(M(K\{T\}), d)$ which is an $\mathbb{R}$-tree (c.f. [BR10, Lemma 1.12]) and whose topology *is finer* than the Berkovich topology on $M(K\{T\})$. However, the Berkovich topology on $M(K\{T\})$ coincides with the *weak topology* on the $\mathbb{R}$-tree $(M(K\{T\}), d)$, which is the topology with sub-basis of open sets given by the connected components of $M(K\{T\}) \setminus |\cdot|_x$, for all $|\cdot|_x \in M(K\{T\})$. See [BR10, Appendix B] for more on $\mathbb{R}$-trees and [BR10, Proposition 1.13] for a proof of this fact.

For our path γ defined above, the distance is given by

$$d(|\cdot|_{B^+(s_1)}, |\cdot|_{B^+(s_2)}) := 2\max\{s_1, s_2\} - s_1 - s_2 = |s_2 - s_1|.$$

[13] If $|\cdot|_x = |\cdot|_{B^+(a_1,r_1)}$ and $|\cdot|_y = |\cdot|_{B^+(a_2,r_2)}$ are points of type I, II or III, then $|\cdot|_x \vee |\cdot|_y$ is just the seminorm corresponding to smallest disk $B^+(a, r)$ containing both $B^+(a_1, r_1)$ and $B^+(a_2, r_2)$.

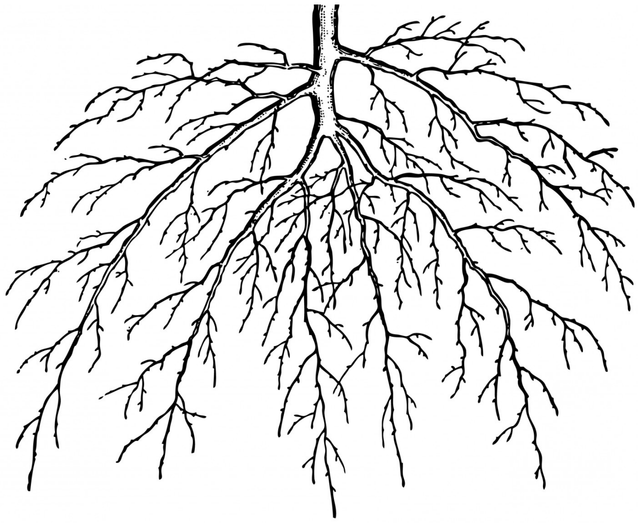


Figure 4: The Berkovich closed ball $\mathcal{B}^+(0,1)$. It has the structure of a tree.

One can also show that type I points are dense in $\mathcal{B}^+(0,1)$: the idea behind this is that an open ball inside $\mathcal{B}^+(0,1)$ contains all the type I points below any point of type II or III of the disk, in the sense that if $|\cdot|_{B^+(b,s)}$ belongs the open ball, then every point $|\cdot|_{(t-b')}$, with $b' \in B^+(b,s)$, belongs to the ball.

Finally, note that the Berkovich closed disk is also contractible, compact and Hausdorff.

For more details and proofs about these facts, we refer the reader to the first Chapter of Baker and Rumely's monograph [BR10].

Remark 2.43. A lot of the properties of the Berkovich closed ball are shared by an arbitrary Berkovich spectrum. In particular, if $(A, \|\cdot\|)$ is a non-zero Banach algebra, then $M(A)$ is non-empty, compact, Hausdorff, locally path connected and locally contractible.

Moreover, if A is also a K-algebra and the valuation on K is non-trivial, we always have an inclusion $\mathrm{Spm}(A) \hookrightarrow M(A)$ and the set $\mathrm{Spm}(A)$ is dense inside $M(A)$. If moreover A doesn't have non-trivial idempotent elements, then $M(A)$ is path-connected.

Example 2.44. The Berkovich spectrum of $(\mathbb{Z}, |\cdot|_\infty)$, denoted $M(\mathbb{Z})$, looks like the broom of Example 2.15. Indeed, all the norms given by Ostrowski's Theorem belong to $M(\mathbb{Z})$, since they are bounded seminorms. Moreover, for every prime p and every $n \in \mathbb{Z}$,

$$|n|_p^\infty := \lim_{t\to\infty} |n|_p^t = \begin{cases} 0 & \text{if } p \text{ divides } n, \\ 1 & \text{else.} \end{cases}$$

defines a bounded seminorm on $\mathbb{Z}$, so it belongs to $M(\mathbb{Z})$. One sees that there are no other bounded seminorms, and that the topology defined in $M(\mathbb{Z})$ looks like Figure 1 (one has be careful when taking open sets around $|\cdot|_0$, because any neighborhood around $|\cdot|_0$ contains almost all hairs of the broom, c.f. [Ber90, Example 1.4.1]; at any other point, $M(\mathbb{Z})$ looks locally as a real interval).

Remark 2.45. Since Berkovich spaces are locally path-connected, they have a topological fundamental group as defined in algebraic topology. We will look into this topological fundamental group later.

Remark 2.46. Given a Banach algebra $(A, \|\cdot\|)$, we have seen how to associate a Berkovich space. In order to globalize this construction, we need first to define a structure sheaf. However, this turns out to be complicated because our building blocks, the Berkovich spectrum of Banach algebras, are compact and Hausdorff. In particular, since compact subsets of Hausdorff spaces are closed, our building blocks are generally not open subsets (unlike in rigid geometry, where our building blocks were closed and simultaneously open). For example, the Banach algebra

$$K\{2t\} := \left\{ f = \sum_{\nu \geq 0} a_\nu t^\nu : \quad |a_\nu|_K \cdot 1/2^\nu \to 0 \right\}$$

defines the Berkovich closed ball of radius 1/2, denoted $\mathcal{B}^+(0,1/2) := M(K\{2t\})$. The inclusion $K\{t\} \hookrightarrow K\{2t\}$ induces the strict inclusion $\mathcal{B}^+(0,1/2) \subsetneq \mathcal{B}^+(0,1)$. Since $\mathcal{B}^+(0,1/2)$ is a compact subset of $\mathcal{B}^+(0,1)$, it is closed, and since $\mathcal{B}^+(0,1)$ is (path-) connected, we see that $\mathcal{B}^+(0,1/2)$ can't be an open subset.

Hence, defining the structure sheaf $\mathcal{O}_{M(A)}$ on $M(A)$ is a bit tricky because we want to define a Grothendieck topology where the "admissible opens" are actually not open subsets (they are called affinoid[14] domains, which are a generalization of the strictly affinoid domains considered in rigid analytic geometry). Once we do this, given an open subset $U \subset M(A)$, $\mathcal{O}_{M(A)}(U)$ is defined as a limit of the algebras A_V corresponding to the affinoid domains V contained in U. The algebra $\mathcal{O}_{M(A)}(U)$ doesn't have in general a norm (because we might have unbounded functions). For more details, see [Ber90, Section 2.3] or [BR10, Appendix C.4].

For the moment, we content ourselves here by saying that given a Banach algebra $(A, \|\cdot\|)$, there is a way to define a structure sheaf $\mathcal{O}_{M(A)}$ on $M(A)$ which gives the expected ring on affinoid domains.

Remark 2.47. Now that we have defined a structure sheaf, one is tempted to define global Berkovich spaces as locally ringed spaces $(X, \mathcal{O}_X)$ locally isomorphic (with respect to a suitable Grothendieck topology) to $(M(A), \mathcal{O}_{M(A)})$, as we did with rigid spaces. This is actually what Berkovich did in [Ber90]. However, it turns out that the spaces obtained by this procedure don't give all the spaces that we would like to consider (in particular, the analytification functor defined in [Ber90] goes from the category of algebraic K-schemes to the category of Berkovich spaces, but this definition doesn't extend to a functor from more general rigid analytic spaces over K to Berkovich spaces).

Berkovich developed in [Ber93, §1] a second approach for defining (Berkovich) analytic spaces via atlases, nets, and morphisms defined up to an equivalence of atlases. This second approach gives analytification functors from the categories of algebraic varieties and from rigid analytic spaces over K to the category of (Berkovich) analytic spaces (sometimes also called non-archimedean analytic spaces). In the present work, we will focus just on the topology of the Berkovich analytification of algebraic schemes, so we will not need all this machinery. We refer the interested reader to [Con08, Section 5], which is a nice survey giving a feeling of what it is occuring, and [Ber93, §1] for the details.

2.4.3 Berkovich analytification of algebraic varieties

We fix for this section a complete non-archimedean field $(K, |\cdot|_K)$ (the valuation may be trivial). Our aim is to define an analytification functor such that for every scheme X locally of finite type over K, we get a Berkovich analytic space X^{an} together with a continuous map $\ker : X^{\text{an}} \to X$. Let's look first at the affine line $\mathbb{A}^1_K := \operatorname{Spec}(K[t])$.

Recall that the Berkovich analytification of $K\{t\}$ is the closed unit disk $\mathcal{B}^+(0,1)$. When K is algebraically closed, type I points of $\mathcal{B}^+(0,1)$ are in one to one correspondence with the closed points $x = (t-a)$ of $\mathbb{A}^1_K$ such that $|a|_K \leq 1$. In general, for $r > 0$, the analytification of $K\{r^{-1}t\}$ gives us the closed disk $\mathcal{B}^+(0,r)$ or radius r, which

[14] Not necessarily strict.

as a set is the set of multiplicative seminorms on $K\{r^{-1}t\}$ bounded by $\|\cdot\|_r$, which is defined as

$$\begin{aligned} \|\cdot\|_r : K\{r^{-1}t\} &\longrightarrow \mathbb{R}_{\geq 0} \\ f = \textstyle\sum_\nu a_\nu t^\nu &\longmapsto \|f\|_r := \max_{x\in B^+(0,r)}\{|f(x)|_K\}. \end{aligned}$$

Type I points of $\mathcal{B}^+(0,r)$ are[15] in one to one correspondence with the closed points $x = (t-a)$ of $\mathbb{A}^1_K$ such that $|a|_K \leq r$. Note that for every $r > 0$, $K[t]$ is a subset of $K\{r^{-1}t\}$, and we can restrict the seminorms of $\mathcal{B}^+(0,r)$ to $K[t]$. All these seminorms restricted to K are just $|\cdot|_K$.

Now, if we want to get the Berkovich space associated to the affine line, the natural candidate is to consider the union of all Berkovich closed disks of increasing radius, i.e.

$$\mathbb{A}^{1,\mathrm{an}}_K := \bigcup_{r>0} \mathcal{B}^+(0,r),$$

which gives us a cover (actually the atlas, in the language of [Ber93]) of the Berkovich affine line by affinoid domains[16]. Note that as a set, $\mathbb{A}^{1,\mathrm{an}}_K$ is just the set of all multiplicative seminorms $|\cdot|_x$ on $K[t]$ extending the given norm $|\cdot|_K$ on K (without any bounded condition). The kernel map is defined as

$$\begin{aligned} \ker : \mathbb{A}^{1,\mathrm{an}}_K &\longrightarrow \mathrm{Spec}(K[t]) \\ |\cdot|_x &\longmapsto \ker(|\cdot|_x) = \{f \in K[t] : \ |f|_x = 0\}. \end{aligned}$$

The topology of $\mathbb{A}^{1,\mathrm{an}}_K$ is the topology induced from the closed balls $\mathcal{B}^+(0,r)$. An equivalent way to describe this topology (c.f. [Nic16, Section 2]) is as the coarsest topology making ker continuous and such that for every Zariski open $U \subset \mathbb{A}^{1,\mathrm{an}}$ and regular function $f \in \mathcal{O}_{\mathbb{A}^{1,\mathrm{an}}_K}(U)$, the map

$$\begin{aligned} \mathrm{ev}_{(U,f)} : \ker^{-1}(U) &\longrightarrow \mathbb{R}_{\geq 0} \\ |\cdot|_x &\longmapsto |f|_x \end{aligned}$$

is continuous. In particular, this gives us an equivalent way of thinking (c.f. [Nic16]) on the elements of $\mathbb{A}^{1,\mathrm{an}}_K$ and on the kernel map: indeed, points would be pairs $(p, |\cdot|_x)$ where p is a (not necessarily closed) point of $\mathrm{Spec}(K[t])$ and $|\cdot|_x$ is a multiplicative *norm* on the residue field $\kappa(p)$ of $\mathrm{Spec}(K[t])$ at p extending the norm $|\cdot|_K$. With this description, the kernel map is the forgetful map sending $(p, |\cdot|_x)$ to p. In particular, we see that if K is algebraically closed, then $\ker^{-1}(p)$ is:

- A unique type I point if p is a closed point;
- The set of all points of type II, III and IV if p is the generic point.

Remark 2.48. Note that there is some parallelism between the procedure of adding generic points in algebraic geometry (to go from old school varieties to schemes) and the procedure of adding points in non-archimedean geometry (to go from rigid analytic geometry to Berkovich geometry). However, when we add points in algebraic geometry, the topology of the scheme becomes more complicated, while in non-archimedean geometry, when we add points the resulting topology is easier.

[15]Here we are still assuming K algebraically closed.
[16]The rigid analytic affine line is defined analogously.

For the case of affine K-schemes $X = \mathrm{Spec}(A)$, where A is a finitely generated algebra over K, the Berkovich analytification X^{an} consists of the multiplicative seminorms on A extending the given norm $|\cdot|_K$ on K (c.f. [Ber90, Remark 3.4.2]). Note that again, as in the case of $\mathbb{A}^{1,\mathrm{an}}$, we don't have to consider any bounded condition because of a similar reason: when considering the Berkovich analytification of a Banach algebra, we have to consider just bounded seminorms, but when we consider the analytification functor of an affine scheme, we first cover our affine scheme with Banach algebras, and then we glue the Berkovich analytification of all these Banach algebras, so at the end of the day we lose the boundedness condition. The topology and the kernel map are defined analogously.

In general, for any scheme X locally of finite type over K, we can glue the Berkovich analytifications of an affine cover of X in order to obtain X^{an}. We can also glue the kernel maps. Since we will not describe the glueing process, a better approach might be to think on X^{an} as the pairs $(p, |\cdot|_x)$ such that p is a point of X, and $|\cdot|_x$ a norm on $\kappa(p)$ extending $|\cdot|_K$.

Remark 2.49 ([Ber90, Corollary 3.4.13 and section 3.5])*.* If X is a proper K-scheme, then our Berkovich analytification functor $X \to X^{\mathrm{an}}$ gives us the typical GAGA properties. For example, it induces an equivalence of categories between the category of finite schemes over X and the category of finite[17] Berkovich K-analytic spaces over X^{an}.

Remark 2.50*.* Under the assumptions of Remark 2.43, the inclusion $\mathrm{Spm}(A) \hookrightarrow M(A)$ extends to an inclusion $X^{\mathrm{rig}} \hookrightarrow X^{\mathrm{an}}$.

Remark 2.51*.* Berkovich spaces have a lot of applications in other areas of mathematics, including Arithmetic Geometry. The original motivation of Berkovich was to understand some phenomena in p-adic spectral theory, as he explains in his funny article [Ber08]. However, there are many more applications, including (but not only):

- A conjecture by Carayol and Drinfeld about the local Langlands program [Boy99; Har97; HT01; Hau05].
- A conjecture by Deligne about vanishing cycles [Ber94].
- Harmonic analysis over the p-adic numbers and Arakelov geometry [BR10; Thu05].
- Tropical geometry [BPR16].
- Birational geometry and Minimal Model Program [Nic16].
- Motivic Zeta functions [NX16].
- Mirror symmetry [KS06].
- Compactification of Bruhat-Tits buildings [RTW15].
- Theory of p-adic period mappings [And03].

Remark 2.52*.* The topology of Berkovich spaces is in general difficult to understand. Their homotopy type, however, is easier to understand because the Berkovich analytification of K-varieties have a deformation retraction onto a simplicial complex, called the *skeleton* of the Berkovich space, which is much simpler.

[17] This is defined in [Ber93, Pages 27-28], and it means that locally, with respect to the Grothendieck topology that we use but that we will not define, we have that $M(B) \to M(A)$ is finite, i.e. that there is an admissible epimorphism of Banach algebras $A^n \twoheadrightarrow B$ for some natural number n.

2.5 Skeleton of Berkovich spaces

From now on, we will restrict our attention to the Berkovich analytification of algebraic varieties (which are, in the terminology of [Ber93], *good* Berkovich spaces).

Let K be a complete non-archimedean field with non-trivial norm, let $\mathcal{O}_K$ be its ring of integers and let X be a connected and smooth K-variety of finite type of dimension n. Assume that X is defined over a discretely valued subfield of K, so that we can also assume that K is discretely valued.

In order to define the skeleton[18] of X^{an}, it is easier to assume that we have an *sncd*-model of X, i.e. a regular scheme $\mathcal{X}$ of finite type over $\mathcal{O}_K$ such that the special fiber $\mathcal{X}_k$ is a divisor with strict normal crossings (such a model always exists for curves, and for arbitrary dimension if the characteristic of the residue field k is zero).

Then, the skeleton $\mathrm{Sk}(\mathcal{X})$ of X^{an} associated to the *sncd*-model $\mathcal{X}$ is defined as the *dual intersection complex*[19] of the special fiber $\mathcal{X}_k$. This complex has a canonical embedding into X^{an}, see Remark 2.55.

Definition 2.53. Recall that the dual intersection complex[20] $\mathrm{Sk}(\mathcal{X})$ of an *sncd*-model $\mathcal{X}$ of X with special fiber $\mathcal{X}_k = \sum_{i \in I} N_i E_i$ is constructed as follows:

For any non-empty subset $J \subset I$, we denote $E_J := \bigcap_{j \in J} E_j$. Then, the complex is given by:

- The faces: for all $d \in \mathbb{N}$, we establish a bijection

$$\begin{array}{ccc} \{\text{Simplices of dimension } d\} & \longleftrightarrow & \left\{\begin{array}{l}\text{Connected components of } E_J, \\ \text{where } |J| = d+1\end{array}\right\} \\ \tau & \longmapsto & C_\tau. \end{array}$$

- The gluing: given τ, τ' simplices of $\mathrm{Sk}(\mathcal{X})$, we establish

$$\tau \subset \tau' \iff C_{\tau'} \subset C_\tau.$$

Remark 2.54. Note that vertices the of $\mathrm{Sk}(\mathcal{X})$ correspond to irreducible components of $\mathcal{X}_k$. Those points are called *divisorial* points in [Nic16]. In terms of valuations, if E is an ireducible component of $\mathcal{X}$ with multiplicity N, it corresponds to the Berkovich point $(\xi, |\cdot|_{NE})$, where $|\cdot|_{NE}$ is the norm defined on the function field $K(X)$ of X corresponding to the valuation giving the order of vanishing on the irreducible component E, i.e. $v_{NE}(f) := \frac{1}{N} \mathrm{ord}_E f$ (c.f. [Nic16, Section 2.3] for more details).

In general, there is such a description for all points in $\mathrm{Sk}(\mathcal{X})$ (c.f. [Nic16, Paragraph 2.4.4]).

Remark 2.55. The idea behind this definition comes from the reduction morphism

$$\mathrm{sp}_{\mathcal{X}} : X^{\mathrm{an}} \to \mathcal{X}_k,$$

which extends the reduction morphism of rigid geometry that maps X^{rig} to the set of closed points of the special fiber, denoted $\mathcal{X}_k^o$, to all the scheme theoretic points of the special fiber $\mathcal{X}_k$:

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{\mathrm{sp}_{\mathcal{X}}} & \mathcal{X}_k \\ \uparrow & & \uparrow \\ X^{\mathrm{rig}} & \xrightarrow{\mathrm{sp}_{\mathcal{X}}|_{X^{\mathrm{rig}}}} & \mathcal{X}_k^o \end{array}$$

---

[18] Here we are following [Nic16] with the extra assumption of properness, but there are similar approaches. Berkovich uses polystable models in [Ber99], which are roughly speaking étale locally semistable models; Gubler et al. use strictly semistable pairs in [GRW16]; Nicaise and Mustaţă define the essential skeleton in [MN15].

[19] One can define a metric on $\mathrm{Sk}(\mathcal{X})$, but we will only consider the skeleton just as a topological space.

[20] It is usually denoted as $|\Delta(\mathcal{X}_k)|$, but since it is homeomorphic to the skeleton $\mathrm{Sk}(\mathcal{X})$ we omit that notation, c.f. [Nic16, Proposition 2.4.6].

If X is a smooth proper curve, the inverse image of smooth closed points under $\mathrm{sp}_{\mathcal{X}}$ is an open ball (in the sense of Berkovich), and the inverse image of a node is an open annulus (also in the sense of Berkovich). The rigid analytic version of the reduction morphism is continuous because of the non-archimedean topology (in other words: rigid analytic open balls and annuli are also closed), while the Berkovich version is anticontinuous. The idea behind this fact is that given a rational function $f \in K(X)$ regular around a point $x \in X^{\mathrm{an}}$, then

$$|f(x)|_{\kappa(x)} < 1 \iff f(x) \in \mathfrak{m}_{\kappa(x)} \iff f(\mathrm{sp}_{\mathcal{X}}(x)) = 0,$$

and we see that an open condition in the Berkovich space corresponds to a closed condition in the special fiber.

The inverse image of the generic point corresponding to an irreducible component of the special fiber is the boundary (in the Berkovich sense) of the open ball corresponding to any smooth closed point of that component, and it coincides with the corresponding point in $\mathrm{Sk}(\mathcal{X})$ described in the previous remark.

Theorem 2.56. *Let X be a proper smooth irreducible K-variety and $\mathcal{X}$ a proper sncd-model of X. Then, there exists a continuous map*

$$H : X^{an} \times [0,1] \to X^{an}$$

such that $H(\cdot,0)$ is the identity, $H(\cdot,1)$ maps onto $\mathrm{Sk}(\mathcal{X})$, and $H(x,t) = x$ for every point x of $\mathrm{Sk}(\mathcal{X})$ and every $t \in [0,1]$. In particular, $\mathrm{Sk}(\mathcal{X})$ is a strong deformation retract of X^{an}.

Moreover, if $U \subset X$ is a dense open subset, then the restriction of H to U induces a deformation retraction of U^{an} onto $\mathrm{Sk}(\mathcal{X})$.

Remark 2.57. For similar results and proofs, see [Ber99, Theorem 5.2], [NX16, p. 3.1.3] and [Nic16]. In the literature, there is still no complete proof of this, but what we will use in this PhD can be found in the above references. We refer to [MN15, (3.1.5)] for more information about the continuity of this map.

Remark 2.58. If we start with a non-proper smooth variety U, then if we have a smooth compactification X and an *sncd*-model $\mathcal{X}$, we can study the skeleton of its analytification. Note that due to the above Theorem, the analytification of different smooth compactifications have the same homotopy type.

On Sections 4 and 5, we will start from a smooth hypersurface U inside a torus $\mathbb{G}_m^n$, and we will study its skeleton via tropicalizations.

Example 2.59. Let $K = \mathbb{C}((t))$, so that $\mathcal{O}_K = \mathbb{C}[[t]]$. Consider the $(n-1)$-dimensional hypersurface X in $\mathbb{P}^n_K = \mathrm{Proj}\,(\mathbb{C}((t))[x_0,\dots,x_n])$ given by the equation

$$f = \sum_{i=0}^{n} x_i^{n+1} + t \cdot x_0 \cdots x_n.$$

Note that the same equation defines a flat proper model $\mathcal{X}$ over $\mathbb{C}[[t]]$. Since the special fiber $\mathcal{X}_{\mathbb{C}}$ is smooth, then $\mathcal{X}$ and X are also smooth.

Lemma 2.60. *Let $f : \mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ be a morphism between schemes, with f locally of finite presentation, proper and flat, and $\mathcal{O}_K$ a discrete valuation ring. If the special fiber $\mathcal{X}_k$ is smooth, then the generic fiber is also smooth.*

Proof. Let s be the closed point of $\mathrm{Spec}(\mathcal{O}_K)$. By [Stacks, Lemma 01V9] (or [EGA IV, Théorème 17.5.1]), we use local finite presentation and smoothness of the special fiber to conclude that f is smooth on an open $U \subset \mathcal{X}$ containing the special fiber $\mathcal{X}_k = f^{-1}(s)$.

Since f is proper, we conclude that $U = \mathcal{X}$, so in particular the generic fiber is smooth. Indeed, $\mathcal{X} \setminus U$ is closed; if it were non-empty, then the properness of f implies that $f(\mathcal{X}\setminus U) \subset \mathrm{Spec}(\mathcal{O}_K)$ is closed and non-empty, and since $\mathcal{O}_K$ is a discrete valuation ring, $f(\mathcal{X}\setminus U)$ has to be equal to s or $\mathrm{Spec}(\mathcal{O}_K)$. In both cases $s \in f(\mathcal{X}\setminus U)$, which implies that $\mathcal{X}\setminus U$ meets $f^{-1}(s)$, hence a contradiction. □

Therefore, in our example both X and $\mathcal{X}$ are smooth and proper, so in particular $\mathcal{X}$ is an *sncd*-model. We can now define the skeleton $\mathrm{Sk}(\mathcal{X})$ of X^{an} with respect to $\mathcal{X}$, which is the dual intersection complex of the special fiber $\mathcal{X}_{\mathbb{C}}$, i.e. a single point. In particular, the analytification X^{an} of the hypersurface given by the equation

$$f = \sum_{i=0}^{n} x_i^{n+1} + t \cdot x_0 \cdots x_n.$$

is contractible.

Example 2.61. Given an elliptic curve E over K, the dual intersection graph of an *sncd*-model can be either contractible (if it has good reduction or additive bad reduction) or homotopy equivalent to a circle (if it has multiplicative bad reduction). Therefore, the homotopy type of the Berkovich analytification of an elliptic curve depends on its reduction.

Remark 2.62 ([BPR13]). If X is a smooth proper curve of genus at least 2 defined over a complete algebraically closed non-archimedean field, the semi-stable reduction theorem guarantees that X has a model $\mathcal{X}$ where the irreducible components of the special fiber are smooth and proper, and each irreducible component intersects transversely the other irreducible components in at most two points. Since we can view a Berkovich open annulus as a real open interval with Berkovich open balls attached to its rational (i.e. type II) points (c.f. [BPR13, Section 2]), we can picture X^{an} as follows: first, one constructs the dual intersection graph of this model, and then one adds Berkovich balls on the type II points.

Remark 2.63 ([BPR16, Section 2.4]). For smooth connected curves X that are not complete, there is an extended notion of skeletons that includes the missing points. Let $\hat{X}$ be its smooth compactification, and denote D the set of missing points $D := \hat{X}\setminus X$. Now, we have to choose a semistable model $\mathcal{X}$ of $\hat{X}$ such that the points of D reduce to different smooth points of the special fiber. Then, we consider inside X^{an} the unique minimal closed connected subset $\mathrm{Sk}(\mathcal{X}, D)$ which contains the skeleton $\mathrm{Sk}(\mathcal{X})$ and whose closure in $\hat{X}^{\mathrm{an}}$ contains D. In other words, we consider inside $\hat{X}^{\mathrm{an}}$ the minimal closed connected set containing the skeleton $\mathrm{Sk}(\mathcal{X})$ and the points D, and we intersect this set with X^{an}.

As in the theorem above, we get a deformation retraction H such that $H(\cdot, 1)$ maps onto $\mathrm{Sk}(\mathcal{X}, D)$.

3 Non-homeomorphic conjugate Berkovich spaces

3.1 Example of the elliptic curve

Let $(K, |\cdot|_K)$ be a complete and algebraically closed non-archimedean field with non-trivial absolute value, and fix a valuation $v_K : K \to \mathbb{Q} \cup \{\infty\}$. For simplicity, we assume that the characteristic of the residue field is different from 2 or 3. Let $t \in K$ be an element of valuation $v_K(t) = 1$. Throughout this section, we fix a compatible set of roots of t, so that we can talk of t^m for any rational number m.

Given an elliptic curve E/K, it is known that its reduction depends only on the valuation of the j-invariant $j(E)$ (c.f. for example [Liu02, Proposition 10.2.33]). Hence, if $|j(E)|_K \le 1$, then E^{an} is contractible; otherwise, E^{an} is homotopic to a circle.

Following an idea of Antoine Chambert-Loir, one can try to take an elliptic curve E and an automorphism σ of K such that $|j(E)|_K \le 1 < |j(E_\sigma)|_K$, where E_σ denotes the fiber product

$$\begin{array}{ccc} E_\sigma & \longrightarrow & E \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\sigma} & \operatorname{Spec} K. \end{array}$$

Remark 3.1. If σ is continuous with respect to the non-archimedean topology, then the absolute value of the j-invariant will not change. This is the case for example if $K = \mathbb{C}_p$ and σ fixes $\mathbb{Q}_p$. Hence, we will have to look to abstract field automorphisms of K.

Now, when we work over a field of characteristic different from 2 or 3, we can always write an elliptic curve in this form (c.f. [Sil09, Section III.1]):

$$E : y^2 = x^3 + ax + b,$$

and its j-invariant is given by

$$j(E) = 1728\frac{4a^3}{4a^3 + 27b^2}.$$

Let's construct an automorphism $\sigma \in \operatorname{Aut}(K)$ that changes this j-invariant.

For simplicity, let's assume that $a = t$. Then,

$$|j(E)|_K = \left|1728\frac{4t^3}{4t^3 + 27b^2}\right|_K = \frac{|t^3|_K}{|4t^3 + 27b^2|_K}.$$

Now, let b be any element in K transcendental over $F(t)$, where F is the prime field[21] of K. By multiplying with an appropriate power of t (for instance, with $t^{1-v_K(b)}$) we can assume that $|b|_K = |t|_K$. In this way,

$$|j(E)|_K = \frac{|t^3|_K}{|4t^3 + 27b^2|_K} = \frac{|t^3|_K}{|b^2|_K} = |t|_K < 1,$$

so its Berkovich analytification E^{an} will be contractible.

Since K is algebraically closed, the element $-\frac{4}{27}t^3 + b^4$ has two square roots. We choose one and denote it by $\sqrt{-\frac{4}{27}t^3 + b^4}$. Let σ_b be an automorphism of K mapping b to $\sqrt{-\frac{4}{27}t^3 + b^4}$. Such an automorphism exists.

Lemma 3.2. *Let C_1 and C_2 be algebraically closed fields with same prime subfield F, and assume that they both have the same transcendence degree (possibly infinite) over F. Then, they are isomorphic.*

[21] Note that t might be inside F. This is the case if $K = \mathbb{C}_p$ and $t = p$, for example.

Moreover, if $(x_1, \dots, x_n)$ *(resp.* $(y_1, \dots, y_n)$*) is an* n*-tuple of algebraically independent elements of* C_1 *(resp. of* C_2*), then the morphism between the fields*

$$F(x_1, \dots, x_n) \longrightarrow F(y_1, \dots, y_n)$$

mapping x_i *to* y_i *can be extended to an isomorphism*

$$\sigma : C_1 \longrightarrow C_2.$$

Proof. Let X be a transcendental basis of C_1 over F, that is, a minimal set of elements inside C_1 such that any element of C_1 is algebraic over the field generated by F and the elements of X, denoted $F(X)$. Such a set exists by the axiom of choice. The cardinality of X is the transcendence degree of C_1.

Let Y be a transcendental basis of C_2. Since C_1 and C_2 have the same transcendental degree, there exists a bijection

$$s : X \longrightarrow Y$$

which induces a field isomorphism

$$F(X) \longrightarrow F(Y).$$

Now, we want to extend this morphism to

$$\sigma : C_1 \longrightarrow C_2.$$

Since $C_1/F(X)$ is an algebraic extension and $C_2/F(Y)$ is an algebraic closure, we can extend the morphism (c.f. [Mil18, Theorem 6.8])

$$\begin{array}{ccc} F(X) & \longrightarrow & F(Y) \\ \downarrow & & \downarrow \\ C_1 & \overset{\sigma}{\dashrightarrow} & C_2 \end{array}$$

Finally, since C_1 is also algebraically closed, we get that σ is indeed an isomorphism of fields (c.f. [Mil18, Theorem 6.8]).

Note that if we have two n-tuples $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ of algebraically independent elements of C_1 and C_2 respectively, then we can extend these tuples to transcendental basis X and Y of C_1 and C_2 respectively, and we can construct a bijection

$$s : X \longrightarrow Y$$

such that $s(x_i) = y_i$ for $i = 1, \dots, n$. The above procedure yields the extension that we were looking for. □

In our case, we extend the morphism[22]

$$F(t, b) \to F\left(t, \sqrt{-\frac{4}{27}t^3 + b^4}\right)$$

to an automorphism σ_b of K. Then, if we pull back this automorphism to E, we get an elliptic curve E_{σ_b}

$$\begin{array}{ccc} E_{\sigma_b} & \longrightarrow & E \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \xrightarrow{\sigma_b} & \operatorname{Spec} K \end{array}$$

[22] If t belongs to the prime field F, then we are just extending the morphism $F(b) \to F\left(\sqrt{-\frac{4}{27}t^3 + b^4}\right)$.

with equation

$$E_{\sigma_b} : y^2 = x^3 + tx + \sqrt{-\frac{4}{27}t^3 + b^4},$$

and we have that

$$|j(E_{\sigma_b})|_t = \frac{|t^3|_K}{|4t^3 + 27\left(-\frac{4}{27}t^3 + b^4\right)|_K} = \frac{|t^3|_K}{|b^4|_K} = \frac{1}{|t|_K} > 1.$$

In particular, its Berkovich analytification $E_{\sigma_b}^{an}$ is not contractible, so this conjugation gives us two non-homeomorphic conjugate Berkovich spaces.

3.2 More examples

We have just constructed a non-continuous automorphism $\sigma \in \operatorname{Aut}_{F(t)} K$ that makes E^{an} and E_σ^{an} non-homeomorphic. But we can ask a converse question: given a non-continuous automorphism $\sigma \in \operatorname{Aut} K$, is there an elliptic curve E such that both E^{an} and E_σ^{an} are non-homeomorphic?

In order to find such an elliptic curve, we first need to make some auxiliary computations.

Lemma 3.3. *Let $\sigma \in \operatorname{Aut} K$ be a non-continuous automorphism such that*[23] $|u|_K = |\sigma(u)|_K$ *for some non-zero $u \in K$ with absolute value $|u|_K$ strictly smaller than* 1. *Then,*

1. *There exists a transcendental element $\mu \in K$ such that*

$$v_K(\mu) < v_K(\sigma(\mu)).$$

2. *There exist $\lambda, \mu \in K$ algebraically independent such that*

$$v_K(\lambda) = v_K(\mu) < v_K(\lambda + \mu)$$

and

$$v_K(\sigma(\lambda)) = v_K(\sigma(\lambda + \mu)) < v_K(\sigma(\mu)).$$

Proof. Let $\sigma \in \operatorname{Aut} K$ be a non-continuous automorphism. This means that there exists $\mu \in K$ such that $v_K(\mu) \neq v_K(\sigma(\mu))$. We can assume that $v_K(\mu) < v_K(\sigma(\mu))$, because if we have the other inequality, we can just consider μ^{-1}.

If μ is transcendental over its residue field, then we obtain the first part of the lemma. Else, we pick any transcendental number ω. If $v_K(\omega) \neq v_K(\sigma(\omega))$ we are done (either ω or ω^{-1} will give us the result), and if we have an equality, then $\mu' := \mu\omega$ works:

$$\begin{aligned} v_K(\mu') &= v_K(\mu) + v_K(\omega) \\ &< v_K(\sigma(\mu)) + v_K(\sigma(\omega)) = v_K(\sigma(\mu')). \end{aligned}$$

Now, for the second part, we first pick μ as above, and let m be the arithmetic mean

$$m := \frac{v_K(\mu) + v_K(\sigma(\mu))}{2},$$

which satisfies $v_K(\mu) < m < v_K(\sigma(\mu))$. Let $s \in K$ be an element such that $s^{v_K(u)} = u$. In particular, $1 = v_K(s) = v_K(\sigma(s))$. Without loss of generality, we can assume that s and μ are algebraically independent: Indeed, if s and μ are not algebraic independent,

[23] If $K = \mathbb{C}_p$, then we always have this assumption, since $\sigma(p) = p$ for every field automorphism σ.

pick an element $\omega \in K$ algebraically independent of s and μ. If $|\omega|_K = |\sigma(\omega)|_K$, then we redefine s as an element such that $s^{v_K(\omega)} = \omega$; else, ω or ω^{-1} will play the role of μ.

Now, we choose an appropriate root of s so that we can consider s^m, and we define

$$\lambda := s^m - \mu.$$

Then, since

$$v_K(\mu) < m = v_K(s^m),$$

we have that

$$v_K(\lambda) = v_K(\mu) < m = v_K(s^m) = v_K(\lambda + \mu),$$

and also that

$$\begin{aligned} v_K(\sigma(\lambda)) &= v_K(\sigma(s^m) - \sigma(\mu)) = m \\ v_K(\sigma(\lambda + \mu)) &= v_K(\sigma(s^m) - \sigma(\mu) + \sigma(\mu)) = m. \end{aligned}$$

Hence, we get the result.

□

Now, we fix a non-continuous automorphism $\sigma \in \operatorname{Aut} K$ such that $v_K(\sigma(u)) = v_K(u)$ for some non-zero u of valuation strictly positive, and λ, μ algebraic independent numbers satisfying Lemma 3.3. Let E be the elliptic curve given by

$$E : y^2 = x^3 + ax + b, \tag{1}$$

where a is a cubic root of $\lambda/4$ and b a square root of $\mu/27$.

Then we have that

$$|j(E)|_K = \frac{|a|_K^3}{|4a^3 + 27b^2|_K} = \frac{|\lambda|_K}{|\lambda + \mu|_K} > 1,$$

so that E^{an} *is not contractible*, and

$$|j(E_\sigma)|_K = \frac{|\sigma(a)|_K^3}{|4\sigma(a)^3 + 27\sigma(b)^2|_K} = \frac{|\sigma(\lambda)|_K}{|\sigma(\lambda) + \sigma(\mu)|_K} = 1,$$

so that E_σ^{an} *is contractible.* In particular, we obtain the following proposition.

Proposition 3.4. *Let K be a complete and algebraically closed non-archimedean field with non-trivial valuation and residue field of characteristic different from 2 or 3. Let $\sigma \in \operatorname{Aut} K$ be a non-continuous automorphism such that $|u|_K = |\sigma(u)|_K$ for some non-zero $u \in K$ with absolute value $|u|_K$ strictly smaller than 1. Then, the elliptic curve E of equation (1) satisfies that E^{an} and E_σ^{an} are non-homeomorphic.*

Corollary 3.5. *Let $p \geq 5$ be a prime number, and $\sigma \in \operatorname{Aut}_{\mathbb{Q}} \mathbb{C}_p$ be a non-continuous automorphism. Then, the elliptic curve E of equation (1) satisfies that E^{an} and E_σ^{an} are non-homeomorphic.*

Remark 3.6. These examples already show how different this problem is in the non-archimedean setting, compared with the complex setting. In complex geometry, the analytification of all conjugate projective smooth curves are homeomorphic as topological spaces (Riemann surfaces are, as topological spaces, classified by their genus, which is invariant under conjugation). In non-archimedean geometry, elliptic curves already give us examples of non-homeomorphic conjugate Berkovich spaces.

3.3 Existence of a contractible conjugate

In this subsection, due to Johannes Nicaise, we show the existence of non-homeomorphic examples in arbitrary dimension.

Proposition 3.7. *Let K be the field of Puiseux series. Every connected smooth and proper variety X over K is conjugate to a smooth and proper variety Y whose Berkovich analytification is contractible.*

Proof. Spread out X over a complex algebra A of finite type, take a general complex arc in the smooth locus of $\mathrm{Spec}(A)$, and let Y be the base change to a completed algebraic closure of the generic point of the arc. Then the Berkovich analytification of Y is contractible. Since both X and Y come from the same variety over the function field of A, they are conjugate.

□

Remark 3.8. For curves, we can do the following: Take any smooth and proper family of curves over a complex connected smooth curve C. For every place x of C (including the closed points of its smooth compactification) we can consider the base change C_x of C to a completed algebraic closure of the fraction field of the completed local ring at x. All these curves are conjugate (because they come from the same curve over the function field of C) but the homotopy type of the Berkovich analytification depends on the reduction at x. So every example where the dual graph of an *sncd*-model at some point x has a loop will give a pair of conjugate curves where one is contractible and the other is not.

Remark 3.9. In the rest of the thesis, we will focus in producing explicit examples of conjugate curves of higher genus with non-homeomorphic analytifications. For this, we will use tropical geometry.

4 Tropicalizing towards explicit examples of curves of higher genus

Our previous explicit examples were constructed because we could use the j-invariant of an elliptic curve, which relates its equation with the homotopy type of its Berkovich analytification. In order to look into higher genus curves, we need a different approach, since we don't have an analogue of the j-invariant.

We fix for this section an algebraically closed field K which is complete with respect to a nontrivial nonarchimedean valuation $\nu : K^* \to \mathbb{R}$. We choose the approach of looking into the tropicalization[24] of a curve. Roughly speaking, the tropicalization of a curve (or more generally a quasi-projective variety) which is inside a torus $\mathbb{G}_m^n$ (for example, we can first embed in $\mathbb{P}_K^n$, and then intersect our variety with a choice of $\mathbb{G}_m^n \hookrightarrow \mathbb{P}_K^n$) consists in the valuation of its points: we obviously lose a lot of information in this process, but if we consider the limit of all (extended) tropicalizations, we can recover the Berkovich analytification.

Theorem 4.1 ([Pay09, Theorem 1.1]). *Let X be an affine variety over K. The Berkovich analytification X^{an} is homeomorphic to the inverse limit in the category of topological spaces of all extended tropicalizations $\pi_\iota : X^{an} \to (\mathbb{R} \cup \infty)^n$, where ι is an affine embedding*

$$\iota : X \hookrightarrow \mathbb{A}^n : y \mapsto (f_1(y), \dots, f_n(y)),$$

and π_ι is given by

$$\pi_\iota(x) = (-\log|f_1|_x, \dots, -\log|f_n|_x),$$

where $|-|_x$ denotes as usual the seminorm associated to $x \in X^{an}$.

The advantage of using tropical geometry is that it is very convenient for computations. But Payne's theorem needs to consider *all* the tropicalizations, which makes the computations again difficult. However, if we restrict to curves and are only interested in the skeleton of the analytification, there exists a *faithful tropicalization*, i.e. a tropicalization that induces a homeomorphism[25] on a finite subgraph Γ containing the skeleton of X^{an} (c.f. [BPR16, Paragraph 5.15.2]).

Theorem 4.2 ([BPR16, Theorem 5.20]). *Let X be a nonsingular curve over K. If Γ is any finite subgraph of X^{an}, then there is a closed immersion $X \hookrightarrow Y_\Delta$ of X into a quasiprojective toric variety Y_Δ such that* $\mathrm{Trop} : X^{an} \to N_{\mathbb{R}}(\Delta)$ *faithfully represents* Γ. *In particular, there exists a faithful tropicalization.*

Remark 4.3. In Theorem 4.2 we are using the standard notation for toric geometry, that is, $N := \mathrm{Hom}(M, \mathbb{R})$ is the dual lattice of a free abelian group of rank n, Δ a fan in $N_{\mathbb{R}}$, $N_{\mathbb{R}}(\Delta)$ is the partial compactification of $N_{\mathbb{R}}$ with respect to Δ (c.f. [Rab12, Definition 3.6]) and Y_Δ the associated toric variety. If for example we start with $M = \mathbb{Z}^n$ and $\Delta = \{0\}$, then $N_{\mathbb{R}}(\Delta) = N_{\mathbb{R}}$ and $Y_\Delta = \mathbb{G}_m^n$. If we start with $M = \mathbb{Z}^2$ and Δ equal to the first quadrant, then $N_{\mathbb{R}}(\Delta) = (\mathbb{R}^2_{\geq 0} \cup \infty)^2$ (c.f. [Rab12, Example 3.7]) and $Y_\Delta = \mathbb{A}^2$, as in Theorem 4.1. Check [Rab12], [BPR13] and [BPR16] for further details with tropical flavor.

Remark 4.4. Theorem 4.2 has also a higher dimensional version, c.f. [GRW16, Theorem 9.5].

In the next subsection, we give all the definitions while we revisit our previous example of non-homeomorphic conjugate elliptic curves in the setting of tropicalizations. Afterwards, we use these tools to construct examples for higher genus curves.

[24] *Tropical geometry* is named in honor of the Brazilian computer scientist Imre Simon, who lived in São Paolo, near the Tropic of Capricorn, c.f. [Kat17] or [MS15, p. 1].

[25] One can define a metric on Berkovich curves and on tropicalizations, and a faithful tropicalization also induces an isometry on the finite subgraph containing the skeleton. In this thesis we will not use this metric, so we don't define it and refer instead the interested reader to [BPR16].

4.1 Tropicalizations

Let $(K, |\cdot|_K)$ be a non-archimedean field with non-trivial valuation. For convenience, let's assume that K is also algebraically closed. In tropical geometry it is more convenient to use valuations and valued groups (rather than norms). Let val be a valuation of our field K normalized so that there exists an element of valuation 1, and let $\Gamma_{val} := \mathrm{val}(K \setminus \{0\}) \subset \mathbb{R}$ denote the valuation group. We denote by k the residue field of K, and we choose an element t of valuation 1 and a compatible collection $\{t^{1/n}\}_{n>0}$ of roots of t.

The tropicalization map in the closed points of $\mathbb{G}_{m,K}^n = \mathrm{Spec}(K[x_1^{\pm 1}, \dots, x_n^{\pm 1}])$ is defined as

$$\begin{aligned} \mathbb{G}_{m,K}^n &\longrightarrow \mathbb{R}^n \\ (x_1, \dots, x_n) &\longmapsto (\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) \end{aligned}$$

Let $X \hookrightarrow \mathbb{G}_{m,K}^n$ be a closed subscheme. Then, the tropicalization of $X \hookrightarrow \mathbb{G}_{m,K}^n$, denoted $\mathrm{Trop}(X)$, is defined as the closure in $\mathbb{R}^n$ of the image of the closed points of X under the previous map, that is,

$$\mathrm{Trop}(X) := \overline{\{(\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) \in \mathbb{R}^n : (x_1, \dots, x_n) \in X\}},$$

that is, the closure[26] of the valuation of the coordinates of the (closed) points of X.

We can also describe $\mathrm{Trop}(X)$ from its Berkovich analytification X^{an}. Indeed, if $X \hookrightarrow \mathbb{G}_{m,K}^n$ is a closed scheme as before, the tropicalization map is defined as

$$\begin{aligned} \mathbb{G}_{m,K}^{n,\mathrm{an}} &\longrightarrow \mathbb{R}^n \\ |\cdot|_x &\longmapsto (-\log(|x_1|_x), \dots, -\log(|x_n|_x)). \end{aligned}$$

Restricting this map to $X^{\mathrm{an}} \subset \mathbb{G}_{m,K}^{n,\mathrm{an}}$ gives us again $\mathrm{Trop}(X)$ (c.f. [Gub13, Proposition 3.8]). We see that the tropicalization map is continuous. Moreover, if $n = 2$, $X \subset \mathbb{G}_m^2$ is a smooth curve, $\hat{X}$ is its smooth compactification and $D := \hat{X} \setminus X$, then for any *sncd*-model $\mathcal{X}$ of $\hat{X}$ such that the points of D reduce to different smooth points of the special fiber, the tropicalization map factors through the retraction onto the extended skeleton $\mathrm{Sk}(\mathcal{X}, D)$ (c.f. [BPR16, Section 2.4] and [BPR13, Section 4]):

$$\begin{array}{ccc} X^{\mathrm{an}} & \xrightarrow{\mathrm{Trop}} & \mathrm{Trop}(X) \\ {\scriptstyle H(\cdot,1)}\big\downarrow & \nearrow_{\mathrm{Trop}|_{\mathrm{Sk}}} & \\ \mathrm{Sk}(\mathcal{X}, D) & & \end{array}$$

We present here a third way to describe the tropicalization, in this case of a hypersurface $X = V(f)$ inside $\mathbb{G}_{m,K}^n$, which corresponds to a morphism $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \to K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]/(f)$ for some Laurent polynomial $f(x_1, \dots, x_n) = \sum_{u \in \mathbb{Z}^n} c_u x^u$.

The tropicalization of this polynomial is defined as the piecewise linear function

$$\mathrm{trop}(f) : \mathbb{R}^n \to \mathbb{R} : w \mapsto \min\{\mathrm{val}(c_u) + w \cdot u : u \in \mathbb{Z}^n \text{ and } c_u \neq 0\},$$

where $w \cdot u$ denotes the scalar product between (the so-called weight vector) w and (the exponent of the monomial $c_u x^u$) u.

---

[26] We don't need to take the closure if $\Gamma_{val} = \mathbb{R}$. This is the case if for example $K = \mathbb{C}(t^{\mathbb{R}})$, the field of Hahn series. In the examples that we will see, we usually have $\Gamma_{val} = \mathbb{Q}$, for example $K = \mathbb{C}_p$ or $K = \mathbb{C}\{t\}$.

Given a weight vector $w \in \mathbb{R}^n$, we denote $W := \mathrm{trop}(f)(w)$. Then, $\mathrm{Trop}(X)$ is exactly the set of $w \in \mathbb{R}^n$ where the minimum W is achieved at least twice, i.e. there are two different $u, u' \in \mathbb{Z}^n$ such that

$$W = \mathrm{val}(c_u) + w \cdot u = \mathrm{val}(c_{u'}) + w \cdot u'.$$

See [MS15, Theorem 3.1.3] for a proof that both definitions coincide.

Example 4.5. Let $K = \mathbb{C}_p$ and consider the elliptic curve E inside $\mathbb{G}^2_{m,K}$ given by $g(x,y) = y^2 - x^3 - x^2 - p$. Then, for $w = (w_1, w_2) \in \mathbb{R}^2$, we have that

u	$\mathrm{val}(c_u) + (w_1, w_2) \cdot u$
$(0,0)$	1
$(2,0)$	$2w_1$
$(3,0)$	$3w_1$
$(0,2)$	$2w_2$

and therefore

$$\mathrm{trop}(g)(w_1,w_2) = \begin{cases} 1 & \text{if } w_1, w_2 \geq 1/2, \\ 2w_1 & \text{if } w_1 \in [0,1/2] \text{ and } w_2 \geq w_1, \\ 3w_1 & \text{if } w_1 \leq 0 \text{ and } w_2 \geq (3/2)w_1, \\ 2w_2 & \text{if } w_2 \leq 0 \text{ and } w_2 \leq (3/2)w_1, \text{ or if } w_2 \in [0,1/2] \text{ and } w_1 \geq w_2 \end{cases}$$

Hence, $\mathrm{Trop}(E)$ is given by the set of (w_1, w_2) where $\mathrm{trop}(g)(w_1, w_2)$ is achieved at least twice, that is,

- If $w_1 = 1/2$ and $w_2 \geq 1/2$, then $u = (0,0)$ and $(2,0)$ achieve the minimum, which is 1. This is the green ray of Figure 5.
- If $w_2 = 1/2$ and $w_1 \geq 1/2$, then $u = (0,0)$ and $(0,2)$ achieve the minimum, which is 1. This is the blue ray of Figure 5.
- If $w_1 = w_2$ and they are in the interval $[0, 1/2]$, then $u = (2,0)$ and $(0,2)$ achieve the minimum, which is $2w_1$. This is the black segment of Figure 5.
- If $w_1 = 0$ and $w_2 \geq 0$, then $u = (2,0)$ and $(3,0)$ achieve the minimum, which is 0. This is the orange ray of Figure 5.
- If $w_1 \leq 0$ and $w_2 = (3/2)w_1$, then $u = (3,0)$ and $(0,2)$ achieve the minimum, which is $3w_1$. This is the red ray of Figure 5.

Note that this is much easier to compute than the image of E^{an} under the tropicalization corresponding to the given embedding $E \subset \mathbb{G}^2_m$.

It is convenient to introduce here the concept of initial form. The *initial form* of f with respect to the weight w is the Laurent polynomial with coefficients in the residue field given by

$$\mathrm{in}_w(f) := \sum_{u : \mathrm{val}(c_u) + w \cdot u = W} \overline{c_u t^{-\mathrm{val}(c_u)}} x^u \in k[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

where $\overline{c_u t^{-\mathrm{val}(c_u)}}$ is just the reduction of $c_u t^{-\mathrm{val}(c_u)}$ in the residue field k.

Example 4.6. Let $K = \mathbb{C}_p$, $k = \mathcal{O}_{\mathbb{C}_p}/\mathfrak{m}_{\mathbb{C}_p}$ and $g(x,y) = y^2 - x^3 - x^2 - p$. We fix the weight $w = (1/4, 1/4)$. Then, the set of $u \in \mathbb{N}^2$ where $c_u \neq 0$ is equal to

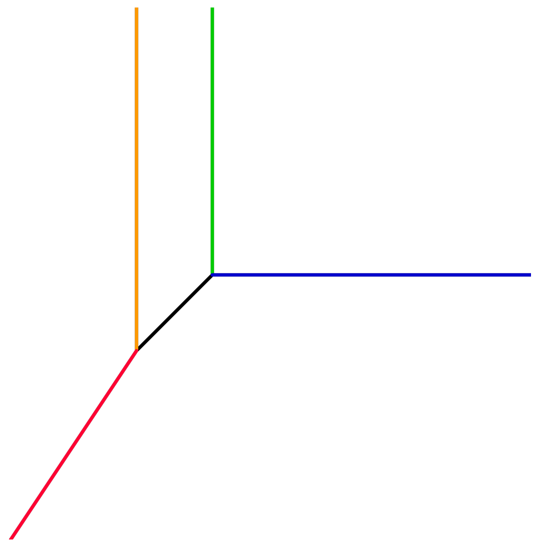


Figure 5: Trop(E).

$\{(0,0),(2,0),(3,0),(0,2)\}$, so W is the minimal value of $\mathrm{val}(c_u)+(1/4,1/4)\cdot u$, where $u\in\{(0,0),(2,0),(3,0),(0,2)\}$. We have

$$\begin{array}{rc} u\mapsto & \mathrm{val}(c_u)+(1/4,1/4)\cdot u \\ (0,0)\mapsto & 1+0=1 \\ (2,0)\mapsto & 0+1/4\cdot 2=1/2 \\ (3,0)\mapsto & 0+1/4\cdot 3=3/4 \\ (0,2)\mapsto & 0+1/4\cdot 2=1/2 \end{array}$$

so we get $W=1/2$, and therefore

$$\begin{aligned} \mathrm{in}_{(1/4,1/4)}(g) &:= \sum\nolimits_{u:\ \mathrm{val}(c_u)+w\cdot u=1/2} \overline{c_u t^{-\mathrm{val}(c_u)}}x^u \\ &= \sum\nolimits_{u\in\{(2,0),(0,2)\}} \overline{c_u t^{-\mathrm{val}(c_u)}}x^u \\ &= -x^2+y^2\in k[x^{\pm1},y^{\pm1}]. \end{aligned}$$

Then, our last definition of tropicalization reads as follows: Trop(X) is the set of $w\in\mathbb{R}^n$ such that $\mathrm{in}_w(f)$ is not a monomial.

In practice, we will also use the Newton polytope for studying tropicalizations. Recall that for a Laurent polynomial $f=\sum c_u x^u\in K[x_1^{\pm1},\dots,x_n^{\pm1}]$ defining an hypersurface X inside $\mathbb{G}_{m,K}^n$, the Newton polytope Newt(f) is defined as the convex hull of the exponents appearing in the monomial, i.e.

$$\mathrm{Newt}(f)=conv(u\in\mathbb{Z}^n:\ c_u\neq 0)\ \subset\mathbb{R}^n.$$

Moreover, we subdivide this polytope using the valuation of K, i.e. we consider the convex hull inside $\mathbb{R}^{n+1}\cup(0,\dots,0,\infty)$ of the set

$$\{(u,\mathrm{val}(c_u))\in\mathbb{R}^{n+1}:\ c_u\neq 0\}\cup(0,\dots,0,\infty),$$

and then we project onto the first n coordinates (the projection lies inside Newt(f)). This subdivision of Newt(f) is called the *regular* subdivision.

Example 4.7. Let $K=\mathbb{C}\{t\}$, and $f(x,y)=x^3+y^3+txy+t^4$. Then we construct the regular subdivision of Newt(f) as follows.

First, we consider the set

$$\{(u, \text{val}(c_u)) \in \mathbb{R}^{2+1} : c_u \neq 0\} = \{(3,0,0), (0,3,0), (1,1,1), (0,0,4)\}.$$

Now, we take the convex hull of this set union the vertical direction $(0, 0, \infty)$. In Figure 6 we drew this on the left, without the vertical faces.

Then, we project to the horizontal plane the edges of the convex hull, and we obtain the regular subdivision of $\text{Newt}(f)$, which is on the right of Figure 6.

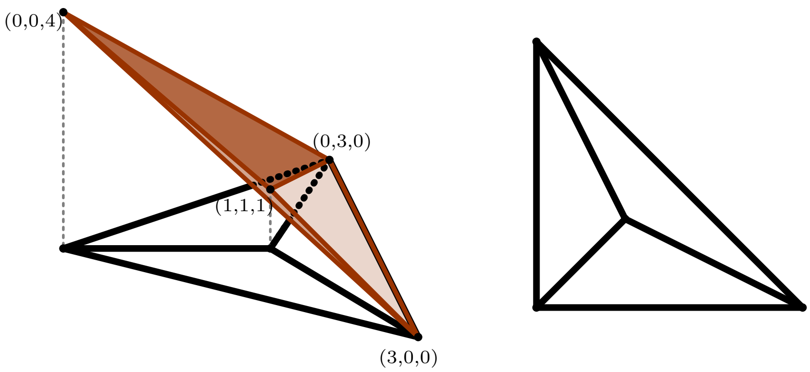


Figure 6: On the left, convex hull induced by the valuation of K; on the right, the corresponding regular subdivision of $\text{Newt}(f)$.

Finally, we present the last way to compute a tropicalization. For any Laurent polynomial $f \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, it turns out that $\text{Trop}(V(f))$ is the $(n-1)$-skeleton of the polyhedral complex dual to the regular subdivision of $\text{Newt}(f)$ (c.f. [MS15, Proposition 3.1.6]). This gives us an easy way to sketch tropicalizations, see Figure 7.

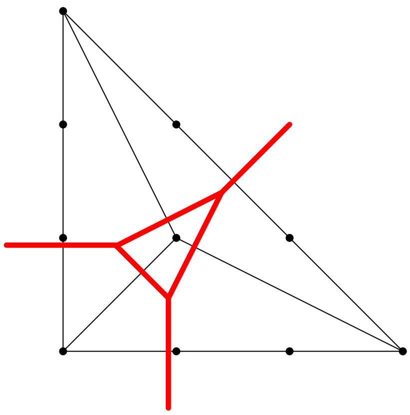


Figure 7: $\text{Trop}(f)$, in red, is dual to $\text{Newt}(f)$, in black. One needs to rotate the red picture 180º to get the usual tropicalization.

Moreover, this approach with the regular subdivision of the Newton polytope allows us to define multiplicities of the maximal cells of the tropicalization. Indeed, any maximal cell σ on $\text{Trop}(f)$, which has dimension $n - 1$, is dual to an edge e_σ of $\text{Newt}(f)$.

Every edge e_σ of $\mathrm{Newt}(f)$ has a lattice length $m(\sigma)$, which is precisely the number of lattice points at e_σ minus one. Then, the *multiplicity* of σ at $\mathrm{Trop}(f)$ is precisely $m(\sigma)$ (c.f. [MS15, p. 112], and also [MS15, Definition 3.4.3 and Lemma 3.4.6] for an algebraic definition).

Example 4.8. In Figure 7, the rays have multiplicity 3 and the segments have multiplicity 1.

Remark 4.9. Tropical hypersurfaces satisfy a balancing condition with respect to the multiplicity defined above (c.f. [MS15, Proposition 3.3.2]). For planar curves, the balancing condition says that for any point $w \in \mathrm{Trop}(f)$, then

$$\sum_{\sigma \in \mathrm{star}(w)} m(\sigma) e_\sigma = 0,$$

where $\mathrm{star}(w)$ is the set of 1 dimensional faces intersecting an arbitrarily small ball around w, and e_σ is the vector lying on the 1 dimensional vector space parallel to σ, with the same orientation as the ray or segment starting[27] on w, such that e_σ lies in the lattice $\mathbb{Z}^2$. Note that such a vector exists because the $\mathrm{Trop}(f)$ is rational with respect to the valuation group Γ_{val} of our base field.

With the notation of Example 4.5, then we have that around $w = (0,0)$, the set $\mathrm{star}(w)$ consists of the black segment, and the orange and red rays. Looking at the Newton polygon, we see that the multiplicities are 2 for the segment, and 1 for the ray. Then, we check that indeed

$$2(1,1) + (0,1) + (-2,-3) = 0.$$

4.2 Faithful and homotopic tropicalizations

We are interested in the homotopy type of the Berkovich analytification of an hypersurface $X \subset \mathbb{G}_m^n$. Assume that we have a smooth compactification $\hat{X}$ of X and a proper *sncd*-model $\mathcal{X}$ of $\hat{X}$ such that the points $D := \hat{X} \setminus X$ reduce to different smooth points of the special fiber (for curves, we can always achieve this), and let $\mathrm{Sk}(\mathcal{X}, D)$ denote the corresponding skeleton. Then, we say that the embedding $X \subset \mathbb{G}_m^n$ induces a *faithful tropicalization* (c.f. [CHW14]) if

$$\mathrm{Trop}\,|_{\mathrm{Sk}(\mathcal{X})} : X^{\mathrm{an}} \longrightarrow \mathbb{R}^n$$

restricted to $\mathrm{Sk}(\mathcal{X})$ is injective and the map $\mathrm{Trop}\,|_{\mathrm{Sk}(\mathcal{X})}$ has a continuous section (in particular, the tropicalization map induces a homeomorphism between $\mathrm{Sk}(\mathcal{X})$ and its image). If X is a planar curve, we also require that the homeomorphism between $\mathrm{Sk}(\mathcal{X})$ and its image are isometric with respect to metrics that we have not defined here (c.f. [BPR16]).

Note that faithful tropicalizations induce in particular an homotopy equivalence between X^{an} and $\mathrm{Trop}(X)$ (c.f. [BPR13, Theorem 5.15]. Indeed, since the tropicalization is faithful, no loop of the skeleton of X^{an} is contracted or created (there is an homeomorphism between the skeleton and its image in the tropicalization), and since the curve is planar, the rays will not intersect creating extra loops in $\mathrm{Trop}(X)$ (for an example of a tropicalization of the projective line inside $\mathbb{P}^3$ where a loop appears, see [Spe07, Example 6.5]).

Since we are only interested in the homotopy type, we will consider *homotopic tropicalizations*.

Definition 4.10. We say that the embedding $X \hookrightarrow \mathbb{G}_m^n$ induces a *homotopic tropicalization* if X^{an} is homotopy equivalent to $\mathrm{Trop}(X)$.

[27] If w is in the interior of the edge, we have two copies of σ at opposite directions, so that the balancing condition trivially holds.

Remark 4.11. Faithful tropicalizations of hypersurfaces are homotopic, but the converse is not true (we could contract the length of a segment, see for a concrete example [CM16, Example 4.9]).

Remark 4.12. If $X \subset \mathbb{G}_m^n$ is not an hypersurface, then faithful tropicalizations may fail to be homotopic. This is because there can be extra intersections of unbounded rays that we wouldn't like to consider. For hypersurfaces, the description via the regular subdivision of the Newton polytope ensures that these phenomena can't happen.

Remark 4.13. Assuming that the residue characteristic of K is not 2 or 3, then any elliptic curve given in Weierstrass form is of the form (c.f. [Sil09, Section III.1]):

$$E : y^2 = x^3 + ax + b,$$

so the regular subdivision of its Newton polygon is of one of these forms:

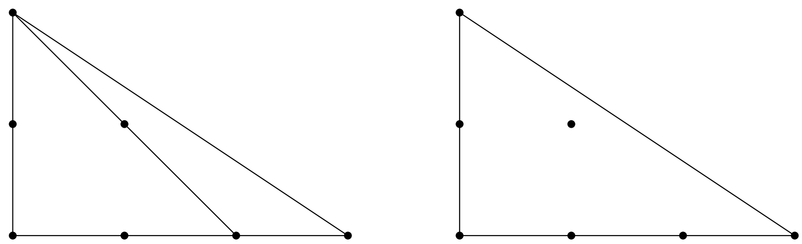


Figure 8: Possible Newton polygons of an elliptic curve written in Weierstrass form.

In particular, we see that there can't be a loop in the tropicalization, so if our elliptic curve has multiplicative bad reduction, the embedding corresponding to the Weierstrass equation can't induce an homotopic tropicalization.

Remark 4.14. Let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ define a hypersurface $X \subset \mathbb{G}_m^n$. We say that the regular subdivision of $\mathrm{Newt}(f)$ is *unimodular* if all the n-dimensional cells are n-simplices with the same minimal volume $1/(n!)$. Note that this implies that all the interior points of $\mathrm{Newt}(f)$ occur as vertices of these cells.

For example, if X is a planar curve, the regular subdivision of $\mathrm{Newt}(f)$ is unimodular if its regular subdivision consists only on triangles of area $1/2$.

One can show that if $\mathrm{Newt}(f)$ is unimodular, then $X \subset \mathbb{G}_m^n$ is smooth [MS15, Proposition 4.5.1].

4.3 Faithful tropicalization of an elliptic curve

In this section we follow [BPR16] and [Hel19] to construct a faithful tropicalization of an elliptic curve. For convenience, we work over $\mathbb{C}_p$, with $p \geq 5$.

If we start with an elliptic curve given in Weierstrass form with multiplicative bad reduction, we've seen that the corresponding tropicalization can't be faithful.

Recall (c.f. [Sil09, Proposition III.3.1]) that the Weierstrass equation of an elliptic curve (E, O) is obtained from the chosen point O (which is the zero element in the group law) by taking a basis of rational functions $\{1, x\}$ of the linear system $\mathcal{L}(2(O))$, and extending it to a basis $\{1, x, y\}$ of $\mathcal{L}(3(O))$. Since the divisor $3(O)$ has degree 3 and an elliptic curve has genus 1, it is very ample[28]; therefore the rational map $E \to \mathbb{P}^2 : P \mapsto [x(P) : y(P) : 1]$ is a closed immersion. Now, the vector space $\mathcal{L}(6(O))$ has dimension 6 by Riemann-Roch (c.f. [Har77, Remark IV.1.3.2 and Example IV.1.3.3]), so there has to be a linear combination of the seven elements $\{1, x, y, x^2, xy, y^2, x^3\}$, which gives the Weierstrass equation.

Now, in order to obtain a new embedding that determines a faithful tropicalization of our elliptic curve, we have to construct rational functions in a different way. This is achieved in [BPR16] (the construction is also studied more explicitly in [Hel19]) in the following way: first, we fix a 3-torsion point $P \in E$, which exists since we are working over an algebraically closed field. Let $Q := P + P$ (which is also a 3-torsion point). Then, we take the rational functions given by the divisors $2(P) - (Q) - (O)$ and $2(Q) - (P) - (O)$, which are principal[29] and therefore correspond to rational functions (f) and (g) respectively. These rational functions define a closed immersion $E \to \mathbb{P}^2 : P \mapsto [f(P) : g(P) : 1]$ because $\{1, f, g\}$ is a basis of $\mathcal{L}((P) + (Q) + (O))$, and the divisor $D = (P) + (Q) + (O)$ is very ample. Now the rational functions $\{1, fg, f^2g, fg^2\}$ lie in $\mathcal{L}(3(O))$, which is 3-dimensional (Riemann-Roch), and therefore there exists a linear combination $af^2g + bfg^2 + cfg = d$. Moreover, as shown in [Hel19], we can choose the coefficients a, b and c to have valuation 0 and simultaneously d with strictly positive valuation. This gives a new equation for the elliptic curve with the following Newton polygon.

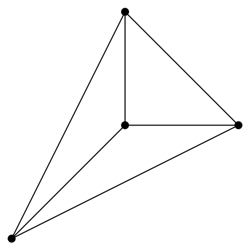


Figure 9: Newton polygon of the new embedding of the elliptic curve. This gives a faithful tropicalization.

Since the Newton polytope has a unimodular triangulation, we conclude that this new embedding of our elliptic curve gives a faithful tropicalization: this is shown in [BPR16, Corollary 5.28] by using the Poincaré-Lelong formula, and without this machinery (by making more explicit computations) in [Hel19]. In particular, it is an homotopic tropicalization.

If we had started with E_σ, the elliptic curve with multiplicative bad reduction conjugate of the elliptic curve E from equation 1 of the previous section, we could apply the inverse automorphism σ^{-1} to obtain an elliptic curve with good reduction (namely E). At the level of the tropicalizations, we would see how the cycle disappears.

This method of constructing faithful tropicalizations, however, relies in the group structure of the elliptic curve, so we can't generalize it straightforward (c.f. [Hel16] for more in this direction). Is there another way of obtaining faithful tropicalizations that we can also use for curves of higher genus?

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[28] See [Har77, Corollary IV.3.2].

[29] A divisor $\sum a_i(P_i)$ on an elliptic curve is principal if and only if $\sum a_i = 0$ and $\sum a_i P_i = O$, where the second sum is the sum from the group law, c.f. [Sil09, Theorem X.3.8].

4.4 Faithful tropicalizations via modifications

Here we will still work over $\mathbb{C}_p$, with p at least 5, and we denote by $\mathbb{F}$ its residue field. Let E be the elliptic curve given by the Weierstrass equation $y^2 = x^3 + x^2 + p$. Reducing modulo p, we see that it has multiplicative bad reduction. In Figure 10 we can see that its tropicalization contains no cycle (c.f. Example 4.6 for an explicit construction of the tropicalization).

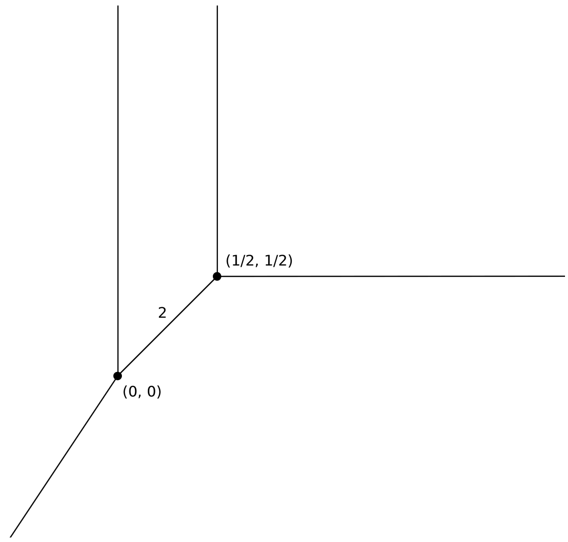

Figure 10: Tropicalization of $g(x, y) = y^2 - x^3 - x^2 - p$.

Note that the segment from $(0, 0)$ to $(1/2, 1/2)$ has multiplicity 2.

In [CM16], they explain how to repair tropicalizations *locally*, in the sense of slightly changing the tropicalization in order to get more information from the skeleton. They do this by using modifications. Instead of explaining this machinery, we will see how does this construction look like with a concrete example.

Theorem 4.15 ([CM16, Theorem 3.4]). *Let e be a vertical bounded edge of* $\text{Trop}(g)$ *of multiplicity at least* 2 *whose endpoints have valency* 3*. If the discriminant of e does not vanish at* $\text{in}_e(g)$*, then the tropicalization map is not faithful at e and we can unfold this edge with a linear re-embedding $I_{g,f}$ of the curve determined by a tropical modification. The new curve* $\text{Trop}(I_{g,f})$ *contains a cycle that maps to e via π_{XY}.*

Here $g(x, y) = y^2 - x^3 - x^2 - p$, e is the only bounded edge of $\text{Trop}(g)$ and $\text{in}_e(g)$ is defined to be the initial degeneration of g at any interior point w inside the edge e; in our case (see Example 4.6),

$$\text{in}_e(g) = \text{in}_{(1/4,1/4)}(g) = \sum_{\substack{u:\, \text{val}(c_u)+(1/4,1/4)\cdot u \\ \text{is minimal}}} \overline{p^{-\text{val}(c_u)} c_u}\, x^u = y^2 - x^2 \in \mathbb{F}[x, y].$$

The discriminant of e is given by $c_{1,1}^2 - 4c_{0,2}c_{2,0}$, so when we evaluate it on $\text{in}_e(g)$ we get $0^2 - 4 \cdot 1 \cdot (-1) = 4 \neq 0$. Considering the modification along $f(x, y) = x + y$, the ideal $I_{g,f} := \langle g, z - f \rangle \subset \mathbb{C}_p[x, y, z]$ defines a curve in $\mathbb{A}^3_{\mathbb{C}_p}$ which is the intersection of $E \times \mathbb{A}^1_{\mathbb{C}_p}$ with the plane $z - x - y = 0$.

Its tropicalization defines a curve $\text{Trop}(I_{g,f})$ in the tropical 3-space that contains a cycle mapping to e via the projection π_{XY}. We can see this cycle by projecting $\text{Trop}(I_{g,f})$ via π_{ZY}. Indeed, this projection is the tropical curve $\text{Trop}(\tilde{g})$ coming from the equation

$$\tilde{g}(z,y) = g(z-y,y) = y^2 - (z-y)^3 - (z-y)^2 - p = -z^3 + 3z^2y - 3zy^2 + y^3 - z^2 + 2zy - p,$$

and we see in Figure 11 that it has a cycle.

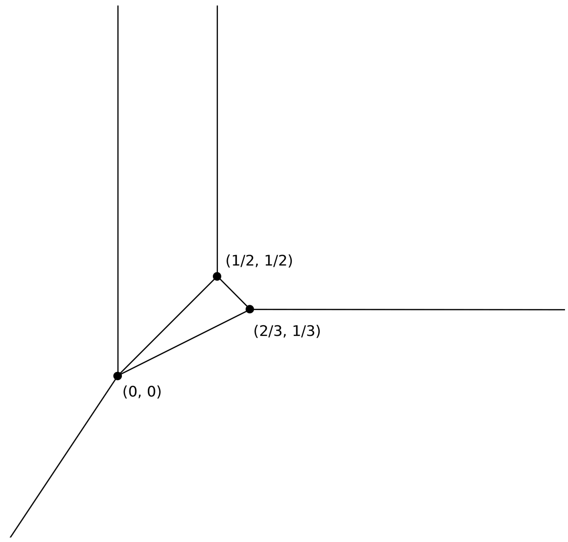

Figure 11: Tropicalization of $\tilde{g}(z,y) = -z^3 + 3z^2y - 3zy^2 + y^3 - z^2 + 2zy - p$.

Remark 4.16. The method of modifications has not been fully developed yet. If we had a way to construct a modification giving faithful tropicalizations in general, we could apply those methods to construct a lot of explicit examples of non-homeomorphic conjugate Berkovich curves. However, we don't have this at our disposal.

Remark 4.17. It would also be interesting to study the method of modifications for obtaining homotopic tropicalizations, since that might be easier than obtaining faithful tropicalizations. In particular, that would help in the study of whether a given smooth curve is Mumford or not.

5 Smooth planar curves of higher genus

In this section we construct explicit examples of conjugate curves ofhigher genus over the field of Puiseux series $\mathbb{C}\{t\}$ such that their Berkovich analytifications are not homeomorphic. For the figures, we use the software Polymake [GJ00]. Let C be the projective curve defined by

$$f(x,y,z) = a_0 z^4 + x^4 + y^4 + t^6 xyz^2 + t^2 x^2 yz + txy^2 z,$$

where a_0 is a transcendental element over $\mathbb{Q}(t)$ with valuation equal to 0. Since the special fiber is smooth[30], we conclude (c.f. Lemma 2.60) that the generic fiber is smooth. Hence, $C = V(f)$ has good reduction and its Berkovich analytification C^{an} is contractible.

Now let a_{21} be a transcendental element over $\mathbb{Q}(t)$ of valuation equal to 21, and let σ be an automorphism of $\mathbb{C}\{t\}$ sending a_0 to a_{21} and fixing t, which exists because of Lemma 3.2. The conjugate curve C_σ is defined by the equation

$$f_\sigma(x,y,z) = a_{21} z^4 + x^4 + y^4 + t^6 xyz^2 + t^2 x^2 yz + txy^2 z.$$

This curve doesn't have good reduction. Figure 12 shows us the Newton polytope of f_σ.

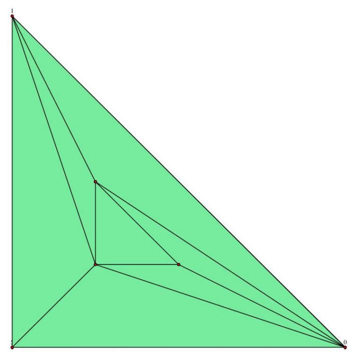

Figure 12: Newton polytope of $f_\sigma(x,y,z) = a_{21} z^4 + x^4 + y^4 + t^6 xyz^2 + t^2 x^2 yz + txy^2 z$.

Since the resultant of f_σ has valuation equal to 100, we conclude that is non-zero and therefore C_σ is smooth.

We see that the three interior points occur, so we conclude (c.f. [BPR16, Proof of Corollary 5.28, (ii)]) that the Berkovich analytification C_σ^{an} has 3 cycles: its topological fundamental group is the free group on 3 letters. In particular, C_σ and C are conjugate curves of genus 3 with non-homeomorphic Berkovich analytification.

[30]The partial derivatives are $4x^3, 4y^3$ and $4a_0 z^3$, so they only vanish at the point $(0,0,0)$. Another way to see this is because the resultant, up to a sign, will be in this case $-18014398509481984\,\overline{a_0}^9$, which is different from zero because a_0 has valuation 0. In Appendix A.1 we include an algorithm written in SAGE that computes the discriminant of plane curves, and that works reasonably well until degree 6.

5.1 Candidates of arbitrary degree

We want to construct candidates of arbitrary degree. We will generalize the idea of the above example to build higher degree examples.

The way we do this is the following: firstly we construct a Newton polytope with where all the interior vertices are vertices of the subdivision, such that when we change the valuation of the coefficient corresponding to the vertex $(0,0)$, all the interior points disappear. The Newton polytope with the interior vertices occurring will correspond to f_σ, and the one without interior vertices, to f.

Secondly, we construct polynomials f and f_σ that define smooth curves and give the desired Newton polytopes. To get the desired Newton polytope is not very difficult, and to prove smoothness we use discriminants. However, we expect them to be smooth because the expected genus of f_σ when looking at the Newton polytope is precisely $(n-1)(n-2)/2$, the number of interior points; if we had a singularity, we would have smaller genus.

Let's describe now how we obtained the example above of degree 4, and afterwards we explain how to obtain a polynomial of arbitrary degree such that the regular subdivision of its Newton polytope has all interior points. In the appendix, we include an algorithm for SAGE that implements this procedure.

For degree 4, the Newton polytope is inside the triangle $(0,0)$, $(0,4)$ and $(4,0)$. The interior points of this polytope are $(1,1)$, $(1,2)$ and $(2,1)$. Since we want our automorphism σ to change the valuation of the coefficient corresponding to the point $(0,0)$ (which corresponds to the monomial z^4), it is more convenient to construct the valuations from the "north-east" direction to the "south-west", which is where the vertex $(0,0)$ is. Hence, we start with $(0,4)$, $(4,0)$ and $(1,2)$, and we give them valuation $v_{0,4}=0$, $v_{4,0}=0$ and $v_{1,2}=1$. The next point in the diagonal is $(2,1)$, and we want it to be over the affine plane generated by $(0,4,0)$, $(4,0,0)$ and $(1,2,1)$, where the last coordinate of $(i,j,v_{i,j})$ is the valuation assigned to the coefficient corresponding to the point (i,j). Since $(2,1,1)$ lies in the plane, we choose $v_{2,1}:=2$, which is the smallest integer strictly bigger than 1 (there is no strict need to choose an integer, we do it just to get later coefficients in $\mathbb{C}[t]$).

The diagonal where $i+j=3$ is done, and now we go to the next diagonal, i.e. where $i+j=2$. The only interior point of the Newton polytope lying in this diagonal is $(1,1)$. In order to compute the valuation $v_{1,1}$, we have to check 4 planes: the first one, which was generated by the points $(4,0,0),(0,4,0)$ and $(1,2,1)$, together with the three new planes given by the 3-tuples

$$\begin{array}{c}(4,0,0),(0,4,0),(2,1,2)\\(4,0,0),(1,2,1),(2,1,2)\\(0,4,0),(1,2,1),(2,1,2)\end{array}$$

A computation shows us that we want $v_{1,1}$ bigger than

$$\max\{v:(1,1,v) \text{ belongs to one of the planes defined above}\}=\max\{2,4,5,1\},$$

so we choose $v_{1,1}=6$. We are done with the interior points, so now we can define f to be

$$\begin{aligned} f(x,y,z) &= a_0z^4+x^4+y^4+\sum_{\substack{1\le i,j\le 2\\ i+j\le 3}} t^{v_{i,j}}x^iy^jz^{4-i-j}\\ &= a_0z^4+x^4+y^4+txy^2z+t^2x^2yz+t^6xyz^2. \end{aligned}$$

A computation with the discriminant shows us that f is indeed smooth. In order to define the automorphism σ of $\mathbb{C}\{t\}$, we need to choose first the valuation of $\sigma(a_0)$ so that f_σ is a Mumford curve. In other words, we want that the point $(0,0,\operatorname{val}(\sigma(a_0)))$

lies above all the planes defined by the 3-tuples

$$\begin{array}{c}(4,0,0),(0,4,0),(1,2,1)\\(4,0,0),(0,4,0),(2,1,2)\\(4,0,0),(1,2,1),(2,1,2)\\(0,4,0),(1,2,1),(2,1,2)\\(4,0,0),(0,4,0),(1,1,6)\\(4,0,0),(1,2,1),(1,1,6)\\(4,0,0),(2,1,2),(1,1,6)\\(0,4,0),(1,2,1),(1,1,6)\\(0,4,0),(2,1,2),(1,1,6)\\(1,2,1),(2,1,2),(1,1,6)\end{array}$$

Another computation shows us that $\mathrm{val}(\sigma(a_0))$ has to be bigger than

$$\max\{4,8,12,0,12,44/3,16,20,40/3,15\},$$

so we look for $\mathrm{val}(\sigma(a_0)) = 21$. Hence we have to choose an element a_{21} of valuation 21 and transcendental over $\mathbb{Q}(t)$, and σ an automorphism fixing t and mapping a_0 to a_{21}. In this way, the conjugate curve

$$\begin{aligned} f_\sigma(x,y,z) &= a_{21}z^4 + x^4 + y^4 + \sum_{\substack{1\le i,j\le 2\\ i+j\le 3}} t^{v_{i,j}} x^i y^j z^{4-i-j} \\ &= a_{21}z^4 + x^4 + y^4 + txy^2z + t^2x^2yz + t^6xyz^2 \end{aligned}$$

will have the desired regular subdivision on its Newton polytope. Again, a computation with the discriminant shows us that f_σ is smooth. This is how we constructed the example of degree 4.

For general degree n bigger than 4, the Newton polytope is inside the triangle $(0,0)$, $(0,n)$ and $(n,0)$. The interior points of this polytope are $\{(i,j):\ 1\le i,j,\le n-2 \text{ and } i+j\le n-1\}$. We give to the vertices $(0,n)$, $(n,0)$ and $(1,n-2)$ the valuations 0, 0 and 1 respectively. As before, we go on in the diagonal where $i+j=n-1$, and for every such point (i,j), we give the valuation $v_{i,j}$ so that the point $(i,j,v_{i,j})$ lies above all the previous (non-vertical) affine planes generated by the the 3-tuples of points $(i',j',v_{i',j'})$. We keep on going diagonal after diagonal, and on each diagonal, from the left to the right (i.e. increasing i) until we are done with all the interior points of the Newton polytope. Finally, we construct the valuation $v_{0,0}$ associated to the vertex $(0,0)$, which allows us to choose an element $a_{v_{0,0}}$ of valuation $v_{0,0}$ and transcendental over $\mathbb{Q}(t)$, and an automorphism σ sending a_0 to $a_{v_{0,0}}$ and fixing t. This automorphism σ will transform our curve C defined by f, which has good reduction[31], to its conjugate C_σ that we expect to be Mumford (it is needed to check whether C_σ is smooth).

In the Appendix A.2 we describe an algorithm written in SAGE that constructs these valuations for any degree n. For example, when $n=5$, we get the following curves:

$$\begin{aligned} f(x,y,z) = \ & a_0z^5 \\ & +x^5+y^5+t^{28}xyz^3+t^{12}x^2yz^2+t^9xy^2z^2+t^4x^3yz+t^2x^2y^2z+txy^3z \\ f_\sigma(x,y,z) = \ & a_{97}z^5 \\ & +x^5+y^5+t^{28}xyz^3+t^{12}x^2yz^2+t^9xy^2z^2+t^4x^3yz+t^2x^2y^2z+txy^3z. \end{aligned}$$

The discriminant of f (up to the sign) equals

$$-171052300930023193359375t^{700}a_0 + \ldots - 3552713678800500929355621337890625a_0^{16},$$

[31] It has good reduction because modulo t, our equation is $\overline{a_0}z^n + x^n + y^n$.

and a computation shows us that the discriminant of f_σ (which is the same, just write a_{97} instead of a_0) has valuation 595, so in particular is different from 0 (hence f_σ is smooth). Figure 13 shows the Newton polytope of f_σ.

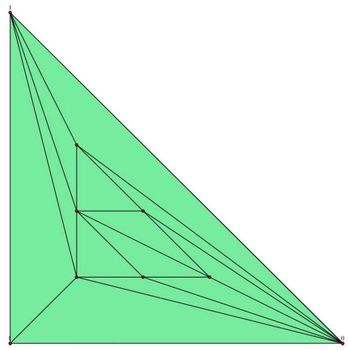

Figure 13: Newton polytope of $f_\sigma(x,y,z) = a_{97}z^5 + x^5 + y^5 + t^{28}xyz^3 + t^{12}x^2yz^2 + t^9xy^2z^2 + t^4x^3yz + t^2x^2y^2z + txy^3z$.

Hence, we have constructed a curve C of genus 10 such that C^{an} and C_σ^{an} are non-homeomorphic.

For degree 6, the polynomial f is given by

$$a_0z^6 + x^6 + y^6 + t^{818}xyz^4 + t^{274}x^2yz^3 + t^{147}xy^2z^3 + t^{32}x^3yz^2 + t^{23}x^2y^2z^2 + t^{15}xy^3z^2 + t^7x^4yz + t^4x^3y^2z + t^2x^2y^3z + txy^4z,$$

and we want to define σ so that a_0 maps to a_{4027}. In order to check smoothness, we have to compute the discriminant. This computation starts getting complicated because it involves a big determinant. It would be easier to check that the discriminant is non-zero, but for the moment we just stick to the algorithm that we have.

Figure 14 shows the Newton polytope of f_σ.

Remark 5.1. In order to overcome the computational difficulties of the discriminant, one can construct in a similar way polynomials f and f_σ such that we have another way to prove smoothness. For example, if f has good reduction and f_σ defines a polynomial whose Newton polygon has a unimodular triangulation, then we can conclude that $V(f_\sigma) \cap \mathbb{G}_m^2$ is smooth.

5.2 Smooth non-homeomorphic planar curves of arbitrary degree

Using the ideas of the previous section, we can construct an explicit smooth curve of arbitrary degree with good reduction such that its conjugate is a Mumford curve.

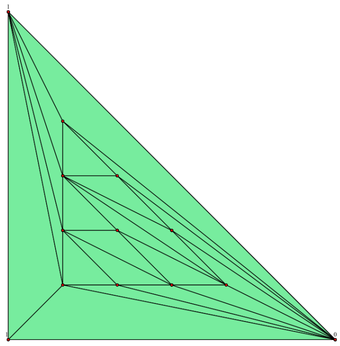

Figure 14: Newton polytope of $f_\sigma(x,y,z) = a_{4027}z^6 + x^6 + y^6 + t^{818}xyz^4 + t^{274}x^2yz^3 + t^{147}xy^2z^3 + t^{32}x^3yz^2 + t^{23}x^2y^2z^2 + t^{15}xy^3z^2 + t^7x^4yz + t^4x^3y^2z + t^2x^2y^3z + txy^4z$.

For degree $n = 4$, consider

$$\begin{aligned} f(x,y,z) = \; & y^4 \\ & + ty^3z + txy^3 \\ & + t^3y^2z^2 + t^4xy^2z + t^7x^2y^2 \\ & + t^{14}yz^3 + t^{24}xyz^2 + t^{46}x^2yz + t^{127}x^3y \\ & + a_0z^4 + t^{2072}xz^3 + t^{4141}x^2yz^2 + t^{12353}x^3z + b_0x^4. \end{aligned}$$

where b_0 and a_0 are algebraically independent elements of $\mathbb{C}\{t\}$ that are transcendental over $\mathbb{Q}(t)$ and have valuation 0. As before, we immediately see that it has good reduction.

Now, let σ be an automorphism of $\mathbb{C}\{t\}$ fixing t and sending a_0 to a_{704} and b_0 to b_{49229}, where a_{704} has valuation 704 and b_{49229} has valuation 49229, and both are algebraically independent elements transcendental over $\mathbb{Q}(t)$. Then, the conjugate of f with respect to σ is

$$\begin{aligned} f_\sigma(x,y,z) = \; & y^4 \\ & + ty^3z + txy^3 \\ & + t^3y^2z^2 + t^4xy^2z + t^7x^2y^2 \\ & + t^{14}yz^3 + t^{24}xyz^2 + t^{46}x^2yz + t^{127}x^3y \\ & + a_{704}z^4 + t^{2072}xz^3 + t^{4141}x^2yz^2 + t^{12353}x^3z + b_{49229}x^4. \end{aligned}$$

Note that setting $z = 1$, we have that $\mathrm{Newt}(f_\sigma)$ is unimodular[32], c.f. Figure 15.

Hence, by [MS15, Proposition 4.5.1], we can conclude that f_σ is smooth inside the torus $\mathbb{G}_m^2$.

Now, let $C \subset \mathbb{G}_m^2$ be the curve defined by $f(x,y,1)$, and $C_\sigma \subset \mathbb{G}_m^2$ its conjugate. We conclude, since both are smooth, that their Berkovich analytification are not homeomorphic.

In Appendix A.3 we give a script written in SAGE that constructs the equation of a smooth curve such that its Berkovich analytification is not not homeomorphic to the Berkovich analytification of its conjugate. Here we list the examples of Mumford curves

[32] This can be checked using Pick's theorem.

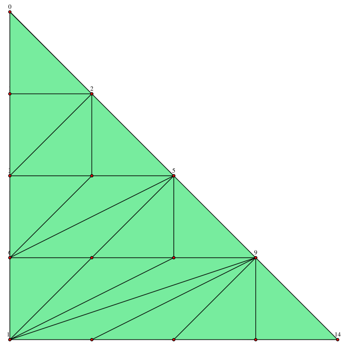

Figure 15: Newton polytope of $f_\sigma(x, y, z)$.

of degree 5 and 6 (the conjugate, which has good reduction, is defined by sending a_ν and $b_{\nu'}$ to a_0 and b_0 resp.):

$$
\begin{aligned}
f(x,y) = \;& y^5 \\
& + ty^4 + txy^4 \\
& + t^3y^3 + t^4xy^3 + t^7x^2y^3 \\
& + t^{14}y^2 + t^{24}xy^2 + t^{46}x^2y^2 + t^{127}x^3y^2 \\
& + t^{704}y + t^{2072}xy + t^{4141}x^2y + t^{12353}x^3y + t^{49229}x^4y \\
& + a_{588722} + t^{2352074}x + t^{7050014}x^2 + t^{28199937}x^3 + t^{112783186}x^4 \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + b_{563854166}x^5
\end{aligned}
$$

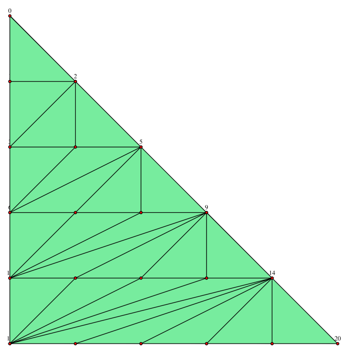

Figure 16: Newton polytope of the Mumford curve of degree 5 $f(x, y)$.

$$
\begin{aligned}
f(x,y) = \quad & y^6 \\
& + ty^5 + txy^5 \\
& + t^3y^4 + t^4xy^4 + t^7x^2y^4 \\
& + t^{14}y^3 + t^{24}xy^3 + t^{46}x^2y^3 + t^{127}x^3y^3 \\
& + t^{704}y^2 + t^{2072}xy^2 + t^{4141}x^2y^2 + t^{12353}x^3y^2 + t^{49229}x^4y^2 \\
& + t^{588722}y + t^{2352074}xy + t^{7050014}x^2y + t^{28199937}x^3y \\
& \qquad + t^{112783186}x^4y + t^{563854166}x^5y \\
& + a_{11275852602} + t^{56376319402}x + t^{225495869320}x^2 + t^{901983466923}x^3 \\
& \qquad + t^{3607933846990}x^4 + t^{18039528235267}x^5 + b_{108236492712487}x^6.
\end{aligned}
$$

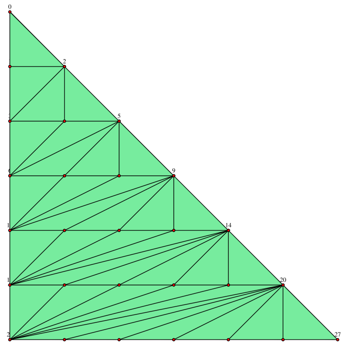

Figure 17: Newton polytope of the Mumford curve of degree 6 $f(x,y)$.

Remark 5.2. The advantage of this second approach is that we don't have to check smoothness, so we actually get explicit examples of non-homeomorphic conjugate Berkovich curves of arbitrary degree. The disadvantage is that the polynomials have many more terms and the valuations appearing are much bigger.

6 Acknowledgments

This thesis has been developed under the supervision of Hélène Esnault. To her go my grateful thanks for opening me the doors to her research group, for suggesting me the problem of this thesis (and also of my master thesis), for creating a great research atmosphere around her with incredibly good mathematicians (without which this thesis could not have been done), for being there when needed and for being a reference and source of inspiration, not just in the (more obvious) mathematical side, but also as a person coming from a working class family who reached the top with her effort and her talent -as well as the rights and social opportunities that existed in France. It has been an honor and a privilege for me to be part of her research group, where my mathematical understanding and vision have evolved thanks to her and to the big number of people who have been around. Thank you very much.

I'm also very thankful to a lot of people who have helped me at various stages of this work: Marta Pieropan gave me careful and detailed feedback and enlightened me about how Academia and funding of projects work, and thanks to her I found financing for my thesis; Wouter Zomervrucht and Marco D'addezio always had interest in all kind of mathematical questions arising when I started studying Berkovich spaces, and their interest and willingness to think together proved to be extremely helpful; Antoine Chambert-Loir suggested me the idea for constructing the first example of non-homeomorphic conjugate elliptic curves using the j-invariant; Peter Jossen oriented me for constructing the appropriate field automorphisms; discussions during the JAVA conferences with Olivier Benoist, Diego Izquierdo and Jérôme Poineau were also very clarifying for me (I would also like to thank the organizers of these "Arbeitsgemeinschaften à la Française" in Tatihou: Giuseppe Ancona, Javier Fresán and Marco Maculan); Daniele Turchetti also oriented me and gave me very useful hints in several conferences; Christian Haase taught me the first notions of tropical geometry, which proved to be very helpful, so I would also want to thank him; I am in debt with Marta Panizzut, who made a huge effort guiding me through the connections between non-archimedean and tropical geometry, and without whose support, enthusiasm, patience and help I couldn't have finished this thesis; I also thank gratefully Raju Krishnamoorthy, whose enthusiasm and interest always motivates me and helped me to come with some examples which unlocked my thesis at some point; Hao Yun had always a lot of temperance, patience and helpfulness towards my numerous small mathematical troubles and, specially, for teaching me so many things about programming in SAGE, LaTeX, etc.; Kay Rülling, who has always been a mathematical reference for me, told me the proof of Lemma 2.60; I learned a lot with Paul Helminck and Luigi Lunardon about different kind of tropicalizations and how to recover information of the Berkovich spaces from their tropicalizations, and we computed a lot together; Grétar Amazeen helpfully sent me lots of English corrections; and Johannes Nicaise helped me at various stages of my thesis, told me the abstract construction of examples on arbitrary dimension and went through an earlier version of this thesis pointing out lots of mistakes, flaws and typos: my sincere gratitude goes to him.

Moreover, not related with this thesis but in general with my mathematical learning, I want to thank the following people from our arithmetic research group (and surroundings) in Freie Universität Berlin: Gabi, Sina, Ángel, Marcel, Elena, Giulia, Lei, Kay, Lars, Valentina, Michael, Raju, Simon, Sinan, Wouter, Yun, Marco, Grétar, Marcin, Tanya, Efstathia, Marta, Fabio, Shuddho, Tohru, Kivanç, Giulio, Fei, Maciek, Xiaoyu and other people visiting us.

My passion for Mathematics was born in Madrid, and there are very many people to whom I owe very much. I will just mention Ignacio Sols, Juan Ramón Delgado, José Javier Etayo and Merche Sánchez for opening me the doors to Mathematics; Angelines and Antonio for guiding me through my Bachelor; my Bachelor classmates with whom I spent countless hours learning mathematics (most of them from the Matgazine team, but not only, like Juanjo) and all the other professors who have taught me all the mathematics I know: I have a pleasant memory of my time and my journey in the mathematics faculty of the UCM; all the people from the cultural association Lewis Carroll and from the Club Deportivo (especially Esteban and Juan); the students and researchers that I met during the summer schools for students in ICMAT (JAE Intro), a wonderful initiative that was the perfect bridge between studies and research (I should here mention José Ignacio Burgos); and, among everything, the family of Matgazine, where I learned how to cooperate and how work and having a nice time shouldn't be contradictory: my deepest thanks go to Víctor, Gabi, Moisés, Eva, Paco, Pablo, Alberto, Juan, Alejandro, Martín, Adrián, Irene(s), Zango, Ángel D., Sheriff, Jaime and many others, what we did (and all the fun we had during our way) was amazing!

Finally, my work as a mathematician is sustained by a much more important and invisible network of love, care, friendship, solidarity and mutual support relationships that is way too big and diverse to describe here. I just want to generically thank all the people (including humans, cyborgs and beyond) belonging to it: if you are not sure if you belong to it, you probably do. Thank you very much for being there, for your patience and the joy of knowing each other.

This thesis was financed by the Hans-Böckler-Stiftung.

A Appendix: Programs to make computations in SAGE

A.1 Program that computes the discriminant

The following script written in SAGE implements the algorithm from [CLO05, Chapter 3, Section 4, Exercise 15]. That algorithm gives the resultant of three homogeneous polynomials F_0, F_1 and F_2 in $\mathbb{C}[x, y, z]$ of degree greater or equal than 2. In the program I wrote, the input is a homogeneous polynomial f of degree greater or equal than 3, with coefficients in $\overline{\mathbb{Q}}[a_0, t]$ and variables x, y and z. Here we see a_0 as an element of $\mathbb{C}\{t\}$ of valuation 0. The partial derivatives play the role of the F_i's, and the output is the discriminant of f up to the sign.

```
# We define the ring R
__tmp__=var("x,y,z,t,a_0")
S = PolynomialRing(QQbar, 't')
R = PolynomialRing(S, 'x,y,z')

f = input()
g(x,y,z,t,a_0) = f
g = g(x,y,z,1,1)
degree = R(g).degree()

F0, F1, F2 = derivative(f,x), derivative(f,y), derivative(f,z)
Fis = [F0,F1,F2]
# We compute the x^(a+1), y^(b+1), z^(c+1) such that a+b+c= l-1
l = degree - 1
List = [(a,b,c) for a in range(l) for b in range(l) for c in range(l) if a+b+c == l-1]
ListOfDivisors = [(x**(List[i][0]+1),y**(List[i][1]+1), z**(List[i][2]+1))
        for i in range(len(List))]
# We perform all the divisions to get the summands of the P_i's
P0Q0R0s,P1Q1R1s,P2Q2R2s = [],[],[]
PiQiRis = [P0Q0R0s,P1Q1R1s,P2Q2R2s]
for n in range(len(PiQiRis)):
    auxlist = []
    for j in range(len(ListOfDivisors)):
        # .iterator() gives the monomials of a symbolic polynomial
        monomialsFn = list(Fis[n].iterator())
        # The 3 below is the number of variables, here x,y,z
        for k in range(3):
            auxlist2 = [(ListOfDivisors[j], ListOfDivisors[j][k])]
            for i in range(len(monomialsFn)):
                if monomialsFn[i] != 0 and
                 monomialsFn[i].maxima_methods().remainder(ListOfDivisors[j][k]) == 0:
                    auxlist2.append(monomialsFn[i]/ListOfDivisors[j][k])
                    monomialsFn[i] = 0
            # This is to add a 0 (seen as a monomial) in case no monomial is divided
            # by ListOfDivisors[j][k]:
            if len(auxlist2) == 1:
                auxlist2.append(0)
            auxlist.append(auxlist2)
    for i in range(0,len(auxlist),3):
        Pn,Qn,Rn = 0,0,0
        #The -1's and +1's below are because in auxlist[i], the first element is the
        # tuple (exponents, variable)
        for k in range(len(auxlist[i])-1):
            Pn = Pn + auxlist[i][k+1]
        for k in range(len(auxlist[i+1])-1):
            Qn = Qn + auxlist[i+1][k+1]
        for k in range(len(auxlist[i+2])-1):
            Rn = Rn + auxlist[i+2][k+1]
        #Here we add the exponents to keep track of things, and the polynomials Pn, Qn,
        # Rn with respect to these exponents:
        PiQiRis[n].append((auxlist[i][0][0], [Pn, Qn, Rn]))
# Now we compute the Fabc's
```

```
Fabc = []
for i in range(len(List)):
    row1 = []
    row2 = []
    row3 = []
    for j in range(3):
        row1.append(P0Q0R0s[i][1][j])
        row2.append(P1Q1R1s[i][1][j])
        row3.append(P2Q2R2s[i][1][j])
    M = matrix([row1, row2, row3])
    Fabc.append(M.determinant().expand())
# The following MonDeg2lMinus2 is a list of all monomials of degree 2l-2.
# They will be our variables afterwards
List2 = [(a,b,c) for a in range(2*l-2+1) for b in range(2*l-2+1) for c
 in range(2*l-2+1) if a+b+c == 2*l-2]
MonDeg2lMinus2 = [x**(List2[i][0])*y**(List2[i][1])* z**(List2[i][2]) for i
 in range(len(List2))]
# Now we define the big system of equations (4.11)
# We want the variables to be the monomials of degree 2l-2
List3 = [(a,b,c) for a in range(l-2+1) for b in range(l-2+1) for c
 in range(l-2+1) if a+b+c == l-2]
x_alphas = [x**List3[i][0] * y**List3[i][1] * z**List3[i][2] for i in range(len(List3))]
system_4_11 = []
for i in Fabc:
    system_4_11.append(i == 0)
for i in range(len(List3)):
    system_4_11.append((x_alphas[i] * F0).expand() == 0)
    system_4_11.append((x_alphas[i] * F1).expand() == 0)
    system_4_11.append((x_alphas[i] * F2).expand() == 0)
# Now we define the matrix C_l of coefficients of the system 4.11, where the variables
# are the monomials of deg 2l-2
C_l = matrix([[equ.lhs().coefficient(v) for v in MonDeg2lMinus2] for equ in system_4_11])
#Define an auxiliary ring for computing the determinant of C_l faster, so that entries
# are polynomials and not symbolic expressions
#See https://ask.sagemath.org/question/8021/large-symbolic-determinant/
AuxRing = PolynomialRing(QQ, ['x', 'y', 'z', 't', 'a_0'])
C_lpoly = []
for i in range(len(C_l[0])):
    C_lpoly.append(C_l[i].list())
    for j in range(len(C_l[0])):
        C_lpoly[i][j] = AuxRing(C_l[i][j])
C_lpoly = matrix([[j for j in C_lpoly[i]] for i in range(len(C_l[0]))])
# The resultant (up to a sign) is the determinant of C_l, and we are done
resultant = C_lpoly.determinant()
print resultant
```

A.2 Program that defines the curve that we expect to be Mumford

The following script written in SAGE implements the algorithm described in section 5.1. The input is the degree, and the output is the polynomial that will give the desired Newton polytope. Note that we have to substitute the coefficient of z^n either by a_0 if we want f, or by a_v, where v is valuations[-1], if we want a Mumford curve.

```
# We want to construct polynomials that will define Mumford curves.
# The idea is to construct them recursively

n = input('Give the degree of the curve that you want to construct (at least 3): ')

# Now we define the (homogeneous) monomials that will appear.
# The coefficients are still not defined

__tmp__=var("x,y,z,t")

# These are the exponents whose coeff will have valuation 0
exponents = [(n,0,0),(0,n,0)]
```

```
# These are the exponents that will be interior points of the Newton polytope.
# I do it in the correct order so that at the end the coefficients have increasing
## valuation when going to the vertex (0,0).

for k in range(1,n-1,1):
    for i in range(1, n-k,1):
        exponents.append(( i, n-i-k, k))

exponents.append((0,0,n))

# Note that in the list 'exponents', the first two elements, together with the last,
## correspond to the convex hull of the Newton polytope.
# The interior vertices correspond to the other elements of 'exponents'.

# Now we define the valuations of the coefficients.
# This is the main difficulty of the program.
# We fix the first 3 valuations

valuations = [0,0,1]

# We define the list Points, which corresponds to the points (i,j,v_ij),
## where i (resp. j) is the exponent of x (resp. y), and v_ij will be the valuation
## of the coefficient of the monomial x^i*y^j*z^(n-1-j)
# We use these points to keep track of the convex hull that defines the Newton polytope.

Points = []
for i in range(len(valuations)):
    Points.append((exponents[i][0], exponents[i][1], valuations[i]))

# The equation of the plane through 3 points is given by the determinant of this matrix:

row1 = [x - Points[0][0], y - Points[0][1], z - Points[0][2]]
row2 = [Points[1][0] - Points[0][0], Points[1][1] - Points[0][1], Points[1][2] -
 Points[0][2]]
row3 = [Points[2][0] - Points[0][0], Points[2][1] - Points[0][1], Points[2][2] -
 Points[0][2]]
M = matrix([row1, row2, row3])
P = M.determinant()

# It is more convenient to keep track of the equation Pz of the plane P in terms of (x,y):

PlaneCoeff = [P.maxima_methods().quotient(x)*x,
    P.maxima_methods().quotient(y)*y,
    P.maxima_methods().quotient(z)*z,
    P - P.maxima_methods().quotient(x)*x - P.maxima_methods().quotient(y)*y -
     P.maxima_methods().quotient(z)*z]
Pz(x,y) = -(PlaneCoeff[0]+PlaneCoeff[1]+PlaneCoeff[3])/(PlaneCoeff[2]/z)

Planes = [Pz]

for i in range(3, len(exponents),1):
    # First, we define the valuation that the next point should have
    val = 0
    for p in Planes:
        if val < p(exponents[i][0], exponents[i][1]):
            val = p(exponents[i][0], exponents[i][1])
    # We put floor() to get integer valuations
    valuations.append(floor(val) + 1)
    newPoint = (exponents[i][0], exponents[i][1], val+1)
    # Here we add the point with the correct valuation
    Points.append(newPoint)

    # Now, we compute all the planes passing through the new point and the pairs of other
    # points. We exclude the vertical planes because they are irrelevant for us.
    for j in range(len(Points)-2):
        for k in range(j+1, len(Points)-1,1):
```

```
            row1 = [x - newPoint[0], y - newPoint[1], z - newPoint[2]]
            row2 = [Points[j][0] - newPoint[0], Points[j][1] - newPoint[1],
             Points[j][2] - newPoint[2]]
            row3 = [Points[k][0] - newPoint[0], Points[k][1] - newPoint[1],
             Points[k][2] - newPoint[2]]
            M = matrix([row1, row2, row3])
            P = M.determinant()
            PlaneCoeff = [P.maxima_methods().quotient(x)*x,
                P.maxima_methods().quotient(y)*y,
                P.maxima_methods().quotient(z)*z,
                P - P.maxima_methods().quotient(x)*x -
                    P.maxima_methods().quotient(y)*y -
                    P.maxima_methods().quotient(z)*z]
            # Here we exclude the vertical planes
            if PlaneCoeff[2] != 0:
                Pz(x,y) = -(PlaneCoeff[0] + PlaneCoeff[1] + PlaneCoeff[3]) /
                 (PlaneCoeff[2]/z)
                Planes.append(Pz)

# Finally, we define the polynomial, which is given by specifying the coefficients
monomials_curve = []

for i in range(len(exponents)):
    monomials_curve.append(t**valuations[i] * x^exponents[i][0] *
     y**exponents[i][1] * z**exponents[i][2])

curve = sum(i for i in monomials_curve)
print curve
```

A.3 Program that defines a Mumford curve of arbitrary degree

The following script written in SAGE implements the algorithm described in section 5.2. The input is the degree, and the output is the polynomial of a smooth Mumford curve of the given degree. Note that we have to substitute the coefficient of z^n by a_v and the coefficient of x^n by $b_{v'}$, where v and v' are the valuations of the coefficient of the corresponding monomial (if we only want to draw the Newton polygon or the tropicalization, we don't need to do that).

```
# We want to construct polynomials that will define Mumford curves.
# The idea is to construct them recursively

n = input('Give the degree of the curve that you want to construct (at least 3): ')

# Now we define the (homogeneous) monomials that will appear.
# The coefficients are still not defined

__tmp__=var("x,y,z,t")

# These are the exponents whose coeff will have valuation 0, 1, 1
exponents = [(0,n,0), (0,n-1,1), (1,n-1,0)]

# We fix the first 3 valuations

valuations = [0,1,1]

# These are the exponents that will be the vertices of the Newton polytope.


for i in range(n-2,-1,-1):
    for k in range(n-i, -1,-1):
        exponents.append((  n-i-k, i, k))


# Now we define the valuations of the coefficients.
# This is the main difficulty of the program.
```

```
# We define the list Points, which corresponds to the points (i,j,v_ij),
## where i (resp. j) is the exponent of x (resp. y), and v_ij will be the valuation of the
## coefficient of the monomial x^i*y^j*z^(n-1-j)
# We use these points to keep track of the convex hull that defines the Newton polytope.

Points = []
for i in range(len(valuations)):
    Points.append((exponents[i][0], exponents[i][1], valuations[i]))

# The equation of the plane through 3 points is given by the determinant of this matrix:

row1 = [x - Points[0][0], y - Points[0][1], z - Points[0][2]]
row2 = [Points[1][0] - Points[0][0], Points[1][1] - Points[0][1], Points[1][2] -
 Points[0][2]]
row3 = [Points[2][0] - Points[0][0], Points[2][1] - Points[0][1], Points[2][2] -
 Points[0][2]]
M = matrix([row1, row2, row3])
P = M.determinant()

# It is more convenient to keep track of the equation Pz of the plane P in terms of (x,y):

PlaneCoeff = [P.maxima_methods().quotient(x)*x,
    P.maxima_methods().quotient(y)*y,
    P.maxima_methods().quotient(z)*z,
    P - P.maxima_methods().quotient(x)*x - P.maxima_methods().quotient(y)*y -
     P.maxima_methods().quotient(z)*z]
Pz(x,y) = -(PlaneCoeff[0]+PlaneCoeff[1]+PlaneCoeff[3])/(PlaneCoeff[2]/z)

Planes = [Pz]

for i in range(3, len(exponents),1):
    # First, we define the valuation that the next point should have
    val = 0
    for p in Planes:
        if val < p(exponents[i][0], exponents[i][1]):
            val = p(exponents[i][0], exponents[i][1])
    # We put floor() to get integer valuations
    valuations.append(floor(val) + 1)
    newPoint = (exponents[i][0], exponents[i][1], val+1)
    # Here we add the point with the correct valuation
    Points.append(newPoint)

    # Now, we compute all the planes passing through the new point and the pairs of other
    # points. We exclude the vertical planes because they are irrelevant for us.
    for j in range(len(Points)-2):
        for k in range(j+1, len(Points)-1,1):
            row1 = [x - newPoint[0], y - newPoint[1], z - newPoint[2]]
            row2 = [Points[j][0] - newPoint[0], Points[j][1] - newPoint[1], Points[j][2] -
             newPoint[2]]
            row3 = [Points[k][0] - newPoint[0], Points[k][1] - newPoint[1], Points[k][2] -
             newPoint[2]]
            M = matrix([row1, row2, row3])
            P = M.determinant()
            PlaneCoeff = [P.maxima_methods().quotient(x)*x,
                P.maxima_methods().quotient(y)*y,
                P.maxima_methods().quotient(z)*z,
                P - P.maxima_methods().quotient(x)*x -
                    P.maxima_methods().quotient(y)*y -
                    P.maxima_methods().quotient(z)*z]
            # Here we exclude the vertical planes
            if PlaneCoeff[2] != 0:
                Pz(x,y) = -(PlaneCoeff[0] + PlaneCoeff[1] + PlaneCoeff[3]) / (PlaneCoeff[2]/z)
                Planes.append(Pz)

# Finally, we define the polynomial, which is given by specifying the coefficients
```

```
monomials_curve = []

for i in range(len(exponents)):
    monomials_curve.append(t**valuations[i] * x^exponents[i][0] *
     y**exponents[i][1] * z**exponents[i][2])

curve = sum(i for i in monomials_curve)

print curve
```

References

[Abe74] Harold Abelson, "Topologically distinct conjugate varieties with finite fundamental group", in: *Topology* 13 (1974), pp. 161–176 (cit. on p. 10).

[ACC07] Enrique Artal Bartolo, Jorge Carmona Ruber, and José Ignacio Cogolludo Agustín, "Effective invariants of braid monodromy", in: *Trans. Amer. Math. Soc.* 359.1 (2007), pp. 165–183 (cit. on p. 10).

[And03] Yves André, *Period mappings and differential equations. From* $\mathbb{C}$ *to* $\mathbb{C}_p$, vol. 12, MSJ Memoirs, Tôhoku-Hokkaidô lectures in arithmetic geometry, With appendices by F. Kato and N. Tsuzuki, Mathematical Society of Japan, Tokyo, 2003, pp. viii+246, ISBN: 4-931469-22-1 (cit. on pp. 21, 22, 33).

[And09] Yves André, "Galois theory, motives and transcendental numbers", in: *Renormalization and Galois theories*, vol. 15, IRMA Lect. Math. Theor. Phys. Eur. Math. Soc., Zürich, 2009, pp. 165–177 (cit. on p. 24).

[BCG15] Ingrid Bauer, Fabrizio Catanese, and Fritz Grunewald, "Faithful actions of the absolute Galois group on connected components of moduli spaces.", English, in: *Invent. Math.* 199.3 (2015), pp. 859–888, ISSN: 0020-9910; 1432-1297/e (cit. on p. 10).

[Ber08] Vladimir Berkovich, "Non-Archimedean analytic geometry: first steps", in: *p-adic geometry*, vol. 45, Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2008, pp. 1–7, DOI: `10.1090/ulect/045/01` (cit. on p. 33).

[Ber90] Vladimir G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, vol. 33, Mathematical Surveys and Monographs, American Mathematical Society, 1990, pp. x+169 (cit. on pp. 23, 25–27, 29–31, 33).

[Ber93] Vladimir G. Berkovich, "Étale cohomology for non-Archimedean analytic spaces", in: *Inst. Hautes Études Sci. Publ. Math.* 78 (1993), 5–161 (1994), ISSN: 0073-8301 (cit. on pp. 31–34).

[Ber94] Vladimir G. Berkovich, "Vanishing cycles for formal schemes", in: *Invent. Math.* 115.3 (1994), pp. 539–571, ISSN: 0020-9910 (cit. on p. 33).

[Ber99] Vladimir G. Berkovich, "Smooth p-adic analytic spaces are locally contractible", in: *Invent. Math.* 137.1 (1999), pp. 1–84, ISSN: 0020-9910, DOI: `10.1007/s002220050323` (cit. on pp. 34, 35).

[BGR84] S. Bosch, U. Güntzer, and R. Remmert, *Non-Archimedean analysis*, vol. 261, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], A systematic approach to rigid analytic geometry, Springer-Verlag, Berlin, 1984, pp. xii+436, ISBN: 3-540-12546-9, DOI: `10.1007/978-3-642-52229-1` (cit. on p. 24).

[Boy99] Pascal Boyer, "Mauvaise réduction des variétés de Drinfeld et correspondance de Langlands locale", in: *Invent. Math.* 138.3 (1999), pp. 573–629, ISSN: 0020-9910, DOI: `10.1007/s002220050354` (cit. on p. 33).

[BPR13] Matthew Baker, Sam Payne, and Joseph Rabinoff, "On the structure of non-Archimedean analytic curves", in: *Tropical and non-Archimedean geometry*, vol. 605, Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 93–121, DOI: `10.1090/conm/605/12113` (cit. on pp. 36, 42, 43, 47).

[BPR16] Matthew Baker, Sam Payne, and Joseph Rabinoff, "Nonarchimedean geometry, tropicalization, and metrics on curves", in: *Algebr. Geom.* 3.1 (2016), pp. 63–105, ISSN: 2214-2584, DOI: `10.14231/AG-2016-004` (cit. on pp. 33, 36, 42, 43, 47–49, 52).

[BR10] Matthew Baker and Robert Rumely, *Potential theory and dynamics on the Berkovich projective line*, vol. 159, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2010, pp. xxxiv+428, ISBN: 978-0-8218-4924-8, DOI: 10.1090/surv/159 (cit. on pp. 27, 29–31, 33).

[CHW14] Maria Angelica Cueto, Mathias Häbich, and Annette Werner, "Faithful tropicalization of the Grassmannian of planes.", English, in: *Math. Ann.* 360.1-2 (2014), pp. 391–437, ISSN: 0025-5831; 1432-1807/e (cit. on p. 47).

[CLO05] David A. Cox, John Little, and Donal O'Shea, *Using algebraic geometry*, Second, vol. 185, Graduate Texts in Mathematics, Springer, New York, 2005, pp. xii+572, ISBN: 0-387-20706-6 (cit. on p. 61).

[CM16] Maria Angelica Cueto and Hannah Markwig, "How to repair tropicalizations of plane curves using modifications", in: *Exp. Math.* 25.2 (2016), pp. 130–164, ISSN: 1058-6458, DOI: 10.1080/10586458.2015.1048013 (cit. on pp. 48, 50).

[Con08] Brian Conrad, "Several approaches to non-Archimedean geometry", in: *p-adic geometry*, vol. 45, Univ. Lecture Ser. Amer. Math. Soc., Providence, RI, 2008, pp. 9–63, DOI: 10.1090/ulect/045/02 (cit. on pp. 22, 24, 31).

[Duc07] Antoine Ducros, "Espaces analytiques p-adiques au sens de Berkovich", in: 311, Séminaire Bourbaki. Vol. 2005/2006, 2007, Exp. No. 958, viii, 137–176, ISBN: 978-2-85629-230-3 (cit. on p. 21).

[EGA IV] Alexander Grothendieck, "Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Quatrième partie", fr, in: *Publications Mathématiques de l'IHÉS* 32 (1967), pp. 5–361 (cit. on p. 35).

[Esn17] Hélène Esnault, "Survey on some aspects of Lefschetz theorems in algebraic geometry", in: *Revista Matemática Complutense* 30.2 (May 2017), pp. 217–232, ISSN: 1988-2807, DOI: 10.1007/s13163-017-0223-8 (cit. on p. 9).

[FP04] Jean Fresnel and Marius van der Put, *Rigid analytic geometry and its applications*, vol. 218, Progress in Mathematics, Birkhäuser Boston, Inc., Boston, MA, 2004, pp. xii+296, ISBN: 0-8176-4206-4, DOI: 10.1007/978-1-4612-0041-3 (cit. on pp. 22, 23).

[GJ00] Ewgenij Gawrilow and Michael Joswig, "polymake: a framework for analyzing convex polytopes", in: *Polytopes—combinatorics and computation (Oberwolfach, 1997)*, vol. 29, DMV Sem. Birkhäuser, Basel, 2000, pp. 43–73 (cit. on p. 52).

[GRW16] Walter Gubler, Joseph Rabinoff, and Annette Werner, "Skeletons and tropicalizations", in: *Adv. Math.* 294 (2016), pp. 150–215, ISSN: 0001-8708, DOI: 10.1016/j.aim.2016.02.022 (cit. on pp. 34, 42).

[Gub13] Walter Gubler, "A guide to tropicalizations", in: *Algebraic and combinatorial aspects of tropical geometry*, vol. 589, Contemp. Math. Amer. Math. Soc., Providence, RI, 2013, pp. 125–189, DOI: 10.1090/conm/589/11745 (cit. on p. 43).

[Har77] Robin Hartshorne, *Algebraic Geometry*, vol. 52, Grad. Texts in Math. Springer Verlag, 1977 (cit. on p. 49).

[Har97] Michael Harris, "Supercuspidal representations in the cohomology of Drinfeld upper half spaces; elaboration of Carayol's program", in: *Invent. Math.* 129.1 (1997), pp. 75–119, ISSN: 0020-9910, DOI: 10.1007/s002220050159 (cit. on p. 33).

[Has24] Helmut Hasse, “Darstellbarkeit von Zahlen durch quadratische Formen in einem beliebigen algebraischen Zahlkörper.”, in: *Journal für die reine und angewandte Mathematik* 153 (1924), pp. 113–130, URL: `http://eudml.org/doc/149539` (cit. on p. 11).

[Hau05] Thomas Hausberger, “Uniformisation des variétés de Laumon-Rapoport-Stuhler et conjecture de Drinfeld-Carayol”, in: *Ann. Inst. Fourier (Grenoble)* 55.4 (2005), pp. 1285–1371, ISSN: 0373-0956 (cit. on p. 33).

[Hea97] Thomas Little Heath, *The Works Of Archimedes*, Cambridge University Press, 1897 (cit. on p. 13).

[Hel16] Paul Alexander Helminck, *Tropical Igusa Invariants and Torsion Embeddings*, 2016, arXiv: `1604.03987 [math.AG]` (cit. on p. 49).

[Hel19] Paul Alexander Helminck, “Faithful tropicalizations of elliptic curves using minimal models and inflection points”, in: *Arnold Mathematical Journal* (September 2019), ISSN: 2199-6806, DOI: `10.1007/s40598-019-00121-y` (cit. on pp. 48, 49).

[HT01] Michael Harris and Richard Taylor, *The geometry and cohomology of some simple Shimura varieties*, vol. 151, Annals of Mathematics Studies, With an appendix by Vladimir G. Berkovich, Princeton University Press, Princeton, NJ, 2001, pp. viii+276, ISBN: 0-691-09090-4 (cit. on p. 33).

[Kat17] Eric Katz, “What is ... tropical geometry?”, in: *Notices Amer. Math. Soc.* 64.4 (2017), pp. 380–382, ISSN: 0002-9920, DOI: `10.1090/noti1507` (cit. on p. 42).

[KS06] Maxim Kontsevich and Yan Soibelman, “Affine structures and non-Archimedean analytic spaces.”, English, in: *The unity of mathematics. In honor of the ninetieth birthday of I. M. Gelfand. Papers from the conference held in Cambridge, MA, USA, August 31–September 4, 2003.* Boston, MA: Birkhäuser, 2006, pp. 321–385, ISBN: 0-8176-4076-2/hbk (cit. on p. 33).

[Kür13] Josef Kürschák, “Über Limesbildung und allgemeine Körpertheorie”, in: *J. Reine Angew. Math.* 142 (1913), pp. 211–253, ISSN: 0075-4102, DOI: `10.1515/crll.1913.142.211` (cit. on p. 12).

[Liu02] Qing Liu, *Algebraic geometry and arithmetic curves*, vol. 6, Oxford Graduate Texts in Mathematics, Oxford University Press, (2002), pp. xvi+576 (cit. on p. 37).

[Mil18] James S. Milne, *Fields and Galois Theory (v4.60)*, Available at www.jmilne.org/math/, 2018 (cit. on p. 38).

[MN15] Mircea Mustaţă and Johannes Nicaise, “Weight functions on non-Archimedean analytic spaces and the Kontsevich-Soibelman skeleton.”, English, in: *Algebr. Geom.* 2.3 (2015), pp. 365–404, ISSN: 2313-1691; 2214-2584/e (cit. on pp. 34, 35).

[MS10] James S. Milne and Junecue Suh, “Nonhomeomorphic conjugates of connected Shimura varieties”, in: *Amer. J. Math.* 132.3 (2010), pp. 731–750 (cit. on p. 10).

[MS15] Diane Maclagan and Bernd Sturmfels, *Introduction to tropical geometry*, vol. 161, Graduate studies in mathematics, Providence, R.I.: American Mathematical Society, 2015, pp. xii, 363 (cit. on pp. 42, 44, 46–48, 56).

[Mum72] David Mumford, “An analytic construction of degenerating curves over complete local rings”, in: *Compositio Math.* 24 (1972), pp. 129–174, ISSN: 0010-437X (cit. on p. 24).

[Nic08] Johannes Nicaise, "Formal and rigid geometry: an intuitive introduction and some applications", in: *Enseign. Math. (2)* 54.3-4 (2008), pp. 213–249, ISSN: 0013-8584 (cit. on p. 24).

[Nic16] Johannes Nicaise, "Berkovich skeleta and birational geometry", in: *Nonarchimedean and tropical geometry*, Simons Symp. Springer, [Cham], 2016, pp. 173–194 (cit. on pp. 32–35).

[NX16] Johannes Nicaise and Chenyang Xu, "Poles of maximal order of motivic zeta functions", in: *Duke Mathematical Journal* 165.2 (2016), pp. 217–243 (cit. on pp. 33, 35).

[Ost16] Alexander Ostrowski, "Über einige Lösungen der Funktionalgleichung $\psi(\mathrm{x})\cdot\psi(\mathrm{x})=\psi(\mathrm{xy})$", in: *Acta Mathematica* 41.1 (December 1916), pp. 271–284, ISSN: 1871-2509, DOI: 10.1007/BF02422947 (cit. on p. 15).

[Pay09] Sam Payne, "Analytification is the limit of all tropicalizations", in: *Math. Res. Lett.* 16.3 (2009), pp. 543–556, ISSN: 1073-2780 (cit. on p. 42).

[Rab12] Joseph Rabinoff, "Tropical analytic geometry, Newton polygons, and tropical intersections", in: *Adv. Math.* 229.6 (2012), pp. 3192–3255, ISSN: 0001-8708, DOI: 10.1016/j.aim.2012.02.003 (cit. on p. 42).

[RTW15] Bertrand Rémy, Amaury Thuillier, and Annette Werner, "Bruhat-Tits buildings and analytic geometry", in: *Berkovich spaces and applications*, vol. 2119, Lecture Notes in Math. Springer, Cham, 2015, pp. 141–202, DOI: 10.1007/978-3-319-11029-5_5 (cit. on p. 33).

[Ser56] Jean-Pierre Serre, "Géométrie algébrique et géométrie analytique", in: *Ann. Inst. Fourier, Grenoble* 6 (1955–1956), pp. 1–42 (cit. on p. 9).

[Ser64] Jean-Pierre Serre, "Exemples de variétés projectives conjuguées non homéomorphes", in: *C. R. Acad. Sci. Paris* 258 (1964), pp. 4194–4196 (cit. on pp. 3, 9).

[Ser65] Jean-Pierre Serre, *Lie algebras and Lie groups*, vol. 1964, Lectures given at Harvard University, W. A. Benjamin, Inc., New York-Amsterdam, 1965, vi+247 pp. (not consecutively paged) (cit. on p. 22).

[Ser79] Jean-Pierre Serre, *Local fields*, Springer Verlag, (1979), pp. viii+241 (cit. on p. 22).

[SGA1] Alexander Grothendieck, *Revêtements étales et groupe fondamental (SGA 1)*, vol. 224, Lecture notes in mathematics, Springer-Verlag, (1971), pp. xxii+447 (cit. on p. 9).

[SGA4] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier, *Théorie de topos et cohomologie étale des schémas I, II, III (SGA 4)*, vol. 269, 270, 305, Lecture Notes in Mathematics, Springer, (1971) (cit. on pp. 9, 19).

[Shi09] Ichiro Shimada, "Non-homeomorphic conjugate complex varieties.", English, in: *Singularities, Niigata-Toyama 2007. Proceedings of the 4th Franco-Japanese symposium, Niigata, Toyama, Japan, August 27–31, 2007*, Tokyo: Mathematical Society of Japan, 2009, pp. 285–301, ISBN: 978-4-931469-55-6/hbk (cit. on p. 10).

[Shi18] Taketo Shirane, "Connected numbers and the embedded topology of plane curves.", English, in: *Can. Math. Bull.* 61.3 (2018), pp. 650–658, ISSN: 0008-4395; 1496-4287/e (cit. on p. 10).

[Shi19] Taketo Shirane, "Galois covers of graphs and embedded topology of plane curves.", English, in: *Topology Appl.* 257 (2019), pp. 122–143, ISSN: 0166-8641 (cit. on p. 10).

[Sil09] Joseph H. Silverman, *The arithmetic of elliptic curves*, Second, vol. 106, Graduate Texts in Mathematics, Springer, Dordrecht, 2009, pp. xx+513 (cit. on pp. 37, 48, 49).

[Spe07] David E. Speyer, *Uniformizing Tropical Curves I: Genus Zero and One*, Available at http://arxiv.org/abs/0711.2677, 2007 (cit. on p. 47).

[Stacks] The Stacks Project Authors, *Stacks Project*, `http://stacks.math.columbia.edu`, (2016) (cit. on p. 35).

[Sto82] Otto Stolz, "Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes", in: *Berichte des naturwissenschaftlichen-medizinischen Verein Innsbruck, 12*, Naturwissenschaftlich-medizinischer Verein Innsbruck, 1882, pp. 74–89 (cit. on p. 13).

[Sza09] Tamás Szamuely, *Galois groups and fundamental groups*, vol. 117, Cambridge Studies in Advanced Mathematics, Cambridge University Press, 2009, pp. x+270 (cit. on pp. 9, 10).

[Thu05] Amaury Thuillier, "Potential theory on curves in non-Archimedean geometry. Applications to Arakelov theory.", Theses, Université Rennes 1, October 2005, URL: `https://tel.archives-ouvertes.fr/tel-00010990` (cit. on p. 33).

[Tol93] Domingo Toledo, "Projective varieties with non-residually finite fundamental group", en, in: *Publications Mathématiques de l'IHÉS* 77 (1993), pp. 103–119, URL: `http://www.numdam.org/item/PMIHES_1993__77__103_0` (cit. on p. 10).

[Ull95] Peter Ullrich, "On the origins of p-adic analysis", in: *Proceedings of the 2nd Gauss Symposium. Conference A: Mathematics and Theoretical Physics (Munich, 1993)*, Sympos. Gaussiana, de Gruyter, Berlin, 1995, pp. 459–473 (cit. on p. 11).

I, Pedro Ángel Castillejo Blasco, declare that this thesis, titled "The topology of conjugate Berkovich spaces", and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this university.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
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Signed:

Date: