Bridgeland stability conditions on the category of holomorphic triples

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Contents

Introduction

Classification problems are of great importance in algebraic geometry. A *moduli space* can been seen as a solution of a geometric classification problem i.e. we classify algebro-geometric objects up to some notion of equivalence.

The notion of stability condition appears naturally in the construction of moduli spaces of vector bundles on smooth projective varieties. In order to construct a reasonable moduli space, we need to restrict to a class of objects which are better behaved and bounded. Therefore, we usually restrict to the class of *semistable objects* whose definition depends on a stability condition.

Rudakov generalized the notion of stability structure in [Rud97, Def. 1.1] to an abelian category $\mathcal{A}$. It is given by a preorder on $\mathcal{A}$ satisfying that if $0 \to E \to F \to G \to 0$ is a short exact sequence, then the middle term F is situated in between the side terms E and G with respect to the preorder. The heart of a bounded t-structure $\mathcal{A}$ in a triangulated category $\mathcal{D}$ is an abelian category, that intuitively breaks up every object in $\mathcal{D}$ in terms of its cohomology (with respect to $\mathcal{A}$) indexed by $\mathbb{Z}$. Moreover the objects in $\mathcal{A}$ are precisely the objects whose cohomology is concentrated in degree 0. Let us consider a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$, from the Grothendieck group $K(\mathcal{A}) = K(\mathcal{D})$ and with image contained in $\mathbb{H} \cup \mathbb{R}_{<0}$. As $K(\mathcal{A})$ is additive with respect to short exact sequences in $\mathcal{A}$, the argument of the $Z([E])$ for $E \in \mathcal{A}$ defines a preorder satisfying Rudakov's condition.

In [Bri07] Bridgeland generalized the notion of stability condition to a triangulated category $\mathcal{D}$. A Bridgeland stability condition consists of a pair $(Z, \mathcal{A})$ where $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a group homomorphism as above and $\mathcal{A}$ is the heart of a bounded t-structure, such that Z satisfies the *Harder-Narasimhan property* on $\mathcal{A}$ and the *support property*. By extending Z via the t-structure, we now can break every object $E \in \mathcal{D}$ into pieces indexed by real numbers.

Moreover, in the last decades there has been a constant interplay between algebraic geometry and physics. Bridgeland stability conditions were introduced as a mathematical formalization of Douglas' work on Π-stability of D-branes for super conformal field theories (SCFT) in [Dou01] and [Dou02] in order to understand homological mirror symmetry.

In [Bri07], Bridgeland proved that the set of stability conditions has a natural topology and is a complex manifold. We are particularly interested in the finite dimensional submanifold of numerical stability conditions, denoted by $\mathrm{Stab}(\mathcal{D})$. The support property allows us to use Bridgeland's deformation result [Bri07, Thm. 7.1] to ensure that $\mathrm{Stab}(\mathcal{D})$ has a well-behaved wall-crossing and chamber decomposition, which is one of the main advantages of this setting. Moreover, the manifold $\mathrm{Stab}(\mathcal{T})$ carries a right action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and a commuting left action by isometries of the group of exact autoequivalences of $\mathcal{T}$.

Stability manifolds of non-singular projective curves were determined in [Bri07], [Mac07] and [Oka06]. In the case $\mathcal{T} = D^b(C)$, for a non-singular curve C of genus $g \geq 1$, we have that $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts on freely and transitively on $\mathrm{Stab}(\mathcal{T})$, which implies that $\mathrm{Stab}(\mathcal{T}) \cong \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. Bridgeland stability conditions have been constructed on projective surfaces as well as a connected component of the stability manifold for K3 surfaces in [Bri08]. As Gieseker semistable and slope semistable sheaves are both particular cases of Bridgeland semistable objects for a stability condition near to the "large-volume limit", see [Bri08, Sec. 14], it gives us also a framework to study the classical moduli spaces. As the original motivation comes from string theory, the construction of stability conditions on Calabi-Yau threefolds is still required. However, the existence of Bridgeland stability conditions has already been proved for surfaces [AB13], for Abelian threefolds in [BMS16] and [MP15], for Fano threefolds with Picard rank one in [Li18] and for smooth quadrics in [Sch14].

One important aspect of Bridgeland stability is its connection with birational geometry. The nef cone of the moduli space of Gieseker stable sheaves on a K3 surface has been described in [BM14] as well as the ample cone of the moduli of Gieseker semistable sheaves on $\mathbb{P}^2$ in [ABCH13].

The existence of moduli spaces of Bridgeland semistable objects as an Artin stack of finite-type over $\mathbb{C}$ has been studied in some cases in [Tod08] and [PT15] via [Lie06]. On some particular surfaces, the moduli spaces exist as projective varieties. One of the difficulties of the study of moduli spaces of Bridgeland semistable objects is that they are not directly related with a GIT problem. However, in [AHLH18] the authors approach the question of the existence of good moduli spaces, as in [Alp13], for discrete stability conditions.

Let $\mathcal{A}$ be a Noetherian abelian category, such that $D^b(\mathcal{A})$, the bounded derived category of $\mathcal{A}$, is $\mathbb{C}$-linear of finite type and saturated and let $Q_{\mathcal{A},n}$ be the abelian category of representations of the n-Kronecker quiver on $\mathcal{A}$, with $n > 0$. Its objects are tuples $(E_1, E_2, (f_j)_{0<j\leq n})$ where $E_i \in \mathcal{A}$, $i = 1, 2$ and $f_j \in \mathrm{Hom}_{\mathcal{A}}(E_1, E_2)$ for $0 < j \leq n$. The main goal of this thesis is to study $\mathrm{Stab}(D^b(Q_{\mathcal{A},n}))$ and as a particular case the stability manifold $\mathrm{Stab}(D^b(\mathrm{TCoh}(X)))$, where $\mathrm{TCoh}(X) := Q_{\mathrm{Coh}(X),n}$, is the abelian category of holomorphic triples $\mathrm{TCoh}(X)$ on a non-singular projective variety X. A holomorphic triple (E_1, E_2, φ) consists of a pair $E_1, E_2 \in \mathrm{Coh}(X)$ and a morphism $\varphi \colon E_1 \to E_2$. Holomorphic triples were introduced by García-Prada et al. in [BGPG04] and [BGP96] for vector bundles over a nonsigular projective curve, it was shown in [BGP96] that moduli spaces of α-semistable holomorphic triples exist and are projective. They also studied wall-crossing and chamber decomposition with respect to a parameter α. A precise construction via GIT of the moduli spaces was given by A. Schmitt in [Sch03]. This category has also played an important role in the study of Higgs bundles. See [BGPG02] and [BGPG03].

One of the main goals of this Ph.D thesis, partly in collaboration with E. Martínez and A. Rüffer, is to describe the stability manifold of the derived category of the abelian category $\mathrm{TCoh}(X)$ of holomorphic triples over smooth projective curves with positive genus. Our main contribution is to give a complete description of the Bridgeland stability manifold $\mathrm{Stab}(D^b(\mathrm{TCoh}(X)))$, where X is a curve of genus ≥ 1, as a 4-dimensional connected complex manifold.

If $\mathcal{A}$ is the category of Vect of vector spaces over $\mathbb{C}$, then $Q_{\mathrm{Vect},n}$ is the category of representations of the n-Kronecker quiver and its stability manifold is completely described in [Mac07]. See Subsection 1.2.4.

Our idea is to generalize the construction for quivers in [Mac07], but instead of using exceptional objects we use semiorthogonal decompositions. The exceptional object S_k, for $k \in \mathbb{Z}$ defined in [Mac07], generates a triangulated subcategory $\langle S_k \rangle$ of $D^b(Q_n)$. It also induces a semiorthogonal decomposition $\langle \langle S_k \rangle, \langle S_{k+1} \rangle \rangle$ of $D^b(Q_n)$. We define the analogous triangulated subcategories D_k of $D^b(Q_{\mathcal{A},n})$, which will also induce semiorthogonal decompositions $\langle D_k, D_{k+1} \rangle$ of $D^b(Q_{\mathcal{A},n})$. The main tools that we use to study $D^b(Q_{\mathcal{A},n})$ are the semiorthogonal decompositions given above and the existence of the Serre functor $\mathcal{S}_{D^b(Q_{\mathcal{A},n})} \colon D^b(Q_{\mathcal{A},n}) \to D^b(Q_{\mathcal{A},n})$.

In the case of $n = 1$, we study the following admissible subcategories

$$D_1 \coloneqq i_*(D^b(\mathcal{A})),\ D_2 \coloneqq j_*(D^b(\mathcal{A})) \text{ and } D_3 \coloneqq l_*(D^b(\mathcal{A})),$$

where

$$\begin{array}{rcl} i_* \colon D^b(\mathcal{A}) & \to & D^b(Q_{\mathcal{A},1}) \\ E & \mapsto & (E, 0, 0) \end{array} \qquad \begin{array}{rcl} j_* \colon D^b(\mathcal{A}) & \to & D^b(Q_{\mathcal{A},1}) \\ E & \mapsto & (0, E, 0) \end{array}$$

and

$$\begin{array}{rcl} l_* \colon D^b(\mathcal{A}) & \to & D^b(Q_{\mathcal{A},1}) \\ E & \mapsto & (E, E, \mathrm{id}). \end{array}$$

It induces the following semiorthogonal decompositions:

$$\langle D_1, D_2 \rangle \ , \ \langle D_2, D_3 \rangle \text{ and } \langle D_3, D_1 \rangle.$$

We follow the construction in [BK90a, Prop. 3.8] to give a precise description of our Serre functor $\mathcal{S}_{\mathcal{T}_\mathcal{A}}$ at the level of objects for $\mathcal{T}_\mathcal{A} := D^b(Q_{\mathcal{A},1})$. As a consequence, we conjecture that

Conjecture 0.1. *If $D^b(\mathcal{A})$ is a n-Calabi–Yau category, then $\mathcal{T}_\mathcal{A}$ is a fractional Calabi–Yau with $q = 3$ and $p = 3n+1$, i.e. $\mathcal{S}^3_{\mathcal{T}_\mathcal{A}.} = [3n+1]$.*

The conjecture is proved at the level of objects, but no yet at the level of morphisms.

Particularly, if X is an n-Calabi-Yau projective variety we obtain a that $D^b(\mathrm{TCoh}(X))$ would be a fractional Calabi-Yau category of dimension $\frac{3n+1}{3}$. Moreover, as we have that $\mathcal{S}_{\mathcal{T}_\mathcal{A}}(^\perp D) = D^\perp$, for an admissible subcategory $\mathcal{D} \subseteq \mathcal{T}_\mathcal{A}$, we obtain the following diagram:

$$\begin{tikzcd}
& D_2 \arrow[dr, bend left, "\mathcal{S}_{\mathcal{T}_\mathcal{A}}"] & \\
D_1 \arrow[ur, bend left, "\mathcal{S}_{\mathcal{T}_\mathcal{A}}"] & & D_3. \arrow[ll, bend left, "\mathcal{S}_{\mathcal{T}_\mathcal{A}}"]
\end{tikzcd}$$

We construct hearts of bounded t-structures in $D^b(Q_{\mathcal{A},n})$ by using the semiorthogonal decompositions described above. They are induced by hearts of t-structures in $D^b(\mathcal{A})$ via recollement as in [BBD82] and gluing as in [CP10]. The difference between these two methods is that applying CP-gluing requires an additional orthogonal condition on the hearts. We proved that all gluing hearts could be also constructed via recollement and moreover the recollement hearts that do not satisfy the orthogonal condition do not admit a stability function.

The definition of the stability function is given by [CP10]. Additionally, we prove the Harder-Narasimhan property (HN) for the all CP-glued pairs, as it does not follow directly from the CP-gluing construction.

In Subsection 2.2.3, we generalize the necessary condition for the existence of σ-semistable objects given in [BGP96, Thm. 6.1].

In Chapter 3, we focus on describing $\mathrm{Stab}(\mathcal{T}_C)$ where C is a complex projective non-singular curve with genus $g \geq 1$. First note that α-stability studied in [BGPG04] appears as a CP-glued prestability condition. We first construct additional hearts in $D^b(\mathrm{TCoh}(C))$, i.e. those which are not given by CP-gluing, by tilting with respect to some torsion theories on $\mathrm{TCoh}(C)$, that also admit stability functions.

One of the main ingredients to describe $\mathrm{Stab}(D^b(X))$ is the fact that for every $\sigma \in \mathrm{Stab}(D^b(X))$, any line bundle $\mathcal{L}$ and every skyscraper $\mathbb{C}(x)$ are σ-stable.

We define

$$\Theta_{12} = \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x)), i_*(\mathcal{L}), j_*(\mathcal{L}) \text{ stable}\},$$

$$\Theta_{23} = \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid j_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), j_*(\mathcal{L}), l_*(\mathcal{L}) \text{ stable}\},$$

$$\Theta_{31} = \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), i_*(\mathcal{L}), l_*(\mathcal{L}) \text{ stable}\}$$

for all line bundles $\mathcal{L}$ and skyscrapers sheaves $\mathbb{C}(x)$ and we prove an analogous statement for $\mathrm{Stab}(\mathcal{T}_C)$.

Theorem 0.2. *If σ is a pre-stability condition on $\mathcal{T}_C$, then*

$$\sigma \in \Theta_{12} \cup \Theta_{23} \cup \Theta_{31}.$$

This result plays an important role in describing explicitly the stablity manifold $\mathrm{Stab}(\mathcal{T}_C)$. We use the stability of the skyscraper sheaves to describe the heart of any $\sigma \in \mathrm{Stab}(\mathcal{T}_C)$ as in [Bri08, Prop. 9.4]. As a consequence they are always given by CP-gluing or by tilting with respect to a torsion theory on $\mathrm{TCoh}(C)$. We define $\Theta_i \subseteq \mathrm{Stab}(\mathcal{T}_C)$, for $i = 1, 2$ or 3, as the set consisting of stability conditions which are, up to the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, CP-glued with respect to the semiorthognal decomposition $\langle D_i, {}^\perp D_i\rangle$.

In Subsection 3.2.4, we classify all stability condition on $\mathrm{Stab}(\mathcal{T}_C)$ in terms of linear algebra. We obtain

Theorem 0.3. *For all pre-stability condition σ on $\mathcal{T}_C$, we have that*

$$\sigma \in \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Gamma,$$

where Γ is the set of pre-stability condition, which up to the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action is given by Lemma 3.14 with $\Delta(M) < 0$. Moreover, note that $\Gamma \subseteq \Theta_{ij}$ and

$$\Theta_i \cap \Theta_j = \emptyset \text{ and } \Theta_i \cap \Gamma = \emptyset$$

for $i, j \in \{1, 2, 3\}$.

In other to prove that the support property is satisfied, we generalize some of the equations in [BGP96] to an arbitrary stability condition on $\mathrm{Stab}(\mathcal{T}_C)$ in Subsection 2.2.3 and we use a Bogomolov-type inequality. The proof of the HN-property goes along the lines of [Bri08]. We produce HN-filtrations in the discrete case and we use Bridgeland's deformation result for a full connected component to extend it.

We obtain our main theorem:

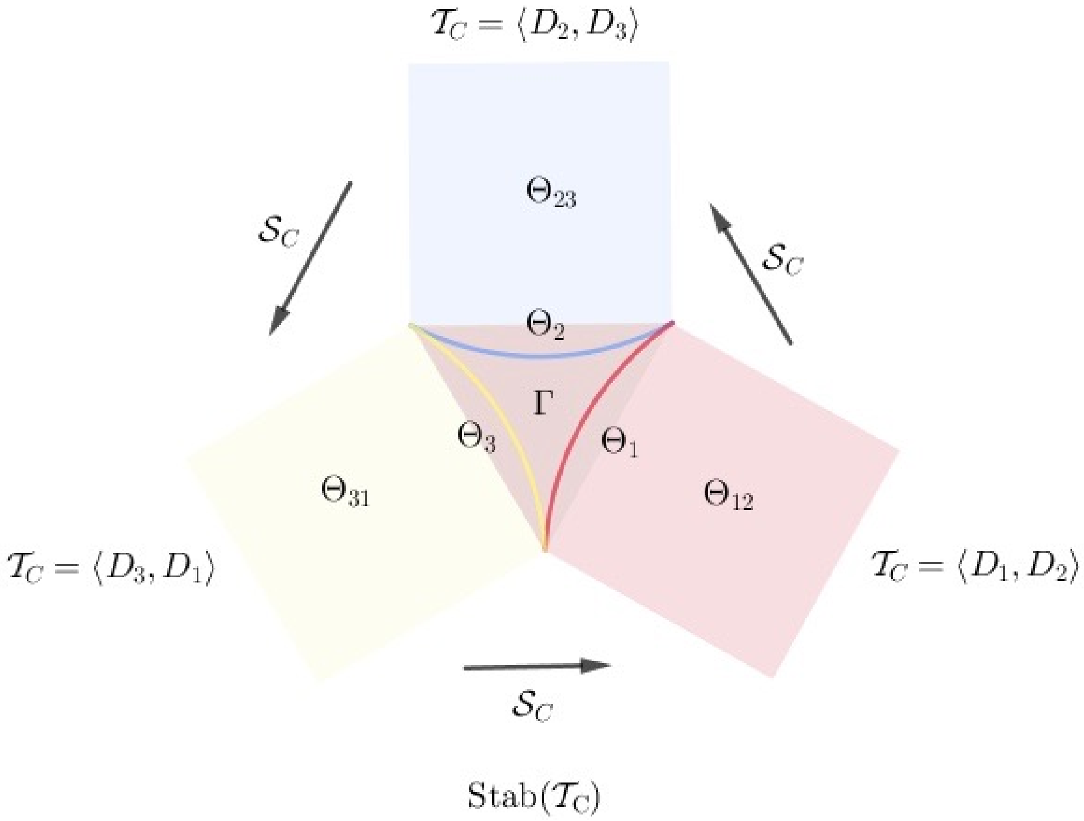
Theorem 0.4. [MRRHR19, Martínez, Rincón, Rüffer] *Let C be a complex projective non-singular curve with $g(C) = 1$, then $\mathrm{Stab}(\mathcal{T}_C)$ is a connected, 4-dimensional complex manifold.*

Under the assumption that all pre-stability conditions constructed above satisfy the support property, we obtain

Theorem 0.5. [MRRHR19, Martínez, Rincón, Rüffer] *Let C be a complex projective non-singular curve with $g(C) > 1$ then* $\mathrm{Stab}(\mathcal{T}_C)$ *is a connected, 4-dimensional complex manifold.*

In Proposition 3.87, we have proved the support property for $\sigma \in \Theta_i$, with $i = 1, 2, 3$ when $g(C) > 1$. For $\sigma \in \Gamma$, we have not proved yet the support property. See Subsection 2.2.3.

We obtain the following diagram of $\mathrm{Stab}(\mathcal{T}_C)$.



Summary

In Chapter 1, we introduce preliminary concepts. We study several features of a triangulated category $\mathcal{D}$ and abelian subcategories of a triangulated category $\mathcal{D}$ via t-structures and torsion pairs. Afterwards, we introduce important concepts to study $\mathcal{D}$, namely semiorthogonal decompositions and the Serre functor. Bridgeland stability conditions are the main object under discussion in Section 1.2. In Chapter 2, we study $\mathrm{Stab}(D^b(Q_{\mathcal{A},n}))$. The object under study in this chapter is $\mathcal{T}_{\mathcal{A},n} \coloneqq D^b(Q_{\mathcal{A},n})$. We construct different semiorthogonal decompositions for $\mathcal{T}_{\mathcal{A},n}$. We prove the existence of the Serre functor and we give an explicit construction of it in the case $n = 1$. In Section 2.2, we construct pre-stability conditions on $\mathcal{T}_{\mathcal{A},n}$ and we prove the Harder-Narasimhan property in some cases and we give explicit examples of CP-glued pre-stability conditions on the category $\mathcal{T}_{\mathrm{Coh}(X)}$, where X is a nonsingular projective curve, surface or a particular threefold. In Chapter 3, we study $\mathrm{TCoh}(C)$ i.e. the category of holomorphic triples over a curve, where C is a nonsingular projective curve over $\mathbb{C}$ with $g(C) \geq 1$. The aim of this chapter is to describe completely the stability manifold $\mathrm{Stab}(\mathcal{T}_C)$. In the process, we prove that all CP-glued pairs σ constructed in Section 2.2 on $\mathcal{T}_C$ are in fact Bridgeland stability conditions. In order to describe $\mathrm{Stab}(\mathcal{T}_C)$ we follow the steps of [Bri08]. In Section 3.1 we first construct additional pairs via tilting. As a consequence we obtain discrete pre-stability conditons. In Section 3.2 we show that all Bridgeland stability condition in $\mathrm{Stab}(\mathcal{T}_C)$ have to be given by the already constructed pairs, either by CP-gluing or by tilting. In Section 3.3 we prove the support property and finally in Section 3.4 we use Bridgeland's deformation result to describe topologically the stability manifold and to extend the HN-property to the non-discrete cases. This chapter appears in [MRRHR19] as joint work with Eva Martínez Romero and Arne Rüffer.

1 Preliminaries

In this chapter we introduce preliminary concepts, which are of great importance for this thesis. We start by studying several features of a triangulated category $\mathcal{D}$. For the definition of a triangulated category and the one of the bounded derived category $D^b(\mathcal{A})$ for an abelian category $\mathcal{A}$, we refer to [GM13], [Huy06] or the original source [Ver96]. If X is a smooth projective variety, we denote $D^b(\mathrm{Coh}(X))$ as $D^b(X)$. In Section 1.1 we follow [BBD82, Ch. 1] to study abelian subcategories of a triangulated category $\mathcal{D}$ via t-structures and torsion pairs. Afterwards, we introduce important concepts to study $\mathcal{D}$, namely semi-orthogonal decompositions and the Serre functor. Bridgeland stability conditions are the main object under discussion in Section 1.2. We follow closely [Bri07] and [Bri08]. We review the definitions, the main results and we study some examples in detail.

1.1 Triangulated and derived categories

1.1.1 t-structures and torsion pairs

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category of finite type. For the general theory of t-structures we suggest [BBD82, Ch. 1].

Definition 1.1. A *t-structure* on a triangulated category $\mathcal{D}$ consists of a pair of full additive subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, with $\mathcal{D}^{\leq i} := \mathcal{D}^{\leq 0}[-i]$ and $\mathcal{D}^{\geq i} := \mathcal{D}^{\geq 0}[-i]$ for $i \in \mathbb{Z}$, such that:

1. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.
2. For all $E \in \mathcal{D}$, there is a distinguished triangle

$$T \longrightarrow E \longrightarrow F \longrightarrow T[1]$$

   with $T \in \mathcal{D}^{\leq 0}$ and $F \in \mathcal{D}^{\geq 1}$.
3. $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

A t-structure is *bounded* if every $E \in \mathcal{D}$ is contained in $D^{\leq n} \cap D^{\geq -n}$ for some $n > 0$.

Definition 1.2. The *heart* of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is defined as

$$\mathcal{A} \coloneqq \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$$

Remark 1.3. [MS17, Lem. 5.2] The heart of a bounded t-structure $\mathcal{A} \subseteq \mathcal{D}$ is an abelian category whose short exact sequences are precisely the exact triangles in $\mathcal{D}$ with objects in $\mathcal{A}$. A morphism $A \to B$ between two objects in $\mathcal{A}$ is defined to be an inclusion if its cone is also in $\mathcal{A}$, and it is defined to be a surjection if the cone is in $\mathcal{A}[1]$.

Example 1.4. [BBD82, Ch. 1]

1. If $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ is a t-structure, then the pair $(\mathcal{D}^{\leq n}, \mathcal{D}^{\geq n})$ is also a t-structure.
2. Let $\mathcal{D} = D^b(\mathcal{A})$, where $\mathcal{A}$ is an abelian category. The standard bounded t-structure of $\mathcal{D}$ is given by $\mathcal{D}^{\leq 0} := \{E \mid H^i(E) = 0 \text{ , } i > 0\}$ and $\mathcal{D}^{\geq 0} := \{E \mid H^i(E) = 0 \text{ , } i < 0\}$. Note that $\mathcal{A}$ is the heart of the standard bounded t-structure.

Definition 1.5. Given triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ endowed with t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$, a functor $F \colon \mathcal{D} \to \mathcal{D}'$ is called *(left)right t-exact* if $(F(\mathcal{D}^{\leq 0}) \subset \mathcal{D}'^{\geq 0})$ $F(\mathcal{D}^{\geq 0}) \subset \mathcal{D}'^{\geq 0}$. We say that F is t-exact if it is left and right t-exact.

Lemma 1.6. [Bri07, Lem. 3.2], [Huy14, Rem. 1.16] *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure if and only if*

1. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$ *for* $k_1 > k_2$.

2. For every nonzero $E \in \mathcal{D}$ there exists a finite sequence of integers

$$k_1 > k_2 > \cdots > k_m$$

and a collection of distinguished triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_m \arrow[ul, dashed] &
\end{tikzcd}$$

with $A_j \in \mathcal{A}[k_j]$ for all j.

Remark 1.7. Let $\mathcal{A} \subseteq \mathcal{D}$ be the heart of a bounded t-structure. For every object $E \in \mathcal{D}$, the objects $A_j \in \mathcal{A}[k_j]$ are its *cohomological objects* with respect to $\mathcal{A}$. They are denoted by $A_i = H_{\mathcal{A}}^{-k_i}(E)$. Moreover, they induce a cohomological functor i.e. they are functorial and induce a long exact sequence of cohomology for any exact triangle. Note that a t-structure is uniquely determined by its heart. See [Huy14, Def. 1.13]

Hearts of bounded t-structures play an important role in the definition of Bridgeland stability conditions. Therefore, we will discuss different ways of giving hearts of bounded t-structures. Namely, tilting in Proposition 1.10 and CP-gluing and BBD-recollement in Subsection 2.2.

Definition 1.8. Let $\mathcal{A}$ be an abelian category. A *torsion pair* for $\mathcal{A}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that

1. $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.
2. For all $E \in \mathcal{A}$, there is a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Example 1.9. Let X be a smooth projective variety. The pair of subcategories

$$\begin{aligned} \mathcal{T} &:= \{\text{Torsion sheaves on X}\} \\ \mathcal{F} &:= \{\text{Torsion-free sheaves on X}\}. \end{aligned}$$

define a torsion pair on $\mathrm{Coh}(X)$.

Tilting

Proposition 1.10. [HRS96, Prop. 2.1] *and* [MS17, Lem. 6.3] *Let $\mathcal{A}$ be the heart of a bounded t-structure on $\mathcal{D}$. Given a torsion pair $(\mathcal{T}, \mathcal{F})$ in $\mathcal{A}$, then the full subcategory*

$$\mathcal{A}^\sharp = \{E \in \mathcal{D} \mid H^i_{\mathcal{A}}(E) = 0 \text{ for } i \notin \{-1, 0\},\ H^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \text{ and } H^0_{\mathcal{A}}(E) \in \mathcal{T}\}$$

is the heart of a bounded t-structure on $\mathcal{D}$. We call $\mathcal{A}^\sharp$ the tilt *of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$. Moreover, the torsion pair $(\mathcal{T}, \mathcal{F})$ gives rise to the torsion pair $(\mathcal{F}[1], \mathcal{T})$ for the tilt $\mathcal{A}^\sharp$.*

Remark 1.11. Note that $\mathcal{A}^\sharp[-1] = (\mathcal{F}, \mathcal{T}[-1])$ is also the heart of a bounded t-structure. The heart $\mathcal{A}^\sharp[-1] = (\mathcal{F}, \mathcal{T}[-1])$ is called the *right tilt* of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$.

1.1.2 Semiorthogonal decomposition and recollement

Definition 1.12. Let $\mathcal{X} \subseteq \mathcal{D}$ be a strictly full triangulated subcategory. The category $\mathcal{X}$ is *(left) right-admissible* if the functor $i \colon \mathcal{X} \hookrightarrow \mathcal{D}$ has a (left) right adjoint $(i^* \colon \mathcal{D} \to \mathcal{X})$ $i^! \colon \mathcal{D} \to \mathcal{X}$. If $\mathcal{X}$ is left and right admissible, we say that $\mathcal{X}$ is *admissible.*

Definition 1.13. An object $E \in \mathcal{D}$ is called *exceptional*, if

$$\mathrm{Hom}_{\mathcal{D}}(E, E) = \mathbb{C} \text{ and } \mathrm{Hom}_{\mathcal{D}}(E, E[t]) = 0,$$

for $t \in \mathbb{Z}$ with $t \neq 0$.

Example 1.14. [Bon90, Thm. 3.2] Let E be an exceptional object in $\mathcal{D}^b(X)$. The triangulated subcategory generated by E, which is denoted by $\langle E \rangle$, is an admissible subcategory that satisfies $\langle E \rangle \cong D^b(\mathrm{Vect}_{\mathbb{C}})$ where $\mathrm{Vect}_{\mathbb{C}}$ is the category of finite dimensional vector spaces. The functors defined as $i^*(F) := E \otimes \mathrm{RHom}(F, E)^*$ and $i^!(F) := E \otimes \mathrm{RHom}(E, F)$ are the left and right adjoint respectively.

Definition 1.15. Let $\mathcal{D}$ be a triangulated category. A *semiorthogonal decomposition* of $\mathcal{D}$ consists of a collection $D_1, \ldots, D_n$ of full triangulated subcategories such that

1. $\mathrm{Hom}_{\mathcal{D}}(D_i, D_j) = 0$ for every $1 \leq j < i \leq n$.
2. $\mathcal{D}$ is generated by the D_i.

We write $\mathcal{D} = \langle D_1, \ldots, D_n \rangle$.

Definition 1.16. An ordered collection $\mathcal{E} = (E_0, E_1, \ldots, E_{n-1}, E_n) \subseteq \mathcal{D}$ is called *exceptional* if every $E_i \in \mathcal{D}$ is an exceptional object and $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[k]) = 0$, for all $k, i, j \in \mathbb{Z}$ and for $i > j$.

An exceptional collection is called *strong* if $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[k]) = 0$, for all i and j, with $k \neq 0$. It is called *Ext* if $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(E_i, E_j) = 0$ for all $i \neq j$. It is called *complete* if $\mathcal{E}$ generates $\mathcal{D}$ by shifts and extensions.

Example 1.17. Let $\mathcal{E} = (E_0, E_1, \ldots, E_{n-1}, E_n) \subseteq \mathcal{D}$ be a complete exceptional collection. Then it induces a semiorthogonal decomposition $\mathcal{D} = \langle \langle E_0 \rangle, \ldots \langle E_n \rangle \rangle$.

Lemma 1.18. [BK90b, Prop. 1.5] *Let $\mathcal{D}$ be a triangulated category. Let D_1 and D_2 be strictly full triangulated subcategories of $\mathcal{D}$. Assume that* $\mathrm{Hom}_{\mathcal{D}}(D_2, D_1) = 0$*. Then, the following are equivalent:*

1. *The category $\mathcal{D}$ is generated by D_1 and D_2 i.e. for each $X \in \mathcal{D}$, there exists a distinguished triangle*
$$X_2 \longrightarrow X \longrightarrow X_1 \longrightarrow X_2[1]$$
with $X_1 \in D_1$ and $X_2 \in D_2$.
2. $D_2 = {}^{\perp}D_1 := \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(X, Y) = 0 \text{ for all } Y \in D_1\}$ *and there exists a functor $i^* \colon \mathcal{D} \to D_1$ which is left adjoint to the inclusion $i \colon D_1 \hookrightarrow \mathcal{D}$, i.e. D_1 is left admissible.*
3. $D_1 = D_2^{\perp} := \{X \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(Y, X) = 0 \text{ for all } Y \in D_2\}$ *and there exists a functor $j^! \colon \mathcal{D} \to D_2$ which is right adjoint to the inclusion $j \colon D_2 \hookrightarrow \mathcal{D}$, i.e. D_2 is right admisible.*

Corollary 1.19. *If $\mathcal{X}$ is an admissible subcategory of $\mathcal{D}$, then we have two semiorthogonal decompositions $\mathcal{D} = \langle \mathcal{X}^{\perp}, \mathcal{X} \rangle$ and $\mathcal{D} = \langle \mathcal{X}, {}^{\perp}\mathcal{X} \rangle$.*

Remark 1.20. [Huy06, Lem. 1.30] Let $F \colon \mathcal{D} \to \mathcal{D}$ be an autoequivalence. If $\mathcal{D}$ has a semiorthogonal decomposition $\mathcal{D} = \langle D_1, D_2 \rangle$, then $\mathcal{D} = \langle F(D_1), F(D_2) \rangle$.

Mutations

We now study the action of the braid group, via mutations, on the set of semiorthogonal decompositions given by admissible subcategories. To study mutations we recommend [Bon90] as the original source. We follow closely the approach given in [Kuz10, Sec. 2] and [Kuz09, Sec. 2].

Let $r \geq 2$ be a natural number, the *braid group* is the group generated by $n-1$ elements, with the following relations:

$$B_r = \langle s_i \mid 1 \leq i;\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \text{ for } |i-j| \geq 2 \rangle.$$

Remark 1.21. Note that $B_2 \cong \mathbb{Z}$.

Notation 1.22. Let $\varphi \in \mathrm{Hom}_{\mathcal{D}}(E, F)$, we denote the cone of φ as $C(\varphi)$.

Lemma 1.23. [Bon90] *Assume that $\mathcal{X} \subseteq \mathcal{D}$ is an admissible subcategory. Then, there are functors*

$$\begin{array}{rcl} \mathbb{L}_{\mathcal{X}} \colon \mathcal{D} & \to & \mathcal{D} \\ F & \to & C(i_* i^! F \to F) \end{array} \qquad \begin{array}{rcl} \mathbb{R}_{\mathcal{X}} \colon \mathcal{D} & \to & \mathcal{D} \\ F & \to & C(F \to i_* i^* F)[-1], \end{array}$$

where $i_* \colon \mathcal{X} \hookrightarrow \mathcal{D}$ is the embedding functor and $i^*, i^!$ are the left and right adjoints respectively. These functors vanish on $\mathcal{X}$ and induce mutually inverse equivalences ${}^{\perp}\mathcal{X} \to \mathcal{X}^{\perp}$ and $\mathcal{X}^{\perp} \to {}^{\perp}\mathcal{X}$, respectively.

The functors $\mathbb{L}_{\mathcal{X}}$ and $\mathbb{R}_{\mathcal{X}}$ are called *left* and *right mutations* respectively.

Corollary 1.24. *Let E be an exceptional object. The left mutation $\mathbb{L}_E F$ induced by E is given by the cone of the evaluation morphism $\bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k_{\mathcal{D}}(E, F)[-k] \otimes E \to F$. We obtain the following triangle*

$$\bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k_{\mathcal{D}}(E, F)[-k] \otimes E \to F \to \mathbb{L}_E F.$$

The right mutation *$\mathbb{R}_E F$ induced by E is given by the cone shifted by -1 of the morphism $F \to \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k_{\mathcal{D}}(E, F)^*[-k] \otimes E$. with the following distinguished triangle*

$$\mathbb{R}_E F \to F \to \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}^k_{\mathcal{D}}(E, F)^*[-k] \otimes E.$$

Note that $V[k] \otimes E$ where V is a complex vector space denotes the direct sum of $\dim V$ *copies of the object $E[k]$. Moreover, if we consider the dual vector space the grading changes the sign.*

Remark 1.25. The definition that we use differs from the one in [Mac07, Def. 3.8] by a ± 1 shift.

Corollary 1.26. [Bon90] *Let $\mathcal{D} = \langle D_1, \ldots D_n \rangle$ with D_i admissible subcategories. Then for each $1 \leq k \leq n-1$ there is a semiorthogonal decomposition*

$$\mathcal{D} = \langle D_1, \ldots, D_{k-1}, \mathbb{L}_{D_k}(D_{k+1}), D_k, D_{k+2}, \ldots, D_n \rangle$$

and for $2 \leq k \leq n$ there is a semiorthogonal decomposition

$$\mathcal{D} = \langle D_1, \ldots, D_{k-2}, D_k, \mathbb{R}_{D_k}(D_{k-1}), D_{k+1}, \ldots, D_n \rangle.$$

Moreover, the braid group relations are satisfied:

$$\mathbb{R}_{D_i}\mathbb{R}_{D_{i+1}}\mathbb{R}_{D_i} = \mathbb{R}_{D_{i+1}}\mathbb{R}_{D_i}\mathbb{R}_{D_{i+1}} \text{ and } \mathbb{L}_{D_i}\mathbb{L}_{D_{i-1}}\mathbb{L}_{D_i} = \mathbb{L}_{D_{i-1}}\mathbb{L}_{D_i}\mathbb{L}_{D_{i-1}}$$

for each $2 \leq 1 \leq m-1$.

Corollary 1.27. [Bon90, Ast. 2.1] *If $\mathcal{E} = (E_0, E_1, \ldots, E_n)$ is an exceptional callection, then*

$$\mathbb{R}_i\mathcal{E} = (E_0, \ldots, E_{i-1}, E_{i+1}, \mathbb{R}_{E_{i+1}}E_i, E_{i+2}, \ldots E_n)$$

$$\mathbb{L}_i\mathcal{E} = (E_0, \ldots, E_{i-1}, \mathbb{L}_{E_i}E_{i+1}, E_i, E_{i+2}, \ldots E_n)$$

are exceptional collections.

Particularly, if we have an exceptional pair (E_1, E_2) then

$$(E_2, \mathbb{R}_{E_2}E_1) \text{ and } (\mathbb{L}_{E_1}E_2, E_1)$$

are also exceptional pairs.

Remark 1.28. [Bon90, Lem. 2.2] If an exceptional collection $\mathcal{E}$ is complete, then the mutated collection is also complete.

Corollary 1.29. [Bon90, Ast. 2.3] *There is action of the braid group B_r, on the set of all exceptional sequences of length r, defined as $s_i\mathcal{E} = \mathbb{L}_i\mathcal{E}$. Let $\mathcal{E} = (E_0, E_1, \ldots, E_{n-1}, E_n)$ be an exceptional collection. Then*

1. $\mathbb{L}_i\mathbb{R}_i\mathcal{E} = \mathbb{R}_i\mathbb{L}_i\mathcal{E} = \mathcal{E}$
2. $\mathbb{L}_i\mathbb{L}_{i+1}\mathbb{L}_i\mathcal{E} \cong \mathbb{L}_{i+1}\mathbb{L}_i\mathbb{L}_{i+1}\mathcal{E}$
3. $\mathbb{L}_i\mathbb{L}_j\mathcal{E} = \mathbb{L}_j\mathbb{L}_i\mathcal{E}$ *and* $\mathbb{R}_i\mathbb{R}_j\mathcal{E} = \mathbb{R}_j\mathbb{R}_i\mathcal{E}$, *for* $|i-j| \geq 2$.

We now connect a bounded derived categories with a strong, exceptional collection with a category of representations of finite-dimensional associative algebras.

Theorem 1.30. [Bon90, Thm. 6.2] *Let $\mathcal{D}$ be the bounded derived category of an abelian category with sufficiently many injective (or projective) objects. Assume that $\mathcal{D}$ is generated by a strong exceptional collection $\{E_0, \dots, E_n\}$. Then, if $E = \oplus_i E_i$ and $A = \mathrm{End}(E)$ we obtain that $R\,\mathrm{Hom}(E, -)\colon \mathcal{D} \to D^b(\mathrm{modA})$ is an exact equivalence, where* modA *is the category of right A-modules of finite rank. Under this identification the objects E_i correspond to the indecomposable projective A-modules. We particularly obtain that the category* modA *under the equivalence becomes a heart of a bounded t-structure on $\mathcal{D}$.*

Example 1.31. The triangulated category $D^b(\mathbb{P}^n)$ has a strong, complete exceptional collection given by $(\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n))$. If $n = 1$, then A is precisely the path algebra of the 2-Kronecker quiver. See Subsection 1.2.4.

Recollement

Semiorthogonal decompositions given by admissible subcategories induce recollements. Having a recollement on $\mathcal{D}$ will be an important tool in the construction of t-structures.

Definition 1.32. Let $\mathcal{X}, \mathcal{Y}$ be full triangulated subcategories. The triangulated category $\mathcal{D}$ is a *recollement* of $\mathcal{X}$ and $\mathcal{Y}$ if there are six triangulated functors

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "i_*"] & \mathcal{D} \arrow[l, bend right=60, "i^*"'] \arrow[l, bend left=60, "i^!"] \arrow[r, "j^*"] & \mathcal{Y}, \arrow[l, bend right=60, "j_!"'] \arrow[l, bend left=60, "j_*"]
\end{tikzcd}$$

with the following properties.

1. $(i^*, i_*), (i_*, i^!), (j_!, j^*), (j^!, j^*)$ and (j^*, j_*) are pairs of adjoint functors with $j_* = j!$ and $i^* = i^!$.
2. $j^* i_* = 0$.
3. $i_*, j_!$ and j_* are fully faithful.
4. Each object $T \in \mathcal{D}$ determines distinguished triangles

$$i_* i^! T \longrightarrow T \longrightarrow j_* j^* T \longrightarrow i_* i^! T[1]$$

$$j_! j^* T \longrightarrow T \longrightarrow i_* i^* T \longrightarrow j_! j^* T[1]$$

where the morphisms into and out of T are counit and unit morphism.

Proposition 1.33. [BBD82], [MRRHR19] *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{X} \subset \mathcal{D}$ be a full triangulated subcategory. Then, $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{X}^\perp$ if and only if $\mathcal{X}$ is admissible.*

Serre functor

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category.

Definition 1.34. A *Serre functor* on $\mathcal{D}$ is an exact autoequivalence $S\colon \mathcal{D} \to \mathcal{D}$ such that for any $E, F \in \mathcal{D}$, there is an isomorphism

$$\eta_{E,F}\colon \operatorname{Hom}_{\mathcal{D}}(E,F) \to \operatorname{Hom}_{\mathcal{D}}(F, S(E))^*$$

(of $\mathbb{C}$-vector spaces) which is functorial in E and F.

Remark 1.35. For $\mathcal{D}$ of finite type, a Serre functor, if it exists, is unique up to isomorphism. Moreover, it commutes with equivalences, i.e. for $F : \mathcal{D} \to \mathcal{D}'$ an equivalence, then we have that $\mathcal{S}_{\mathcal{D}'} \circ F \cong F \circ \mathcal{S}_{\mathcal{D}}$. See [Huy14].

Example 1.36. Let X be a smooth projective variety defined over $\mathbb{C}$, then the autoequivalence

$$\begin{array}{rcl} \mathcal{S}_X\colon D^b(X) & \to & D^b(X) \\ E & \mapsto & E \otimes \omega_X[\dim X]. \end{array}$$

where ω_X is the dualizing line bundle, is a Serre functor on $D^b(X)$.

Furthermore, given an admissible subcategory $\mathcal{X} \subset \mathcal{D}$ it is easy to see that

$$\mathcal{S}_{\mathcal{D}}({}^{\perp}\mathcal{X}) = \mathcal{X}^{\perp} \text{ and } \mathcal{S}_{\mathcal{D}}^{-1}(\mathcal{X}^{\perp}) = {}^{\perp}\mathcal{X}.$$

In terms of mutations, we obtain the following lemma.

Lemma 1.37. [Kuz09, Lem. 2.11] *Let $\mathcal{D} = \langle D_1, D_2 \rangle$ be a triangulated category with Serre functor $\mathcal{S}_{\mathcal{D}}$. Then we obtain*

$$\mathcal{S}_{\mathcal{D}}(D_2) = \mathbb{L}_{D_1}(D_2) \text{ and } \mathcal{S}_{\mathcal{D}}^{-1}(D_1) = \mathbb{R}_{D_2}(D_1).$$

Moreover, the Serre functor will play an important role in the understanding and construction of the adjoint functors.

Lemma 1.38. *If $\mathcal{D}$ admits a Serre functor $\mathcal{S}_{\mathcal{D}}$ and $\mathcal{X}$ is an admissible triangulated subcategory of $\mathcal{D}$, then $\mathcal{X}$ admits a Serre functor given by*

$$\mathcal{S}_{\mathcal{X}} = i^!\mathcal{S}_{\mathcal{D}}i_* \text{ and } \mathcal{S}_{\mathcal{X}}^{-1} = i^*\mathcal{S}_{\mathcal{D}}^{-1}i_*.$$

Proposition 1.39. [Huy06, Rem. 1.31] *Let $\mathcal{T}_1, \mathcal{T}_2$ be triangulated subcategories which admit Serre functors $\mathcal{S}_{\mathcal{T}_1}$ and $\mathcal{S}_{\mathcal{T}_2}$ respectively and let $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ be a functor.*

1. *If F admits a left adjoint $G\colon \mathcal{T}_2 \to \mathcal{T}_1$, then it also admits a right adjoint $H\colon \mathcal{T}_2 \to \mathcal{T}_1$, given as*
$$H = \mathcal{S}_{\mathcal{T}_1} \circ G \circ \mathcal{S}_{\mathcal{T}_2}^{-1}.$$

2. *If F admits a right adjoint $H\colon \mathcal{T}_2 \to \mathcal{T}_1$, then it also admits a left adjoint $G\colon \mathcal{T}_2 \to \mathcal{T}_1$, given as*
$$G = \mathcal{S}_{\mathcal{T}_1}^{-1} \circ H \circ \mathcal{S}_{\mathcal{T}_2}.$$

1.2 Bridgeland stability conditions

In this section we define Bridgeland stability conditions. We follow Bridgeland's papers [Bri07] and [Bri08]. We also recommend the following lecture notes on Bridgeland stability theory [MS17], [Bay11], [Huy14] and [BM11, Appex B.].

Definition 1.40. The *Grothendieck group* $K(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the abelian group generated by the isomorphism classes of objects in $\mathcal{D}$ subject to the relation $[A] = [C] + [B]$, where $C \to A \to B \to C[1]$ is an exact triangle. As $\mathcal{D}$ is $\mathbb{C}$-linear of finite type, we consider the Euler bilinear form given by
$$\chi(E, F) = \sum_i (-1)^i \operatorname{Hom}_{\mathcal{D}}(E, F[i]).$$

We define the *numerical Grothendieck group* $\mathcal{N}(\mathcal{D})$ as the quotient $K(\mathcal{D})/K(D)^{\perp}$, where $K(\mathcal{D})^{\perp}$ denotes the right orthogonal with respect to the Euler form. Moreover, if $\mathcal{N}(D)$ has finite rank then $\mathcal{D}$ is called *numerically finite*.

Example 1.41. 1. If $\mathcal{D} = \langle D_1, \ldots, D_n \rangle$, then $K(\mathcal{D}) = \oplus_i K(D_i)$.

2. If $\mathcal{D}$ is generated by a complete exceptional collection. Then the Grothendieck group $K(\mathcal{D}) \cong \mathbb{Z}^{n+1}$ is the free abelian group generated by $[E_i]$, for $i = 0, \ldots, n$.

Example 1.42. 1. If A is a finite-dimensional algebra over $\mathbb{C}$, then the bounded derived category $D^b(A)$ of finite-dimensional left A-modules is numerically finite.

2. If X is a smooth projective variety over $\mathbb{C}$, then the bounded derived category $D^b(X)$ of coherent sheaves on X is numerically finite.

Throughout all this chapter, we assume that $\mathcal{D}$ is numerically finite.

Remark 1.43. If $\mathcal{A}$ is an abelian category, we define the Grothendieck group $K(\mathcal{A})$ as the abelian group generated by isomorphism classes of objects of $\mathcal{A}$ subject to the relation $[A] = [C] + [B]$, where $0 \to C \to A \to B \to 0$ is a short exact sequence in $\mathcal{A}$. If $\mathcal{A} \subseteq \mathcal{D}$ is a heart of a bounded t-structure then $K(\mathcal{D}) \cong K(\mathcal{A})$.

We now fix a finite rank $\mathbb{Z}$-lattice Λ and a surjective homomorphism

$$v\colon K(\mathcal{D}) \twoheadrightarrow \Lambda.$$

If $\mathcal{D}$ is numerically finite, then $\mathcal{N}(\mathcal{D})$ is a finite rank $\mathbb{Z}$-lattice. We often choose $\Lambda = \mathcal{N}(\mathcal{D})$ and v as the natural projection.

The definition of a Bridgeland stability condition has two main ingredients the heart of a bounded t-structure and a stability function.

Definition 1.44. Let $\mathcal{A}$ be an abelian category. We say that a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is a *weak stability function* on $\mathcal{A}$ if, for all $E \in \mathcal{A}$, we have $\Im(Z([E])) \geq 0$, with $\Im(Z([E])) = 0$ then $\Re(Z(E)) \leq 0$. If additionally, for $E \neq 0$, we have that if $\Im Z(E) = 0$ then $\Re Z(E) < 0$, we say that Z is a *stability function* on $\mathcal{A}$. Note that in this case, the image of Z is contained in the semi-closed upper half plane

$$\overline{\mathbb{H}} = \{\alpha \in \mathbb{C} \mid \Im(\alpha) \geq 0 \text{ if } \Im(\alpha) = 0, \Re(\alpha) < 0\}.$$

We consider a group homorphism $Z\colon \Lambda \to \mathbb{C}$, such that $Z \circ v\colon K(\mathcal{A})(= K(\mathcal{D})) \to \mathbb{C}$ is a stability function on $\mathcal{A}$. We define the slope $\mu_Z(E)\colon K(\mathcal{A}) \to \mathbb{R} \cup \infty$

$$\mu_Z(E) = \begin{cases} -\frac{\Re(Z(E))}{\Im(Z(E))} & \text{if } \Im(Z(E)) \neq 0 \\ +\infty & \text{otherwise,} \end{cases}$$

where $Z(E) := Z(v([E]))$. We say that $0 \neq E \in \mathcal{A}$ is *Z-semistable* if for all proper subobjects $F \subseteq E$, we have that $\mu(F) \leq \mu(E)$. We also define the phase of $0 \neq E$ as

$$\phi(E) = \arg(Z(E))\frac{1}{\pi} \in (0, 1].$$

Note that E is Z-semistable if and only if for all proper subobjects $F \subseteq E$, we have that $\phi(F) \leq \phi(E)$. We will constantly use the correspondence between slope and phase given for the complex numbers in the semi-closed upper half plane.

Remark 1.45. Note that this definition of stability function agrees with the definition of stability given in [Rud97, Def. 1.1], because it satisfies that for a short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we have that

$$\begin{aligned} \text{either } \phi(A) < \phi(B) &\iff \phi(A) < \phi(C) \iff \phi(B) < \phi(C), \\ \text{or } \phi(A) > \phi(B) &\iff \phi(A) > \phi(C) \iff \phi(B) > \phi(C), \\ \text{or } \phi(A) = \phi(B) &\iff \phi(A) = \phi(C) \iff \phi(B) = \phi(C). \end{aligned}$$

Definition 1.46. A stability function $Z\colon K(\mathcal{A}) \to \mathbb{C}$ satisfies the *Harder-Narasimhan property* (HN-property, for short) on $\mathcal{A}$ if for every $0 \neq E \in \mathcal{A}$, there is a filtration

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{m-1} \subseteq E_m = E$$

on $\mathcal{A}$, such that E_i/E_{i-1} is Z-semistable for $i = 1, \dots, m$ and

$$\phi(E_1/E_0) > \cdots > \phi(E_m/E_{m-1}).$$

Moreover, as the HN-filtration is unique, the quotients E_i/E_{i-1} are the *HN-factors* of E.

Definition 1.47. A *(weak) pre-stability condition* on $\mathcal{D}$ is pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A} \subseteq \mathcal{D}$ is the heart of a bounded t-structure and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism such that $Z \circ v\colon K(\mathcal{A})(= K(\mathcal{D})) \to \mathbb{C}$ is a (weak) stability function on $\mathcal{A}$ satisfying the HN-property. The homomorphism Z is also called a central charge.

Remark 1.48. [BLMS17, Prop. 2.9] Let $\sigma = (Z, \mathcal{A})$ be a weak stability condition, and let $\alpha \in \mathbb{R}$. We form the following subcategories of $\mathcal{A}$

$$\mathcal{T}_\sigma^\alpha \coloneqq \{E \in \mathcal{A} \mid \text{The HN-factors } F \text{ of } E \text{ satisfy } \mu_Z(F) > \alpha\},$$
$$\mathcal{F}_\sigma^\alpha \coloneqq \{E \in \mathcal{A} \mid \text{The HN-factors } F \text{ of } E \text{ satisfy } \mu_Z(F) \leq \alpha\}.$$

By the HN-property, we obtain $(\mathcal{T}_\sigma^\alpha, \mathcal{F}_\sigma^\alpha)$ is a torsion pair on $\mathcal{A}$. Therefore, by Proposition 1.10, we obtain a heart of a bounded t-structure $\mathcal{A}_\sigma^\alpha$ with a torsion pair $(\mathcal{F}_\sigma^\alpha[1], \mathcal{T}_\sigma^\alpha)$.

We will use several examples of hearts of bounded t-structures given by tilting as above in Subsection 1.2.2 and Sec 3.1.

We now define a slicing. Intuitively, a heart of a bounded t-structure $\mathcal{A} \subseteq \mathcal{D}$ breaks up every object in $\mathcal{D}$ in terms of its cohomology index by $\mathbb{Z}$, a slicing further refines the heart of a bounded t-structure, which allows us to break up each object into pieces indexed by the real numbers.

Definition 1.49. [Bri07] A *slicing* $\mathcal{P}$ on $\mathcal{D}$ is a collection of full subcategories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ satisfying:

1. $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$, for all $\phi \in \mathbb{R}$.
2. If $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$, $i = 1, 2$, then $\mathrm{Hom}_{\mathcal{D}}(E_1, E_2) = 0$.
3. For every nonzero object $E \in \mathcal{D}$ there exists a finite sequence of maps

$$0 = E_0 \xrightarrow{f_0} E_1 \xrightarrow{f_1} \cdot \to E_{m-1} \xrightarrow{f_{m-1}} E_m = E$$

and of real numbers $\phi_0 > \cdots > \phi_m$ such that the cone of f_j is in $\mathcal{P}(\phi_j)$ for $j = 0, \cdots, m-1$.

For every interval $I \subseteq \mathbb{R}$ we define $\mathcal{P}(I)$ as the extension-closed subcategory generated by the subcategories $\mathcal{P}(\phi)$ with $\phi \in I$.

Proposition 1.50. [Bri07, Prop. 5.3] *To give a pre-stability condition σ on $\mathcal{D}$ is equivalent to giving a slicing $\mathcal{P}$ and a group homomorphism $Z\colon \Lambda \to \mathbb{C}$ such that for every $0 \neq E \in \mathcal{P}(\phi)$, we have that $Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$.*

Remark 1.51. [Bri07, Lem. 5.2] If σ is a pre-stability condition, then $\mathcal{P}(\phi)$ is an abelian subcategory.

Remark 1.52. The key point of the proof of [Bri07, Prop. 5.3] is to show that if $\mathcal{P}$ is a slicing, then $\mathcal{A} = \mathcal{P}(0,1]$ is the heart of a bounded t-structure satisfying the HN-property. If we have a pre-stability condition $\sigma = (Z, \mathcal{A})$, we define $\mathcal{P}(\phi)$, for $\phi \in (0,1]$ as the set of all Z-semistable objects in $\mathcal{A}$.

Notation 1.53. Let $\sigma = (Z, \mathcal{P})$ be a pre-stability condition. By Definition 1.49, for every $E \in \mathcal{D}$, there is a filtration associated to E, that we also call the Harder-Narasimhan filtration. The semistable objects in the filtration are called *Harder-Narasimhan factors* (HN-factors, for short). Moreover, we write $\phi^+(E), \phi^-(E)$ for the largest and the smallest phase appearing in this filtration respectively. If E is σ-semistable, $\phi^+(E) = \phi^-(E) = \phi(E)$.

Remark 1.54. 1. Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition. By definition if $E \in \mathcal{D}$ is σ-semistable, then there exists $n \in \mathbb{Z}$ such that $E[n] \in \mathcal{A}$.

2. If $E, A \in \mathcal{D}$ and $\phi^-(E) > \phi^+(A)$, then $\mathrm{Hom}_{\mathcal{D}}(E, A) = 0$.
3. Consider the last triangle $E_{m-1} \to E \to A_m \to E_{m-1}[1]$ of the HN-filtration of $E \in \mathcal{D}$, where A_m is the cone of f_{m-1}. We have that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(E_{m-1}, A_m) = 0$.

Definition 1.55. The simple objects of $\mathcal{P}(\phi)$ are called *σ-stable* objects.

Let $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ be the universal covering of $\mathrm{GL}^+(2,\mathbb{R})$, whose objects can be given by pairs (T, f) where $T \in \mathrm{GL}^+(2,\mathbb{R})$ and $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous increasing function that satisfies $f(x+1) = f(x) + 1$ for all $x \in \mathbb{R}$ such that the induced maps of T and f on $S^1 = \mathbb{R}/2\mathbb{Z} = (\mathbb{R}^2 - \{(0,0)\})/\mathbb{R}_{>0}$ coincide. In the next section we study in detail $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and its action on the set of pre-stability conditions.

We define a right action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on the set of pre-stability conditions. If $\sigma = (Z, \mathcal{A})$ is a pre-stability condition and $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, then we define $\sigma' = \sigma g = (Z', \mathcal{P}')$ as $Z = T^{-1} \circ Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$, where $\mathcal{P}$ and $\mathcal{P}'$ are the slicing of Z and Z' respectively. Note that the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action preserves the semistable objects, but relabels their phases.

In the next section, we study $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and its action on the set of pre-stability conditions in detail.

Let us consider the group $\mathrm{Aut}_\Lambda(\mathcal{D})$ of autoequivalences Φ on $\mathcal{D}$ whose induced automorphism ϕ_* of $K(\mathcal{D})$ is compatible with the map $v\colon K(\mathcal{D}) \to \Lambda$, i.e. there is a group isomorphism $v(\phi_*)\colon \Lambda \to \Lambda$ such that the following diagram commutes

$$\begin{tikzcd} K(\mathcal{D}) \arrow[r, "\phi_*"] \arrow[d, "v"'] & K(\mathcal{D}) \arrow[d, "v"] \\ \Lambda \arrow[r, "v(\phi_*)"] & \Lambda. \end{tikzcd} \tag{1.1}$$

We define a left action of the group $\mathrm{Aut}_\Lambda(\mathcal{D})$ on the set of pre-stability conditions. For $\Phi \in \mathrm{Aut}_\Lambda(\mathcal{D})$, we define $\Phi\sigma = (Z', \mathcal{P}')$ as $Z' = Z \circ v(\phi_*)^{-1}$ and $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$. Note that if E is a σ-semistable object, then $\Phi(E)$ is $\Phi(\sigma)$-semistable.

Definition 1.56. A pre-stability condition σ is *locally finite* if there is some $\epsilon > 0$ such that each category $\mathcal{P}((\phi-\epsilon, \phi+\epsilon))$, for $\phi \in \mathbb{R}$, is of finite length.

Definition 1.57. A pre-stability condition is *discrete* if the image of Z is a discrete subgroup of $\mathbb{C}$.

Lemma 1.58. [Bri08, Lem. 4.5] *Suppose that $\sigma = (Z, \mathcal{P})$ is a discrete pre-stability condition and fix $0 < \epsilon < \frac{1}{2}$. Then for each $\phi \in \mathbb{R}$ the category $\mathcal{P}((\phi-\epsilon, \phi+\epsilon))$ is of finite length. In particular σ is locally finite.*

Remark 1.59. [Bri07, Lem. 5.2] The categories $\mathcal{P}(\phi)$ with $\phi \in \mathbb{R}$ are abelian. If σ is locally finite, then $\mathcal{P}(\phi)$ has finite length. Therefore, a σ-semistable object $E \in \mathcal{P}(\phi)$ admits a finite Jordan-Hölder filtrations, i.e. a finite filtration

$$E_0 \subset E_1 \subset \ldots \subset E_n = E$$

with stable quotient $E_{i+1}/E_i \in \mathcal{P}(\phi)$, as the stable objects are the simple objects in $\mathcal{P}(\phi)$.

We now introduce the support property. It plays an important role in proving good deformation properties and a well-behaved wall and chamber decomposition. We suggest [BMS16, App. A] to understand better the relation between the support property and effective deformations of Bridgeland stability conditions.

Definition 1.60. A pre-stability condition $\sigma = (Z, \mathcal{A})$ satisfies the *support property* if there is a symmetric bilinear form Q on $\Lambda \otimes \mathbb{R} := \Lambda_{\mathbb{R}}$ which satisfies

1. All σ-semistable objects $E \in \mathcal{A}$, satisfy $Q(v(E), v(E)) \geq 0$.
2. All non zero vectors $v \in \Lambda_{\mathbb{R}}$ with $Z(v) = 0$ satisfy $Q(v, v) < 0$.

Lemma 1.61. [KS08, Sec. 2.1], [BMS16, Lem. 11.4] *The following statement are equivalent*

1. *The pre-stability condition $\sigma = (Z, \mathcal{A})$ satisfies the support property.*
2. *The pre-stability condition $\sigma = (Z, \mathcal{A})$ satisfies that there is a constant $C \in \mathbb{R}_{>0}$ and a norm $||.||$ on $\Lambda_{\mathbb{R}}$, such that for all σ-stable objects $0 \neq E \in \mathcal{D}$, we have*

$$||v(E)|| < C|Z(v(E))|.$$

Remark 1.62. If $\mathrm{rk}(\Lambda) = 2$ and $Z\colon \Lambda \to \mathbb{C}$ is injective, then every pre-stability condition $\sigma = (Z, \mathcal{A})$ trivially satisfies the support property with respect to any positive semidefinite quadratic form.

Definition 1.63. A pre-stability condition $\sigma = (Z, \mathcal{A})$ that satisfies the support property is called a *Bridgeland stability condition*. The set of Bridgeland stability conditions with respect to (Λ, v) is denoted by $\mathrm{Stab}_\Lambda(\mathcal{D})$. If $\Lambda = \mathcal{N}(\mathcal{D})$ and v the natural projection, then the set of stability conditions is denoted by $\mathrm{Stab}(\mathcal{D})$.

We now follow [Bri07] to define a topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$ and to prove that it is a complex manifold.

There is a generalized metric on the set of slicings $\mathrm{Slice}(\mathcal{D})$, i.e. a metric that does not need to be finite: given two slicings $\mathcal{P}$ and $\mathcal{Q}$, we define

$$d(\mathcal{P}, \mathcal{Q}) = \sup_{0 \neq E \in \mathcal{D}} \{|\phi^+_{\mathcal{P}}(E) - \phi^+_{\mathcal{Q}}(E)|, |\phi^-_{\mathcal{P}}(E) - \phi^-_{\mathcal{Q}}(E)|\} \in [0, \infty].$$

Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition. As Λ has finite rank, we define the following norm

$$||.||_\sigma \colon \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \to [0, \infty]$$

by sending a group homomorphism $W\colon \Lambda \to \mathbb{C}$ to

$$||W||_\sigma = \sup\{\frac{|W(E)|}{|Z(E)|} \colon E\ \sigma-\text{semistable}\}.$$

Note that the norm on $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$ depends on σ, however after fixing a connected component Σ and $\tau, \sigma \in \Sigma$ then $||.||_\sigma$ and $||.||_\tau$ are equivalent. See [Bri07, Lem. 6.2].

We define a topology on $\mathrm{Stab}_\Lambda(\mathcal{D})$ as the coarsest topology such that both forgetful maps

$$\begin{array}{rcl} \mathrm{Stab}_\Lambda(\mathcal{D}) & \to & \mathrm{Slice}(\mathcal{D}) \\ (Z, \mathcal{P}) & \to & \mathcal{P} \end{array} \qquad \begin{array}{rrcl} \mathcal{Z}\colon & \mathrm{Stab}_\Lambda(\mathcal{D}) & \to & \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C}) \\ & (Z, \mathcal{P}) & \to & Z \end{array}$$

are continuous.

In order to study $\mathrm{Stab}_\Lambda(\mathcal{D})$, we study the projection $\mathcal{Z}\colon \mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$.

Lemma 1.64. [Bri07, Lem. 6.4] *Suppose that $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{Q})$ are stability conditions on $\mathcal{D}$ with the same central charge Z. Suppose also that $d(\mathcal{P}, \mathcal{Q}) < 1$. The $\sigma = \tau$.*

Theorem 1.65. [Bri07, Thm. 7.1] *Let $\sigma = (Z, \mathcal{P})$ be a stability condition. If $1/8 > \epsilon > 0$, then for any group homomorphism $W : K(\mathcal{D}) \to \mathbb{C}$ with*

$$||W - Z||_\sigma < \sin(\pi\epsilon),$$

there exists a stability condition $\tau = (W, \mathcal{Q})$ on $\mathcal{D}$ with $d(\mathcal{P}, \mathcal{Q}) < \epsilon$.

Theorem 1.66. [Bay16, Thm. 1.2] and [BMS16, Prop. A.5] *Let Q be a quadratic form on $\Lambda \otimes \mathbb{R}$. Assume that the stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to Q. Then:*

1. *There is an open neighbourhood $\sigma \in U_\sigma \subseteq \mathrm{Stab}_\Lambda(\mathcal{D})$ such that $\mathcal{Z} : U_\sigma \to \mathrm{Hom}(\Lambda_\mathbb{R}, \mathbb{C})$ is a covering of the set of Z' such that Q is negative definite on $\mathrm{Ker}\, Z'$.*
2. *All stability conditions in U_σ satisfy the support property with respect to Q.*

Lemma 1.67. [Bri08, Lem. 4.5] and [BM11, Prop. B.4] *If σ is a Bridgeland stability condition and we fix $0 < \epsilon < \frac{1}{2}$, then for each $\phi \in \mathbb{R}$ the quasi-abelian subcategory $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length. As a consequence, Bridgeland stability conditions are locally-finite.*

The main point of the proof of the last lemma is that for a Bridgeland stability condition σ there exist discrete stability conditions arbitrarily close to σ.

Theorem 1.68. [Bri07] *The map $\mathcal{Z} : \mathrm{Stab}_\Lambda(\mathcal{D}) \to \mathrm{Hom}(\Lambda, \mathbb{C})$ is a local homeomorphism. Particularly, it implies that $\mathrm{Stab}_\Lambda(\mathcal{D})$ is a complex manifold of dimension $\mathrm{rk}(\Lambda)$.*

For a complete proof see [Bay11, Sec 5.5].

Remark 1.69. Let $\sigma \in \mathrm{Stab}_\Lambda(\mathcal{D})$ and $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, then σg is also a Bridgeland stability condition. Indeed, if σ satisfies the support property with respect to Q, then σg also satisfies the support property with respect to Q. If $\Phi \in \mathrm{Aut}_\Lambda(\mathcal{D})$, then $\Phi\sigma$ satisfies the support property with respect to $Q \circ \phi_*^{-1}$. By [Bri07, Lem. 8.2], the right action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and the left action of $\mathrm{Aut}_\Lambda(\mathcal{D})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$ commute.

Example 1.70. Let us consider

$$\begin{array}{rcl} \mathbb{C} & \hookrightarrow & \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \\ \alpha = a + bi & \mapsto & T_\alpha := e^a \begin{bmatrix} \cos(b) & -\sin(b) \\ \sin(b) & \cos(b) \end{bmatrix}, \end{array}$$

and $f_\alpha(x) = x + \frac{b}{\pi}$. We now check that T_α and f_α agree on S^1. If we consider polar coordinates we characterize the objects of $\mathbb{R}^2 - (0,0)/\mathbb{R}_{>0}$ by $(\cos(\theta), \sin(\theta))$ with $\theta \in (0, 2\pi]$

then

$$T_\alpha(\cos(\theta), \sin(\theta)) = e^a((\cos(\theta + b), \sin(\theta + b))),$$

which coincides with the restriction of f_α to S^1. As a consequence we have an action of the additive group $\mathbb{C} \subseteq \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\mathrm{Stab}_\Lambda(\mathcal{D})$. By [Oka06, Prop. 4.1] this action is holomorphic, free and contains the shifts. Moreover the quotient $\mathrm{Stab}_\Lambda(\mathcal{D})/\mathbb{C}$ is a complex manifold.

Wall and chamber structure

The support property is one of the key ingredients to ensure that we have a well-behaved wall and chamber decomposition. We obtain the following theorem presented in [Bri08, Sec. 9].

Proposition 1.71. [BM11, Prop. 3.3] *Let* $\mathrm{Stab}^\dagger(\mathcal{D}) \subseteq \mathrm{Stab}_\Lambda(\mathcal{D})$ *be a connected component of the space of Bridgeland stability conditions. Fix a primitive class* $\alpha \in \Lambda$ *and an arbitrary set* $S \subset \mathcal{D}$ *of objects of class* α*. Then, there exists a collection of walls* W^S_β *with* $\beta \in \Lambda$*, with the following properties:*

1. *Every wall* W^S_β *is a close submanifold with boundary of real codimension one.*
2. *The collection* W^S_β *is locally finite, i.e. every compact subset* $K \subseteq \mathrm{Stab}^\dagger(\mathcal{D})$ *intersects only a finite numer of walls.*
3. *For every stability condition* $\sigma = (Z, \mathcal{P}) \in W^S_\beta$*, there exists a phase* ϕ *and an inclusion* $F_\beta \hookrightarrow E_\alpha$ *in* $\mathcal{P}(\phi)$ *with* $v([F_\beta]) = \beta$ *and some* $E_\alpha \in S$*.*
4. *If* $C \subseteq \mathrm{Stab}^\dagger(\mathcal{D})$ *is a connected component of the complement of* $\cup_{\beta\in\Lambda} W^S_\beta$ *and* $\sigma_1, \sigma_2 \in C$ *then an object* $E_\alpha \in S$ *is* σ_1*-stable if and only if it is* σ_2*-stable.*

1.2.1 Examples

1.2.2 Bridgeland stability conditions on curves with $g(C) \geq 1$

We now study $\mathrm{Stab}(C)$, where C is a non-singular projective curve with $g(C) \geq 1$. Theorem 1.75 asserts that $\mathrm{Stab}(C) \cong \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. We prove several simple results in order to fix some facts and the notation that we use throughout the thesis.

To describe $\mathrm{Stab}(C) := \mathrm{Stab}(D^b(C))$, we follow closely [Bri07, Sec. 9] and [Mac07].

First of all, note that there is an isomorphism $K(D^b(C)) \cong \mathrm{Pic}(C) \oplus \mathbb{Z}$ given by the $(\det(E), \mathrm{rk}(E))$ for every $E \in \mathcal{D}^b(C)$, which implies that

$$\begin{array}{rcl} \mathcal{N}(D^b(C)) & \cong & \mathbb{Z}^2 \\ {[E]} & \mapsto & (\deg(E), \mathrm{rk}(E)). \end{array}$$

As before, let $v\colon K(D^b(C)) \to \mathcal{N}(D^b(C))$ be the natural projection.

Note that the classical slope stability induces a Bridgeland stability condition in this case. Indeed we define $\sigma_\mu := (Z_\mu, \mathrm{Coh}(C))$ with

$$\begin{array}{rcl} Z_\mu\colon \mathbb{Z}^2 & \to & \mathbb{C} \\ (d,r) & \mapsto & -d+ir. \end{array}$$

Remark 1.72. If X is a smooth projective variety with dimension ≥ 2, then the classical slope stability does not induce a Bridgeland stability condition. Moreover, there is no stability condition with heart $\mathrm{Coh}(X)$. See [Huy14, Cor. 3.3]. However, we can use slope stability to induce torsion pairs and new hearts by using tilting.

To describe $\mathrm{Stab}(C)$, we start by studying whether the skyscraper sheaves $\mathbb{C}(x)$, for a closed point $x \in C$, and a line bundle $\mathcal{L}$ are stable. We assume that they are not semistable and we study the last triangle of their HN-filtrations $E \to X \to A \to E[1]$, where X is either $\mathbb{C}(x)$ or $\mathcal{L}$, note that by Remark 1.54, we have that $\mathrm{Hom}^{\leq 0}(E,A) = 0$.

Lemma 1.73. (GKR for curve)[GKR04, Lem. 7.2] *Let*

$$E \to X \to A \to E[1]$$

be a distinguished triangle in $D^b(C)$ *with* $X \in \mathrm{Coh}(C)$ *and* $\mathrm{Hom}^{\leq 0}_{D^b(C)}(E,A) = 0$, *then* $E, A \in \mathrm{Coh}(C)$.

After applying directly the last lemma, we obtain the following result, which is proved in the first lines of [Mac07, Thm. 2.7].

Corollary 1.74. *If* $\sigma \in \mathrm{Stab}(C)$, *then* $\mathbb{C}(x)$ *and* $\mathcal{L}$ *are* σ*-stable for all points* $x \in C$ *and all line bundles* $\mathcal{L} \in \mathrm{Coh}(C)$.

Theorem 1.75. [Mac07, Thm. 2.7] *The action of* $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ *on* $\mathrm{Stab}(C)$ *is free and transitive, so that*

$$\mathrm{Stab}(C) \cong \widetilde{\mathrm{GL}}^+(2,\mathbb{R}).$$

Remark 1.76. Since $\mathcal{L}$ and $\mathbb{C}(x)$ are σ-stable and there are non-zero morphisms $\mathcal{L} \to \mathbb{C}(x)$ and $\mathbb{C}(x) \to \mathcal{L}[1]$, we obtain by Remark 1.54 that

$$\phi_\sigma(\mathcal{L}) < \phi_\sigma(\mathbb{C}(x)) < \phi_\sigma(\mathcal{L}) + 1.$$

For every $\sigma \in \mathrm{Stab}(C)$, there is $g = (T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $\sigma = \sigma_\mu g$. We now study this correspondence.

Iwasawa decomposition

We define the following matrices in $\mathrm{SL}(2,\mathbb{R})$

$$K_\phi = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi), \end{bmatrix},\ A_a = \begin{bmatrix} a & 0 \\ 0 & \frac{1}{a} \end{bmatrix} \text{ and } N_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix},$$

for $\phi \in [0, 2\pi)$, $x, a \in \mathbb{R}$ and $a > 0$.

Lemma 1.77. (Iwasawa decomposition)[HJJ$^+$07, Sec. 16.3] *For every $T \in \mathrm{GL}^+(2,\mathbb{R})$, there are real numbers $\phi \in [0, 2\pi)$, $k, a \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$, such that $T = kK_\phi A_a N_x$. Moreover, this representation is unique.*

Remark 1.78. From the decomposition above it follows that $\mathrm{GL}^+(2,\mathbb{R}) \cong \mathbb{C}^* \times \mathbb{H}$ and $\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \cong \mathbb{C} \times \mathbb{H}$. Moreover, we clearly obtain

$$1 \to \mathbb{Z} = \pi_1(\mathrm{GL}^+(2,\mathbb{R})) \to \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathrm{GL}^+(2,\mathbb{R}) \to 1.$$

Remark 1.79. Let us consider $\sigma_\mu = (Z_\mu, \mathrm{Coh}(C))$ and its corresponding slicing $\mathcal{P}_\mu$. By Example 1.70 we have that $\mathrm{Coh}^r(C) := \mathcal{P}_\mu(r, r+1]$ for $r \in \mathbb{R}$ is a heart of a bounded t-structure. Clearly all the hearts appearing in the stability conditions $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(C)$ are of this form. Indeed, if $\sigma = \sigma_\mu g$ with $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, then by the definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action, we obtain that

$$\mathcal{A} = \mathcal{P}_\mu(f(0), f(1) = f(0) + 1].$$

If $f(0) = n + \theta$, with $n \in \mathbb{Z}$ and $\theta \in [0, 1)$ then we also have $\mathcal{A} = \mathrm{Coh}^\theta(C)[n]$.

Let us consider $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ where $T = kK_\phi A_a N_x$ and $f(0) = n + \theta$, as above. We now relate ϕ and θ.

Lemma 1.80. *Let us consider $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ where $T = kK_\phi A_a N_x$ as above, then there is $m \in \mathbb{Z}$, such that $2m + \frac{\phi}{\pi} = f(0)$. Moreover, we have two cases*

$$n = 2m \text{ with } \theta = \frac{\phi}{\pi} \text{ or } n = 2m + 1 \text{ with } \theta = \frac{\phi}{\pi} - 1.$$

Proof. As f and T restricted to S^1 agree, in order to compute $f(0)$, we compute

$$T(1, 0) = ka(\cos(\phi), \sin(\phi)),$$

in $(\mathbb{R}^2 - \{(0, 0)\})/\mathbb{R}_{>0}$. It is just given by $(\cos(\phi), \sin(\phi))$ and it implies that

$f\big|_{[0,2)}(0) = \frac{\phi}{\pi} \in [0,2)$. Therefore, there is $m \in \mathbb{Z}$ with

$$f(0) = \frac{\phi}{\pi} + 2m.$$

$\square$

Corollary 1.81. *Let $\sigma = (Z, \mathcal{A}) \in \operatorname{Stab}(C)$. If $\sigma = \sigma_\mu g$ with $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, where $T = kK_\phi A_a N_x$ then $\mathcal{A} = \operatorname{Coh}^{\frac{\phi}{\pi}}(C)[2m]$.*

As a corollary, we obtain that, if we do not rotate σ the heart will not change up to a shift.

Corollary 1.82. *Let $\sigma = (Z, \mathcal{A}) \in \operatorname{Stab}(C)$. If $\sigma' = \sigma g$ with $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ where $T = \pm k A_a N_x$, then there is $l \in Z$ with $\mathcal{A}' = \mathcal{A}[l]$ where $\sigma' = (Z', \mathcal{A}')$.*

Proof. As $\phi = 0$ or $\phi = \pi$, then $f(0) = 2m$ or $f(0) = 2m+1$. Therefore, we have that $\mathcal{A}' = \mathcal{P}'(0,1] = \mathcal{P}(f(0), f(1)] = \mathcal{A}[l]$ with $l = 2m$ or $l = 2m+1$. $\square$

Remark 1.83. 1. Let $\sigma = \sigma_\mu g = (Z, \mathcal{A})$ with $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$. We obtain that $Z = T^{-1} Z_\mu$, i.e. $Z(d,r) = Ad + Br + i(Cr + Dd)$. Moreover, if $T = kK_\phi A_a N_x$, then $\arg(C + Di) = \phi$. Indeed, as

$$C = \cos(\phi)\frac{a}{k} \text{ and } D = \sin(\phi)\frac{a}{k},$$

then $\arg(C + Di) = \arg(\cos(\phi) + \sin(\phi)i) = \phi$.

2. Let $\theta \in (0,1)$. First note, that by the HN-property, if $\mathcal{T}_\theta = \mathcal{P}_\mu(\theta, 1]$ and $\mathcal{F}_\theta = \mathcal{P}_\mu(0, \theta]$, then $(\mathcal{T}_\theta, \mathcal{F}_\theta)$ is a torsion pair of $\operatorname{Coh}(C)$. By Proposition 1.10 it induces a heart of a bounded t-structure which is precisely $\operatorname{Coh}^\theta(C)$. Moreover, if we define $\alpha = -\cot(\pi\theta)$, then by Remark 1.48 we obtain that

$$\operatorname{Coh}^\theta(C) = \operatorname{Coh}(C)^\alpha_{\sigma_\mu}.$$

Indeed, it is enough to notice that $\mathcal{T}_\theta = \mathcal{T}^\alpha_{\sigma_\mu}$ and that $\mathcal{F}_\theta = \mathcal{F}^\alpha_{\sigma_\mu}$, because

$$\mu(E) > \alpha \text{ if and only if } \phi(E) > \theta.$$

Let $\sigma = \sigma_\mu g = (Z, \mathcal{A})$ with $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $\mathcal{P}$ its slicing. By Corollary 1.74, the objects $\mathbb{C}(x)$ and $\mathcal{L}$ are σ-stable. We now study their phases. Note that if we have two different points $x, y \in C$, with $x \neq y$, then $\phi_{\sigma_\mu}(\mathbb{C}(x)) = \phi_{\sigma_\mu}(\mathbb{C}(y)) = 1$, by the definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action, it implies that for every $\sigma \in \operatorname{Stab}(C)$ we obtain

$$\phi_0 := \phi_\sigma(\mathbb{C}(x)) = \phi_\sigma(\mathbb{C}(y)).$$

By definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action, we obtain that $\mathcal{P}(\phi_0) = \mathcal{P}_\mu(f(\phi_0)) = \mathcal{P}_\mu(1)$, i.e. $f(\phi_0) = 1$. Analogously, if $\phi_1 = \phi_\sigma(\mathcal{O}_C)$, then $f(\phi_1) = \frac{1}{2}$.

Remark 1.84. We have that

$$-n < \phi_0 = f^{-1}(1) \leq -n+1 \text{ if and only if } f(0) = n + \theta,$$

with $n \in \mathbb{Z}$ and $\theta \in [0,1)$. Indeed, after applying f to the inequality $-n < \phi_0 = f^{-1}(1) \leq -n+1$, we obtain $-n + f(0) < 1 \leq f(0) - n + 1$, it follows that $n \leq f(0) < n+1$. As a consequence, if $f(0) = n+\theta$ it implies that $\mathbb{C}(x)[n] \in \mathcal{A}$.

The values $Z([\mathbb{C}(x)]), Z([\mathcal{O}_C]) \in \mathbb{C}$ and $\phi_0, \phi_1 \in \mathbb{R}$ describe σ completely.

Lemma 1.85. *There is a homeomorphism*

$$\begin{aligned} \rho\colon \mathrm{Stab}(C) &\rightarrow \{(m_0, m_1, \phi_0, \phi_1) \in \mathbb{R}^4 \mid \phi_1 < \phi_0 < \phi_1 + 1, \text{ and } m_0, m_1 > 0\} =: \mathcal{M} \\ \sigma = (Z, \mathcal{A}) &\mapsto (m_0, m_1, \phi_0, \phi_1), \end{aligned}$$

where $m_0 = |Z([\mathbb{C}(x)])|, m_1 = |Z([\mathcal{O}_C])|$.

Proof. By the stability of $\mathbb{C}(x)$ and $\mathcal{O}_C$, we obtain $\phi_1 < \phi_0 < \phi_1 + 1$ and by definition $m_0, m_1 > 0$. The map ρ is clearly continuous because it is defined in terms of the slicing and Z. Let $\sigma = \sigma_\mu(T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. To characterize f it is enough to give $f\colon \mathbb{R}/2\mathbb{Z} \to \mathbb{R}/2\mathbb{Z}$ and n, where n is the integer part of $f(0)$. By Remark 1.84 we have that

$$-n < \phi_0 = f^{-1}(1) \leq -n+1, \text{ if and only if } n \leq f(0) < n+1.$$

We now prove that we have a bijection. Note that T^{-1} is completely characterized by its image. If $m_0 e^{\pi\phi_0 i} = A + Di$ and $m_1 e^{\pi\phi_1 i} = B + Ci$, then $T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$ and

$$m_0^2 = A^2 + D^2 \text{ and } m_1^2 = B^2 + C^2.$$

Moreover, the function $f\colon \mathbb{R}/2\mathbb{Z} \to \mathbb{R}/2\mathbb{Z}$ is given by sending t to

$$\frac{\arg(m_1(\sin(\phi_1\pi)\cos(t\pi) - \cos(\phi_1\pi)\sin(t\pi)) + i m_0(\sin(\phi_0\pi)\cos(t\pi) - \cos(\phi_0\pi)\sin(t\pi)))}{\pi}$$

for $t \in \mathbb{R}/2\mathbb{Z}$ and n the integer part of $1 - \phi_0$. Consequently, we characterized completely (T, f) with the information in the tuple $(m_0, m_1, \phi_0, \phi_1)$ satisfying the conditions above, and as a consequence we also characterized σ.

We just need to prove that ρ is open. We define

$$\begin{aligned} \nu\colon \mathcal{M} &\to \mathrm{GL}^+(2,\mathbb{R}) \\ (m_0,m_1,\phi_0,\phi_1) &\mapsto \begin{bmatrix} -m_0\cos(\phi_0\pi) & m_1\cos(\phi_1\pi) \\ -m_0\sin(\phi_0\pi) & m_1\sin(\phi_1\pi) \end{bmatrix}^{-1}. \end{aligned}$$

Note that $\frac{1}{\det(\nu(m_0,m_1,\phi_0,\phi_1))} = m_0m_1\sin((\phi_0-\phi_1)\pi) > 0$, as $0 < \phi_0 - \phi_1 < 1$. Moreover, ν is clearly a covering and

$$\nu \circ \rho = \mathcal{Z},$$

where $\mathcal{Z}\colon \mathrm{Stab}(C) \cong \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathrm{GL}^+(2,\mathbb{R})$ is the universal covering. It implies that ρ is a covering and therefore it is also open. □

For the rest of the thesis if $\sigma \in \mathrm{Stab}(C)$, we mix the different forms that we use to represent σ. Namely as an object $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, as $\rho(\sigma)$ and of course as $(Z,\mathcal{A})$.

Remark 1.86. In [BK06], Kreussler and Burban generalized the description of the stability manifold to irreducible singular curves of arithmetic genus one.

1.2.3 Bridgeland stability conditions induced by exceptional objects

We now review the construction of pre-stability conditions on triangulated categories generated by exceptional collections done in [Mac07]. We recommend [Bon90] for the general theorems about exceptional collections.

We assume now that $\mathcal{D}$ has a complete exceptional collection $\mathcal{E}$ and we study pre-stability conditions on $\mathcal{D}$.

Lemma 1.87. [Mac07, Lem. 3.14] *Let $\mathcal{E} = (E_0, E_1, \dots, E_n)$ be a complete Ext-exceptional collection on $\mathcal{D}$. Then $\langle E_0, \dots, E_n\rangle$, the extension-closed subcategory generated by $\mathcal{E}$, is the heart of a bounded t-structure.*

Remark 1.88. By [Mac07, Cor. 3.15] we also obtain, that $\langle E_i, E_j\rangle$ is a full abelian subcategory of $\mathcal{D}$.

Lemma 1.89. [Mac07, Lem. 3.16] *Let $\mathcal{E} = (E_0, E_1, \dots, E_n)$ be a complete Ext-exceptional collection on $\mathcal{D}$ and let $\sigma = (Z,\mathcal{A}) \in \mathcal{D}$. Assume that $E_0, \dots, E_n \in \mathcal{A}$.*

Then $\mathcal{A} = \langle E_0, \dots, E_n\rangle$ and E_i is stable for each $i = 0, \dots, n$.

Proposition 1.90. [Mac07, Lem. 3.17] *Let $(E_0, \dots, E_n)$ be a complete Ext-exceptional collection on $\mathcal{D}$ and let $\sigma = (Z,\mathcal{A})$ be a numerical pre-stability condition on $\mathcal{D}$ such that $\langle E_0, \dots, E_n\rangle = \mathcal{A}$. Fix $j > i$. Then σ induces a stability condition σ_{ij} on the full triangulated subcategory D_{ij} generated by E_i, E_j, in such way that every semistable object in σ_{ij} with phase ϕ corresponds to a semistable object in σ with the same phase.*

Let us assume that $\mathcal{D} = (E_0, \dots, E_n)$. If there is $p \in \mathbb{Z}^{n+1}$, such that $\mathcal{A}_p = (E_0[p_0], \dots, E_n[p_n])$ is an Ext-exceptional collection, then $\mathcal{A}_p$ is the heart of a bounded t-structure. We now define

$$Z_p \colon \mathcal{N}(\mathcal{A}_p) \cong \mathbb{Z}^{n+1} \to \mathbb{C},$$

as follows: since $\mathcal{N}(\mathcal{A}_p)$ is generated by the classes $[E_i[p_i]]$, we choose $z_i \in \overline{\mathbb{H}}$ and we extend Z_p to $\mathbb{Z}^{n+1}$. Note that the pair $\sigma_p = (Z_p, \mathcal{A}_p)$ is a locally finite pre-stability condition, as $\mathcal{A}_p$ is an abelian category of finite length.

Definition 1.91. Let $\mathcal{E}$ be a complete, Ext-exceptional collection. We define

$$\mathbb{H}^{\mathcal{E}} = \{\sigma = (Z, \mathcal{A}) \mid \text{pre-stability conditions with } \langle \mathcal{E} \rangle = \mathcal{A}\}.$$

Definition 1.92. Let $\mathcal{E}$ be a complete, Ext-exceptional collection. We define $\Theta'_{\mathcal{E}}$ as the smallest set closed under the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ containing $\mathbb{H}^{\mathcal{E}}$.

Definition 1.93. We define

$$\Theta_{\mathcal{E}} = \bigcup_{\{p \in \mathbb{Z}^{n+1} \mid \mathcal{E}[p] \text{ is Ext }\}} \Theta'_{\mathcal{E}[p]},$$

where $\mathcal{E}[p] = (E_0[p_0], E_1[p_1], \dots, E_n[p_n])$.

Lemma 1.94. [Mac07, Lem. 3.19] *The set $\Theta_{\mathcal{E}}$ is an open, connected and simply connected $(n+1)$-dimensional complex submanifold.*

Remark 1.95. In Section 2.2, we present analogous statements proved in [CP10] that use semiorthogonal decompositions instead of exceptional objects.

1.2.4 Bridgeland stability conditions on the n-Kronecker quivers

Definition 1.96. A *quiver* $Q = (V, A, s, t)$ is a quadruple consisting of two sets: V the set of vertices and A the set of arrows and two maps $s, t \colon A \to V$ which associate to each arrow $\alpha \in A$ its source $s(\alpha) \in V$ and its target $t(\alpha) \in V$. We denote it just by Q.

Let $K(n)$ be the n-Kronecker quiver $1 \underset{\alpha_n}{\overset{\alpha_1}{\vdots}} 2$ given by two vertices $V = \{1, 2\}$ and n-arrows $A = \{\alpha_i \mid i = 1, \dots, n\}$ from 1 to 2.

To study quivers and quiver representations, we recommend [ASS06].

Let $Q_n \coloneqq \mathrm{Rep}(K(n))$ be the category of representations of the n-Kronecker quiver, with $n \in \mathbb{N}_{\geq 1}$.

The aim of this subsection is to study the Bridgeland stability manifold of

$\mathrm{Stab}(Q_n) \coloneqq \mathrm{Stab}(D^b(Q_n))$. We follow closely [Mac07]. For the case $n = 2$, we recommend [Oka06] and for a more general picture [DK16].

Let S_1 and S_2 be the two simple representations, with $\mathbb{C}$ at the source and $\mathbb{C}$ at the target respectively.

Note that S_1, S_2 are exceptional objects and that (S_1, S_2) gives us an Ext-exceptional pair. See [ASS06, Sec. 3.2]. We now follow [Rin94] to show that we can get all exceptional objects in $\mathcal{D}^b(Q_n)$ via the action of the braid group $B_2 \cong \mathbb{Z}$, i.e. via mutations.

First note that if $E \in D^b(Q_n)$ is exceptional, then $E \in Q_n[m]$, for some $m \in \mathbb{Z}$. Indeed, as Q_n is hereditary, then $E \cong \sum H^i(E)[-i]$ and since $\mathrm{Hom}(E, E) = \mathbb{C}$, it implies that there is just one $m \in \mathbb{Z}$ that satisfies $H^{-m}(E) \neq 0$ and then $E \in Q_n[m]$. Therefore, we just need to study exceptional representations of Q_n.

By definition, any simple representation of Q_n is an exceptional objects of $D^b(Q_n)$ and any exceptional representation is indecomposable. If $n = 1$, then Q_1 has only finitely many isomorphism classes of indecomposable representation. As a consequence, all the indecomposable representations of Q_1 are exceptional. If $n \geq 2$, then only the preinjective and preprojective indecomposable representations are exceptional. See [Rin94, Sec. 2].

Lemma 1.97. [Rin94, Sec. 4] *If E is an exceptional representation in Q_n, there are exceptional representations E^- and E^+ such that (E^-, E) and (E, E^+) are complete exceptional pairs.*

In [CB93] and [Rin94], the authors prove that the action of the braid group on the set of complete exceptional collections of a quiver Q is transitive. In our case, we obtain the theorem.

Theorem 1.98. [CB93, Thm. 10] *The action of B_2 is transitive on the set of complete exceptional sequences.*

We define

$$S_{k+2} \coloneqq R_{S_{k+1}} S_k \text{ for } k \geq 0$$

and

$$S_{k-1} \coloneqq L_{S_k} S_{k+1} \text{ for } k \leq 1.$$

Corollary 1.99. *The set $\{S_i\}_{i \in \mathbb{Z}}$ gives us all exceptional objects in $D^b(Q_n)$ up to shifts.*

Proof. Let E be an exceptional representation in Q_n. Then, there is an exceptional representation E^-, such that (E^-, E) is a complete exceptional collection. As the action of B_2 is transitive, then there is $s \in B_2$ such that $s(S_1, S_2) = (E^-, E)$. It implies that $E \cong S_i[m]$, for some $i, m \in \mathbb{Z}$. □

Remark 1.100. See [Dim15, Sec. 2.C.1]. In general the action of the braid group B_n is not free. However, the action of B_2 on the set of complete exceptional collections of Q_n with $n \geq 2$ is in fact free. Therefore, for $n \geq 2$ the set of S_i with $i \in \mathbb{Z}$, give as complete set of exceptional representations. In the case $n = 1$, the action is clearly not free as we just have three indecomposable representations, namely

$$S_1 = \mathbb{C} \to 0 \ , \ S_2 = 0 \to \mathbb{C} \text{ and } S_3 = \mathbb{C} \to \mathbb{C}.$$

Lemma 1.101. [Mac07, Lem. 4.2] *Let $\sigma \in \mathrm{Stab}(Q_n)$, then there is an exceptional pair of the form (S_i, S_{i+1}) where S_i, S_{i+1} are σ-stable objects.*

Remark 1.102. The proof of Lemma 1.101 goes along the lines of Lemma 1.73, i.e. we assume that an exceptional object is not semistable and we study the last triangle of the HN-filtration and we use the fact that Q_n is hereditary.

We now define

$$\mathcal{C}_i := \{\sigma \in \mathrm{Stab}(Q_n) \mid S_i, S_{i+1} \ \sigma- \text{ stable } \},$$

for $i \in \mathbb{Z}$, if $n \geq 2$ or $i = 1, 2$ and 3 if $n = 1$. In this case $\mathcal{C}_i$ is precisely $\Theta_{\mathcal{E}}$, where $\mathcal{E} = (S_i, S_{i+1})$. By Lemma 1.94, the set $\mathcal{C}_i$ is a connected, simply connected two dimensional manifold.

We now shortly focus on the case $n = 1$. The Serre functor of the triangulated category $D^b(Q_1)$ is defined as follows

$$\mathcal{S}_{Q_1}(E_1 \xrightarrow{\varphi} E_2) = E_2 \xrightarrow{\psi} C(\varphi),$$

where $E_1 \xrightarrow{\varphi} E_2 \xrightarrow{\psi} C(\varphi) \to E[1]$ is a distinguished triangle and satisfies that $\mathcal{S}^3 = [1]$. See Lemma 2.57 for the generalization and the proof of this result.

Note that as $\mathcal{S}_{Q_1}$ acts as an autoequivalence, we obtain precisely that

$$\mathcal{S}_{Q_1}(\mathcal{C}_1) = \mathcal{C}_2 \ , \ \mathcal{S}_{Q_1}(\mathcal{C}_2) = \mathcal{C}_3 \text{ and } \mathcal{S}_{Q_1}(\mathcal{C}_3) = \mathcal{C}_1.$$

Therefore it is enough to study one of the $\mathcal{C}_i$. Note that the exceptional pair $(S_1[n], S_2)$ is Ext if and only if $n \geq 0$. Indeed, we have that $\mathrm{Hom}^i(S_1[n], S_2) = \mathrm{Hom}(S_1, S_2[i-n])$, which is different from zero only if $i - n = 1$, then $i = 1 + n \geq 0$. Therefore, we obtain $\mathrm{Hom}^{\leq 0}(S_1[n], S_2) = 0$. Let us consider $\sigma \in \mathcal{C}_1$, then S_1 and S_2 are σ-stable. As there is a non-zero map $S_1 \to S_2[1]$, it implies that $\phi(S_1) < \phi(S_2) + 1$.

We define the following map

$$\begin{array}{rcl} \rho \colon \mathcal{C}_1 & \to & \{(m_1, m_2, \phi_1, \phi_2) \in \mathbb{R}^4 \mid m_1, m_2 > 0,\ \phi_1 < \phi_2 + 1\} \\ \sigma & \mapsto & (|Z([S_1]|, |Z(S_2)|, \phi(S_1), \phi(S_2)). \end{array}$$

Lemma 1.103. [Mac07, Lem. 3.19] *The map ρ is a homeomorphism.*

Proof. It is clearly continuous. We now show that it is a bijection. Let $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathcal{C}_1$, for $i = 1, 2$, with $\rho(\sigma_1) = \rho(\sigma_2)$. As $\mathcal{N}(Q_1)$ is generated by $[S_1]$ and $[S_2]$, then Z is completely determined by $\rho(\sigma_i)$, as a consequence we obtain $Z_1 = Z_2$.

Moreover, as S_1 and S_2 are σ-stable there are $n_1, n_2 \in \mathbb{Z}$ with $S_1[n_1] \in \mathcal{A}_1$ and $S_2[n_2] \in \mathcal{A}_2$. Note that n_1 and n_2 are the same for both stability conditions as they are characterized by ϕ_1 are ϕ_2. Precisely, an object E satisfies that $E[n] \in \mathcal{A}$ if and only if $-n < \phi(E) \leq -n+1$.

If $\phi_1 \leq \phi_2$, as $S_2 \in \mathcal{A}_i[n_2]$, then $\phi_2 \leq -n_2+1$, which implies that $-n_1 < \phi_1 < -n_2+1$ and as a consequence, we obtain $n_2 \leq n_1$. By Lemma 1.87 the set $\langle S_1[n_1], S_2[n_2]\rangle \subseteq \mathcal{A}_i$ is the heart of a bounded t-structure and it implies $\mathcal{A}_1 = \mathcal{A}_2 = \langle S_1[n_1], S_2[n_2]\rangle$.

In the case $\phi_2 < \phi_1 < \phi_2 + 1$, without losing generality, we assume $n_2 = 0$, then we obtain $n_1 = -1$. As the last triangle of the HN-filtration of S_3, is precisely

$$S_2 \to S_3 \to S_1 \to S_2[1],$$

we obtain that if S_3 is not σ-stable then $\phi_2 \geq \phi_1$. Then, in this case S_3 is σ-stable and there is $n_3 \in \mathbb{Z}$ with $S_3[n_3] \in \mathcal{A}_i$, for $i = 1, 2$.

Since we have non-zero morphisms $S_2 \to S_3 \to S_1$, it implies that $\phi_2 < \phi(S_3) < \phi_1$ and therefore $n_3 = 0$ or $n_3 = -1$. If $S_3 \in \mathcal{A}_i$, then $(S_3, S_1[-1])$ is an Ext-exceptional pair and $\langle S_3, S_1[-1]\rangle \subseteq \mathcal{A}_i$, which implies $\mathcal{A}_1 = \mathcal{A}_2$, or if $S_3[-1] \in \mathcal{A}_i$, then $(S_2, S_3[-1])$ is an Ext-exceptional pair and we argue as above.

We now prove the surjectivity. We have a 4-tuple $(m_1, m_2, \phi_1, \phi_2)$ and we want to construct a stability condition. We first give the heart. If $\phi_1 \leq \phi_2$, then if $-n_1, -n_2$ are the integer parts of ϕ_1, ϕ_2 respectively, we have that $n_1 \geq n_2$, and we define the heart as $\langle S_1[n_1], S_2[n_2]\rangle$ and $Z([S_1]) = Z((1,0)) = m_1 e^{\phi_1 \pi}$ and $Z([S_2]) = Z(0,1) = m_2 e^{\phi_2 \pi}$, which is enough to define $Z\colon \mathbb{Z} \to \mathbb{C}$. Note that $Z([S_i[n_i]]) \in \overline{\mathbb{H}}$ as $0 < \phi_1 + n_i \leq 1$.

In the case $\phi_2 < \phi_1 < \phi_2 + 1$, we assume that $n_2 = 0$, and it implies that $n_1 = -1$. We define Z as above. We now define the heart. If $\Im(Z(1,1)) > 0$ or if $\Im(Z(1,1)) = 0$ and $\Re(Z(1,1)) < 0$, we define the heart as $\langle S_3, S_1[-1]\rangle$. If $\Im(Z(1,1)) < 0$ or $\Im(Z(1,1)) = 0$ with $\Re(Z(1,1)) > 0$, then we define the heart $\langle S_2, S_3[-1]\rangle$. By the same argument of Lemma 1.85 ρ is also open. □

Proposition 1.104. [Mac07, Thm. 4.5], [DK16, Thm 1.5], [Oka06] *We have that* $\mathrm{Stab}(Q_n) \cong \cup_{i\in\mathbb{Z}} \mathcal{C}_i$ *and moreover* $\mathrm{Stab}(Q_n)$ *is a connected, contractible* 2*-dimensional complex manifold.*

Remark 1.105. In [Mac07, Thm. 4.5], Macrì showed that $\mathrm{Stab}(Q_n)$ is simply connected. In [Qiu11, Thm. 7.5.1], Qiu proved that $\mathrm{Stab}(Q_1) \cong \mathbb{C}^2$. In [Oka06, Thm. 1.1], Okada showed that for $n = 2$, we have that $\mathrm{Stab}(Q_2) \cong \mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$. In [DK16, Thm. 1.5], Dimitrov proved that $\mathrm{Stab}(Q_n) \cong \mathbb{C} \times \mathbb{H}$ for $n \geq 3$. See also [Shi11].

Bridgeland stability conditions on $D^b(\mathbb{P}^1)$.

We now study $\mathrm{Stab}(D^b(\mathbb{P}^1))$. We follow [Oka06]. Note that $(\mathcal{O}, \mathcal{O}(1))$ is a complete, strong exceptional collection. We define $A = \mathrm{End}(\mathcal{O} \oplus \mathcal{O}(1), \mathcal{O} \oplus \mathcal{O}(1)) = \begin{bmatrix} \mathbb{C} & 0 \\ \mathbb{C}^2 & \mathbb{C} \end{bmatrix}$, which is precisely the path algebra of the 2-Kronecker quiver. We study the autoequivalence induced by Theorem 1.30

$$\begin{aligned} F \coloneqq \mathrm{RHom}(A, -)\colon D^b(\mathbb{P}^1) &\to D^b(Q_2) \\ E &\mapsto \mathrm{RHom}(\mathcal{O} \oplus \mathcal{O}(1), E) \end{aligned}$$

and whether the skyscraper sheaves $\mathbb{C}(x)$ for $x \in \mathbb{P}^1$, are σ-stable for $\sigma \in \mathrm{Stab}(\mathbb{P}^1)$. Note that $F(\mathcal{O}(1)) = \mathbb{C} \rightrightarrows \mathbb{C}^2$ and $F(\mathcal{O}) = 0 \rightrightarrows \mathbb{C}$ are the projective indecomposables of $\mathrm{Rep}(Q_2)$. We also have that $F(\mathcal{O}(-1)[1]) = \mathbb{C} \rightrightarrows 0$. Therefore, the Ext-exceptional collection given by (S_1, S_2) is precisely $(\mathcal{O}(-1)[1], \mathcal{O})$. By Lemma 1.87, we have that $\langle \mathcal{O}(-1)[1], \mathcal{O} \rangle$ is the heart of a bounded t-structure. It is called the *Kronecker heart*, as its image under F is precisely $\mathrm{Rep}(Q_2)$. Moreover for $k \in \mathbb{Z}$, we have that

$$F(\mathcal{O}(k)[1]) = \mathbb{C}^{-k} \rightrightarrows \mathbb{C}^{-k-1} \text{ for } k \leq -1$$

and

$$F(\mathcal{O}(k)) = \mathbb{C}^{k} \rightrightarrows \mathbb{C}^{k+1} \text{ for } k \geq 1,$$

which are precisely the exceptional representations S_k. See [DW17, Exm. 11.2.4]

Remark 1.106. By Lemma 1.101, for every $\sigma \in \mathrm{Stab}(\mathbb{P}^1)$ there is k, such that $\mathcal{O}(k), \mathcal{O}(k+1)$ are σ-stable.

Remark 1.107. Note that $\sigma_\mu = (\mathbb{Z}_\mu, \mathrm{Coh}(\mathbb{P}^1)) \in \mathrm{Stab}(\mathbb{P}^1)$ satisfies that $\mathbb{C}(x)$ and $\mathcal{O}(k)$ are stable for all points $x \in \mathbb{P}^1$ and all $k \in Z$. Moreover, the heart $\mathrm{Coh}(\mathbb{P}^1)$ is not given by any Ext-exceptional pair.

We now study skyscraper sheaves $\mathbb{C}(x)$ for $x \in \mathbb{P}^1$ and under which conditions they are stable for $\sigma \in \mathrm{Stab}(\mathbb{P}^1)$. As in Lemma 1.73, we assume that $\mathbb{C}(x)$ is not σ-stable, for a $x \in \mathbb{P}^1$ and we study the last triangle of its HN-filtration.

Lemma 1.108. [GKR04, Lem. 6.3], [Oka06, Lem. 3.1] *If $\mathbb{C}(x)$ is not σ-semistable, then there is $k \in \mathbb{Z}$ such that its HN-filtration is given by*

$$\mathcal{O}(k+1) \to \mathbb{C}(x) \to \mathcal{O}(k)[1] \to \mathcal{O}(k+1)$$

and $\mathcal{O}(k)$, $\mathcal{O}(k+1)$ are σ-stable and $\phi(\mathcal{O}(k+1)) > \phi(\mathcal{O}(k)) + 1$

Lemma 1.109. [GKR04, Lem. 6.3], [Oka06, Lem. 3.1] *If $\mathcal{O}(n)$ is not σ-semistable for some $n \in \mathbb{Z}$, then there is $k \neq n, n-1 \in \mathbb{Z}$ such that its HN-filtration is given by*

$$\mathcal{O}(k+1)^{\oplus n-k} \to \mathcal{O}(n) \to \mathcal{O}(k)^{\oplus n-k-1}[1] \text{ if } n > k+1,$$

or

$$\mathcal{O}(k+1)^{\oplus k-n} \to \mathcal{O}(n) \to \mathcal{O}(k)^{\oplus k-n+1}[1] \text{ if } n < k$$

and $\mathcal{O}(k), \mathcal{O}(k+1)$ are σ-stable and $\phi(\mathcal{O}(k+1)) > \phi(\mathcal{O}(k)) + 1$.

By Proposition 1.104, we have already that $\mathrm{Stab}(\mathbb{P}^1) = \cup_k \mathcal{C}_k$. If

$$\sigma = (Z, \mathcal{A}) \in \mathcal{C}_k = \{\sigma \in \mathrm{Stab}(\mathbb{P}^1) \mid \mathcal{O}(k), \mathcal{O}(k+1) \text{ stable}\}$$

with $\phi_k := \phi(\mathcal{O}(k))$ and $\phi_{k+1} := \phi(\mathcal{O}(k+1))$, then as there is a non-zero morphism $\mathcal{O}(k) \to \mathcal{O}(k+1)$, we obtain that $\phi_k < \phi_{k+1}$ and moreover these values characterize σ as follows:

Lemma 1.110. *1. If $\phi_k + 1 < \phi_{k+1}$, then neither $\mathbb{C}(x)$ nor $\mathcal{O}(n)$ are σ-stable, for $n \neq k, k+1 \in \mathbb{Z}$. Moreover, there are $p, q \in \mathbb{Z}$ with $q \geq 2$ such that*

$$\mathcal{A}[p] = \langle \mathcal{O}(k)[q], \mathcal{O}(k+1) \rangle.$$

2. If $\phi_k + 1 > \phi_{k+1} > \phi_k$, then $\mathbb{C}(x)$ and $\mathcal{O}(n)$ are σ-stable for every $n \in \mathbb{Z}$. Moreover, there is $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\sigma = \sigma_\mu g$.

After studying the $\mathbb{C}$-action defined in on $\mathrm{Stab}(\mathbb{P}^1)$, we obtain the following theorem.

Theorem 1.111. [Oka06, Thm. 1.1] $\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2$.

Remark 1.112. Note that by Lemma 1.110, under the assumption that $\mathbb{C}(x)$ is σ-stable for all $x \in \mathbb{P}^1$, there is $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that $\sigma = \sigma_\mu g$ as in the case of a curve C with $g(C) \geq 1$.

2 Bridgeland stability conditions on $\mathcal{T}_{\mathcal{A},n}$

Let $\mathcal{A}$ be a Noetherian abelian category, such that $D^b(\mathcal{A})$ is $\mathbb{C}$-linear of finite type. We consider the category $Q_{\mathcal{A},n}$ of representations of the n-Kronecker quiver over $\mathcal{A}$ with $n \in \mathbb{Z}_{>0}$, i.e. the objects of $Q_{\mathcal{A},n}$ are tuples $(E_1, E_2, (f_j)_{0<j\leq n})$ where $E_i \in \mathcal{A}$, $i = 1, 2$ and $f_j \in \mathrm{Hom}_{\mathcal{A}}(E_1, E_2)$, $j = 1, \ldots, n$. A morphism between $(E_1, E_2, (f_j)_{0<j\leq n})$ and $(F_1, F_2, (t_j)_{0<j\leq n})$ is given by pair $g_1 \in \mathrm{Hom}(E_1, F_1)$ and $g_2 \in \mathrm{Hom}(E_2, F_2)$, such that $g_2 \circ f_j = t_j \circ g_1$ for all $j = 1, 2, \ldots, n$.

Remark 2.1. The category $Q_{\mathcal{A},n}$ is abelian. See [JMP10, Thm. 1].

Notation 2.2. If $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$ is the category of vector spaces over $\mathbb{C}$, we denote $Q_{\mathcal{A},n}$ by Q_n.

The aim of this chapter is to study $\mathrm{Stab}(D^b(Q_{\mathcal{A},n}))$. If $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, then Q_n is the category of representations of the n-Kronecker quiver and its stability manifold is completely described in [Mac07], as explained in Subsection 1.2.4. If $\mathcal{A}$ is $\mathrm{Coh}(X)$, where X is a smooth projective variety and $n = 1$, we obtain the category $\mathrm{TCoh}(X)$ of holomorphic triples over X. In Chapter 3, we give a precise description of the stability manifold $\mathrm{Stab}(\mathcal{D}^b(\mathrm{TCoh}(X)))$ for a smooth projective curve of genus ≥ 1. In this case, the study of $\mathcal{D}^b(\mathrm{TCoh}(X))$ will be appearing in [MRRHR19].

2.1 The triangulated category $\mathcal{T}_{\mathcal{A},n}$

The object under study in this section is $\mathcal{T}_{\mathcal{A},n} \coloneqq D^b(Q_{\mathcal{A},n})$. We construct different semiorthogonal decompositions for $\mathcal{T}_{\mathcal{A},n}$. We prove the existence of the Serre functor and we give an explicit construction of it in the case $n = 1$.

Our idea is to generalize the construction for quivers in [Mac07], but instead of using exceptional objects we use semiorthogonal decompositions. Often we separate the case $n = 1$ and $n \geq 2$, as Q_1 has finitely many exceptional representations and Q_n, with $n \geq 2$, has infinitely many ones.

Semiorthogonal decompositions of $\mathcal{T}_{\mathcal{A},n}$

Let S_k be an exceptional representation of the n-Kronecker quiver for $n \geq 2$ and $k \in \mathbb{Z}$ as in Subsection 1.2.4. The representation S_k generates a triangulated subcategory $\langle S_k \rangle \cong \mathcal{D}^b(\mathrm{Vect}_{\mathbb{C}})$ of $D^b(Q_n)$. It also induces a semiorthogonal decomposition $\langle \langle S_k \rangle, \langle S_{k+1} \rangle \rangle$ of $D^b(Q_n)$. We now define the analogous triangulated subcategories D_k of $\mathcal{T}_{\mathcal{A},n}$, which will also induce semiorthogonal decompositions $\langle D_k, D_{k+1} \rangle$ of $\mathcal{T}_{\mathcal{A},n}$.

Consider the following functors:

$$\begin{array}{rcl} i_1 \colon \mathcal{A} & \hookrightarrow & Q_{\mathcal{A},n} \\ E & \mapsto & (E,0,0), \end{array}$$

$$\begin{array}{rcl} j_2 \colon \mathcal{A} & \hookrightarrow & Q_{\mathcal{A},n} \\ E & \mapsto & (0,E,0). \end{array}$$

The definition of the functor at the level of morphisms is the expected one. Since the functors i_1 and j_2 are exact, we can extend them to the level of derived categories. We obtain the following functors:

$$\begin{array}{rcl} i_{1*} \colon D^b(\mathcal{A}) & \to & \mathcal{T}_{\mathcal{A},n} \\ E & \mapsto & (E,0,0), \end{array}$$

$$\begin{array}{rcl} j_{2*} \colon D^b(\mathcal{A}) & \to & \mathcal{T}_{\mathcal{A},n} \\ E & \mapsto & (0,E,0). \end{array}$$

Lemma 2.3. [BDG17a, Thm. 2.4], [BDG17b, Cor. 2.4] *Let $F \colon \mathcal{A} \to \mathcal{B}$ be an exact functor between abelian categories, $G \colon \mathcal{B} \to \mathcal{A}$ be the right (left) adjoint such that $F \circ G \to \mathrm{Id}_{\mathcal{B}}$ ($\mathrm{Id}_{\mathcal{B}} \to F \circ G$) is an isomorphism. Then, if DF is the derived functor of F and G has a right (left) derived functor RG (LG), we obtain that RG (LG) is the right (left) adjoint of DF.*

Lemma 2.4. *The functors i_{1*} and j_{2*} are exact embeddings of $D^b(\mathcal{A})$ into $\mathcal{T}_{\mathcal{A},n}$.*

Proof. We first show that i_{1*} is an exact embedding. Note that the functor i_1 is clearly an exact fully-faithful functor. Moreover, we can define the functor

$$\begin{array}{rcl} i_1^* \colon Q_{\mathcal{A},n} & \to & \mathcal{A} \\ (E_1, E_2, f_l) & \mapsto & E_1, \end{array}$$

which at the level of morphisms is defined as expected. It follows

$$\mathrm{Hom}_{Q_{\mathcal{A},n}}((F_1, F_2, f_l), i_1(E_1)) \cong \mathrm{Hom}_{Q_{\mathcal{A},n}}((F_1, F_2, f_l), (E_1, 0, 0)) \cong \mathrm{Hom}_{\mathcal{A}}(F_1, E_1)$$

i.e. i_1^* is the left adjoint of i_1. Note that by definition i_1^* is exact. Therefore, we directly consider the induced functor at the level of bounded derived categories

$$\begin{aligned} i_1^* \colon \mathcal{T}_{\mathcal{A},n} &\to D^b(\mathcal{A}) \\ (E_1, E_2, f_j) &\mapsto E_1. \end{aligned}$$

By Lemma 2.3, after replacing G by i_1 and F by i_1^*, we have that (i_1^*, i_{1*}) is an adjoint pair at the level of derived categories. We now prove that i_{1*} is a fully-faithful exact functor. Indeed, it follows directly from the fact that the counit

$$\epsilon \colon i_1^* i_{1*} \to \mathrm{Id}_{D^b(\mathcal{A})}$$

is a natural isomorphism, since by the definition of the adjunction we have that

$$\epsilon_G \colon i_1^* i_{1*}(G) \to G$$

is precisely the identity for every $G \in D^b(\mathcal{A})$.

Therefore, we obtain that i_{1*} is an exact embedding. We analogously prove that j_{2*} is an exact embedding. We define the functor

$$\begin{aligned} j_2^! \colon Q_{\mathcal{A},n} &\to \mathcal{A} \\ (E_1, E_2, f_j) &\mapsto E_2, \end{aligned}$$

which at the level of morphisms is defined as expected. Note that $j_2^!$ is the right adjoint of j_2. Indeed, we have

$$\mathrm{Hom}_{D^b(\mathcal{A})}(F, E_2) \cong \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}((0, F, 0), (E_1, E_2, f_j)),$$

and $j_2(F) = (0, F, 0)$, with $F \in \mathcal{A}$. Since $j_2^!$ is exact, then we take the induced derived functor. As $(j_{2*}, j_2^!)$ is also an adjoint pair at the level of derived categories, as for the functor i_{1*}, it implies that j_{2*} is an exact embedding. □

Definition 2.5. We define $D_1 := i_{1*}(D^b(\mathcal{A}))$ and $D_2 := j_{2*}(D^b(\mathcal{A}))$.

Remark 2.6. The categories D_i, with $i = 1, 2$, are equivalent to $D^b(\mathcal{A})$.

Definition 2.7. [BVdB03, Def. 1.2] Let $\mathcal{T}$ be a triangulated category of finite type. We say that $\mathcal{T}$ is *saturated* if every covariant and contravariant cohomology functor $\mathcal{T} \to \mathrm{Vect}_{\mathbb{C}}$ of finite type is representable.

Example 2.8. [BVdB03, Cor. 3.1.5] and [BK90b, Sec. 2] The triangulated categories $D^b(X)$, for a smooth projective variety X, and $D^b(\mathrm{Vect}_{\mathbb{C}})$ are saturated.

Lemma 2.9. *If $D^b(\mathcal{A})$ is a saturated category, then D_1 and D_2 are admissible triangulated subcategories of $\mathcal{T}_{\mathcal{A},n}$.*

Proof. Since the functors i_{1*} and j_{2*} are exact embeddings, it is clear that each subcategory D_i with $i = 1, 2$ is a full triangulated subcategory of $\mathcal{T}_{\mathcal{A},n}$. By [BK90b, Prop. 2.6], we obtain that they are also admissible. □

We assume throughout all this section that the category $D^b(\mathcal{A})$ is saturated. This condition allows to easily prove the admissibility of the triangulated subcategories and the existence of the Serre functor.

Lemma 2.10. *We have $D_2^{\perp} = D_1$.*

Proof. Let us consider $j_2^!$ as defined in Lemma 2.4. By definition, we have that $\mathrm{Ker}(j_2^!) = D_2^{\perp}$, then $D_2^{\perp} = \{(E_1, E_2, f_j) \in \mathcal{T}_{\mathcal{A},n} \mid j_2^!(E_1, E_2, f_j) = 0\}$. As $j^!(E_1, E_2, f_j) = E_2$, then by definition

$$D_2^{\perp} = D_1.$$

□

Remark 2.11. Note that if $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, then $D_1 = \langle S_1 \rangle$ and $D_2 = \langle S_2 \rangle$, where S_1, S_2 are the simple representation of the n-Kronecker quiver.

We now study the categories $D_3 := {}^{\perp}D_2$ and $D_0 := D_1^{\perp}$.

Corollary 2.12. *We obtain three semiorthogonal decompositions*

$$\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle \ , \ \mathcal{T}_{\mathcal{A},n} = \langle D_2, D_3 \rangle \text{ and } \mathcal{T}_{\mathcal{A},n} = \langle D_0, D_1 \rangle.$$

Proof. Since D_1, D_2 are admissible triangulated subcategories, the argument follows directly from Corollary 1.19. □

To describe D_0 and D_3 explicitly, we start by considering the following functor

$$\begin{aligned} j_3 \colon \mathcal{A} &\to Q_{\mathcal{A},n} \\ E &\mapsto (E, E^{\oplus n}, (i_l)_{0<l\leq n}), \end{aligned} \tag{2.1}$$

where the i_l is a canonical inclusion for each l and it is defined at the level of morphisms as expected. As the functor j_3 is an exact embedding, we can consider directly the induced functor at the level of bounded derived categories j_{3*}.

Lemma 2.13. *The functor j_{3*} is an exact embedding.*

Proof. Let $E = (E_1, E_2, f_{l, 0<l\leq n}) \in Q_{\mathcal{A},n}$ and let us consider the functor

$$\begin{array}{rcl} j_3^! \colon Q_{\mathcal{A},n} & \to & \mathcal{A} \\ E & \mapsto & E_1. \end{array} \tag{2.2}$$

We now prove that $j_3^!$ is a right adjoint of j_3. Note that

$$\mathrm{Hom}_{\mathcal{A}}(F_1, E_1) \cong \mathrm{Hom}_{Q_{A,n}}(j_3(F_1), E).$$

Indeed, if $t_1 \colon F_1 \to E_1$, then we define

$$t_2 := \sum_{l=1}^{n} f_l \circ t_1 \circ \pi_l,$$

where π_j is the canonical projection, for each l we obtain

$$\begin{tikzcd} F_1 \arrow[r, "t_1"] \arrow[d, "i_l"'] & E_1 \arrow[d, "f_l"] \\ F_1^{\oplus n} \arrow[r, "t_2"] & E_2. \end{tikzcd} \tag{2.3}$$

We clearly get the desired bijection. Moreover, by definition $j_3^!$ is also exact and $(j_{3*}, {j_3}^!)$ is an adjoint pair at the level of bounded derived categories. Therefore, as in Lemma 2.4 we obtain that j_{3*} is an exact embedding. □

Remark 2.14. The category $j_{3*}(D^b(\mathcal{A}))$ is a full triangulated subcategory of $\mathcal{T}_{\mathcal{A},n}$ and it is also equivalent to $D^b(\mathcal{A})$. Once again as $D^b(\mathcal{A})$ is saturated, by [BK90b, Prop. 2.6], we obtain that $j_{3*}(D^b(\mathcal{A}))$ is a full triangulated admissible subcategory of $\mathcal{T}_{\mathcal{A},n}$.

Lemma 2.15. $j_{3*}(D^b(\mathcal{A})) = D_3$

Proof. First note that by [BK90b, Lem. 1.7], it is enough to prove that $j_{3*}(D^b(\mathcal{A}))^\perp = D_2$. By adjointness, if $E \in j_{3*}(D^b(\mathcal{A}))^\perp$ then

$$0 = \mathrm{Hom}_{\mathcal{T}_{A,n}}(j_{3*}(F_1), E) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(F_1, E_1)$$

for all $F_1 \in D^b(\mathcal{A})$, which implies that $E_1 = 0$. □

Let us now consider the functor

$$\begin{array}{rcl} i_0 \colon \mathcal{A} & \to & Q_{\mathcal{A},n} \\ E & \mapsto & (E^{\oplus n}, E, (\pi_l)_{0<l\leq n}), \end{array} \tag{2.4}$$

where π_l is a canonical projection for each j and i_0 is defined at the level of morphisms as expected.

As the functor i_0 is an exact embedding, we can consider directly the induced functor at the level of bounded derived categories i_{0*}.

Lemma 2.16. *The functor i_{0*} is an exact embedding.*

Proof. Let $E = (E_1, E_2, (f_l)_{0<l\leq n}) \in \mathcal{T}_{\mathcal{A},n}$ and let us consider the functor

$$\begin{aligned} i_0^*\colon Q_{\mathcal{A},n} &\to \mathcal{A} \\ E &\mapsto E_2. \end{aligned} \tag{2.5}$$

We now prove that i_0^* is a left adjoint of i_0. Note that

$$\mathrm{Hom}_{\mathcal{A}}(E_2, F_2) \cong \mathrm{Hom}_{Q_{A,n}}(E, i_0(F_2)).$$

Indeed, if $t_2\colon F_2 \to E_2$, then we define

$$t_1 := \sum_{l=1}^{n} i_l \circ t_2 \circ f_l$$

and for each j we obtain

$$\begin{tikzcd} E_1 \arrow[r, "t_1"] \arrow[d, "f_l"'] & F_1^{\oplus n} \arrow[d, "\pi_l"] \\ E_2 \arrow[r, "t_2"] & F_2. \end{tikzcd} \tag{2.6}$$

We clearly get the desired bijection. Moreover, by definition i_0^* is also exact and $(i_0{}^*, i_{0*})$ is an adjoint pair at the level of bounded derived categories. Therefore, as in Lemma 2.4 we obtain that i_{0*} is an exact embedding. □

Remark 2.17. The category $i_{0*}(D^b(\mathcal{A}))$ is a full triangulated subcategory of $\mathcal{T}_{\mathcal{A},n}$ and it is also equivalent to $D^b(\mathcal{A})$. Once again as $D^b(\mathcal{A})$ is saturated, by [BK90b, Prop. 2.6], we obtain that $i_{0*}(D^b(\mathcal{A}))$ is an admissible subcategory of $\mathcal{T}_{\mathcal{A},n}$.

Claim 2.18. $i_{0*}(D^b(\mathcal{A})) = D_0$

Proof. First note that it is enough to prove that ${}^{\perp}i_{0*}(D^b(\mathcal{A})) = D_0$. By adjointness, if $E \in {}^{\perp}i_{0*}(D^b(\mathcal{A}))$ then

$$0 = \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(E, i_{0*}(F_2)) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(E_2, F_2)$$

for all $F_2 \in D^b(\mathcal{A})$, which implies that $E_2 = 0$. □

Moreover, the semiorthogonal decompositions

$$\mathcal{T}_{\mathcal{A},n} = \langle D_0, D_1\rangle \text{ and } \mathcal{T}_{\mathcal{A},n} = \langle D_2, D_3\rangle$$

induce respectively the following triangles for each $E \in \mathcal{T}_{\mathcal{A},n}$

$$\begin{aligned} {i_1}^*{i_1}^!(E) &\to E \to i_{0*}i_0^*(E), \\ j_{3*}j_3^!(E) &\to E \to j_{2*}{j_2}^*(E). \end{aligned} \tag{2.7}$$

Example 2.19. By the definition of i_{0*} and j_{3*}, we obtain the following triangles for every $G \in D^b(\mathcal{A})$:

$$\begin{aligned} i_{1*}(G^{\oplus n}[-1]) &\to j_{2*}(G) \to i_{0*}(G), \\ j_{3*}(G) &\to i_{1*}(G) \to j_{2*}(G^{\oplus n}[1]). \end{aligned}$$

Remark 2.20. Note that in the case $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, then $D_0 = \langle S_0\rangle$ and $D_3 = \langle S_3\rangle$, where S_0 is the exceptional representation with dimension vector $(n, 1)$ and S_3 is the exceptional representaion with dimension vector $(1, n)$.

Lemma 2.21. *If $D^b(\mathcal{A})$ is saturated, then the triangulated category $\mathcal{T}_{\mathcal{A},n}$ is also saturated.*

Proof. As D_1 and D_2 are equivalent to $D^b(A)$, they are both saturated triangulated subcategories of $\mathcal{T}_{\mathcal{A},n}$. Moreover by [BK90b, Prop. 2.6] they are also admissible. Therefore, by [BK90b, Prop. 2.10], we obtain that $\mathcal{T}_{\mathcal{A},n}$ is saturated. □

Lemma 2.22. *If $D^b(\mathcal{A})$ has a complete exceptional collection, then $\mathcal{T}_{\mathcal{A},n}$ also has a complete exceptional collection.*

Proof. Let $\mathcal{E} = \{E_1, E_2, \dots, E_n\}$ be a complete exceptional collection of $D^b(\mathcal{A})$. We claim that $\mathcal{F} = \{i_{1*}(E_1), \dots, i_{1*}(E_n), j_{2*}(E_1), \dots, j_{2*}(E_n)\}$ is a complete exceptional collection of $\mathcal{T}_{\mathcal{A},n}$. Indeed, it is clearly complete, because $\langle D_1, D_2\rangle$ is a semiorthogonal decomposition. Since the functors i_{1*} and j_{2*} are fully faithful, then when E is an exceptional object of $D^b(\mathcal{A})$, we get that $i_{1*}(E)$ and $j_{2*}(E)$ are exceptional objects in $\mathcal{T}_{\mathcal{A},n}$. To show that $\mathcal{F}$ is exceptional collection, we use that

$$\mathrm{Hom}(j_{2*}(E_i), i_{1*}(E_j)[l]) = 0 \quad \text{for all } l, i \text{ and } j,$$

because $D_2^\perp = D_1$ and $\mathcal{E}$ is an exceptional collection. □

Example 2.23. The triangulated category $D^b(\mathbb{P}^1)$ has a complete, strong, exceptional collection given by $(\mathcal{O}, \mathcal{O}(1))$. By Lemma 2.22, the category $\mathcal{T}_{\mathrm{Coh}(\mathbb{P}^1),1}$ also has a complete exceptional collection given by

$$\mathcal{E} = \{i_{1*}(\mathcal{O}), i_{1*}(\mathcal{O}(1)), j_{2*}(\mathcal{O}), j_{2*}(\mathcal{O}(1))\}.$$

Moreover, the complete exceptional collection

$$\mathcal{E} = \{i_{1*}(\mathcal{O}), i_{1*}(\mathcal{O}(1)), j_{2*}(\mathcal{O})[1], j_{2*}(\mathcal{O}(1))[1]\}.$$

is strong.

Let Q be the quiver

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] \arrow[d, "l_1"] & \bullet \arrow[d, "l_2"] \\ \bullet \arrow[r, bend left, "\gamma"] \arrow[r, bend right, "\delta"'] & \bullet \end{tikzcd}$$

and let A be the path algebra of Q under the relations $l_2 \circ \alpha = \gamma \circ l_1$ and $ł_2 \circ \beta = \delta \circ l_1$.

Lemma 2.24. *There is an equivalence of categories*

$$\mathcal{T}_{\mathrm{Coh}(\mathbb{P}^1),1} \cong D^b(\mathrm{modA})$$

Proof. Since we have that

$$\mathcal{E} = \{i_{1*}(\mathcal{O}), i_{1*}(\mathcal{O}(1)), j_{2*}(\mathcal{O})[1], j_{2*}(\mathcal{O}(1))[1]\}$$

is a complete, strong exceptional collection, by Theorem 1.30 we have that

$$\mathcal{T}_{\mathrm{Coh}(\mathbb{P}^1),1} \cong D^b(\mathrm{mod}(\mathrm{End}(\mathcal{P}))),$$

where

$$\begin{tikzcd} & & \mathcal{O} \arrow[dd] & & \mathcal{O}(1) \arrow[dd] & & 0 \arrow[dd] & & 0 \arrow[dd] \\ \mathcal{P} & = & & \oplus & & \oplus & & \oplus & \\ & & 0 & & 0 & & \mathcal{O}[1] & & \mathcal{O}(1)[1] \end{tikzcd}.$$

We notice that

$$\mathrm{End}(\mathcal{P}) = \begin{pmatrix} \mathbb{C} & 0 & 0 & 0 \\ \mathbb{C}^2 & \mathbb{C} & 0 & 0 \\ \mathbb{C} & 0 & \mathbb{C} & 0 \\ \mathbb{C}^2 & \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \end{pmatrix},$$

which is precisely the path algebra of the quiver Q with the relations described above. $\square$

Remark 2.25. There is also an equivalence of categories $\mathcal{T}_{\mathrm{Coh}(\mathbb{P}^1),1} \cong \mathcal{T}_{Q_{\mathrm{Vect}_{\mathbb{C}}},2,1}$. Indeed, we have that $\mathcal{E} = \{i_{1*}(S_1), i_{1*}(S_2)[1], j_{2*}(S_1)[1], j_*(S_2)[2]\}$ is a complete, strong, exceptional collection. By Theorem 1.30, we obtain that

$$\mathcal{T}_{Q_{\mathrm{Vect}_{\mathbb{C}}},2,1} \cong D^b(\mathrm{modA}).$$

Lemma 2.26. *The Grothendieck group $K(\mathcal{T}_{\mathcal{A},n})$ is isomorphic to $K(\mathcal{A}) \bigoplus K(\mathcal{A})$.*

Proof. The categories D_1 and D_2 are equivalent to $D^b(\mathcal{A})$. Considering the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2\rangle$ and the fact that $K(\mathcal{A}) \cong K(D^b(\mathcal{A}))$, it follows that $K(\mathcal{T}_{\mathcal{A},n}) \cong K(\mathcal{A}) \bigoplus K(\mathcal{A})$. $\square$

Corollary 2.27. *The numerical Grothendieck group $\mathcal{N}(\mathcal{T}_{\mathcal{A},n})$ is isomorphic to the group $\mathcal{N}(\mathcal{A}) \bigoplus \mathcal{N}(\mathcal{A})$.*

Proof. Let $E = (E_1, E_2, f_j)$ and $F = (F_1, F_2, g_j) \in \mathcal{T}_{\mathcal{A},n}$. We want to show that $E = (E_1, E_2, f_j) \in K(\mathcal{T}_{\mathcal{A},n})^{\perp}$ if and only if E_1 and $E_2 \in K(\mathcal{A})^{\perp}$.

As the Euler form is linear in the Grothendieck group and as $[E] = ([i_*E_1], [j_*E_2])$, then

$$\chi(E, F) = \chi(i_{1*}E_1, i_{1*}F_1) + \chi(i_{1*}E_1, j_{2*}F_2) + \chi(j_{2*}E_2, i_{1*}F_1) + \chi(j_{2*}E_2, i_{1*}F_2).$$

As $D_2^{\perp} = D_1$, then $\chi(j_{2*}E_2, i_{1*}F_1) = 0$. Now, by Example 2.19 and by adjointness we have that

$$\chi(i_{1*}E_1, j_{2*}F_2) = \chi_{D^b(\mathcal{A})}(E_1, F_2^{\oplus n}[-1]) = -n\chi_{D^b(\mathcal{A})}(E_1, F_2).$$

Therefore, we obtain that

$$\chi(E, F) = \chi_{D^b(\mathcal{A})}(E_1, F_1) + \chi_{D^b(\mathcal{A})}(E_2, F_2) - n\chi_{D^b(\mathcal{A})}(E_1, F_2).$$

We clearly obtain that if E_1 and E_2 are in $K(\mathcal{A})^{\perp}$, then $E \in K(\mathcal{T}_{\mathcal{A},n})^{\perp}$.

We now prove the other direction. Let us consider $E \in K(\mathcal{T}_{\mathcal{A},n})^{\perp}$ i.e.

$$\chi_{\mathcal{T}_{\mathcal{A},n}}(E, F) = 0$$

for all $F \in K(\mathcal{T}_{\mathcal{A},n})$, in particular for any $i_{1*}(G)$, $j_{2*}(G)$, where $G \in D^b(\mathcal{A})$. As a consequence, we have that

$$0 = \chi(E, i_{1*}(G)) = \chi_{D^b(\mathcal{A})}(E_1, G)$$

and

$$0 = \chi(E, j_{2*}(G)) = \chi_{D^b(\mathcal{A})}(E_2, G) - n\chi_{D^b(\mathcal{A})}(E_1, G).$$

It implies that for all $G \in D^b(\mathcal{A})$, we get that $\chi_{D^b(\mathcal{A})}(E_1, G) = \chi_{D^b(\mathcal{A})}(E_2, G) = 0$. $\square$

Example 2.28. In the case that $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, as the isomorphism $\mathcal{N}(D^b(\mathrm{Vect}_{\mathbb{C}})) \cong \mathbb{Z}$ is given by the dimension, we get the following description of the numerical Grothendieck group induced by the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle$. Let $E = (E_1, E_2, f_{j_{0<j\leq n}})$, then

$$\begin{aligned} \mathcal{N}(\mathbb{Q}_n) \to& \quad \mathbb{Z}^2 \\ E \mapsto& \quad (\dim(E_1), \dim(E_2)). \end{aligned}$$

Example 2.29. In the case that $\mathcal{A}$ is $\mathrm{Coh}(C)$, where C is a smooth projective curve, we have that $\mathcal{N}(D^b(C)) \cong \mathbb{Z}^2$ by considering the rank and the degree, then we get

$$\begin{aligned} \mathcal{N}(\mathcal{T}_{C,n}) \to& \quad \mathbb{Z}^4 \\ E \mapsto& \quad (\mathrm{rk}(E_1), \deg(E_1), \mathrm{rk}(E_2), \deg(E_2)). \end{aligned}$$

Remark 2.30. [JMP10, Prop. 3] Let $F \in \mathrm{Aut}(\mathcal{A})$ be an autoequivalence, then the functor F_{Q_n}, defined as follows at the level of objects

$$\begin{aligned} F_{Q_n} \colon Q_{\mathcal{A},n} &\to Q_{\mathcal{A},n} \\ (E_1, E_2, (f_j)_{0<j\leq n}) &\mapsto (F(E_1), F(E_2), F(f_j)_{0<j\leq n}) \end{aligned}$$

and for morphisms $\varphi = (\varphi_1, \varphi_2) \in \mathrm{Hom}_{Q_{\mathcal{A},n}}$ as $F_{Q_n}(\varphi) = (F(\varphi_1), F(\varphi_2))$, is also an autoequivalence. Moreover, if F is exact, then F_{Q_n} is also exact.

Example 2.31. If $\mathcal{A} = \mathrm{Coh}(X)$, where X is a nonsingular projective variety. We have the exact autoequivalence

$$\begin{aligned} - \otimes \mathcal{L} \colon \mathrm{Coh}(X) &\to \mathrm{Coh}(X) \\ E &\to E \otimes \mathcal{L}, \end{aligned}$$

where $\mathcal{L}$ is a line bundle. By Remark 2.30, it induces an exact autoequivalence on $Q_{\mathcal{A},n}$. Therefore, by passing to the derived category, we obtain an autoequivalence of $\mathcal{T}_{\mathcal{A},n}$.

Theorem 2.32. *If the triangulated category $D^b(\mathcal{A})$ is saturated, then the category $\mathcal{T}_{\mathcal{A},n}$ admits a Serre functor.*

Proof. Since $D^b(\mathcal{A})$ is a saturated triangulated category of finite type, by [BK90b, Cor. 3.5] the triangulated category $D^b(\mathcal{A})$ has a Serre functor $\mathcal{S}_{\mathcal{A}}$. Since $D_2 \cong D^b(\mathcal{A})$ and $D_2^{\perp} = D_1$ are admissible and have Serre functors $\mathcal{S}_{\mathcal{A}}$, by [BK90b, Prop. 3.8], we deduce that $\mathcal{T}_{\mathcal{A},n}$ also has a Serre functor. □

We now follow the constructive proof in [BK90b, Prop. 3.8] to give explicitly the isomorphism induced by the Serre functor in the numerical Grothendieck group. For a complete construction of the Serre functor, we suggest to consult the original source. To make the notation simple during the construction we assume

$$i_* \coloneqq i_{1*} \text{ and } j_* \coloneqq j_{2*}.$$

First of all we want to construct a representative of the contravariant functor

$$h = \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, -)^*,$$

with $X = (X_1, X_2, f_{j_{0<j\leq n}}) \in \mathcal{T}_{\mathcal{A},n}$, i.e. an object $\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(X)$ such that

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(-, \mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(X)) = \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, -)^*.$$

In order to construct a representing object for h, we need to construct a representing object for $h\big|_{D_2}$ and a representing object for $h\big|_{D_1}$. Let us consider

$$\begin{array}{rcl} h\big|_{D_1} \colon D^b(\mathcal{A}) & \to & \mathrm{Vect}_{\mathbb{C}} \\ E & \mapsto & \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, i_*E)^*. \end{array}$$

By adjointness we have

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, i_*E) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(i^*X, E).$$

By Serre duality on $D^b(\mathcal{A})$ we get

$$\mathrm{Hom}_{D^b(\mathcal{A})}(i^*X, E) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(E, \mathcal{S}_A i^*X)^*.$$

Therefore, we get

$$\mathrm{Hom}_{D^b(\mathcal{A})}(E, \mathcal{S}_A i^*X)^* \cong \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(i_*E, i_*\mathcal{S}_A i^*X)^*,$$

which finally implies that

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, i_*E)^* \cong \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(i_*E, i_*\mathcal{S}_A i^*X),$$

and the representing object of $h\big|_{D_1}$ is given by

$$F_1 = i_*\mathcal{S}_A i^*X = (\mathcal{S}_{\mathcal{A}}(X_1), 0, 0).$$

Let us consider

$$\begin{array}{rcl} h\big|_{D_2} \colon D^b(\mathcal{A}) & \to & \mathrm{Vect}_{\mathbb{C}} \\ E & \mapsto & \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, j_*E)^*. \end{array}$$

By adjointness we have

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, j_*E) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(j^*X, E).$$

By Serre duality on $D^b(\mathcal{A})$ we get

$$\mathrm{Hom}_{D^b(\mathcal{A})}(j^*X, E) \cong \mathrm{Hom}_{D^b(\mathcal{A})}(E, \mathcal{S}_{\mathcal{A}}j^*X)^*.$$

As before, we obtain

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, j_*E)^* \cong \mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(j_*E, j_*\mathcal{S}_A j^*X),$$

and the representing object of $h\big|_{D_2}$ is given by

$$F_2 = (0, \mathcal{S}_A j^*X, 0).$$

We also need to find a representing object for the following functor

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{T}_{A,n}}(-, F_2)\colon D^b(\mathcal{A}) & \to & \mathrm{Vect}_{\mathbb{C}} \\ E & \mapsto & \mathrm{Hom}_{\mathcal{T}_{A,n}}(i_*(E), F_2). \end{array}$$

By adjointness

$$\mathrm{Hom}_{\mathcal{T}_{A,n}}(i_*E, F_2) \cong \mathrm{Hom}_{\mathcal{T}_{A,n}}(i_*E, i_*i^!F_2).$$

Therefore, by Example 2.19, the representing object is given by $i_*i^!F_2 = (j^!(F_2)^{\oplus n}[-1], 0, 0)$. We obtain that

$$\mathrm{Hom}_{\mathcal{T}_{A,n}}(i_*i^!F_2, i_*i^!F_2) \cong \mathrm{Hom}_{\mathcal{T}_{A,n}}(i_*i^!F_2, F_2).$$

The identity induces a morphism

$$\gamma\colon i_*i^!F_2 \to F_2,$$

which is precisely the morphism that appears in

$$i_*i^!F_2 \to F_2 \to i_0 i_0^* F_2$$

induced by the semiorthogonal decomposition $\langle D_0, D_1\rangle$ as in Example 2.19. Therefore, we obtain the morphism

$$h(\gamma)\colon \mathrm{Hom}_{\mathcal{T}_{A,n}}(X, F_2)^* \to \mathrm{Hom}_{\mathcal{T}_{A,n}}(X, i_*i^!F_2)^*,$$

as

$$\mathrm{Hom}_{\mathcal{T}_{A,n}}(X, F_2)^* = \mathrm{Hom}_{D^b(\mathcal{A})}(F_2, F_2)$$

and
$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(X, i_*i^!F_2)^* = \mathrm{Hom}_{D^b(\mathcal{A})}(i_*i^!F_2, F_1).$$

Then, the identity induces a morphism $\delta\colon i_*i^!F_2 \to F_1$ in D_1. We obtain a triangle

$$i_*i^!F_2 \to F_1 \to C(\delta) \to i_*i^!F_2[1]$$

in D_1. After composing with γ, we obtain the following morphism

$$C(\delta) \to i_*i^!F_2[1] \xrightarrow{\gamma} F_2[1].$$

After taking once again the cone

$$C(t)[-1] \to C(\delta) \xrightarrow{t} F_2[1].$$

And we define $\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(X) = C(t)[-1]$.

As the Serre functor is an autoequivalence, it induces an isomorphism of the numerical Grothendieck group. We make explicit this homomorphism.

Proposition 2.33. *The automorphism induced by the Serre functor is defined as follows:*

$$\begin{array}{rcl} \mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}\colon \mathcal{N}(D^b(\mathcal{A}))^{\oplus 2} & \to & \mathcal{N}(D^b(\mathcal{A}))^{\oplus 2} \\ ([X_1],[X_2]) & \mapsto & ((1-n^2)[\mathcal{S}_{\mathcal{A}}(X_1)] + n[\mathcal{S}_{\mathcal{A}}(X_2)], [\mathcal{S}_{\mathcal{A}}(X_2)] - n[\mathcal{S}_{\mathcal{A}}(X_1)]) \end{array}$$

where $\mathcal{S}_{\mathcal{A}}$ is the group automorphism of $\mathcal{N}(D^b(\mathcal{A}))$ induced by the Serre functor in $D^b(\mathcal{A})$.

Proof. First note that by

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}([X_1],[X_2]) = (0, \mathcal{S}_{\mathcal{A}}(j^*[X])) + ([C(\delta))], 0),$$

where $j^*\colon \mathcal{N}(\mathcal{T}_{\mathcal{A},n}) \to \mathcal{N}(D^b(\mathcal{A}))$ is the group homomorphism induced by j^*.

As by Example 2.19 we have the following triangle

$$X_1^{\oplus n} \to X_2 \to j^*(X) \to X \to X_1^{\oplus n}[1],$$

we obtain
$$\mathcal{S}_{\mathcal{A}}([j^*(X)]) = [\mathcal{S}_{\mathcal{A}}(X_2)] - n[\mathcal{S}_{\mathcal{A}}(X_1)].$$

By the triangle
$$i_*(i^!(F_2)) \to F_1 \to C(\delta) \to i_*(i^!(F_2))[1],$$

we obtain
$$[C(\delta)] = [\mathcal{S}_{\mathcal{A}}(X_1)] + n[\mathcal{S}_{\mathcal{A}}([j^*(X_2)]),$$

then

$$[C(\delta)] = [\mathcal{S}_{\mathcal{A}}(X_1)] + n[\mathcal{S}_{\mathcal{A}}(X_2)] - n^2[\mathcal{S}_{\mathcal{A}}(X_1)].$$

Therefore, we have

$$[\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(X)] = ((1-n^2)[\mathcal{S}_{\mathcal{A}}(X_1)] + n[\mathcal{S}_{\mathcal{A}}(X_2)], [\mathcal{S}_{\mathcal{A}}(X_2)] - n[\mathcal{S}_{\mathcal{A}}(X_1)]).$$

□

Remark 2.34. Note that the morphism is given precisely by multiplying the Coxeter matrix $\begin{bmatrix} 1-n^2 & n \\ -n & 1 \end{bmatrix}$ by the pair $([\mathcal{S}_A(X_1)], [\mathcal{S}_A(X_2)])$.

Example 2.35. If we take $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, and $n = 1$. The induced automorphism of the numerical Grothendieck group is given by

$$\begin{aligned} \mathcal{S}_{Q_1} \colon \mathbb{Z}^2 &\to \mathbb{Z}^2 \\ (r_1, r_2) &\mapsto (r_2, r_2 - r_1). \end{aligned}$$

If $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve, we obtain that the induced automorphism is given by

$$\begin{aligned} \mathcal{S}_{\mathcal{T}_{Coh(C),1}} \colon \mathbb{Z}^4 &\to \mathbb{Z}^4 \\ (r_1, d_1, r_2, d_2) &\mapsto (-r_2, -d_2, r_1 - r_2, d_1 - d_2). \end{aligned}$$

If we take $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, and $n = 2$. By Example 2.27, we have $\mathcal{N}(Q_n) = \mathbb{Z}_2$ and

$$\begin{aligned} \mathcal{S}_{Q_2} \colon \mathbb{Z}^2 &\to \mathbb{Z}^2 \\ (r_1, r_2) &\mapsto (2r_2 - 3r_1, r_2 - 2r_1). \end{aligned}$$

Moreover, if we consider the equivalence of triangulated categories $D^b(Q_2) \cong D^b(\mathbb{P}^1)$, we use the well-known description of the Serre functor for $D^b(\mathbb{P}^1)$. The semiorthogonal decomposition $\langle D_1, D_2 \rangle$ on $\mathcal{D}^b(Q_2)$ is given by $\langle \mathcal{O}(-1)[1], \mathcal{O} \rangle$ on $D^b(\mathbb{P}^1)$ via the equivalence given by $\mathrm{RHom}(\mathcal{O} \oplus \mathcal{O}(1), -) \colon D^b(\mathbb{P}^1) \to D^b(Q_2)$. See [Bon90, Ex. 6.3.].

For example for $\mathcal{O}(3)$ we obtain that its triangle with respect to the semiorthogonal decomposition $\langle \mathcal{O}(-1)[1], \mathcal{O} \rangle$ is given by

$$\mathcal{O}^{\oplus 4} \to \mathcal{O}(3) \to \mathcal{O}(-1)^{\oplus 3}[1].$$

It means that $[\mathcal{O}(3)]$ is represented by $(3, 4)$. After applying the Serre functor, we obtain

$\mathcal{S}(\mathcal{O}(3)) = \mathcal{O}(1)[1]$, numerically we obtain $(-1,-2)$, which corresponds to

$$\mathcal{O}^{\oplus 2}[1] \to \mathcal{O}(1)[1] \to \mathcal{O}(-1)[2].$$

If we take $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve and $n = 2$,

$$\begin{array}{rcl} \mathcal{S}_{\mathcal{T}_{\mathrm{Coh}(C),2}} \colon \mathbb{Z}^4 & \to & \mathbb{Z}^4 \\ (r_1, d_1, r_2, d_2) & \mapsto & (-2r_2 + 3r_1, -2d_2 + 3d_1, -r_2 + 2r_1, -d_2 + 2d_1). \end{array}$$

Remark 2.36. Note that from Corollary 2.12 we can also conclude that D_1 and D_2 are admissible without the assumption of being saturated. From Lemma 2.4, we have that D_1 is left-admissible and from the semiorthogonal decomposition $\langle D_0, D_1\rangle$, as D_0 is left-admissible by Lemma 1.18, we obtain that D_1 is right-admissible, as a consequence D_1 is admissible. We prove analogously the admissibility of D_2. Moreover, by [BK90b, Prop. 3.8], if D_1 and D_2 have Serre functors, we can prove that $\mathcal{T}_{\mathcal{A},n}$ also has a Serre functor and we can prove that D_0 and D_3 are also admissible. We assumed from the beginning that $D^b(\mathcal{A})$ is saturated to directly have the existence of the Serre functor $\mathcal{S}_{\mathcal{A}}$ and to make the arguments cleaner and easier.

The Serre functor on the triangulated category $\mathcal{T}_A$

We now study $\mathcal{T}_{\mathcal{A},1}$. From now on we denote it by $\mathcal{T}_{\mathcal{A}}$. Analogously to the description of $D^b(Q_1)$ via the exceptional pairs

$$(S_1, S_2)\ ,\ (S_2, S_3) \text{ and } (S_3, S_1),$$

we obtain three semiorthogonal decomposition of $\mathcal{T}_A$. Moreover, we give an explicit description of the Serre functor. As a consequence we obtain several examples of fractional Calabi–Yau categories.

First note that since $n = 1$, we have $l_* := j_{3*} = i_{0*}$, therefore

$$D_0 = D_3.$$

Moreover, as the functors i_0^* and $j_3^!$, as defined in Lemma 2.16 and Lemma 2.13, are the left and right adjoint of l_* respectively, we rename them as

$$l^! := j_3^! \text{ and } l^* := i_0^*.$$

Remark 2.37. By the definition of $l^!$ and l^* given in Lemma 2.16 and Lemma 2.13 we obtain that

$$l^* = j^! \text{ and } l^! = i^*.$$

Remark 2.38. In many cases, we express a morphism $\Phi \in \mathrm{Hom}_{\mathcal{T}_\mathcal{A}}(E, F)$ as two horizontal arrows, but note that they just represent $i^*(\Phi)$ and $j^!(\Phi)$ and they do not characterize the morphism Φ.

Proposition 2.39. *We have the following three semiorthogonal decompositions on $\mathcal{T}_\mathcal{A}$*

$$\mathcal{T}_\mathcal{A} = \langle D_1, D_2\rangle,\ \mathcal{T}_\mathcal{A} = \langle D_2, D_3\rangle \text{ and } \mathcal{T}_\mathcal{A} = \langle D_3, D_1\rangle.$$

Proof. It follows directly from Corollary 2.12. □

The semiorthogonal decompositions $\mathcal{T}_\mathcal{A} = \langle D_3, D_1\rangle$ and $\mathcal{T}_\mathcal{A} = \langle D_2, D_3\rangle$ define $i^!$ as the right adjoint of i_* and j^* as the left adjoint of j_*. We obtain the following triangles for $X = (X_1, X_2, \varphi)$

$$\begin{tikzcd}
i^!(X) \arrow[r, "\pi_X"] \arrow[d] & X_1 \arrow[r, "\varphi"'] \arrow[d, "\varphi"] & X_2 \arrow[r] \arrow[d] & i^!(X)[1] \arrow[d] \\
0 \arrow[r] & X_2 \arrow[r, "\mathrm{id}"'] & X_2 \arrow[r] & 0
\end{tikzcd} \tag{2.8}$$

and

$$\begin{tikzcd}
X_1 \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & X_1 \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & X_1[1] \arrow[d, "\mathrm{id}"'] \\
X_1 \arrow[r, "\varphi"] & X_2 \arrow[r, "\tau_X"] & j^*(X) \arrow[r] & X_1[1].
\end{tikzcd} \tag{2.9}$$

As a consequence, we can now describe the functors $i^!$ and j^* at the level of objects and some features at the level of morphisms.

Corollary 2.40. *Let $X = (X_1, X_2, \varphi_X)$ and $Y = (Y_1, Y_2, \varphi_Y) \in \mathcal{T}_\mathcal{A}$. The functor $i^!$ is given as follows*

$$\begin{aligned} i^! \colon \mathcal{T}_\mathcal{A} &\to D^b(\mathcal{A}) \\ X &\mapsto C(\varphi_X)[-1] \end{aligned} \tag{2.10}$$

at the level of objects. If $\psi \colon X \to Y$ is a morphism of triples, then the following diagram commutes

$$\begin{tikzcd}
C(\varphi_X)[-1] \arrow[r, "i^!(\psi)"] \arrow[d, "\pi_X"] & C(\varphi_Y)[-1] \arrow[d, "\pi_Y"] \\
X_1 \arrow[r, "i^*(\psi)"] & Y_1.
\end{tikzcd} \tag{2.11}$$

Proof. As we have the (triangle 2.9) and i^* is an exact functor, we obtain the triangle

$$i^!(X) \xrightarrow{\pi_X} X_1 \xrightarrow{\varphi} X_2 \to i^!(X),$$

therefore $i^!(X) \cong C(\varphi_X)[-1]$. We use the naturality of the adjunction and we obtain

$$\psi \circ \pi_X = i_* i^!(\psi) \circ \pi_Y$$

and by taking i^* in both sides we obtain the (square 2.11). Indeed, we have the following diagram, where the two inner squares commute:

$$\begin{tikzcd}
\mathrm{Hom}_{D^b(\mathcal{A})}(i^!(X), i^!(X)) \arrow[r] \arrow[d, "i^!(\psi)"'] & \mathrm{Hom}_{\mathcal{T}_\mathcal{A}}(i_*(i^!(X)), X) \arrow[d, "\psi"'] \\
\mathrm{Hom}_{D^b(\mathcal{A})}(i^!(X), i^!(Y)) \arrow[r, "i^!(\psi)"] & \mathrm{Hom}_{\mathcal{T}_\mathcal{A}}(i_*(i^!(X)), Y) \\
\mathrm{Hom}_{D^b(\mathcal{A})}(i^!(Y), i^!(Y)) \arrow[r] \arrow[u, "i^!(\psi)"] & \mathrm{Hom}_{\mathcal{T}_\mathcal{A}}(i_*(i^!(Y)), Y). \arrow[u, "i^!(\psi)"]
\end{tikzcd} \tag{2.12}$$

Note that $i^!(\psi) \circ id_{i^!(X)} = \mathrm{id}_{i^!(Y)} \circ i^!(\psi)$. □

Corollary 2.41. *Let $X = (X_1, X_2, \varphi_X)$ and $Y = (Y_1, Y_2, \varphi_Y) \in \mathcal{T}_\mathcal{A}$. The functor j^* is given as follows*

$$\begin{aligned} j^* \colon \mathcal{T}_\mathcal{A} &\to D^b(\mathcal{A}) \\ X &\mapsto C(\varphi_X) \end{aligned} \tag{2.13}$$

at the level of objects. If $\psi \colon X \to Y$ is a morphism of triples, then the following diagram commutes

$$\begin{tikzcd}
X_2 \arrow[r, "j^!(\psi)"] \arrow[d, "\tau_X"'] & Y_2 \arrow[d, "\tau_Y"] \\
C(\varphi_X) \arrow[r, "j^*(\psi)"] & C(\varphi_Y)
\end{tikzcd} \tag{2.14}$$

Proof. The proof goes along the lines of Corollary 2.40. □

As we have already proved the existence of the Serre functor in Lemma 2.32, we now describe it explicitly. We first compute the image under the Serre functor for some precise objects.

Lemma 2.42. *Let $X \in D^b(\mathcal{A})$, then*

1. $\mathcal{S}_{\mathcal{T}_\mathcal{A}} j_*(X) = l_* \mathcal{S}_\mathcal{A}(X)$.

2. $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}} i_*(X) = j_* \mathcal{S}_{\mathcal{A}}(X)[1]$.
3. $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}} l_*(X) = i_* \mathcal{S}_{\mathcal{A}}(X)$.

Proof. We use the fact that

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}({}^{\perp}D) = D^{\perp}$$

and that

$$\mathcal{S}_{\mathcal{A}} = F^! \mathcal{S}_{\mathcal{T}_{\mathcal{A}}} F_*,$$

where F is one of the functors i, j or l. See [Shi, Prop. 3.3].
As ${}^{\perp}D_1 = D_2$ and $D_1^{\perp} = D_3$, we obtain that $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}} j_*(X) \in D_3$, for all $X \in D^b(\mathcal{A})$. Now as

$$j^! \mathcal{S}_{\mathcal{T}_{\mathcal{A}}} j_* = \mathcal{S}_{\mathcal{A}}$$

and by the definition of $j^!$ in Lemma 2.10 and of l_* we obtain that

$$\mathcal{S}_{\mathcal{T}_{A,n}} j_*(X) = l_* \mathcal{S}_{\mathcal{A}}(X).$$

As ${}^{\perp}D_2 = D_3$ and $D_2^{\perp} = D_1$, we obtain that $\mathcal{S}_{\mathcal{T}_{A}} l_*(X) \in D_1$, for all $X \in D^b(\mathcal{A})$. Now as

$$l^! \mathcal{S}_{\mathcal{T}_{A}} l_* = \mathcal{S}_{\mathcal{A}}$$

and by the definition of $l^!$ in Remark 2.37, we obtain that

$$\mathcal{S}_{\mathcal{T}_{A}} l_*(X) = i_* \mathcal{S}_{A}(X).$$

As ${}^{\perp}D_3 = D_1$ and $D_3^{\perp} = D_2$, we obtain that $\mathcal{S}_{\mathcal{T}_{A}} i_*(X) \in D_2$, for all $X \in D^b(\mathcal{A})$. By the exactness of the rows of the following triangle

$$\begin{tikzcd}
i^! j_*(X) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "\mathrm{id}"'] & X \arrow[d] \\
0 \arrow[r] & X \arrow[r, "\mathrm{id}"'] & X \arrow[r] & 0,
\end{tikzcd} \tag{2.15}$$

induced by the semiorthogonal decomposition $\langle D_3, D_1 \rangle$, we obtain

$$i^! j_*(X) = X[-1].$$

By the definition of $i^!$ on D_2 and as we have that

$$i^! \mathcal{S}_{\mathcal{T}_{\mathcal{A}}} i_* = \mathcal{S}_{\mathcal{A}},$$

for all $X \in D^b(\mathcal{A})$ and we obtain

$$\mathcal{S}_{\mathcal{T}_\mathcal{A}} i_*(X) = j_* \mathcal{S}_\mathcal{A}[1](X).$$

□

Lemma 2.43. *Let $X \in D^b(\mathcal{A})$, then*

1. $\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1} j_*(X) = i_* \mathcal{S}_\mathcal{A}^{-1}(X)[-1]$.
2. $\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1} i_*(X) = l_* \mathcal{S}_\mathcal{A}^{-1}(X)$.
3. $\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1} l_*(X) = j_* \mathcal{S}_\mathcal{A}^{-1}(X)$.

Proof. It follows directly from Lemma 2.42. □

Remark 2.44. Let $X = X_1 \xrightarrow{\varphi} X_2$ be an element in $\mathcal{T}_\mathcal{A}$. As we have $F^! \mathcal{S}_{\mathcal{T}_\mathcal{A}} = \mathcal{S}_\mathcal{A} F^*$ for $F = i, j$ or l, then we obtain directly

$$\begin{aligned} l^!(\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X)) &= \mathcal{S}_\mathcal{A}(X_2), \\ i^!(\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X)) &= \mathcal{S}_\mathcal{A}(X_1), \\ j^!(\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X)) &= \mathcal{S}_\mathcal{A}(C(\varphi)). \end{aligned}$$

Remark 2.45. Let $X = X_1 \xrightarrow{\varphi} X_2$ be an element in $\mathcal{T}_\mathcal{A}$. As we have $F^* \mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1} = \mathcal{S}_\mathcal{A}^{-1} F^*$ for $F = i, j$ or l, then we obtain directly

$$\begin{aligned} l^*(\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1}(X)) &= \mathcal{S}_\mathcal{A}^{-1}(X_1), \\ i^*(\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1}(X)) &= \mathcal{S}_\mathcal{A}^{-1}(C(\varphi)[-1]), \\ j^*(\mathcal{S}_{\mathcal{T}_\mathcal{A}}^{-1}(X)) &= \mathcal{S}_\mathcal{A}^{-1}(X_2). \end{aligned}$$

Let us consider an element $X = X_1 \xrightarrow{\varphi} X_2$ and its decomposition

$$j_* j^!(X) \to X \to i_* i^*(X) \xrightarrow{t_X} j_* j^!(X)[1]$$

with respect to $\mathcal{T}_\mathcal{A} = \langle D_1, D_2 \rangle$. We now study the morphism $t_X \colon i_* i^*(X) \to j_* j^!(X)[1]$. More explicitly, we obtain

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & X_1 \arrow[r, "\mathrm{id}"] \arrow[d, "\varphi"'] & X_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ X_2 \arrow[r, "\mathrm{id}"'] & X_2 \arrow[r] & 0 \arrow[r, "t_X"] & X_2[1]. \end{tikzcd} \tag{2.16}$$

Remark 2.46. See [BBD82, Prop. 1.3.3] and [BBD82, Cor. 1.1.10]. Note that t_X characterizes the triangle, i.e. it is the unique morphism representing the isomorphism class of the triangle induced by the semiorthogonal decomposition $\langle D_1, D_2\rangle$ for X. Moreover, note that $C(t_X)[-1]$ is isomorphic to X up to a non unique isomorphism in $\mathcal{T}_{\mathcal{A}}$.

Remark 2.47. Let us consider the triangle

$$j_*j^!(X) \to X \to i_*i^*(X) \xrightarrow{t_X} j_*j^!(X)[1],$$

after applying the Serre functor we obtain a triangle

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(j_*j^!(X)) \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X) \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(i_*i^*(X)) \xrightarrow{\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(t_X)} \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(j_*j^!(X)[1].$$

As $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}({}^{\perp}D) = D^{\perp}$, then

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(j_*j^!(X)) \in D_3 \text{ and } \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(i_*i^*(X)) \in D_2.$$

By the uniqueness of the triangle induced by the semiorthogonal decomposition, it implies that we obtain precisely the corresponding triangle of $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X)$ induced by the semiorthogonal decomposition

$$\mathcal{T}_{\mathcal{A}} = \langle D_2, D_3\rangle$$

up to isomoprhism. Moreover $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(t_X)$ is the unique morphism that characterizes the triangle as mentioned in Remark 2.46. See [Kuz19, Lem. 2.3] and [BBD82, Prop. 1.3.3].

Notation 2.48. Let us consider the canonical functor

$$C(\mathcal{A}) \to D^b(\mathcal{A}),$$

where $C(\mathcal{A})$ is the category of complexes over $\mathcal{A}$. If $\varphi \in \mathrm{Mor}(C(\mathcal{A}), C(\mathcal{A}))$ we denote by $[\varphi]$ its image in $\mathrm{Mor}(D^b(\mathcal{A}), D^b(\mathcal{A}))$ given by the roof (id, φ).

The following technical lemma connects the triple X with the roof (id, φ), i.e with $[\varphi]$ and it plays a role in the explicit construction of the Serre functor. We first consider the decomposition of $l_*(X_2)$ with respect to the $\mathcal{T}_{\mathcal{A}} = \langle D_1, D_2\rangle$, that it is in fact given by

$$j_*(X_2) \to l_*(X_2) \to i_*(X_2) \xrightarrow{t_{l_*(X_2)}} j_*(X_2)[1].$$

We study two morphisms in $\mathcal{T}_{\mathcal{A}}$, the first on given by $t_{l_*(X_2)} \circ i_*([\varphi])$

$$\begin{tikzcd} X_1 \arrow[r, "{[\varphi]}"] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r, "t_{l_*(X_2)}"] & X_2[1]. \end{tikzcd} \tag{2.17}$$

The second one given by t_X.

Lemma 2.49.

$$t_X = t_{l_*(X_2)} \circ i_*([\varphi]).$$

Proof. First note that t_X is given by a roof

$$\begin{tikzcd} & C(l_{2X}) \arrow[dl] \arrow[dr] & \\ i_*i^*(X) & & j_*j^!(X)[1]. \end{tikzcd}$$

Note that $l_{2X} \colon j_*j^!(X) \to X$ is a morphism of complexes and $C(j_*j^!(X)) \to X)$ its cone. We study the composition $t_{l_*(X_2)} \circ i_*([\varphi])$ by composing the following roofs

$$\begin{tikzcd} & i_*(X_1) \arrow[dl] \arrow[dr] & & C(j_*(X_2) \to l_*(X_2)) \arrow[dl] \arrow[dr] & \\ i_*(X_1) & & i_*(X_2) & & j_*(X_2)[1]. \end{tikzcd}$$

We first define a morphism $C(l_{2X}) \to C(j_*(X_2) \to l_*(X_2))$, induced by φ, such that the following diagram commutes

$$\begin{tikzcd} C(l_{2X}) \arrow[r] \arrow[d, "\mathrm{qis}"'] & C(j_*(X_2) \to l_*(X_2)) \arrow[d, "\mathrm{qis}"'] \\ i_*(X_1) \arrow[r, "i_*(\varphi)"'] & i_*(X_2). \end{tikzcd} \tag{2.18}$$

We do it explicitly for the case where $X \in Q_{\mathcal{A}}$, the general case is done analogously. In this case $C(l_{2X})$ is the complex given by

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[d] \\ 0 \arrow[r] & X_2 \arrow[r, "\mathrm{id}"] & X_2 \arrow[r] & 0 \end{tikzcd} \tag{2.19}$$

in position -1 and 0. Moreover, the complex given by $C(j_*(X_2) \to l_*(X_2))$ is

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[d] \\ 0 \arrow[r] & X_2 \arrow[r, "\mathrm{id}"] & X_2 \arrow[r] & 0, \end{tikzcd} \tag{2.20}$$

it has objects only in position -1 and 0.

We define the morphism in position 0 by

$$\begin{tikzcd}
X_1 \arrow[r, "\varphi"] \arrow[d, "\varphi"'] & X_2 \arrow[d, "\mathrm{id}"] \\
X_2 \arrow[r, "\mathrm{id}"'] & X_2.
\end{tikzcd} \tag{2.21}$$

In position -1 the morphism is given by the identity.

This morphism clearly makes the diagram commute.

We obtain the following roof

$$\begin{tikzcd}
& & C(l_{2X}) \arrow[dl] \arrow[dr] & & \\
& i_*(X_1) \arrow[dl] \arrow[dr] & & C(j_*(X_2) \to l_*(X_2)) \arrow[dl] \arrow[dr] & \\
i_*(X_1) & & i_*(X_2) & & j_*(X_2)[1].
\end{tikzcd}$$

Therefore, the object $C(l_{2X})$ will give us a common roof for both morphisms and it implies that

$$t_X = t_{l_*(X_2)} \circ i_*([\varphi]).$$

□

Remark 2.50. Note that from the $[\varphi]$ we can recover X up to isomorphism. Indeed, by Lemma 2.49, we obtain that after taking the cone of the composition of $i_*([\varphi])$ with $t_{l_*(X)}$ we get an object isomorphic to $X[1]$.

We now describe the Serre functor at the level of objects.

Lemma 2.51. *Let $X = X_1 \xrightarrow{\varphi} X_2$ be an element of $\mathcal{T}_A$. If $\mathcal{S}_{\mathcal{A}}(i_X)$ is a morphism of complexes, where $X_2 \xrightarrow{i_X} C(\varphi)$ is the morphism of complexes given by the injection, then $\mathcal{S}_{\mathcal{T}_A}(X)$ is isomorphic to the triple $Y := \mathcal{S}_{\mathcal{A}}(X_2) \xrightarrow{\mathcal{S}_{\mathcal{A}}(i_X)} \mathcal{S}_{\mathcal{A}}(C(\varphi))$ in $\mathcal{T}_A$.*

Proof. Let us consider the decomposition of X with respect to $\mathcal{T}_{\mathcal{A}} = \langle D_3, D_1 \rangle$

$$\begin{tikzcd}
C(\varphi)(-1) \arrow[r] \arrow[d] & X_1 \arrow[r, "{[\varphi]}"] \arrow[d, "\varphi"'] & X_2 \arrow[r, "\tau_X"] \arrow[d, "\mathrm{id}"'] & C(\varphi) \arrow[d] \\
0 \arrow[r] & X_2 \arrow[r, "\mathrm{id}"'] & X_2 \arrow[r] & 0.
\end{tikzcd} \tag{2.22}$$

Note that $\tau_X = q \circ [i_X] \in D^b(\mathcal{A})$ where $q \in \mathrm{Hom}_{D^b(\mathcal{A})}(C(\varphi), C(\varphi))$ is an isomorphism.

As the Serre functor is an autoequivalence and behaves well with respect to semiorthogonal decompositions (see Remark 2.47), we obtain the triangle given by the decomposition of $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X)$ with respect to $\langle D_1, D_2 \rangle$

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(i_* C(\varphi)[-1]) \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X) \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}} l_*(X_2) \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(i_* C(\varphi)).$$

We obtain precisely

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathcal{S}_{\mathcal{A}}(X_2) \arrow[r, "\mathrm{id}"] \arrow[d] & \mathcal{S}_{\mathcal{A}}(X_2) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathcal{S}_{\mathcal{A}}(C(\varphi)) \arrow[r, "\mathrm{id}"] & \mathcal{S}_{\mathcal{A}}(C(\varphi)) \arrow[r] & 0 \arrow[r, "t_{\mathcal{S}(X)}"] & \mathcal{S}_{\mathcal{A}}(C(\varphi))[1].
\end{tikzcd} \tag{2.23}$$

Note that by Remark 2.50 we have that $C(t_{\mathcal{S}(X)})[-1]$ is isomorphic to $\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X)$. Then, we now focus on studying the morphism $t_{\mathcal{S}(X)}$.

We first consider the triangle

$$l_*(C(\varphi)) \xrightarrow{l_{\mathrm{id}}} i_*(C(\varphi)) \xrightarrow{t_{l_*(C(\varphi)}} j_*(C(\varphi))[1] \to l_*(C(\varphi))[1]$$

given by the decomposition of $i_*(C(\varphi))$ with respect to $\langle D_2, D_3 \rangle$, i.e.

$$\begin{tikzcd}
C(\varphi) \arrow[r, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & C(\varphi) \arrow[r] \arrow[d] & 0 \arrow[d] \\
C(\varphi) \arrow[r, "l_{\mathrm{id}}"] & 0 \arrow[r, "t_{l_*(C(\varphi))}"] & C(\varphi)[1].
\end{tikzcd} \tag{2.24}$$

As the Serre functor is exact and behaves well with respect to semiorthogonal decompositions, by Lemma 2.42 we obtain

$$\begin{tikzcd}
\mathcal{S}_{\mathcal{A}}(C(\varphi)) \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[r] \arrow[d] & \mathcal{S}_{\mathcal{A}}(C(\varphi))[1] \arrow[d, "\mathrm{id}"'] \\
0 \arrow[r] & \mathcal{S}_{\mathcal{A}}(C(\varphi))[1] \arrow[r, "\mathrm{id}"'] & \mathcal{S}_{\mathcal{A}}(C(\varphi))[1].
\end{tikzcd}$$

Therefore, we have that $C(j_* \mathcal{S}_{\mathcal{A}}(C(\varphi)) \to l_* \mathcal{S}_{\mathcal{A}}(C(\varphi))) = i_*(\mathcal{S}_{\mathcal{A}}(C(\varphi))$ and

$$\begin{tikzcd}
\mathcal{S}_{\mathcal{A}}(C(\varphi)) \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[d] \\
0 \arrow[r, "t_{\mathcal{S}_{\mathcal{A}}(l_*(C(\varphi)))}"] & \mathcal{S}_{\mathcal{A}}(C(\varphi))[1].
\end{tikzcd}$$

Let us now consider the morphism $l_*(X_2) \to i_*(C(\varphi))$ as the last morphism of the (triangle 2.22) and $t_{\mathcal{S}(X)} = \mathcal{S}_{\mathcal{T}_\mathcal{A}}(l_*(X_2) \to i_*(C(\varphi)))$. Note that it is given by direct composition of the roof $l_*(\tau_X)$ with the roof l_{id}, as follows:

$$\begin{tikzcd} X_2 \arrow[r, "\tau_X"] \arrow[d, "\mathrm{id}"'] & C(\varphi) \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & C(\varphi) \arrow[d] \\ X_2 \arrow[r, "\tau_X"] & C(\varphi) \arrow[r, "l_{\mathrm{id}}"] & 0. \end{tikzcd} \tag{2.25}$$

After applying the Serre functor and by Lemma 2.42 we obtain

$$\begin{tikzcd} \mathcal{S}_\mathcal{A}(X_2) \arrow[r, "\mathcal{S}_\mathcal{A}(\tau_X)"] \arrow[d] & \mathcal{S}_\mathcal{A}(C(\varphi)) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r, "t_{l_*(\mathcal{S}_\mathcal{A}(C(\varphi))}"] & \mathcal{S}_\mathcal{A}(C(\varphi)). \end{tikzcd} \tag{2.26}$$

Since this morphism is precisely $t_{\mathcal{S}(X)}$, we get

$$C(t_{\mathcal{S}(X)})[-1] = C(t_{l_*(\mathcal{S}_\mathcal{A}(C(\varphi))} \circ i_*(\mathcal{S}_\mathcal{A}(\tau_X))[-1] \cong \mathcal{S}_{\mathcal{T}_\mathcal{A}}(X).$$

On the other hand, note that

$$C(t_{l_*(\mathcal{S}_\mathcal{A}(C(\varphi))} \circ i_*(\mathcal{S}_\mathcal{A}(\tau_X)) \cong C(t_{l_*(\mathcal{S}_\mathcal{A}(C(\varphi))} \circ i_*([\mathcal{S}_\mathcal{A}(i_X)]),$$

because $\tau_X = q \circ [i_X]$ where q is an isomorphism in $D^b(\mathcal{A})$. As $\mathcal{S}_\mathcal{A}(i_X)$ is a morphism of complexes, we get $[\mathcal{S}_\mathcal{A}(i_X)] = (\mathrm{id}, \mathcal{S}_\mathcal{A}(i_X))$ as a roof.

By Lemma 2.49, we obtain

$$t_{l_*(\mathcal{S}_\mathcal{A}(C(\varphi))} \circ i_*([\mathcal{S}_\mathcal{A}(i_X)]) = t_Y.$$

As a consequence $C(t_Y)[-1] = C(t_{\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X)})[-1]$ which is isomorphic to Y and to $\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X)$. Therefore, we obtain

$$\mathcal{S}_{\mathcal{T}_\mathcal{A}}(X) \cong Y$$

in $\mathcal{T}_\mathcal{A}$. □

Proposition 2.52. *Let $X = X_1 \xrightarrow{\varphi} X_2$ be an element of $\mathcal{T}_\mathcal{A}$. If $\mathcal{S}_\mathcal{A}^{-1}(p_X)$ is a morphism of complexes, where*

$$C(\varphi)[-1] \xrightarrow{p_X} X_1$$

is the projection, then $\mathcal{S}_{\mathcal{T}_A}^{-1}(X)$ *is isomorphic to* $\mathcal{S}_A^{-1}(C(\varphi)(-1)) \xrightarrow{\mathcal{S}_A^{-1}(p_X)} \mathcal{S}_{\mathcal{A}}^{-1}(X_1)$.

Proof. The proof follows along the lines of Proposition 2.51. □

We present the following examples:

Example 2.53. If $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$ and $n = 1$,

$$\mathcal{S}_{Q_1}(E_1, E_2, \varphi) = (E_2, \mathrm{C}(\varphi), i_X).$$

Example 2.54. If $\mathcal{A} = \mathrm{Coh}(X)$ and $n = 1$, where X is a smooth projective variety, then

$$\mathcal{S}_{\mathrm{TCoh}(X)}(E_1, E_2, \varphi) = (E_2 \otimes \omega_X[n], \mathrm{C}(\varphi) \otimes \omega_X[n], \psi),$$

where $\psi = i_X \otimes \mathrm{id}_{\omega_X}[n]$. If X is a n-Calabi–Yau variety, then

$$\mathcal{S}_{\mathrm{TCoh}(X)}(E_1, E_2, \varphi) = (E_2[n], \mathrm{C}(\varphi)[n], i_X[n]).$$

Definition 2.55. A triangulated category $\mathcal{T}$ is *n-Calabi–Yau* if $\mathcal{T}$ has a Serre functor $S_{\mathcal{T}}$ and there is an integer $n \in \mathbb{Z}$ such that $S_{\mathcal{T}} \cong [n]$.

Definition 2.56. A triangulated category $\mathcal{T}$ is a *fractional Calabi–Yau* if it has a Serre functor $S_{\mathcal{T}}$ and there are integers p and $q \neq 0$ such that $S_{\mathcal{T}}^q \cong [p]$.

Conjecture 2.57. *If $D^b(\mathcal{A})$ is a n-Calabi–Yau category, then $\mathcal{T}_{\mathcal{A}}$ is a fractional Calabi–Yau with $q = 3$ and $p = 3n + 1$.*

By Lemma 2.51, we have that

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}^3(X_1, X_2, \varphi) = (\mathcal{S}_{\mathcal{A}}^3(X_1[1]), \mathcal{S}_{\mathcal{A}}^3(X_2[1]), \mathcal{S}_{\mathcal{A}}^3(\varphi[1])).$$

Then, as $\mathcal{D}^b(\mathcal{A})$ is a n-Calabi–Yau category, we have that

$$\mathcal{S}_{\mathcal{T}_{\mathcal{A}}}^3(X_1, X_2, \varphi) = (X_1[3n + 1], X_2[3n + 1], \varphi[3n + 1]).$$

Therefore, we have already proved Conjecture 2.57 at the level of objects.

Under the assumption that the last conjecture is satisfied, we obtain the following examples:

Example 2.58. Let $\mathcal{A} = \mathrm{Vect}_{\mathbb{C}}$, then in $\mathcal{T}_{\mathrm{Vect}_{\mathbb{C}}}$ we have that $\mathcal{S}_{\mathcal{T}_{\mathrm{Vect}_{\mathbb{C}},1}}^3 = [1]$, which is a well-known result. See [Kel05, Ex. 8.3].

Example 2.59. Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is an elliptic curve. Then $D^b(\mathrm{TCoh}(C))$ is a fractional Calabi–Yau and $\mathcal{S}^3 = [4]$.

Example 2.60. Let $\mathcal{A} = \mathrm{Coh}(S)$, where S is a K3 surface. Then $D^b(\mathrm{TCoh}(S))$ is a fractional Calabi–Yau and $\mathcal{S}^3 = [7]$.

Conjecture 2.61. *Let Q be the Dynkin quiver A_m. We consider the category of representations $Q_{\mathcal{A}}$ over $\mathcal{A}$. Assume that $D^b(\mathcal{A})$ is a n-Calabi–Yau category, then $D^b(Q_{\mathcal{A}})$ admits a Serre functor and is a fractional Calabi–Yau category with $q = m+1$ and $p = (m+1)n + (m-1)$. See [Kel05, Ex. 8.3].*

Further semiorthogonal decompositions for $\mathcal{T}_{\mathcal{A},n}$, with $n \geq 2$

In the case $n \geq 2$, we now define inductively the categories D_k by using D_1 and D_2, for $k \in \mathbb{Z}$. We define the subcategories D_k inductively

$$D_k^{\perp} = D_{k-1}$$
$$^{\perp}D_k = D_{k+1}.$$

Moreover, as $\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(^{\perp}D) = D^{\perp}$, we get

$$D_k = \mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(D_{k+2}) \quad D_k = \mathcal{S}^{-1}_{\mathcal{T}_{\mathcal{A},n}}(D_{k-2}).$$

Remark 2.62. Since D_k is the orthogonal complement of an admissible subcategory and $\mathcal{T}_{\mathcal{A},n}$ is saturated, then D_k is also admissible.

Remark 2.63. By [BK90b, Lem. 1.9] and Lemma 1.23 we have $D_k{}^{\perp}$ and $^{\perp}D_k$ are equivalent categories. As D_1 and D_2 are equivalent to $D^b(\mathcal{A})$, we obtain that each of the subcategories D_k is equivalent to $D^b(\mathcal{A})$.

Remark 2.64. Let $X \in \mathcal{T}_{\mathcal{A},n}$. By the definition of D_k, we obtain semiorthogonal decompositions $\mathcal{T}_{\mathcal{A},n} = \langle D_k, D_{k+1}\rangle$ for $k \in \mathbb{Z}$. Let us consider the triangle of X induced by the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_k, D_{k+1}\rangle$,

$$X_{k+1} \to X \to X_k \to X_{k+1}[1].$$

After applying the Serre functor we obtain the triangle given for $\mathcal{S}_{\mathcal{T}_{\mathcal{A},n}}(X)$ induced by the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_{k-2}, D_{k-1}\rangle$

$$Y_{k-2} \to \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}(X) \to Y_{k-1} \to Y_{k-2}[1],$$

with $Y_{k-i} \in D_{k-i}$. The argument goes along the lines of Remark 2.47.

Remark 2.65. Note that to obtain the categories D_k, we are applying the braid group action on two strands $B_2 \cong \mathbb{Z}$ on the set of 2-term semiorthogonal decomposition as explained in Corollary 1.27 via left and right mutations.

If $\mathcal{A}$ is the category of vector spaces over $\mathbb{C}$, then $D_1 = \langle S_1 \rangle$ and $D_2 = \langle S_2 \rangle$ as mentioned in Remark 2.11 and (S_1, S_2) is a complete exceptional collection. As explained in [Rin94, Thm. 3], the action of the braid group is transitive on the set of complete exceptional sequences. See Corollary 1.99.

As A consequence of Remark 2.64, we focus on studying the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle$, and we will use the Serre functor and its inverse to study the semiorthogonal decompositions $\mathcal{T}_{\mathcal{A},n} = \langle D_k, D_{k+1} \rangle$.

2.2 Constructing stability conditions on $\mathcal{T}_{\mathcal{A},n}$

The aim of this section is to use the semiorthogonal decompositions described in Section 2.1 to construct pre-stability conditions on $\mathcal{T}_{\mathcal{A},n}$. The construction given in [CP10] applied to $\mathcal{T}_{\mathcal{A},n}$ heavily depends on the stability conditions in $\mathrm{Stab}(D^b(\mathcal{A}))$. For the convenience of the reader we review relevant material from [CP10, Sec. 2] without proofs to make our exposition self-contained. In addition, we prove the Harder-Narasimhan property in some cases and we give explicit examples of CP-glued pre-stability conditions on the category $\mathcal{T}_{\mathrm{Coh}(X)}$, where X is a nonsingular projective curve, surface or a particular threefold.

Let $\mathcal{T}$ be an arbitrary triangulated category.

We start by constructing hearts of bounded t-structures. In this thesis we use three different ways of constructing hearts on $\mathcal{T}$, namely CP-gluing, BBD-recollement and tilting with respect to a torsion pair on an existent heart as in Proposition 1.10. We now briefly explain and compare CP-gluing and BBD-recollement. For more details we recommend [CP10, Sec. 2] and [BBD82, Sec. 1.4].

We start by reviewing the construction of hearts of bounded t-structures coming from semiorthogonal decompositions. Let $\mathcal{T} = \langle D_1, D_2 \rangle$ be a semiorthogonal decomposition and let $i^*\colon \mathcal{T} \to D_1$ be the left adjoint of the inclusion $i_*\colon D_1 \to \mathcal{T}$, and $j^!\colon \mathcal{T} \to D_2$ be the right adjoint of the inclusion $j_*\colon D_2 \to \mathcal{T}$.

Lemma 2.66. [CP10, Lem. 2.1] *Let $\mathcal{A}_i$ on D_i be hearts of bounded t-structures such that*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0.$$

Then, there is a t-structure on $\mathcal{T}$ with the heart

$$\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2) = \{T \in \mathcal{T} \mid i^*(T) \in \mathcal{A}_1, j^!(T) \in \mathcal{A}_2\}. \tag{2.27}$$

With respect to this t-structure on $\mathcal{T}$ the functors i_* and $j^!$ are t-exact.

Remark 2.67. t-exactness of the functors i^* and $j^!$ implies that

$$i_*(\mathcal{A}_1) \subseteq \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2) \text{ and } j_*(\mathcal{A}_2) \subseteq \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2).$$

As a consequence, by Definition 1.2 of a heart of a bounded t-structure it follows directly that $\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(i_*A_1, j_*A_2) = 0$.

Theorem 2.68. [BBD82, Thm. 1.4.10] *Let $\mathcal{X}, \mathcal{Y}, \mathcal{T}$ be triangulated categories such that $\mathcal{T}$ is a recollement of $\mathcal{X}$ and $\mathcal{Y}$ and assume the notation of Definition 1.32 . Let $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ and $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ be t-structures in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then there is a t-structure $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$ in $\mathcal{T}$ defined by:*

$$\mathcal{T}^{\leq 0} := \{T \in \mathcal{T} \mid i^*T \in \mathcal{Y}^{\leq 0}, j^*T \in \mathcal{X}^{\leq 0}\}$$
$$\mathcal{T}^{\geq 0} := \{T \in \mathcal{T} \mid i^*T \in \mathcal{Y}^{\geq 0}, j^!T \in \mathcal{X}^{\geq 0}\}.$$

When $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ are the corresponding hearts in $\mathcal{X}$ and $\mathcal{Y}$ respectively, we denote by $\mathrm{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}}) := \mathcal{T}^{\leq 0} \cap \mathcal{T}^{\geq 0}$*.*

We now relate the hearts given by CP-gluing and by BBD-recollement. If the hearts used in Theorem 2.68 satisfy the additional orthogonal condition of Lemma 2.66, we have that both constructions coincide.

Lemma 2.69. [MR18, Prop. 1.16.12] *Let $\mathcal{T} = \langle D_1, D_2 \rangle$. Let $(D_1^{\leq 0}, D_1^{\geq 0})$ and $(D_2^{\leq 0}, D_2^{\geq 0})$ be t-structures with hearts $\mathcal{A}_1$ and $\mathcal{A}_2$ in D_1 and D_2 respectively. If* $\mathrm{Hom}^{\leq 0}(i_*A_1, j_*A_2) = 0$*, then*

$$\mathrm{rec}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2).$$

In Section 2.1 we have already described several semiorthogonal decompositions of $\mathcal{T}_{\mathcal{A},n}$. We now give necessary and sufficient conditions to apply Lemma 2.66 to $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle$.

Notation 2.70. Since throughout this section we are going to use only the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle$, for simplicity, we take $i_* := i_{1*}$ and $j_* := j_{2*}$.

Lemma 2.71. *Let $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2 \rangle$ be the semiorthogonal decomposition given in Corollary 2.12 for $n \geq 1$ and let $\sigma_2 = (Z_2, \mathcal{A}_2)$ be a stability condition on* $\mathrm{Stab}(D^b(\mathcal{A}))$*. If $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $\sigma_1 := \sigma_2 g = (Z_1, \mathcal{A}_1) \in$* $\mathrm{Stab}(D^b(\mathcal{A}))$*, then*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A},n}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$$

if and only if $f(0) \geq 0$.

Proof. First note that if we consider the slicing of the stability conditions, we obtain $\mathcal{A}_1 = \mathcal{P}_1(0,1] \subseteq D^b(\mathcal{A})$ and $\mathcal{A}_2 = \mathcal{P}_2(0,1] \subseteq D^b(\mathcal{A})$. By the definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action we get that $\mathcal{P}_1(0,1] = \mathcal{P}_2(f(0), f(1)]$. By adjointness, we have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A},n}}(i_*(E_1), j_*(E_2)) = \mathrm{Hom}^{\leq 0}_{D^b(\mathcal{A})}(E_1, i^!(j_*(E_2))),$$

for every $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. The triangle induced by the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_0, D_1 \rangle$ for $j_*(E_2)$ is given by

$$\begin{tikzcd} i^!(j_*(E_2)) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_2^{\oplus n} \arrow[r] \arrow[d] & E_2^{\oplus n} \arrow[d] \\ 0 \arrow[r] & E_2 \arrow[r, "\mathrm{id}"'] & E_2 \arrow[r] & 0, \end{tikzcd} \tag{2.28}$$

as shown in Remark 2.19. As a consequence, we obtain

$$\mathrm{Hom}_{D^b(\mathcal{A})}(E_1, i^!(j_*(E_2))) = \mathrm{Hom}_{D^b(\mathcal{A})}(E_1, E_2^{\oplus n}[-1]).$$

Note that $E_1 \in \mathcal{P}_2(f(0), f(1)]$ and $E_2^{\oplus n}[-1] \in \mathcal{P}_2(-1, 0]$. If $f(0) \geq 0$, then by the definition of slicing we obtain that

$$\mathrm{Hom}^{\leq 0}_{D^b(\mathcal{A})}(E_1, E_2^{\oplus n}[-1]) = 0.$$

If $f(0) < 0$, then $f(0) = -m + \theta$, with $-m \in \mathbb{Z}_{<0}$ and $\theta \in \mathbb{R}$ with $0 \leq \theta < 1$. There is $F \in \mathcal{P}_2(f(0), f(1)]$, such that $F[m] \in \mathcal{P}_2(0, 1]$.

Since $-m + 1 \leq 0$ then

$$\mathrm{Hom}^{-m+1}_{D^b(\mathcal{A})}(F, F^{\oplus n}[m-1]) \neq 0,$$

as the canonical inclusion is a non-zero morphism. Therefore, if

$$\mathrm{Hom}_{D^b(\mathcal{A})}(E_1, E_2^{\oplus n}[-1]) = 0,$$

for all $E_1 \in \mathcal{P}_2(f(0), f(1)]$ and E_2, with $E_2^{\oplus n}[-1] \in \mathcal{P}_2(-1, 0]$, then $f(0) \geq 0$. □

Remark 2.72. Let C be a smooth projective curve with $g(C) \geq 1$. Since the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action is transitive, there is always $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\sigma_1 = \sigma_2 g$, where $\sigma_i \in \mathrm{Stab}(C)$, for $i = 1, 2$. As a consequence, we can always apply Lemma 2.71.

Let us consider $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(D^b(\mathcal{A}))$. We take $g_i = (T_i, f_i) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and we define $\sigma_i = \sigma g_i = (Z_i, \mathcal{A}_i)$ for $i = 1, 2$.

Corollary 2.73.

$$\mathrm{Hom}_{\mathcal{T}_{\mathcal{A},n}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$$

if and only if $f_1(0) \geq f_2(0)$.

Proof. Note that $\sigma_1 = \sigma_2(g_2^{-1} g_1)$. By Lemma 2.71, it is enough to show that $f_2^{-1} \circ f_1(0) \geq 0$ if and only if $f_1(0) \geq f_2(0)$, which clearly follows from the fact that $f_2 \colon \mathbb{R} \to \mathbb{R}$ is continuous and increasing. □

By applying the Serre functor, we can obtain Corollary 2.73 for the other semiorthogonal decompositions. We do it explicitly for $\mathcal{T}_{\mathcal{A}}$, because we will use these conditions in the next chapter.

Corollary 2.74. *1. If $\mathcal{T}_{\mathcal{A}} = \langle D_3, D_1\rangle$, then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}}(l_*\mathcal{A}_1, i_*\mathcal{A}_2) = 0$ if and only if $f_1(0) \geq f_2(0) + 1$.*

2. If $\mathcal{T}_{\mathcal{A}} = \langle D_2, D_3\rangle$, then $\mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}}(j_*\mathcal{A}_1, l_*\mathcal{A}_2) = 0$ if and only if $f_1(0) \geq f_2(0) + 1$.

Proof. Let us consider the functor $G \coloneqq (- \otimes \omega_C^*[-1])_{Q_1}$, which by Remark 2.31 is an autoequivalence. We define the following autoequivalence $F \coloneqq G \circ \mathcal{S}_{\mathcal{T}_{\mathcal{A}}}$. Note that from Lemma 2.42, we obtain

$$F(l_*\mathcal{A}_i) = i_*\mathcal{A}_i \ , \ F(i_*\mathcal{A}_i) = j_*\mathcal{A}_i[1] \text{ and } F(j_*\mathcal{A}_i) = l_*\mathcal{A}_i,$$

for $i = 1, 2$. Therefore, we have that

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}{}_{\mathcal{T}_{\mathcal{A}}}(l_*\mathcal{A}_1, i_*\mathcal{A}_2) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(F(l_*\mathcal{A}_1), F(i_*(\mathcal{A}_2)) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2[1]).$$

After applying Corollary 2.73 to σ_1 and $\sigma_2[1] = (-T_2, f_2 + 1)$, we get that

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}{}_{\mathcal{T}_{\mathcal{A}}}(l_*\mathcal{A}_1, i_*\mathcal{A}_2) = 0 \text{ if and only if } f_1(0) \geq f_2(0) + 1.$$

Analogously, we have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(j_*\mathcal{A}_1, l_*\mathcal{A}_2) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(F(j_*\mathcal{A}_1), F(l_*\mathcal{A}_2)) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(l_*\mathcal{A}_1, i_*\mathcal{A}_2).$$

Then, as above $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(j_*\mathcal{A}_1, l_*\mathcal{A}_2) = 0$ if and only if $f_1(0) \geq f_2(0) + 1$. □

Notation 2.75. If X is a nonsingular projective variety, then we denote $\mathcal{T}_{\mathrm{Coh}(X)}$ as $\mathcal{T}_X$.

Remark 2.76. Consider $\mathcal{T}_X = \langle D_1, D_2\rangle$. By definition we have that

$$\mathrm{TCoh}(X) = \mathrm{gl}_{12}(\mathrm{Coh}(X), \mathrm{Coh}(X)).$$

Consider $\mathcal{T}_X = \langle D_2, D_3\rangle$, then we define

$$\mathcal{H}_{23} \coloneqq \mathrm{gl}_{23}(\mathrm{Coh}(X)[1], \mathrm{Coh}(X)).$$

Consider $\mathcal{T}_X = \langle D_3, D_1\rangle$, then we define

$$\mathcal{H}_{31} \coloneqq \mathrm{gl}_{31}(\mathrm{Coh}(X)[1], \mathrm{Coh}(X)).$$

Notation 2.77. When working in $\mathcal{T}_X$, we use the following notation

$$i_* \operatorname{Coh}(X) = \operatorname{Coh}_1(X) \text{ and } j_*(\operatorname{Coh}(X)) = \operatorname{Coh}_2(X) \text{ and } l_*(\operatorname{Coh}(X)) = \operatorname{Coh}_3(X).$$

Lemma 2.78. [MR18, Prop. 1.15.16] *The hearts* $\operatorname{TCoh}(C)$, $\mathcal{H}_{23}$ *and* $\mathcal{H}_{23}$ *on* $\mathcal{T}_C$ *have cohomological dimension two.*

We now define suitable stability functions on the hearts given in Lemma 2.66. We consider two Bridgeland stability conditions $\sigma_1 = (Z_1, \mathcal{A}_1)$ and $\sigma_2 = (Z_2, \mathcal{A}_2)$ on D_i, with $i = 1, 2$, respectively, such that they satisfy the orthogonality condition

$$\operatorname{Hom}_{\mathcal{T}}^{\leq 0}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0.$$

We define Z as

$$\begin{aligned} Z\colon \mathcal{N}(\mathcal{T}) &\to \mathbb{C} \\ [E] &\to Z_1([i^*(E)]) + Z_2([j^!(E)]), \end{aligned} \tag{2.29}$$

where $[E]$ denotes the class of E in $\mathcal{N}(\mathcal{T})$.

Our next step is to study whether the pair $\sigma = (Z, \operatorname{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2))$ is a Bridgeland stability condition on $\mathcal{T}$ i.e. if σ satisfies the support property and Z satisfies the Harder-Narasimhan property on $\operatorname{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$.

Remark 2.79. The homomorhpism Z is clearly a stability function on $\operatorname{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$. Indeed, as Z_1 and Z_2 are stability functions and $i^*(E) \in \mathcal{A}_1$ and $j^!(E) \in \mathcal{A}_2$ then $Z_1([i^*(E)])$ and $Z_2([j^!(E)])$ are in $\mathbb{H} \cup \mathbb{R}_{<0}$.

Remark 2.80. If we have a group homomorphism $v\colon \mathcal{N}(D_i) \twoheadrightarrow \Lambda_i$ and the central charges Z_i are defined on Λ_i, as in Definition 1.47, then we define the central charge $Z\colon \Lambda = \Lambda_1 \oplus \Lambda_2 \to \mathbb{C}$ as in Equation (2.29).

Definition 2.81. Let $\sigma = (Z, \mathcal{A})$ be a pair given by a heart of a bounded t-structure $\mathcal{A}$ and a stability function $Z\colon \mathcal{N}(\mathcal{A}) \to \mathbb{C}$. We say that σ is a *CP-glued pair* if there is a semiorthogonal decomposition of $\mathcal{T} = \langle D_1, D_2 \rangle$ and Bridgeland stability conditions $\sigma_1 = (Z_1, \mathcal{A}_1) \in \operatorname{Stab}(D_1)$ and $\sigma_2 = (Z_2, \mathcal{A}_2) \in \operatorname{Stab}(D_2)$ such that $\mathcal{A} = \operatorname{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$ and Z is given by Equation (2.29).

Definition 2.82. Let $\sigma = (Z, \mathcal{A})$ be a pair given by the heart of a bounded t-structure $\mathcal{A}$ and a stability function $Z\colon \mathcal{N}(\mathcal{A}) \to \mathbb{C}$. We say that σ is a *CP-glued pre-stability condition* if σ is a CP-glued pair and it is also a pre-stability condition i.e. Z satisfies the Harder-Narasimhan property on $\mathcal{A}$.

Definition 2.83. We define Θ_k the set of pre-stability conditions on $\mathcal{T}_{\mathcal{A},n}$ which are, up to the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, CP-glued induced by the semiorthogonal decomposition $\langle D_k, D_{k+1}\rangle$, for $k \in \mathbb{Z}$.

Notation 2.84. If σ is a CP-glued pair of $\sigma_i \in \mathrm{Stab}(D^b(\mathcal{A}))$, for $i = 1,2$, given by the semiorthogonal decomposition $\mathcal{T}_{\mathcal{A},n} = \langle D_k, D_{k+1}\rangle$. Then $\sigma = \mathrm{gl}_{k,k+1}(\sigma_1,\sigma_2)$, for $k \in \mathbb{Z}$.

The next lemma provides a characterization of the CP-glued pre-stability conditions.

Proposition 2.85. [CP10, Prop. 2.2]

1. *A pre-stability condition $\sigma = (Z,\mathcal{A})$ on $\mathcal{T}$ is CP-glued from $\sigma_1 = (Z_1,\mathcal{A}_1)$ on D_1 and $\sigma_2 = (Z_2,\mathcal{A}_2)$ on D_2 if and only if $Z_i = Z|_{D_i}$ for $i = 1,2$ and $\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$ with $i_*\mathcal{A}_1, j_*A_2 \subset \mathcal{A}$.*
2. *Let $\sigma = (Z,\mathcal{A})$ be a pre-stability condition on $\mathcal{T}$. Assume that the heart $\mathcal{A}$ is glued from hearts $\mathcal{A}_1 \subset D_1$ and $\mathcal{A}_2 \subset D_2$, with $\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$. Then, there exists pre-stability conditions $\sigma_i = (Z_i = Z|_{D_i}, \mathcal{A}_i)$ on D_i, for $i = 1,2$, such that σ is CP-glued from σ_1 and σ_2.*
3. *If $\sigma = (Z,\mathcal{P})$ is CP-glued from $\sigma_1 = (Z_1,\mathcal{P}_1)$ and $\sigma_2 = (Z_2,\mathcal{P}_2)$, then $i_*\mathcal{P}_1(\phi) \subset \mathcal{P}(\phi)$ and $j_*\mathcal{P}_2(\phi) \subset \mathcal{P}(\phi)$ for every $\phi \in \mathbb{R}$.*

We now explain the behaviour of the CP-glued pre-stability conditions under the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action. First of all, it is important to mention that if $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is a CP-glued pre-stability condition σg is not necessarily a CP-glued pre-stability condition. We explain this statement in detail in Chapter 3.

Lemma 2.86. *Let $\sigma = \mathrm{gl}_{12}(\sigma_1,\sigma_2)$ be a CP-glued pre-stability condition on $\mathcal{T}$ with respect to the semiorthogonal decomposition $\mathcal{T} = \langle D_1, D_2\rangle$. Let $g = (T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, then $\sigma g = (W,\mathcal{B})$ satisfies the following conditions: If $\sigma_i g = (W_i,\mathcal{B}_i)$, for $i = 1,2$ then*

1. *$i_*\mathcal{B}_1 \subseteq \mathcal{B}$ and $j_*\mathcal{B}_2 \subseteq \mathcal{B}$*
2. *$W\big|_{D_i} = W_i$, for i=1,2.*

Moreover, if $\mathrm{Hom}^{\leq 0}(i_*\mathcal{B}_1, j_*\mathcal{B}_2)$ then $\sigma g = \mathrm{gl}_{12}(\sigma_1 g, \sigma_2 g)$

Proof. Let us consider the slicing $\mathcal{P}_i$ of σ_i, for $i = 1,2$, and $\mathcal{P}$ the slicing of σ. By definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action we have $\mathcal{B}_i = \mathcal{P}_i(f(0), f(1)]$ and $\mathcal{B} = \mathcal{P}(f(0), f(1)]$. By the third part of Proposition 2.85 we obtain directly that $i_*\mathcal{B}_1 \subseteq \mathcal{B}$ and $j_*\mathcal{B}_2 \subseteq \mathcal{B}$. By definition we have that $W = T^{-1} \circ Z$, then if $E \in D_i$, we obtain that

$$W([E]) = T^{-1} \circ Z_i([E]) = W_i,$$

because if $E \in D_1$, then $i^*(E) = E$ and $j^!(E) = 0$ and if $E \in D_2$, then $i^*(E) = 0$ and $j^!(E) = E$. If we also assume that $\mathrm{Hom}^{\leq 0}(i_*\mathcal{B}_1, j_*\mathcal{B}_2)$, by the first part of Proposition 2.85 we obtain that $\sigma g = \mathrm{gl}_{12}(\sigma_1 g, \sigma_2 g)$. □

We now consider the hearts of $\mathcal{T} = \langle D_1, D_2 \rangle$ constructed via BBD-recollement from hearts $\mathcal{A}_i \subseteq D_i$, with $i = 1, 2$, which do not satisfy $\mathrm{Hom}^{\leq 0}_{\mathcal{T}}(i_* A_1, j_* A_2) = 0$. We claim that $\mathrm{rec}(i_* A_1, j_* A_2)$ does not admit a stability function. More precisely,

Proposition 2.87 (Jealousy lemma)**.** [MRRHR19] *Let $\mathcal{T}_{\mathcal{A}} = \langle D_1, D_2 \rangle$ and $\sigma_2 \in \mathrm{Stab}(D^b(\mathcal{A}))$. if $\sigma_1 = (\mathcal{A}_1, Z_1) = \sigma_2 g \in \mathrm{Stab}(D^b(\mathcal{A}))$, for $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. If $\mathrm{Hom}_{\mathcal{T}_{\mathcal{A}}}(i_* A_1, j_* A_2) \neq 0$, but $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(i_* \mathcal{A}_1, j_* \mathcal{A}_2) = 0$. Then, the heart $\mathrm{rec}(i_* \mathcal{A}, j_* \mathcal{A}_2)$ does not admit a stability function.*

2.2.1 Harder-Narasimhan property

The aim of this section is to study the Harder-Narasimhan (HN) property for the CP-glued pairs $\sigma = (Z, \mathcal{A})$ on $\mathcal{T}_{\mathcal{A},n}$. The idea behind proving the HN-property goes along the lines of [Bri08], i.e. the Harder-Narasimhan filtrations are produced first for the discrete stability functions and the general case is deduced by Bridgeland's deformation theorem for a connected component. We consider the general case just for $\mathcal{T}_C$. However, whenever we have that a CP-glued pair satisfies a stronger orthogonality condition, the HN-property is always satisfied. In this section, we study the discrete cases and in the next chapter we study the general case for $\mathcal{A} = \mathrm{Coh}(C)$. Note that the existence of stability conditions in $\mathrm{Stab}(\mathcal{T}_{\mathcal{A},n})$ heavily depends on the existence of stability conditions in $\mathrm{Stab}(D^b(\mathcal{A}))$.

We first recall useful statements to study the HN-property. We also recommend [BM11, Prop. B.2] for further details.

Lemma 2.88. [CP10, Lem. 3.4] *Let Z be a stability function on an abelian category $\mathcal{A}$. Assume that 0 is an isolated point of $\Im Z(\mathcal{A}) \subseteq \mathbb{R}_{\geq 0}$ and that $\mathcal{A}_0 = \{E \in \mathcal{A} \mid \Im(Z(E)) = 0\}$ is Noetherian. Then Z satisfies the Harder-Narasimhan property on $\mathcal{A}$ if and only if $\mathcal{A}$ is Noetherian.*

Corollary 2.89. [BLMS17, Rem. 2.5] *Let us consider a pair $\sigma = (Z, \mathcal{A})$ consisting of a heart of a bounded t-structure $\mathcal{A}$ and a stability function Z. If the image of Z is a discrete subset of $\mathbb{C}$ and $\mathcal{A}$ is Noetherian, then Z satisfies the Harder-Narasimhan property on $\mathcal{A}$.*

Proof. If Z is discrete, then it satisfies that 0 is an isolated point of $\Im Z(\mathcal{A}) \subseteq \mathbb{R}_{\geq 0}$. As $\mathcal{A}$ is Noetherian then $\mathcal{A}_0$ is also Noetherian and by Lemma 2.88, we obtain that Z satisfies the HN-property on $\mathcal{A}$. □

As we are taking stability conditions $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(D^b(\mathcal{A}))$, for $i = 1, 2$, i.e they satisfy the support property and the HN-property, this implies that they are locally finite. It follows that

$$\mathcal{A}_0 = \{E \in \mathcal{A} \mid \Im(Z(E)) = 0\}$$

is already Noetherian. As a consequence, we obtain the following result.

Corollary 2.90. [CP10, Prop. 3.5] *Let $\mathcal{T} = \langle D_1, D_2\rangle$ and $\sigma_1, \sigma_2 \in \mathrm{Stab}(D^b(\mathcal{A}))$ be discrete stability conditions such that $\mathrm{Hom}_{\mathcal{T}}^{\leq 0}(i_*A_1, j_*A_2) \leq 0$. Then Z, as defined in Equation (2.29), has the HN-property on $\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$.*

Proof. Since the σ_i are discrete, they satisfy that 0 is an isolated point of $\Im Z_i(\mathcal{A}_i) \subset \mathbb{R}_{\geq 0}$ for $i = 1, 2$. As σ_i has the HN-property and, by the support property of σ_i, the set $\mathcal{A}_0$ is already Noetherian. By Lemma 2.88 it implies that $\mathcal{A}_i$ is Noetherian. As a consequence, we have that $\mathcal{A}$ is Noetherian and by Lemma 2.89 we have that Z has the HN-property on $\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$. □

Remark 2.91. There are discrete stability conditions constructed in [Bri07] for curves, in [Bri08] and [AB13] for surfaces and in [BMS16] for abelian threefolds. Moreover, for curves and surfaces there exists a connected component in the space of stability conditions in which discrete stability conditions are dense.

Remark 2.92. Recall that by Lemma 1.58 all discrete pre-stability conditions are locally finite.

The next lemma asserts that whenever we have CP-glued pairs, whose hearts satisfy a stronger orthogonality assumption, then the Harder-Narasimhan property will be always satisfied.

Theorem 2.93. [CP10, Thm. 3.6] *Let $\mathcal{T} = \langle D_1, D_2\rangle$ and $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(D_i)$, with slicings $\mathcal{P}_i$, for $i = 1, 2$. If*

$$\mathrm{Hom}_{\mathcal{T}}^{\leq 0}(i_*A_1, j_*A_2) = 0$$

and there is $\theta \in (0, 1)$ such that

$$\mathrm{Hom}_{\mathcal{T}}^{\leq 0}(i_*\mathcal{P}_1(\theta, \theta + 1], j_*\mathcal{P}_2(\theta, \theta + 1)) = 0,$$

then Z, as defined in Equation (2.29), satisfies the Harder-Narasimhan property on $\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$.

2.2.2 CP-gluing pre-stability conditions on the same orbit

We now study CP-glued pairs $\sigma = (Z, \mathcal{B}) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ on $\mathcal{T}_{\mathcal{A},n} = \langle D_1, D_2\rangle$, where $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(D^b(\mathcal{A}))$, for $i = 1, 2$, and there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, with

$$T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}$$

and $\sigma_1 = \sigma_2 g$, i.e. if σ_1 and σ_2 are in the same $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit. In this section we study whether σ is a CP-glued pre-stability condition.

Proposition 2.94. *Under the assumption that $f(0) > 0$ or that $f(0) = 0$ with the condition that if $(A + C) = 0$ then $B \geq 0$, we obtain that σ is a CP-glued pre-stability condition.*

Proof. Let $\mathcal{P}_i$ be the slicing of σ_i, with $i = 1, 2$. By Lemma 2.71 we have that $f(0) \geq 0$. Note that by applying the same argument as in Lemma 2.71, after rotating σ_i by θ, we obtain that

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(i_*\mathcal{P}_1(\theta, \theta + 1], j_*\mathcal{P}_2(\theta.\theta + 1]) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_{\mathcal{A}}}(i_*(\mathcal{P}_2(f(\theta), f(\theta) + 1], j_*\mathcal{P}_2(\theta, \theta + 1]) = 0$$

if and only if $f(\theta) \geq \theta$. Therefore, we just need to show that there is $\theta \in (0, 1)$ such that $f(\theta) \geq \theta$. The proof falls naturally into three cases:

Case 1: If $f(0) \geq 1$. Then, if $\theta \in (0, 1)$ then $f(\theta) > f(0) \geq 1 > \theta$.

Case 2: If $1 > f(0) > 0$, by the correspondence between f and T we have that

$$f(\theta) = \arg(C\cos(\theta\pi) - B\sin(\theta\pi) + (-A\sin(\theta\pi) + D\cos(\theta\pi)i).$$

Since $f(0) > 0$, then $D > 0$. As a consequence, there is always a $\theta \in (0, 1)$ such that $-A\sin(\theta\pi) + D\cos(\theta\pi) = 0$. Indeed, it is equivalent to prove that there is $x \in \mathbb{R}$ with $-A + Dx = 0$, which follows from the fact that the image of $\cot(y)$ between 0 and π covers all the reals.

If $-A\sin(\theta\pi) + D\cos(\theta\pi) = 0$, then $f(\theta) = \arg(-\det(T^{-1})\sin(\theta\pi)) = 1 > \theta$, because $\det(T^{-1}) \geq 0$. Therefore, we found θ with $f(\theta) \geq \theta$

Case 3: If $f(0) = 0$, i.e $D = 0$ and $C \geq 0$. Since $\det(T^{-1}) = -AC > 0$ then $-A \geq 0$. By the correspondence between slope and argument, we have that that $f(\theta) \geq \theta$ if and only if

$$\frac{-\cos(\theta\pi)}{\sin(\theta\pi)} \leq \frac{-C\cos(\theta\pi) + B\sin(\theta\pi)}{-A\sin(\theta\pi)},$$

i.e. if and only if

$$(A + C)\cos(\theta\pi) - B\sin(\theta\pi) \leq 0.$$

Once again if $(A+C) \neq 0$, by using $\cot(y)$, it is enough to find $x \in \mathbb{R}$ with $(A+C)x - B \leq 0$, which is always possible. If $(A + C) = 0$, then it is only possible to find θ if $B \geq 0$. □

So far we have shown that the discrete CP-glued pairs in $\mathcal{T}_{\mathcal{A},n}$ and the CP-glued pairs induced by stability conditions on the same orbit with $f(0) > 0$ are pre-stability conditions.

Remark 2.95. Note that Lemma 2.94 implies that if $\mathrm{Stab}(D^b(\mathcal{A}))$ is not empty then there always exist CP-glued pre-stability conditions on $\mathcal{T}_{\mathcal{A},n}$.

Remark 2.96. Let C be a non-singular projective curve with positive genus. As $\mathrm{Stab}(\mathcal{C}) \cong \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, then if $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ is a CP-glued pair, there is always

$g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\sigma_1 g = \sigma_2$. Then, by Lemma 2.94 if $f(0) > 0$, or $f(0) = 0$ together with the assumptions of that lemma, σ satisfy the Harder-Narasimhan property.

Let us make the last remark more explicit.

Example 2.97. Let $\mathcal{A} = \mathrm{Coh}(C)$, where C is a smooth projective curve of genus $g \geq 1$.

If $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(C)$, then under the isomorphism given by Theorem 1.75, they are given by $(T_i, f_i) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that

$$Z_i(d, r) = A_i d + B_i r + i(C_i r + D_i d),$$

where $i = 1, 2$. If $f_1(0) \geq f_2(0)$, by Lemma 2.71 we obtain that σ is a CP-glued pair. We split our example into two cases:

Case 1: If $f_1(0) > f_2(0)$. By Lemma 2.94 we obtain that σ is a CP-glued pre-stability condition on $\mathcal{T}_C$.

Case 2: If $f_1(0) = f_2(0)$. There is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ with $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C. \end{bmatrix}$ If $(A + C) \neq 0$, then by Lemma 2.94, we have that σ is a CP-glued pre-stability condition. If $C \in \mathbb{Q}$, then by Lemma 2.90, we obtain that σ is a CP-glued pre-stability condition.

Note that if $g(C) = 0$ and if σ_i, for $i = 1, 2$ are in the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit of σ_μ, the same example holds.

We recover the notion of α-stability for holomorphic triples given by Álvarez-Cónsul and García-Prada in [ACGP01] as a CP-glued pre-stability condition.

Remark 2.98 ($\alpha-$stability)**.** Let $\sigma = (Z_1(r, d) = -d - r\alpha + ir, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$, which corresponds to $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, where $T^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$ and $f(0) = 0$. then $\mathrm{gl}_{12}(\sigma, \sigma_\mu) = (Z, \mathrm{TCoh}(C))$ is a discrete CP-glued pre-stability conditon and

$$Z(r_1, d_1, r_2, d_2) = -d_1 - d_2 - \alpha r_1 + i(r_1 + r_2).$$

Remark 2.99. Pre-stability conditions $\sigma = (Z, \mathrm{TCoh}(C))$ on elliptic curves were studied in [Dal07, Sec. 5.3].

Remark 2.100. Let $\mathcal{A} = \mathrm{Coh}(X)$, where X is a smooth projective variety. The existence of Bridgeland stability conditions has already been proved if X is a surface in [AB13], if X an Abelian threefold in [BMS16] and [MP15], for Fano threefolds with Picard rank one in [Li18] and for smooth quadrics in [Sch14]. Note that if there is $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(X)$, then

$$\mathrm{gl}_{12}(\sigma g, \sigma)$$

is a CP-glued pre-stability condition in $\mathcal{T}_X$, where $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, satisfies the conditions of Lemma 2.94.

The Discriminant

Definition 2.101. We define

$$\Delta(M) = (A + C)^2 - 4BD,$$

where $M = T^{-1}$.

For the notation above see Subsection 2.2.2.

We now classify pre-stability conditions $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ as above, by using its discriminant. We also study the behaviour of the pre-stability conditions up to the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action. This classification will play an important role in the proof of the support property of the pre-stability conditions on $\mathcal{T}_C$. See Sec. 3.3. We start by studying stability conditions with $\Delta(M) \geq 0$.

Remark 2.102. As $\det(T) > 0$, if $\Delta(M) \geq 0$ we only have two options, either the eigenvalues are both positive or the eigenvalues are both negative.

Lemma 2.103. *Let σ be a CP-glued pre-stability condition given as above with $\Delta(M) \geq 0$ and positive eigenvalues, then there is $h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that $\sigma h = \mathrm{gl}_{12}(\sigma_1 h, \sigma_2 h)$ with $\sigma_i h = (Z'_i, \mathcal{A}'_i)$ and*

$$\mathcal{A}'_1 = \mathcal{A}'_2.$$

Proof. As $\Delta(M) \geq 0$, it guarantees the existence of real eigenvalues for M and therefore for T. Let λ be an eigenvalue. Let us consider the corresponding eigenvector v. We obtain $Tv = \lambda v$. We give v in polar coordinates $v = m(\cos(\phi, \sin(\phi))$, with $\phi \in (-\pi, \pi]$ and $m \in \mathbb{R}_{>}0$. We define $h = (K_\phi, f_\phi) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.

First of all, we consider $g \circ h = (TK_\phi, f \circ f_\phi)$. Let us show that $f \circ f_\phi(0) = \phi/\pi$. By the correspondence between $f \circ f_\phi$ and TK_ϕ over S^1, it suffices to compute $TK_\phi v_0$, where $v_0 = (1, 0)$. We have that $(TK_\phi)v_0 = T(1/m)v$. Since $(\cos(\phi), \sin(\phi))$ is an eigenvector, we get $(TK_\phi)v_0 = \frac{\lambda v}{m}$. Consequently, if we study the induced map $f\colon S^1 \to S^1$, where $S^1 = (-1, 1]$, we obtain $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi$ on S^1. Therefore, we get $f \circ f_\phi(0) = \phi/\pi + 2k$ over $\mathbb{R}$ with $k \in \mathbb{Z}$. Due to the fact that f is an increasing continuous function and $-1 < \phi/\pi \leq 1$, it implies $-1 \leq f(-1) < \phi/\pi + 2k < f(1) = f(0) + 1 < 2$ and $k = 0$, $k = 1$ or $k = -1$.

If $k = 1$, then $-1 \leq \phi/\pi + 2 < 2$ and $1 < \phi/\pi + 2 \leq 3$. It implies that $-1 < \phi/\pi < 0$ and $-1 \leq f(\phi/\pi) < 1$, i.e. $-1 < \phi/\pi + 2 < 1$, this clearly forces $-3 < \phi/\pi < -1$, which is impossible.

If $k = -1$, then $-1 < \phi/\pi - 2 < 2$ and $-3 < \phi/\pi - 2 \leq -1$, it implies that $\phi/\pi = 1$ and $1 \leq f(\phi/\pi) < 2$, which is impossible.

Then $k = 0$ and $f(\phi/\pi) = \phi/\pi$.

If $\sigma_i h = (Z'_i, \mathcal{A}'_i)$, for $i = 1, 2$, we have that

$$\mathcal{A}'_1 = \mathcal{P}_2(f \circ f_\phi(0), f \circ f_\phi(1)] = \mathcal{P}_2(\phi/\pi, \phi/\pi + 1]$$

and

$$\mathcal{A}'_2 = \mathcal{P}_2(f_\phi(0), f_\phi(1)] = \mathcal{P}_2(\phi/\pi, \phi/\pi + 1],$$

so $\mathcal{A}'_1 = \mathcal{A}'_2$ and by Lemma 2.86, we obtain $\sigma h = \mathrm{gl}_{12}(\sigma_2 g h, \sigma_2 h)$. □

Lemma 2.104. *Let σ be a CP-glued pre-stability condition given as above with $\Delta(M) \geq 0$ and negative eigenvalues, then there is $h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that $\sigma h = \mathrm{gl}_{12}(\sigma_1 h, \sigma_2 h)$ with $\sigma_i h = (Z'_i, \mathcal{A}'_i)$ and*

$$\mathrm{Hom}^{\leq 1}(i_* \mathcal{A}'_1, j_* \mathcal{A}'_2) = 0.$$

Proof. As $\Delta(M) \geq 0$, it guarantees the existence of real eigenvalues for T^{-1} and therefore for T. Let $\beta < 0$ be an eigenvalue of T. Let us consider the corresponding eigenvector v. We obtain $Tv = \beta v$. We give v in polar coordinates $v = m(\cos(\phi), \sin(\phi))$, with $\phi \in (-\pi, \pi]$ and $m \in \mathbb{R}_{>0}$.

We study now σh where $h = (K_\phi, f_\phi) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.

First of all we consider $gh = (TK_\phi, f \circ f_\phi)$. By the correspondence between $f \circ f_\phi$ and TK_ϕ over S^1, to compute $f \circ f_\phi(0)$ it suffices to compute $TK_\phi v_0$, where $v_0 = (1, 0)$. We have $(TK_\phi)v_0 = T(1/m)v$. Since $(\cos(\phi), \sin(\phi))$ is an eigenvector, we get $(TK_\phi)v_0 = \frac{\beta v}{m}$. Consequently, if we study the induced map $f \colon S^1 \to S^1$, where $S^1 = (-1, 1]$, as $\beta < 0$, we show that $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$. We obtain two cases:

Case 1: If $-1 < \phi/\pi \leq 0$, then $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$ on S^1. Consequently, we obtain $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1 + 2k$, for $k \in \mathbb{Z}$. Due to the fact that f is an increasing continuous function, we have that $-1 < \phi/\pi \leq 0$, implies

$$-1 < f(-1) < \phi/\pi + 1 + 2k \leq f(0) < 1,$$

and the only possible option is $k = 0$.

Case 2: If $0 < \phi/\pi \leq 1$, then $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi - 1$ on S^1. Consequently, we obtain $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi - 1 + 2k$, for $k \in \mathbb{Z}$. Due to the fact that f is an increasing continuous function, we have that $0 < \phi/\pi \leq 1$ implies $0 < f(0) < \phi/\pi - 1 + 2k \leq f(1) < 2$, and the only possible option is $k = 1$.

Therefore, we have $f \circ f_\phi(0) = f(\phi/\pi) = \phi/\pi + 1$ We now consider $\sigma_2 h$. As $\sigma_i h = (Z'_i, \mathcal{A}'_i)$, for $i = 1, 2$, we show that $\mathrm{Hom}^{\leq 1}_{\mathcal{T}_{\mathcal{A}}}(i_*(\mathcal{A}'_1), j_*(\mathcal{A}'_2)) = 0$. Indeed, as

$$\mathcal{A}'_1 = \mathcal{P}_2(f \circ f_\phi(0), f \circ f_\phi(1)] = \mathcal{P}_2(\phi/\pi + 1, \phi/\pi + 2]$$

and

$$\mathcal{A}'_2 = \mathcal{P}_2(f_\phi(0), f_\phi(1)] = \mathcal{P}_2(\phi/\pi, \phi/\pi + 1].$$

By Lemma 2.86 we have that $\sigma h = \mathrm{gl}_{12}(\sigma_1 h, \sigma_2 h)$

Remark 2.105. If $\Delta(M) \geq 0$, then we reduce our study, up to the action, to two types of pre-stability conditions $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$, either it satisfies a stronger orthogonality condition

$$\mathrm{Hom}^{\leq 1}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$$

or

$$\mathcal{A}_1 = \mathcal{A}_2.$$

We now consider pre-stability conditons with $\Delta(M) < 0$.

Lemma 2.106. *Let $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ be a CP-glued pre-stability condition given as above. For all $h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, we have that $\sigma h = \mathrm{gl}_{12}(\sigma_1 h, \sigma_2 h)$ is a CP-glued pre-stability condition.*

Proof. We assume that there is a $h = (K_r, f_r) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, with $r \in \mathbb{R}$, such that σh is not a CP-glued stability condition. Note that by Corollary 1.81 it is enough to show our statement for h of this form. If $\sigma_i h = (Z'_i, \mathcal{A}'_i)$ then

$$\mathcal{A}'_2 = \mathcal{P}_2(r, r + 1]$$

and

$$\mathcal{A}'_1 = \mathcal{P}_2(f(r), f(r) + 1].$$

By Lemma 2.71, it implies that there is $r \in \mathbb{R}$ such that $f(r) < r$. But $f(0) \geq 0$, therefore there is $x' \in \mathbb{R}$ with $f(x') = x'$. As the restriction of f to S^1 agrees with the restriction of T to S^1, we obtain that T has a real eigenvalues and it contradicts the assumption that $\Delta(M) < 0$. $\square$

2.2.3 Semistability on $\mathcal{T}_{\mathcal{A}}$

We now follow the steps of [BGP96, Sec. 3] in order to study the σ-semistable objects in the CP-glued pre-stability conditions $\sigma = (Z, \mathcal{B}) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$, on $\mathcal{T}_{\mathcal{A}}$ such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, that satisfies $\sigma_1 = \sigma_2 g$ with $1 > f(0) \geq 0$.

We start by the case $f(0) = 0$ and by Remark 2.105 the pre-stability conditions with $1 > f(0) \geq 0$ and positive discriminant.

We have that $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ has the following form

$$T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix},$$

with $C > 0$ and $\det(T) > 0$. It implies that $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{H}$. We obtain that $\sigma_1 = (Z_1, \mathcal{H})$ and $\sigma_2 = (Z_2, \mathcal{H})$. By definition we have

$$\begin{aligned} \Re(Z_1(w)) &= -A\Re(Z_2(w))) + B\Im(Z_2(w)), \\ \Im(Z_1(w)) &= C\Im(Z_2(w)), \end{aligned} \tag{2.30}$$

where $w \in \mathcal{N}(D^b(\mathcal{A}))$.

We use the following notation

$$d_2 = -\Re(Z_2([j^!(E])) \ , \ d_1 = -\Re(Z_2([i^*(E)])),$$

and

$$r_2 = \Im(Z_2([j^!(E)])) \text{ and } r_1 = \Im(Z_2([i^*(E)])).$$

Therefore,

$$Z_2([j^!(E)]) = -d_2 + ir_2 \text{ and } Z_1([i^*(E)]) = Ad_1 + Br_1 + i(Cr_1).$$

Note that for every $E \in \mathcal{B}$, we have that $r_1, r_2 \geq 0$. If $r_1, r_2 \neq 0$, we define

$$\mu_\sigma(E) = \frac{-Ad_1 - Br_1 + d_2}{Cr_1 + r_2}.$$

We obtain the following inequalities.

Lemma 2.107. *If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{B}$ with $r_1, r_2 \neq 0$ is a σ-semistable object and $[\varphi] \neq 0$, then*

$$-B \geq (A + C)\mu_\sigma(E)$$

Proof. As $\mathcal{H}$ is an abelian category, we can compute $\mathrm{Ker}(\varphi), \mathrm{Coker}(\varphi) \in \mathcal{H}$ and by the definition of $\mathcal{B}$ we obtain the following short exact sequence in $\mathcal{B}$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & \mathrm{Ker}(\varphi) \arrow[r] \arrow[d, "0"'] & E_1 \arrow[r] \arrow[d, "\varphi"'] & \mathrm{Img}(\varphi) \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd} \tag{2.31}$$

and

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & E_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Img}(\varphi) \arrow[r] & E_2 \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0.
\end{tikzcd} \tag{2.32}$$

Let $Z_2(\mathrm{Img}(\varphi)) = -d_1'' + r_1''i$. Note that $r_1'' \neq 0$. Indeed if $r_1'' = 0$, then $\phi(j_*(\mathrm{Img}(\varphi))) = 1$, and by the σ-semistability $\phi(j_*(\mathrm{Img}(\varphi))) \leq \phi(E) \leq 1$, therefore $\mathrm{Img}(Z(E)) = 0$, which implies that $r_1, r_2 = 0$, which is a contradiction.

From these short exact sequences and the correspondence between slope and phase, we obtain that

$$\frac{d_1''}{r_1''} \leq \mu_\sigma(E) \text{ and } \mu_\sigma(E) \leq \frac{-Ad_1'' - Br_1''}{Cr_1''}.$$

It follows that

$$\begin{aligned}
\mu_\sigma(E) &\leq \frac{-A}{C}\frac{d_1''}{r_1''} - \frac{B}{C}, \\
\mu_\sigma(E) &\leq \frac{-A}{C}\mu_\sigma(E) - \frac{B}{C}.
\end{aligned}$$

As we have that $-A, C > 0$, therefore we get

$$\mu_\sigma(E)(A + C) \leq -B.$$

$\square$

Example 2.108. Consider σ_α as in Remark 2.98 on $\mathcal{T}_C$. As

$$\sigma_\alpha = \mathrm{gl}_{12}(\sigma_\mu g, \sigma_\mu)$$

with $g = (T, f)$ where $T^{-1} = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$ and $f(0) = 0$. If there are σ_α-semistable objects, as $A + C = 0$ and $-B = \alpha$, then $\alpha \geq 0$. See [BGP96, Prop. 3.13].

Lemma 2.109. *If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$ with $r_1, r_2 \neq 0$ and $[\varphi] \neq 0$ is σ-semistable, then*

$$-Ad_1 + d_2 - \mu_\sigma(E)(r_2 - Ar_1) \leq 0.$$

Proof. We clearly have

$$\mu_\sigma(E) = \frac{-Ad_1 + d_2 - Br_1}{Cr_1 + r_2} = \frac{-Ad_1 + d_2}{Cr_1 + r_2} + \frac{-Br_1}{Cr_1 + r_2}.$$

By Lemma 2.107, we obtain

$$\begin{aligned} \mu_\sigma(E) &\geq \frac{-Ad_1 + d_2}{Cr_1 + r_2} + (A + C)\frac{\mu_\sigma(E)r_1}{Cr_1 + r_2}, \\ \mu_\sigma(E)(r_2 - Ar_1) &\geq -Ad_1 + d_2. \end{aligned}$$

□

Lemma 2.110. *If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$ with $r_1, r_2 \neq 0$ and $[\varphi] \neq 0$ is a σ-semistable object, then*

$$(r_2 - r_1)\mu_\sigma(E) \leq d_2 - d_1$$

Proof. If $\mathrm{Ker}(\varphi) = 0$, then we have the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E_1 \arrow[r] & E_2 \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0. \end{tikzcd} \tag{2.33}$$

By the σ-semistability of E, we obtain directly $\mu_\sigma(E) \leq \frac{d_2 - d_1}{r_2 - r_1}$.

If $\mathrm{Coker}(\varphi) = 0$, then we have the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathrm{Ker}(\varphi) \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & E_2 \arrow[r] \arrow[d, "\mathrm{id}"'] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0 \end{tikzcd} . \tag{2.34}$$

We obtain

$$\mu_\sigma(E) \geq \frac{-A(d_1 - d_2) - B(r_1 - r_2)}{C(r_1 - r_2)} = \frac{-A(d_1 - d_2)}{C(r_1 - r_2)} + \frac{-B}{C},$$

By Lemma 2.107, we have the following inequality

$$\mu_\sigma(E) \geq \mu_\sigma(E) + \mu_\sigma(E)\frac{A}{C} - \frac{A(d_1 - d_2)}{C(r_1 - r_2)}.$$

Since we assume $-A, C \geq 0$, it implies directly that

$$\mu_\sigma(E) \geq \frac{d_1 - d_2}{r_1 - r_2}.$$

As in this case we get that $r_1 - r_2 \geq 0$, because $\mathrm{rk}(\mathrm{Ker}(\varphi)) = r_1 - r_2$ and $\mathrm{Ker}(\varphi) \in \mathcal{H}$, we obtain that

$$\mu_\sigma(E)(r_2 - r_1) \leq (d_2 - d_1).$$

We now assume that $\mathrm{Coker}(\varphi) \neq 0$ and $\mathrm{Ker}(\varphi) \neq 0$. From the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \mathrm{Ker}(\varphi) \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & \mathrm{Img}(\varphi) \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0, \end{tikzcd} \tag{2.35}$$

it follows that

$$\frac{-Ad_1' - Br_1'}{Cr_1'} \leq \mu_\sigma(E),$$

where $Z_2(\mathrm{Ker}(\varphi)) = -d_1' + r_1' i$. Note that we get $r_1' \neq 0$. If $r_1' = 0$, then $\phi(i_*(\mathrm{Ker}(\varphi))) = 1$ and by σ-semistability we have that $1 = \phi(i_*(\mathrm{Ker}(\varphi))) \leq \phi(E) \leq 1$, which implies that $r_1, r_2 = 0$, and it gives us a contradiction.

We now make some computations

$$\begin{aligned} \frac{-Ad_1'}{Cr_1'} &\leq \mu_\sigma(E) + \frac{B}{C} \\ \frac{-Ad_1' r_1}{r_1'} &\leq \mu_\sigma(E) C r_1 + B r_1 \\ &\leq \frac{-ACd_1 r_1 - BCr_1^2 + Cd_2 r_1 + BCr_1^2 + Br_1 r_2}{Cr_1 + r_2} \\ &\leq \frac{-ACd_1 r_1 + Cd_2 r_1 + Br_1 r_2}{Cr_1 + r_2} \\ &\leq \frac{(Cr_1 + r_2)(-Ad_1 + d_2) + Ad_1 r_2 + Br_1 r_2 - d_2 r_2}{Cr_1 + r_2} \\ &\leq -Ad_1 + d_2 - \mu_\sigma(E) r_2. \end{aligned}$$

We obtain

$$-Ad_1' r_1 + \mu_\sigma(E) r_2 r_1' - r_1'(-Ad_1 + d_2) \leq 0. \tag{2.36}$$

Note that if it is satisfied that $r_2 - r_1'' \neq 0$, then from the exact sequence below

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\phi"] & E_1 \arrow[r] \arrow[d, "\phi"] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Img}(\varphi) \arrow[r] & E_2 \arrow[r] & \mathrm{Coker}(\varphi) \arrow[r] & 0
\end{tikzcd} \tag{2.37}$$

it follows that

$$\mu_\sigma(E) \leq \frac{d_2 - d_1''}{r_2 - r_1''}, \tag{2.38}$$

where $Z_2(\mathrm{Img}(\phi)) = -d_1'' + ir_1''$, as before.

As $d_1'' = d_1 - d_1'$ and $r_1'' = r_1 - r_1'$, after multiplying 2.38 by $-Ar_1$ and adding 2.36 we obtain

$$Ar_1(d_2 - d_1) - r_1'(-Ad_1 + d_2) + \mu_\sigma(E)(-Ar_1(r_2 - r_1) + r_1'(-Ar_1 + r_2)) \leq 0.$$

By Lemma 2.109, we have

$$d_1 - d_2 + \mu_\sigma(E)(r_2 - r_1) \leq 0.$$

As a consequence, we get

$$\mu_\sigma(E)(r_2 - r_1) \leq d_2 - d_1.$$

If $r_2 - r_1'' = 0$, then we obtain that $d_2 - d_1'' \geq 0$. Due to the fact that $d_1'' = d_1 - d_1'$ and $r_1'' = r_1 - r_1'$, we obtain that

$$-Ar_1(d_1 - d_1' - d_2) \leq 0.$$

Adding the last inequality with Equation (2.36) we have that

$$\mu_\sigma(E)r_2r_1' - r_1'(-Ad_1 + d_2) - Ar_1(d_1 - d_2) \quad \leq \quad 0.$$

By Lemma 2.109, we obtain

$$\begin{aligned}
\mu_\sigma(E)r_2r_1' - r_1'(\mu_\sigma(E)(r_2 - Ar_1) - Ar_1(d_1 - d_2)) &\leq 0 \\
-Ar_1(-\mu_\sigma(E)r_1' - (d_1 - d_2)) &\leq 0 \\
-Ar_1(\mu_\sigma(E)(r_2 - r_1) - (d_1 - d_2)) &\leq 0.
\end{aligned}$$

As $-A > 0$, we conclude that

$$\mu_\sigma(E)(r_2 - r_1) \leq (d_2 - d_1).$$

$\square$

Lemma 2.111. *Let $\sigma = (Z, \mathcal{A})$ be as above with $Z = Ad_1 + Br_1 - d_2 + i(Cr_1 + r_2)$. If there is a σ-semistable object with $r_2 > r_1 > 0$ then*

$$Cy + Ax \leq -B,$$

where $x = \frac{d_1}{r_1}$ and $y = \frac{d_2}{r_2}$. Moreover, if $[\varphi] \neq 0$ then

$$y - x \geq 0,$$

and

$$-B \in [Cy + Ax, (Ax + Cy + y - x - \frac{r_1}{r_2}x(A + C))\frac{r_2}{r_2 - r_1}].$$

Proof. Let $E = E_1 \xrightarrow{\varphi} E_2$ be a σ-semistable object. Let us consider the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & E_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{2.39}$$

By the semistability of E and the correspondence between slope and phase it follows that

$$\frac{d_2}{r_2} \leq \mu_\sigma(E) \leq \frac{-Ad_1 - Br_1}{Cr_1}. \tag{2.40}$$

Thus it implies

$$Cr_1d_2 + Ad_1r_2 \leq -\frac{B}{C}r_1r_2,$$

as $r_1, r_2, C > 0$. We obtain

$$Cy + Ax \leq -B.$$

As $\varphi \neq 0$, by Lemma 2.110, we have the following inequality

$$\frac{-Ad_1 - Br_1 + d_2}{Cr_1 + r_2} \leq \frac{d_2 - d_1}{r_2 - r_1}.$$

From this equation we obtain

$$-B \leq \frac{Cd_2r_1 + Ad_1r_2 - (C + A)d_1r_1 + d_2r_1 - d_1r_2}{(r_2 - r_1)r_1.}$$

Since we have that $x = \frac{d_1}{r_1}$ and $y = \frac{d_2}{r_2}$, we obtain

$$-B \leq (Ax + Cy + y - x - \frac{r_1}{r_2}x(A+C))\frac{r_2}{r_2 - r_1}.$$

Therefore, we have

$$-B \in [Cy + Ax, (Ax + Cy + y - x - \frac{r_1}{r_2}x(A+C))\frac{r_2}{r_2 - r_1}].$$

From the equation

$$Cy + Ax \leq -B,$$

we obtain

$$\begin{aligned}
Cy + Ax &\leq (Ax + Cy + y - x - \frac{r_1}{r_2}x(-A-C))\frac{r_2}{r_2 - r_1}, \\
(Cy + Ax)(r_2 - r_1) &\leq (Ax + Cy + y - x)r_2 - x(C+A)r_1, \\
0 &\leq (Ax + Cy + y - x)r_2 - x(C+A)r_1 - (Cy + Ax)r_2 + (Cy + Ax)r_1, \\
0 &\leq (y - x)r_2 + (y - x)Cr_1, \\
0 &\leq (y - x)(r_2 + Cr_1).
\end{aligned} \tag{2.41}$$

As $r_2 + Cr_1 > 0$, then we obtain

$$y - x \geq 0.$$

□

We now write Lemma 2.111 explicitly for $\mathcal{T}_C$.

Lemma 2.112. *Let $\sigma = (Z, \mathrm{TCoh}^\theta(C)) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ be a pre-stability condition on $\mathcal{T}_C$ with $\sigma_i = (Z_i, \mathrm{Coh}^\theta(C))$ and $\theta \in [0, 1)$. If $Z_2(d_2, r_2) = A_2 d_2 + B r_2 + i(C_2 r_2 + D_2 d_2)$ and $\sigma_1 = \sigma_2 g$, were $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and*

$$T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$$

with $C > 0$ and there is a σ-semistable object $E \in \mathrm{TCoh}^\theta(C)$ with $[E] = [r_1, d_1, r_2, d_2]$ and $C_2 r_2 + D_2 d_2 > C_2 r_1 + D_2 d_1 > 0$, then

$$Cy + Ax \leq -B,$$

where $x = \frac{-A_2 d_1 - B_2 r_1}{C_2 r_1 + D_2 d_1}$ and $y = \frac{-A_2 d_2 - B r_2}{C_2 r_2 + D_2 d_2}$. Moreover, if $\varphi \neq 0$ then

$$y - x \geq 0,$$

and

$$-B \in [Cy + Ax, (Ax + Cy + y - x - \frac{C_2 r_1 + D_2 d_1}{C_2 r_2 + D_2 d_2} y(A + C)) \frac{C_2 r_2 + D_2 d_2}{C_2 r_2 + D_2 d_2 - C_2 r_1 - D_2 d_1}].$$

2.2.4 Duality and semistability on $\mathcal{T}_C$

The aim of this subsection is to prove an analogous statement to Lemma 2.111 for σ-stable objects E with $r_1 > r_2 > 0$, where σ is a pre-stability condition on $\mathcal{T}_C$. We follow closely [GJ12] to define the derived dual in this case and to show that it induces an anti-autoequivalence. We define the duality functor for $Q_{\mathrm{Coh}(X),n}$, where X is a non-singular projective variety. As in [BGP96, Sec. 3.2], we use the duality to study σ-stable objects for pre-stability conditions σ on $\mathcal{T}_C$.

We now consider the following functor

$$\mathcal{H}om(, \mathcal{O}_X) \colon \mathrm{Coh}(X) \to \mathrm{Coh}(X),$$

which is a left exact and contravariant functor. As $\mathrm{Coh}(X)$ does not have enough projectives, in order to compute the right derived functor, we use an adapted class for $\mathcal{H}om(, \mathcal{O}_X)$. For the definition of adapted class see [Huy06, Rem. 2.51]

Lemma 2.113. [Huy06, Prop. 3.26] *Every bounded complex E of coherent sheaves is isomorphic to a bounded complex of locally free sheaves in $D^b(X)$.*

Remark 2.114. If E is a bounded acyclic complex of locally free sheaves, then $\mathcal{H}om(E, \mathcal{O}_X)$ is acyclic.

As a consequence, the class of locally free sheaves in $\mathrm{Coh}(X)$ is adapted for the left exact functor $\mathcal{H}om(-, \mathcal{O}_X)$. By [Huy06, Rem. 2.51] we can define the right derived functor. Moreover, if $F \in \mathrm{Kom}^b(X)$, then $\mathcal{H}om(F, \mathcal{O}_X)$ is also bounded. As a consequence, we obtain the following functor.

$$\mathbb{D} = R\mathcal{H}om(-, \mathcal{O}_X) \colon D^b(X)^{\mathrm{op}} \to D^b(X). \tag{2.42}$$

Lemma 2.115. [Kuz17, Sec. 2.3] *The functor*

$$\mathbb{D} = R\mathcal{H}om(-, \mathcal{O}_X) \colon D^b(X)^{\mathrm{op}} \to D^b(X)$$

is an equivalence of categories and $\mathbb{D}^2 = \mathrm{Id}$ *for every* $E \in D^b(X)$.

Let us consider the functor

$$\begin{aligned} \mathcal{H}om(-, \mathcal{O}_X)_{Q_n} \colon Q_{\mathrm{Coh}(X),n} &\to Q_{\mathrm{Coh}(X),n} \\ (E_1, E_2, \varphi_l) &\mapsto (\mathcal{H}om(E_2, \mathcal{O}_X), \mathcal{H}om(E_1, \mathcal{O}_X)), \mathcal{H}om(\varphi_l)). \end{aligned}$$

Lemma 2.116. *Every bounded object E in $\operatorname{Kom}^b(\operatorname{TCoh}(X))$ is quasi-isomorphic to a complex $F \in \operatorname{Kom}^b(\operatorname{TCoh}(X))$ of locally free sheaves.*

Proof. It is enough to show that for $E = E_1 \xrightarrow{\varphi} E_2 \in \operatorname{TCoh}(C)$ there is $F \in \operatorname{TCoh}(X)$ of locally free sheaves with $F \twoheadrightarrow E$. There are locally-free sheaves F_i and surjective morphisms $F_i \xrightarrow{\pi_i} E_i$ for $i = 1, 2$. Let us consider the triple $F_1 \to F_1 \oplus F_2$ as the inclusion of the first component. Note that $F_1 \oplus F_2$ is also locally free. We have the following morphism in $\operatorname{TCoh}(C)$

$$\begin{tikzcd} F_1 \arrow[r, "\pi_1"] \arrow[d] & E_1 \arrow[d] \\ F_1 \oplus F_2 \arrow[r, "\varphi\circ\pi_1+\pi_2"] & E_2 \end{tikzcd}, \tag{2.43}$$

which is clearly surjective. □

Remark 2.117. Note that the last lemma holds for $Q_{\operatorname{Coh}(C),n}$ by choosing $(F_1, F_1^{\oplus n} \oplus F_2, \tau_i)$ where τ_i are the canonical inclusions.

Since for any bounded acyclic object $E \in \operatorname{Kom}^b(Q_{\operatorname{Coh}(X),n})$ of locally free sheaves, we have that $\mathcal{H}om(E, \mathcal{O}_X)_{Q_n}$ is acyclic. As a consequence, the class of objects with locally free sheaves as components in $Q_{\operatorname{Coh}(X),n}$ is adapted for the left exact functor $\mathcal{H}om(-, \mathcal{O}_X)_{Q_n}$ and by [Huy06, Rem. 2.51], we can define the right derived functor

$$R\mathcal{H}om(-, \mathcal{O}_X)_{Q_n} \colon D^b(Q_{\operatorname{Coh}(X),n})^{\mathrm{op}} \to D^b(Q_{\operatorname{Coh}(X),n}).$$

Remark 2.118. By [GJ12, Cor. 5] if $\mathcal{A}$ has enough injecives then $Q_{\mathcal{A},n}$ has also enough injectives. Therefore, we could have define $R\mathcal{H}om(-,-)_{Q_n}$ as a bifunctor as in [Huy06, Lem. 3.25].

Proposition 2.119. *The functor*

$$\mathbb{D}_1 := R\mathcal{H}om(-, \mathcal{O}_X)_{Q_1} \colon \mathcal{T}_X^{\mathrm{op}} \to \mathcal{T}_X$$

is an equivalence of categories.

Proof. Note that $\mathcal{T}_X^{\mathrm{op}} = \langle D_2^{\mathrm{op}}, D_1^{\mathrm{op}} \rangle$. Moreover, we have that

$$\mathbb{D}_1|_{D_2^{\mathrm{op}}} \colon D_2^{\mathrm{op}} \to D_1$$

and

$$\mathbb{D}_1|_{D_1^{\mathrm{op}}} \colon D_1^{\mathrm{op}} \to D_2$$

are equivalences of categories by Lemma 2.115. As $\mathbb{D}_1(D_3^{\mathrm{op}}) \subseteq D_3$, by [Kal11, Lem. 1.3] we obtain that $\mathbb{D}_1$ is an equivalence of categories. □

We first study the functor $\mathbb{D}$ on $D^b(C)$, where C is a curve with $g(C) \geq 1$. Let us start with $\mathbb{C}(x)$. By [Huy06, Cor. 3.40], we obtain that $\mathbb{D}(\mathbb{C}(x)) = \mathbb{C}(x)[-1]$ and for a locally free sheaf $E \in \mathrm{Coh}(C)$, we obtain that $\mathbb{D}(E) = E^\vee = \mathcal{H}om(E, \mathcal{O}_C)$ wich satisfies that

$$\deg(E^\vee) = -\deg(E) \text{ and } \mathrm{rank}(E) = \mathrm{rank}(E^\vee).$$

Moreover, in general it follows that if $E \in D^b(C)$ and $[E] = [r, d]$ then $[\mathbb{D}(E)] = [r, -d]$.

Let us consider the following torsion pair

Remark 2.120. Let $\sigma = (Z, \mathcal{A}) \in \mathrm{Stab}(D^b(C))$. We have the following torsion pair on $\mathcal{A}$

$$\mathcal{T} = \{E \in \mathcal{A} \mid \phi(E) = 1\},$$

$$\mathcal{F} = \{E \in \mathcal{A} \mid \text{Its } \sigma\text{-semistable factors } F_i \text{ satisfy } \phi_\sigma(F_i) < 1\}.$$

If $\mathrm{Coh}^\theta(C) \subseteq \mathcal{T}_C$ with $\theta \in [0, 1)$, we rename the torsion pair given in Remark 2.120 as $\mathrm{Coh}^\theta(C) = (T^\theta_\sigma, F^\theta_\sigma)$. Recall that by Remark 1.83, we also have that $\mathrm{Coh}^\theta(C) = (\mathcal{F}_\theta[1], \mathcal{T}_\theta)$ where

$$\mathcal{T}_\theta = \mathcal{P}_\mu(\theta, 1] \text{ and } \mathcal{F}_\theta = \mathcal{P}_\mu(0, \theta]$$

in $\mathrm{Coh}(C)$.

Remark 2.121. Note that if we have $\sigma_1 = (Z_1, \mathrm{Coh}^\theta(C))$ and $\sigma_2 = (Z_2, \mathrm{Coh}^\theta(C))$ with $0 \leq \theta < 1$, then $F^\theta_{\sigma_1} = F^\theta_{\sigma_2}$. Therefore, for any stability condition σ with heart $\mathrm{Coh}^\theta(C)$, we denote F^θ_σ by F^θ.

Definition 2.122. Let $\sigma = (Z(r, d) = Ad + Br + i(Cr + Dd), \mathrm{Coh}^\theta(C)) = \mathrm{Stab}(C)$. We define

$$\sigma^\vee = (Z(r, d) = Ad - Br + i(Cr - Dd), \mathrm{Coh}^{1-\theta}(C)[-1]) \in \mathrm{Stab}(C).$$

Remark 2.123. Note that $\mathbb{D}(\mathrm{Coh}^\theta(C)) \neq \mathrm{Coh}^{1-\theta}(C)[-1]$.

Remark 2.124. If $E \in F^\theta$, then $\mu_\sigma(E) = -\mu_{\sigma^\vee}(\mathbb{D}(E))$. Precisely, we have $Z_\sigma(E) = -\overline{Z_{\sigma^\vee}(\mathbb{D}(E))}$.

Example 2.125. Note that $\sigma_\mu^\vee = \sigma_\mu$.

Remark 2.126. Let E a μ-stable object in $\mathrm{Coh}(C)$ with $\phi(E) \neq 1$. It implies that E is locally free. Indeed, if $E = T(E) \oplus F(E)$, where $T(E)$ and $F(E)$ are the torsion and torsion-free parts respectively. If $T(E) \neq 0$ is a subobject of E, then $1 = \phi(T(E)) < \phi(E) < 1$ which is a contradiction to the stability of E.

Lemma 2.127. *Let $\sigma = (Z, \mathrm{Coh}^\theta(C))$, then $\mathbb{D}(F^\theta) = F^{1-\theta}[-1]$, where $F^{1-\theta}[-1]$ is given with respect to $\sigma^\vee$.*

Proof. We start by studying the image of $\mathcal{F}_\theta$ and $\mathcal{T}_\theta$ under $\mathbb{D}$. Let E be a σ-stable object with $E \in \mathcal{T}_\theta \subseteq \mathrm{Coh}(C)$. Then as E is locally free, we obtain that $\mathcal{H}om(E, \mathcal{O}_C) = E^\vee$. By definition, $\frac{\deg(E)}{\mathrm{rk}(E)} > -\cot(\pi\theta)$ which implies that

$$\frac{-\deg(E)}{\mathrm{rk}(E)} < \cot(\pi\theta) = -\cot(\pi - \pi\theta)$$

and $E^\vee \in \mathcal{F}_{1-\theta}$. If E is a σ-stable object with $E \in \mathcal{F}_\theta \subseteq \mathrm{Coh}(C)$ then as E is locally free, we obtain that $\mathcal{H}om(E[1], \mathcal{O}_C) = E^\vee[-1]$. By definition, $\frac{\deg(E)}{\mathrm{rk}(E)} \leq -\cot(\pi\theta)$ which implies that

$$\frac{-\deg(E)}{\mathrm{rk}(E)} \geq \cot(\pi\theta) = -\cot(\pi - \pi\theta).$$

Moreover, if $\frac{\deg(E)}{\mathrm{rk}(E)} < -\cot(\pi\theta)$, then $E^\vee[-1] \in \mathcal{T}_{1-\theta}[-1]$. By the Jordan-Hölder filtration, we can extend this result to any object in $\mathcal{F}_\theta$ and $\mathcal{T}_\theta$. Let $E \in F^\theta \subseteq \mathrm{Coh}^\theta(C)$. We have that a short exact sequence in $\mathrm{Coh}^\theta(C)$ given by

$$0 \to F[1] \to E \to T \to 0,$$

with $F \in \mathcal{F}_\theta$ and $T \in \mathcal{T}_\theta$. As $E \in F^\theta$ and F^θ is closed under subobjects,i.e. any subobject of an object in F^θ, also belongs to F^θ. we obtain that $F[1] \in F^\theta$ and by definition $T \in F^\theta$. After applying $\mathbb{D}$, we obtain a triangle

$$\mathbb{D}(T) \to \mathbb{D}(E) \to \mathbb{D}(F[1]) \to \mathbb{D}(T)[1],$$

as shown above $\mathbb{D}(T) \in \mathcal{F}_{1-\theta}$ and $\mathbb{D}(F[1]) \in \mathcal{T}_{1-\theta}[-1]$, as a consequence

$$\mathbb{D}(E) \in \mathrm{Coh}^{1-\theta}(C)[-1] = (\mathcal{F}_{1-\theta}, \mathcal{T}_{1-\theta}[-1]).$$

Moreover, we get that $\mathbb{D}(E) \in F^{1-\theta}[-1]$ as $\mathbb{D}(T), \mathbb{D}(F[1]) \in F^{1-\theta}[-1]$. □

Example 2.128. If $\sigma_\mu = (Z_\mu, \mathrm{Coh}(C))$, the torsion pair T^0 and F^0 is given precisely by the subcategory of torsion sheaves and torsion-free sheaves respectively. We also have that

$$\mathbb{D}(\mathrm{Coh}(C)) = \langle \mathbb{C}(x)[-1], \mathcal{L} \rangle,$$

for any line bundle $\mathcal{L}$ in $\mathrm{Coh}(C)$ and any point $x \in C$. Note that $\langle \mathbb{C}(x)[-1], \mathcal{L} \rangle$ is a heart of a bounded t-structure on $D^b(C)$ which does not admit a stability function. Moreover, we trivially have that $\mathbb{D}(F^0) = F^0$.

We now study $\mathbb{D}_1$ on $\mathcal{T}_C$. Let $\sigma = (Z, \mathrm{TCoh}^\theta(C)) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ be a pre-stability condition on $\mathcal{T}_C$ with $\sigma_1 = \sigma_2 g$ where $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ with $f(0) = 0$ and $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$.

We now examine the behaviour of σ-stable objects in $\mathrm{TCoh}^{\theta}(C)$ under the anti-autoequivalence $\mathbb{D}_1$.

Lemma 2.129. *Let $\sigma = (Z, \mathrm{TCoh}^{\theta}(C))$ be a pre-stability condition as above. If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathrm{TCoh}^{\theta}(C))$ is σ-stable with $\phi_\sigma(E) < 1$, i.e. $\Im(Z(E)) \neq 0$, then $E_1, E_2 \in F^{\theta}$.*

Proof. If $E_1 = 0$ or $E_2 = 0$, then it follows from the definition. We assume that $E_1 \neq 0$ and $E_2 \neq 0$, therefore by the stability of E, we get that $\varphi \neq 0$. As $E_2 \in \mathrm{Coh}^{\theta}(C)$, then we have a short exact sequence

$$0 \to T_2 \to E_2 \to F_2 \to 0$$

in $\mathrm{Coh}^{\theta}(C)$ with $T_2 \in T^{\theta}$ and $F_2 \in F^{\theta}$. If $T_2 \neq 0$, then we have a subobject of the form $0 \to T_2$ of E with $\phi(0 \to T_2) = 1$, which contradicts the stability of E. Therefore, we have that $T_2 = 0$ and $E_2 \in F^{\theta}$. For E_1 we also have a short exact sequence

$$0 \to T_1 \to E \to F_1 \to 0$$

with $T_1 \in T^{\theta}$ and $F_1 \in F^{\theta}$. If $T_1 \neq 0$, we have a subobject of E of the form $T_1 \to 0$, since there are no morphisms from T^{θ} to F^{θ}. Once again, it contradicts the stability of E and $E_1 \in F^{\theta}$. $\square$

Definition 2.130. Let $\sigma = (Z, \mathrm{TCoh}^{\theta}(C)) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$, we define the dual stability condition on $\mathcal{T}_C$ as

$$\sigma^* = \mathrm{gl}_{12}(\sigma_2^{\vee}, \sigma_1^{\vee}) = (Z^*, \mathrm{TCoh}^{1-\theta}(C)[-1]).$$

Remark 2.131. Let $E = E_1 \to E_2 \in \mathrm{TCoh}^{\theta}(C)$, with $E_1, E_2 \in F^{\theta}$ then

$$\mu_\sigma(E) = -\mu_{\sigma*}(\mathbb{D}_1(E)).$$

Moreover, we also have that $\sigma^* = \mathrm{gl}_{12}(\sigma_2^{\vee}, \sigma_2^{\vee} g N_{\frac{2B}{C}})$.

Example 2.132. If $\sigma = \mathrm{gl}_{12}(\sigma_\mu g, \sigma_\mu)$, with $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, where

$$T^{-1} = \begin{bmatrix} 1 & -\alpha \\ 0 & 1 \end{bmatrix}$$

and $f(0) = 0$, then $\sigma^* = \mathrm{gl}_{12}(\sigma_\mu, \sigma_\mu g')$ with $g' = gN_{-2\alpha}$. Note that $g' = g^{-1}$. By the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, an object $E \in \mathrm{TCoh}(C)$ is σ^*-stable if and only if it is $\sigma^* g$-stable, where $\sigma^* g = \mathrm{gl}_{12}(\sigma_\mu g, \sigma_\mu)$.

Lemma 2.133. *Let us consider $\sigma = (Z, \mathrm{TCoh}^{\theta}(C)) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$. An object E is σ-stable with $\phi_\sigma(E) < 1$ if and only if $\mathbb{D}_1(E)$ is σ^*-stable and $\phi_{\sigma^*}(\mathbb{D}(E)) < 1$*

Proof. As in Remark 2.129 we have that $E_1, E_2 \in F^{\theta}$. By Remark 2.127, it follows that $\mathbb{D}(E_1), \mathbb{D}(E_2) \in F^{1-\theta}[-1]$. Then $\mathbb{D}_1(E) = \mathbb{D}(E_2) \to \mathbb{D}(E_1) \in \mathrm{TCoh}(C)^{1-\theta}[-1]$ and $\phi_{\sigma*}(\mathbb{D}_1(E)) < 1$. Let us consider $Q = Q_1 \to Q_2 \in \mathrm{TCoh}^{1-\theta}(C)[-1]$, satisfying $Q_1, Q_2 \in F^{1-\theta}[-1]$ and $0 \to G \to \mathbb{D}_1(E) \to Q \to 0$ a short exact sequence in $\mathrm{TCoh}^{1-\theta}(C)[-1]$, where $G = G_1 \to G_2$. Note that it is enough to prove that $\phi_{\sigma*}(\mathbb{D}_1(E)) < \phi_{\sigma*}(Q)$ to show the stability of $\mathbb{D}_1(E)$. Indeed, if $P = P_1 \to P_2$ is an arbitrary quotient of $\mathbb{D}_1(E)$, either there is a object $Q = Q_1 \to Q_2$ with $Q_1, Q_2 \in F^{1-\theta}[-1]$ with $P \twoheadrightarrow Q$ and $\phi_{\sigma*}(Q) < \phi_{\sigma^*}(P)$ or $\phi_{\sigma^*}(P) = 1$ and we trivially obtain that $\phi_{\sigma^*}(\mathbb{D}_1(E)) < \phi_{\sigma^*}(P)$. We also have that $G_1, G_2 \in F^{1-\theta}[-1]$ as $F^{1-\theta}[-1]$ is closed under subobjects. Moreover, the duality gives us a correspondence between short exact sequences on $F^{1-\theta}[-1]$ and F^{θ}. We obtain a short exact sequence in $\mathrm{TCoh}^{\theta}(C)$ given by $0 \to \mathbb{D}_1(Q) \to E \to \mathbb{D}_1(G) \to 0$. By the stability of E we obtain that $\mu_{\sigma}(\mathbb{D}_1(Q)) < \mu_{\sigma}(E)$ and it follows if and only if $\mu_{\sigma*}(\mathbb{D}_1(E)) < \mu_{\sigma*}(Q)$. As a consequence $\mathbb{D}_1(E)$ is σ^*-stable. □

Let $\sigma = \mathrm{gl}_{12}(\sigma_2 g, \sigma_2) = (Z, \mathrm{TCoh}^{\theta}(C))$ as above and $E = E_1 \xrightarrow{\varphi} E_2$ a σ-stable object with $\phi(E) < 1$, $\varphi \neq 0$ and $0 < C_2 r_2 + D_2 d_2 < C_2 r_1 + D_2 d_1$, where $[E] = (r_1, d_1, r_2, d_2)$. Note that we cannot apply Lemma 2.112 directly.

Note that $\sigma = \mathrm{gl}_{12}(\sigma_2^{\vee}, \sigma_2^{\vee} g N_{\frac{2B}{C}})$. After applying the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action and by Lemma 2.133, we have that E is σ-stable if and only $\mathbb{D}_1(E)$ is σ'-stable, where $\sigma' = \sigma^* g' = (\sigma_2^{\vee} g', \sigma_2^{\vee})$ and

$$g' = N_{\frac{-2B}{C}} g^{-1} = (T', f') \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$$

with $T'^{-1} = \frac{1}{\det(T^{-1})} \begin{bmatrix} C & B \\ 0 & -A \end{bmatrix}$.

If $\sigma_2 = (Z_2(r,d) = A_2 d_2 + B_2 r_2 + i(C_2 r_2 + D_2 d_2), \mathrm{Coh}^{\theta}(C))$, then

$$\sigma_2^{\vee} = (Z_2^{\vee}(r,d) = A_2 d_2 - B_2 r_2 + i(C_2 r_2 - D_2 d_2), \mathrm{Coh}^{1-\theta}[-1](C)).$$

If we apply Lemma 2.112 to σ', we have that if $F = F_1 \xrightarrow{\varphi_F} F_2$ is σ-stable with

$$C_2 \operatorname{rk}(F_2) - D_2 \deg(F_2) > C_2 \operatorname{rk}(F_1) - D_2 \deg(F_1)$$

and $\varphi_F \neq 0$, then

$$\frac{-A_2 \deg(F_1) + B_2 \operatorname{rk}(F_1)}{C_2 \operatorname{rk}(F_1) - D_2 \deg(F_1)} \leq \frac{-A_2 \deg(F_2) + B_2 \operatorname{rk}(F_2)}{C_2 \operatorname{rk}(F_2) - D_2 \deg(F_2)}.$$

Note that $[\mathbb{D}_1(E)] = [r_2, -d_2, r_1, -d_1]$. As a consequence, we obtain that $\mathbb{D}_1(E)$ satisfies

$$C_2 \operatorname{rk}(\mathbb{D}(E_1)) - D_2 \deg(\mathbb{D}(E_1)) > C_2 \operatorname{rk}(\mathbb{D}(E_2)) - D_2 \deg(\mathbb{D}(E_2)),$$

because $C_2r_2 + D_2d_2 < C_2r_1 + D_2d_1$. We now can apply Lemma 2.112 to σ' and we obtain

$$\frac{A_2d_2 + B_2r_2}{C_2r_2 + D_2d_2} \leq \frac{A_2d_1 + B_2r_1}{C_2r_1 + D_2d_1}$$

which is precisely

$$\frac{-A_2d_1 - B_2r_1}{C_2r_1 + D_2d_1} \leq \frac{-A_2d_2 - B_2r_2}{C_2r_1 + D_2d_2}.$$

Moreover, we obtain the following result, which gives us necessary conditions to have σ-stable objects.

Lemma 2.134. *Let $\sigma = (Z, \mathrm{TCoh}^{\theta}(C)) = \mathrm{gl}_{12}(\sigma, \sigma_2)$ be a pre-stability condition on $\mathcal{T}_C$ with $Z_2(d_2, r_2) = A_2d_2 + Br_2 + i(C_2r_2 + D_2r_2)$ and $\sigma_1 = \sigma_2 g$, were $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$.*

If there is a σ-semistable object with $C_2r_1 + D_2d_1 > C_2r_2 + D_2d_2 > 0$ then

$$Cy + Ax \leq -B,$$

where $x = \frac{-A_2d_1 - B_2r_1}{C_2r_1 + D_2d_1}$ and $y = \frac{-A_2d_2 - Br_2}{C_2r_2 + D_2d_2}$. Moreover, if $\varphi \neq 0$ then

$$y - x \geq 0,$$

and

$$-B \in [Cy + Ax, Ax + Cy + y - x - \frac{C_2r_2 + D_2d_2}{C_2r_1 + D_2d_1} y(A + C) \frac{C_2r_1 + D_2d_1}{C_2r_1 + D_2d_1 - C_2r_2 - D_2d_2}].$$

Corollary 2.135. *If σ_α is defined as in Remark 2.98 on $\mathcal{T}_C$ and E is a α-stable object with $[E] = (r_1.d_1, r_2, d_2)$ and $r_1 \neq r_2$, then $\alpha \in [\frac{d_2}{r_2} - \frac{d_1}{r_1}, (\frac{d_2}{r_2} - \frac{d_1}{r_1})(1 + \frac{r_1+r_2}{|r_2-r_1|})]$.*

In [BGP96] and [Sch03], it was shown that quasi-projective moduli spaces $\mathcal{M}_\alpha(r_1, d_1, r_2, d_2)$ of σ_α-stable holomorphic triples of exist. Moreover, if $r_1 + r_2$ and $d_1 + d_2$ are coprime and α is generic then $\mathcal{M}_\alpha(r_1, d_1, r_2, d_2)$ is projective.

Proposition 2.136. [BGP96, Thm. 6.1] *A necessary condition for $\mathcal{M}_\alpha(r_1, r_2, d_1, d_2)$ to be non-empty is*

1. $0 \leq \alpha_m \leq \alpha \leq \alpha_M$ *if* $r_1 \neq r_2$,

2. $0 \leq \alpha_m \leq \alpha$ *if* $r_1 = r_2$, *where*

$$\begin{aligned}\alpha_m &= \frac{d_2}{r_2} - \frac{d_1}{r_1} \text{ and}\\ \alpha_M &= (1 + \frac{r_2 + r_1}{|r_2 - r_1|})\alpha_m, \text{ for } r_1 \neq r_2.\end{aligned}$$

Remark 2.137. Note that Corollary 2.135 agrees with the necessary conditions for the existence of σ_α-stable objects of Proposition 2.136.

Remark 2.138. In the next chapter, we show that all the CP-glued pairs on $\mathcal{T}_C$, where $g(C) \geq 1$ satisfy the HN-property and we use Lemma 2.111 and Lemma 2.134 to prove the support property.

3 Bridgeland stability conditions of holomorphic triples over curves

Let C be a nonsingular projective curve over $\mathbb{C}$ with $g(C) \geq 1$. We study $\mathrm{TCoh}(C)$ i.e. the category of holomorphic triples over C. The aim of this section is to describe completely the stability manifold $\mathrm{Stab}(\mathcal{T}_C)$. In the process, we prove that all CP-glued pairs σ constructed in Section 2.2 on $\mathcal{T}_C$ are in fact Bridgeland stability conditions. In order to describe $\mathrm{Stab}(\mathcal{T}_C)$ we follow the steps of [Bri08]. In Section 3.1 we first construct additional pairs via tilting, as a consequence we obtain discrete pre-stability conditons. In Section 3.2 we show that all Bridgeland stability conditions in $\mathrm{Stab}(\mathcal{T}_C)$ have to be given by the already constructed pairs, either by CP-gluing of by tilting. In Section 3.3 we prove the support property and finally in Section 3.4 we use Bridgeland's deformation result to describe topologically the stability manifold and to extend the HN-property to the non-discrete cases. This chapter appears in [MRRHR19] as joint work with Eva Martínez Romero and Arne Rüffer.

3.1 Constructing pre-stability conditions via tilting

In this section we construct pre-stability conditions in $\mathrm{Stab}(\mathcal{T}_C)$ whose hearts are not given by Proposition 2.66. We follow the steps of [Bri08, Lem. 6.1], i.e. we are going to use weak stability functions on $\mathrm{TCoh}(C)$, in the sense of [Rud97], to obtain torsion pairs on $\mathrm{TCoh}(C)$ via truncation of the HN-filtrations. After tilting, we will obtain hearts that admit Bridgeland stability functions.

Remark 3.1. The intuition of this constructions comes from Proposition 3.40. This propositions gives us a description of the torsion pair of $\mathrm{TCoh}(C)$, which after tilting will give us a heart of a pre-stability condition. We decided to start with the construction of the torsion pair to follow the order of [Bri08] and to make the structure of this chapter cleaner.

We define the following homomorphism:

$$\begin{aligned} Z\colon \mathbb{Z}^4 \quad &\to \quad \mathbb{C} \\ (r_1, d_1, r_2, d_2) \quad &\mapsto \quad D_1 d_1 + (C_1 - 1) r_1 + i(r_1 + r_2), \end{aligned} \tag{3.1}$$

where $D_1, C_1 \in \mathbb{R}$, and $D_1 < 0$. We define the phase of an element E, for every $E \in \mathrm{TCoh}(C)$ with $E \neq 0 \to T$, where T is a torsion sheaf, as

$$\lambda(E) = (1/\pi)\arg(Z([E])) \in (0, 1].$$

Lemma 3.2. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence where $A, B, C \in \mathrm{TCoh}(C)$ and $A, B, C \neq 0 \to T$, where T is a torsion sheaf, then*

$$\lambda(A) < \lambda(B) \iff \lambda(B) < \lambda(C) \text{ and } \lambda(A) > \lambda(B) \iff \lambda(B) > \lambda(C).$$

Proof. As the Grothendieck group is additive with respect to short exact sequences in $\mathrm{TCoh}(C)$, after applying the homomorphism Z, we obtain that $Z([B]) = Z([A]) + Z([C])$. In addition, the image of the classes of the triples A, B and C under Z lies inside $\mathbb{H} \cup \mathbb{R}_{<0}$. Consequently, the argument of a sum of complex numbers in $\mathbb{H} \cup \mathbb{R}_{<0}$ satisfies precisely the conditions above. □

For a triple $E = E_1 \xrightarrow{\varphi} E_2 \in \mathrm{TCoh}(C)$, let $T(E)_i$ be the torsion part and $F(E_i)$ is the torsion-free part of of E_i for $i = 1, 2$. By the functoriality of the torsion part, we obtain $T(E) = T(E_1) \to T(E_2)$ and the following short exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T(E_1) \arrow[r] \arrow[d, "t"'] & E_1 \arrow[r] \arrow[d, "\varphi"'] & F(E_1) \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
0 \arrow[r] & T(E_2) \arrow[r] & E_2 \arrow[r] & F(E_2) \arrow[r] & 0,
\end{tikzcd} \tag{3.2}$$

where $F(E) = F(E_1) \to F(E_2) = E/T(E)$.

Definition 3.3. An triple $E = E_1 \xrightarrow{\varphi} E_2 \in \mathrm{TCoh}(C)$ is called *torsion-free* if $T(E)_i = 0$, for $i = 1, 2$.

Definition 3.4. Let $E \in \mathrm{TCoh}(C)$ a triple. We define the *torsion-free triple* of

$$E = E_1 \xrightarrow{\varphi} E_2 \text{ as } F(E) = F(E_1) \xrightarrow{f} F(E_2).$$

Definition 3.5. A torsion-free triple $E \in \mathrm{TCoh}(C)$ is called λ-semistable if for all non-zero subobjects $F \subseteq E$ we have $\lambda(F) \leq \lambda(E)$.

We now show that the λ-semistable objects admit HN-filtration. The proof goes along the line of the classical proof for μ-stability on curves.

Lemma 3.6 (HN-filtration for λ-stability.)**.** *Let $F = F_1 \xrightarrow{\varphi} F_2 \in \mathrm{TCoh}(C)$ be a torsion-free object, then there is a unique Harder-Narasimhan filtration i.e. there is an increasing filtration*

$$0 \subseteq E_1 \subseteq E_2 \cdots \subseteq E_{n-1} \subseteq E_n = F$$

where $G_i = E_i/E_{i-1}$ is λ-semistable for each $i = 0, \dots, n$ and

$$\lambda(G_1) > \lambda(G_2) > \cdots > \lambda(G_{n-1}) > \lambda(G_n).$$

Moreover, this filtration is unique.

Proof. If F is λ-semistable, there is nothing to prove. Let us consider the objects $E \subseteq F$. Note that E is also torsion-free. Take the object E_1 with maximal $\lambda(E_1)$ among all the subobjects of F and with maximal imaginary part among all the subobjects of F with maximal λ-phase. This object exists because the phase is bounded and the fact that there are only finitely many options for $\Im(\lambda(G))$ with $G \subseteq F$. Indeed, the boundedness follows from Riemann–Roch and the correspondence between slope and phase.

As a consequence, the subobject E_1 is necessarily λ-semistable. Moreover, note that F/E_1 is torsion free. Indeed, If F/E_1 is not torsion free we could find a subobject $E' \subseteq F$, with F/E' torsion-free and $\lambda(E) < \lambda(E')$. We also have that for all E with $0 \neq E/E_1 \subseteq F/E_1$, we get $\lambda(E/E_1) < \lambda(E_1)$. If not, we will have that $\lambda(E/E_1) \geq \lambda(E_1)$, by Lemma 3.2 it follows that $\lambda(E_1) \leq \lambda(E)$ with $\Im(Z(E)) > \Im(Z(E_1))$, which is a contradiction. We now apply the same construction to F/E_1. We get a filtration

$$0 \subseteq E_1 \subseteq E_2 \cdots \subseteq E_{n-1} \subseteq E_n = F$$

such that the $G_i = E_i/E_{i-1}$ are λ-semistable. □

We recall the following torsion pair in $\mathrm{Coh}(C)$, because it plays a role in the proof of the following lemma.

Remark 3.7. We consider the torsion pair $(\mathcal{T}_1, \mathcal{F}_1) = \mathrm{Coh}(C)$, where $B \in \mathcal{T}_1$ if the HN-factors of its torsion-free part satisfy

$$\frac{-D_1 d_1 - (C_1 - 1) r_1}{r_1} > -\cot(3\pi/4)$$

and $B \in \mathcal{F}_1$ if it is a torsion-free sheaf, whose HN-factors satisfy

$$\frac{-D_1 d_1 - (C_1 - 1) r_1}{r_1} \leq -\cot(3\pi/4).$$

This torsion pair is given by truncating the Harder-Narasimhan filtration with respect to $\sigma_1 = (Z(r,d) = D_1 d + (C_1 - 1) r + ir, \mathrm{Coh}(C))$ in $\mathrm{Stab}(C)$.

Under the same assumptions of Lemma 3.6, we prove the following lemma.

Lemma 3.8. *Let $\phi = 3/4$. There is a torsion pair $(\mathcal{T}, \mathcal{F})$ on the category* $\mathrm{TCoh}(C)$ *defined as follows: $E \in \mathcal{T}$ if the Harder-Narasimhan λ-semistable factors A_i of $F(E)$ satisfy*

$\lambda(A_i) > \phi$ and $i^!(E) \in \mathrm{Coh}(C)$. We say that $E \in \mathcal{F}$ if $i^*(E)$ is torsion-free and the Harder-Narasimhan factors A_i of $F(E)$ satisfy $\lambda(A_i) \leq \phi$ or $i^*(E) = 0$.

Proof. Note that if $E \in \mathcal{T}$, by our definition of $\mathcal{T}$ and the correspondence between slope and phase, we have that $F(E)$ satisfies

$$\frac{-D_1 d_1 - (C_1 - 1) r_1}{r_1 + r_2} > -\cot(3\pi/4),$$

i.e.

$$-D_1 d_1 - C_1 r_1 - r_2 > 0. \tag{3.3}$$

where here $d_i = \deg(F(E)_i)$ and $r_i = \mathrm{rank}(F(E)_i)$, for $i = 1, 2$.

We show that $(\mathcal{T}, \mathcal{F})$ is a torsion pair of $\mathrm{TCoh}(C)$. We first prove that $\mathrm{Hom}_{\mathcal{T}_C}(\mathcal{T}, \mathcal{F}) = 0$. By our definition of stability we have that $\mathrm{Hom}_{\mathcal{T}_C}(E, F) = 0$, for all objects $E \in \mathcal{T}$ and $F \in \mathcal{F}$ that are torsion-free.

Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}$ and $G = G_1 \to G_2 \in \mathcal{F}$.

Let us consider the following short exact sequences as in the triangle (3.2)

$$0 \to T(E) \to E \to F(E) \to 0$$

and

$$0 \to T(G) \to G \to F(G) \to 0.$$

By definition of $\mathcal{F}$, we have that $F(G) \in \mathcal{F}$ and as $i^*(G)$ is torsion-free, we get $i^*(T(G)) = 0$ and $T(G) \in \mathcal{F}$. Then it is enough to show $\mathrm{Hom}_{\mathrm{TCoh}(C)}(E, G) = 0$ for $G = 0 \to H$, for any $H \in \mathrm{Coh}(C)$ and $G = G_1 \to G_2 \in \mathcal{F}$ where G_1, G_2 are torsion-free.

Case 1: $G \in \mathcal{F}$ a torsion-free triple. By definition $F(E) \in \mathcal{T}$, then by stability we have $\mathrm{Hom}_{\mathrm{TCoh}(C)}(F(E), G) = 0$. Also

$$\mathrm{Hom}_{\mathrm{TCoh}(C)}(T(E), G) = 0$$

as G is torsion-free. Therefore, it follows $\mathrm{Hom}_{\mathrm{TCoh}(C)}(E, G) = 0$.

Case 2: $G = 0 \to H$.

We have

$$\mathrm{Hom}_{\mathcal{T}_C}(E, G) = \mathrm{Hom}_{\mathcal{T}_C}(E, j_*(j^!(G))),$$

by adjointness

$$\mathrm{Hom}_{\mathcal{T}_C}(E, j_*(j^!(G))) = \mathrm{Hom}_{D^b(C)}(j^*(E), j^!(G)) = \mathrm{Hom}_{D^b(C)}(\mathrm{Ker}(\varphi)[1], H) = 0,$$

because $j^*(E)[1] = i^!(E) = \mathrm{Ker}(\varphi) \oplus \mathrm{Coker}(\varphi)[1]$ is in $\mathrm{Coh}(C)$, which implies that

$\mathrm{Coker}(\varphi) = 0$.

We now prove that for every $E \in \mathrm{TCoh}(C)$ there is a short exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$ respectively. Note that for a torsion-free object E, by Lemma 3.6, there is a short exact sequence

$$0 \to T \to E \to F \to 0,$$

with T torsion-free such that the HN-factors A_i of T satisfy that $\lambda(A_i) > \frac{3}{4}$ and $F = F_1 \xrightarrow{f'} F_2 \in \mathcal{F}$ also torsion-free. Note that $T = T_1 \xrightarrow{t'} T_2$ is not necessarily surjective, however we have the following claim.

Claim 3.9. *Either* $\mathrm{Coker}(t') = 0$ *or* $\mathrm{Coker}(t')$ *is a torsion sheaf.*

Proof. Because of Lemma 3.6. It is enough to show the statement for a λ-semistable object $T \in \mathcal{T}$. Let us assume that $\mathrm{Coker}(t') \neq 0$. Note that $t' \neq 0$. Indeed, if $t' = 0$, then as $T_2 \neq 0$, we have that $0 \to T_2$ is a subobject and a quotient, which implies by semistability that $\lambda(T) = \frac{1}{2} < \frac{3}{4}$. It gives us a contradiction. Then $\mathrm{Img}(t'), \mathrm{Coker}(t') \neq 0$, and as a consequence we have two short exact sequences:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d, "t'"'] & T_1 \arrow[r] \arrow[d, "0"'] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Img}(t') \arrow[r] & T_2 \arrow[r] & \mathrm{Coker}(t') \arrow[r] & 0 \end{tikzcd} \tag{3.4}$$

and

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d, "t'"'] & T_1 \arrow[r] \arrow[d, "t'"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Img}(t') \arrow[r] & T_2 \arrow[r] & \mathrm{Coker}(t') \arrow[r] & 0 \end{tikzcd} . \tag{3.5}$$

If $\mathrm{rk}(\mathrm{Coker}(t')) > 0$, then by λ-semistability of T, we obtain $\lambda(T) = \frac{1}{2} < \frac{3}{4}$, which gives us a contradiction. Therefore, we have that $\mathrm{Coker}(t')$ is a torsion sheaf. $\square$

We obtain the following short exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T_1 \arrow[r] \arrow[d, "t'"'] & E_1 \arrow[r] \arrow[d, "\varphi"'] & F_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Img}(t') \arrow[r] & E_2 \arrow[r] & F_2' \arrow[r] & 0
\end{tikzcd} \tag{3.6}$$

Note that $T' \coloneqq T_1 \xrightarrow{t'} \mathrm{Img}(t')$ is in $\mathcal{T}$. Indeed, let us consider the last short exact sequence in its HN-decomposition $0 \to E \to T' \twoheadrightarrow A \to 0$ with $A = A_1 \to A_2$ a λ-semistable torsion free sheaf. We want to show that $\lambda(A) > \frac{3}{4}$. Note that E is also a subobject of T, as a consequence we consider the short exact sequence $0 \to E \to T \to T/E \to 0$. We have that T/E is a quotient of T and therefore $\lambda(F(T/E)) > \frac{3}{4}$. We also have that $\frac{3}{4} < \lambda(F(T/E)) = \lambda(A)$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F_1 \arrow[r] \arrow[d, "f'"'] & F_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & \mathrm{Coker}(t') \arrow[r] & F_2' \arrow[r, "g"] & F_2 \arrow[r] & 0
\end{tikzcd} \tag{3.7}$$

and F is the torsion-free part of $F_1 \xrightarrow{f'} F_2'$ and it implies that $F_1 \xrightarrow{f'} F_2' \in \mathcal{F}$.

Therefore, if $E_1 \to E_2$ is torsion-free. The triangle (3.6) gives us the decomposition of E in $\langle \mathcal{T}, \mathcal{F} \rangle$.

Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathrm{TCoh}(C)$. Let us consider again the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T(E_1) \arrow[r] \arrow[d, "t"'] & E_1 \arrow[r] \arrow[d, "\varphi"'] & F(E_1) \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\
0 \arrow[r] & T(E_2) \arrow[r] & E_2 \arrow[r] & F(E_2) \arrow[r] & 0.
\end{tikzcd} \tag{3.8}$$

Since $F(E)$ is torsion-free, as mentioned before there is a short exact sequence

$$0 \to T^{'} \to F(E) \to F' \to 0,$$

with $T' \in \mathcal{T}$ and $F^{'} \in \mathcal{F}$. Explicitely

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & T_1' \arrow[r, "l_1"] \arrow[d, "t'"'] & F(E_1) \arrow[r, "g_1"] \arrow[d, "f"'] & F_1' \arrow[r] \arrow[d, "f'"'] & 0 \arrow[d] \\
0 \arrow[r] & T_2' \arrow[r, "l_2"] & F(E_2) \arrow[r, "g_2"] & F_2' \arrow[r] & 0.
\end{tikzcd} \tag{3.9}$$

After choosing a splitting $E_i = F(E_i) \oplus T(E_i)$, for $i = 1, 2$, we have a morphism

$$l\colon F(E_1) \hookrightarrow E_1 \xrightarrow{\varphi} E_2 \to T(E_2).$$

We now define the following morphism:

$$\beta_1\colon T_1^{'} \oplus T(E_1) \xrightarrow{\begin{bmatrix} l_1 & 0 \\ 0 & \mathrm{id} \end{bmatrix}} F(E_1) \oplus T(E_1),$$

We obtain the following short exact sequence:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & T_1' \oplus T(E_1) \arrow[r, "\beta_1"] \arrow[d, "\varphi\circ\beta_1"'] & F(E_1) \oplus T(E_1) \arrow[r, "{(g_1,0)}"] \arrow[d, "\varphi"] & F_1^{'} \arrow[r] \arrow[d, "l'"'] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Img}(\varphi \circ \beta_1) \arrow[r, "i"] & F(E)_2 \oplus T(E_2) \arrow[r] & G \arrow[r] & 0. \end{tikzcd} \tag{3.10}$$

Note that $\varphi(x, y) = (f(x), l(x) + t(y))$, for $(x, y) \in F(E_1) \oplus T(E_1)$.

We claim that

$$T_1' \oplus T(E_1) \xrightarrow{\varphi\circ\beta_1} \mathrm{Img}(\varphi \circ \beta_1) \in \mathcal{T} \text{ and } F_1' \xrightarrow{l'} G \in \mathcal{F}.$$

Indeed, note that we have the following decompositions:

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & T(E_1) \arrow[r] \arrow[d, "t"'] & T_1' \oplus T(E_1) \arrow[r] \arrow[d, "\varphi\circ\beta_1"'] & T_1' \arrow[r] \arrow[d, "t'"'] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Ker}(\pi) \arrow[r] & \mathrm{Img}(\varphi \circ \beta_1) \arrow[r, "\pi"] & T_2' \arrow[r] & 0. \end{tikzcd} \tag{3.11}$$

Note that, by the triangle (3.9), we have that $\pi\colon \mathrm{Img}(\varphi \circ \beta_1) \subseteq T_2' \oplus T(E_2) \to T_2'$ is just the projection. Note that we abused the notation by ignoring the inclusion l_2. As $\mathrm{Ker}(\pi) = \mathrm{Img}(\varphi \circ \beta_1 \mid_{\mathrm{Ker}\, t' \oplus T(E_1)})$ is given by the points $(0, x) \in \mathrm{Img}(\varphi \circ \beta_1) \subseteq T_2' \oplus T(E_2)$ and we can see it as a subset of $\subseteq T(E)_2$. Thus, $\mathrm{Ker}(\pi)$ is a torsion sheaf and by definition T_2' is torsion-free. As a consequence, the torsion-free part of

$$T_1' \oplus T(E_1) \xrightarrow{\varphi\circ\beta_1} \mathrm{Img}(\varphi \circ \beta_1) \in \mathcal{T}$$

is $T_1' \xrightarrow{t'} T_2'$.

Analogously, we have the following decomposition:

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F_1' \arrow[r] \arrow[d, "l'"'] & F_1' \arrow[r] \arrow[d, "f'"'] & 0 \arrow[d] \\
0 \arrow[r] & \operatorname{Ker}([g_2,0]) \arrow[r, "i"] & G \arrow[r, "{[g\circ g_2,0]}"] & F_2' \arrow[r] & 0.
\end{tikzcd} \tag{3.12}$$

We first check that $[g_2, 0]$ is well-defined. As $G = F(E_2) \oplus T(E_2)/\operatorname{Img}(\varphi \circ \beta_1)$, then if $(x, y) \in \operatorname{Img}(\varphi \circ \beta_1)$, there are $(x', y') \in T_1' \oplus T(E_1)$ which satisfy $(f(x'), l(x') + t(y')) = (x, y)$. Note that $f(x') \in T_2'$, therefore $g_2(f(x')) = 0$. Since g_2 is surjective, we have that $[g_2, 0]$ is clearly surjective.

We now check that

$$\operatorname{Ker}([g \circ g_2, 0]) = T_2 \oplus T(E_2)/\operatorname{Img}(\varphi \circ \beta_1)$$

and $i([x]) = ([0, x])$. Let $[(x, y)] \in G$. If $(g \circ g_2)(y) = 0$, then $g_2(y) \in \operatorname{Coker}(t')$. As the standard projection $p\colon T_2' \to \operatorname{Coker}(t')$ is surjective there is $y' \in T_2'$ such that $p(y') = g_2(y)$. Moreover, we have that $(f(y'), x) - (0, l(y') - x) \in \operatorname{Img}(\varphi \circ \beta_1)$, as

$$\varphi \circ \beta_1(y', 0) = (f(y'), l(y')).$$

As a consequence, $\operatorname{Ker}([g_2])$ is a torsion sheaf. We now obtain that the torsion-free part of $F_1' \xrightarrow{l'} G$ is the same as the one $F_1' \to F_2'$ and it implies that $F_1' \xrightarrow{l'} G \in \mathcal{F}$. $\square$

After tilting with respect to the torsion pair of Lemma 3.8 we obtain the following heart

$$\mathcal{A}_r = \{E \in \mathcal{T}_C \mid H^i(E) = 0 \text{ for } i \neq 1, 0, H^1(E) \in \mathcal{T} \text{ and } H^0(E) \in \mathcal{F}\},$$

where $r = \frac{\arg(C_1 + D_1 i)}{\pi} \in (-1, 0]$. It has a corresponding torsion pair $\mathcal{A}_r = \langle \mathcal{F}, \mathcal{T}[-1] \rangle$.

We now define a stability function Z_r on $\mathcal{A}_r$ such that the pair $(Z_r, \mathcal{A}_r)$ is a Bridgeland stability condition.

Remark 3.10. The CP-glued property for pre-stability conditions is not invariant under the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action. If we have $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ a pre-stability condition and we act by $\sigma_2^{-1} = g_2 \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and we we obtain a non CP-glued pre-stability condition, then the heart of σg_2 is going to be given by $\mathcal{A}_r$. See Subsection 3.2.1.

We now define

$$\begin{aligned} Z_r\colon \mathbb{Z}^4 &\to \mathbb{C} \\ (r_1, d_1, r_2, d_2) &\mapsto A_1 d_1 + B_1 r_1 - d_2 + i(D_1 d_1 + r_1 C_1 + r_2), \end{aligned} \tag{3.13}$$

where $M = \begin{bmatrix} -A_1 & B_1 \\ -D_1 & C_1 \end{bmatrix}$ with $\det(M) > 0$, $\det(M+I) > 0$, $A_1, B_1, C_1, D_1 \in \mathbb{R}$ and $D_1 < 0$.

Remark 3.11. Note that if $E \in \mathrm{TCoh}(C)$ is a λ-semistable torsion-free triple, then $\lambda(E) > \frac{3}{4}$ if and only if $\Im(Z_r(E)) < 0$. Moreover, $\lambda(E) \leq \frac{3}{4}$ if and only if $\Im(Z_r(E)) \geq 0$.

If we consider the same C_1, D_1 as above for the construction of $\mathcal{A}_r$, we obtain the following lemma.

Lemma 3.12. *The group homomorphism Z_r is a stability function on $\mathcal{A}_r$*

Proof. First of all, we show that the image of $E \in \mathcal{A}_r$ under Z_r lies in $\mathbb{H} \cup \mathbb{R}_{<0}$.

Let $E \in \mathcal{T}$, then we consider the short exact sequence $0 \to T(E) \to E \to F(E) \to 0$. Note that by definition $F(E) \in \mathcal{T}$. Indeed, because of the right exactness of $\mathrm{Coker}(-)$ we obtain that $i^!(E) \in \mathrm{Coh}(C)$ implies that $i^!(F(E)) \in \mathrm{Coh}(C)$.

We prove now that $Z_r(E[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. We first show that $Z_r(F(E)[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. It is enough to assume that $F(E)$ is λ-semistable and then it follows directly from the analogous equation to 3.3. Note that $Z_r(T(E)[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Indeed, as $\mathrm{rk}(T(E_1)) = \mathrm{rk}(T(E_2)) = 0$, then $\deg(T(E_1)) \geq 0$ if $T(E_1) \neq 0$ and $\Im(Z_r((T(E)[-1])) = -\deg(T(E_1))D_1 > 0$, as $D_1 < 0$. Since Z is additive with respect to short exact sequences we obtain that $Z(E[-1]) \in \mathbb{H} \cup \mathbb{R}_{<0}$. If $\deg(T(E_1)) = 0$, then $T(E_1) = 0$, and $F(E_1) \cong E_1$, thus

$$\Im(Z_r(F(E)[-1])) = \Im Z_r(E[-1]) > 0.$$

We now show that if $E \in \mathcal{F}$, then $Z_r(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$. Once again we consider the short exact sequence $0 \to T(E) \to E \to F(E) \to 0$. As $i^*(E)$ is a torsion-free sheaf, we get $i^*(T(E)) = 0$ and $T(E) = 0 \to T(E_2)$, where $T(E_2)$ is a torsion sheaf. Once again note that by definition $T(E) \in \mathcal{F}$ and $F(E) \in \mathcal{F}$. Then, it suffices to show that our claim follows for $T(E)$ and $F(E)$. Clearly $Z_r(T(E)) = -\deg(T(E_2)) < 0$, as $T(E_2)$ is a torsion-sheaf.

Let $F(E) \in \mathcal{F}$ a torsion-free $\lambda-$semistable object. If $F(E)$ satisfies $\lambda(F(E)) < 3/4$, then $-D_1 d_1 - C_1 r_1 - r_2 < 0$ and $Z(E)$ lies in the upper-half plane.

We now assume that $F(E) = F(E_1) \xrightarrow{f} F(E_2)$ in $\mathcal{F}$ is a torsion-free object with $D_1 d_1 + r_1 C_1 + r_2 = 0$. It suffices to show our statement for $F(E)$ a λ-semistable object. We now prove that $A_1 d_1 + B_1 r_1 - d_2 < 0$.

First note that if $F(E_1) = 0$, then $0 = D_1 d_1 + r_1 C_1 + r_2 = r_2$ and it implies $F(E) = 0$.

If $F(E_2) = 0$, then $D_1 d_1 + r_1 C_1 = 0$, and

$$A_1 d_1 + B_1 r_1 - d_2 = A_1 d_1 + B_1 r_1 < 0,$$

because $\det(M) > 0$.

Therefore, we assume $F(E_1)$ and $F(E_2) \neq 0$.

Claim 3.13. *If* $F(E_1), F(E_2) \neq 0$, *then* $\mathrm{rk}(\mathrm{Coker}(f)) = 0$.

Proof. If $\mathrm{Coker}(f) \neq 0$, then we have the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F(E_1) \arrow[r, "id"] \arrow[d, "f"'] & F(E_1) \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & \mathrm{Img}(f) \arrow[r] & F(E_2) \arrow[r] & \mathrm{Coker}(f) \arrow[r] & 0. \end{tikzcd} \tag{3.14}$$

If $\mathrm{rk}(\mathrm{Coker}(f)) \neq 0$, then by Lemma 3.2 and by the λ-semistability of $F(E)$, we have that $3/4 = \lambda(F(E)) \leq 1/2$, which gives us a contradiction. $\square$

Note that $\lambda(F(E_1) \to \mathrm{Img}(f)) = \lambda(F(E))$, and moreover

$$[F(E)] = [F(E_1) \to \mathrm{Img}(f)] + (0, 0, 0, d_2''),$$

where $d_2'' = \deg(\mathrm{Coker}(f)) > 0$, $d_1'' = \deg(\mathrm{Img}(f))$ and $d_2 = \deg(F(E_2))$. As

$$A_1 d_1 + B_1 r_1 - d_2 = A_1 d_1 + B_1 r_1 - d_1'' - d_2'' < A_1 d_1 + B_1 r_1 - d_1''$$

and $F(E_1) \to \mathrm{Img}(f) \in \mathcal{F}$, as it is a subobject of $F(E)$, the it is enough to show our statement for torsion-free objects $F(E_1) \xrightarrow{f} F(E_2) \in \mathcal{F}$ with $\mathrm{Coker}(f) = 0$.

Therefore, we assume that $\mathrm{Coker}(f) = 0$. Hence, we have $F(E) = F(E_1) \twoheadrightarrow F(E_2)$. As a consequence, we get

$$r_1 = \mathrm{rk}(F(E_1)) \geq \mathrm{rk}(F(E_2)) = r_2.$$

Since $K = i_*(i^!(F(E))) = \mathrm{Ker}(f) \to 0$ is a subobject of $F(E)$ in $\mathrm{TCoh}(C)$ and $\mathcal{F}$ is closed under subobjects, it implies that $K \in \mathcal{F}$. Since $[i_*(i^!(E))] = [K] = (r_1 - r_2, d_1 - d_2, 0, 0)$, where $d_1 = \deg(F(E_1))$ and $d_2 = \deg(F(E_2))$, it follows that

$$-D_1(d_1 - d_2) - C_1(r_1 - r_2) \quad \leq \quad 0. \tag{3.15}$$

By hypothesis

$$-D_1 d_1 - r_1 C_1 = r_2.$$

Therefore, after replacing $-D_1 d_1 - C_1 r_1$ in Equation (3.15), we obtain

$$D_1 d_2 \quad \leq \quad -(C_1 + 1) r_2. \tag{3.16}$$

We want now to show that $A_1 d_1 + B_1 r_1 - d_2 < 0$. First note that $d_1 = \frac{r_2 + C_1 r_1}{-D_1}$, we obtain

$$\begin{aligned} A_1d_1 + B_1r_1 - d_2 &= A_1\frac{r_2 + r_1C_1}{-D_1} + r_1B_1 - d_2 \\ &= (\frac{1}{-D_1})(A_1r_2 + r_1A_1C_1 - r_1B_1D_1 + D_1d_2). \end{aligned}$$

Since $-D_1 > 0$, it is enough to show that $A_1r_2+r_1A_1C_1-r_1B_1D_1+D_1d_2 < 0$. By Equation (3.16), we obtain

$$\begin{aligned} A_1r_2 + r_1A_1C_1 - r_1B_1D_1 + D_1d_2 &\overset{3.16}{\leq} A_1r_2 + r_1C_1A_1 - r_1B_1D_1 - (C_1+1)r_2 \\ &= r_2(A_1 - C_1 - 1) - r_1(\det(M)). \end{aligned}$$

Since $-r_1 \leq -r_2$ and $\det(M) > 0$, we obtain

$$\begin{aligned} r_2(A_1 - C_1 - 1) - r_1(\det(M)) &\leq r_2(A_1 - C_1 - 1) - r_2(\det(M)) \\ &= r_2(-\operatorname{Tr}(M) - 1 - \det(M)) \\ &= r_2(-\det(M+I)) < 0, \end{aligned}$$

as $\det(M+I) > 0$.

Since Z_r is additive with respect to short exact sequences we obtain that $Z_r(E) \in \mathbb{H} \cup \mathbb{R}_{<0}$ for $E \in \mathcal{F}$. □

We now show that the pair defined above satisfies the Harder-Narasimhan property whenever $A_1, B_1, C_1, D_1 \in \mathbb{Q}$. In Section 3.4, we prove that the Harder-Narsimhan property holds for all $A_1, B_1, C_1, D_1 \in \mathbb{R}$.

Lemma 3.14. *If $A_1, B_1, C_1, D_1 \in \mathbb{Q}$, the pair $(Z_r, \mathcal{A}_r)$ is a pre-stability condition.*

Proof. We follow the steps of [Bri08, Prop. 7.1]. First note that by [BM11, Prop. B.2], it is enough to show that if $E \in \mathcal{A}_r$ and

$$0 \subset L_1 \subset L_2 \ldots \subset L_i \subset \ldots \subset E,$$

where L_i belongs to the full subcategory $\mathcal{P}'(1)$ of objects with phase one, the sequence stabilizes. As $L_i \in \mathcal{A}_r = (\mathcal{F}, \mathcal{T}[-1])$, we consider the short exact sequence

$$0 \to F_i \to L_i \to T_i[-1] \to 0,$$

where $F_i \in \mathcal{F}$ and $T_i \in \mathcal{T}$. As $\Im Z_r(L_i) = 0$, we obtain that $\Im Z_r(T_i[-1]) = 0$. Note that it implies that $T_i = 0$. Indeed, by definition the torsion part $T(T_i)$ and the torsion-free part

$F(T_i)$ of T_i satisfy that $\Im Z_r(T(T_i)) < 0$ and $\Im Z_r(F(T_i)) < 0$, if $T(T_i)_1 \neq 0$, then they are non-zero. If $T(T_i)_1 = 0$, then $\Im(Z_r(T_i)) = \Im(Z(F(T_i))) < 0$. Therefore, we obtain that $\mathcal{P}'(1) \subseteq \mathcal{F} \subseteq \mathrm{TCoh}(C)$, as $\mathrm{TCoh}(C)$ is Noetherian, our result follows. □

Remark 3.15. Note that the pre-stability condition $\sigma_r = (Z_r, \mathcal{A}_r)$ given above is discrete. By Lemma 1.58 it is also locally finite.

Remark 3.16. Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14. Note that $i_*(\mathbb{C}(x))[-1]$, $l_*(\mathbb{C}(x))[-1]$ and $j_*(\mathbb{C}(x))$ are in $\mathcal{A}_r$ and $j_*(\mathbb{C}(x))$ is stable of phase one. Indeed, as $Z_r(j_*(\mathbb{C}(x))) = -1$, and $j_*(\mathbb{C}(x))$ is a simple object in $\mathcal{F}$, then $j_*(\mathbb{C}(x))$ is σ-stable of phase one.

Lemma 3.17. *We have that*

$$\mathrm{Coh}_2(C) \subseteq \mathcal{A}_r \text{ and } \mathrm{Coh}_1^r(C) \subseteq \mathcal{A}_r \text{ and } \mathrm{Coh}_3^{r_3}(C) \subseteq \mathcal{A}_r$$

where $r_3 = \dfrac{\mathrm{acot}(\frac{C_1+1}{D_1})}{\pi}$.

Proof. First of all, note that $\mathrm{Coh}_2(C) \subseteq \mathcal{F} \subseteq \mathcal{A}_r$. We consider the torsion pair $\mathrm{Coh}(C) = (\mathcal{T}_1, \mathcal{F}_1)$ of Remark 3.7, which also induces the heart $\mathrm{Coh}_1^r(C)$ after a right tilting, i.e. $\mathrm{Coh}_1^r(C) = (\mathcal{F}_1, \mathcal{T}_1[-1])$. Let $E \in \mathrm{Coh}_1(C)$ be a μ-semistable object. If $E = i_*(\mathbb{C}(x))$, then by Remark 3.16, we have $E[-1] \in \mathcal{A}_r$. We assume that E is torsion free. As the only possible subobjects or quotients of E are in $\mathrm{Coh}_1(C)$, we have that E is μ-semistable if and only if it is λ-semistable. It follows directly from the definition of $\mathcal{T}$ and $\mathcal{F}$ that

$$i_*(\mathcal{T}_1) \subseteq \mathcal{T} \text{ and } i_*(\mathcal{F}_1) \subseteq \mathcal{F}.$$

We also consider a torsion pair $\mathrm{Coh}(C) = (\mathcal{T}_3, \mathcal{F}_3)$, as in Remark 1.83, such that after taking the right tilt $\mathrm{Coh}^{r_3}(C) = (\mathcal{F}_3, \mathcal{T}_3[-1])$. Let $E \in \mathrm{Coh}_3(C)$ be a μ-semistable object. We have that $E = l_*(\mathbb{C}(x)) \in \mathcal{A}_r$ by Remark 3.16. We assume that E is torsion free. We start with $E \in l_*(\mathcal{T}_3)$. We consider its HN-factors A_i, where $i = 0, \dots, m$, and

$$\lambda(A_1) > \lambda(A_2) > \ldots > \lambda(A_m).$$

To show that $E \in \mathcal{T}$, it is enough to show that $\lambda(A_m) > \frac{3}{4}$, i.e. $-D_1 d_1' - C_1 r_1' - r_2' > 0$, where $\mathrm{rk}((A_m)_i) = r_i'$ and $\deg((A_m)_i) = d_i'$ and $A_m = (A_m)_1 \twoheadrightarrow (A_m)_2$, because A_m is a quotient of E. We have that $E \twoheadrightarrow l_*((A_m)_1)$. By the μ-semistability

$$\mu((A_m)_1) = \frac{d_1'}{r_1'} \geq \frac{d_1}{r_1} = \mu(E)$$

equivalently

$$\frac{-D_1 d_1' - C_1 r_1'}{r_1'} \geq \frac{-D_1 d_1 - C_1 r_1}{r_1}.$$

As $E \in \mathcal{T}_3$, we have that $\frac{-D_1 d_1 - C_1 r_1}{r_1} > 1$. Since $r_2' \leq r_1'$, we obtain

$$-D_1 d_1' - C_1 r_1' - r_2' \geq -D_1 d_1' - C_1 r_1' - r_1' > 0.$$

As a consequence $l_*(\mathcal{T}_3) \subseteq \mathcal{T}$. Analogously, we prove that $l_*(\mathcal{F}_3) \subseteq \mathcal{F}$. □

3.2 The stability manifold $\mathrm{Stab}(\mathcal{T}_C)$

Lemma 1.73 is the main tool to prove that $\mathbb{C}(x)$ is a σ-stable object for every $\sigma \in \mathrm{Stab}(C)$ and every $x \in C$. In this section, we prove the analogous statement for $\mathcal{T}_C$, to give a characterization of every $\sigma \in \mathrm{Stab}(\mathcal{T}_C)$ in terms of $\mathbb{C}(x)$. In Subsection 3.2.1, we follow closely the steps of [Bri08, Lem. 10.1] to describe all the possible hearts appearing on pre-stability conditions on $\mathcal{T}_C$. We finally prove that every pre-stability condition on $\mathcal{T}_C$ has to be given by one of the already constructed pairs in Lemma 3.14 or Section 2.2 i.e. either by CP-gluing or by tilting.

Remark 3.18. Due to the assumption that $g(C) \geq 1$, if $\mathrm{Hom}_{\mathrm{Coh}(C)}(E, A) \neq 0$, then $\mathrm{Hom}_{\mathrm{Coh}(C)}(E, A \otimes \omega_C) \neq 0$. If f is a non-zero morphism in $\mathrm{Hom}_{\mathrm{Coh}(C)}(E, A)$, we denote by f_C the non-zero morphism in $\mathrm{Hom}_{\mathrm{Coh}(C)}(E, A \otimes \omega_C)$ associated to f.

Lemma 3.19. *If there is a distinguished triangle in $\mathcal{T}_C$ of the form*

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\varphi_E"'] & X \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\varphi_A"] & E_1[1] \arrow[d, "\varphi_E[1]"] \\ E_2 \arrow[r] & 0 \arrow[r] & A_2 \arrow[r, "l"] & E_2[1] \end{tikzcd} \tag{3.17}$$

with $X \in \mathrm{Coh}(C)$ and $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, A) = 0$, then $E_1, A_1 \in \mathrm{Coh}(C)$.

Proof. First of all note that $l_2 := j^!(l)$ is an isomorphism in $D^b(C)$. We now consider the functor $F \colon \mathcal{T}_C \to \mathrm{Mor}(D^b(C))$ defined as $F(E_1 \xrightarrow{\varphi_E} E_2) = E_1 \xrightarrow{[\varphi_E]} E_2$ at the level of objects. For Φ in $\mathrm{Hom}_{\mathcal{T}_C}(E, F)$, by 2.40, we get the commutative square

$$\begin{tikzcd} E_1 \arrow[r, "i^*(\Phi)"] \arrow[d, "{[\varphi_E]}"'] & F_1 \arrow[d, "{[\varphi_F]}"] \\ E_2 \arrow[r, "j^!(\Phi)"] & F_2 \end{tikzcd} \tag{3.18}$$

in $D^b(C)$, which is in fact a morphism in $\mathrm{Mor}(D^b(C))$.

Claim 3.20. *The functor $F \colon \mathcal{T}_C \to \mathrm{Mor}(D^b(C))$ is full.*

Proof. Let $(\varphi_1, \varphi_2) \in \mathrm{Hom}_{\mathrm{Mor}(D^b(C))}(F(E), F(G))$, with $E, G \in \mathcal{T}_C$. By definition we have the commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "\varphi_1"] \arrow[d, "{[\varphi_E]}"'] & G_1 \arrow[d, "{[\varphi_G]}"] \\ E_2 \arrow[r, "\varphi_2"] & G_2. \end{tikzcd} \tag{3.19}$$

Note that we obtain a commutative square in $\mathcal{T}_C$ given by

$$\begin{tikzcd} i_*(E_1) \arrow[r, "t_E"] \arrow[d, "\varphi_1"'] & j_*(E_2)[1] \arrow[d, "\varphi_2"] \\ i_*(G_1) \arrow[r, "t_F"] & j_*(G_2)[1]. \end{tikzcd} \tag{3.20}$$

By taking cones of the horizontal arrows and using Remark 2.46, we obtain that there is a morphism $\Phi \in \mathrm{Hom}_{\mathcal{T}_C}(E, F)$ with $F(\Phi) = (\varphi_1, \varphi_2)$. $\square$

By [Huy06, Corollary 3.15], for every $G \in D^b(C)$, we have that $G = \oplus_{i\in\mathbb{Z}} G_i[-i]$ with $G_i = H^i(G) \in \mathrm{Coh}(C)$. We also have the canonical morphisms (which come from the genuine chain maps)

$$G^\alpha[-\alpha] \xrightarrow{i_\alpha} G \xrightarrow{\pi_\beta} G^\beta[-\beta],$$

for $\alpha, \beta \in \mathbb{Z}$. Moreover, note that if we have a morphism $\psi\colon G_1 \to G_2$ in $D^b(C)$, then $H^i(\psi)[-i] = \pi_i^2 \circ \psi \circ i_i^1 \colon G_1^i[-i] \to G_2^i[-i]$.

For $G = G_1 \xrightarrow{\varphi_G} G_2 \in \mathcal{T}_C$ we get that $F(G) \in \mathrm{Hom}_{D^b(C)}(\oplus_i G_1^i[-i], \oplus_i G_2^i[-i])$, with $G_j^i \in \mathrm{Coh}(C)$, for $j = 1, 2$ and $i \in \mathbb{Z}$. Due to the fact that $\mathrm{Coh}(C)$ has homological dimension one, there is a non-zero morphism ${t_G}^i$ from $G_1^i[-i] \to G_2^i[-i] \oplus G_2^{i-1}[-i+1]$ to $G_1 \xrightarrow{\varphi_G} G_2$ in $\mathrm{Mor}(D^b(C))$. We construct the morphism

$$\begin{tikzcd} A_1 \arrow[r, "\pi_i\circ{[\varphi_A]}"] \arrow[d, "{[\varphi_A]}"'] & A_2^i[-i] \arrow[r, "id_C{[-i]}"] \arrow[d, "\mathrm{id}"] & A_2^i[-i] \otimes \omega_C \arrow[r, "l{[-i]}\otimes\omega_C"] \arrow[d, "l{[-i]}\otimes\omega_C"] & E_2^{i+1}[-i] \otimes \omega_C \arrow[r, "t^i_{\mathcal{S}_{\mathcal{T}_C}(E)}"'] \arrow[d] & E_2 \otimes \omega_C[1] \arrow[d, "{[i_E]}"] \\ A_2 \arrow[r, "\pi_i"] & A_2^i[-i] \arrow[r] & E_2^{i+1}[-i] \otimes \omega_C \arrow[r] & \mathcal{C}(\varphi_E)^{i+1}[-i] \oplus \mathcal{C}(\varphi_E)^i[-i+1] \otimes \omega_C \arrow[r] & C(\varphi_E)[1] \end{tikzcd}$$

in $\mathrm{Mor}(D^b(C))$. Note that the morphism id_C is given by Remark 3.18.

We obtain a morphishm $A \to \mathcal{S}_{\mathcal{T}_C}(E)$ in $\mathrm{Mor}(D^b(C))$. Since F is full, we have a morphism in $\mathrm{Hom}_{\mathcal{T}_C}(A, \mathcal{S}_{\mathcal{T}_C}(E))$. By Serre duality, we obtain

$$\mathrm{Hom}_{\mathcal{T}_C}(A, \mathcal{S}_{\mathcal{T}_C}(E)) \cong \mathrm{Hom}_{\mathcal{T}_C}(E, A)^*.$$

By hypothesis, this morphism has to be zero. This directly implies that $\mathrm{id}_C[-i]\circ(\pi_i\circ[\varphi_A]) = 0$, because $l \otimes \omega_C$ is an isomorphism. Since the construction does not depend on the $i \in \mathbb{Z}$, we obtain that the induced morphism $A_1^i[-i] \xrightarrow{\mathrm{id}_C[-i]\circ\pi_i\circ[\varphi_A]\circ i_i} A_2^i[-i]\otimes\omega_C$ is zero. Due to the fact that it is precisely the morphism $(\pi_i\circ[\varphi_A]\circ i_i)_C[-i]$ in $\mathrm{Coh}(C)[-i]$, as in Remark 3.18, we obtain that $A_1^i \xrightarrow{\pi_i\circ[\varphi_A]\circ i_1} A_2^i$ is also zero. The triangle (3.17) in $\mathcal{T}_C$ induces a long exact sequence in cohomology and we get

$$\begin{tikzcd} A_1^i \arrow[r, "\cong"] \arrow[d, "\pi_i\circ[\varphi_A]\circ i_i"'] & E_1^{i+1} \arrow[d, "\pi_{i+1}\circ[\varphi_E]\circ i_{i+1}"] \\ A_2^i \arrow[r, "\cong"] & E_2^{i+1}, \end{tikzcd}$$

as the morphism in the component i in $\mathrm{Mor}(D^b(C))$ is precisely the one given by the cohomology. Therefore $\pi_i\circ[\varphi_E]\circ i_1 = 0$, for all $i \neq -1, 0$.

This implies that there is a morphism from $E_1^i[-i] \to E_2^{i-1}[-i+1]$ to $E_1 \xrightarrow{\varphi_E} E_2$, which, moreover, is a split homorphism for $i \neq 0, 1$. In addition, it provides a split monomorphism from $A_1^i[-i] \to A_2^{i-1}[-i+1]$ to $A_1 \xrightarrow{\varphi_A} A_2$ for $i \neq -1, 0$.

We get the isomorphism

$$\begin{tikzcd} A_1^i[-i] \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\varphi_A"] & E_1[1] \arrow[r] \arrow[d, "\varphi_E[1]"] & E_1^{i+1}[-i] \arrow[d] \\ A_2^{i-1}[-i+1] \arrow[r] & A_2 \arrow[r, "l"] & E_2[1] \arrow[r] & E_2^i[-i+1] \end{tikzcd}$$

in $\mathrm{Mor}(D^b(C))$, were the morphism of the first row is the isomorphism induced by the long exact sequence of cohomology of the triangle (3.17).

From the isomorphism above, we construct the morphism

$$\begin{tikzcd} E_1[1] \arrow[r] \arrow[d, "{[\varphi_E]}"'] & E_1^{i+1}[-i] \arrow[r, "\cong"] \arrow[d] & A_1^i[-i] \arrow[r] \arrow[d] & A_1 \arrow[d, "{[\varphi_A]}"'] \\ E_2[1] \arrow[r] & E_2^i[-i+1] \arrow[r, "\cong"] & A_2^{i-1}[-i+1] \arrow[r] & A_2 \end{tikzcd}$$

in $\mathrm{Mor}(D^b(C))$. Once again, since the functor F is full, this needs to be the zero morphism. As a consequence, we obtain that $A_1^i[-i], A_2^i[-i]$ are zero for all $i \neq 0, -1$. We now study the remaining object in the long exact sequence of cohomology

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & A_1^{-1} \arrow[r, "t_1"'] \arrow[d, "0"'] & E_1^0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & A_1^0 \arrow[r, "t_2"'] \arrow[d, "0"'] & E_1^1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & A_2^{-1} \arrow[r, "\cong"] & E_2^0 \arrow[r] & 0 \arrow[r] & A_2^0 \arrow[r, "\cong"] & E_2^1 \arrow[r] & 0. \end{tikzcd} \tag{3.21}$$

We now show that $t_i = 0$, for $i = 1, 2$. If $t_1 \neq 0$, by Remark 3.18, we obtain a non-zero morphism $t_{1C} \colon A_1^{-1} \to E_1^0 \otimes \omega_C$. By Serre duality

$$\mathrm{Hom}_{D^b(C)}(A_1^{-1}, E_1^0 \otimes \omega_C) \cong \mathrm{Hom}_{D^b(C)}(E_1^0[-1], A_1^{-1})$$

and there is a non-zero morphism $t_1' \colon E_1^0[-1] \to A_1^{-1}$. As $E_1^0 \to E_2^0$ is a direct summand of E, we get the non-zero morphism

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\varphi_E"'] & E_1^0 \arrow[r, "t_1'[1]"] \arrow[d] & A_1^{-1}[1] \arrow[r] \arrow[d, "0"'] & A_1 \arrow[d, "\varphi_A"'] \\ E_2 \arrow[r] & E_2^0 \arrow[r, "0"'] & A_2^{-1}[1] \arrow[r] & A_2 \end{tikzcd}$$

in $\mathrm{Mor}(D^b(C))$. As above, this implies that $t_1' = 0$ and therefore $t_1 = 0$. We now prove that $t_2 = 0$.

The triangle $A_1 \to A_2 \to C(\varphi_A)$ induces a long exact sequence of cohomology. It follows that $H^i(C(\varphi_A)) = 0$ unless $i = -1$, i.e. $C(\varphi_A) \in \mathrm{Coh}(C)[1]$.

Note that $A_2[-1] \to C(\varphi_A)[-1] \to A_1$ is a short exact sequence in $\mathrm{Coh}(C)$, then the morphism $C(\varphi_A)[-1] \xrightarrow{\pi_A} A_1$ is surjective in $\mathrm{Coh}(C)$.

Let us assume that $t_2 \neq 0$. By Remark 3.18, there is a non-zero morphism $t_2' \colon A_1 \to E_1^1 \otimes \omega_C$. We now consider $t_3 := t_2' \circ \pi_A$ in $\mathrm{Coh}(C)$. Since π_A is surjective and $t_2' \neq 0$, then $t_3 \neq 0$.

This induces the morphism $\Phi \colon A \to \mathcal{S}_{\mathcal{T}_C}(E)$ given by

$$\begin{tikzcd} A_1 \arrow[r, "t_2'"] \arrow[d, "\varphi_A"'] & E_1^1 \otimes \omega_C \arrow[r, "i"] \arrow[d] & E_1[1] \otimes \omega_C \arrow[r, "\varphi_E \otimes \mathrm{id}"] \arrow[d] & E_2[1] \otimes \omega_C \arrow[d] \\ A_2 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathcal{C}(\varphi_E)[1] \otimes \omega_C \end{tikzcd}$$

in $\mathrm{Hom}_{\mathcal{T}_C}(A, \mathcal{S}_{\mathcal{T}_C}(E))$. Note that i is a split monomorphism. By Serre duality,

$$\mathrm{Hom}_{\mathcal{T}_C}(A, \mathcal{S}_{\mathcal{T}_C}(E)) \cong \mathrm{Hom}_{\mathcal{T}_C}(E, A)^* = 0.$$

Therefore, we have that $\Phi = 0$. After applying $i^!$, we obtain

$$i^!(\Phi) = \mathcal{C}(\varphi_A)[-1] \xrightarrow{t_3} E_1^1 \otimes \omega_C \xrightarrow{i} E_1[1] \otimes \omega_C,$$

which has to be zero. As consequence $t_3 = 0$ and we have a contradiction. Hence, we get $A_1^{-1} = E_1^1 = 0$, which implies that $A_1, E_1 \in \mathrm{Coh}(C)$. $\square$

Remark 3.21. From now one we assume that all pre-stability conditions on $\mathcal{T}_C$ that we considering are locally finite.

Proposition 3.22. *Let X be either a skyscraper sheaf $\mathbb{C}(x)$ or a line bundle $\mathcal{L}$ on C. For any pre-stability condition σ on $\mathcal{T}_C$, if $i_*(X)$ is not σ-semistable, then $j_*(X)$ and $l_*(X)$ are σ-stable.*

Proof. We assume that $i_*(X)$ is not σ-semistable. Therefore, we consider the last triangle of its Harder-Narasimhan filtration

$$E \longrightarrow i_*(X) \longrightarrow A \longrightarrow E[1] ,$$

explicitly

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\varphi_E"'] & X \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\varphi_A"] & E_1[1] \arrow[d, "\varphi_E[1]"] \\ E_2 \arrow[r] & 0 \arrow[r] & A_2 \arrow[r] & E_2[1] \end{tikzcd} , \tag{3.22}$$

with $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, A) = 0$ and $A \in \mathcal{T}_C$ semistable. By Lemma 3.19, we have $E_1, A_1 \in \mathrm{Coh}(C)$, thus we obtain a short exact sequence

$$0 \to E_1 \to X \to A_1 \to 0$$

in $\mathrm{Coh}(C)$ which is not possible. Hence, either $E_1 = 0$ or $A_1 = 0$.

If $E_1 = 0$, then by adjointness we have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, A) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(j_*(E_2), A) = \mathrm{Hom}^{\leq 0}_{D^b(C)}(E_2, A_2) = 0.$$

As $E_2[1] \cong \mathcal{A}_2$, we have $\mathrm{Hom}^{\leq 0}_{D^b(C)}(E_2, E_2[1]) = 0$, which is a contradiction.

Therefore, we obtain $A_1 = 0$ and $E_1 \cong X$. By adjointness we have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, A) = \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, j_*(A_2)) = \mathrm{Hom}^{\leq 0}_{D^b(C)}(C(\varphi_E), A_2) = 0.$$

Exactness of the functor j^* yields the following triangle

$$C(\varphi_E) \longrightarrow X[1] \longrightarrow A_2 \longrightarrow C(\varphi_E)[1] .$$

Since $\mathrm{Hom}^{\leq 0}(C(\varphi_E), A_2) = 0$, due to Lemma 1.73, the classical GKR lemma for curves, we get that

$$C(\varphi_E)[-1], A_2[-1] \in \mathrm{Coh}(C)$$

which once again it is not possible. As A_2 cannot be zero, we get $C(\varphi_E) = 0$ and $A_2 \cong X[1]$.

This implies that $A \cong j_*(X)[1]$ and that $E \cong l_*(X) \in D_3$. As a consequence $j_*(X)$ is σ-semistable.

We now show that $l_*(X)$ is σ-semistable. We proceed by contradiction. If $l_*(X)$ is not σ-semistable, we examine the last triangle of its HN-filtration

$$F \to l_*(X) \to B \to F[1]$$

where $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(F, B) = 0$ and B is σ-semistable. We first apply the Serre functor and then we apply the autoequivalence given by tensoring by $l_*(\omega_C^*)$. We obtain

$$\begin{tikzcd} F_2[1] \arrow[r] \arrow[d, "\varphi'_E"'] & X[1] \arrow[r] \arrow[d] & B_2[1] \arrow[r] \arrow[d, "\varphi'_B"] & F_2[1] \arrow[d, "\varphi'_F[2]"] \\ C(\varphi_F)[1] \arrow[r] & 0 \arrow[r] & C(\varphi_B)[1] \arrow[r] & C(\varphi_F)[2] \end{tikzcd}$$

with

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathcal{S}_{\mathcal{T}_C}(F), \mathcal{S}_{\mathcal{T}_C}(B)) = 0.$$

Arguing as above, $B_2 = 0$ and $F_2 \cong X$. Then $B \in D_1$ and by adjointness

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(F, B) \cong \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(F, i_*(B_1)) = \mathrm{Hom}^{\leq 0}(F_1, B_1) = 0.$$

By applying the classical GKR lemma for curves to the triangle

$$F_1 \to X \to B_1,$$

we get that $B_1 = 0$ or $F_1 = 0$. As $B \neq 0$, we get $F_1 = 0$ and $B_1 \cong X$. Consequently $B \cong i_*(X)$ and $i_*(X)$ is σ-semistable, which contradicts our assumption. Therefore $l_*(X)$ is σ-semistable.

We prove now that $l_*(X)$ and $j_*(X)$ are σ-stable. We start by proving by contradiction that $l_*(X)$ is stable. If $l_*(X)$ is not σ-stable, we consider its Jordan-Hölder filtration. Note that all its σ-stable factors A_i have the same phase ϕ. We assume that $\mathrm{Hom}_{\mathcal{T}_C}(A_{i_0}, l_*(X)) \neq 0$ for a σ-stable factor A_{i_0}. Therefore by [Huy14, Ex. 1.6], there is a short exact sequence

$$0 \to E^{'} \to l_*(X) \to E^{''} \to 0,$$

with $E^{'}$ and $E^{''}$ semistable of phase ϕ, such that all stable factors of $E^{'}$ are isomorphic to A_{i_0} and $\mathrm{Hom}_{\mathcal{T}_C}(E^{'}, E^{''}) = 0$. By the semistability of $E^{'}$ and $E^{''}$, we get

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E^{'}, E^{''}) = 0.$$

Arguing as above, i.e. by applying the Serre functor and by tensoring by $l_*(\omega_C^*)$, we obtain $E_2^{'} \cong X$ and $E_2^{''} = 0$ and

$$\mathrm{Hom}^{\leq 0}(E_1^{'}, E_1^{''}) = 0.$$

Once again, the classical GKR lemma for curves, implies that $E_1^{''} = 0$ or $E_1^{'} = 0$. Since $i_*(X)$ is not σ-semistable we have that $E_1^{'} \neq 0$. Therefore, we obtain $E_1^{''} = 0$ and $E_1^{'} \cong X$. As a consequence, we have $E^{''} = 0$ which implies that all the stable factors of $l_*(X)$ are isomorphic to A_{i_0}. Hence, $[l_*(X)] = n[A_{i_0}]$, where n is the number of stable factors. Since $[l_*(\mathbb{C}(x))] = (0,1,0,1)$ and $[\mathcal{L}] = (1, \deg(\mathcal{L}), 1, \deg(\mathcal{L}))$, we must have $n = 1$, i.e. $l_*(\mathbb{C}(x))$ and $l_*(\mathcal{L})$ are stable.

An analogous proof works for the stability of $j_*(X)$. Instead of using the Serre functor, we use $\mathcal{S}^{-1}$. Consequently, we obtain that $j_*(\mathbb{C}(x))$ and $j_*(\mathcal{L})$ are stable. □

We now give several consequences of the last proposition.

Remark 3.23. If X is either $\mathbb{C}(x)$ or $\mathcal{L}$ as above, we use Proposition 3.22 to prove that if $j_*(X)$ ($l_*(X)$) is not σ-semistable, then $i_*(X)$ and $l_*(X)$ ($j_*(X)$ and $i_*(X)$) are σ-stable. Meaning that if one of the objects $i_*(X), j_*(X), l_*(X)$ is not σ-semistable then the other two have to be σ-stable.

Remark 3.24. If $i_*(X)$ is not σ-semistable, where X is either $\mathbb{C}(x)$ or $\mathcal{L}$, then by Proposition 3.22, we obtain the HN-filtration for $i_*(X)$. It is given precisely by

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "\varphi[1]"] \\ X \arrow[r] & 0 \arrow[r] & X[1] \arrow[r] & X[1] \end{tikzcd} .$$

After applying the Serre functor, we obtain the corresponding HN-filtration for $j_*(X)$ and $l_*(X)$.

Moreover, we define

$$\phi_x^0 \coloneqq \phi_\sigma(i_*(\mathbb{C}(x))) \text{ and } \phi_{\mathcal{L}}^1 \coloneqq \phi_\sigma(i_*(\mathcal{L})),$$
$$\phi_x^2 \coloneqq \phi_\sigma(j_*(\mathbb{C}(x))) \text{ and } \phi_{\mathcal{L}}^3 \coloneqq \phi_\sigma(j_*(\mathcal{L})),$$
$$\phi_x^4 \coloneqq \phi_\sigma(l_*(\mathbb{C}(x))) \text{ and } \phi_{\mathcal{L}}^5 \coloneqq \phi_\sigma(l_*(\mathcal{L})).$$

If $\mathcal{L} = \mathcal{O}_C$, then $\phi_1 = \phi_\sigma(i_*(\mathcal{O}_C))$, $\phi_3 = \phi_\sigma(j_*(\mathcal{O}_C))$, and $\phi_5 = \phi_\sigma(l_*(\mathcal{O}_C))$.

Lemma 3.25. *If $i_*(\mathbb{C}(x))$ is not σ-semistable, then $\phi_x^4 > \phi_x^2 + 1$. Similarly, if $i_*(\mathcal{L})$ is not σ-semistable, then $\phi_{\mathcal{L}}^5 > \phi_{\mathcal{L}}^3 + 1$.*

Proof. If $i_*(\mathbb{C}(x))$ is not σ-semistable, then Remark 3.24 gives explicitly the last triangle of its HN-filtration, which implies $\phi_x^4 > \phi_x^2 + 1$. The same conclusion can be drawn for $i_*(\mathcal{L})$. □

Remark 3.26. After applying the Serre functor to the HN-filtration of $i_*(X)$, we obtain the analogous results for $j_*(X)$ and $l_*(X)$, i.e.

If $l_*(\mathbb{C}(x))$ is not σ-semistable, then $\phi_x^2 > \phi_x^0$.

If $l_*(\mathcal{L})$ is not σ-semistable, then $\phi_{\mathcal{L}}^3 > \phi_{\mathcal{L}}^1$.

If $j_*(\mathbb{C}(x))$ is not σ-semistable, then $\phi_x^0 - 1 > \phi_x^4$.

If $j_*(\mathcal{L})$ is not σ-semistable, then $\phi_{\mathcal{L}}^1 - 1 > \phi_{\mathcal{L}}^5$.

Lemma 3.27. *If $i_*(X)$ is strictly σ-semistable, where X is either $\mathbb{C}(x)$ or $\mathcal{L}$, then $j_*(X)$ and $l_*(X)$ are σ-stable. Moreover, a Jordan-Hölder filtration is given by*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\mathrm{id}"'] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "\mathrm{id}[1]"] \\
X \arrow[r] & 0 \arrow[r] & X[1] \arrow[r] & X[1]
\end{tikzcd} .$$

Proof. First note that if $i_*(X)$ is strictly σ-semistable, it implies that $j_*(X)$ and $l_*(X)$ are σ-semistable. Indeed, if one of them is not σ-semistable by Proposition 3.22 follows that $i_*(X)$ is σ-stable, which contradicts our assumption.

Let A_{i_0} be a stable factor such that $\mathrm{Hom}(A_{i_0}, i_*(X)) \neq 0$. By [Huy14, Ex. 1.6], there is a short exact sequence

$$E^{'} \to i_*(X) \to E^{''},$$

with $E^{'}, E^{''}$ σ-semistable, such that $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E^{'}, E^{''}) = 0$ and all the stable factors of $E^{'}$ are isomorphic to A_{i_0}. By Lemma 3.19, we have $E^{'} \cong l_*(X)$ and $E^{''} \cong j_*(X)$. Since all the stable factors of $l_*(X)$ are isomorphic to A_{i_0}, there is a natural number n, such that $[l_*(X)] = n[A_{i_0}]$ in the Grothendieck group. As explained before, it implies that $n = 1$ and $l_*(X)$ is stable.

We now prove by contradiction that $j_*(X)$ is stable. We assume that $j_*(X)$ is strictly σ-semistable. We apply the same reasoning as above, in this case we obtain a short exact sequence

$$F^{'} \to j_*(X) \to F^{''},$$

where $F^{'} \cong i_*(X)$, since all the stable factors of $F^{'}$ are isomorphic, we obtain that $i_*(X)$ is stable, which contradicts our hypothesis. □

It now makes sense to define the following sets:

Definition 3.28. We define the set Θ_{ij} of pre-stability conditions on $\mathcal{T}_C$ for $ij = 12, 23$ or

31 as follows:

$$\Theta_{12} = \{\sigma \mid i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x)), i_*(\mathcal{L}), j_*(\mathcal{L}) \text{ stable for all line bundles } \mathcal{L} \in \mathrm{Coh}(C) \text{ and all } x \in C\ \},$$
$$\Theta_{23} = \{\sigma \mid j_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), j_*(\mathcal{L}), l_*(\mathcal{L}) \text{ stable for all line bundles } \mathcal{L} \in \mathrm{Coh}(C) \text{ and all } x \in C\ \},$$
$$\Theta_{31} = \{\sigma \mid i_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), i_*(\mathcal{L}), l_*(\mathcal{L}) \text{ stable for all line bundles } \mathcal{L} \in \mathrm{Coh}(C) \text{ and all } x \in C\ \}.$$

Recall that we assumed that all the pre-stability conditions are locally finite.

Remark 3.29. Note that

$$\mathcal{S}_{\mathcal{T}_C}(\Theta_{12}) = \Theta_{23} \text{ and } \mathcal{S}_{\mathcal{T}_C}(\Theta_{23}) = \Theta_{31} \text{ and } \mathcal{S}_{\mathcal{T}_C}(\Theta_{31}) = \Theta_{12}.$$

Theorem 3.30. *If σ is a pre-stability condition on $\mathcal{T}_C$, then*

$$\sigma \in \Theta_{12} \cup \Theta_{23} \cup \Theta_{31}.$$

Proof. Let σ be an arbitrary pre-stability condition. We first assume that $\sigma \notin \Theta_{23}$ and we prove that $\sigma \in \Theta_{12}$ or $\sigma \in \Theta_{31}$.

Thus, there is a line bundle $\mathcal{L}$ such that $j_*(\mathcal{L})$ or $l_*(\mathcal{L})$ is not σ-stable, or either $j_*(\mathbb{C}(x))$ or $l_*(\mathbb{C}(x))$ is not σ-stable for some $x \in C$.

Assume that there is $x \in C$ such that $l_*(\mathbb{C}(x))$ is not σ-stable. We need to show that $\sigma \in \Theta_{12}$, as it cannot be in Θ_{31}. By Remark 3.23, it follows that $j_*(\mathbb{C}(x))$ and $i_*(\mathbb{C}(x))$ are σ-stable. We now show that for every line bundle $\mathcal{L}$ and every $x \in C$, we have that $j_*(X)$ and $i_*(X)$ are σ-stable, where $X = \mathcal{L}$ or $X = \mathbb{C}(x)$.

We prove it by contradiction, assume that there is a line bundle $\mathcal{L}$, such that $i_*(\mathcal{L})$ is not σ-stable, which implies that $j_*(\mathcal{L})$ and $l_*(\mathcal{L})$ are σ-stable.

Since $l_*(\mathbb{C}(x))$ is not σ-stable, by Remark 3.24 we obtain that its HN-filtration (or its JH-filtration) is given by

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\varphi"'] & \mathbb{C}(x) \arrow[r] \arrow[d] & \mathbb{C}(x) \arrow[r] \arrow[d] & 0 \arrow[d] \\
\mathbb{C}(x) \arrow[r] & \mathbb{C}(x) \arrow[r] & 0 \arrow[r] & \mathbb{C}(x)[1]
\end{tikzcd},$$

which implies that $\phi^2_x \geq \phi^0_x$. Analogously, since $i_*(\mathcal{L})$ is not σ-semistable, then we get $\phi^3_{\mathcal{L}} + 1 \leq \phi^5_{\mathcal{L}}$. Let us consider the morphism $\mathbb{C}(x) \to \mathcal{L}[1]$ in $D^b(C)$, it induces a morphism $j_*(\mathbb{C}(x)) \to j_*(\mathcal{L})[1]$. Since by hypothesis both are stable and not isomorphic, we obtain that $\phi^2_x < \phi^3_{\mathcal{L}} + 1$.

After applying the Serre functor to the morphism $l_*(\mathbb{C}(x)) \to l_*(\mathcal{L}[1])$, we obtain a morphism $l_*(\mathcal{L})[1] \to i_*(\mathbb{C}(x))[1]$. As both objects are σ-stable and not isomorphic, we get

$\phi^5_{\mathcal{L}} < \phi^0_x$. We now put all the inequalities together, yielding

$$\phi^0_x \leq \phi^2_x < \phi^3_{\mathcal{L}} + 1 \leq \phi^5_{\mathcal{L}} < \phi^0_x,$$

which is a contradiction. Therefore $i_*(\mathcal{L})$ has to be σ-stable for all line bundles. Analogously we prove that $j_*(\mathcal{L})$ has to be also stable for all line bundles $\mathcal{L}$.

We now assume that there is a point $y \in C$, such that $j_*(\mathbb{C}(y))$ is not σ-stable. Then, by Remark 3.23 it implies that $i_*(\mathbb{C}(y))$ and $l_*(\mathbb{C}(y))$ are σ-stable and by the triangle

$$\begin{tikzcd}
\mathbb{C}(y)[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \mathbb{C}(y) \arrow[r] \arrow[d] & \mathbb{C}(y) \arrow[d] \\
0 \arrow[r] & \mathbb{C}(y) \arrow[r] & \mathbb{C}(y) \arrow[r] & 0
\end{tikzcd},$$

we obtain $\phi^0_y - 1 \geq \phi^4_y$.

Note that $[i_*(\mathbb{C}(x))] = [i_*(\mathbb{C}(y))]$ in the Grothendieck group. As a consequence, we obtain

$$\phi^0_x = \phi^0_y + m,$$

with $m \in \mathbb{Z}$. But as $i_*(\mathcal{O}_C)$ is σ-semistable and we have non-zero morphisms from $i_*(\mathcal{O}_C)$ to $i_*(\mathbb{C}(x))$ to $i_*(\mathbb{C}(y)$ and from $i_*(\mathbb{C}(x))$ and $i_*(\mathbb{C}(y))$ to $i_*(\mathcal{O}_C)[1]$, we obtain

$$\phi_1 < \phi^0_y < \phi_1 + 1 \text{ and } \phi_1 < \phi^0_x < \phi_1 + 1,$$

which implies that

$$\phi^0_y = \phi^0_x.$$

Since we have a non-zero morphism $j_*(\mathcal{O}_C) \to l_*(\mathbb{C}(y))$, we obtain $\phi_3 < \phi^4_y$ and from the morphism $j_*(\mathbb{C}(x)) \to j_*(\mathcal{O}_C)[1]$ we obtain $\phi^2_x < \phi_3 + 1$.

As a consequence, we obtain

$$\phi_3 + 1 < \phi^4_y + 1 \leq \phi^0_y = \phi^0_x \leq \phi^2_x < \phi_3 + 1,$$

which is a contradiction. Therefore, we obtain that $j_*(\mathbb{C}(y))$ is σ-stable. We analogously prove it for $i^*(\mathbb{C}(y))$ for all $y \in C$. Then $\sigma \in \Theta_{12}$. The other cases follow analogously. $\square$

Corollary 3.31. *Let σ be a pre-stability condition on $\mathcal{T}_C$, with $i_*(\mathbb{C}(x))$ σ-stable for $x \in C$. Then, for all $y \in C$, we have that $i_*(\mathbb{C}(y))$ is σ-stable and*

$$\phi^0_x = \phi^0_y.$$

Proof. By Theorem 3.30, we have that if $\sigma \in \Theta_{12}$ or $\sigma \in \Theta_{31}$, it follows directly that for all $y \in C$ we have that $i_*(\mathbb{C}(y))$ is σ-stable. Note that $[i_*(\mathbb{C}(x))] = [i_*(\mathbb{C}(y))]$ in the Grothendieck group. This implies that

$$\phi_x^0 = \phi_y^0 + m,$$

with $m \in \mathbb{Z}$. But as $i_*(\mathcal{O}_C)$ is σ-semistable and we have non-zero morphisms from $i_*(\mathcal{O}_C)$ to $i_*(\mathbb{C}(x))$ to $i_*(\mathbb{C}(y)$ and from $i_*(\mathbb{C}(x))$ and $i_*(\mathbb{C}(y))$ to $i_*(\mathcal{O}_C)[1]$, we obtain

$$\phi_1 < \phi_y^0 < \phi_1 + 1 \text{ and } \phi_1 < \phi_x^0 < \phi_1 + 1,$$

which implies that

$$\phi_y^0 = \phi_x^0.$$

If $\sigma \in \Theta_{23}$ and there is $y \in C$, such that $i_*(\mathbb{C}(y))$ is not σ-stable. It implies that $\phi_y^4 \geq \phi_y^2 + 1$. As $i_*(\mathbb{C}(x)$, $j_*(\mathbb{C}(x))$ and $l_*(\mathbb{C}(x))$ are σ-stable, we get

$$\phi_x^4 < \phi_x^0 < \phi_x^2 + 1 = \phi_y^2 + 1 \leq \phi_y^4,$$

which is a contradiction, because as explained before $\phi_x^4 = \phi_y^4$. Then $i_*(\mathbb{C}(y))$ is σ-semistable for every $y \in C$. As before

$$\phi_x^0 = \phi_y^0 + m,$$

but

$$\phi_x^2 < \phi_x^0 < \phi_x^2 + 1 \text{ and } \phi_x^2 < \phi_y^0 < \phi_x^2 + 1,$$

which implies

$$\phi_x^0 = \phi_y^0.$$

□

Remark 3.32. Analogously, by using Serre duality, we prove Corollary 3.31 for $j_*(\mathbb{C}(x))$ and $l_*(\mathbb{C}(x))$.

By Corollary 3.31, if $i_*(\mathbb{C}(x))$ is σ-stable for some $x \in C$, then $\phi_0(x)$ does not depend on x. We analogously prove the same statement for ϕ_2 and ϕ_4. Therefore, we define

$$\phi_0 \coloneqq \phi_x^0,$$

$$\phi_2 \coloneqq \phi_x^2,$$

$$\phi_4 \coloneqq \phi_x^4.$$

Lemma 3.33. *Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14. Then $i_*(\mathbb{C}(x))$ $l_*(\mathbb{C}(x))$ and $j_*(\mathbb{C}(x))$ are σ-stable.*

Proof. By Remark 3.16, we have that $i_*(\mathbb{C}(x))[-1]$, $l_*(\mathbb{C}(x))[-1]$ and $j_*(\mathbb{C}(x))$ are in $\mathcal{A}_r$ and that $j_*(\mathbb{C}(x))$ is stable of phase one. We now show that $i_*(\mathbb{C}(x))[-1]$ is σ-stable. By contradiction we first assume that $i_*(\mathbb{C}(x))$ is not σ-semistable. As a consequence of Remark 3.26, we have that $l_*(\mathbb{C}(x))$ is σ-stable and $\phi_4 > \phi_2 + 1$, but $\phi_2 = 1$ and $1 < \phi_4 < 2$, which gives a contradiction. If $i_*(\mathbb{C}(x))$ is σ-semistable but not stable, by Lemma 3.27 $\phi_4 \geq \phi_2 + 1$ and we have again a contradiction. The same reasoning works to prove that $l_*(\mathbb{C}(x))[-1]$ is stable. □

Lemma 3.34. *Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14. Then $i_*(\mathcal{O}_C)$ and $j_*(\mathcal{O}_C)$ are σ-stable if and only if $\phi_1 < \frac{3}{2}$.*

Proof. First note that $\phi_1 = \phi(i_*(\mathcal{O}_C))$ and $\phi_3 = \phi(j_*(\mathcal{O}_C)) = \frac{1}{2}$ makes sense, as Lemma 3.17 implies that $j_*(\mathcal{O}_C) \in \mathcal{A}_r$ and that $i_*(\mathcal{O}_C)$ is in $\mathcal{A}_r$ or in $\mathcal{A}_r[1]$. Moreover, if $i_*(\mathcal{O}_C)$ and $j_*(\mathcal{O}_C)$ are σ-stable, then $\phi_1 < \phi_3 + 1 = \frac{3}{2}$, because there is a non-zero morphism $i_*(\mathcal{O}_C) \to j_*(\mathcal{O}_C)[1]$.

We now prove the other direction. We assume that $\phi_1 < \frac{3}{2}$ and that $i_*(\mathcal{O}_C)$ is not stable. Then, by Remark 3.26 $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are stable and

$$\phi_5 \geq \phi_1 \geq \phi_3 + 1 = \frac{3}{2},$$

which gives us a contradiction. Analogously, if $j_*(\mathcal{O}_C)$ is not stable. Then by Remark 3.26 $i_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are stable and

$$\phi_1 - 1 \geq \phi_3 = \frac{1}{2} \geq \phi_5,$$

which gives us a contradiction. □

Remark 3.35. Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14. We have that $i_*(\mathcal{O}_C)$, $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable if and only if

$$\frac{1}{2} = \phi_3 < \phi_5 < \phi_1 < \frac{3}{2}.$$

Lemma 3.36. *Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14. Then $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable and $i_*(\mathcal{O}_C)$ is not σ-stable if and only if $\phi_1 \geq \frac{3}{2}$ and $\phi_5 > \frac{1}{2}$.*

Proof. As $i_*(\mathcal{O}_C)$ is not stable, by Lemma 3.34 we get $\phi_1 \geq \frac{3}{2}$, and since $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable and there is a non-zero morphism $j_*(\mathcal{O}_C) \to l_*(\mathcal{O}_C)$ and $\frac{1}{2} < \phi_5$. We now prove the other direction. By Lemma 3.34, we have that either $i_*(\mathcal{O}_C)$ or $j_*(\mathcal{O}_C)$ are not σ-stable. If $j_*(\mathcal{O}_C)$ is not stable, then, by Remark 3.26 we have that $i_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are stable and $\phi_1 - 1 \geq \phi_3 = \frac{1}{2} \geq \phi_5$, which contradicts that $\phi_5 > \frac{1}{2}$. Therefore, we have that $i_*(\mathcal{O}_C)$ is not stable and as a consequence $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable. □

Lemma 3.37. *Let $\sigma = (Z_r, \mathcal{A}_r)$ be a pre-stability condition constructed in Lemma 3.14, then $i_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable and $j_*(\mathcal{O}_C)$ is not σ-stable if and only if $\phi_1 \geq \frac{3}{2}$ and $\phi_5 \leq \frac{1}{2}$.*

Proof. The proof follows the steps of Lemma 3.36. □

3.2.1 Pre-stability conditions in Θ_{12}

We now study pre-stability conditions $\sigma \in \Theta_{12}$. We are going to show that they are given as the ones the constructed in Lemma 2.85 or in Lemma 3.14.

We first characterize the hearts of the pre-stability conditions in terms of the stability of the skyscraper sheaves. We study pre-stability conditions satisfying that $j_*(\mathbb{C}(x))$ is stable of phase one. We separate them into two cases when $l_*(\mathbb{C}(x))$ is not σ-stable and when it is. If $l_*(\mathbb{C}(x))$ is not σ-stable, then the pre-stability conditions are CP-glued. If $l_*(\mathbb{C}(x))$ is σ-stable, we obtain stability conditions of the form of Lemma 3.12. We follow closely [Bri08, Prop. 10.1].

We first introduce a lemma that will play a role in the characterization of the heart.

Lemma 3.38. [BM02, Prop. 5.4] *Let X be a quasi-projective scheme, take a non-zero object E of $D(X)$ and let $s \geq 0$ be an integer such that for all points $x \in X$,*

$$\mathrm{Hom}^i_{D(X)}(E, \mathbb{C}(x))^* = 0 \text{ unless } 0 \leq i \leq s.$$

Then E is quasi-isomorphic to a complex of locally free sheaves of the form

$$0 \to L_s \to L_{s-1} \to \cdots \to L_0 \to 0.$$

In particular, E has homological dimension at most s.

Lemma 3.39. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition such that $i_*(\mathbb{C}(x))$ is σ-stable, $j_*(\mathbb{C}(x))$ is σ-stable of phase one and $l_*(\mathbb{C}(x))$ is not σ-stable.*

We assume that $i_*(\mathbb{C}(x))[n] \in \mathcal{A}$. If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$, then $H^i(E_1) = 0$, unless $i = -n-1, -n$. Also $H^i(E_2) = 0$ unless $i = 0$ and $H^i(C(\varphi)) = 0$, unless $i = -n-1, -n-2, 0$.

Proof. First note that $n \geq 0$, because $l_*(\mathbb{C}(x))$ is not σ-stable. Indeed, it implies that $\phi_0 - \phi_2 \leq 0$. As $\phi_2 = 1$ we have $\phi_0 \leq 1$, also $0 < \phi(i_*(\mathbb{C}(x))[n]) = \phi_0 + n \leq 1$. Combining the inequalities above, we get $n \geq 0$.

Let $E \in \mathcal{A}$ be a stable element with phase $0 < \phi(E) < 1$. As $j_*(\mathbb{C}(x))$ is stable we have that $\mathrm{Hom}^i_{\mathcal{T}_C}(E, j_*(\mathbb{C}(x))) = 0$, for all $i < 0$. By adjointness, we have

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, j_*(\mathbb{C}(x))) = \mathrm{Hom}^i_{D^b(C)}(C(\varphi), \mathbb{C}(x)).$$

Moreover, by stability we have $\mathrm{Hom}^i(i_*(\mathbb{C}(x))[n], E) = 0$, for all $i < 0$. By adjointness and Serre duality in $D^b(C)$, we get

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(i_*(\mathbb{C}(x))[n], E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x)[n], C(\varphi)[-1]) = \mathrm{Hom}^{n-i+2}_{D^b(C)}(C(\varphi), \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(C(\varphi), \mathbb{C}(x)) = 0$ for $j > n+2$. Therefore, we obtain $\mathrm{Hom}^i(C(\varphi), \mathbb{C}(x)) = 0$, unless $0 \leq i \leq n+2$. By Lemma 3.38, it follows that $C(\varphi)$ is isomorphic to a complex of locally-free sheaves and

$$H^i(C(\varphi)) = 0 \text{ unless } -n-2 \leq i \leq 0. \tag{3.23}$$

Similarly, by the stability $\mathrm{Hom}^i(E, i_*(\mathbb{C}(x))[n]) = 0$, for all $i < 0$. By adjointness,

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, i_*(\mathbb{C}(x))[n]) = \mathrm{Hom}^i_{D^b(C)}(E_1, \mathbb{C}(x)[n]) = \mathrm{Hom}^{i+n}_{D^b(C)}(E_1, \mathbb{C}(x)),$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(E_1, \mathbb{C}(x)) = 0$ for $j < n$.

Since $l_*(\mathbb{C}(x))$ is not σ-semistable then, from its HN-filtration, it follows that $\phi_2 \geq \phi_4 \geq \phi_0$, it implies that

$$l_*(\mathbb{C}(x))[n] \in \mathcal{P}(0, n+1],$$

because $j_*(\mathbb{C}(x)) \in \mathcal{P}(0,1]$ and $i_*(\mathbb{C}(x))[n] \in \mathcal{P}(0,1]$. As a consequence, we have

$$\mathrm{Hom}^i(l_*(\mathbb{C}(x))[n], E) = 0,$$

for all $i < 0$. By adjointness and Serre duality in $D^b(C)$, we get

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(l_*(\mathbb{C}(x))[n], E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x)[n], E_1) = \mathrm{Hom}^{n+1-i}_{D^b(C)}(E_1, \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(E_1, \mathbb{C}(x)) = 0$ for $j > n+1$. Note that $\mathcal{S}_C(\mathbb{C}(x)) = \mathbb{C}(x)[1]$ for every curve. Consequently, we obtain $\mathrm{Hom}^i(E_1, \mathbb{C}(x)) = 0$ unless $n \leq i \leq n+1$. By Lemma 3.38, we have that E_1 is isomorphic to a length two complex of locally-free sheaves and

$$H^i(E_1) = 0 \text{ unless } i = -n, -n-1. \tag{3.24}$$

We now prove a similar result for E_2. In this case we use that $j_*(\mathbb{C}(x))$ is stable of phase one.

First of all, by the same reasoning as above $l_*(\mathbb{C}(x)) \in \mathcal{P}(-n, 1]$. Hence, we get

$$\mathrm{Hom}^i(E, l_*(\mathbb{C}(x))) = 0,$$

for all $i < 0$. By adjointness, we obtain

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, l_*(\mathbb{C}(x))) = \mathrm{Hom}^i_{D^b(C)}(E_2, \mathbb{C}(x)).$$

In addition, since $j_*(\mathbb{C}(x))$ is stable of phase 1, we get $\mathrm{Hom}^i_{\mathcal{T}_C}(j_*(\mathbb{C}(x)), E) = 0$ for $i \leq 0$. By adjointness and Serre duality in $D^b(C)$, we have

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(j_*(\mathbb{C}(x)), E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x), E_2) = \mathrm{Hom}^{1-i}_{D^b(C)}(E_2, \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(E_2, \mathbb{C}(x)) = 0$ for $j > 0$. Thus, we obtain $\mathrm{Hom}^i(E_2, \mathbb{C}(x)) = 0$, unless $i = 0$. By Lemma 3.38, we have that E_2 is isomorphic to a length one complex of locally-free sheaves and

$$H^i(E_2) = 0 \text{ unless } i = 0. \tag{3.25}$$

The triangle $E_1 \to E_2 \to C(\varphi)$ induces a long exact sequence in cohomology

$$\cdots \to H^i(E_1) \to H^i(E_2) \to H^i(C(\varphi)) \to \cdots .$$

As a consequence, we have that

$$H^i(C(\varphi)) = 0 \text{ unless } -n-2, -n-1, 0. \tag{3.26}$$

If $E \ncong j_*(\mathbb{C}(x))$ is σ-stable with $\phi(E) = 1$, we analogously obtain that

$$H^i(E_1) = 0 \text{ unless } i = -n-1 \text{ and } H^i(E_2) = 0 \text{ for all } i.$$

It implies that

$$H^i(C(\varphi)) = 0 \text{ unless } i = -n-2$$

and that E_1 is torsion free. $\square$

Lemma 3.40. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition such that $l_*(\mathbb{C}(x)), i_*(\mathbb{C}(x))$ are σ-stable and $j_*(\mathbb{C}(x))$ is σ-stable of phase one. Then, for $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}_C$ we have*

1. *If $E \in \mathcal{A}$, then $H^i(E_j) = 0$, unless $i = 0, 1$, for $j = 1, 2$. Also $H^i(C(\varphi)) = 0$, unless $i = -1, 0$. Moreover, $H^{-1}(C(\varphi)), H^0(E_1)$ are torsion-free.*
2. *If E is stable of phase one, then either $E = j_*(T)$, where $T \in \mathrm{Coh}(C)$ a torsion sheaf, or $E \in \mathrm{TCoh}(C)$ with $H^0(C(\varphi)) = 0$. We have that E_1 and E_2 are torsion-free.*
3. $\mathrm{TCoh}(C) \subseteq \mathcal{P}(0, 2]$
4. *The pair*

$$\mathcal{T} = \mathrm{TCoh}(C) \cap \mathcal{P}(1, 2] \text{ and } \mathcal{F} = \mathrm{TCoh}(C) \cap \mathcal{P}(0, 1]$$

defines a torsion pair of $\mathrm{TCoh}(C)$. Moreover, the heart $\mathcal{A}$ is the corresponding tilt.

Proof. First note that $i_*(\mathbb{C}(x))[-1], l_*(\mathbb{C}(x))[-1] \in \mathcal{A}$. Indeed, since there is a non-zero morphism $i_*(\mathbb{C}(x)) \to j_*(\mathbb{C}(x)[1])$ and both elements are stable, we obtain $\phi_0 < \phi_2 + 1 = 2$. We now consider the non-zero morphism $j_*(\mathbb{C}(x)) \to l_*(\mathbb{C}(x))$. Since both objects are stable, as above it implies $1 = \phi_2 < \phi_4$. Analogously we have a non-zero map $l_*(\mathbb{C}(x)) \to i_*(\mathbb{C}(x))$. It follows that $1 < \phi_4 < \phi_0 < 2$. Therefore, we obtain $1 < \phi_4 < 2$ and $1 < \phi_0 < 2$.

We start by proving part one. Let $E \in \mathcal{A}$ be a stable element with phase $0 < \phi(E) < 1$. As $j_*(\mathbb{C}(x))$ is stable we have that $\mathrm{Hom}^i_{\mathcal{T}_C}(E, j_*(\mathbb{C}(x))) = 0$, for all $i < 0$. By adjointness, we have

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, j_*(\mathbb{C}(x))) = \mathrm{Hom}^i_{D^b(C)}(C(\varphi), \mathbb{C}(x)).$$

Moreover, by stability we have $\mathrm{Hom}^i(i_*(\mathbb{C}(x))[-1], E) = 0$, for all $i < 0$. By adjointness and Serre duality in $D^b(C)$, we get

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(i_*(\mathbb{C}(x))[-1], E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x)[-1], C(\varphi)[-1]) = \mathrm{Hom}^{1-i}_{D^b(C)}(C(\varphi), \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(C(\varphi), \mathbb{C}(x)) = 0$ for $j > 1$. Therefore, we obtain $\mathrm{Hom}^i(C(\varphi), \mathbb{C}(x)) = 0$, unless $0 \leq i \leq 1$. By Lemma 3.38, it follows that $C(\varphi)$ is isomorphic to a length two complex of locally-free sheaves and

$$H^i(C(\varphi)) = 0 \text{ unless } i = -1, 0. \tag{3.27}$$

Similarly, by stability $\mathrm{Hom}^i(E, i_*(\mathbb{C}(x))[-1]) = 0$, for all $i < 0$. By adjointness,

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, i_*(\mathbb{C}(x))[-1]) = \mathrm{Hom}^i_{D^b(C)}(E_1, \mathbb{C}(x)[-1]) = \mathrm{Hom}^{i-1}_{D^b(C)}(E_1, \mathbb{C}(x)).$$

Once again, by stability $\mathrm{Hom}^i(l_*(\mathbb{C}(x))[-1], E) = 0$, for all $i < 0$. By adjointness and Serre duality in $D^b(C)$, we get

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(l_*(\mathbb{C}(x))[-1], E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x)[-1], E_1) = \mathrm{Hom}^{-i}_{D^b(C)}(E_1, \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(E_1, \mathbb{C}(x)) = 0$ for $j > 0$. Note that $\mathcal{S}_C(\mathbb{C}(x)) = \mathbb{C}(x)[1]$ for every curve.

Consequently, we obtain $\mathrm{Hom}^i(E_1, \mathbb{C}(x)) = 0$ unless $-1 \leq i \leq 0$. By Lemma 3.38, we have that E_1 is isomorphic to a length two complex of locally-free sheaves and

$$H^i(E_1) = 0 \text{ unless } i = 0, 1. \tag{3.28}$$

We prove a similar result for E_2. In this case we use that $j_*(\mathbb{C}(x))$ is stable of phase one. First of all, as above by stability $\mathrm{Hom}^i(E, l_*(\mathbb{C}(x))[-1]) = 0$, for all $i < 0$. By adjointness, we obtain

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(E, l_*(\mathbb{C}(x))[-1]) = \mathrm{Hom}^i_{D^b(C)}(E_2, \mathbb{C}(x)[-1]) = \mathrm{Hom}^{i-1}_{D^b(C)}(E_2, \mathbb{C}(x)).$$

In addition, since $j_*(\mathbb{C}(x))$ is stable of phase 1, we get $\mathrm{Hom}^i_{\mathcal{T}_C}(j_*(\mathbb{C}(x)), E) = 0$ for $i \leq 0$. By adjointness and Serre duality in $D^b(C)$, we have

$$0 = \mathrm{Hom}^i_{\mathcal{T}_C}(j_*(\mathbb{C}(x)), E) = \mathrm{Hom}^i_{D^b(C)}(\mathbb{C}(x), E_2) = \mathrm{Hom}^{1-i}_{D^b(C)}(E_2, \mathbb{C}(x))^*,$$

i.e. $\mathrm{Hom}^j_{D^b(C)}(E_2, \mathbb{C}(x)) = 0$ for $j > 0$. Thus, we obtain $\mathrm{Hom}^i(E_2, \mathbb{C}(x)) = 0$, unless $-1 \leq i \leq 0$. By Lemma 3.38, we have that E_2 is isomorphic to a length two complex of locally-free sheaves and

$$H^i(E_2) = 0 \text{ unless } i = 0, 1. \tag{3.29}$$

This concludes the proof of the first part.

We now proceed to prove the second part. Let $E \in \mathcal{P}(1)$ be a stable object, which is not isomorphic to $j_*(T)$, where T is a torsion sheaf. We have that $\mathrm{Hom}_{\mathcal{T}_C}(E, j_*(\mathbb{C}(x))) = 0$, which implies $\mathrm{Hom}^i_{D^b(C)}(C(\varphi), \mathbb{C}(x)) = 0$, unless $i = 1$. By Lemma 3.38,

$$H^i(C(\varphi)) = 0 \text{ unless } i = -1.$$

Since the phase of E is one, we have that $\mathrm{Hom}^i_{\mathcal{T}_C}(E, i_*(\mathbb{C}(x))[-1]) = 0$, for $i \leq 0$. It follows that $\mathrm{Hom}^i_{\mathcal{T}_C}(E_1, \mathbb{C}(x)) = 0$ unless $i = 0$. By Lemma 3.38,

$$H^i(E_1) = 0 \text{ unless } i = 0.$$

Once again, by stability $\mathrm{Hom}^i(E, l_*(\mathbb{C}(x))[-1]) = 0$, for $i \leq 0$. It implies that $\mathrm{Hom}^i(E_2, \mathbb{C}(x)) = 0$ unless $i = 0$. By Lemma 3.38,

$$H^i(E_2) = 0 \text{ unless } i = 0.$$

This completes the proof of the second part.

For the third part, we assume that $E \in \mathrm{TCoh}(C)$. If $F \in \mathcal{P}((2, \infty))$, then by the first part $F \in D^{\leq -2} \subseteq D^{\leq -1}$, where $(D^{\leq 0}, D^{\geq 0})$ is the standard t-structure. Consequently, we have $0 = \mathrm{Hom}_{\mathcal{T}_C}(D^{\leq 0}, D^{\geq 1}) = \mathrm{Hom}(D^{\leq -1}, D^{\geq 0})$, therefore $\mathrm{Hom}_{\mathcal{T}_C}(F, E) = 0$. Analogously, we have that if $B \in \mathcal{P}(\leq 0)$, then $B \in D^{\geq 1}$. As $\mathrm{Hom}_{\mathcal{T}_C}(D^{\leq 0}, D^{\geq 1}) = 0$, we obtain $\mathrm{Hom}_{\mathcal{T}_C}(E, B) = 0$. It follows that $E \in \mathcal{P}(0, 2]$.

We now prove the fourth part. Let $E \in \mathrm{TCoh}(C)$. By the third part of the statement, there is a triangle

$$A \to E \to B \to A[1],$$

where $A \in \mathcal{P}(1, 2]$ and $B \in \mathcal{P}(0, 1]$. It induces triangles

$$i^*(A) \to i^*(E) \to i^*(B) \to i^*(A)[1],$$

$$j^!(A) \to j^!(E) \to i^!(B) \to i^!(A)[1],$$

and

$$j^*(A) \to j^*(E) \to j^*(B) \to j^*(A)[1].$$

We obtain a long exact sequence

$$\to H^{-1}(i^*(A)) \to H^{-1}(i^*(E)) \to H^{-1}(i^*(B)) \to H^0(i^*(A)) \to H^0(i^*(E)) \to H^0(i^*(B))$$

$$\to H^1(i^*(A)) \to H^1(i^*(E)) \to H^1(i^*(B)) \to 0.$$

By (3.28), we have $H^i(i^*(B)) = 0$ unless $i = 0, 1$ and $H^i(i^*(A)) = 0$ unless $i = -1, 0$. Taking the long exact sequence, we obtain that $H^{-1}(i^*(A)) = H^1(i^*(B)) = 0$. This implies that $i^*(A), i^*(B) \in \mathrm{Coh}(C)$.

Analogously, by (3.29), we have $j^!(A), j^!(B) \in \mathrm{Coh}(C)$ and we obtain $A, B \in \mathrm{TCoh}(C)$.

Moreover, if $A \in \mathcal{T}$, we have additional information. Indeed, by (3.27), we have $H^i(j^*(B)) = 0$ unless $i = -1, 0$ and $H^i(j^*(A)) = 0$ unless $i = -2, -1$. After taking the long exact sequence we obtain $H^i(j^*(A)) = 0$ unless $i = -1$. As a consequence, if $A = A_1 \xrightarrow{g} A_2$, it follows that $\mathrm{Coker}(g) = 0$ and

$$A = A_1 \twoheadrightarrow A_2.$$

□

By applying Serre duality of doing exactly the same proof under the new hypothesis, we obtain the following results.

Lemma 3.41. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition such that $j_*(\mathbb{C}(x)), i_*(\mathbb{C}(x))$ are σ-stable and $i_*(\mathbb{C}(x))$ is σ-stable of phase one. Then, for $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}_C$ we have*

1. *If $E \in \mathcal{A}$, then $H^i(E_j) = 0$, unless $i = -1, 0$, for $j = 1, 2$. Also $H^i(C(\varphi)) = 0$, unless $i = -1, 0$. We also have that $H^{-1}(E_1), H^{-1}(E_2)$ are torsion-free.*
2. *If E is stable of phase one, then either $E = i_*(T)$, where $T \in \mathrm{Coh}(C)$ is a torsion sheaf, or $E \in \mathcal{H}_{31}$ with $H^0(E_1) = 0$. Moreover, we have that E_2 and $C(\varphi)$ are torsion-free.*
3. *$\mathcal{H}_{31} \subseteq \mathcal{P}(0, 2]$*
4. *The pair*

$$\mathcal{T} = \mathcal{H}_{31} \cap \mathcal{P}(1, 2] \text{ and } \mathcal{F} = \mathcal{H}_{31} \cap \mathcal{P}(0, 1]$$

defines a torsion pair of $\mathcal{H}_{31}$. Moreover, the heart $\mathcal{A}$ is the corresponding tilt.

Lemma 3.42. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition such that $i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x))$ are σ-stable and $l_*(\mathbb{C}(x))$ is σ-stable of phase one. Then, for $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}_C$ we have*

1. *If $E \in \mathcal{A}$, then $H^i(E_1) = 0$, unless $i = 0, 1$ and $H^i(E_2) = 0$, unless $i = 0, -1$. Also*

$H^i(C(\varphi)) = 0$, unless $i = -1, 0$. We also have that $H^{-1}(C(\varphi)), H^{-1}(E_2)$ are torsion-free.

2. *If E is stable of phase one, then either $E = l_*(T)$, where $T \in \mathrm{Coh}(C)$ a torsion sheaf, or $E \in \mathcal{H}_{23}$ with $H^0(E_2) = 0$ and $H^0(E_1)$ torsion-free sheaves.*
3. *$\mathcal{H}_{23} \subseteq \mathcal{P}(0, 2]$*
4. *The pair*
$$\mathcal{T} = \mathcal{H}_{23} \cap \mathcal{P}(1, 2] \text{ and } \mathcal{F} = \mathcal{H}_{23} \cap \mathcal{P}(0, 1]$$
defines a torsion pair of $\mathcal{H}_{23}$. Moreover, the heart $\mathcal{A}$ is the corresponding tilt.

Remark 3.43. See Remark 2.76 for the definition of $\mathcal{H}_{23}$ and $\mathcal{H}_{31}$.

We study now the orbit of $\sigma \in \Theta_{12}$ under the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ in order to choose a simpler representative.

Proposition 3.44. *For every $\sigma \in \Theta_{12}$, there is $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that for $\sigma g = (Z, \mathcal{A})$ we can find stability conditions*

$$\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$ and $r > -1$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, $Z\big|_{D_1} = Z_1$ and $Z\big|_{D_2} = Z_\mu$.

Proof. By the stability of $i_*(\mathbb{C}(x))$ and $i_*(\mathcal{O}_C)$, we have $\phi_1 < \phi_0 < \phi_1 + 1$. Therefore we can find an orientation preserving transformation $M\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfying that

$$\begin{aligned} (A, D) &\mapsto (-1, 0) \\ (B, C) &\mapsto (0, 1), \end{aligned}$$

where $Z(i_*(\mathbb{C}(x)) = A+Di$ and $Z(i_*(\mathcal{O}_C)) = B+Ci$. There is an increasing function $f\colon \mathbb{R} \to \mathbb{R}$ that satisfies $f(x + 1) = f(x) + 1$ with $f(1) = \phi_0$, $f(1/2) = \phi_1$, whose restriction to S^1 agrees with the restriction of $T \coloneqq M^{-1}$ to S^1. We obtain
$(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. The stability condition $\sigma' \coloneqq \sigma(T, f)$ satisfies

$$Z'(r_1, d_1, 0, 0) = -d_1 + r_1 i \quad \text{and} \quad \mathrm{Coh}_1(C) \subseteq \mathcal{A}',$$

where $\sigma' = (Z', \mathcal{A}')$. Indeed, we have

$$\begin{aligned} i_*(\mathbb{C}(x)) \in \mathcal{P}(\phi_0) &= \mathcal{P}(f(1)) &= \mathcal{P}'(1), \\ i_*(\mathcal{O}_C) \in \mathcal{P}(\phi_1) &= \mathcal{P}(f(1/2)) &= \mathcal{P}'(1/2), \\ i_*(\mathcal{L}) \in \mathcal{P}(\phi_\mathcal{L}) &= \mathcal{P}(f(t_\mathcal{L})) &= \mathcal{P}'(t_\mathcal{L}), \end{aligned}$$

with $t_{\mathcal{L}} \in (0,1)$. Therefore, all skyscraper sheaves and line bundles in $\mathrm{Coh}_1(C)$ are in $\mathcal{P}^{'}(0,1]$. Since any object in $\mathrm{Coh}_1(C)$ admits a filtration with quotients either isomorphic to skyscraper sheaves or to line bundles, we obtain $\mathrm{Coh}_1(C) \subseteq \mathcal{A}^{'}$.

As the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ on the set of pre-stability conditions preserves the stability of the objects and $j_*(\mathbb{C}(x))$, $j_*(\mathcal{O}_C)$ are σ-stable, we have $\phi_3 < \phi_2 < \phi_3 + 1$ in $\sigma^{'}$. Therefore, we can find $g_1 = (T_1, f_1) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, such that $\sigma^{''} := \sigma'(T_1, f_1)$ satisfies $Z^{''}\Big|_{D2} = Z_\mu$ and $\mathrm{Coh}_2(C) \subseteq \mathcal{A}^{''}$.

Moreover, if we consider $\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) \in \mathrm{Stab}(C)$ with $f_1(0) = r \in \mathbb{R}$, which under our correspondence with $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is $(T_1, f_1) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, then $\sigma^{''}$ also satisfies that $Z^{''}\Big|_{D1} = Z_1$ and $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}^{''}$. Indeed, by definition $Z^{''} = T_1^{-1} \circ Z'$. Consequently, we can assert that $Z^{''}\Big|_{D1} = T_1^{-1} \circ Z_\mu$.

We now show that $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}^{''}$, where $f_1(0) = r = n + \theta$, and $n \in \mathbb{Z}$ and $\theta \in [0,1)$. We prove it in several steps. We first show that $\mathrm{Coh}_1(C)[n] \subseteq \mathcal{P}(-1,1]$. Then, we construct a torsion pair of $\mathrm{Coh}_1(C)[n]$ and we compare it with $\mathrm{Coh}(C)[n] = \langle \mathcal{T}_\theta[n], \mathcal{F}_\theta[n] \rangle$, which is the torsion pair given in Remark 1.83.

We first show that $i_*(\mathbb{C}(x)[n]) \in \mathcal{A}^{''}$. Note that $f_1(\phi_0) = 1$, because

$$i_*(\mathbb{C}(x)) \in \mathcal{P}^{'}(1) = \mathcal{P}^{''}(f_1^{-1}(1)) = \mathcal{P}^{''}(\phi_0)$$

and $f_1(-n) = \theta$.

We apply f_1^{-1} to the following inequality

$$\theta < 1 \leq \theta + 1$$

and we obtain

$$-n < \phi_0 \leq -n + 1, \tag{3.30}$$

which is equivalent to $i_*(\mathbb{C}(x)[n]) \in \mathcal{A}^{''}$.

Note that $-n < 2$. Indeed, let us consider the non-zero morphism $i_*(\mathbb{C}(x)) \to j_*(\mathbb{C}(x))[1]$. By the stability of $i_*(\mathbb{C}(x))$ and $j_*(\mathbb{C}(x))$ in $\sigma^{''}$, we get $\phi_0 < \phi_2 + 1$. As $\phi_2 = 1$, it implies that $\phi_0 < 2$. By Equation (3.30), we obtain

$$-n \leq 1.$$

Claim 3.45. $f_1(0) = r > -1$

Proof. Since $r = n + \theta$ and $n \geq -1$, we obtain $r \geq -1$. We just need to prove that $r \neq -1$.

We prove it by contradiction. Assume that $r=-1$. Then if

$$Z_1(r_1,d_1)=A_1d_1+Br_1+(C_1r_1+D_1d_1)i,$$

we get $D_1=0$. By the stability of $i_*(\mathbb{C}(x))$ and $j_*(\mathbb{C}(x))$ in σ'', we get $\phi_0<2$. But, by the definition of Z_1, we obtain $\phi_0=\phi(i_*(\mathbb{C}(x))[-1])+1=2$, which is a contradiction. □

We now study line bundles $\mathcal{L}$.

Note that $f_1(\phi_{\mathcal{L}})=t_{\mathcal{L}}$, with $t_{\mathcal{L}}\in(0,1)$ because

$$i_*(\mathcal{L})\in\mathcal{P}'(t_{\mathcal{L}})=\mathcal{P}''(f^{-1}(t_{\mathcal{L}}))=\mathcal{P}''(\phi_{\mathcal{L}}).$$

The proof falls naturally into two cases:

Case 1: $\theta<t_{\mathcal{L}}\leq\theta+1$

We apply f_1^{-1} to the following inequality

$$\theta<t_{\mathcal{L}}\leq\theta+1$$

and we obtain

$$-n<\phi_{\mathcal{L}}\leq-n+1, \tag{3.31}$$

which is equivalent to $i_*(\mathcal{L}[n])\in\mathcal{A}''$.

Case 2: $\theta-1<t_{\mathcal{L}}\leq\theta$

We apply f_1^{-1} to the following inequality

$$\theta-1<t_{\mathcal{L}}\leq\theta$$

and we obtain

$$-n-1<\phi_{\mathcal{L}}\leq-n, \tag{3.32}$$

which is equivalent to $i_*(\mathcal{L}[n+1])\in\mathcal{A}''$.

Claim 3.46. $\mathrm{Coh}_1(C)\subseteq\mathcal{P}(-1,1]$

Proof. We just proved that all line bundles and the the skyscraper sheaves are in $\mathcal{P}(-n-1,-n+1]$, therefore $\mathrm{Coh}_1(C)[n]\subseteq\mathcal{P}(-1,1]$. We set

$$\mathcal{T}_1=\mathrm{Coh}_1(C)[n]\cap\mathcal{P}(0,1] \text{ and } \mathcal{F}_1=\mathrm{Coh}_1(C)[n]\cap\mathcal{P}(-1,0].$$

□

Claim 3.47. *$\langle\mathcal{T}_1,\mathcal{F}_1\rangle$ is a torsion pair of* $\mathrm{Coh}_1(C)[n]$.

Proof. The proof falls naturally into two cases:

Case 1: $l_*(\mathbb{C}(x))$ is not σ''-stable.

Let $E \in \mathrm{Coh}(C)$. We have $E[n] \in \mathrm{Coh}(C)[n]$. Since $\mathrm{Coh}_1(C)[n] \subseteq \mathcal{P}(-1,1]$, there are elements $T \in \mathcal{P}(0,1]$ and $F \in \mathcal{P}(-1,1]$ such that

$$0 \to T \to i_*(E)[n] \to F \to 0.$$

We want to show that $T, F \in \mathrm{Coh}_1(C)[n]$.

We apply the long exact sequence in cohomology and we obtain

$$\cdots \to H^i(T_1) \to H^i(E[n]) \to H^i(F_1) \to \cdots .$$

By Lemma 3.39, we have that $H^i(T_1) = 0$ unless $i = -n, -n-1$ and $H^i(F_1) = 0$ unless $i = -n, -n+1$. Then, we obtain that $H^i(T_1) = H^i(F_1) = 0$ unless $i = -n$.

We also obtain the long exact sequence

$$\cdots \to H^i(T_2) \to 0 \to H^i(F_2) \to \cdots .$$

Since $H^i(T_2) = 0$ unless $i = 0$ and $H^i(F_2) = 0$ unless $i = 1$, we obtain that $T_2 = F_2 = 0$.

Therefore, we obtain $T, F \in \mathrm{Coh}_1(C)[n]$ as desired. As a consequence, we have that $\mathrm{Coh}_1(C)[n] = (\mathcal{T}_1, \mathcal{F}_1)$ and $\mathcal{T}_1, \mathcal{F}_1[1] \subseteq \mathcal{A}''$.

Case 2: $l_*(\mathbb{C}(x))$ is σ''-stable.

Note that the stability of $l_*(\mathbb{C}(x))$ implies, as proved in Lemma 3.40, that $n = -1$ and that $l_*(\mathbb{C}(x))[-1] \in A''$.

Let $E \in \mathrm{Coh}(C)$. We have $E[-1] \in \mathrm{Coh}(C)[-1]$. Since $\mathrm{Coh}_1(C)[-1] \in \mathcal{P}(-1,1]$, there are elements $T \in \mathcal{P}(0,1]$ and $F \in \mathcal{P}(-1,1]$ such that

$$0 \to T \to i_*(E)[-1] \to F \to 0.$$

We want to show that $T, F \in \mathrm{Coh}_1(C)[-1]$.

We apply the long exact sequence in cohomology and we obtain

$$\cdots \to H^i(T_1) \to H^i(E[-1]) \to H^i(F_1) \to \cdots .$$

By Lemma 3.40, we have $H^i(T_1) = 0$ unless $i = 0, 1$ and $H^i(F_1) = 0$ unless $i = 1, 2$. Then, we obtain that $H^i(T_1) = H^i(F_1) = 0$ unless $i = 1$.

We also obtain the long exact sequence

$$\cdots \to H^i(T_2) \to 0 \to H^i(F_2) \to \cdots .$$

Since $H^i(T_2) = 0$ unless $i = 0, 1$ and $H^i(F_2) = 0$ unless $i = 1, 2$ we obtain that $T_2 = F_2 = 0$. □

We now proceed to prove that $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}''$.

We now show that $\mathcal{T}_1 = \mathcal{T}_\theta$ and $\mathcal{F}_1 = \mathcal{F}_\theta$ It is enough to show that $\mathcal{T}_\theta \subseteq \mathcal{T}_1$ and $\mathcal{F}_\theta \subseteq \mathcal{F}_1$, where $\mathrm{Coh}(C) = (\mathcal{T}_\theta, \mathcal{F}_\theta)$ is the torsion pair described in Remark 1.83 (up to shift) and $\theta \neq 0$.

Let $E \in \mathrm{Coh}(C)$ be a slope stable torsion free sheaf, such that $E[n] \in \mathrm{Coh}(C)[n]$. Let

$$0 \to T \to i_*(E)[n] \to F \to 0$$

be the triangle induced by the torsion pair $(\mathcal{T}_1, \mathcal{F}_1)$. Since $i_*(F[1]) \in \mathcal{A}''$ and $i_*(T) \in \mathcal{A}''$, then $\Im(Z)(i_*(T)) > 0$, which implies $\mu(T[-n]) > -\cot(\theta\pi)$ and $\Im(Z)(i_*(F)) \leq 0$, which implies $\mu(F[-n]) \leq -\cot(\theta\pi)$. Since E is stable and

$$0 \to T[-n] \to E \to F[-n] \to 0$$

is a triangle in $\mathrm{Coh}(C)$, we get

$$\mu(T[-n]) < \mu(i_*(E)) < \mu(F[-n])$$

which is a contradiction. Then, either $T = 0$ or $F = 0$. it follows that $E[n] \in \mathcal{T}_1$ or $E[n] \in \mathcal{F}_1$.

Clearly if $\mu(E) > -\cot(\theta\pi)$, then $E[n] \in \mathcal{T}_1$ and if $\mu(E) < -\cot(\theta\pi)$, then $i_*(E)[n] \in \mathcal{F}_1$. The only missing case is if $\mu(E) = -\cot(\theta\pi)$. By the stability of $i_*(\mathbb{C}(x))$ and $i_*(\mathcal{O}_C)$, we obtain that

$$\det \begin{bmatrix} -A_1 & B_1 \\ -D_1 & C_1 \end{bmatrix} > 0.$$

It follows that $Z(i_*(E)[n]) \in \mathbb{R}_{>0}$ and $i_*(E)[n] \in \mathcal{F}_1$.

If $\theta = 0$, then $\mathcal{T}_1^\theta = \mathrm{Coh}(C)[n]$ and $\mathcal{F}_1^\theta = 0$. Since $r \in \mathbb{Z}$, we obtain

$$Z_1(r, d) = A_1 d_1 + B r_1 + (C r_1) i,$$

i.e. $D_1 = 0$. It implies that for all $E \in \mathrm{Coh}(C)[n]$ we have $\Im(Z)(i_*(E)) \geq 0$. As a consequence $\mathcal{T}_1 = \mathrm{Coh}_1(C)[n]$. Therefore $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}''$. □

Remark 3.48. Let us consider $\sigma \in \Theta_{12}$ and $g' = (T', f') \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. By Proposition

3.44, there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that for $\sigma g = (Z, \mathcal{A})$ we can find stability conditions

$$\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$ and $r > -1$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$. Note that the proof of the inclusion of the hearts in $\mathcal{A}$ depends only on the stability of $i_*(\mathbb{C}(x))$ and $j_*(\mathbb{C}(x))$ for all $x \in C$ and $i_*(\mathcal{L})$ and $j_*(\mathcal{L})$, for every line bundle $\mathcal{L}$. As a consequence, we obtain that $\sigma g g' = (Z', \mathcal{A}')$ satisfies that

$$\mathrm{Coh}_1^{f \circ f'(0)}(C) \subseteq A' \text{ and } \mathrm{Coh}_2^{f'(0)}(C) \subseteq \mathcal{A}'.$$

The following lemma gives us some CP-glued stability conditions in Θ_{12}.

Lemma 3.49. *Let $\sigma = (Z, \mathcal{A}) \in \Theta_{12}$, such that there are stability conditions*

$$\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) = (T_1, f_1) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, and $Z\big|_{D1} = Z_1$ and $Z\big|_{D2} = Z_\mu$. If $f_1(0) = r \geq 0$, then $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_\mu)$.

Proof. Since σ_1 and σ_2 satisfy gluing conditions i.e $f_1(0) \geq f_2(0) = 0$, then there is $\sigma_{12} = \mathrm{gl}_{12}(\sigma_1, \sigma_2) \in \mathrm{Stab}(\mathcal{T}_C)$. It follows directly from Proposition 2.85 that $\sigma = \sigma_{12}$. □

We now study pre-stability conditions $\sigma = (Z, \mathcal{A}) \in \Theta_{12}$. For

$$\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

we have $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, $Z\big|_{D1} = Z_1$, $Z\big|_{D2} = Z_\mu$, and

$$-1 < r = n + \theta = f_1(0) < 0,$$

where $n \in \mathbb{Z}$ and $\theta \in [0, 1)$ and σ_1 is given by (T_1, f_1) under the correspondence with $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.

Claim 3.50. *$l_*(\mathbb{C}(x))$ is σ-stable.*

Proof. We prove it by contradiction. Assume that $l_*(\mathbb{C}(x))$ is not stable. By Remark 3.26 it implies that $\phi_0 - \phi_2 \leq 0$. Since $\phi_2 = 1$ in σ, then $\phi_0 \leq 1$. We also know that $-n < \phi_0 \leq -n + 1$, because $\mathrm{Coh}_1^r \subseteq \mathcal{A}$. After combining these equations, we have $n \geq 0$. Therefore $n + \theta = f_1(0) \geq 0$, which contradicts our assumption. □

3.2.2 Pre-stability conditions in Θ_{23} and Θ_{31}

We have the analogous statement of Proposition 3.44 for Θ_{23} and Θ_{31}. We leave the proof to the reader, since it follows exactly the same steps as the result for Θ_{12}.

Proposition 3.51. *For every $\sigma \in \Theta_{23}$, there is a $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that for $\sigma g = (Z, \mathcal{A})$ we can find stability conditions*

$$\sigma_3 = (Z_3, \mathrm{Coh}^r(C)) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_3^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, $r < 0$, $Z\big|_{D3} = Z_3$ *and* $Z\big|_{D2} = Z_\mu$.

Proposition 3.52. *For every $\sigma \in \Theta_{31}$, there is a $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that for $\sigma g = (Z, \mathcal{A})$ we can find stability conditions*

$$\sigma_3 = (Z_3, \mathrm{Coh}^{r_3}(C)) \text{ and } \sigma_1 = (Z_1, \mathrm{Coh}^{r_1}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_3^{r_3}(C) \subseteq \mathcal{A}$, $\mathrm{Coh}_1^{r_1}(C) \subseteq \mathcal{A}$, *with* $r_3 - r_1 > 0$, $Z\big|_{D3} = Z_3$, $Z\big|_{D1} = Z_1$ *and* $M_3 - M_1 = I$.

3.2.3 Non-gluing pre-stability conditions

We now study pre-stability conditions σ on $\mathcal{T}_C$ satisfying that $i_*(\mathbb{C}(x))$ and $l_*(\mathbb{C}(x))$ are stable, where $j_*(\mathbb{C}(x))$ is stable with phase one and $j_*(\mathcal{O}_C)$ has phase 1/2. Note that as proved in Lemma 3.40, we have $l_*(\mathbb{C}(x))[-1] \in \mathcal{A}$. We use Lemma 3.40 to prove that these stability coditions are precisely given by the pair $(Z, \mathcal{A}_r)$ constructed in Lemma 3.14.

Lemma 3.53. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition on $\mathcal{T}_C$ such that $i_*(\mathbb{C}(x)), l_*(\mathbb{C}(x))$ are σ-stable and the object $j_*(\mathbb{C}(x))$ and $j_*(\mathcal{O}_C)$ are in $\mathcal{A}$ and are also σ-stable with*

$$Z([j_*(\mathbb{C}(x))]) = -1 \text{ and } Z([j_*(\mathcal{O}_C)]) = i.$$

Then σ is given by the pairs constructed in Lemma 3.14.

Proof. It suffices to show the statement for $\sigma \in \Theta_{12}$, the other two cases follow analogously. By Proposition 3.44, there are stability conditions

$$\sigma_1 = (Z_1, \mathrm{Coh}^r(C)) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C),$$

such that $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$ and $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, with $Z\big|_{D1} = Z_1$ and $Z\big|_{D2} = Z_\mu$. Note that $-1 < f_1(0) < 0$, where $\sigma_1 = (T_1, f_1) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. First of all we show that Z is given by Proposition 3.13. Our stability function Z is completely determined by Z_1 and Z_μ, therefore it has the following form

$$Z(r_1, d_1, r_2, d_2) = Ad_1 + Br_1 - d_2 + i(Cr_1 + Dd_1 + r_2).$$

We define

$$M := T_1^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}.$$

Since σ_1 is a stability condition, we have that $\det(M) > 0$. As $-1 < f_1(0) < 0$, then $D = m_0 \sin(\phi_0 \pi)$, where $m_0^2 = (A^2 + D^2)$. By Lemma 3.40, we obtain $1 < \phi_0 < 2$ and $D < 0$.

We now show that $\det(M+I) > 0$. If $\mathrm{Tr}(M) \geq 0$, there is nothing to prove as $\det(M+I) = \det(M) + \mathrm{Tr}(M) + 1 > 0$.

Claim 3.54. *If* $\mathrm{Tr}(M) < 0$, *then* $l_*(\mathcal{O}_C)$ *is* σ*-stable.*

Proof. We assume that $-A + C < 0$ and that $l_*(\mathcal{O}_C)$ is not σ-stable. Then $\phi_1 - \phi_3 \leq 0$. By hypothesis $\phi_3 = 1/2$ and $0 < \phi_1 < 2$. Therefore, we have $0 < \phi_1 \leq 1/2$, which implies $B > 0, C \geq 0$, as $B = m_1 \cos(\phi_1 \pi)$ and $C = m_1 \sin(\phi_1 \pi)$, with $m_1^2 = B^2 + C^2$. Since $D < 0$, it implies $BD \leq 0$. As $\det(M) = -AC + BD > 0$, we obtain $-AC > 0$. As a consequence $-A > 0$ and $-A + C > 0$, which contradicts our assumption. □

Under the assumption that $\mathrm{Tr}(M) < 0$, we have that $l_*(\mathcal{O}_C)$ is σ-stable. Since $l_*(\mathbb{C}(x))$ and $l_*(\mathcal{O}_C)$ are stable, we obtain $\phi_5 < \phi_4 < \phi_5 + 1$ which is equivalent to the fact that $\det(M+I) > 0$, as we have shown in Lemma 1.85.

We consider the torsion pair $(\mathcal{T}, \mathcal{F}) = \mathrm{TCoh}(C)$ given in Lemma 3.40. We are going to show that it is equal to the torsion pair $(\mathcal{T}', \mathcal{F}')$ given by Lemma 3.8. Note that $j_*(\mathbb{C}(x)) \in \mathcal{F}$. It is enough to show that $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{F}' \subseteq \mathcal{F}$.

Let us take a torsion-free λ-semistable object $E = E_1 \to E_2 \in \mathcal{T}'$, which by definition satisfies $\lambda(E) > 3/4$. There is a short exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}$ and $F = F_1 \to F_2 \in \mathcal{F}$. By Lemma 3.40 and Lemma 3.8 we have that $i^!(E), i^!(T) \in \mathrm{Coh}(C)$. It follows that $i^!(F) \in \mathrm{Coh}(C)$.

As F_1 is torsion-free and $F_1 \twoheadrightarrow F_2$, then $F \neq 0 \to G_2$, where G_2 is a torsion sheaf. We apply Lemma 3.2 and we obtain that

$$3/4 < \lambda(E) \leq \lambda(F) \leq 3/4,$$

which gives us a contradiction and it implies that $E \in \mathcal{T}$.

Let us take a λ-semistable torsion-free object $E = E_1 \to E_2 \in \mathcal{F}'$, and by definition $\lambda(E) \leq 3/4$. We start with the case $\lambda(E) < 3/4$. There is a short exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. We get $\lambda(T) \geq 3/4$ and $\lambda(F) \leq 3/4$. If $T \neq 0$, then by λ-semistability

$$3/4 \leq \lambda(T) \leq \lambda(E) < 3/4,$$

which give us a contradiction and it implies that $E \in F$.

Let us take a torsion-free λ-semistable object $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{F}'$ with $\lambda(E) = 3/4$.

Consider the short exact sequence

$$0 \to T \to E \to F \to 0,$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$. Note that

$$3/4 \leq \lambda(T) \leq \lambda(E) = 3/4,$$

so that

$$3/4 = \lambda(T).$$

We show that $T = 0$.

If $T = T_1 \xrightarrow{\varphi'} T_2$, we get $T[-1] \in \mathcal{P}'(1)$. Moreover by Lemma 3.40, we have that $\mathcal{P}'(1) \subseteq \mathrm{TCoh}(C)$ which implies directly that $T = 0$.

Note that for all torsion-free objects $E \in \mathcal{T}'$, after using its Harder-Narasimhan filtration, we obtain that $E \in \mathcal{T}$. Analogously for $E \in \mathcal{F}'$.

If $E \in \mathcal{T}'$ is not a torsion-free object, we consider its decomposition

$$0 \to T \to E \to F \to 0$$

in $(\mathcal{T}, \mathcal{F})$. As $E \in \mathcal{T}'$ and $\mathcal{T}'$ is closed under quotients, we get $F \in \mathcal{T}'$. But as we proved in Lemma 3.12 $\Im(Z_r(E)) = \Im(Z(E)) < 0$ for every object $E \in \mathcal{T}'$, then $\Im(Z(F)) < 0$. But $F \in \mathcal{A}$ then $\Im(Z(F)) \geq 0$. Which gives us a contradiction. As a consequence $F = 0$, and we obtain that $E \in \mathcal{T}$.

If $E \in \mathcal{F}'$, by definition of $\mathcal{F}'$, we have that $F(E) \in \mathcal{F}'$ and therefore $F(E) \in \mathcal{F}$. Moreover, as $i^*(E)$ is torsion-free, we obtain that $T(E) \in \mathrm{Coh}_2(C) \subseteq \mathcal{F}$. Therefore, we obtain that $E \in \mathcal{F}$.

Consequently $\mathcal{T} = \mathcal{T}'$ and $\mathcal{F} = \mathcal{F}'$. □

Remark 3.55. Under the assumption of the last proposition, for $\sigma \in \Theta_{12}$ we can define a third stability condition $\sigma_3 = ((M + I)^{-1}, f_3) \in \mathrm{Stab}(C)$, where the integer part of $f_3(0) = r_3$ is -1. As explained before to characterize σ_3 it is enough to give $f_3 \colon \mathbb{R}/2\mathbb{Z} \to \mathbb{R}/2\mathbb{Z}$, which is determined by $(M+I)^{-1}$ and the integer part of $f_3(0)$. Note that by Lemma 3.17 $\mathrm{Coh}_3^{r_3}(C) \subseteq \mathcal{A}$.

Proposition 3.56. *Let σ be pre-stability condition in Θ_{12}. There is an element $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that σg is given by a CP-glued pre-stability condition or one constructed by tilting in Subsection 3.1.*

Proof. After applying Proposition 3.44, it follows directly from Lemma 3.49 and Lemma 3.53. □

3.2.4 Characterising CP-glued pre-stability conditions in Θ_{12}

Note that the CP-glued pre-stability conditions are not invariant under the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action. As a consequence, it makes sense to define the following sets. Recall that Θ_i is the set consisiting of pre-stability conditions, for $i = 1, 2$ or 3, which are, up to the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, CP-glued with respect to the semiorthognal decomposition $\langle D_i, {}^{\perp}D_i\rangle$.

Claim 3.57. *The set* $\Theta_1 \subseteq \Theta_{12}$

Proof. It follows directly from Proposition 2.85. □

The aim of this section is to characterize the sets Θ_i inside Θ_{12} i.e. to study CP-glued pre-stability conditions up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action. It is a remarkable fact that the description is completely given in terms of linear algebra. Moreover, we use this classification to prove the support property, i.e. to prove that already constructed pre-stability conditions are Bridgeland stability conditions.

Condition (*): If $\sigma \in \Theta_{12}$ satisfies that there are stability conditions $\sigma_1 = (\mathrm{Coh}_1^r(C), Z_1)$ and $\sigma_2 = (\mathrm{Coh}_2(C), Z_\mu) \in \mathrm{Stab}(C)$ with $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}(C)_2 \subseteq \mathcal{A}$, and $Z\big|_{\mathcal{C}1} = Z_1$ and $Z\big|_{\mathcal{C}2} = Z_\mu$. Let $f_1(0) = n + \theta$, with $n \in \mathbb{Z}$ and $\theta \in [0, 1)$, we say that it satisfies condition (*). Let $\sigma \in \Theta_{12}$. By Proposition 3.44, we have that there is $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, such that σg satisfies condition (*).

Under condition (*), we define $f_{12}(\sigma)(x) = f_1(x) - x$.

From now on we assume that σ satisfies condition (*) and $\sigma_1 = (T, f_1)$. If $f_{12}(\sigma)(0) \geq 0$ by Proposition 3.49, we already know that $\sigma \in \Theta_1$. We now assume $f_{12}(\sigma)(0) < 0$. We would like to understand when $\sigma \in \Theta_{12}$ is in Θ_i, for $i = 1, 2, 3$.

Recall that by Remark 3.55 there is also $\sigma_3 = (Z_3, \mathrm{Coh}^{r_3}(C))$ with $Z\big|_{D3} = Z_3$ and $\mathrm{Coh}_3^{r_3}(C) \subseteq \mathcal{A}$, and we can also define

$$f_{23}(\sigma)(x) = x - f_3(x) \text{ and } f_{31}(\sigma)(x) = f_2(x) - f_1(x).$$

The following lemma characterizes Θ_1 inside Θ_{12}.

Lemma 3.58. *Let σ as in (*), such that $f_{12}(\sigma)(0) < 0$. There is a $t \in \mathbb{R}$ such that $f_{12}(\sigma)(t) = 0$ if and only if $\sigma \in \Theta_1$.*

Proof. Assume that there is $t \in \mathbb{R}$ such that $f_{12}(\sigma)(t) = 0$. Since $f_2(t) = t$, it implies that $f_1(t) = t$. Let $g = (K_{t\pi}, f_{t\pi}) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. We would like to show that $\sigma g = \mathrm{gl}_{12}(\sigma_1 g, \sigma_2 g)$. Since $f_1 \circ f_{t\pi}(0) = f_1(t) = t$ and $f_2 \circ f_{t\pi}(0) = f_2(t) = t$, as a consequence $f_{12}(\sigma g)(0) = 0$ and it follows directly from Lemma 2.86 that $\sigma g = \mathrm{gl}_{12}(\sigma_1 g, \sigma_2 g)$. Therefore, we obtain $\sigma \in \Theta_1$.

We now assume that $\sigma \in \Theta_1$, i.e. that there is $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that σg satisfies gluing conditions for $\sigma_1 g$ and $\sigma_2 g$. Without losing generality, we take $g = (K_{l\pi}, f_{l\pi}) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, with $l \in \mathbb{R}$ Thus, $f_1(l) = (f_1 \circ f_{l\pi})(0) \geq (f_2 \circ f_{l\pi})(0) = f_{l\pi}(0) = l$. By hypothesis $f_{12}(\sigma)(0) < 0$, but also $f_{12}(\sigma)(l) \geq 0$. Since $f_{12}(\sigma)(x)$ is a continuous function, there is a $t \in \mathbb{R}$ that satisfies $f_{12}(\sigma)(t) = 0$, as we desired. $\square$

Remark 3.59. [Fixed point] Since in our case $f_{12}(\sigma)(x) = f_1(x) - x$, then $f_{12}(\sigma)(x) = 0$ if and only if there is $x \in \mathbb{R}$ such that $f_1(x) = x$. After using that the restrictions of f_1 and T to S^1 agree, the search of points $x \in \mathbb{R}$ such that $f_1(x) = x$ reduces to the study of the eigenvectors of T.

Let

$$M := T^{-1} = \begin{bmatrix} -A & B \\ -D & C \end{bmatrix}.$$

We now study the characteristic polynomial $p(x) = x^2 - \mathrm{Tr}(M)x + \det(M)$ of M. It plays an important role in determining if $\sigma \in \Theta_i$ for some $i = 1, 2, 3$. The sign of the discriminant $\Delta(M) = \mathrm{Tr}(M)^2 - 4\det(M)$ of $p(x)$ tells us about the existence of real eigenvalues.

Proposition 3.60. *If $f_{12}(\sigma)(0) < 1$, the discriminant of $\Delta(M) = \mathrm{Tr}(M)^2 - 4\det(M)$ is non-negative and the eigenvalues are positive, then $\sigma = (Z, \mathrm{TCoh}^l(C))$, with $l \in \mathbb{R}$ and $-1 < l \leq 1$. Moreover $\sigma \in \Theta_1$.*

Proof. Since $\Delta(M) \geq 0$, it guarantees the existence of real eigenvalues. By hypothesis the eigenvalues are positive. The same follows for T_1. Let $\lambda \in \mathbb{R}$ be an eigenvalue of T_1 and $v \in \mathbb{R}^2$ its corresponding eigenvector. We obtain $T_1 v = \lambda v$. Let us consider the polar coordinates of $v = (m\cos(\phi), m\sin(\phi))$ with $\phi \in (-\pi, \pi]$ and $m \in \mathbb{R}_{>0}$.

We claim now that $\sigma g \in \Theta_1$, where $g = (K_\phi, f_\phi) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$.

First of all we consider $\sigma_1 g = (T_1 K_\phi, f_1 \circ f_\phi)$. Let us show that $f_1 \circ f_\phi(0) = \phi/\pi$. By the correspondence between $f_1 \circ f_\phi$ and $T_1 K_\phi$ over S^1, it suffices to compute $T_1 K_\phi v_0$, where $v_0 = (1, 0)$. We have $(T_1 K_\phi) v_0 = T_1 (1/m) v$. Since $(\cos(\phi), \sin(\phi))$ is an eigenvector, we get $(T_1 K_\phi) v_0 = \lambda/m v$. Consequently, if we study the induced map $f_1 \colon S^1 \to S^1$, where $S^1 = (-1, 1]$, we obtain $f = (f_1 \circ f_\phi)(0) = f_1(\phi/\pi) = \phi/\pi$. Therefore, $(f_1 \circ f_\phi)(0) = \phi/\pi + 2k$ over $\mathbb{R}$ with $k \in \mathbb{Z}$.

Due to the fact that f is an increasing continuous function and $-1 < \phi/\pi < 1$, we have that $-2 < f(-1) < \phi/\pi + 2k < f(1) = f(0) + 1 < 2$ and it implies $k = 0$, $k = 1$ or $k = -1$. If $k = 1$, then $-2 < \phi/\pi + 2 < 2$ and $1 < \phi/\pi + 2 < 3$. It follows that $-1 < \phi/\pi < 0$ and $-2 < f(\phi/\pi) < 1$, i.e. $-2 < \phi/\pi + 2 < 1$. This clearly forces $-4 < \phi/\pi < -1$, which is impossible. If $k = -1$, then $-2 < \phi/\pi - 2 < 2$ and $-3 < \phi/\pi - 2 < -1$. It implies that $0 < \phi/\pi < 1$ and $-1 < f(\phi/\pi) < 2$, i.e. $-1 < \phi/\pi - 2 < 2$. This forces $1 < \phi/\pi < 4$, which is impossible. As a consequence $k = 0$.

We now consider $\sigma_2 g = (K_\phi, f_\phi)$. By definition $f_\phi(0) = \phi/\pi$.

As a consequence $f_{12}(\sigma g) = 0$. By Lemma 2.85, it is clear that

$$\sigma g = (Z, \mathrm{TCoh}^\phi(C)) = \mathrm{gl}_{12}(\sigma_1 g, \sigma_2 g).$$

□

Remark 3.61. Note that since $\det(M) > 0$, we have that both eigenvalues have the same sign, i.e. they are both positive or both negative.

Remark 3.62. Note that from Lemma 3.60 and Remark 3.59, we obtain that if $\sigma \in \Theta_{12}$ satisfies (*), then $\sigma \in \Theta_1$ if and only if the eigenvalues of M are positive.

Remark 3.63. Note that by Lemma 3.58, if $\Delta(M) < 0$, then σ can never be in Θ_1.

The study of negative eigenvalues of M is closely related to studying the other gluing cases appearing in Θ_{12}.

Under the assumption that the discriminant $\Delta(M) = \mathrm{Tr}(M)^2 - 4\det(M)$ of M is non-negative and that the eigenvalues λ_1, λ_2 of M are both negative we prove the following lemmas.

Lemma 3.64. *If* $\det(M + I) > 0$, *then there are just two options*

$$\lambda_1, \lambda_2 < -1 \text{ and } \lambda_1, \lambda_2 > -1.$$

Proof. It is well-known that $\det(M + I) = \det(M) + 1 + \mathrm{Tr}(M)$. Since the eigenvalues of M are negative, we also know that $\mathrm{Tr}(M) < 0$. The proof falls naturally into two cases:

Case 1: $\mathrm{Tr}(M)/2 < -1$.

From the quadratic formula, it follows that one of the eigenvalues is $\lambda_1 = \frac{\mathrm{Tr}(M) - \sqrt{\Delta(M)}}{2} < -1$. We want to show that the other eingenvalue staisfies $\lambda_2 = \frac{\mathrm{Tr}(M) + \sqrt{\Delta(M)}}{2} < -1$. Note that $-\mathrm{Tr}(M) - 2 > 0$. As $-\det(M) < \mathrm{Tr}(M) + 1$, after multiplying by 4 and adding $\mathrm{Tr}(M)^2$, we obtain

$$\mathrm{Tr}(M)^2 - 4\det(M) \quad < \quad \mathrm{Tr}(M)^2 + 4\,\mathrm{Tr}(M) + 4.$$

After taking square roots, we obtain $\frac{\mathrm{Tr}(M)+\sqrt{\Delta(M)}}{2} < -1$.

Case 2: $0 > \mathrm{Tr}(M)/2 > -1$.

We already know that λ_1 and λ_2 are negative. Therefore, it suffices to show that $\lambda_2 = \frac{\mathrm{Tr}(M)-\sqrt{\Delta(M)}}{2} > -1$. Note that $\mathrm{Tr}(M)+2 > 0$. Analogously, we get $\frac{\mathrm{Tr}(M)-\sqrt{\Delta(M)}}{2} > -1$.

Case 3: If $\mathrm{Tr}(M)/-2 = -1$ As before we obtain

$$\begin{aligned} \mathrm{Tr}(M)^2 - 4\det(M) &< \mathrm{Tr}(M)^2 + 4\,\mathrm{Tr}(M) + 4 \\ \Delta(M) &< 0, \end{aligned}$$

which contradicts our hypothesis. Therefore, this case does not appear. □

Since the pre-stability conditions in Θ_2 or Θ_3 satisfy that $l_*(\mathcal{L})$ is stable for all line bundles $\mathcal{L} \in \mathrm{Coh}(C)$. In order to determine which $\sigma \in \Theta_{12}$ are in Θ_2 or Θ_3 we need to study the stability of $l_*(\mathcal{L})$.

Lemma 3.65. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition as in (*). Let $\mathcal{L} \in \mathrm{Coh}(C)$ be a line bundle with $\deg(\mathcal{L}) = d \leq \frac{-C}{D}$. The object $l_*(\mathcal{L})$ is stable in σ if and only if*

$$-Dd^2 - (A+C)d - B > 0 \tag{3.33}$$

holds.

Proof. Since $\deg(\mathcal{L}) = d \leq \frac{-C}{D}$ we have $i_*(\mathcal{L}) \in \mathcal{A}$.

Assume that $l_*(\mathcal{L})$ is stable. By considering the triangle

$$j_*(\mathcal{L}) \longrightarrow l_*(\mathcal{L}) \longrightarrow i_*(\mathcal{L}) \longrightarrow j_*(\mathcal{L})[1]$$

we obtain $\phi^3_{\mathcal{L}} < \phi^1_{\mathcal{L}}$. By the correspondence between slope and phase, the equation

$$d < \frac{-Ad - B}{Dd + C}$$

holds for all $d \leq \frac{-C}{D}$.

We now assume inequality (3.33) holds for $\deg(\mathcal{L}) = d \leq -\cot(\pi\theta)$. By the triangle

$$j_*(\mathcal{L}) \longrightarrow l_*(\mathcal{L}) \longrightarrow i_*(\mathcal{L}) \longrightarrow j_*(\mathcal{L})[1] \ ,$$

we have that $l_*(\mathcal{L}) \in \mathcal{A}$. By using inequiality (3.33) and the correspondence between phases and slope we obtain

$$\phi^3_{\mathcal{L}} < \phi^1_{\mathcal{L}}.$$

By Lemma 3.27, we have that $l_*(\mathcal{L})$ is stable. □

Remark 3.66. The discriminant of the quadratic equation $Dd^2 + (A+C)d + B$ is given by $(A+C)^2 - 4BD = \Delta(M)$, i.e. it is the same discriminant of $p(x)$.

Claim 3.67. *Let $\mathcal{L}$ be a line bundle. If $i_*(\mathcal{L})[-1] \in \mathcal{A}$, then $l_*(\mathcal{L})$ is stable.*

Proof. Assume that $l_*(\mathcal{L})$ is not stable. Therefore by Lemma 3.27 we have that $\phi^1_{\mathcal{L}} - \phi^3_{\mathcal{L}} \leq 0$. As $0 < \phi^3_{\mathcal{L}} \leq 1$, it implies $\phi^1_{\mathcal{L}} \leq 1$, which contradicts our hypothesis. Indeed, if $i_*(\mathcal{L})[-1] \in \mathcal{A}$, then $1 < \phi^1_{\mathcal{L}} \leq 2$. □

Lemma 3.68. *If $\lambda_1, \lambda_2 < 0$, then inequality (3.33) holds for all $d \leq -\cot(\theta\pi)$.*

Proof. Let us consider the polynomial

$$q(x) = Dx^2 + (A+C)x + B.$$

As the discriminant of $q(x)$ is also $\Delta(M) \geq 0$, we get that $q(x)$ has real roots $\mu_1, \mu_2 \in \mathbb{R}$. We assume that $\mu_1 \leq \mu_2$. It is enough to show that for $\mu_1 \geq \frac{-C}{D}$. Indeed, if $\frac{-C}{D} \leq \mu_1$ and $d < \frac{-C}{D}$ then $q(x) < 0$. As $0 > 2\lambda_1 = (-A+C) - \sqrt{\Delta(M)}$, we obtain the following inequality

$$\frac{C}{-D} \quad < \quad \frac{-(A+C) - \sqrt{\Delta(M)}}{2D} = \mu_1,$$

as we wanted to prove. □

Corollary 3.69. *If $\lambda_1, \lambda_2 < 0$, then $l_*(\mathcal{L})$ is σ-stable for all line bundles $\mathcal{L}$. Moreover, σ is in Θ_{23} and Θ_{31}.*

Proof. It follows directly from Lemma 3.65, Lemma 3.67 and Lemma 3.68. □

Lemma 3.70. *There is a $t \in \mathbb{R}$ such that $f_{31}(\sigma)(t) = 1$ if and only if $\sigma \in \Theta_3$.*

Proof. Assume that there is $t \in \mathbb{R}$ such that $f_{31}(\sigma)(t) = 1$. Note that $f_{12}(\sigma)(0) < 0$ implies that $f_{31}(\sigma)(0) < 1$.

We have that $f_3(t) - f_1(t) = 1$. Let $g = (K_{t\pi}, f_{t\pi}) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. We would like to show that $\sigma g \in \Theta_3$. Since $f_1 \circ f_{t\pi}(0) = f_1(t) = f_3(t) - 1$ and $f_3 \circ f_{t\pi}(0) = f_3(t)$, then $f_{31}(\sigma g)(0) = 1$. As $l_*(\mathcal{L})$ is stable for any line bundle $\mathcal{L} \in \mathrm{Coh}(C)$, by Lemma 3.69, it follows directly from Lemma 2.86 that $\sigma g = \mathrm{gl}_{31}(\sigma_3 g, \sigma_1 g)$. Therefore, we obtain $\sigma \in \Theta_3$.

We now assume that $\sigma \in \Theta_3$, i.e. that there is $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that σg satisfies the gluing conditions. Without losing generality, we take $g = (K_{l\pi}, f_{l\pi}) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, with $l \in \mathbb{R}$. Thus, we obtain $f_3(l) = (f_3 \circ f_{l\pi})(0) \geq (f_1 \circ f_{l\pi})(0) = f_{l\pi}(0)+1 = l+1$. By hypothesis $f_{31}(\sigma)(0) < 1$, but also $f_{31}(\sigma)(l) \geq 1$. Since we have that $f_{31}(\sigma)(x)$ is a continuous function, there is a $t \in \mathbb{R}$ that satisfies $f_{31}(\sigma)(t) = 1$, as we desired. □

Proposition 3.71. *If $\lambda_1, \lambda_2 < -1$, then $\sigma \in \Theta_3$.*

Proof. By Lemma 3.53, we have that σ is one of the stability conditions constructed in Lemma 3.14. By Lemma 3.17 and by Remark 3.55 we have that there is $r_3 \in (-1,0)$ and a stability condition $\sigma_3 = (\mathrm{Coh}^{r_3}(C), Z_3) \in \mathrm{Stab}(C)$ such that $\mathrm{Coh}_3^{r_3}(C) \subseteq \mathcal{A}$ and $Z\big|_{D_3} = Z_3$.

We want to rotate σ to obtain a stability condition in Θ_3. By hypothesis, the eigenvalues of T are negative and $-1 < \frac{1}{\lambda_1}, \frac{1}{\lambda_2} < 0$. Therefore, the eigenvalues of T_3 are positive. We denote them by β_1, β_2. Indeed, as the eigenvalues of $M_3 = M_1 + I$ are precisely $\frac{1}{\lambda_i} + 1$ and they are positive, then β_1, β_2 are positive.

Let $v_i \in \mathbb{R}^2$ be the corresponding eigenvector of β_i. We obtain $T_3 v_i = \beta_i v_i$. Let us consider the polar coordinates of $v_i = (m_i \cos(\theta_i), m_i \sin(\theta_i))$ with $\theta_i \in [-\pi, \pi)$ and $m_i \in \mathbb{R}_{>0}$.

We now claim that $\sigma g \in \Theta_3$, where $g = (K_{\theta_i}, f_{\theta_i}) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

First of all, consider $\sigma_3 g = (T_3 K_{\theta_i}, (f_3 \circ f_{\theta_i}))$. Let us show that $f_3 \circ f_{\theta_i}(0) = \theta_i/\pi$. By the correspondence between $(f_3 \circ f_{\theta_i})$ and $T_3 K_{\theta_i}$ over S^1, it suffices to compute $T_3 K_{\theta_i} v_0$, where $v_0 = (1,0)$. We have $(T_3 K_{\theta_i}) v_0 = T_3 (1/m_i) v_i$. Since $(\cos(\theta_i \pi), \sin(\theta_i \pi))$ is an eigenvector, we get

$$(T_3 K_{\theta_i}) v_0 = \beta_i 1/m v_i.$$

As $\beta_i \geq 0$, it implies that if we study the induced map $f_3 : S^1 \to S^1$, where $S^1 = (-1,1]$, we obtain $(f_3 \circ f_{\theta_i})(0) = f_3(\theta_i/\pi) = \theta_i/\pi$.

Consequently, we get $(f_3 \circ f_\theta)(0) = f_3(\theta/\pi) = \theta/\pi + 2k$ with $k \in \mathbb{Z}$. Due to the fact that f_3 is an increasing continuous function, we have that $-1 < \theta_i/\pi < 1$ implies $-1 < \theta_i/\pi + 2k < 2$ and $k = 0$. Therefore, we have $(f_3 \circ f_{\theta_i})(0) = f_3(\theta_i/\pi) = \theta_i/\pi$.

We now consider $\sigma_1 g$. Similarly we show that $(f_1 \circ f_{\theta_i})(0) = \theta_1 - 1$. By the correspondence between $(f_1 \circ f_\theta)$ and $T_1 K_{\theta_i}$ over S^1, it suffices to compute $T_1 K_{\theta_i} v_0$, where $v_0 = (1,0)$. We have $(T_1 K_{\theta_i}) v_0 = T_1 (1/m_i) v_i$. Since $(\cos(\theta_i), \sin(\theta_i))$ is an eigenvector, we get

$$(T_1 K_{\theta_i}) v_0 = \frac{\alpha_i m}{v_i}.$$

As $\alpha_i < 0$, if we study the induced map $f_1 : S^1 \to S^1$, where $S^1 = (-1,1]$, we have two cases:

Case 1: If $-1 < \theta_i/\pi < 0$, then $(f_1 \circ f_{\theta_i}(0)) = f_1(\theta_i/\pi) = \theta_i + 1$. Consequently, we obtain $(f_1 \circ f_\theta)(0) = f_1(\theta_i/\pi) = \theta_i/\pi + 1 + 2k$ with $k \in \mathbb{Z}$. Due to the fact that f_1 is an increasing continuous function, we have that $-1 < \theta_i/\pi < 0$ implies $-2 < \theta_i/\pi + 2k + 1 < 0$ and the only possible option is $k = -1$.

Case 2: If $0 < \theta_i/\pi < 1$, then $(f_1 \circ f_{\theta_i})(0) = f_1(\theta_i/\pi) = \theta_i - 1$. Consequently, we obtain $(f_1 \circ f_\theta)(0) = f_1(\theta_i/\pi) = \theta_i/\pi - 1 + 2k$ with $k \in \mathbb{Z}$. Due to the fact that f_1 is an increasing continuous function, we have that $0 < \theta_i/\pi < 1$ implies $-1 < \theta_i/\pi + 2k - 1 < 1$ and the only possible option is $k = 0$.

Therefore, we have $(f_1 \circ f_{\theta_i})(0) = f_1(\theta_i/\pi) = \theta_i/\pi - 1$. It implies that $f_{31}(\sigma g) = 1$. By Lemma 3.70 we have that $\sigma \in \Theta_3$. We compute $f_{12}(\sigma g) = -1$ and $f_{23}(\sigma g) = 0$. We now have gluing conditions for the semiorthogonal decomposition $\langle D_3, D_1 \rangle$ and by Lemma 2.86 we obtain $\sigma g = g_{31}(\sigma_3 g, \sigma_1 g)$. □

Remark 3.72. Note that from Proposition 3.71 and Lemma 3.70, we obtain that if $\sigma \in \Theta_{12}$ satisfies (*), then $\sigma \in \Theta_3$ if and only if the eigenvalues λ_1, λ_2 of M are < -1.

Lemma 3.73. *Let σ as above, i.e. such that $f_{12}(\sigma)(0) < 0$. There is a $t \in \mathbb{R}$ such that $f_{23}(\sigma)(t) = 1$ if and only if $\sigma \in \Theta_2$.*

Proof. The proof goes exactly as the one of Lemma 3.70. □

Proposition 3.74. *If $0 > \lambda_1, \lambda_2 > -1$, then $\sigma \in \Theta_2$.*

Proof. The proof goes exactly as the one of Proposition 3.71. □

Remark 3.75. Note that from Proposition 3.74 and Lemma 3.73, we obtain that if $\sigma \in \Theta_{12}$ satisfies (*), then $\sigma \in \Theta_2$ if and only if the eigenvalues λ_1, λ_2 of M satisfy $0 > \lambda_1, \lambda_2 > -1$.

Remark 3.76. Note that if $\sigma \in \Theta_{12}$ satisfies (*), then either $\sigma \in \Theta_1$ or $\sigma \in \Theta_{12} \cap \Theta_{23}$. Indeed, if the eigenvalues of M are positive, we have that $\sigma \in \Theta_1$. If the eigenvalues are negative, it follows from Corollary 3.69. If $\Delta(M) < 0$, if follows from Lemma 3.65. This remark is useful when studying Θ_{23}(or Θ_{31}); because either $\sigma \in \Theta_2$ or $\sigma \in \Theta_{12} \cap \Theta_{23}$. Thus we can use our classification inside Θ_{12} to understand Θ_{23}.

Remark 3.77. Note that if $\sigma \in \Theta_{23}$, then by Proposition 3.52, up to the action we can find stability conditions

$$\sigma_3 = (Z_3, \mathrm{Coh}^r(C)) = (T_3, f_3) \text{ and } \sigma_2 = (Z_\mu, \mathrm{Coh}(C)) \in \mathrm{Stab}(C)$$

with $\mathrm{Coh}_3^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$, $r < 0$, $Z\big|_{D3} = Z_3$ and $Z\big|_{D2} = Z_\mu$. If $i_*(\mathbb{C}(x))$ is σ-stable for $x \in C$ and $\Delta(M_3) < 0$, with $M_3 := T_3^{-1}$, by following the steps of Lemma 3.65 we can show that $i_*(\mathcal{L})$ is σ-stable for all line bundles $\mathcal{L}$. As a consequence, by Lemma 3.53, we have that σ is given, up to the action, by a pre-stability condition constructed in Lemma 3.14. Moreover, $M_3 = I + M$, the M appearing in Lemma 3.14 and $\Delta(M) = \Delta(M+3) < 0$. Analogously for the case $\sigma \in \Theta_{31}$.

Theorem 3.78. *For all pre-stability conditions σ on $\mathcal{T}_C$, we have that*

$$\sigma \in \Theta_1 \cup \Theta_2 \cup \Theta_3 \cup \Gamma,$$

where Γ is the set of pre-stability condition, which up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action is given by Lemma 3.14 with $\Delta(M) < 0$. Moreover, note that $\Gamma \subseteq \Theta_{ij}$, where $ij = 12, 23, 31$.

Proof. By Theorem 3.30, we have that $\sigma \in \Theta_{12} \cup \Theta_{23} \cup \Theta_{31}$. By Serre duality, it is enough to check for $\sigma \in \Theta_{12}$. By Proposition 3.44, we have that up to the action of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, there are $\sigma_1 = (\mathrm{Coh}_1^r(C), Z_1) = (T, f)$ and $\sigma_2 = (\mathrm{Coh}_2(C), Z_\mu) \in \mathrm{Stab}(C)$ with $\mathrm{Coh}_1^r(C) \subseteq \mathcal{A}$, $\mathrm{Coh}(C)_2 \subseteq \mathcal{A}$, and $Z|_{D1} = Z_1$, $Z|_{D2} = Z_\mu$ with $f(0) \geq -1$. We reduce to the study of these pre-stability conditions, because the sets Θ_i and Γ, for $i = 1, 2, 3$, are defined up to the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action. If $f(0) \geq 0$, then by Lemma 3.49 $\sigma \in \Theta_1$. If $-1 < f(0) < 0$, we classify these pre-stability conditions in two sets if $\Delta(M) \geq 0$ or $\Delta(M) < 0$. If $\Delta(M) < 0$ then $\sigma \in \Gamma$. If $\Delta(M) \geq 0$, with positive eigenvalues then by Lemma 3.60 we get $\sigma \in \Theta_1$. If the eigenvalues are smaller that -1, then by Proposition 3.71 we have that $\sigma \in \Theta_3$ and if the eigenvalues are between 0 and -1 by Proposition 3.74, we obtain that $\sigma \in \Theta_2$. $\square$

Remark 3.79. Note that by Lemma 3.58, Lemma 3.70 and Lemma 3.73 we have that

$$\Theta_i \cap \Theta_j = \emptyset \text{ and } \Theta_i \cap \Gamma = \emptyset$$

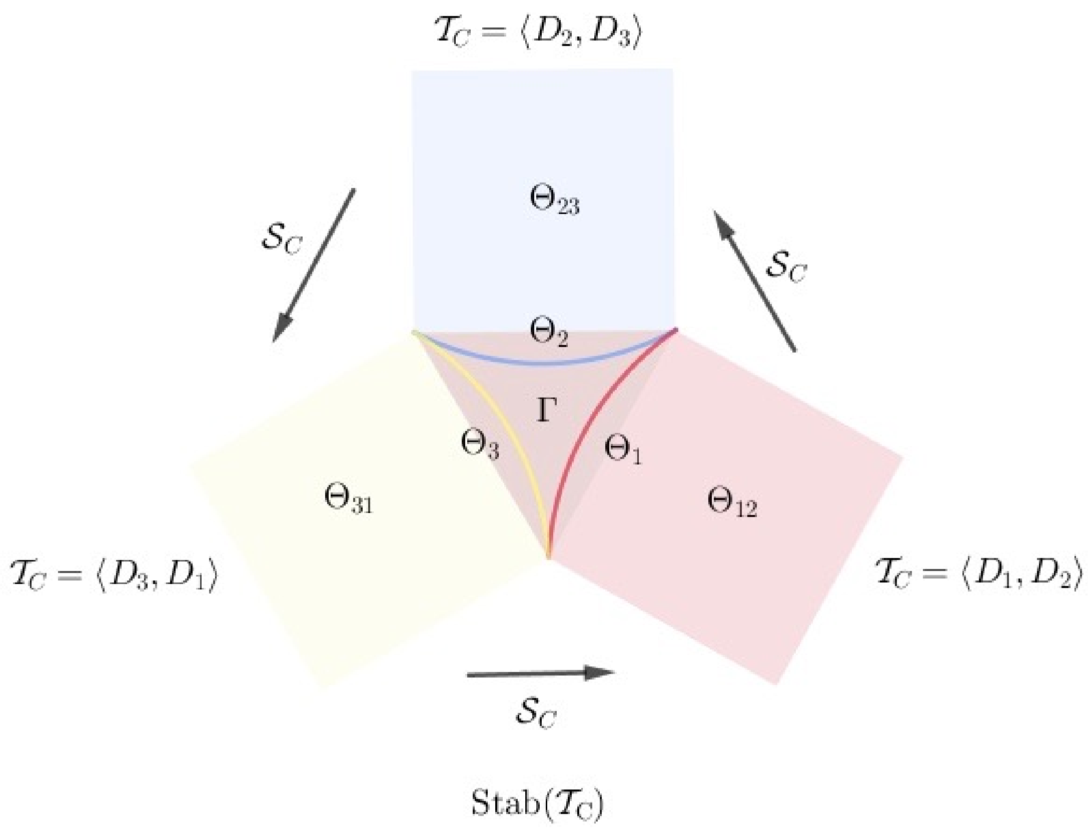
for $i, j \in \{1, 2, 3\}$.

If we see the set Θ_{ij} as the i vertex of a triangle, where $ij = 12, 23, 31$, the Serre functor $\mathcal{S}_C$ acts as a rotation. The following diagram describes $\mathrm{Stab}(C)$.

3.3 Support property for $\mathcal{T}_\mathcal{C}$, with $g(C) \geq 1$

In this section, we prove the support property for the already constructed pre-stability conditions. We start by studying CP-glued pre-stability conditions $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ on $\mathcal{T}_\mathcal{C} = \langle D_1, D_2 \rangle$, with $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(D^b(C))$, for $i = 1, 2$. By Theorem 1.75 there is always $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, such that $\sigma_1 = \sigma_2 g$ that satisfies $f(0) \geq 0$. Therefore, the proof of the support property falls naturally into the cases $f(0) \geq 1$ and $1 > f(0) \geq 0$. More precisely by Remark 2.105 and Theorem 3.78 we reduce the proof of the support property to three cases. Namely, in Lemma 3.80 we prove it for pre-stability conditions satisfying that $f(0) \geq 1$, in Lemma 3.83 we prove it for CP-glued pre-stability condition with $\mathcal{A}_1 = \mathcal{A}_2$, and in Lemma 3.86 for CP-glued pre-stability conditions with negative discriminant. Finally, in Subsection 3.3.2 we prove it for $\sigma \in \Gamma$.

We use the inequalities proved in Subsection 2.2.3 to study the σ-semistable objects in our pre-stability conditions. These inequalities follow closely the steps of [BGP96, Sec. 3]. In



Subsection 3.3.2 we prove the support property under the assumption that $g(C) = 1$, for non-gluing pre-stability conditions with $\Delta(M) < 0$. We conjecture that for $g(C) > 1$ the same result holds.

3.3.1 Support property for CP-glued pre-stability conditions

We start by proving the support property for pre-stability conditions with a stronger orthogonality condition.

Lemma 3.80. *If the pair $\sigma = (Z, \mathcal{A}) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ is a CP-glued pre-stability condition satisfying that*

$$\mathrm{Hom}^{\leq 1}_{\mathcal{T}_C}(i_*\mathcal{A}_1, j_*\mathcal{A}_2) = 0$$

or equivalently $f(0) \geq 1$, then it satisfies the support property and as consequence it is a Bridgeland stability condition.

Proof. We use the following notation $Z_1([E]) = Z_1([i^*(E)])$ and $Z_2([E]) = Z_2([j^!(E)])$.

First note that we can linearly extend Z and the homomorphism induced by the exact functors $i^*, j^!$ to $\mathcal{N}(\mathcal{T}_C) \otimes \mathbb{R} \cong \mathbb{R}^4$. We define the quadratic form

$$Q\colon \mathcal{N}(\mathcal{T}_{\mathcal{C}}) \otimes \mathbb{R} \to \mathbb{R},$$

as

$$Q(v) = \Im(Z_1(v))\Im(Z_2(v)) + \Re(Z_1(v))\Re(Z_2(v)),$$

where $\Re(\alpha)$ and $\Im(\alpha)$ are the real and the imaginary parts of $\alpha \in \mathbb{R}^4$ respectively.

By the linearity of Z, it is clear that Q is a quadratic form. We first show that it is negative definite on

$$\operatorname{Ker}(Z) = \{v \in \mathbb{R}^4 \mid \Re(Z_1(v)) = -\Re(Z_2(v)) \text{ and } \Im(Z_1(v)) = -\Im(Z_2(v))\}.$$

Indeed,

$$Q(v) = -\Im(Z_1(v))^2 - \Re(Z_1(v))^2 \leq 0,$$

for $v \in \operatorname{Ker}(Z)$. Note that $Q(v) = 0$, implies that $Z_i(v) = 0$, for $i = 1, 2$. Then $v = 0$ as $\operatorname{rank}(\mathcal{N}(D^b(C)) = 2$.

Let $E = E_1 \xrightarrow{\varphi} E_2$ be a σ-semistable object.

Claim 3.81. $\varphi = 0$.

Proof. As σ is a CP-glued pre-stability condition, by the definition of $\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$, we have that $E_1 \in \mathcal{A}_1$ and $E_2 \in \mathcal{A}_2$. As $\varphi \in \operatorname{Hom}_{D^b(C)}(E_1, E_2) = \operatorname{Hom}_{\mathcal{T}_C}(i_*(E_1), j_*(E_2)[1])$ and by hypothesis $\operatorname{Hom}_{\mathcal{T}_C}(i_*(E_1), j_*(E_2)[1]) = 0$, we obtain that $\varphi = 0$. $\square$

Note that by the definition of a CP-glued heart, we have $i_*(E_1), j_*(E_2) \in \mathcal{A}$. As $\varphi = 0$, we have $i_*(E_1) \subseteq E$ and also $E \twoheadrightarrow i_*(E)$. It follows that $\phi(i_*(E)) = \phi(E)$. Analogously for $j_*(E_2)$. Then, we get $\phi(i_*(E_1)) = \phi(j_*(E_2))$. If $\Im(Z_1(E)) = 0$, then it implies that $\Im(Z(E)) = 0$ and $\Im(Z_2(E)) = 0$. It follows that $\Re(Z_1(E)), \Re(Z_2(E)) < 0$ and

$$Q(E) = \Re(Z_1(E))\Re(Z_2(E)) > 0.$$

Analogously for $\Im(Z_2(E)) = 0$.

We now assume that $\Im(Z_1(E)), \Im(Z_2(E)) \neq 0$. By the correspondence between slope and phase, we obtain

$$\frac{-\Re(Z_1(E))}{\Im(Z_1(E))} = \frac{-\Re(Z_2(E))}{\Im(Z_2(E))}.$$

As a consequence

$$Q(E) = \Im(Z_1(E))\Im(Z_2(E)) + \Re(Z_1(E))\Re(Z_2(E)) \geq 0.$$

Indeed, as $\Im(Z_1(E)), \Im(Z_2(E)) > 0$, by hypothesis, then $\Re(Z_1(E)), \Re(Z_2(E))$ have the same sign and $\Re(Z_1(E))\Re(Z_2(E)) \geq 0$. $\square$

Let $\sigma = (Z, \mathcal{A}) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$, with $\sigma_i = (Z_i, \mathcal{A}_i) \in \mathrm{Stab}(C)$, be a pre-stability condition, such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ with $\sigma_1 = \sigma_2 g$ and $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$. Note that by the definition of the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action we obtain $\mathcal{H} = \mathcal{A}_1 = \mathcal{A}_2$. Moreover, we have that

$$\begin{aligned} \Re(Z_1(F)) &= -A\Re(Z_2(F)) + B\Im(Z_2(F)) \\ \Im(Z_1(F)) &= C\Im(Z_2(F)) \end{aligned}$$

for all $F \in \mathcal{H}$.

Remark 3.82. Under the following notation

$$d_2 = -\Re(Z_2(j^!(E)))\ ,\ d_1 = -\Re(Z_2(i^*(E))),$$

and

$$r_2 = \Im(Z_2(j^!(E))) \text{ and } r_1 = \Im(Z_2(i^*(E))),$$

for $E \in \mathcal{A}$, we obtain

$$Z_2([E]) = -d_2 + r_2 \text{ and } Z_1([E]) = Ad_1 + Br_1 + i(Cr_1).$$

We now prove the support property for this type of pre-stability conditions. We first recall a torsion pair that plays a role in the proof.

Lemma 3.83. *Let $\sigma = (Z, \mathcal{A}) = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$ be a pre-stability condition, such that there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ with $\sigma_1 = \sigma_2 g$ and $T^{-1} = \begin{bmatrix} -A & B \\ 0 & C \end{bmatrix}$. Then σ satisfies the support property and therefore it is a Bridgeland stability condition.*

Proof. First note that we can linearly extend Z to $\mathcal{N}(\mathcal{T}_C) \otimes \mathbb{R} \cong \mathbb{R}^4$.

We denote $Z_1([E]) = Z_1([i^*(E)])$ and $Z_2([E]) = Z_2([j^!(E)])$. As mentioned before, since i^* and $j^!$ are exact, they induce homomorphisms on the Grothendieck groups. We use the notation given above for elements $v \in \mathcal{N}(\mathcal{T}_{\mathcal{A}}) \otimes \mathbb{R}$, i.e. $Z_1(v)$ and $Z_2(v)$. We claim that for $\delta = \frac{-CA}{B^2} > 0$, the pre-stability condition σ satisfies the support property with the quadratic form $Q \colon \mathcal{N}(\mathcal{T}_{\mathcal{A}}) \otimes \mathbb{R} \to \mathbb{R}$, defined as

$$Q(v) = -\Re(Z_1(v))\Im(Z_2(v) + \Im(Z_1(v))\Re(Z_2(v)) + \Im(Z_1(v))\Im(Z_2(v)) + \delta(\Re(Z_1(v))\Re(Z_2(v))$$

where $\Re(\alpha)$ and $\Im(\alpha)$ are the real and the imaginary parts respectively and $v \in \mathbb{R}^n$.

Remark 3.84. Under the notation of Remark 3.82, we obtain that

$$Q(v) = -Ad_1r_2 - Cd_2r_1 - Br_1r_2 + Cr_1r_2 + \delta(-Ad_1d_2 - Bd_2r_1).$$

We first show that it is negative definite on

$$\operatorname{Ker}(Z) = \{v \in \mathbb{R}^n \mid \Re(Z_1(v)) = -\Re(Z_2(v)) \text{ and } \Im(Z_1(v)) = -\Im(Z_2(v))\}.$$

In fact

$$\begin{aligned} Q(v) &= \Re(Z_2(v))\Im(Z_2(v)) - \Im(Z_2(v))\Re(Z_2(v)) - \Im(Z_2(v))\Im(Z_2(v)) - \delta(\Re(Z_2(v))\Re(Z_2(v)) \\ &= -\Im(Z_2(v))^2 - \delta(\Re(Z_2(v))^2) < 0 \end{aligned}$$

if $v \neq 0$. Note that if $Q(v) = 0$, for $v \in \operatorname{Ker}(Z)$, then $Z_i(v) = 0$ for $i = 1, 2$. It follows that $v = 0$, as $\operatorname{rank}(\mathcal{N}(D^b(C))) = 2$.

Let $E = E_1 \xrightarrow{\varphi} E_2$ be a σ-semistable object. We consider the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & E_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{3.34}$$

If $E_1 = 0$ or $E_2 = 0$, we clearly have that $Q([E]) = 0$.

If $\Im(Z_1(E)) = 0$, it follows that $-\Re(Z_1(E)) > 0$. We claim that $\Im(Z_2(E)) = 0$. Indeed, after considering the decomposition of E_2 with respect to the standard torsion $(\mathcal{T}, \mathcal{F})$ pair given in Remark 2.120 on $\mathcal{A}$, we have that there is a subtriple $T_2 = 0 \to T(E_2)$ of E, such that $\Im(Z_2(T_2)) = 0$. If $T_2 \neq 0$

$$1 = \phi(T_2) \leq \phi(E) \leq 1,$$

which implies that $\phi(E) = 1$, thus $\Im(Z_2(E)) = 0$. If $T_2 = 0$, then $E_2 \in F$ and the morphism $\varphi = 0$, as E_1 is in T. It implies that $E_1 \to 0$ is a subtriple and once again we obtain that

$$1 = \phi(E_1 \to 0) \leq \phi(E) \leq 1.$$

So, $\phi(E) = 1$ and $\Im(Z_2(E)) = 0$ and it implies that $E = 0$. We obtain that $\Im(Z_2(E)) = 0$ and $-\Re(Z_2(E)) > 0$. Hence, we get

$$Q(E) = \delta(\Re(Z_1(E))\Re(Z_2(E))) > 0.$$

If $\Im(Z_2(E)) = 0$, then $-\Re(Z_2(E)) > 0$. By the short exact sequence above, we obtain that

$$1 = \phi(0 \to E_2) \leq \phi(E_1 \to 0) \leq 1.$$

It implies $\Im(Z_1(E)) = 0$ and $-\Re(Z_1(E)) > 0$. As a consequence, we get

$$Q(E) = \delta(\Re(Z_1(E))\Re(Z_2(E))) > 0.$$

We now assume $\Im(Z_1(E)), \Im(Z_2(E)) \neq 0$. If $\varphi = 0$, then $E = i_*i^*(E) \oplus j_*(j^!(E))$. By the semistability of E, we have

$$\frac{-\Re(Z_2(E))}{\Im(Z_2(E))} = \frac{-\Re(Z_1(E))}{\Im(Z_1(E))}.$$

Then

$$-\Re(Z_1(v))\Im(Z_2(v) + \Im(Z_1(v))\Re(Z_2(v)) = 0$$

and $\Re(Z_1(v))\Re(Z_2(v)) > 0$. As a consequence, we have

$$Q(v) = \Im(Z_1(v))\Im(Z_2(v)) + \delta(\Re(Z_1(v))\Re(Z_2(v))) \geq 0.$$

Therefore, we also assume $\varphi \neq 0$

Let $y = \frac{-\Re(Z_2(E)))}{\Im(Z_2(E))}$ and $x = \frac{-\Re(Z_2(j_*i^*(E))}{\Im(Z_2(j_*i^*(E)))}$. Note that $\Im(Z_1(E)) \neq 0$, implies $\Im(Z_2(j_*i^*(E))) \neq 0$. After dividing $Q([E])$ by $\Im(Z_2(E))\Im(Z_2(j_*i^*(E)))$ and by Equation (2.30), we obtain that $Q(v) \geq 0$ if and only if

$$-Ax - Cy - B + C + \delta(-Axy - By) \geq 0.$$

First note that by the semistability of E and Lemma 2.111 we have that $Cy + Ax \leq -B$ and $0 \leq y - x$.

Our proof falls naturally into two cases:

Case 1: $y \geq 0$. As $C > 0$, it implies that $0 \leq Cy \leq -Ax - B$. Then $-Axy - By \geq 0$ and since $-Ax - Cy - B + C \geq 0$, we get that $Q([E]) \geq 0$.

Case 2 If $y < 0$, then $x < y < 0$. Moreover if $-Ax - B \leq 0$, we have that $-Axy - By \geq 0$, and we argue as before. If $0 < -Ax-B$, as $-A > 0$ we get $0 < -Ax-B \leq -Ay-B < -B$. Then $Cy \leq -Ax - B \leq -B$ and $\frac{B}{-A} < y < 0$. Let us consider the following function

$$\begin{aligned} f\colon (0,-B)\times(\frac{B}{-A},0) &\to \mathbb{R} \\ (x,y) &\mapsto \frac{-Ax-Cy-B+C}{-Axy-By}. \end{aligned}$$

We show now that $f(x,y) < -\delta$ for all $(x,y)\in(0,-B)\times(\frac{B}{-A},0)$. Note $B\leq -Ax-Cy$. Then $0<C\leq -Ax-Cy-B+C$. Also $\frac{B}{-A}\leq y$, after multiplying both sides of the equation by $(-Ax-B)>0$, we obtain

$$\begin{aligned} \frac{B}{-A}(-Ax-B) &\leq y(-Ax-B), \\ Bx+\frac{B^2}{A} &< y(-Ax-B). \end{aligned}$$

As $xB>0$, we get $\frac{B^2}{A}<y(-Ax-B)$. Therefore, we obtain

$$\frac{-A}{-B^2} > \frac{1}{y(x+\alpha)}.$$

It follows that

$$f(x,y) < \frac{C}{y(x+\alpha)} < \frac{-CA}{-B^2}.$$

Since $\delta=\frac{-CA}{B^2}$, we have that $f(x,y)<-\delta$. Hence, we get

$$\frac{-Ax-Cy-B+C}{y(-Ax-B)} \quad < \quad -\delta.$$

As a consequence, we finally obtain $-Ax-Cy-B+C+\delta(-Axy-By)>0$ $\square$

Support property for CP-glued stability conditions with negative discriminant

Let $\sigma=(Z,\mathcal{A})=\mathrm{gl}_{12}(\sigma_1,\sigma_2)$ be a pre-stability condition, where $\sigma_1=(Z_1,\mathcal{A}_1)$ and $\sigma_2=(Z_2,\mathcal{A}_2)$, such that there is $g=(T,f)\in\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ with $\sigma_1=\sigma_2 g$ where (T,f) satisfies that $M:=T^{-1}=\begin{bmatrix}-A & B\\ -D & C\end{bmatrix}$ with

$$\Delta(M)=\mathrm{Tr}(M)^2-4\det(M)=(A+C)^2-4BD<0$$

and $0<f(0)<1$.

Claim 3.85. *Let $E=E_1\xrightarrow{\varphi}E_2$ be a σ-semistable object with $\Im(Z_2(E_1))<0$. Then $E_1\in\mathcal{A}_2[1]$*

Proof. Note that by the definition of gluing $E_2 \in \mathcal{A}_2$. By Claim 2.106, we have that for all $h \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, the object σh is a CP-glued pre-stability condition. As $\sigma_1 = \sigma_2 g$, if we act by g_1^{-1}, then $\sigma' = \sigma g^{-1} = \mathrm{gl}_{12}(\sigma_2, \sigma_2 g_1^{-1}) = (Z', \mathcal{A}')$. Since E is σ'-semistable, then $E \in A'[n]$ for some n. By Lemma 2.66, we obtain $E_1 \in \mathcal{A}_2[n]$, because $\mathcal{A}_2$ is equal to $\mathcal{P}_1(f^{-1}(0), f^{-1}(1)] = \mathcal{P}_2(0,1]$, where $\mathcal{P}_i$ gives us the slicing of σ_i, for $i = 1,2$. As $0 \leq f(0) < 1$, then $-1 < f^{-1}(0) < 0$ and $0 < f^{-1}(1) < 1$. Then we obtain that $\mathcal{A}_2 \subseteq \mathcal{P}_1(-1,1]$, which implies $n = 0,1$. As

$$\Im(Z_2(E_1)) < 0,$$

we get that $E_1 \in \mathcal{A}_2[1]$. □

Lemma 3.86. *Let $\sigma = (Z, \mathcal{A})$ be as above. Then σ is a Bridgeland stability condition.*

Proof. We just need to prove that it satisfies the support property. Note that we can extend Z to $\mathcal{N}(\mathcal{T}_\mathcal{A}) \otimes \mathbb{R} \cong \mathbb{R}^4$.

We claim that σ satisfies the support property with respect to the following quadratic form

$$\begin{array}{rcl} Q\colon \mathbb{R}^4 & \to & \mathbb{R} \\ v & \to & -\Im(Z_2(j^!(v))))\Re(Z_2(i^*(v))) + \Im(Z_2(i^*v)))\Re(Z_2(j^!(v)). \end{array}$$

We use the following notation

$$d_1 = -\Re(Z_2(i^*(v))) \text{ and } r_1 = \Im(Z_2(i^*(v)))$$

and

$$d_2 = -\Re(Z_2(j^!(v))) \text{ and } r_2 = \Im(Z_2(j^!(v))).$$

Note that $r_2 \geq 0$.

We first show that Q is negative definite on

$$\mathrm{Ker}(Z) = \{v \in \mathbb{R}^4 \mid \Re(Z_2(j^!v)) = -\Re(Z_1(i^*v)) \text{ and } \Im(Z_2(i*v)) = -\Im(Z_1(j^!v))\}.$$

Note that

$$Z_1(w) = -A\Re(Z_2(w)) + B\Im(Z_2(w)) + i(-D\Re(Z_2(w)) + C\Im(Z_2(w))),$$

with $w \in \mathbb{R}^2$

$$\begin{array}{rcl} Q(v) & = & \Im(Z_1(i^*v))\Re(Z_2(i^*(v))) - \Im(Z_2(i^*(v)))\Re(Z_1(i^*(v)) \\ & = & -(Cr_1 + Dd_1)d_1 - (Ad_1 + Br_1)r_1 \\ & = & -Dd_1^2 + (A+C)d_1 r_1 + Br_1^2 < 0. \end{array}$$

Indeed, as $0 \leq f(0) = t < 1$ and $t < 1 < t+1$, it implies that $D > 0$. Morever since $\Delta(M) = (A+C)^2 - 4BD < 0$, then $B > 0$ and $-Dd^2 - (A+C)dr - Br^2 < 0$ for all $(r,d) \in \mathbb{R}^2$. If $(r,d) = 0$, it implies that $v = 0$.

Now let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$ be a σ-semistable object. We show that $Q([E]) \geq 0$. First of all, if $[E] = v \in \mathcal{N}(\mathcal{T}_{\mathcal{A}})$ with $\Im(Z_2(v)) = 0$, then $\Re(Z_2(v)) < 0$ and after considering the following short exact sequence in $\mathcal{A}$

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"'] & E_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd} \tag{3.35}$$

we obtain $1 = \phi(0 \to E_2) \leq \phi(E_1 \to 0) \leq 1$. Then $\phi(E_1 \to 0) = 1$, and it implies that $Cr_1 + Dd_1 = 0$ and $r_1 \leq 0$, because of Remark 1.83. As $-d_2 < 0$ then

$$Q([E]) = -d_2 r_1 \geq 0.$$

If $r_1 = 0$, as $Cr_1 + Dd_1 \geq 0$, then $Dd_1 \geq 0$ and $d_1 \geq 0$. Therefore, we have

$$Q([E]) = d_1 r_2 \geq 0.$$

Let us now assume that $r_2, r_1 \neq 0$. From the exact sequence (3.35) and the correspondence between slope and phase, we obtain

$$\frac{d_2}{r_2} \leq \frac{-Ad_1 + Br_1 + d_2}{Cr_1 + Dd_1 + r_2},$$

which is equivalent to

$$-Dd_1d_2 - Cr_1d_2 - Ad_1r_2 - Br_1r_2 \geq 0.$$

We define $x = \frac{d_1}{r_1}$ and $y = \frac{d_2}{r_2}$. Let us consider two cases:

Case 1: $r_1 > 0$. Since by definition $Cr_1 + Dd_1 \geq 0$, then $C + Dx \geq 0$. Moreover we have $C + Dx > 0$. Indeed, if $C + Dx = 0$, we get $E_1 \in \mathcal{A}_2[1]$ and $r_1 \leq 0$, which contradicts our assumption. By (3.35), we obtain

$$y \leq \frac{-Ax - B}{Dx + C}.$$

As $\Delta(M) < 0$ and $D > 0$, then for all $t \in \mathbb{R}$ we have that $Dt^2 + (A+C)t + B > 0$. Thus, we obtain $Dx^2 + (A+C)x + B > 0$, which implies

$$x > \frac{-Ax + B}{Dx + C}.$$

Finally, we get

$$x > y$$

or equivalently $Q([E]) = d_1 r_2 - d_2 r_1 > 0$.

Case 2: $r_1 < 0$. We claim that $\varphi = 0$. Indeed, by Claim 3.85, we have that $E_1 \in \mathcal{A}_2[1]$. Then

$$\varphi \in \mathrm{Hom}_{D^b(\mathcal{A})}(E_1, E_2) = 0,$$

as $\mathcal{A}_2$ is the heart of a bounded t-structure in $D^b(\mathcal{A})$.

It implies that $i_*(E_1) \subseteq E$, but it is also true that $E \twoheadrightarrow i_*(E_1)$. Therefore $\phi(E) = \phi(E_1 \to 0)$. Analogously $j_*(E_2) \subseteq E$ and $E \twoheadrightarrow j_*(E_2)$, it follows that $\phi(E) = \phi(0 \to E_2)$. As a consequence, we have $\phi(E_1 \to 0) = \phi(0 \to E_2)$.

If $\phi(E_1 \to 0) = 1$, then $\phi(0 \to E_2) = 1$ and $r_2 = 0$. Note that we considered this case already. Therefore, we assume $\phi(E_1 \to 0) < 1$ and $Cr_1 + Dd_1 > 0$. We obtain

$$\frac{d_2}{r_2} = \frac{-Ad_1 - Br_1}{Cr_1 + Dd_1}.$$

As before, for all $t \in \mathbb{R}$, we obtain $Dt^2 + (A+C)t + B > 0$ and it implies that $Dx^2 + (A+C)x + B > 0$. Then $(Dx + C)x > -Ax - B$. Since $Dd_1 + Cr_1 > 0$, we obtain $Dx + C < 0$ and

$$x < \frac{-Ax - B}{Dx + C} = y.$$

Hence $x < y$ or equivalently $Q([E]) = d_1 r_2 - d_2 r_2 > 0$. $\square$

We now use Remark 2.105 to prove that all CP-glued prestability conditions on $\mathcal{T}_C$ satisfy the support property.

Proposition 3.87. *If the pair $\sigma = (Z, \mathcal{A})$ on $\mathcal{T}_C$ is a pre-stability condition with $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$, then it satisfies the support property and therefore, it is a Bridgeland stability condition.*

Proof. By the transitivity of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action on $\mathrm{Stab}(C)$, there is $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ with $(Z_1, \mathcal{A}_1) = \sigma_1 = \sigma_2 g$ and $\sigma_2 = (Z_2, \mathcal{A}_2)$.

The proof falls naturally into the following cases:

Case 1: If $f(0) \geq 1$, then it follows directly from Lemma 3.80.

Case 2: If $0 \leq f(0) < 1$ and $\Delta(M) \geq 0$, then we have the existence of real eigenvalues

$\lambda_1, \lambda_2 \in \mathbb{R}$. As $\det(T^{-1}) > 0$, then we have that the eigenvalues are both positive or both negative.

If $\lambda_1, \lambda_2 \geq 0$, then by Lemma 2.103 there is $h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that σh satisfies the conditions of Lemma 3.83. Since the support property is stable under the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, we have that σ has the support property.

If $\lambda_1, \lambda_2 \leq 0$, then by Lemma 2.104 there is $h \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that σh satisfies the conditions of Lemma 3.80. Since the support property is stable under the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, we have that σ has the support property.

Case 3: If $0 < f(0) < 1$ and $\Delta(M) < 0$, then this case follows directly from Lemma 3.86.

□

Remark 3.88. By Remark 1.69, after applying Serre duality, we have that any pre-stability condition $\sigma \in \Theta_i$, with $i \in \{1, 2, 3\}$ satisfies the support property. By Theorem 3.78, we just need to prove the support property for $\sigma \in \Gamma$.

We showed that all the CP-glued pre-stability conditions, up to the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action, satisfy the support property. We now prove the support property for $\sigma \in \Gamma$ just when $g(C) = 1$. For $g(C) > 1$, we conjecture that it is also satisfied. We start by studying the σ-semistable objects in the pre-stability conditions in Lemma 3.14 under the assumption that $g(C) \geq 1$.

Semistability on non-gluing pre-stability conditions

We now study σ-semistable objects, where $\sigma = (Z, \mathcal{A})$ satisfies that $i_*(\mathbb{C}(x)), l_*(\mathbb{C}(x))$ are σ-stable and $j_*(\mathbb{C}(x))$ is σ-stable of phase one. After applying the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action on σ the objects $i_*(\mathbb{C}(x))$, $j_*(\mathbb{C}(x))$ and $l_*(\mathbb{C}(x))$ are always σ-stable. Therefore, we can use the description of the hearts given in Lemma 3.40, Lemma 3.41 and Lemma 3.42 to prove that all σ-semistable objects have a particular form.

Lemma 3.89. *If $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{A}$ is σ-semistable then*

1. *$E \in \mathrm{TCoh}(C)$ or*
2. *$E \in \mathcal{H}_{23}[-1]$ or*
3. *$E \in \mathcal{H}_{31}[-1]$.*

Proof. First note that there are elements $g_1, g_2 \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ with $\delta_i = \sigma g_i = (W_i, \mathcal{B}_i)$ for $i = 1, 2$, such that δ_1 satisfies that $i_*(\mathbb{C}(x))$ is δ_1-stable of phase one, $j_*(\mathbb{C}(x)), l_*(\mathbb{C}(x))$ are δ_1-stable and δ_2 satisfies that $l_*(\mathbb{C}(x))$ is δ_2-stable of phase one and $j_*(\mathbb{C}(x)), i_*(\mathbb{C}(x))$ are δ_2-stable. As $E \in \mathcal{A}$, by Lemma 3.40, we obtain that E satisfies that its cohomology has the following form

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_1) \arrow[r] \arrow[d] & H^1(E_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_2) \arrow[r] \arrow[d] & H^1(E_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(C(\varphi)) \arrow[r] & H^0(C(\varphi)) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{3.36}$$

Note that for an object $G = G_1 \xrightarrow{\psi} G_2 \in \mathcal{B}_1$, by Lemma 3.41 its cohomology has the following form:

$$\begin{tikzcd}
0 \arrow[r] & H^{-1}(G_1) \arrow[r] \arrow[d] & H^0(G_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(G_2) \arrow[r] \arrow[d] & H^0(G_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(C(\psi)) \arrow[r] & H^0(C(\psi)) \arrow[r] & 0
\end{tikzcd} \tag{3.37}$$

and if $G = G_1 \xrightarrow{\psi} G_2 \in \mathcal{B}_2$, by Lemma 3.42 its cohomology has the following form:

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(G_1) \arrow[r] \arrow[d] & H^1(G_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(G_2) \arrow[r] \arrow[d] & H^0(G_2) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(C(\psi)) \arrow[r] & H^0(C(\psi)) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{3.38}$$

Remark 3.90. The diagrams above are non-commutative. We just use them as it is easier to visualize the cohomology of the objects.

As E is δ_i-semistable we have that $E \in \mathcal{B}_i[n]$, for $i = 1, 2$ and $n \in \mathbb{Z}$, where the only possible cases are $n = 1, 0, -1, -2$. We study all the different cases.

Case 1: $E \in \mathcal{A} \cap B_1[-2] \cap \mathcal{B}_2[-2]$. Since the intersection is trivial, then $E = 0$.

Case 2: $E \in \mathcal{A} \cap \mathcal{B}_1[-2] \cap \mathcal{B}_2[-1]$. The intersection is contained in $\mathrm{Coh}_3(C)[-1]$, it implies $E \in \mathcal{H}_{23[-1]}$.

Case 3: $E \in \mathcal{A} \cap \mathcal{B}_1[-2] \cap \mathcal{B}_2$. The intersection is trivial.

Case 4: $E \in \mathcal{A} \cap \mathcal{B}_1[-2] \cap \mathcal{B}_2[1]$.The intersection is trivial.

Case 5: $E \in \mathcal{A} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2[-2]$. The intersection is trivial.

Case 6: $E \in \mathcal{A} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2[-1]$. The intersection has the following form

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H^1(E_1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_2) \arrow[r] \arrow[d] & H^1(E_2) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & H^0(C(\varphi)) \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{3.39}$$

This implies that $E \in \mathcal{H}_{23}[-1]$.

Case 7: $E \in \mathcal{A} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2$. The intersection has the following form

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_1) \arrow[r] \arrow[d] & H^1(E_1) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_2) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] & H^0(C(\varphi)) \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd} \tag{3.40}$$

and $E \in \mathcal{H}_{31}[-1]$.

Case 8: $E \in \mathcal{A} \cap \mathcal{B}_1[-1] \cap \mathcal{B}_2[1]$. The intersection is trivial.

Case 9: $E \in \mathcal{A} \cap \mathcal{B}_1 \cap \mathcal{B}_2[-2]$. The intersection is trivial.

Case 10: $E \in \mathcal{A} \cap \mathcal{B}_1 \cap \mathcal{B}_2[-1]$. The intersection is contained in $\mathrm{Coh}_2(C)$ and therefore $E \in \mathrm{TCoh}(C)$.

Case 11: $E \in \mathcal{A} \cap \mathcal{B}_1 \cap \mathcal{B}_2$. We have that E has the following form

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_1) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_2) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & H^{-1}(C(\varphi)) \arrow[r] & H^0(C(\varphi)) \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{3.41}$$

It implies that $E \in \mathrm{TCoh}(C)$.

Case 12: If $E \in \mathcal{A} \cap \mathcal{B}_1 \cap \mathcal{B}_2[1]$. The intersection is contained in $\mathrm{Coh}_1(C)$, then

$E \in \mathrm{TCoh}(C)$.

Case 13: If $E \in \mathcal{A} \cap \mathcal{B}_1[1] \cap \mathcal{B}_2[i]$, with $i = -2, -1, 0, 1$. The intersection $\mathcal{A} \cap \mathcal{B}_1[1]$ is trivial, therefore $E = 0$. $\square$

3.3.2 Support property for non-gluing pre-stability conditions with negative discriminant and $g = 1$

We consider pre-stability conditions $\sigma = (Z, \mathcal{A})$ as constructed in Lemma 3.14 with $\Delta < 0$, i.e. there is $\sigma_1 = (Z_1, \mathrm{Coh}_1^{r_1}(C)) = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ such that $\mathrm{Coh}_1^{r_1}(C) \subseteq \mathcal{A}$ and $Z|_{\mathrm{Coh}_1^{r_1}(C)} = Z_1$, with $-1 < f(0) < 0$. We assume $\Delta(M) < 0$, where $M = T^{-1}$. Under the assumption that $g(C) = 1$, these pre-stability conditions satisfy the support property and as a consequence they are Bridgeland stability conditions.

Since $\Delta(M) < 0$, after applying the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action we never obtain a CP-glued pre-stability condition, because $f(\theta) < \theta$ for all $\theta \in \mathbb{R}$. However, under the assumption $g = 1$, the quadratic form induced by the Euler bilinear form

$$-\chi(E, E) = d_2 r_1 - d_1 r_2$$

is negative definite on $\mathrm{Ker}(Z)$ as in Lemma 3.86. Therefore, it is a good candidate for being a quadratic form appearing in the support property.

We now prove some useful statements about $\mathcal{A}$.

Lemma 3.91. *If $F \in \mathrm{Coh}(C)$ is μ-stable, then $i_*(F)$, $j_*(F)$ and $l_*(F)$ are σ-stable.*

Proof. The proof goes along the lines of Proposition 3.22. If F is μ-stable, it is either a skyscraper sheaf of a torsion-free sheaf. In Proposition 3.22, we have already proved the statement for skyscraper sheaves, therefore we assume that F is torsion-free and $r > 0$. Let us assume that $i_*(F)$ is not σ-semistable. Then, we consider the last triangle of its Harder-Narasimhan filtration

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\varphi_E"'] & F \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\varphi_A"] & E_1[1] \arrow[d, "\varphi_E[1]"] \\ E_2 \arrow[r] & 0 \arrow[r] & A_2 \arrow[r] & E_2[1] \end{tikzcd}, \tag{3.42}$$

which satisfies $\mathrm{Hom}^{\leq n}_{\mathcal{T}_C}(E, A) = 0$, with $n \leq 0$. By Lemma 3.17 we have that $\mathrm{Hom}_{D^b(C)}(E_1, A_1) = 0$ and $E_1, A_1 \in \mathrm{Coh}(C)$. By Serre duality on C, we get $\mathrm{Hom}_{D^b(C)}(A_1, E[1]) = 0$, which implies that the short exact sequence

$$0 \to E_1 \to F \to A_1 \to 0$$

splits. As F is μ-stable, it is indecomposable. Therefore, either $E_1 = 0$ or $A_1 = 0$. Following exactly the same steps as in Lemma 3.22 with $X = F$, we show that $A_1 = 0$ and $j_*(F), l_*(F)$ are σ-stable. As in Lemma 3.24, the last triangle of the HN-filtration of $i_*(F)$ is given by

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\varphi"'] & F \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & F[1] \arrow[d, "\varphi[1]"] \\ F \arrow[r] & 0 \arrow[r] & F[1] \arrow[r] & F[1] \end{tikzcd} ,$$

and it implies that $\phi(l_*(F)) > \phi(j_*(F)) + 1$. As $j_*(F) \in \mathrm{Coh}_2(C) \subseteq \mathcal{A}$, then

$$1 < \phi(j_*(F)) + 1 \leq 2,$$

which implies that $1 < \phi(l_*(F))$. Moreover, note that $\phi(l_*(F)) < 2$. By the stability of $l_*(\mathbb{C}(x))$ we have that $1 < \phi_4 < 2$ and we also have a non-zero morphism $l_*(F) \to l_*(\mathbb{C}(x))$. By Lemma 3.17 we have that $\mathrm{Coh}^{r_3}(C) \subseteq \mathcal{A}$, so that $l_*(F)[-1] \in \mathcal{A}$. Moreover, by the correspondence between slope and phase, we have that

$$\phi(j_*(F)) = \frac{d}{r} < \frac{Ad + Br - d}{-Cr - Dd - r} = \phi(l_*(F)[-1]),$$

which implies

$$-Dd^2 - (A + C)dr - Br^2 < 0$$

and induces a contradiction, because $\Delta(M) = (A+C)^2 - 4BD < 0$ and $D, B < 0$. Therefore, we obtain that $i_*(F)$ is σ-semistable. We now assume that $i_*(F)$ is striclty-semistable, by Remark 3.27 we obtain exactly the same contradiction. Therefore, we get that $i_*(F)$ is σ-stable. Analogously, we prove that $j_*(F)$ and $l_*(F)$ are σ-stable. □

Lemma 3.92. *If $F \in \mathrm{Coh}(C)$ is μ-semistable, then $i_*(F)$, $j_*(F)$ and $l_*(F)$ are σ-semistable.*

Proof. If $F = \mathbb{C}(x)$, the statement is already proved by Lemma 3.33. Therefore, we assume that F is torsion-free. Let us consider a JH-filtration of F with respect to μ. Note that all the μ-stable factors A_i, for $i = 0, \cdots, n$, have the same slope $\mu(F)$. By Lemma 3.91, we obtain that $j_*(A_i)$ is σ-stable in $\mathcal{A}$. As $Z|_{\mathrm{Coh}_2(C)} = Z_\mu$, we have that $\phi(j_*(A_i)) = \phi(j_*(F)) = \lambda$, with $\lambda \in \mathbb{R}$. Since the category $\mathcal{P}(\lambda)$ is closed under extensions, we obtain that $j_*(F)$ is σ-semistable. Note that since F is μ-semistable, then $i_*(F)$ is in $\mathcal{A}$ or in $\mathcal{A}[1]$ and $l_*(F)$ is in $\mathcal{A}$ or in $\mathcal{A}[1]$. Analogously, the same conclusion can be drawn for $i_*(F)$ and $l_*(F)$. □

Lemma 3.93. *We have that $\mathcal{A} \cap D_2 = \mathrm{Coh}_2(C)$.*

Proof. Let $E \in \mathcal{A}$. By Lemma 3.40, we obtain that the cohomology of E has the following form

$$\begin{tikzcd}
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_1) \arrow[r] \arrow[d] & H^1(E_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & H^0(E_2) \arrow[r] \arrow[d] & H^1(E_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^{-1}(C(\varphi)) \arrow[r] & H^0(C(\varphi)) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd} \tag{3.43}$$

If $E \in D_2$, then $H^0(E_1) = H^1(E_1) = 0$. By considering the long exact sequence of cohomology $0 \to H^{-1}(C(\varphi)) \to H^0(E_1) \to H^0(E_2) \to H^0(C(\varphi)) \to H^1(E_1) \to H^1(E_2) \to H^1(C(\varphi)) \to 0$, induced by the triangle $E_1 \xrightarrow{\varphi} E_2 \to C(\varphi) \to E_1[1]$, we obtain that $H^{-1}(C(\varphi)) = H^1(E_2) = 0$. It implies that $E \in \mathrm{Coh}_2(C)$. $\square$

In order to prove the support property, we study for every $\lambda \in (0, 1]$ the abelian category $\mathcal{P}(\lambda)$.

Lemma 3.94. *Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathrm{TCoh}(C)$ be a σ-semistable object in $\mathcal{A}$. Then*

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = 0.$$

Proof. First note that $E \in \mathrm{TCoh}(C) \cap \mathcal{A} = \mathcal{F}$, where $\mathcal{A} = (\mathcal{F}, \mathcal{T}[-1])$ as in Lemma 3.40. By Serre duality

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = \mathrm{Hom}_{\mathcal{T}_C}(E[2], \mathcal{S}_{\mathcal{T}_C}(E))^* = \mathrm{Hom}(E[1], E_2 \to C(\varphi))^*.$$

It suffices to prove that $\mathrm{Hom}(E[1], E_2 \to C(\varphi)) = 0$.

Let us consider the following triangle

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & E_2 \arrow[r] \arrow[d] & 0 \arrow[d] \\
C(\varphi) \arrow[r] & C(\varphi) \arrow[r] & 0 \arrow[r] & C(\varphi)[1].
\end{tikzcd} \tag{3.44}$$

It induces a long exact sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{T}_C}(E[1], j_*(C(\varphi))) \to \mathrm{Hom}_{\mathcal{T}_C}(E[1], \mathcal{S}_{\mathcal{T}_C}(E)[-1]) \to \mathrm{Hom}_{\mathcal{T}_C}(E[1], i_*(E_2))) \to \cdots.$$

Therefore, it is enough to prove that

$$\mathrm{Hom}_{\mathcal{T}_C}(E[1], i_*(E_2)) = 0 \text{ and } \mathrm{Hom}_{\mathcal{T}_C}(E[1], j_*(C(\varphi))) = 0.$$

By adjointness

$$\mathrm{Hom}_{\mathcal{T}_C}(E[1], i_*(E_2)) = \mathrm{Hom}_{D^b(C)}(E_1[1], E_2) = 0$$

as E_1, $E_2 \in \mathrm{Coh}(C)$ and $\mathrm{Coh}(C)$ is the heart of a bounded t-structure in $D^b(C)$.

We now prove that

$$\mathrm{Hom}_{\mathcal{T}_C}(E[1], j_*(C(\varphi))) = 0.$$

Case 1: If $\mathrm{Ker}(\varphi) = 0$, we obtain that $C(\varphi) = \mathrm{Coker}(\varphi)$ and by adjointeness we get

$$\mathrm{Hom}_{\mathcal{T}_C}(E[1], j_*(C(\varphi))) = \mathrm{Hom}_{\mathcal{T}_C} = \mathrm{Hom}_{D^b(C)}(\mathrm{Coker}(\varphi)[1], \mathrm{Coker}(\varphi)) = 0,$$

as $\mathrm{Coker}(\varphi) \in \mathrm{Coh}(C)$ and $\mathrm{Coh}(C)$ is the heart of a bounded t-structure.

Case 2: $\mathrm{Ker}(\varphi) \neq 0$

It is enough to show that $\phi(E) + 1 > \phi^+(j_*(C(\varphi)))$. Let us compute $\phi^+(j_*(C(\varphi)))$. By Claim 3.93 we have that $\mathcal{A} \cap D_2 = \mathrm{Coh}_2(C)$, so $j_*(C(\varphi)) \notin \mathcal{A}$. As a consequence, we first need to consider its filtration in the t-structure induced by $\mathcal{A}$, given by

$$\begin{tikzcd}
0 \arrow[rr] & & j_*(\mathrm{Ker}(\varphi))[1] \arrow[rr] \arrow[dl] & & j_*(C(\varphi)) \arrow[dl] \\
& j_*(\mathrm{Ker}(\varphi))[1] \arrow[ul, dashed] & & j_*(\mathrm{Coker}(\varphi)) \arrow[ul, dashed] &
\end{tikzcd}.$$

By definition, $\phi^+(j_*(C(\varphi))) = \phi^+(j_*(\mathrm{Ker}(\varphi))) + 1$. Let us consider the HN-filtration

$$0 \subseteq H_1 \subseteq H_2 \cdots \subseteq H_{n-1} \subseteq H_n = j_*(\mathrm{Ker}(\varphi))$$

of $j_*(\mathrm{Ker}(\varphi))$ in $\mathcal{A}$ with respect to σ. Note that by Lemma 3.92 if $F \in \mathrm{Coh}(C)$ is μ-semistable then $j_*(F)$ is also σ-semistable. Yherefore if we consider the HN-filtration of $\mathrm{Ker}(\varphi)$ with respect to μ, as $\mathrm{Coh}_2(C) \subseteq \mathcal{A}$ and $Z|_{\mathrm{Coh}_2(C)} = Z_\mu$, it will give us the HN-filtration of $j_*(\mathrm{Ker}(\varphi))$ in $\mathcal{A}$ with respect to σ. By the uniqueness of the HN-filtration, we deduce that $H_i \in \mathrm{Coh}_2(C)$, for all $i = 0, \cdots, n$. Moreover, we have that $H_1 \neq 0$ is σ-semistable and

$$\phi(H_1) = \phi^+(j_*(\mathrm{Ker}(\varphi))) = \phi^+(j_*(C(\varphi))) - 1.$$

Let $H_1 = 0 \to F_1$, with $F_1 \in \mathrm{Coh}(C)$. By definition $F_1 \subseteq \mathrm{Ker}(\varphi)$.

As $\mathrm{Ker}(\varphi) \to 0$ is a subobject of E in $\mathrm{TCoh}(C)$ and $\mathcal{F}$ is closed under subobjects, we obtain that $F_1 \to 0 \in \mathcal{F} \subseteq \mathcal{A}$. By Lemma 3.92, as F_1 is μ-semistable, $i_*(F_1)$ is also σ-semistable. Moreover, we have a non-zero morphism $i_*(F_1) \to E$. As they are both σ-semistable it implies that

$$\phi_\sigma(i_*(F_1)) \leq \phi_\sigma(E).$$

Let $d = \deg(F_1)$ and $r = \operatorname{rank}(F_1)$. By the definition of $\mathcal{F}$, we get that $\operatorname{Ker}(\varphi)$ and therefore F_1 is torsion-free and $r > 0$. As $i_*(F_1) \in \mathcal{F}$, we also have that $Cr + Dd \geq 0$.

Claim 3.95. $\phi(H_1) < \phi(i_*(F_1))$.

Proof. We consider two cases, the first one is $Cr_1 + Dd_1 = 0$ In this case $\phi(i_*(F_1)) = 1$. As $r_1 > 0$, it implies $\phi(H_1) < 1$. Therefore, we obtain

$$\phi(H_1) < \phi(i_*(F_1)).$$

The second case is $Cr_1 + Dd_1 > 0$. In this case

$$\frac{d}{r} < \frac{-Ad - Br}{Cr + Dd}$$

if and only if $Dd^2 + (A + C)dr + Br^2 < 0$. Due to the fact that $\Delta(M) < 0$ and $D, B < 0$, we obtain that

$$Dx^2 + (A + C)xy + By^2 < 0$$

for all $x, y \in \mathbb{R}$. By the correspondence between slope and phase, we obtain that

$$\phi(H_1) < \phi(i_*(F_1)).$$

$\square$

As a consequence, we have

$$\phi^+(j_*(C(\varphi))) - 1 = \phi(j_*(F_1)) < \phi(E)$$

as we wanted to prove. $\square$

We now use Serre duality to obtain the same results for the σ-stable objects $E \in \mathcal{H}_{23}[-1]$, $\mathcal{H}_{31}[-1]$.

Lemma 3.96. *Let* $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}_{31}[-1]$ *be* σ*-semistable object in* $\mathcal{A}$*. Then*

$$\operatorname{Hom}_{\mathcal{T}_C}(E, E[2]) = 0.$$

Proof. Let us consider the $\mathcal{S}_{\mathcal{T}_C}[-1] \in \operatorname{Aut}(\mathcal{T}_C)$. We compute $\beta = \mathcal{S}_{\mathcal{T}_C}[-1](\sigma) = (W', \mathcal{B}_2')$.

The Serre functor induces an isomorphism of the Grothendieck group given by

$$\begin{aligned} \mathcal{S}_{\mathcal{T}_C}[-1]\colon \mathbb{Z}_4 &\to \mathbb{Z}_4 \\ (r_1, d_1, r_2, d_2) &\mapsto (r_2, d_2, r_2 - r_1, d_2 - d_1) \end{aligned}$$

and

$$W'(r_1, d_1, r_2, d_2) = Z \circ (\mathcal{S}_{\mathcal{T}_C}[-1])^{-1}(r_1, d_1, r_2, d_2).$$

Therefore, we obtain

$$W'(r_1, d_1, r_2, d_2) = A(d_1 - d_2) + B(r_1 - r_2) - d_1 + i(D(d_1 - d_2) + C(r_1 - r_2) + r_1).$$

Note that β satisfies that $l_*(\mathbb{C}(x))$ is β-stable of phase one and $j_*(\mathbb{C}(x))$ and $i_*(\mathbb{C}(x))$ are β-stable. The shape of $\mathcal{B}_2'$ is given in Lemma 3.42. Moreover $\mathcal{S}_{\mathcal{T}_C}[-1](E) \in \mathrm{TCoh}(C)$. There is $g \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, such that $\beta' := \beta g = (Z', \mathcal{A}')$ satisfies that $j_*(\mathbb{C}(x))$ is β'-stable of phase one. We have that $\mathcal{S}_{\mathcal{T}_C}[-1](E) \in \mathcal{A}'$ or $\mathcal{S}_{\mathcal{T}_C}[-1](E) \in \mathcal{A}'[-1]$, because $\mathcal{S}_{\mathcal{T}_C}[-1](E)$ is β'-semistable and $\mathrm{TCoh}(C) \subseteq \mathcal{P}'(0, 2]$, where $\mathcal{P}'$ is the slicing induced by β'. Since $\mathcal{S}_{\mathcal{T}_C}[-1](E) \in \mathcal{B}_2'$, then by Lemma 3.89, Case 11, we obtain that $\mathcal{S}_{\mathcal{T}_C}[-1](E) \in \mathcal{A}'$. We compute Z' explicitly.

Let $g = (\begin{bmatrix} A & -B \\ D & -C \end{bmatrix}, f')$ with $f'(0) = \arg(A + iD)$.

Then, we obtain that $Z' = -M^{-1} \circ W' = -T \circ W'$. Therefore

$$Z'(r_1, d_1, r_2, d_2) = \frac{C + \alpha}{\alpha} d_1 + \frac{B}{\alpha} r_1 - d_2 + i(\frac{A - \alpha}{\alpha} r_1 + \frac{D}{\alpha} d_1 + r_2).$$

As a consequence, we obtain that β' is one of the stability conditions constructed in Lemma 3.13 with

$$M' = \begin{bmatrix} \frac{-C-\alpha}{\alpha} & \frac{B}{\alpha} \\ \frac{-D}{\alpha} & \frac{A-\alpha}{\alpha} \end{bmatrix},$$

where $\alpha = \det(M) > 0$. Note that $\Delta(M') = ((A + C)^2 - 4BD)\frac{1}{\alpha^2} < 0$.

Thus, we now can apply Lemma 3.94 to β' and $\mathcal{S}_{\mathcal{T}_C}[-1](E)$. We obtain that

$$\mathrm{Hom}_{\mathcal{T}_C}(\mathcal{S}_{\mathcal{T}_C}[-1](E), \mathcal{S}_{\mathcal{T}_C}[-1](E)[2]) = 0.$$

Since $\mathcal{S}_{\mathcal{T}_C}[-1]$ is an autoequivalence, we finally obtain

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = 0.$$

□

Lemma 3.97. *Let $E = E_1 \xrightarrow{\varphi} E_2 \in \mathcal{H}_{31}[-1]$ be a σ-semistable object in $\mathcal{A}$. Then*

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = 0.$$

Proof. The proof goes along the lines of Lemma 3.96. □

Corollary 3.98. *If E is σ-stable, then*

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = 0.$$

Proof. It follows directly from Lemma 3.89, Lemma 3.94, Lemma 3.96, and Lemma 3.97. □

Now we compute the Euler form for all σ-stable objects.

Lemma 3.99. *If E is a σ-stable object, then $-\chi(E, E) = d_2 r_1 - d_1 r_2 \geq 0$.*

Proof. The proof falls naturally into two cases:

Case 1: $[\varphi] = 0$. It implies that either $E_1 = 0$ or $E_2 = 0$. If not it would contradict that E is σ-stable. It clearly follows that $-\chi(E, E) = 0$.

Case 2: $[\varphi] \neq 0$. As $E \in \mathcal{A}$ and $\mathcal{A}$ is the heart of a bounded t-structure, we have that $\mathrm{Hom}_{\mathcal{T}_C}(E, E[n]) = 0$ for all $n < 0$. By Corollary 3.98 we have that $\mathrm{Hom}_{\mathcal{T}_C}(E, E[2]) = 0$ and by Lemma 2.78 we have that $\mathrm{TCoh}(C)$ has homological dimension 2, which implies, after applying the Serre functor, that $\mathcal{H}_{23}[-1]$ and $\mathcal{H}_{31}[-1]$ also have homological dimension 2. Therefore, it follows that $\mathrm{Hom}_{\mathcal{T}_C}(E, E[n]) = 0$ for $n \geq 2$. As a consequence, we obtain

$$-\chi(E, E) = -\hom_{\mathcal{T}_C}(E, E) + \hom_{\mathcal{T}_C}(E, E[1]).$$

As E is σ-stable, it follows that $-\hom_{\mathcal{T}_C}(E, E) = -1$. To prove our claim, it suffices to show that $\hom_{\mathcal{T}_C}(E, E[1]) > 0$.

By Serre duality

$$\mathrm{Hom}_{\mathcal{T}_C}(E, E[1]) = \mathrm{Hom}(E[1], \mathcal{S}_{\mathcal{T}_C}(E))^*$$

where $\mathcal{S}_{\mathcal{T}_C}(E) = E_2[1] \xrightarrow{i_E[1]} C(\varphi)[1]$. Then, it implies that there is a non-zero morphism $E \to \mathcal{S}_{\mathcal{T}_C}(E)[-1]$, given by

$$\begin{tikzcd} E_1 \arrow[r, "{[\varphi]}"] \arrow[d, "\varphi"'] & E_2 \arrow[d, "i_E"] \\ E_2 \arrow[r, "i_E"] & C(\varphi). \end{tikzcd} \tag{3.45}$$

As a consequence, we have $\hom_{\mathcal{T}_C}(E, \mathcal{S}_{\mathcal{T}_C}(E)) > 0$, and therefore $\hom_{\mathcal{T}_C}(E, E[1]) > 0$ and

$$-\chi(E, E) = d_2 r_1 - d_1 r_2 > 0.$$

□

Proposition 3.100. *Let $\sigma = (Z, \mathcal{A})$ be a pre-stability condition as in Lemma 3.14 with $\Delta(M) < 0$. Then it satisfies the support property and therefore it is a Bridgeland stability condition.*

Proof. We claim that σ satisfies the support property with respect to the following quadratic form

$$\begin{array}{rcl} Q\colon \mathbb{R}^4 & \to & \mathbb{R} \\ (r_1, d_1, r_2, d_2) & \mapsto & d_2 r_1 - d_1 r_2. \end{array}$$

We first show that Q is negative definite on

$$\operatorname{Ker}(Z) = \{(r_1, d_1, r_2, d_2) \mid d_2 = Ad_1 + Br_1 \text{ and } r_2 = -Cr_1 - Dd_1\}.$$

Let $(r_1, d_1, r_2, d_2) \in \operatorname{Ker}(Z)$, then

$$\begin{array}{rcl} Q(r_1, d_1, r_2, d_2) & = & (Ad_1 + Br_1)r_1 - d_1(-Cr_1 - Dd_1), \\ & = & Dd_1^2 + (A + C)d_1 r + Br_1^2 < 0 \end{array}$$

as $-1 < f(0) = r < 0$, we have that $1 < \phi_0 \leq 2$. Since $D = m\sin(\phi_0 \pi)$, with $m \in \mathbb{R}_{>0}$, then $D < 0$. Morever since $\Delta(M) = (A + C)^2 - 4BD < 0$, then $B < 0$ and $Dd_1^2 + (A + C)d_1 r_1 + Br_1^2 < 0$ for all $(r_1, d_1) \in \mathbb{R}^2$. Let $E = E_1 \xrightarrow{\varphi} E_2$ be a σ-semistable object. By [BMS16, Lem. A.6] it is enough to show that $Q(E) \geq 0$ for σ-stable objects. By Claim 3.99 we have that $d_2 r_1 - d_1 r_2 \geq 0$. □

Remark 3.101. For the support property for $g > 1$, we would need to prove Lemma 3.94. But it would not be enough as we cannot use directly the Euler form

$$-\chi(E, E) = d_2 r_1 - d_1 r_2 - (1 - g)(r_1^2 + r_2^2 - r_1 r_2),$$

because it is not negative definite on Ker_Z.

Conjecture 3.102. *Let $g > 1$ and σ as above, then the pre-stability condition σ satisfies the support property with respect to the quadratic form*

$$Q(r_1, d_1, r_2, d_2) = d_2 r_1 - d_1 r_2.$$

Theorem 3.103. *Let $g = 1$ and $\sigma \in \Theta_{12}$ be a pre-stability condition. Then it satisfies the support property and therefore it is a Bridgeland stability condition.*

Proof. If $\sigma \in \Theta_i$, it follows directly from Proposition 3.87 and if $\sigma \in \Gamma$, then it follows from

Proposition 3.100.

3.4 Topological description of S_{12}

It is now our purpose to study the topology of $\mathrm{Stab}(\mathcal{T}_C)$, we proceed by defining the following sets

$$\begin{aligned} S_{12} =& \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x)), i_*(\mathcal{O}_C), j_*(\mathcal{O}_C) \ \sigma\text{-stable for all closed points } x \in C\},\\ S_{23} =& \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid j_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), j_*(\mathcal{O}_C), l_*(\mathcal{O}_C) \ \sigma\text{-stable for all closed points } x \in C\},\\ S_{31} =& \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), l_*(\mathbb{C}(x)), i_*(\mathcal{O}_C), l_*(\mathcal{O}_C) \ \sigma\text{-stable for all closed points } x \in C\}. \end{aligned}$$

Throughout the whole section, we assume that every pre-stability $\sigma \in \Gamma$ is also a Bridgeland stability condition, i.e. it satisfies the support property. The aim of this section is to prove that S_{12} is an open, connected four dimensional complex manifold. The proof goes along the lines of [Bri08, Prop. 9.4]. It is based on the well-behaved wall and chamber decompositions of the space of stability conditions. See [BM11, Prop. 3.3] or [Bri08, Prop. 9.3]. We first proof that $S_{ij} \cap \mathrm{Stab}^\dagger(\mathcal{T}_C)$ is open in a connected component $\mathrm{Stab}^\dagger(\mathcal{T}_C)$ of $\mathrm{Stab}(\mathcal{T}_C)$ Afterwards, we prove that $\mathrm{Stab}(\mathcal{T}_C)$ is connected, thus the sets S_{ij} are open in $\mathrm{Stab}(\mathcal{T}_C)$.

Lemma 3.104. *The set $S_{12} \cap \mathrm{Stab}^\dagger(\mathcal{T}_C)$ is open in $\mathrm{Stab}^\dagger(\mathcal{T}_C)$.*

Proof. Let $S = \{i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x)), i_*(\mathcal{O}_C), j_*(\mathcal{O}_C) \mid x \in C\} \subseteq \mathcal{T}_C$. First note that the classes of $i_*(\mathbb{C}(x))$, $j_*(\mathbb{C}(x))$, and $i_*(\mathcal{O}_C)$ and $j_*(\mathcal{O}_C)$ in $K(\mathcal{T}_C)$ are primitive. By [BM11, Prop. 3.3], we have a well-behaved wall and chamber decomposition. We consider the set $\Theta^\dagger$ of points $\sigma \in \mathrm{Stab}^\dagger(\mathcal{T}_C)$ at which all objects of S are σ-stable. We now prove that $\Theta^\dagger$ is open. Let $B \subseteq \mathrm{Stab}^\dagger(\mathcal{T}_C)$ be a compact set, we show that

$$F = \{\sigma \in B \mid \text{ not every } E \in S \text{ is stable in } \sigma\}$$

is a closed set. As in [Bri08, Prop. 9.4] we show that $F = \cup_{j=0}^n \bar{C}_j$, where each C_j is a chamber in which some $E \in S$ is not stable.

Take a chamber $C \subseteq B$ in which some $E \in S$ is not stable, we want to prove that $\bar{C} \subseteq F$. By [BM11, Prop. 3.3] for every stability condition σ in the closure of C, the object E cannot be σ-stable then $\sigma \in F$.

Take now $\sigma \in F$. There is an object $E \in S$ which is not stable. We have two cases to consider: If E is stricly semistable, we can find stability conditions λ arbitrarily close to σ such that E is unstable. Then σ lies in the closure of a chamber where E is unstable. If E is not semistable, then there is an open neighbourhood of σ where E is not semistable and then it lies in a chamber where E is not stable. Therefore, we have $\Theta^\dagger$ is open. □

We now introduce the following notation:

Let $g_1, g_2 \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$, with $g_1 = (T_1, f_1)$ and $g_2 = (T_2, f_2)$. We denote by

$$M_i = T_i^{-1} = \begin{bmatrix} -A_i & B_i \\ -D_i & C_i \end{bmatrix} \text{ for } i = 1, 2.$$

Let us consider the subset

$$\mathcal{P}_{12} = \{(\sigma_1, \sigma_2) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})^2 \mid \phi_0 < \phi_2 + 1, \phi_1 < \phi_3 + 1 \text{ and if } \phi_0 > \phi_2, \text{ then } \det(M_1 + M_2) > 0\},$$

where $f_1(0) = n + \theta_1$, $f_2(0) = m + \theta_2$ where $n, m \in \mathbb{Z}$ and $\theta_1, \theta_2 \in [0, 1)$, where $\rho(\sigma_1) = (m_0, m_1, \phi_0, \phi_1)$ and $\rho(\sigma_2) = (m_2, m_3, \phi_2, \phi_3)$. As we explained in Lemma 1.85.

Remark 3.105. For $\sigma \in S_{12}$, we consider

$$\begin{aligned} \phi_0 &= \phi(i_*(\mathbb{C}(x))) \quad \text{and} \quad \phi_1 = \phi(i_*(\mathcal{O}_C)) \\ \phi_2 &= \phi(j_*(\mathbb{C}(x))) \quad \text{and} \quad \phi_3 = \phi(j_*(\mathcal{O}_C)). \end{aligned} \tag{3.46}$$

Lemma 3.106. *For every $\sigma \in S_{12}$, we have that*

$$\phi_1 < \phi_0 < \phi_1 + 1 \text{ and } \phi_3 < \phi_2 < \phi_3 + 1.$$

Proof. It follows directly from the stability of $i_*(\mathbb{C}(x)), i_*(\mathcal{O}_C)$ and $j_*(\mathbb{C}(x)), j_*(\mathcal{O}_C)$. □

Since every $\sigma \in S_{12}$ satisfies

$$\phi_1 < \phi_0 < \phi_1 + 1 \text{ and } \phi_3 < \phi_2 < \phi_3 + 1,$$

then by Lemma 1.85 for $(m_0, m_1, \phi_0, \phi_1)$, where

$$m_0 = \mid Z(i_*(\mathbb{C}(x))) \mid \text{ and } m_1 = \mid Z(i_*(\mathcal{O}_C)) \mid$$

and for $(m_2, m_3, \phi_2, \phi_3)$, where

$$m_2 = \mid Z(j_*(\mathbb{C}(x))) \mid \text{ and } m_3 = \mid Z(j_*(\mathcal{O}_C)) \mid$$

we obtain two stability conditions $\sigma_1 = (Z_1, \mathcal{A}_1) = (T_1, f_1)$ and

$$\sigma_2 = (Z_2, \mathcal{A}_2) = (T_2, f_2) \in \mathrm{Stab}(C)$$

respectively. We define $\pi(\sigma) = (\sigma_1, \sigma_2)$.

We define the following map

$$\begin{aligned} \pi\colon \quad S_{12} &\to \mathcal{P}_{12} \\ \sigma &\mapsto (\sigma_1, \sigma_2). \end{aligned} \tag{3.47}$$

Note that

$$Z\big|_{D_1} = Z_1 \text{ and } Z\big|_{D_2} = Z_2,$$

as m_0 and ϕ_0 characterizes $Z(i_*(\mathbb{C}(x)))$ and $Z_1(\mathbb{C}(x))$. Analogously, it works for the remaining objects.

Remark 3.107. Note that $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ acts freely on $\mathcal{P}_{12}$. It is enough to check that $(\sigma_1 g, \sigma_2 g) \in \mathcal{P}_{12}$, where $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $(\sigma_1, \sigma_2) \in \mathcal{P}_{12}$. We define $\sigma_i' = (T_i', f_i') = \sigma_i g$, for $i = 1, 2$. Let $f_1'^{-1}(1) = \phi_0'$ and $f_2'^{-1}(1) = \phi_2'$. Since $\phi_0 > \phi_2 + 1$ and by definition of the action we have that $f(\phi_0') = \phi_0$ and $f(\phi_2') = \phi_2$, then $\phi_0' > \phi_2'$, as f^{-1} is an increasing continuous function. We analogously prove that $\phi_1' < \phi_3' + 1$. We now have $T_i' = T_i \circ T$ for $i = 1, 2$, and

$$\det(T^{-1} \circ M_1 + T^{-1} \circ M_2) = \det(T^{-1}) \det(M_1 + M_2) > 0.$$

Then, we get that $(\sigma_1 g, \sigma_2 g) \in \mathcal{P}_{12}$.

Lemma 3.108. *The map π is well-defined, continuous, open and $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-equivariant.*

Proof. First note that $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ also acts freely on S_{12}. As π is defined in terms of the slicing we clearly have a continuous map from S_{12} to $\widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \times \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$. We now show that π is also $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-equivariant. Indeed, let $g = (T, f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ and $\sigma = (Z, \mathcal{A}) \in S_{12}$. We show that $\pi(\sigma g) = (\sigma_1 g, \sigma_2 g)$.

Let $\sigma' = \sigma g$. By definition of the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action,

$$\phi_{\sigma'}(i_*(\mathbb{C}(x))) = f^{-1}(\phi_0) \text{ and } \phi_{\sigma'}(i_*(\mathcal{O}_C)) = f^{-1}(\phi_1).$$

As $i_*(\mathbb{C}(x)) \in \mathcal{A}[n]$, for some $n \in \mathbb{Z}$, then $m_0' =\mid Z'(i_*(\mathbb{C}(x)) \mid$ and $m_1' =\mid Z'(i_*(\mathcal{O}_C)) \mid$. Under the isomorphism given in Lemma 1.85 the preimage of $(m_0', m_1', f^{-1}(\phi_0), f^{-1}(\phi_1))$ is precisely $\sigma_1 g$. We prove analogously that the preimage of $(m_2', m_3', f^{-1}(\phi_2), f^{-1}(\phi_3))$ is σ_2.

We now show that $(\sigma_1, \sigma_2) \in \mathcal{P}_{12}$. First we prove that $n - m \geq -1$. Since $i_*(\mathbb{C}(x)), j_*(\mathbb{C}(x))$ are stable and we have a non-zero morphism $i_*(\mathbb{C}(x)) \to j_*(\mathbb{C}(x))[1]$, it follows that $\phi_0 - \phi_2 < 1$.

If $\phi_0 > \phi_2$, then by Lemma 3.27 we get that $l_*(\mathbb{C}(x))$ is stable. We show now that in this case $\det(M_1 + M_2) > 0$. By Proposition 3.44 and the analogous proposition for Θ_{23} and Θ_{31}, we obtain that there is $g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that by acting by g we obtain a stability condition $\sigma^{'} = \sigma g$ such that $\pi(\sigma^{'}) = (\sigma_1 g, \sigma_\mu)$. Let $\sigma_1 g = (T^{'}, f^{'})$ and $M^{'} = T^{'^{-1}}$. By Lemma 3.53, we have $\det(M^{'} + I) > 0$.

Note that $M^{'} = M_2^{-1} M_1$, therefore

$$0 < \det(M_2^{-1} M_1 + I) = \det(M_2^{-1}) \det(M_1 + M_2).$$

As $\phi_3 < \phi_2 < \phi_3 + 1$, we obtain $\det(M_2) > 0$. It implies $\det(M_1 + M_2) > 0$. Moreover, as $i_*(\mathcal{O}_C)$ and $j_*(\mathcal{O}_C)$ are σ-stable and there is a non-zero morphism $i_*(\mathcal{O}_C) \to j_*(\mathcal{O}_C)[1]$, it directly implies that $\phi_1 < \phi_3 + 1$. Consequently, we obtain that π is well defined.

We now show that π is a local homeomorphism. We have already that

$$\pi' \colon S_{12} \to \mathrm{GL}^+(2, \mathbb{R})^2$$

is a local homeomorphism, where π' maps each stability condition to its stability function. We consider the following covering

$$\begin{array}{rcl} p \colon \widetilde{\mathrm{GL}}^+(2, \mathbb{R})^2 & \to & \mathrm{GL}^+(2, \mathbb{R})^2 \\ (\sigma_1, \sigma_2) & \mapsto & (Z_1, Z_2). \end{array}$$

By Theorem 1.66 for every $\sigma \in S_{12}$ there is an open set U_σ, such that $\pi' \colon U_\sigma \to \pi'(U_\sigma)$ is a covering. As $p \colon p^{-1}(\pi'(U_\sigma)) \to \pi'(U_\sigma)$ is also a covering and $p \circ \pi = \pi'$, then $\pi \colon U_\sigma \to p^{-1}(\pi'(U_\sigma))$ has to be a covering and as a consequence a local homeomorphism. As $\mathcal{P}_{12}$ is an open subset of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})^2$, then we get that π is also a local homeomorphism. □

In order to prove that the map π is in fact a homeomorphism, we study the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on S_{12}. As $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ acts freely on S_{12}, we define a section of the action

$$\mathcal{V}_{12} = \{\sigma \in S_{12} \mid \pi(\sigma) = (\sigma_1, \sigma_2) \text{ such that } \sigma_2 = \sigma_\mu\}$$

Claim 3.109. *If $\sigma \in \mathcal{V}_{12}$ and $\pi(\sigma) = (\sigma_1, \sigma_\mu)$ with $0 \leq f(0)$, then $\sigma \in \Theta_{12}$.*

Proof. Indeed, if $l_*(\mathbb{C}(x))$ is not σ-stable, then by Theorem 3.30 we have $\sigma \in \Theta_{12}$. We assume that $l_*(\mathbb{C}(x))$ is σ-stable, since $i_*(\mathbb{C}(x))$ and $j_*(\mathbb{C}(x))$ are also σ-stable, we obtain that $\phi_2 < \phi_4 < \phi_0 < \phi_2 + 1$. By the definition of σ, we have $\phi_2 = 1$, then $1 < \phi_0 < 2$ and it follows that $n = -1$ and $f(0) < 0$, which is a contradiction. □

Claim 3.110. *If $\sigma \in \mathcal{V}_{12}$ and $\pi(\sigma) = (\sigma_1, \sigma_\mu)$, with $\sigma_1 = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ and $0 > f(0) > -1$, then $l_*(\mathbb{C}(x))$ is σ-stable.*

Proof. If $l_*(\mathbb{C}(x))$ is not σ-stable, then by Remark 3.24 we have $\phi_0 - \phi_2 \leq 0$. Since $\phi_2 = 1$, we obtain $\phi_0 \leq 1$. If n is the integer part of $f(0)$, then $-n < \phi_0 \leq 1$, and it follows $n \geq 0$, which is a contradiction. $\square$

The image of $\mathcal{V}_{12}$ under π is contained in

$$\mathcal{L}_{12} = \{(\sigma, \sigma_\mu) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})^2 \mid f(0) > -1,\ 3/2 > f^{-1}(1/2) \text{ and if} f(0) < 0 \text{ then } \det(M+I) > 0\}.$$

Abusing the notation we see $\mathcal{L}_{12}$ as a subset of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

Claim 3.111. *The subset $\mathcal{L}_{12} \subset \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ is open and connected.*

Proof. We define

$$U_1 = \{g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \mid f(0) > 0\}$$

and

$$U_2 = \{g \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \mid 1/2 > f(0) > -1\ ,\ \det(M+I) > 0 \text{ and } f^{-1}(1/2) < 3/2\},$$

which are open sets in $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ as $\det\colon \widetilde{\mathrm{GL}}^+(2,\mathbb{R}) \to \mathbb{R}_{>0}$ is a continuous function. We obtain $\mathcal{L}_{12} = U_1 \cup U_2$. Indeed, $U_1 \cup U_2 \subseteq \mathcal{L}_{12}$. Now if $g \in \mathcal{L}_{12}$, the only case that is not trivial is if $f(0) = 0$. In this case $-A, C \in \mathbb{R}_{>0}$. It follows that $\mathrm{Tr}(M) \geq 0$. As a consequence, we get $\det(M+I) > 0$ and $g \in U_2$, hence that $\mathcal{L}_{12}$ is an open subset of $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

We define $Y \subseteq \mathbb{R}^4$ as follows: We say that $(m_0, m_1, \phi_0, \phi_1) \in Y$ if

$$m_i > 0,\ \phi_0 < 2\ ,\ \phi_1 < \frac{3}{2}\ ,\ \phi_1 < \phi_0 < \phi_1 + 1$$

and if

$$1 \leq \phi_0 < 2 \text{ and } 0 < \phi_1 < \frac{3}{2}, \text{ then } \delta(m_0, m_1, \phi_0, \phi_1) > -1,$$

where

$$\begin{aligned} \delta\colon \mathbb{R}_{>0} \times \mathbb{R}_{>0} \times (1,2) \times (0, \tfrac{3}{2}) &\to \mathbb{R} \\ (m_0, m_1, \phi_0, \phi_1) &\mapsto m_0 m_1 \sin((\phi_0 - \phi_1)\pi) - m_0 \cos(\phi_0 \pi) + m_1 \sin(\phi_1 \pi). \end{aligned} \tag{3.48}$$

Note that Y is connected, because although f is defined in terms of trigonometric functions, it is restricted to intervals where it behaves well.

Claim 3.112. *The following map*

$$\begin{aligned} \rho\colon \mathcal{L}_{12} &\to Y \\ (T, f) &\mapsto (m_0, m_1, f^{-1}(1), f^{-1}(\tfrac{1}{2})) \end{aligned} \tag{3.49}$$

where $m_0 = |A + Di|, m_1 = |B + Ci|$ is a homeomorphism.

Proof. Since we defined ρ in the same way as the homeomorphism given in Lemma 1.85, in order to show that ρ is a homeomorphism, it suffices to show that Y is the image of $\mathcal{L}_{12}$ under the map defined in Lemma 1.85. First of all we prove that it is well-defined. Let $\sigma_1 \in \mathcal{L}_{12}$. Clearly $m_0, m_1 > 0$. Since $f(0) > -1$ and f^{-1} is an increasing map, as a consequence ß0 $> f^{-1}(-1)$ and we have

$$2 > f^{-1}(1).$$

It also implies

$$f^{-1}(1/2) < f^{-1}(1) < 2 \text{ and } f^{-1}(1/2) < f^{-1}(1) < f^{-1}(3/2) = f^{-1}(1/2) + 1.$$

By assumption $f^{-1}(1/2) < 3/2$.

If $f(0) \geq 0$, then $1 \leq f^{-1}(1)$. We just need to consider the case $-1 < f(0) < 0$. In this case $1 < f^{-1}(1) < 2$ and $0 < f^{-1}(1/2) < 3/2$. By the correspondence between f^{-1} and M, we have that $\det(M+I) > 0$ if and only if $\delta(m_0, m_1, f^{-1}(1), f^{-1}(1/2)) > -1$. Indeed, since

$$A + Di = m_0(\cos(f^{-1}(1)\pi) + i\sin(f^{-1}(1)\pi) \tag{3.50}$$

and

$$B + Ci = m_1(\cos(f^{-1}(1/2)\pi) + i\sin(f^{-1}(1/2)\pi),$$

we obtain

$$\det(M+I) = \delta(m_0, m_1, f^{-1}(1), f^{-1}(1/2)) + 1.$$

It follows that $\mathcal{L}_{12}$ is connected. □

□

Proposition 3.113. *The map*

$$\begin{aligned} \pi\colon \quad \mathcal{V}_{12} &\to \mathcal{L}_{12} \\ \sigma &\mapsto \sigma_1, \end{aligned} \tag{3.51}$$

is a homeomorphism.

Proof. First we prove that π is injective. Let $\sigma = (Z, \mathcal{A})$, $\tau = (W, \mathcal{B}) \in \mathcal{V}_{12}$, such that $\pi(\sigma) = \pi(\tau) = \sigma_1$. If $0 \leq f(0)$, by Claim 3.109, we obtain $\sigma, \tau \in \Theta_{12}$. By Lemma 3.49, the stability conditions are completely characterized by σ_1 and $\sigma_2 = \sigma_\mu$ i.e. $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_2) = \tau$.

If $0 > f(0) > -1$, by Claim 3.110, we have that $l_*(\mathbb{C}(x))$ is stable. In Lemma 3.53 we described this type of stability conditions and its hearts. As by definition $Z = W$ and $d(\mathcal{P}, \mathcal{Q}) < 1$, where $\mathcal{P}$ and $\mathcal{Q}$ are the slicing of σ and τ respectively, then $\sigma = \tau$.

Thus, by Lemma 3.108 we already have a homeomorphism onto the image of $\mathcal{V}_{12}$. We now prove that it is in fact onto. By Lemma 3.104 and Lemma 3.108 the image of $\mathcal{V}_{12}$ is open. Since $\mathcal{L}_{12}$ is also connected, it is enough to prove that $\pi(\mathcal{V}_{12})$ is closed. Moreover, it contains a dense subset as the image of the discrete stability conditions constructed in Section 2.2 and Lemma 3.14. We prove it by contradiction. Assume that $\pi(\mathcal{V}_{12})$ is not close and let us take a $\tau_1 = (Z, \mathcal{A})$ in the boundary of $\pi(\mathcal{V}_{12})$ which does not belong to the image. Note that there is $\tau^{'} = (Z^{'}, \mathcal{A}^{'}) = \pi(\sigma^{'}) \in \pi(\mathcal{V}_{12})$, where $\sigma' = (W', \mathcal{B}')$, sufficiently close to τ_1 such for $W = Z(r_1, d_1) + Z_\mu(r_2, d_2)$, we have that $Q_{W'}$ restricted to $\operatorname{Ker} W$ is negative definite, where $Q_{W'}$ is the quadratic form given by the support property satisfied by σ'.

Consider the open subset of $\operatorname{Hom}(\mathbb{Z}^4, \mathbb{C})$, consisting in homomorphisms whose kernel is negative definite with respect to $Q_{W'}$ and let U be the connected component containing W'. Then as in [BMS16, Proposition A.5], there is a continuos function $C\colon U \to \mathbb{R}_{>0}$ such that $C(Y) \in \mathbb{R}_{>0}$ satisfies that $||v||C(Y) < |Y(v)|$ for $v \in \mathbb{R}^4$ with $Q_{W'}(v) \geq 0$, an appropiate norm in $\mathbb{R}^4$ and $|-|$ the Euclidean norm in $\mathbb{C}$. Then there is $0 < \epsilon < \frac{1}{8}$, such that $|W - W^{'}|_\infty \leq \sin(\pi\epsilon)C(W')$. As a consequence we have that $|W - W^{'}|_\infty < \sin(\epsilon\pi)\frac{|W'(E)|}{||[E]||}$, which implies $||W - W'||_{\sigma'} < \sin(\pi\epsilon)$.

By Bridgeland's deformation Theorem 1.65, there is a stability condition $\sigma = (W, \mathcal{B})$ in the neighbourhood of $\sigma^{'}$. This also implies that σ belongs to

$$\overline{S_{12}} \coloneqq \{\sigma \in \operatorname{Stab}(\mathcal{T}_C) : j_*(\mathbb{C}(x)), i_*(\mathbb{C}(x)), j_*(\mathcal{O}_C), i_*(\mathcal{O}_C) \text{ are } \sigma\text{-semistable}\}.$$

We now show that σ is in fact in S_{12}. It is possible to choose $\tau^{'}$ appropriately, such that σ is in a desired wall. We now assume the object $j_*(\mathbb{C}(x))$ is σ-semistable but not stable and $j_*(\mathbb{C}(x)) \in \mathcal{P}_\sigma(1)$. By Lemma 3.27 the Jordan-Hölder filtration is given by

$$i_*(\mathbb{C}(x))[-1] \to j_*(\mathbb{C}(x)) \to l_*(\mathbb{C}(x)),$$

all with the same phase. Therefore, we obtain $i_*(\mathbb{C}(x))[-1] \in \mathcal{P}_\sigma(1)$. It implies that $W\big|_{D_1} = Z = Ad + Br + iCr$. It follows that τ_1 is a stability condition given by $(Z, \operatorname{Coh}(C)[n])$, for $n \in Z$. By the definition of $\mathcal{L}_{12}$ we have that $f(0) = n \geq 0$. As a consequence τ_1 and σ_μ satisfy the CP-gluing conditions. Moreover $\sigma_1 = \mathrm{gl}_{12}(\tau_1, \sigma_\mu)$ is a discrete stability condition, by Corollary 2.90 and by Lemma 3.83 a Bridgeland stability condition in $\mathcal{V}_{12}$ satisfying $\pi(\sigma_1) = \tau_1$. We obtain a contradiction.

We now choose $\tau^{'}$ such that σ satisfies that $i_*(\mathbb{C}(x))$ is semistable, but not stable. By Lemma 3.27 the Jordan-Hölder filtration is given by

$$l_*(\mathbb{C}(x)) \to i_*(\mathbb{C}(x)) \to j_*(\mathbb{C}(x))[1],$$

all with the same phase. By continuity $j_*(\mathbb{C}(x)) \in \mathcal{P}(1)$, as a consequence $i_*(\mathbb{C}(x)) \in \mathcal{P}(2)$. It implies τ_1 is a stability condition over a curve is given by $(Z, \operatorname{Coh}(C)[n])$, with $n \geq 0$, which gives us a contradiction.

We now choose $\tau^{'}$ such that σ satisfies that $i_*(\mathcal{O}_C)$ is semistable, but not stable. By Lemma 3.27 the Jordan-Hölder filtration is given by

$$l_*(\mathcal{O}_C) \to i_*(\mathcal{O}_C) \to j_*(\mathcal{O}_C))[1],$$

all with the same phase. By continuity $j_*(\mathcal{O}_C) \in \mathcal{P}(1/2)$, as a consequence $i_*(\mathcal{O}_C) \in \mathcal{P}(3/2)$. It implies that $W|_{D_1} = Z = Ad + i(Cr + Dd)$ with $C < 0$. We go back to $\tau_1 = (T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. If $f(0) \geq 0$, the CP-gluing conditions are satisfied and $\sigma_1 = \mathrm{gl}_{12}(\tau_1, \sigma_\mu)$ is a CP-glued pair. As $B = 0$, the conditions of Proposition 2.94 are also satisfied and σ_1 is a stability condition. Therefore, $\pi(\sigma_1) = \tau_1$ and we obtain a contradiction. If $0 > f(0) > -1$, the fact that $\phi(i_*(\mathcal{O}_C)) = 3/2$, contradicts $f^{-1}(\frac{1}{2}) < \frac{3}{2}$.

We now choose $\tau^{'}$ such that σ satisfies that $j_*(\mathcal{O}_C)$ is semistable, but not stable. By Lemma 3.27 the Jordan-Hölder filtration is given by

$$i_*(\mathcal{O}_C)[-1] \to j_*(\mathcal{O}_C) \to l_*(\mathcal{O}_C)),$$

all with the same phase. By continuity $j_*(\mathcal{O}_C) \in \mathcal{P}(1/2)$, as a consequence $i_*(\mathcal{O}_C)[-1] \in \mathcal{P}(1/2)$ and we conclude the proof along the lines of the last paragraph.

We finally obtain $\pi(\mathcal{V}_{12}) = \mathcal{L}_{12}$. □

Corollary 3.114. *The map*

$$\begin{aligned} \pi\colon \quad S_{12} &\to \mathcal{P}_{12} \\ \sigma &\mapsto (\sigma_1, \sigma_2) \end{aligned} \tag{3.52}$$

is a homeomorphism.

We can finally prove the Harder-Narasimhan property for non-discrete stability conditions.

Proposition 3.115. *The pairs $\sigma = (Z, \mathcal{A})$ constructed in Remark 2.79 with the semiorthogonal decomposition $\langle D_1, D_2\rangle$ and the pairs $\sigma = (Z_r, \mathcal{A}_r)$ given in Lemma 3.12 with $f^{-1}(\frac{1}{2}) < \frac{3}{2}$ are Bridgeland stability conditions.*

Proof. Whenever $\sigma_1 = (T_1, f_1) = (Z_1, \mathcal{A}_1) \in \mathrm{Stab}(C)$ and σ_2 are discrete we have already shown that it gives us a Bridgeland stability condition. It is enough to show it whenever $\sigma_2 = \sigma_\mu$. Let $\sigma = \mathrm{gl}_{12}(\sigma_1, \sigma_\mu)$ be a CP-glued pair. As $f_1(0) \geq 0$, we get $(\sigma_1, \sigma_\mu) \in \mathcal{L}_{12}$. Therefore, there is a stability condition $\tau \in \mathcal{V}_{12}$ such that $\pi(\tau) = (\sigma_1, \sigma_2)$. By Claim 3.109, we obtain $\tau \in \Theta_{12}$ and by Lemma 3.49, we have that $\tau = \mathrm{gl}_{12}(\sigma_1, \sigma_2) = (Z, \mathcal{A})$. As a consequence the pair $(Z, \mathcal{A})$ gives a Bridgeland stability condition.

Let $\sigma = (Z_r, \mathcal{A}_r)$. If we consider σ_1 as in Lemma 3.12 with $f^{-1}(\frac{1}{2}) < \frac{3}{2}$. We get that $-1 < f_1(0) < 0$ and by hypothesis $\det(M_1 + I) > 0$, we have that $(\sigma_1, \sigma_\mu) \in \mathcal{L}_{12}$. As a consequence, there is a stability condition $\tau \in \mathcal{V}_{12}$ such that $\pi(\tau) = (\sigma_1, \sigma_2)$. By Claim 3.110, we obtain that $l_*(\mathbb{C}(x))$ is τ-stable. Therefore by Lemma 3.53, we have that τ is given precisely by the construction in

Remark 3.116. If $\sigma = (Z_r, \mathcal{A}_r)$ is a pre-stability condition given in Lemma 3.12 with $f^{-1}(\frac{1}{2}) \geq \frac{3}{2}$, then either $i_*(\mathcal{O}_C)$ is not stable or $j_*(\mathcal{O}_C)$ is not stable. Then σ is in S_{23} or in S_{31}. Precisely, by Lemma 3.34, Lemma 3.36 and Lemma 3.37 if $\phi_5 > \frac{3}{2}$ then $j_*(\mathcal{O}_C)$ and $l_*(\mathcal{O}_C)$ are σ-stable, and if $\phi_5 < 1/2$ then $l_*(\mathcal{O}_C)$ and $i_*(\mathcal{O}_C)$ are σ-stable. As a consequence, all the already constructed pairs in Lemma 3.12 are Bridgeland stability conditions.

Theorem 3.117. *The space of stability conditions* $\mathrm{Stab}(\mathcal{T}_C) = S_{12} \cup S_{23} \cup S_{31}$ *is a connected, four dimensional complex manifold.*

Proof. Since $\mathcal{V}_{12}$ is connected, it implies that S_{12} is also connected. Moreover $S_{12} \cap S_{23} = S_{23} \cap S_{31} = S_{12} \cap S_{31}$ is not empty, so $\mathrm{Stab}(\mathcal{T}_C)$ is connected. □

Zusammenfassung

In Kapitel 1 geben wir eine Einführung in grundlegende Konzepte. Wir untersuchen verschiedene Eigenschaften einer triangulierten Kategorie $\mathcal{D}$ und abelscher Unterkategorien von $\mathcal{D}$ mit Hilfe von T-Strukturen und Torsionspaaren. Als nächstes geben wir einen Überblick über die wichtigen Konzepte der semiorthogonalen Zerlegung und des Serre-Funktors. Das Hauptaugenmerk von Abschnitt 1.2 liegt auf einer umfassenden Diskussion der Bridgeland-Stabilitätsbedingungen. In Kapitel 2 studieren wir die Stabilitätsmannigfaltigkeit $\mathrm{Stab}(D^b(Q_{\mathcal{A},n}))$. In Abschnitt 2.1 beschäftigen wir uns mit der beschränkten abgeleiteten Kategorie der Darstellungen des n-Kronecker-Köchers über einer abelschen Kategorie $\mathcal{T}_{\mathcal{A},n} := D^b(Q_{\mathcal{A},n})$ und konstruieren verschiedene semiorthogonale Zerlegungen von $\mathcal{T}_{\mathcal{A},n}$. Wir beweisen die Existenz des Serre-Funktors und geben eine explizite Beschreibung für den Fall $n = 1$. In Abschnitt 2.2 konstruieren wir Prestabilitäts-Bedingungen auf $\mathcal{T}_{\mathcal{A},n}$ und geben konkrete Beispiele von CP-verklebten Prestabilitäts-Bedingungen auf der Kategorie $\mathcal{T}_{\mathrm{Coh}(X)}$, wobei hier X eine glatte projektive Kurve, Fläche oder 3-Mannigfaltigkeit bezeichnet. In Kapitel 3 untersuchen wir $\mathcal{T}_{\mathrm{Coh}(C)}$, die Kategorie der holomorphen Tripel über einer glatten projektiven Kurve C über $\mathbb{C}$ mit Geschlecht $g(C) \geq 1$.Das Ziel dieses Kapitels ist es eine vollständige Beschreibung der Stabilitätsmannigfaltigkeit $\mathrm{Stab}(\mathcal{T}_C)$ zu geben. Darüber hinaus beweisen wir das alle CP-verklebten Paare σ sogar Bridgeland-Stabilitätsbedingungen sind. Um $\mathrm{Stab}(\mathcal{T}_C)$ zu beschreiben folgen wir der Konstruktion in [Bri08]. In Abschnitt 3.1 konstruieren wir zusätzliche Paare mit Hilfe von Kipp-Theorie und erhalten als Konsequenz diskrete Prestabilitäts-Bedingungen. In Abschnitt 3.2 zeigen wir, dass alle Bridgeland Stabilitätsbedingungen in $\mathrm{Stab}(\mathcal{T}_C)$ durch bereits konstruierte Paare gegeben sind, entweder durch CP-kleben oder kippen. InAbschnitt 3.3 beweisen wir die Unterstützungs-Eigenschaft und schlussendlich in benutzen wir in Abschnitt 3.4 Bridgelands Deformationsresultat um eine topologische Beschreibung der Stabilitätsmannigfaltigkeit zu erhalten und die HN-Eigenschaft auf den nicht-diskreten Fall zu übertragen. Dieses Kapitel erscheint in [MRRHR19] als kollaborative Arbeit mit Eva Martínez Romero und Arne Rüffer.

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