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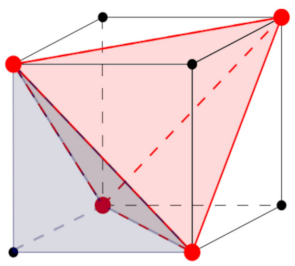
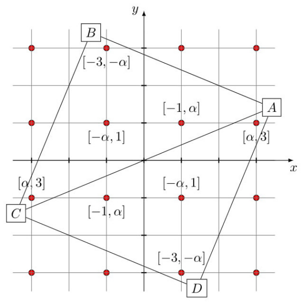
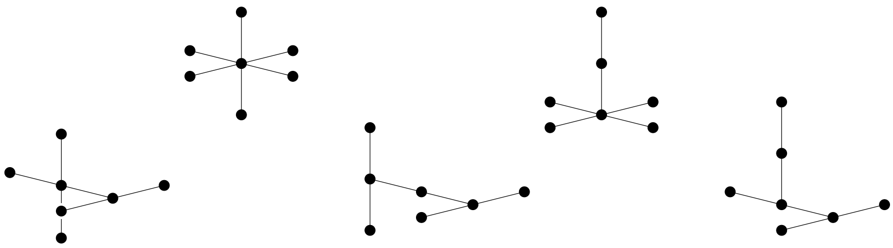
Covering properties of lattice polytopes







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Summary

This thesis studies problems concerning the interaction between polytopes and lattices. Motivation for the study of lattice polytopes comes from two very different fields: discrete optimization, in particular integer linear programming, and algebraic geometry, specifically the study of toric varieties.

The first topic we study is the existence of unimodular covers for certain interesting families of 3-dimensional lattice polytopes. A unimodular cover of a lattice polytope is a collection of unimodular simplices whose union equals the polytope. Admitting a unimodular cover is a weaker property than admitting a unimodular triangulation, and stronger than having the integer decomposition property (IDP). This last property is particularly interesting in the algebraic context, and there are various conjectures relating it to smoothness ([Oda97]). We show that unimodular covers exist for all 3-dimensional parallelepipeds and for all Cayley sums of polygons where one polygon is a weak Minkowski summand of the other. For both classes of polytopes only the IDP property was previously known.

We then explore questions related to the so-called flatness constant, the largest width that a hollow convex body can have in a given dimension. A hollow convex body is one that contains no lattice points in its interior. The flatness constant was shown to be finite in work of Kinchin ([Khi48]), and upper bounds for it have been well studied, among other reasons because it played a crucial role in the first polynomial time algorithm for integer linear programs in fixed dimension ([Len83]). In this thesis, we focus on lower bounds for the flatness constant, and for some specializations of it. In particular, we construct a wide hollow tetrahedron, and conjecture that in dimension three there are no hollow convex bodies of larger width. As evidence for this conjecture, we can show that a local version of it: every small perturbation of our tetrahedron that maintains hollowness decreases in width. We further construct the first known examples of lattice polytopes of width larger than their dimensions. We then exploit these explicit constructions to obtain asymptotic lower bounds for the flatness constant.

Finally, we study a somehow opposite question: how small can the covering radius of a non-hollow lattice polytope be. We conjecture that a certain explicit family of polytopes achieves the minimum covering radius, and show that this conjecture can be translated into the language of covering minima, introduced by Kannan and Lovász ([KL88]) with the purpose of finding upper bounds for the flatness constant mentioned above. From our investigation of this conjecture, a natural definition of a discrete surface area emerges. We then formulate a stronger conjecture, which proposes an upper bound for the covering radius of simplices in terms of the newly defined discrete surface area and the normalized volume.

Zusammenfassung

Diese Disseration beschäftigt sich mit dem Zusammenspiel von Polytopen und Gittern. Die Motivation für die Untersuchung von Gitterpolytopen kommt dabei aus zwei sehr unterschiedlichen Gebieten: der diskreten Optimierung, insbesondere der ganzzahligen linearen Optimierung und der algebraischen Geometrie, genauer dem Studium von torischen Varietäten.

Das erste Thema, das wir untersuchen, ist die Existenz von unimodularen Überdeckungen für bestimmte Familien von 3-dimensionalen Gitterpolytopen. Eine unimodulare Überdeckung eines Gitterpolytopes ist eine Menge, bestehend aus unimodularen Simplizes, deren Vereinigung gleich dem Polytop ist. Die Eigenschaft eine solche unimodulare Überdeckung zu besitzen ist schwächer, als eine unimodulare Triangulierung zu besitzen und impliziert die Integer Decomposition Property (IDP). Letztere Eigenschaft ist von besonderem Interesse hinsichtlich ihrer algebraischen Bedeutung und es gibt verschiedene Vermutungen, die diese Eigenschaft mit dem Konzept der Glattheit eines Gitterpolytopes verbinden ([Oda97]). Wir zeigen, dass für alle 3-dimensionalen Parallelepipede und alle Cayley-Summen von Polygonen, von denen das Eine ein schwacher Minkowski-Summand des Anderen ist, eine unimodulare Überdeckung existiert. Bisher war nur bekannt, dass diese beiden Klassen die IDP-Eigenschaft erfüllen.

Als Nächstes widmen wir uns verschiedenen Fragen die sogenannte Flatness-Konstante betreffend. Dies ist die größte Gitterweite, die ein hohler konvexer Körper in gegebener Dimension annehmen kann. Hierbei ist ein hohler konvexer Körper ein konvexer Körper, der keine Gitterpunkte in seinem Inneren hat. In einer Arbeit von Kinchin ([Khi48]) wurde die Endlichkeit dieser Konstanten gezeigt. Darüber hinaus gab es viele Untersuchungen hinsichtlich oberer Schranken, unter anderem, da die Flatness-Konstante eine entscheidende Rolle in dem ersten Algorithmus zur Lösung ganzzahliger linearer Optimierungsprobleme in fester Dimension in polynomieller Zeit spielt ([Len83]). In dieser Arbeit konzentrieren wir uns auf untere Schranken der Flatness-Konstanten und auf einige ihrer Spezialisierungen. Wir konstruieren einen breiten hohlen Tetraeder, von dem wir vermuten, dass es keinen hohlen konvexen Körper mit größerer Weite in Dimension 3 gibt. Als Indiz für Richtigkeit dieser Vermutung können wir eine lokale Version beweisen: jede kleine lokale Modifikation dieses Tetraeders, die die Eigenschaft der Hohlheit erhält, verringert die Weite. Darüber hinaus konstruieren wir die ersten bekannten Beispiele von Gitterpolytopen, deren Weite größer als deren Dimension ist und nutzen diese expliziten Konstruktionen um asymptotische untere Schranken für die Flatness-Konstante zu zeigen.

Zuletzt untersuchen wir wie klein der Überdeckungsradius eines nicht-hohlen Gitterpolytop sein kann. Wir vermuten, dass der minimale Überdeckungsradius von einer bestimmten, expliziten Familie von Polytopen angenommen wird und übersetzen diese Vermutung in eine Vermutung über Überdeckungsminima. Überdeckungsminima wurden von Kannan und Lovasz ([KL88]) eingeführt um obere Schranken für die Flatness-Konstante zu finden. Unsere Untersuchungen führen auf natürliche Art und Weise zu der Definition eines diskreten Oberflächenmaßes. Dadurch können wir eine stärkere Vermutung formulieren, mit deren Hilfe wir eine obere Schranke für den Überdeckungsradius von Simplizes vorschlagen, die von der neu eingeführten diskreten Oberfläche und dem normalisierten Volumen des Simplizes abhängt.

Acknowledgements

The first, very heartfelt, thank you goes to Francisco Santos; an adviser is also a role-model, and I have learned invaluable lessons, not just about work, but also about community. You see the person, not just the math, and you always take (a lot of) time to help out. Thank you.

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Leo, since I don't thank you often enough, let me write here a very broad thank you, for a lot of LaTeX, mathematical, food-related, sleep-related, and letting-me-blow-off-steam support. The last few weeks especially would have been incredibly tough to get through without you.

Contents

Notation

$\mathbb{Z}, \mathbb{Z}_{\geq 0}, \mathbb{R}$	the integer, nonnegative integer, and real numbers
$[n]$	the set of natural numbers $\{1, 2, \ldots, n\}$
$e_1, \ldots, e_d$	the canonical basis of $\mathbb{R}^d$
$\mathbf{1}_d$	the vector in $\mathbb{R}^d$ with all entries equal to 1
$\mathrm{lin}\,(S)$	the linear span of the set S
$\mathrm{aff}\,(S)$	the affine span of the set S
$\mathrm{cone}\,(S)$	the conical hull of the set S
$\mathrm{conv}\,(S)$	the convex hull of the set S
P	a polytope
$f(P)$	the f-vector of the polytope P
$\partial(P)$	the boundary of P
$\mathrm{int}(P)$	the interior of P
$\mathrm{relint}(P)$	the relative interior of$P$
$\mathcal{N}(P)$	the normal fan of P
$\mathcal{T}$	a triangulation
$\mathrm{Cay}(P, Q)$	the Cayley sum of polytopes P and Q
$P \oplus Q$	the direct sum of polytopes P and Q
Λ	a lattice
$\mathrm{vol}(C)$	the Euclidean volume of C
$\mathrm{Vol}(C)$	the normalized volume of C
$\mathrm{width}(C, \Lambda)$	the lattice width of C with respect to the lattice Λ
$\mu_i(C, \Lambda)$	the i-th covering minimum of C with respect to Λ
$\mu(C, \Lambda)$	the covering radius of C with respect to Λ

Chapter 1

Introduction

This thesis deals with questions regarding polytopes (and generalizations or restrictions thereof) in the presence of a *lattice*. In this first chapter, we introduce the precise topics we are interested in and some background. We start with Section 1.1, dedicated to an overview of basic concepts regarding polytopes, which will be useful throughout. In Section 1.2, we truly enter into the spirit of this thesis: we define and give basic properties of *lattices* and of *lattice polytopes*, that is, polytopes whose vertices lie on the lattice, and discuss some of the many contexts in which lattice polytopes appear. In Section 1.3 we define *unimodular triangulations and covers* of lattice politopes and the IDP property, emphasising their relevance in Ehrhart theory and in toric geometry. In Section 1.4 we will broaden our focus and consider the interaction of more general *convex bodies* with lattices. There we will see definitions of *lattice width*, *covering radius* and *covering minima* of convex bodies, and state important theorems which play key roles in Chapters 3 to 5. We give an overview of the results presented in the following chapters in Section 1.5.

1.1 Polytopes

A polytope P is the *convex hull* of a finite set of points X in Euclidean space, that is, it is the smallest convex body containing X; this is denoted by $P = \operatorname{conv}(X)$. Polytopes have been the subject of ample study, touching on different mathematical topics, such as commutative algebra, toric geometry, discrete optimization, convex geometry and many others. For a thorough introduction to polytope theory, and an overview of many topics of current interest, we point the reader for example to [Zie95]. Here we will limit ourselves to recalling some basic notions that will be useful throughout this thesis.

The definition we have given above of polytopes is sometimes called the V-description of a polytope. An equivalent definition of a polytope is that it is the intersection of finetely many closed halfspaces, whenever the intersection is bounded. A closed halfspace H is of the form $H^- = \{x \in \mathbb{R}^d | a_i^\intercal x \leq b_i\}$, for some $a_i \in \mathbb{R}^d$, $b_i \in \mathbb{R}$. This representation is called the H-description. Any polytope P can be expressed via both the V-description and H-description,

$$P = \operatorname{conv}(X) = \{x \in \mathbb{R}^d | Ax \leq b\}.$$

The *affine span* $\operatorname{aff}(P)$ of the polytope P is the smallest affine space containing it. The *dimension* of P is the dimension of $\operatorname{aff}(P)$. A *face* of a polytope is any set of the form $H \cap P$, where H is an affine hyperplane in $\mathbb{R}^d$ such that P is contained in one of the closed halfspaces H defines. By convention, the polytope itself is considered a face. A face of a polytope is itself a polytope. A *proper* face is any face which is not empty and not the whole polytope. The *boundary* ∂P of the polytope is the union of all its proper faces, and the *interior* of P is its complement, $\operatorname{int}(P) = P \setminus \partial P$.

Faces of dimension 0 are called *vertices*, of dimension 1 *edges*, of dimension $d-1$ *facets*, and of dimension $d-2$ *ridges*. The f*-vector* of the polytope is a vector $f(P) = (f_{-1}(P), f_0(P), \ldots, f_d(P)) \in \mathbb{Z}^d_{\geq 0}$ whose entry $f_i(P)$ records the number of i-dimensional faces of P.

The *normal cone* of a face F of P is the cone of all linear functions which achieve their maximum over the polytope on F:

$$N_F = \{c \in (\mathbb{R}^d)^* |\ c(x) \geq c(y)\ \forall x \in F, y \in P\}.$$

Then the *normal fan* of P is the collection of all its normal cones, that is,

$$\mathcal{N}(P) = \{N_F : F \text{ is a face of } P\}.$$

The first and most well-known examples of polytopes are probably the standard simplex and the hypercube, and to these two we add the cross-polytope:

Example 1.1. Let $e_1 = (1, 0, \ldots, 0), \ldots, e_d = (0, \ldots, 0, 1)$ be the coordinate points in $\mathbb{R}^d$. The d-dimensional...
...*standard simplex* Δ_d is the convex hull of the origin and all the coordinate points $e_i \in \mathbb{R}^d$.
... *cube* C_d is the convex hull of all points of the form $(\pm 1, \ldots, \pm 1)$.
... *cross-polytope* C_d^* is the convex hull of all points of the form $\pm e_i$.

The simplex and cube are examples of *simple* polytopes, while simplex and cross-polytope are *simplicial*. Simple polytopes are characterized by having exactly d edges

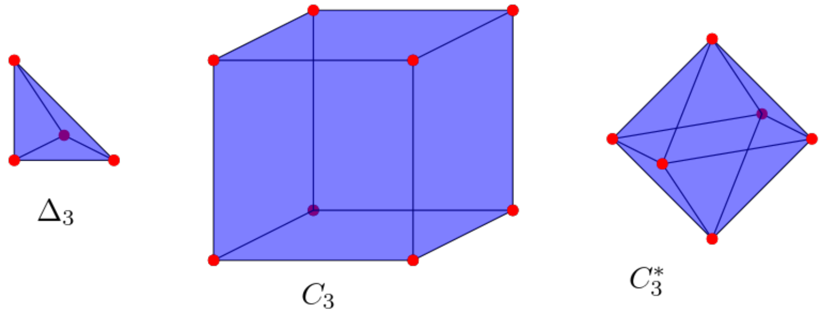


FIGURE 1.1: The standard simplex, cube, and cross-polytope in dimension three

meet in each vertex, if d is the dimension, while simplicial polytopes are those whose proper faces are all simplices, or equivalently, those whose facets contain exactly d ridges.

Many interesting questions arise when one investigates how polytopes can be decomposed. Subdivisions and triangulations are a special kind of decomposition of polytopes into subpolytopes. More details can be found in [DLRS10].

A *subdivision* of a polytope P is a collection $\mathcal{S}$ of polytopes contained in P such that

(a) if $\sigma \in \mathcal{T}$ then $\tau \in \mathcal{T}$ for any face τ of σ,

(b) if $\sigma_1, \sigma_2 \in \mathcal{T}$, then $\sigma_1 \cap \sigma_2$ is a face of both.

A *triangulation* of P is a subdivision containing only simplices. In other words, a triangulation of P is a geometric simplicial complex whose support is equal to P. We can talk of the *f-vector* $f_{\mathcal{T}}$ of the triangulation, defined in the same way as for simplicial polytopes.

The Minkowski sum and Cayley sum are important (and closely related, as we will see) operations on polytopes.

Definition 1.2. Let P and Q be two polytopes in $\mathbb{R}^d$. The *Minkowski sum* of P and Q is the polytope

$$P + Q := \{p + q \in \mathbb{R}^d : p \in P, q \in Q\} \subset \mathbb{R}^d.$$

See Figure 1.2.

Definition 1.3. The *Cayley sum* of two polytopes $P, Q \subset \mathbb{R}^d$ is the polytope in $\mathbb{R}^{d+1}$ defined as

$$\mathrm{Cay}(P, Q) = \mathrm{conv}(P \times \{0\} \cup Q \times \{1\}).$$

See Figure 1.3.

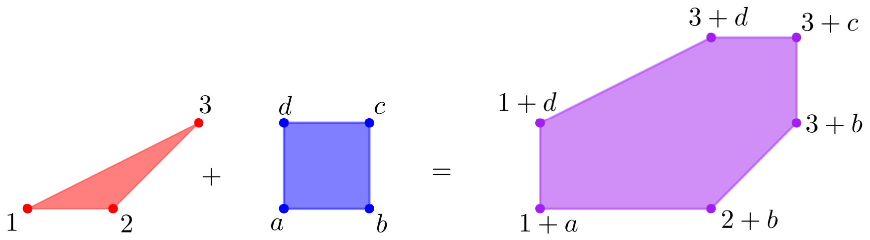


FIGURE 1.2: The Minkowski sum of a triangle and a square

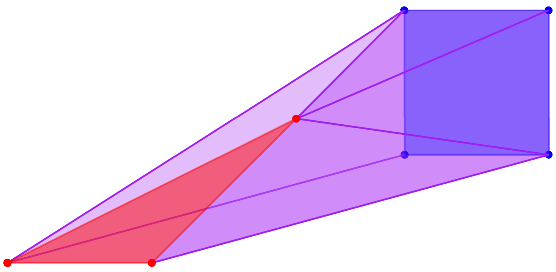


FIGURE 1.3: The Cayley sum of a triangle and a square

The *Cayley Trick* explains the connection between these two constructions: intersecting the Cayley sum of two polytopes with a hyperplane separating the embedded polytopes, one sees (a scaled copy of) their Minkowski sum. In formula, we have the following:

$$2\,\mathrm{Cay}(P,Q) \cap (\mathbb{R}^d \times \{1\}) \cong P + Q.$$

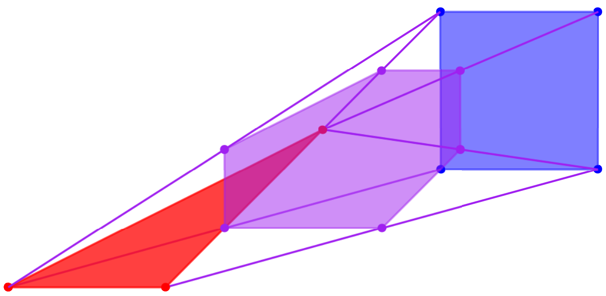


FIGURE 1.4: "Cutting" the Cayley sum down the middle we obtain a scaled copy of the Minkowski sum

A *mixed subdivision* of a Minkowski sum $P + Q$ is a subdivision which respects the structure of the summands, or more precisely, it is a subdivision where all cells are of the form $F + G$, with $F \subset P$ and $G \subset Q$, and the intersection of two cells $F + G$ and $F' + G'$ satisfies $(F + G) \cap (F' + G') = (F \cap F') + (G \cap G')$. We say that a mixed subdivision is *fine* if it cannot be properly refined by a mixed subdivision.

The Cayley Trick is particularly useful when discussing subdivisions of Cayley and Minkowski sums, since it provides the following canonical bijections:

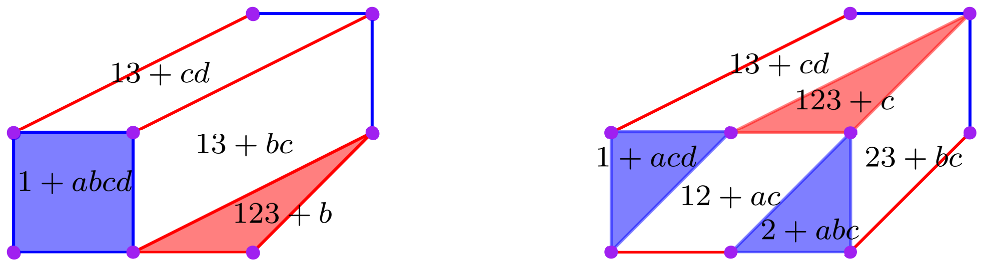


FIGURE 1.5: Two examples of mixed subdivisions of the Minkowski sum of Figure 1.2, with full-dimensional cells labeled; the mixed subdivision on the right is fine, while the one on the left is not, since the cell $1 + abcd$ could be replaced by cells $1 + abc$ and $1 + acd$.

$$\begin{array}{ccc} \text{polyhedral subdivisions of } \mathrm{Cay}(P,Q) & \leftrightarrow & \text{mixed subdivisions of } P+Q, \\ \text{triangulations of } \mathrm{Cay}(P,Q) & \leftrightarrow & \text{fine mixed subdivisions of } P+Q. \end{array}$$

In [DLRS10] one can find more details on the Cayley Trick and on triangulations and mixed subdivisions.

Another important operation on polytopes is the direct sum, defined only for polytopes containing the origin.

Definition 1.4. Let $P \subset \mathbb{R}^d$ and $P' \subset \mathbb{R}^{d'}$ be two polytopes, each containing the origin. Then their *direct sum* is

$$P \oplus P' = \operatorname{conv}\left(\{(p,0) \in \mathbb{R}^{d+d'} | p \in P\} \cup \{(0,p') \in \mathbb{R}^{d+d'} | p' \in P'\}\right).$$

Remark 1.5. In the definition of direct sum, it is important where the origin lies inside of each summand. In Figure 1.6, we see how the direct sum of two segments can yield three different polytopes depending on where the origin lies in each of the segments.

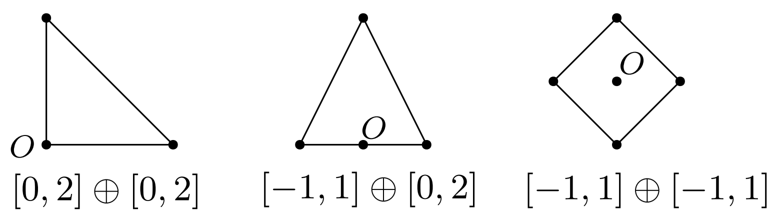


FIGURE 1.6: Three examples of direct sums of two segments of length two.

1.2 Lattices and lattice polytopes

We begin with the definition of the other central player of this thesis, the *lattice*, and give some properties and further definition needed in the following chapters. A more complete introduction to lattices and lattice polytopes can be found in the upcoming book on lattice polytopes by Haase, Nill and Paffenholz ([HNP2x]), or in lecture notes by Carsten Lange ([Lan16]).

Definition 1.6. A *linear lattice* is a discrete additive subgroup of a finite real vector space (which we usually assume to be $\mathbb{R}^d$). An *affine lattice* is any translation of a linear lattice.

We will usually say lattice to mean linear lattice, and will explicitely say so when we consider affine lattices.

The *rank* of the lattice Λ is the dimension of its linear span $\mathrm{lin}(\Lambda)$. It is not hard to show that if a lattice has rank r, it has a *lattice basis* of cardinality r, that is, there are $v_1, \dots v_r \in \mathbb{R}^d$ such that

$$\Lambda = \left\{ \sum_{i \in [r]} \lambda_i v_i | \lambda_i \in \mathbb{Z} \text{ for all } i \in [r] \right\}.$$

This shows in particular that any linear lattice is linearly isomorphic to the *standard lattice* $\mathbb{Z}^r$, for some $r \in \mathbb{Z}_{\geq 0}$.

A vector $v \in \Lambda$ is said to be *primitive* if it is not an integer multiple of any other vector in Λ. Clearly a basis is a set of affinely independent primitive vectors, but the converse is not always true.

Given a full-dimensional lattice $\Lambda \subseteq \mathbb{R}^d$, its dual lattice Λ^* is

$$\Lambda^* = \{f \in (\mathbb{R}^d)^* | f(v) \in \mathbb{Z} \text{ for all } v \in \Lambda\},$$

which is itself a lattice. If $\mathcal{B}$ is a lattice basis of Λ, then the basis of $(\mathbb{R}^d)^*$ dual to $\mathcal{B}$ is also a lattice basis for Λ^*.

Associated to a lattice basis $\mathcal{B} = \{v_1, \dots, v_r\}$ is a *fundamental parallelepiped* $\Pi(\mathcal{B})$, defined as

$$\Pi(\mathcal{B}) = \left\{ \sum_{i \in [r]} \lambda_i v_i | 0 \leq \lambda_i < 1 \text{ for all } i \in [r] \right\}.$$

It is called fundamental parallelepiped because any point of $\mathrm{lin}(\Lambda)$ can be written uniquely as the sum of a lattice point and a point in $\Pi(\mathcal{B})$. That is, its closure is a *fundamental domain* of $\mathrm{lin}(\Lambda)$ with respect to the action of Λ by translations.

Given lattices Λ and Λ', a *unimodular transformation* (or lattice transformation) is a linear map $\tilde{\phi} : \mathrm{lin}(\Lambda) \to \mathrm{lin}(\Lambda')$ which restricts to a bijection $\Lambda \to \Lambda'$. A linear map is unimodular if and only if the matrix representing it with respect to lattice bases has determinant equal to one.

In particular, we now see that given a lattice basis $\mathcal{B}$ of Λ, $\mathcal{B}'$ is also a lattice basis if and only if there is a unimodular transformation of Λ which restricts to a bijection between $\mathcal{B}$ and $\mathcal{B}'$. This also shows that the volume of any fundamental parallelepiped, which is the determinant of the the corresponding basis, is independent of the chosen basis. This is called the *determinant* of Λ.

A *sublattice* Γ of a lattice Λ is a subgroup $\Gamma \leq \Lambda$. The *index* of Γ in Λ is $|\Lambda/\Gamma|$, the cardinality of the set of cosets Λ/Γ. This cardinality is infinite if $\operatorname{rank}(\Gamma) < \operatorname{rank}(\Lambda)$. If instead the ranks are equal, one can show that the index is equal to the determinant of the matrix representing the identity map $\operatorname{lin}(\Lambda) \to \operatorname{lin}(\Gamma)$ with respect to a basis of Λ and a basis of Γ; that is, it equals $\det(\Lambda)/\det(\Gamma)$.

We say that a subspace $\mathcal{L}$ of $\mathbb{R}^d$ is a Λ-*rational subspace* (or *lattice subspace*, when Λ is understood) if a lattice basis of $\mathcal{L} \cap \Lambda$ is a linear basis of $\mathcal{L}$, or equivalently, if

$$\operatorname{rank}(\mathcal{L} \cap \Lambda) = \dim(\mathcal{L}).$$

We can now give the definition of lattice polytope:

Definition 1.7. A *lattice polytope* is a polytope in $\mathbb{R}^d$ whose vertices lie in a given lattice $\Lambda \subseteq \mathbb{R}^d$.

The most basic lattice polytope is a *unimodular simplex*, a lattice simplex whose vertices form an affine basis for the lattice. This includes the standard simplex $\Delta_d = \operatorname{conv}(0, e_1, \dots, e_d)$ of Example 1.1, with respect to the standard lattice $\mathbb{Z}^d$, which we call the *standard simplex*. We say that two lattice polytopes P and P' with respect to lattices Λ and Λ' are unimodularly equivalent if there is an (affine) unimodular transformation of the lattices which sends vertices of P to vertices of P'. For example, all unimodular simplices are unimodularly equivalent.

Unimodular simplices are the lattice polytopes with smallest possible volume, equal to $\frac{1}{d!}\operatorname{vol}(\Pi_\Lambda)$, where vol() denotes the usual euclidean volume, and Π_Λ is any fundamental parallelepiped of Λ. The volume of any other lattice polytope is an integer multiple of this; this suggests the following definition of *normalized volume* of a lattice polytope P:

$$\operatorname{Vol}_\Lambda(P) := \frac{d!}{\operatorname{vol}(\Pi(\mathcal{B}_\Lambda))}\operatorname{vol}(P),$$

which is an integer for any lattice polytope and is equal to 1 for a unimodular simplex.

In dimension 2, there is a special relationship between the number of lattice points in a lattice polytope and its volume, as described by the following theorem, known as Pick's theorem (for more details see for example [BR15]):

Theorem 1.8 (Pick's theorem). *Let P be a lattice polygon with i interior lattice points, b lattice points on the boundary, and (Euclidean) volume a. Then $a = i + \frac{b}{2} - 1$.*

In particular, a lattice triangle which contains no other lattice points except its vertices must be unimodular. There is no such direct relation between volume and lattice points in higher dimension: lattice simplices which contain no lattice points other than its vertices are called *empty* simplices, and already in dimension 3 there are empty simplices which are not unimodular simplices. Indeed, the following theorem, due to White [Whi64], is a complete classification of empty tetrahedra up to unimodular equivalence, and shows that there are empty simplices with (normalized) volume equal to any natural number:

Theorem 1.9 (White 1964). *Every empty tetrahedron of (normalized) volume $q \in \mathbb{N}$ is unimodularly equivalent to*

$$T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\},$$

for some $p \in \mathbb{Z}$ with $\gcd(p,q) = 1$. Moreover, $T(p,q)$ is equivalent to $T(p',q)$ if and only if $p' = \pm p^{\pm 1} \mod q$.

There are many different contexts in which lattice polytopes appear naturally. This will be visible in the following sections: in Section 1.3 we discuss unimodular triangulations, covers and the IDP property, which have strong connections to toric geometry and Ehrhart theory. Section 1.4 highlights connections to integer linear programming, via certain parameters such as width.

1.3 Unimodular triangulations, covers, and the IDP property

We now highlight certain properties of lattice polytopes that we will need in Chapter 2. For more details, see [BR15] and [BS18].

A lattice triangulation of a lattice polytope P is a triangulation with the additional property that all simplices in the triangulation are themselves lattice polytopes. An important class of such triangulations is the following.

A *unimodular triangulation* of a lattice polytope is a triangulation composed only of unimodular simplices. Not all lattice polytopes admit a unimodular triangulation. What one can always do is triangulate a lattice polytope into empty simplices. In dimensions one and two, this implies that any lattice polytope has a unimodular triangulation, since all empty simplices in these dimensions are unimodular. However, as we saw in Theorem 1.9, starting in dimension 3 many empty simplices are not unimodular. Since the only lattice points in an empty simplex are the vertices, these clearly only admit a unimodular triangulation if they are themselves unimodular.

Having a unimodular triangulation is a much-studied property of lattice polytopes, which falls into a hierarchy of several covering properties, see, e.g., [BG09, Section 2.D], [HPPS14, Sect. 1.2.5], [mfo04, p. 2097], [mfo07, p. 2313].

In particular, having a unimodular triangulation implies having a unimodular cover:

Definition 1.10. A *unimodular cover* of a lattice polytope P is a collection of unimodular simplices whose union equals P.

Having a unimodular cover is in turn stronger than having the *integer decomposition property*, or abbreviated, being IDP: a lattice polytope P is IDP if for every positive integer n, every lattice point $p \in nP \cap \mathbb{Z}^d$ can be written as the sum of n lattice points in P.

Both having a unimodular triangulation and being IDP have important consequences in Ehrhart theory and in algebraic geometry. One example is the following theorem, due to Stanley.

Theorem 1.11. *Let P be a lattice polytope which admits a unimodular triangulation $\mathcal{T}$. Then*

$$h^*_P(x) = h_{\mathcal{T}}(x),$$

where $h^*_P(x)$, the h^*-polynomial of P, is a central object of study in Ehrhart theory: it is the numerator of rational function of the Ehrhart series, which encodes the information of how many lattice points are contained in all the integer dilations of the polytope.

The IDP property is also of great interest in Ehrhart theory; it appears for example in long-standing conjectures about classes of polytopes whose h^*-polynomial may be unimodal (see [Bra16]). It can be translated into algebraic terms for the toric variety associated to the lattice polytope, and is studied in toric geometry. It is sometimes hard to show that certain classes of lattice polytopes have the integer decomposition property directly, and it can be more convenient to show the stronger property of having a unimodular cover.

Following [HH17, Tsu18], we say that a pair (P, Q) of lattice polytopes has the integer decomposition property, or that *the pair* (P, Q) *is IDP*, if

$$(P + Q) \cap \mathbb{Z}^d = P \cap \mathbb{Z}^d + Q \cap \mathbb{Z}^d.$$

A lattice polytope Q is called *smooth* if it is simple and the primitive edge directions at every vertex form a linear basis for the lattice; equivalently, if the projective toric variety defined by the normal fan of Q is smooth.

Oda posed several questions regarding smoothness and the IDP property for lattice polytopes ([Oda97]). The following versions of Oda's questions are now considered conjectures [HNPS08, mfo07], and they are open even in dimension three:

Conjecture. (i) (Related to problems 2 and 5 in [Oda97]) Every smooth lattice polytope is IDP.

(ii) (Related to problems 1, 3, 4, 6 in [Oda97]) Every pair (P, Q) of lattice polytopes with Q smooth and the normal fan of Q refining that of P is IDP.

When the normal fan of a polytope Q refines that of another polytope P, as in the second conjecture, we say that *P is a weak Minkowski summand of Q*, since this is easily seen to be equivalent to the existence of a polytope P' such that $P + P' = kQ$ for some dilation constant $k > 0$. This property has the following algebraic implication for the projective toric variety X_Q: P is a weak Minkowski summand of Q if and only if the Cartier divisor defined by P on X_Q is *numerically effective*, or "nef" (see [CLS11, Cor. 6.2.15, Prop. 6.3.12], but observe that what we here call "weak Minkowski summand" is simply called "Minkowski summand" there).

1.4 Width and covering minima

We will now broaden our focus to *convex bodies*, that is, convex and bounded sets, and their interaction with a lattice Λ. We often assume that Λ is the integer lattice, which is no loss of generality since the properties we study are invariant under affine transformations.

Geometry of numbers is a field of mathematics where geometry is employed to obtain results in number theory. Hermann Minkowski, who pioneered the subject, studied in particular O-centrally symmetric convex bodies; a convex body C is called *O-centrally symmetric* if $x \in C$ implies $-x \in C$. A fundamental result in geometry of numbers is Minkowski's First Theorem, which states that any O-centrally symmetric convex body

FIGURE 1.7: With respect to the lattice functional $(-1, 1)$, the polygon has width 4; with respect to $(1, 0)$, it has width 2. Because of the interior lattice point, there is no lattice functional that can give width less than 2, and therefore the lattice width of the polytope is 2.

in $\mathbb{R}^d$ which contains no lattice point other than the origin in its interior has volume bounded above by $2^d \det(\Lambda)$. It is easy to see that this upper bound is tight: the cube $[-1, 1]^d$ achieves the bound for $\Lambda = \mathbb{Z}^d$.

If one drops the O-centrally-symmetric requirement, it is natural to study convex bodies containing no interior lattice points at all:

Definition 1.12. A convex body is *hollow* (or lattice-point-free) if it contains no lattice point in its interior.

For (non-symmetric) hollow convex bodies, we cannot hope to prove an analogue of Minkowski's theorem, since no upper bound on the volume exists: we can fit a convex body of arbitrarily large volume between two consecutive lattice hyperplanes; and this will clearly contain no lattice points in its interior. The 'right' parameter to study in this case is the width, which, informally speaking, tells us how 'flat' the convex body is.

Definition 1.13. The *width* of a convex body $C \subset \mathbb{R}^d$ with respect to a linear functional $f \in (\mathbb{R}^d)^*$ is

$$\operatorname{width}(C, f) := \max_{x \in C} f(x) - \min_{x \in C} f(x).$$

Equivalently, it is the length of the segment $f(C)$.

The *lattice width* of C is the smallest width with respect to any lattice functional:

$$\operatorname{width}(C, \Lambda) := \inf_{f \in \Lambda^* \setminus 0} \operatorname{width}(C, f).$$

The infimum is achieved for some lattice functional, and we can thus exchange it in the definition with min. See Figure 1.7 for an example of calculation of lattice width of a polygon.

The celebrated *flatness theorem* ([Khi48]) states that hollow bodies in fixed dimension d have bounded lattice width. That is,

Theorem 1.14 (Kinchin (1948))**.** *For each fixed d, the supremum of* $\mathrm{width}(C)$ *among all hollow bodies $C \subset \mathbb{R}^d$ is a certain constant $w_c(d) < \infty$.*

Since we approach from the perspective of lattice polytopes, the following specializations of the flatness constant w_c are also of interest: $w_p(d)$, defined as the maximum width among all hollow *lattice d-polytopes*; $w_s(d)$, the maximum width among hollow *lattice d-simplices*; and $w_e(d)$, the maximum width among *empty d-simplices*; here a lattice simplex is *empty* if its only lattice points are its vertices. Clearly, since $w_e(d) \leq w_s(d) \leq w_p(d) \leq w_c(d)$, all of these are finite.

The flatness theorem is well known today thanks in part to its role in the 1983 polynomial time algorithm to solve the integer linear programming problem in fixed dimension, due to Lenstra ([Len83]). An integer linear programming problem is an optimization problem where one seeks to optimize a linear functional c over a polyhedron (called feasible set) defined by the constraints $Ax \leq b$, for some matrix A and vector b, subject to the additional constraint that the solution should be an integer point. If one does not fix the dimension, feasibility of integer linear programs is known to be NP-complete. Lenstra proved an algorithmic version of the flatness theorem, which allows one to explicitly find either a lattice point in the feasible set, or a direction in which the polyhedron is flat. Iterating this procedure provides a polynomial time algorithm for integer progamming in fixed dimension.

Lenstra's result in fixed dimension rekindled interest in the flatness constant, and upper bounds for $w_c(d)$ are thus well studied (see references, e.g., in the introductions to [KL88, BLPS99]). The current best upper bound is $w_c(d) \in O^*(d^{4/3})$ [Rud00], where the notation $O(\ )^*$ denotes that a polylog factor is neglected. Better upper bounds are known for restricted classes of convex bodies. For example, it is known that the maximum width of hollow (not necessarily lattice) simplices [BLPS99] and of centrally symmetric hollow bodies [Ban96] is in $O(n \log n)$.

Exact values are much harder to come by, and work on lower bounds for the flatness constants is very scarce. Chapter 4 is dedicated to the study of such lower bounds, that is, to constructions of explicit hollow convex bodies (and hollow lattice polytopes, hollow lattice simplices, empty lattice simplices) of large width.

We have presented the flatness theorem as a natural counterpart to Minkowski's first theorem for non-symmetric convex bodies. This analogy was recognized and pushed further by Kannan and Lovász in their seminal paper [KL88]. Here, Kannan and Lovász define a sequence of numbers, called the *covering minima*, which are reminiscent of *successive minima*, defined and used by Minkowski to refine his first theorem.

For any O-symmetric convex body K and any lattice Λ in $\mathbb{R}^d$, the *i-th successive minimum* $\lambda_i(K,\Lambda)$ is the smallest real number λ by which it is necessary to dilate K so that λK contains i linearly independent lattice points. Since an O-symmetric convex body K contains no lattice point other than the origin if and only if $\lambda_1(K) \geq 1$, Minkowski's first theorem can be reformulated to say that $\lambda_1(K,\Lambda)^d \operatorname{Vol}_\Lambda(K) \leq 2^d$. Using the successive minima, Minkowski was able to strengthen the bound to $\Pi_1^d \lambda_i(K,\Lambda) \operatorname{Vol}_\Lambda(K) \leq 2^d$.

Kannan and Lovász defined the covering minima as follows.

Definition 1.15. The *j-th covering minimum* of a convex body $K \subseteq \mathbb{R}^d$ with respect to a lattice $\Lambda \subseteq \mathbb{R}^d$ is defined as

$$\mu_j(K,\Lambda) := \max_{\pi} \mu(\pi(K), \pi(\Lambda)),$$

where π runs over all linear projections $\pi : \mathbb{R}^d \to \mathbb{R}^j$ such that $\pi(\Lambda)$ is a lattice.

Whenever the lattice the operator refers to is clear from the context, we will drop it from our notation: for example, we will use $\operatorname{width}(K)$ to indicate $\operatorname{width}(K,\Lambda)$. It is easy to see that an equivalent definition of the covering minima is that μ_j is the smallest number μ such that any $(d-j)$-dimensional affine subspace intersects some lattice translation of μK.

Kannan and Lovász exploited relations between the covering minima to give an upper bound of $\mathcal{O}(d^2)$ for the flatness constant. The first step is to establish the following easy connection between the first covering minimum and the width of a convex body:

$$\mu_1(K,\Lambda) = (\operatorname{width}(K,\Lambda))^{-1}.$$

Thus the covering minima interpolate between two well known and studied quantities: the width $\mu_1^{-1}(K)$ on the one hand, and the covering radius $\mu_d(K)$ (highest dimensional covering minima) on the other.

The covering minima are interesting in their own right. In the same paper ([KL88]), Kannan and Lovász also searched for a statement regarding covering minima similar to Minkowski's second theorem. As we have already noted, it is not possible to bound the volume of hollow convex bodies. What Kannan and Lovász could prove is that there is a projection which is 'almost' hollow and whose volume normalized to the projected lattice is bounded by a function of the dimension.

The covering minima have not been studied as extensively as the successive minima, and many basic questions about them remain unanswered. As an example, consider the simplex

$$S(\mathbf{1}_{d+1}) := \operatorname{conv}(\{-\mathbf{1}_d, e_1, \ldots, e_d\}),$$

where $\mathbf{1}_d = (1, \ldots, 1)$ denotes the all-one vector in dimension d, and e_i denotes the ith coordinate unit vector. It contains a unique interior lattice point, the origin. Its covering radius, that is, the d-th covering minimum, was computed in [GMS17, Prop. 4.9]:

$$\mu(S(\mathbf{1}_{d+1}), \mathbb{Z}^d) = \frac{d}{2}. \tag{1.1}$$

We can use this equality, along with the fact that $S(\mathbf{1}_{n+1})$ projects to $S(\mathbf{1}_{d+1})$ for every $d < n$, to obtain

$$\mu_d(S(\mathbf{1}_{n+1})) \geq \mu_d(S(\mathbf{1}_{d+1})) = \frac{d}{2}. \tag{1.2}$$

The converse inequality was conjectured in [GMS17]:

Conjecture ([GMS17, Rem. 4.10], Conjecture B in Chapter 5)**.** For every $n \in \mathbb{N}$ and $d \leq n$,

$$\mu_d(S(\mathbf{1}_{n+1})) = \frac{d}{2}. \tag{1.3}$$

It might be surprising at first that the covering minima are not known even for this specific simplex. Indeed, this seemingly simple question was the starting point for the work presented in Chapter 5, where we show the equivalence of Conjecture B to a conjecture regarding the covering radius of all non-hollow lattice polytopes.

1.5 Summary of results

In this section, we give an overview of the results presented in the rest of the thesis. After the introduction, the thesis is divided into four chapters, reflecting four projects I have worked on regarding lattices and polytopes.

Chapter 2 deals with unimodular covers of 3-dimensional polytopes. It consists of joint work with F. Santos and is based on the paper [CS19b]. We show the existance of unimodular covers of two classes of polytopes, which were previously known to be IDP.

Theorem (Corollary 2.3)**.** Any 3-dimensional lattice parallelepiped admits a unimodular cover.

Theorem (Theorem 2.7)**.** If Q is a lattice polygon, and P is a weak Minkowski summand of Q, then their Cayley sum $\operatorname{Cay}(P, Q)$ has a unimodular cover.

Chapters 3 and 4 concern lower bounds for the flatness constant.

Chapter 3 focuses on dimension three. Sections 3.1 and 3.2 of Chapter 3 are based on joint work with Francisco Santos ([CS19a]), while Section 3.3 is based on joint work with Gennadiy Averkov, Antonio Macchia and F. Santos ([ACMS19]). We construct an explicit example of a wide hollow tetrahedron, thus providing a lower bound for $w_c(3)$.

Theorem (Theorem 3.1)**.** There is a hollow (non-lattice) 3-simplex of width $2+\sqrt{2} \simeq 3.4142$.

We further conjecture that this tetrahedron is the hollow convex body in dimension three of largest width (Conjecture 3.2), and provide some evidence for this conjecture. In particular, we show that a local version of it holds:

Theorem 1.16 (Corollary 3.10)**.** *The tetrahedron of Theorem 3.1 is a strict local maximizer for width among hollow convex* 3*-bodies.*

Chapter 4 is joint work with Francisco Santos, and is based on [CS19a]. In it, we study lower bounds, both asymptotically and in fixed dimension, for the flatness constants $w_c(d), w_p(d), w_s(d), w_e(d)$ defined in Section 1.4. We show certain lower bounds in fixed dimension for w_p (resp. w_s) by constructing an explicit example of a hollow lattice polytope (resp. simplex) of width larger than its dimension:

Theorem (Theorem 4.1)**.** There is a hollow lattice 14-polytope of width 15 and a hollow lattice 404-simplex of width 408.

In personal communication, F. Santos informed us that he has found an empty 10-simplex of width 11. Combining these results in fixed dimensions with the technical tool of direct sums (introduced in section 5.1) allows us also to prove the following asymptotic lower bounds:

Theorem (Theorem 4.4)**.**

$$\lim_{d\to\infty} \frac{w_p(d)}{d} = \lim_{d\to\infty} \frac{w_c(d)}{d} \geq \frac{2+\sqrt{2}}{3} = 1.138\ldots$$
$$\lim_{d\to\infty} \frac{w_s(d)}{d} \geq \frac{11}{10} = 1.1$$

For empty simplices, our results are slightly weaker:

Theorem (Theorem 4.6)**.** For every $d, m \in \mathbb{N}$ we have

$$w_e(dm) \geq (m-3)\, w_e(d).$$

In particular,

$$\limsup_{d\to\infty} \frac{w_e(d)}{d} = \sup_{d\in\mathbb{N}} \frac{w_e(d)}{d} \geq 1.1$$

In Chapter 5, based on the paper [CSS19] joint with F. Santos and Matthias Schymura, we explore how small the covering radius of non-hollow lattice polytopes can be, a somewhat opposite problem to that of Chapter 4.

We propose several conjectures.

Conjecture (Conjecture A)**.** Let $P \subseteq \mathbb{R}^d$ be a non-hollow lattice d-polytope. Then

$$\mu(P) \leq \frac{d}{2},$$

with equality if and only if P is obtained by direct sums and/or translations of simplices of the form $S(\mathbf{1}_l)$.

We show that this is equivalent to Conjecture B stated at the end of the previous section regarding the covering minima of the simplex $S(\mathbf{1}_{d+1})$ (Theorem 5.2), and that it holds in dimensions two and three (Theorem 5.3).

We then propose the following definition:

Definition 1.17. Let $S = \operatorname{conv}(\{v_0, \ldots, v_d\})$ be a d-simplex with the origin in its interior. We say that S has *rational vertex directions* if the line through the origin and the vertex v_i has rational direction, for every $0 \leq i \leq d$.

Writing $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ for the linear projection vanishing at v_i, we define the *discrete surface area* of such a simplex S as

$$\operatorname{Surf}_{\mathbb{Z}^d}(S) := \sum_{i=0}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)).$$

This allows us to formulate the following conjecture.

Conjecture (Conjecture C)**.** Let S be a d-simplex with the origin in its interior and with rational vertex directions. Then

$$\mu(S) \leq \frac{1}{2} \frac{\operatorname{Surf}_{\mathbb{Z}^d}(S)}{\operatorname{Vol}_{\mathbb{Z}^d}(S)}. \tag{1.4}$$

We prove that this conjecture implies Conjecture A (Corollary 5.35), that it holds in dimension two (Corollary 5.43) and in Theorem 5.4 show that it holds with equality for the following simplices $S(\omega)$ (which include $S(\mathbf{1}_{d+1})$ as a special case): For each $\omega = (\omega_0, \ldots, \omega_d) \in \mathbb{R}^{d+1}_{>0}$,

$$S(\omega) := \operatorname{conv}(\{-\omega_0 \mathbf{1}_d, \omega_1 e_1, \ldots, \omega_d e_d\}).$$

A different way to extend conjecture A is to ask for the maximal covering radius among lattice polytopes with at least $k \geq 1$ interior lattice points. The natural conjecture in this setting is:

Conjecture (Conjecture D)**.** Let $k, d \in \mathbb{N}$ be nonnegative integers. Then, for every lattice d-polytope P with k interior lattice points we have

$$\mu(P) \leq \frac{d-1}{2} + \frac{1}{k+1}.$$

Equality holds for $k = 1$ if and only if P is obtained by direct sums and/or translations of simplices of the form $S(\mathbf{1}_l)$, and for $k \geq 2$, if and only if P is obtained by direct sums and/or translations of the segment $[0, k+1]$ and simplices $S(\mathbf{1}_l)$.

We prove that this conjecture holds in dimension two (Theorem 5.55).

Chapter 2

Unimodular covers of 3-polytopes

This chapter is dedicated to showing that certain classes of lattice polytopes have unimodular covers. A *unimodular cover* of a polytope P is a collection of unimodular simplices whose union equals P. Having a unimodular cover is one of the properties of lattice polytopes discussed in Section 1.3. Recall that it is stronger than the IDP property: a lattice polytope $P \subset \mathbb{R}^d$ is said to have the *integer decomposition property* if for every positive integer n, every lattice point $p \in nP \cap \mathbb{Z}^d$ can be written as a sum of n lattice points in P.

We recall here Oda's conjectures regarding smoothness and the IDP property, also introduced in Section 1.3.

Conjecture 2.1. *(i) (Related to problems 2 and 5 in [Oda97]) Every smooth lattice polytope is IDP.*

(ii) (Related to problems 1, 3, 4, 6 in [Oda97]) Every pair (P, Q) of lattice polytopes with Q smooth and the normal fan of Q refining that of P is IDP.

Motivated by these and other questions, several authors have studied the IDP property for different classes of lattice polytopes. For example, very recently Beck et al. [BHH$^+$19] proved that all smooth centrally symmetric 3-polytopes are IDP. More precisely, they show that any such polytope can be covered by lattice parallelepipeds and unimodular simplices, both of which are trivially IDP.

In Section 2.1 we show:

Theorem 2.2. *Every 3-dimensional lattice parallelepiped has a unimodular cover.*

This, together with the mentioned result from [BHH$^+$19], gives:

Corollary 2.3. *Every smooth centrally symmetric lattice 3-polytope has a unimodular cover.* □

These results leave open the following important questions:

Question 2.4. Do 3-dimensional parallelepipeds have unimodular triangulations?

Question 2.5. Higher dimensional parallelotopes (affine images of cubes) are IDP. Do they have unimodular covers?

The two-dimensional case of Conjecture 2.1(ii) is known to hold, with three different proofs by Fakhruddin [Fak02], Ogata [Oga06] and Haase et al. [HNPS08]. This last one actually shows that smoothness of Q is not needed. In dimension three, however, the conjecture fails without the smoothness assumption. Indeed, if we let $P = Q$ be any non-unimodular *empty tetrahedron*, then P is obviously a weak Minkowski summand of Q but the pair (P, Q) is not IDP. By an empty tetrahedron we mean a lattice tetrahedron containing no lattice points other than its vertices (see the proof of Lemma 2.11 for a classification of them).

An alternative approach to Conjecture 2.1(ii) is via Cayley sums, which we discuss in Section 2.2. Recall from Section 1.1 that the *Cayley sum* of two lattice polytopes $P, Q \subset \mathbb{R}^d$ is the lattice polytope

$$\mathrm{Cay}(P, Q) := \mathrm{conv}\,(P \times \{0\} \cup Q \times \{1\}) \subset \mathbb{R}^3.$$

We normally require $\mathrm{Cay}(P, Q)$ to be full-dimensional (otherwise we can delete coordinates) but this does not require P and Q to be full-dimensional. All that is needed is that the linear subspaces parallel to them to span $\mathbb{R}^d$.

From the Cayley trick (also explained in Section 1.1), it is easy to prove the following theorem.

Proposition 2.6 (see, e.g. [Tsu18, Thm. 0.4])**.** *If* $\mathrm{Cay}(P, Q)$ *is IDP then the pair* (P, Q) *is mixed IDP.*

In particular, the following statement from Section 2.2 is stronger than the aforementioned result of [Fak02, HNPS08, Oga06]:

Theorem 2.7. *Let Q be lattice polygon, and P a weak Minkowski summand of Q. Then the Cayley sum* $\mathrm{Cay}(P, Q)$ *has a unimodular cover.*

This has the following two corollaries, also proved in Section 2.2. A *prismatoid* is a polytope whose vertices all lie in two parallel facets. A polytope has width 1 if its vertices lie in two *consecutive* parallel lattice hyperplanes. Observe that this is the same as being ($SL(\mathbb{Z}, d)$-equivalent to) a Cayley sum.

Corollary 2.8. *Every smooth* 3*-dimensional lattice prismatoid has a unimodular cover.*

Corollary 2.9. *Every integer dilation kP, $k \geq 2$, of a lattice* 3*-polytope P of width* 1 *has a unimodular cover.*

A special case of the latter are integer dilations of empty tetrahedra. That their dilations have unimodular covers is [SZ13, Cor. 4.2] (and is also implicit in [KS03]).

We believe that the 3-polytopes in all these statements have unimodular triangulations, but this remains an open question.

2.1 Parallelepipeds

The main tool for the proof of Theorem 2.2 is what we call the parallelepiped circumscribed to a given tetrahedron, defined as follows:

Definition 2.10. Let T be a tetrahedron with vertices p_1, p_2, p_3, and p_4. Consider the points $q_i = \frac{1}{2}(p_1 + p_2 + p_3 + p_4) - p_i$, $i \in [4]$, and let

$$C(T) = \operatorname{conv}(p_i, q_i : i \in [4]).$$

$C(T)$ is a parallelepiped with facets $\operatorname{conv}(p_i, p_j, q_k, q_l)$ for all choices of $\{i, j, k, l\} = [4]$. We call it the *parallelepiped circumscribed* to T.

For each $i \in [4]$, let $T_i = \operatorname{conv}(q_i, p_j, p_k, p_l)$, with $\{i, j, k, l\} = [4]$; we call these T_i the *corner tetrahedra* of $C(T)$. Together with T they triangulate $C(T)$.

Modulo an affine transformation, the situation of T and $C(T)$ is exactly that of the regular tetrahedron inscribed in a cube; see Figure 2.1.

Lemma 2.11. *Let $T = \operatorname{conv}\{p_1, p_2, p_3, p_4\}$ be an empty lattice tetrahedron that is not unimodular. Let $C(T)$ be the parallelepiped circumscribed to T and let T_1, T_2, T_3 and T_4 be the corresponding corner tetrahedra in $C(T)$. Then, every T_i contains at least one lattice point different from $\{p_1, \ldots, p_4\}$.*

Proof. By White's classification of empty tetrahedra ([Whi64], see also, e. g. [HPPS14, Sect. 4.1]), there is no loss of generality in assuming $T = \operatorname{conv}(p_1, p_2, p_3, p_4)$ with

$$p_1 = (0, 0, 0), \quad p_2 = (1, 0, 0), \quad p_3 = (0, 0, 1), \quad p_4 = (a, b, 1).$$

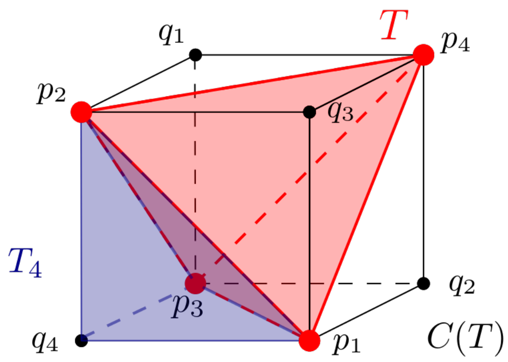


FIGURE 2.1: In red we have a tetrahedron T, in black its circumscribed parallelepiped $C(T)$, and in blue the corner simplex T_4.

where $b \geq 2$ is the (normalized) volume of T, and $a \in \{1, \ldots, b-1\}$ satisfies $\gcd(a, b) = 1$. This gives

$$q_1 = \left(\frac{1+a}{2}, \frac{b}{2}, 1\right), \qquad q_2 = \left(\frac{a-1}{2}, \frac{b}{2}, 1\right),$$
$$q_3 = \left(\frac{1+a}{2}, \frac{b}{2}, 0\right), \qquad q_4 = \left(\frac{1-a}{2}, -\frac{b}{2}, 0\right).$$

Then, the inequalities $b \geq 1 + a \geq 2$ imply:

$$u := (1,1,0) \in \operatorname{conv}(p_1p_2q_3) \subset T_4, \quad v := (0,-1,0) \in \operatorname{conv}(p_1p_2q_4) \subset T_3.$$

Observe that $u + v = p_1 + p_2 = q_3 + q_4$. Now, this implies that the quadrilateral $\operatorname{conv}(p_1q_4p_2q_3)$ contains a fundamental domain for the lattice $\mathbb{Z}^2 \times \{0\}$. Hence, its translate $\operatorname{conv}(q_2p_3q_1p_4)$ contains a fundamental domain for $\mathbb{Z}^2 \times \{1\}$ and, in particular, it contains at least one lattice point other than p_3 and p_4. By central symmetry around its center $\left(\frac{a}{2}, -\frac{b}{2}, 1\right)$, $\operatorname{conv}(q_2p_3q_1p_4)$ must contain lattice points in both triangles $\operatorname{conv}(q_2p_3p_4) \subset T_1$ and $\operatorname{conv}(q_1p_3p_4) \subset T_2$.

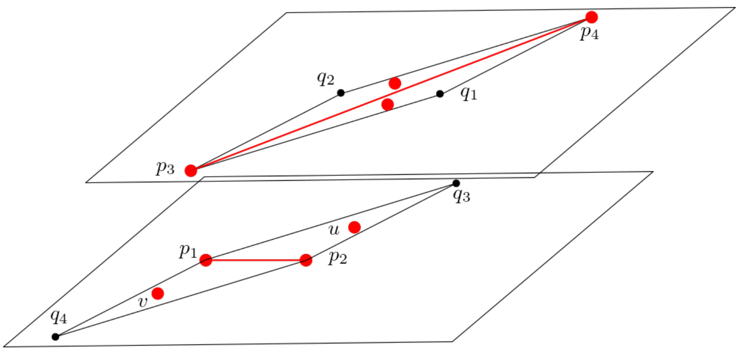


FIGURE 2.2: The parallelograms $\operatorname{conv}(p_1p_2q_3q_4)$ and $\operatorname{conv}(q_2q_1p_4p_3)$. The large red dots are lattice points.

□

Lemma 2.12. *Let P be a lattice parallelepiped and let $T \subset P$ be a tetrahedron. Then, at least one of the four corner tetrahedra T_i of the circumscribed parallelogram $C(T)$ is fully contained in P.*

Proof. Let us denote the vertices of T by p_1, p_2, p_3, p_4 and the vertices of $C(T)$ not in T by q_1, q_2, q_3, q_4, with the conventions of Definition 2.10.

We call *band* any region of the form $f^{-1}([\alpha, \beta])$ for some functional $f \in (\mathbb{R}^3)^*$ and closed interval $[\alpha, \beta] \subset \mathbb{R}$. We claim that any band containing T must contain at least three of the q_is. This claim implies that the parallelepiped P, which is the intersection of three bands, contains at least one of the q_is and hence it fully contains the corresponding T_i.

To prove the claim, suppose that $q_1 \notin B := f^{-1}([\alpha, \beta])$ for a certain band $B \supset T$. Without loss of generality, say $f(q_1) < \alpha$. Then the equalities $q_1 + q_2 = p_3 + p_4$ and $q_1 + p_1 = q_2 + p_2$ respectively give:

$$f(q_2) = f(p_3 + p_4 - q_1) = f(p_3) + f(p_4) - f(q_1) > 2\alpha - \alpha = \alpha, \tag{2.1}$$

$$f(q_2) = f(q_2 + p_2 - p_1) = f(q_2) + (f(p_2) - f(p_1)) < \alpha + (\beta - \alpha) = \beta, \tag{2.2}$$

so that $q_2 \in B$.

Inequality (2.1) also implies

$$f(q_1) < f(p_i) < f(q_2), \qquad \text{for } i = 3, 4. \tag{2.3}$$

The translation of vector $\frac{1}{2}(p_1 + p_2 - p_3 - p_4)$ sends q_1, q_2, p_3, p_4 to p_2, p_1, q_4, q_3 (in this order). By applying this to inequality (2.3), we obtain

$$\alpha \leq f(p_2) < f(q_i) < f(p_1) \leq \beta, \qquad \text{for } i = 3, 4,$$

so that $q_3, q_4 \in B$. This finishes the proof of the claim, and of the lemma. □

Corollary 2.13. *Let T be an empty lattice tetrahedron contained in a lattice parallelepiped P. Then, T can be covered by unimodular tetrahedra contained in P.*

Proof. We proceed by induction on the (normalized) volume of T, which is a positive integer. If this volume equals 1 then T is unimodular and there is nothing to prove, so we assume T is not unimodular. Let p_1, p_2, p_3, p_4 denote the vertices of T.

Lemma 2.12 guarantees that one of the corner tetrahedra T_i of the parallelepiped $C(T)$ is contained in P. Without loss of generality, suppose $T_4 = \operatorname{conv}(p_1, p_2, p_3, q_4)$ is in P. By Lemma 2.11, we know that T_4 contains a lattice point other than the p_is, which we denote by u. Then $S = \mathrm{conv}(T \cup \{u\})$ can be triangulated in two different ways: $S = T \cup T_4'$, where $T_4' = \mathrm{conv}(p_1, p_2, p_3, u) \subseteq T_4$ and $S = S_1 \cup S_2 \cup S_3$, with

$$S_1 = \mathrm{conv}(p_2, p_3, p_4, u), S_2 = \mathrm{conv}(p_1, p_3, p_4, u), S_3 = \mathrm{conv}(p_1, p_2, p_4, u).$$

Each of the tetrahedra S_i has lattice volume strictly smaller than T because, for each i, p_i is the unique point of $C(T)$ maximizing the distance to the opposite facet $\mathrm{conv}(p_j, p_k, p_l)$ of T. Thus, S_1, S_2 and S_3 cover T and have volume strictly smaller than T. The S_i may not be empty, but we can triangulate them into empty tetrahedra, which by inductive hypothesis they can be covered unimodularly. □

Proof of Theorem 2.2. Arbitrarily triangulate the parallelepiped into empty lattice tetrahedra and apply Corollary 2.13 to these tetrahedra. □

Let us say that a lattice 3-polytope P *has the circumscribed parallelepiped property* if it satisfies the conclusion of Lemma 2.12: "for every empty tetrahedron T contained in P at least one of the four corner tetrahedra in $C(T)$ is contained in P". If this holds then P has a unimodular cover, since then the proofs of Corollary 2.13 and Theorem 2.2 work for P. Hence, a positive answer to the following question would imply that every smooth 3-polytope has a unimodular cover, which in turn implies Conjecture 2.1(i) in dimension three.

Question 2.14. Does every smooth 3-polytope have the circumscribed parallelepiped property?

Our proof that parallelepipeds have the property (Lemma 2.12) is based on the fact that they have only three (pairs of) normal vectors. The proof, and the property of being IDP, fail if there are four of them:

Example 2.15 (Non-IDP octahedron and triangular prism)**.** The following lattice octahedron Q and triangular prism P are not IDP:

$$Q = \mathrm{conv}((0,1,1), (1,0,1), (1,1,0), (0,-1,-1), (-1,0,-1), (-1,-1,0)), \tag{2.4}$$
$$P = \mathrm{conv}((0,1,1), (1,0,1), (1,1,0), (-1,0,0), (0,-1,0), (0,0,-1)). \tag{2.5}$$

Indeed, in both polytopes the only lattice points are the six vertices and the origin. The point $(1,1,1)$ lies in the second dilation but is not the sum of two lattice points in the polytope. Hence, they are not IDP, which implies they do not admit unimodular covers.

2.2 Cayley sums

We now turn our attention to Theorem 2.7. A triangulation of $\mathrm{Cay}(P,Q) \subset \mathbb{R}^3$ consists of tetrahedra of types $(1,3)$, $(2,2)$ and $(3,1)$, where the type denotes how many vertices they have in P and in Q. Empty tetrahedra of types $(1,3)$ or $(3,1)$, which are Cayley sums of a triangle in P and a point in Q, or viceversa, are automatically unimodular. The case that we need to study are therefore tetrahedra of type $(2,2)$, which are Cayley sums of a segment $p \subset P$ and a segment $q \subset Q$. The following lemma, whose proof we postpone to Section 2.3, is crucial to understand how to unimodularly cover these tetrahedra. We use the following conventions: if a, b are points, we denote by $[a,b]$ and (a,b) respectively the closed and open segments with endpoints a, b. Given a segment $s = [a,b]$, we denote the vector $\overrightarrow{s} := b - a$ and the line spanned by $\overrightarrow{s}$ by $\langle \overrightarrow{s} \rangle$.

Lemma 2.16. *Let Q be a two-dimensional lattice polytope and P a weak Minkowski summand of it. Let $p = [p_1, p_2] \subset P$ and $q = [q_1, q_2] \subset Q$ be two primitive and non-parallel lattice segments, and let $\langle \overrightarrow{p} \rangle$ and $\langle \overrightarrow{q} \rangle$ be the lines spanned by them. If the parallelogram $p + q$ is not unimodular, then at least one of the regions*

$$((p_1, p_2) + \langle \overrightarrow{q} \rangle) \cap P, \qquad \text{and} \qquad ((q_1, q_2) + \langle \overrightarrow{p} \rangle) \cap Q$$

contains a lattice point. See Figure 2.3.

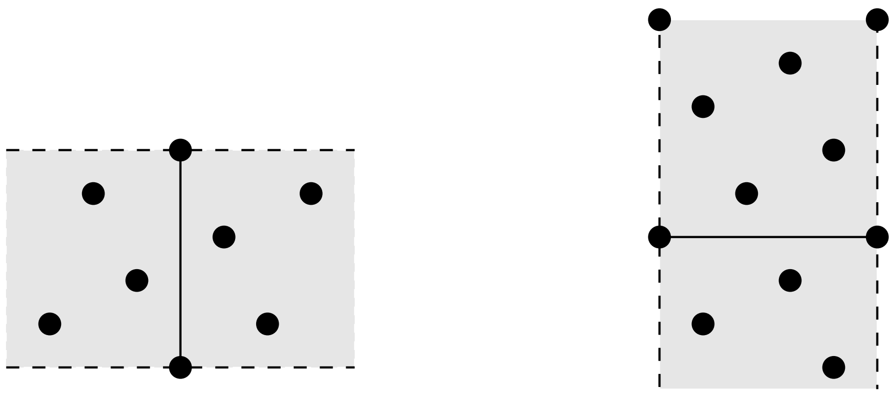


FIGURE 2.3: The strips of Lemma 2.16

Corollary 2.17. *Let T be an empty lattice tetrahedron contained in the Cayley sum* $\mathrm{Cay}(P,Q)$*, where Q is a lattice polygon and P is a weak Minkowski summand of Q. Then, T can be covered by unimodular tetrahedra contained in* $\mathrm{Cay}(P,Q)$.

Proof. The proof is by induction on the normalized volume of T, which we assume to be at least 2. This implies that T is of type $(2,2)$, since empty tetrahedra of types $(1,3)$ and $(3,1)$ are unimodular. Thus, T is the Cayley sum of primitive segments $p = [p_1, p_2] \subset P$

and $q = [q_1, q_2] \subset Q$. Let u be the lattice point whose existence is guaranteed by Lemma 2.16. Assume (the other case is similar) that

$$u \in ((p_1, p_2) + \langle \overrightarrow{q} \rangle) \cap P,$$

and call t the triangle $t = \operatorname{conv}(u, p_1, p_2) \subset P$.

Let us denote $\tilde{u}$, $\tilde{p}_1$, $\tilde{p}_2$, $\tilde{q}_1$, $\tilde{q}_2$ the points corresponding to u, p_1, p_2, q_1, q_2 in $\mathrm{Cay}(P, Q)$. That is, $\tilde{p}_i = p \times \{0\}$, $\tilde{q}_i = p \times \{1\}$, and $\tilde{u} = u \times \{0\}$. Observe that the assumption $u \in ((p_1, p_2) + \langle \overrightarrow{q} \rangle$ implies that of the segments $[u, q_i]$ crosses the triangle $\operatorname{conv}(p_1, p_2, q_j)$, where $\{i, j\} = \{1, 2\}$.

In turn, this means that the polytope $\operatorname{conv}(\tilde{u}, \tilde{p}_1, \tilde{p}_2, \tilde{q}_1, \tilde{q}_2) = \mathrm{Cay}(t, q)$ has the following two triangulations:

$$\mathcal{T}^+ := \{\mathrm{Cay}(p, q), \mathrm{Cay}(t, \{q_i\})\},$$
$$\mathcal{T}^- := \{\mathrm{Cay}([p_1, u], q), \mathrm{Cay}([p_2, u], q), \mathrm{Cay}(t, \{q_j\})\}.$$

The tetrahedra $\mathrm{Cay}(t, \{q_j\})$ and $\mathrm{Cay}(t, \{q_i\})$ are unimodular, which implies that $T = \mathrm{Cay}(p, q)$ has volume equal to the sum of the volumes of $\mathrm{Cay}([p_1, u], q)$ and $\mathrm{Cay}([p_2, u], q)$. In particular, we have covered T by the three tetrahedra in $\mathcal{T}^-$, which are of smaller volume and hence have unimodular covers by inductive assumption. □

Proof of Theorem 2.7. Arbitrarily triangulate $\mathrm{Cay}(P, Q)$ into empty lattice tetrahedra and apply Corollary 2.17 to these tetrahedra. □

Let us now show how to derive Corollaries 2.8 and 2.9 from this theorem. *Prismatoids* were defined in [San12] as polytopes whose vertices all lie in two parallel facets. In particular, a *lattice prismatoid* is any d-polytope $SL(\mathbb{Z}, d)$-equivalent to one of the form

$$\operatorname{conv}(Q_1 \times \{0\} \cup Q_2 \times \{k\}),$$

where Q_1, Q_2 are lattice $(d-1)$-polytopes and $k \in \mathbb{Z}_{>0}$. This is almost a generalization of Cayley sums, which would be the case $k = 1$, except the definition of prismatoid requires Q_1 and Q_2 to be full-dimensional, while the Cayley sum only requires this for $Q_1 + Q_2$.

Proposition 2.18. *Let Q_1, Q_2 be two lattice polygons and consider the prismatoid*

$$P := \operatorname{conv}(Q_1 \times \{0\} \cup Q_2 \times \{k\}),$$

with $k \geq 2$. If $P \cap (\mathbb{R}^2 \times \{1\})$ is a lattice polygon then P has a unimodular cover.

Proof. The condition that $P \cap (\mathbb{R}^2 \times \{1\})$ is a lattice polygon implies the same for $P \cap (\mathbb{R}^2 \times \{i\})$, for every i. Indeed, the condition implies that every edge of $\mathrm{Cay}(P, Q)$ of the form $[u \times \{0\}, v \times \{k\}]$ has a lattice point in $\mathbb{R}^2 \times \{i\}$, and hence it has a lattice point in $P \cap (\mathbb{R}^2 \times \{i\})$, for every i.

Observe that for every $i \in \{1, \ldots, k-1\}$ the intersection $P \cap (\mathbb{R}^2 \times \{i\})$ has the same normal fan as $Q_1 + Q_2$. Thus, each slice

$$P \cap (\mathbb{R}^2 \times [i-1, i])$$

is a Cayley polytope. For $i \in \{2, \ldots, k-1\}$, both bases have the same normal fan (and therefore each is a weak Minkowski summand of the other); for $i \in \{1, k\}$ one base is a weak Minkowski summand of the other. We can therefore apply Theorem 2.7 to each slice and combine the covers thus obtained to get a unimodular cover of P. □

Proof of Corollaries 2.8 and 2.9. In both cases the polytope under study satisfies the hypotheses of Proposition 2.18: in Corollary 2.8, the smoothness of the prismatoid implies that every edge of the form $[u \times \{0\}, v \times \{k\}]$ has lattice points in all slices. In Corollary 2.9, since P has width one, $P \cong \mathrm{Cay}(Q_1, Q_2)$ for some Q_1 and Q_2. Hence,

$$kP \cap (\mathbb{R}^2 \times \{1\}) = (k-1)Q_1 + Q_2. \qquad \square$$

2.3 Proof of Lemma 2.16

Let f_q be the primitive lattice functional constant on q and f_p the one constant on p. We assume that $f_q(p_1) < f_q(p_2)$ and $f_p(q_1) < f_p(q_2)$.

Observe that in the strip $q + \langle \overrightarrow{p} \rangle$, there is a unique lattice point on the line $f_q(x) = -1$; indeed, since q is primitive, the only way that in the strip there could be two lattice points on $f_q(x) = -1$ is if they were on the boundary of the strip, which would however imply that $p + q$ is a unimodular paralellogram, against our assumptions. Since translating the polytopes by lattice vectors will not result in any loss of generality, we can assume that p_1 is that unique lattice point. That is, $f_q(p_1) = -1$, or equivalently, the triangle $\mathrm{conv}(q_1, q_2, p_1)$ is unimodular. Similarly, the unique lattice point in the strip on the line $f_q(x) = 1$ is then $q_1 + q_2 - p_1$.

We let $H_1 = \{f_q(x) \leq 0\}$ and $H_2 = \{f_q(x) \geq 0\}$; similarly let $V_1 = \{f_p(x) \leq 0\}$ and $V_2 = \{f_p(x) \geq 0\}$. In the figures, we draw p as a vertical segment and q as a horizontal one, so that $H_i \cap V_j$ are the four quadrants. See Figure 2.4.

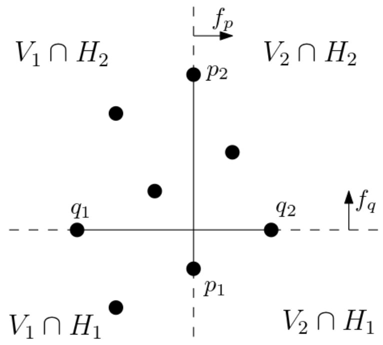


FIGURE 2.4: Setup for the proof of Lemma 2.16

Let $w = \text{area}(p+q) \geq 2$, where area denotes the area normalized to a fundamental domain. Then:

$$w = \text{width}_{f_q}(p + \langle \overrightarrow{q} \rangle) = \text{width}_{f_q}(p) = \text{width}_{f_p}(q) = \text{width}_{f_p}(q + \langle \overrightarrow{p} \rangle).$$

Proof of Lemma 2.16. Suppose by contradiction that there is no lattice point as described in the lemma. In particular, no lattice point on the boundary of Q can be in the interior of the strip $q + \langle \overrightarrow{p} \rangle$. Thus the boundary of Q contains two primitive segments which each have one vertex on each side of the strip $q + \langle \overrightarrow{p} \rangle$; we will call these $b = [b_1, b_2], t = [t_1, t_2]$, with b and t crossing the strip in H_1 and H_2 respectively and the convention that $f_p(b_2) > f_p(b_1)$ and $f_p(t_2) > f_p(t_1)$. This readily implies

$$\begin{aligned} f_p(t_1) \leq f_p(q_1), \quad f_p(t_2) \geq f_p(q_2), \\ f_p(b_1) \leq f_p(q_1), \quad f_p(b_2) \geq f_p(q_2). \end{aligned} \tag{2.6}$$

The same holds for P and the strip $p + \langle \overrightarrow{q} \rangle$, and we call the segments $\ell = [l_1, l_2]$ and $r = [r_1, r_2]$, with ℓ and r crossing the strip $p + \langle \overrightarrow{q} \rangle$ in V_1 and V_2 respectively. The only difference is that in the case that P is one dimensional we have $\ell = r = p$. Again we have

$$\begin{aligned} f_q(l_1) \leq f_q(p_1), \quad f_q(l_2) \geq f_q(p_2), \\ f_q(r_1) \leq f_q(p_1), \quad f_q(r_2) \geq f_q(p_2). \end{aligned} \tag{2.7}$$

Observe that a priori one of l and r can coincide with p, if this is on the boundary of P, and similarly one of t, b might be q, if this is on the boundary of Q.

Claim 2.19. *The following inequalities hold,*

$$\text{width}_{f_q}(\ell), \text{width}_{f_q}(r), \text{width}_{f_p}(t), \text{width}_{f_p}(b) \geq w.$$

Each inequality is strict, unless the segment in question coincides with p or q.

Proof. The inequality $\geq w$ follows in each case from equations (2.7) and (2.6).

If one of the inequalities, say the one for ℓ, is not strict, then ℓ has one endpoint on each of the boundary lines of $(p + \langle \overrightarrow{q} \rangle)$. Unless $\ell = p$, one of the endpoints of ℓ is not an endpoint of p, say $l_1 \neq p_1$. Thus the triangle $T = \operatorname{conv}(p_2, p_1, l_1)$ is contained in P and its edge $[p_1, l_1]$ is an integer dilation of q. Since $\operatorname{width}_{f_q}(T) = w \geq 2$, T must contain a lattice point in the interior of the strip. $\square$

Claim 2.20. *$f_q(b_2 - b_1)$ and $f_q(t_2 - t_1)$ are non-zero and have the same sign. That is, f_q achieves its maximum over b and over t on the same halfplane V_1 or V_2.*

Proof. Suppose by contradiction that the maximum of f_q on t lies in V_1 and that the maximum on b lies in V_2.

Then $Q \cap V_2$ is contained in the open strip $\{-1 < f_q(x) < w - 1\}$, of width w. This cannot contain a translated copy of r, since $\operatorname{width}_{f_q}(r) \geq w$, see Figure 2.5. This is a contradiction, since P is a Minkowski summand of Q and therefore Q must have an edge parallel to t. $\square$

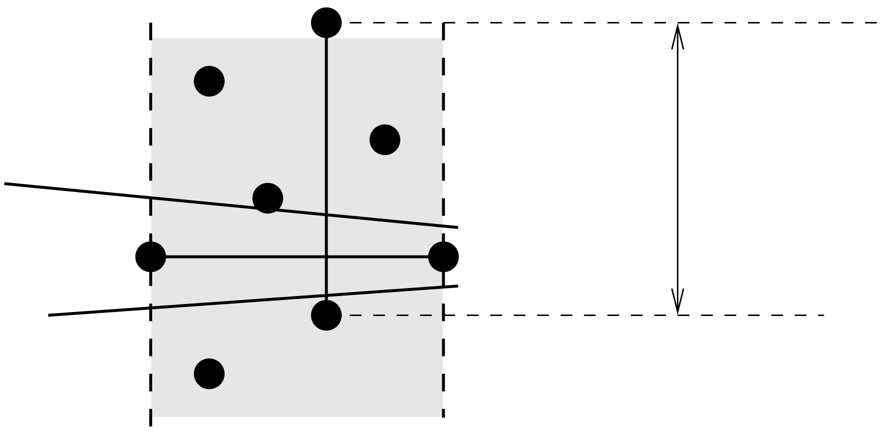


FIGURE 2.5: Illustration of the proof of Claim 2.20

We assume w.l.o.g. that the maximum on t (and hence on b) is achieved in V_2, that is to say, f_p and f_q increase in the same direction along t (and hence along b).

Claim 2.21. *Assume w.l.o.g. that b and t either are parallel or their affine spans cross in V_2. Then,*

1. *The intersection of Q with any line parallel to p in V_2 has width w.r.t. f_q strictly smaller than w.*

2. *$f_p(r_2) > f_p(r_1)$, that is, f_p achieves its maximum over r in H_2.*

Proof. Both t and b must intersect p, otherwise p_1 or p_2 are the lattice points we are looking for in Q. Their intersections with p are thus endpoints of a segment of width w.r.t f_q less than w, the width of p. Since t and b cross in V_2, the same is true for any segment parallel to p contained in $Q \cap V_2$.

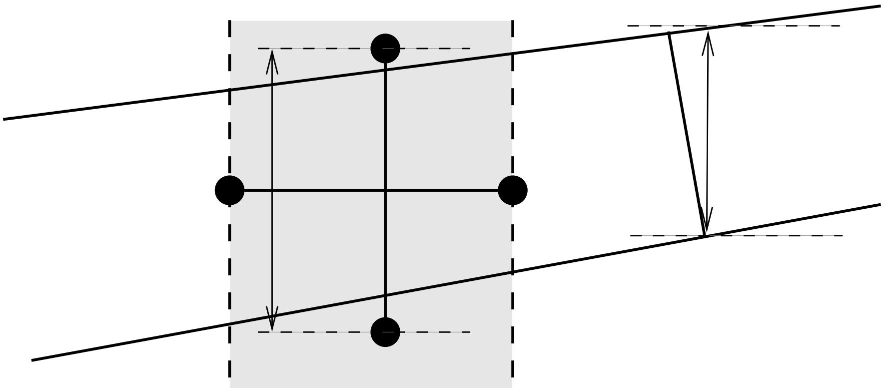


FIGURE 2.6: Illustration of the proof of Claim 2.21

For part (b), If $f_p(r_2) \leq f_p(r_1)$, it would be impossible to fit a translated copy r' of r in the correct side of Q: r would need to lie inside the triangle delimited by the affine line $\langle t \rangle$ and the inequalities $f_q(x) \geq f_q(r_1)$, $f_p(x) \leq f_p(r_1)$. However, this triangular region has width less than w w.r.t. f_q, by combining part (a) with the fact that f_p and f_q increase in the same direction along t, see Figure 2.6. □

The last two claims can be summarized as saying that in the pictures b, t and r have positive slope. Observe that this implies that q is not in the boundary of Q and $p \neq r$, so both P and Q are full dimensional.

Let g be the primitive lattice functional constant on $[p_1, q_2]$ (and therefore constant also on $[q_1, q_1 + q_2 - p_1]$). By the assumption on p_1, the values of g on these segments differer by 1. We choose the sign of g so that

$$g([p_1, q_2]) = g([q_1, q_1 + q_2 - p_1]) + 1.$$

Claim 2.22. *$g(t_1) > g(t_2)$, $g(b_1) > g(b_2)$, and $g(r_1) < g(r_2)$.*

Proof. Since b and t must respectively separate p_1 and $q_1 + q_2 - p_1$ from the other two vertices of the parallelogram $\operatorname{conv}(q_1, p_1, q_2, q_1 + q_2 - p_1)$, they must respectively intersect its (parallel) edges $[p_1, q_2]$ and $[q_1, q_1 + q_2 - p_1]$, which implies the stated inequalities for b and t. The same argument applied to the parallelogram $\operatorname{conv}(p_1, q_2, p_2, p_1 + p_2 - p_2)$, yields the inequalities for ℓ and r. □

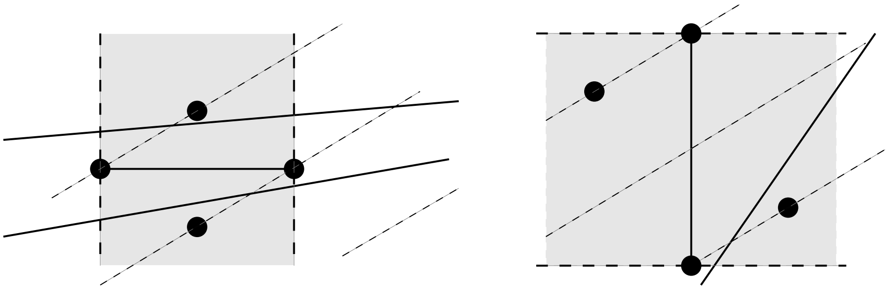


FIGURE 2.7: Illustration of the proof of Claim 2.22

We are now ready to show a contradiction. Since the normal fan of Q refines that of P, Q must have an edge r' which is a translated copy of r. Let r'_1 and r'_2 be its endpoints. Now consider the lattice line d through r'_1 parallel to $[p_1, q_2]$, that is, g is constant on d. Let d' be the parallel line defined by $g(d') = g(d) + 1$.

Consider the segment s contained in $r'_1 + \langle \overrightarrow{p} \rangle$ with endpoints $s_1 = r'_1$ on d and s_2 on d'. Since t separates q_1 and $q_1 + q_2 - p_1$ and g decreases from t_1 to t_2 (by Claim 2.22), the inequality $g(x) < g(d')$ holds on $Q \cap V_2$, and in particular for r'_2. Since r'_2 is a lattice point, $g(r'_2) \leq g(d) = g(r'_1)$, which contradicts Claim 2.22). $\square$

Chapter 3

The flatness constant in dimension three

This chapter focuses on the flatness constant in dimension three. Recall from Section 1.4 that the flatness constant in dimension d is

$$w_c(d) := \sup_{\substack{C \text{ hollow } d\text{-dim} \\ \text{convex body}}} \text{width}(C, \Lambda),$$

and was shown to be finite in each dimension by Khinchin (Theorem 1.14).

Upper bounds of the flatness constant have been widely studied (for an overview of some of these results see Section 1.4). However, the only values of the flatness constant which are known exactly are the trivial $w_c(1) = 1$, and $w_c(2) = 1 + \frac{\sqrt{3}}{2}$, proved by Hurkens in [Hur90]. In Chapter 4, we will see his construction of the hollow triangle which achieves this width in detail.

We show that the flatness constant in dimension three is at least $2+\sqrt{2}$ by constructing an explicit example (section 3.2):

Theorem 3.1. *There is a hollow (non-lattice) tetrahedron Δ of width $2+\sqrt{2} \simeq 3.4142$.*

The tetrahedron Δ of Theorem 3.1 is symmetric with respect to the fcc-cubic lattice and has width $2+\sqrt{2}$ for seven different linear functionals (the three coordinates and four diagonals of the cube). To certify that no integer functional gives smaller width to it we develop in Section 3.1 a method which may be of independent interest, based on existence of long piecewise-linear paths of rational directions.

This tetrahedron maximizes width among a two-parameter family of hollow tetrahedra that contains *two* of the five existing hollow 3-polytopes of width 3 (see Theorem 3.7

in Section 3.2), in much the same way as the value of $w_c(2) = 1 + 2/\sqrt{3}$ is attained by optimizing a perturbation of the second dilation of the unimodular triangle (see details in Section 4.2). This makes us conjecture that:

Conjecture 3.2. *The tetrahedron Δ in Theorem 3.1 is the convex 3-body of largest width; that is, $w_c(3) = 2 + \sqrt{2}$.*

As further evidence for this conjecture, in Section 3.3 we show that a local version of it holds:

Theorem 3.3. *Δ is a strict local maximizer for width among hollow convex 3-bodies.*

3.1 A certificate for width

In Sections 4.2–4.3, we construct explicit examples of polytopes of width larger than their dimension. Before that, we show a heuristic method to certify the width of a convex body. This method indirectly takes advantage of the fact that in our examples the width is attained with respect to *several* different functionals.

By a *rational path* Γ in $\mathbb{R}^d$, with respect to a certain lattice Λ, we mean a concatenation of segments in rational directions. That is, Γ is given as a sequence $p_0, p_1, \dots, p_t$ of points in $\mathbb{R}^d$ such that for every i the vector $p_{i+1} - p_i$ is parallel to a lattice vector. This allows us to define the *lattice length* of each segment $[p_i, p_{i+1}]$ as the scalar $\lambda > 0$ such that $\frac{1}{\lambda}(p_{i+1} - p_i)$ is *primitive*, meaning that it is the shortest integer vector in its direction. The *lattice length* of the rational path Γ is the sum of the lattice lengths of the individual segments; we denote this by $\text{length}_\Lambda(\Gamma)$.

We say that a functional f is *strictly increasing* along Γ if

$$f(p_0) < f(p_1) < \cdots < f(p_t).$$

The *open polar cone* of Γ, denoted $\text{cone}(\Gamma)^\circ$, is the set of functionals $f \in (\mathbb{R}^d)^*$ that are strictly increasing along Γ.[1]

Lemma 3.4. *Let $P \subset \mathbb{R}^d$ be a convex body. Let Γ be a rational path of lattice length w for a certain lattice Λ, with the first and last points of Γ in P. Then any lattice functional in the open polar cone of Γ gives width at least w to P.*

[1]The notation $\text{cone}(\Gamma)^\circ$ comes from the fact that this cone is the (open) polar, in the standard sense, of the cone generated by the vectors $p_{i+1} - p_i$, $i = 1, \dots, t$.

Proof. If $f \in \operatorname{cone}(\Gamma)^\circ$ then

$$\operatorname{width}(P, f) \geq \operatorname{length}_\Lambda(\Gamma) = w,$$

since f takes an integer positive value in the primitive vector parallel to each segment of Γ. $\square$

Remark 3.5. As a consequence of the lemma, if $\Gamma_1, \ldots, \Gamma_k$ is a collection of rational paths with end-points in P, all of length at least w, and with the property that

$$\bigcup_{i=1}^{k} \operatorname{cone}(\Gamma_i)^\circ = \mathbb{R}^d \setminus \{0\},$$

then the lattice width of P is at least w.

Example 3.6. The necessity for using the *open* polar cone $\operatorname{cone}(\Gamma)^\circ$ and not the closed one in Lemma 3.4 can be illustrated by considering P to be the square $[0,1]^2$. The two boundary paths between opposite vertices in P have lattice length two and by Lemma 3.4 this guarantees that the width of P with respect to any functional in $\Lambda^* \setminus (\langle e_1^* \rangle \cup \langle e_2^* \rangle)$ is at least two. But, of course, the width of P with respect to the functionals e_1^* and e_2^* is 1, and these two functionals are weakly increasing along the boundary paths.

3.2 A hollow 3-simplex of width 3.4142

Consider the (dilated) face-centered cubic lattice

$$\Lambda := \{(a, b, c) : a, b, c \in 2\mathbb{Z}, a + b + c \in 4\mathbb{Z}\},$$

with dual

$$\Lambda^* = \left\{(a, b, c) \in \frac{1}{4}\mathbb{Z} : a + b, a + c, b + c \in \frac{1}{2}\mathbb{Z}^3\right\}.$$

Here and in what follows we use the standard coordinates in $(\mathbb{R}^3)^*$, so that (a, b, c) denotes the functional $(x, y, z) \mapsto ax + by + cz$.

For the sake of symmetry, all constructions in this section are with respect to the following *affine* lattice, which is a translation of Λ:

$$\Lambda_{\mathbf{1}} := \Lambda - (1, 1, 1) = \{(a, b, c) : a, b, c \in 1 + 2\mathbb{Z}, a + b + c \in 1 + 4\mathbb{Z}\}.$$

By lattice width with respect to $\Lambda_{\mathbf{1}}$ we mean the lattice width with respect to Λ.

Consider the following lattice tetrahedron (see Figure 3.1) of width three in $\Lambda_{\mathbf{1}}$:

$$\Delta_0 := \operatorname{conv}\{(3,1,5), (-1,3,-5), (-3,-1,5), (1,-3,-5)\}.$$

Δ_0 is (modulo unimodular transformation) the hollow 3-simplex of normalized volume 25 and width 3 that appears in [AKW17, Figure 2] and [AWW11, Figure 1(h)]. We

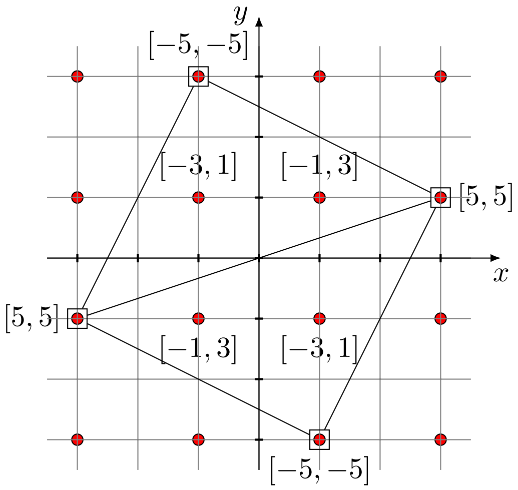


FIGURE 3.1: Projection along the z-axis of the hollow lattice 3-simplex Δ_0 of width three in the lattice $\Lambda_{\mathbf{1}} := \{(a,b,c) : a,b,c \in 1+2\mathbb{Z}, a+b+c \in 1+4\mathbb{Z}\}$. Dots represent (projections of) vertical lattice lines. For those that intersect Δ_0, next to the dot we show the interval of values of z in the intersection. For example, the interval $[-1,3]$ next to the dot with coordinates $(1,1)$ indicates that the points $(1,1,-1)$ and $(1,1,3)$ are in the boundary of Δ_0. At the four vertices of Δ_0 the interval degenerates to a point.

want to modify Δ_0 to a non-hollow simplex of larger width, in the spirit of the previous section. We chose this Δ_0 because it achieves its lattice width only with respect to two lattice functionals, namely $x/4$ and $y/4$. This gives a certain freedom to scale down the z coordinate and enlarge the other two, thus increasing the minimum width. We can simultaneously rotate the whole tetrahedron around the z axis.

To formalize this, we consider the family of tetrahedra that share the following properties with Δ_0: they are circumscribed around the unimodular simplex

$$\operatorname{conv}\{(-1,1,1), (-1,-1,-1), (1,-1,1), (1,1,-1)\},$$

and they are invariant under the order four isometry $(x,y,z) \mapsto (-y,x,-z)$. Put differently, for each $(x,y,z) \in \mathbb{R}^3$ we define $\Delta(x,y,z)$ to be the tetrahedron with vertices

$$A = (x,y,z), \qquad B = (-y,x,-z), \qquad C = (-x,-y,z), \qquad D = (y,-x,-z).$$

We constrain (x,y,z) to satisfy that $(-1,1,1)$, $(-1,-1,-1)$, $(1,-1,1)$ and $(1,1,-1)$ lie, respectively, in the planes containing ABC, BCD, ACD and ABD. By symmetry, these four conditions are equivalent to one another and an easy computation shows that they translate to the equality

$$z = \frac{x^2+y^2}{x^2+y^2-2x-2y}. \tag{3.1}$$

In the rest of this section we show the following, which implies Theorem 3.1:

Theorem 3.7. *Let $(x,y,z) \in \mathbb{R}^3$ be a point satisfying the constraint of Equation (3.1). Then, the width of $\Delta(x,y,z)$ with respect to Λ is at most $2+\sqrt{2}$, with equality if and only if*

$$(x,y,z) \in \left\{ \left(2+\sqrt{2},\ \sqrt{2},\ 2+\sqrt{2}\right),\ \ \left(\sqrt{2},\ 2+\sqrt{2},\ 2+\sqrt{2}\right) \right\}.$$

Proof of the upper bound in Theorem 3.7. Let us consider the functionals $(\frac{1}{2},0,0)$, $(0,\frac{1}{2},0)$, and $(0,0,\frac{1}{2})$, which are in Λ^*. The width of $\Delta(x,y,z)$ with respect to first two equals $\max\{|x|,|y|\}$, and with respect to the third equals $|z|$. We are going to show that whenever $\max\{|x|,|y|\} \geq 2+\sqrt{2}$ we have $|z| \leq 2+\sqrt{2}$. Let

$$f(x,y) := \frac{x^2+y^2}{x^2+y^2-2x-2y}$$

be the function giving z in terms of x and y. The assumption $\max\{|x|,|y|\} \geq 2+\sqrt{2}$ implies that the denominator of f is positive, since it is only negative (or zero) inside (or on) the circle with center $(1,1)$ and radius $\sqrt{2}$. The numerator is also obviously positive, and thus z is positive. The equation

$$f(x,y) = \frac{x^2+y^2}{x^2+y^2-2x-2y} = 2+\sqrt{2}$$

defines again a circle, with center $(\sqrt{2},\sqrt{2})$ and radius 2. Outside the circle z is smaller than $2+\sqrt{2}$ and inside the circle at least one of $|x|$ and $|y|$ is. $\square$

Thus, for the rest of the section we fix $\Delta = \Delta(2+\sqrt{2},\ \ \sqrt{2},\ \ 2+\sqrt{2})$, which has the following vertices and is depicted in Figure 3.2:

$$\begin{aligned} A &= (\ 2+\sqrt{2},\ \ \sqrt{2},\ \ 2+\sqrt{2}), & B &= (-\sqrt{2},\ \ 2+\sqrt{2},\ \ -2-\sqrt{2}), \\ C &= (-2-\sqrt{2},\ \ -\sqrt{2},\ \ 2+\sqrt{2}), & D &= (\ \sqrt{2},\ \ -2-\sqrt{2},\ \ -2-\sqrt{2}). \end{aligned} \tag{3.2}$$

Observe that the width of Δ with respect to the following 14 lattice functionals equals $2+\sqrt{2}$:

$$\pm\tfrac{1}{2}(1,0,0),\quad \pm\tfrac{1}{2}(0,1,0),\quad \pm\tfrac{1}{2}(0,0,1),\quad \tfrac{1}{4}(\pm1,\pm1,\pm1). \tag{3.3}$$

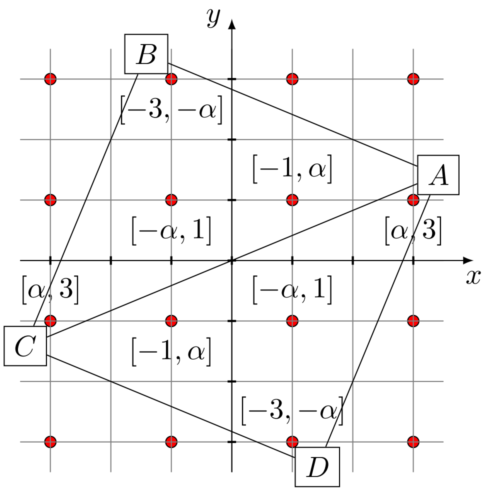


FIGURE 3.2: The hollow 3-simplex Δ of width $2+\sqrt{2}$, drawn with the same conventions as in Figure 3.1. We abbreviate $\alpha = 1+\sqrt{2}$.

We now prove that this is the width of Δ.

Proof of the equality in Theorem 3.7. In Figure 3.2 we have written, next to each vertical lattice line $\ell = (x_0, y_0) \times \mathbb{R}$ intersected by Δ, the interval $\{z \in \mathbb{R} : (x_0, y_0, z) \in \Delta\}$. Hollowness follows from this information, since the intervals do not contain points of Λ in their interior. To check correctness of these computations observe that the facet-defining inequality for triangle ABC is

$$z \leq \frac{x-y}{\sqrt{2}} - y + 2 + \sqrt{2}.$$

Plugging in the coordinates $(x_0, y_0) \in \{(-3,-1), (-1,1), (-1,3), (1,1)\}$ of the four vertical lines meeting the triangle ABC we get that the highest points of Δ on each are indeed

$$(-3,-1,\mathbf{3}),\ (-1,1,\mathbf{1}),\ (-1,3,\mathbf{-1-\sqrt{2}}),\ (1,1,\mathbf{1+\sqrt{2}}).$$

The rest of upper and lower bounds for the intervals in Figure 3.2 follow by symmetry.

To show that the width is at least $2+\sqrt{2}$ we apply Lemma 3.4 to various paths. For example, the expression

$$D = C + \left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right)(4,0,0) + \frac{1}{2}(0,-4,0) + \left(1 + \frac{\sqrt{2}}{2}\right)(0,0,-4).$$

gives a rational path from vertex C to vertex D with directions $(4,0,0)$, $(0,-4,0)$ and $(0,0,-4)$ and of length

$$\left(\frac{1}{2} + \frac{\sqrt{2}}{2}\right) + \frac{1}{2} + \left(1 + \frac{\sqrt{2}}{2}\right) = 2 + \sqrt{2}.$$

The open polar cone of this rational path is the octant $\{(a,b,c) \in (\mathbb{R}^3)^* : a > 0, b < 0, c < 0\}$, so all lattice functionals in the interior of the octant give width at least $2+\sqrt{2}$ to Δ. The same path in reverse implies the same for the opposite octant, and the symmetry of order 4 in Δ implies it for the eight open octants.

We now define a second family of paths whose open polar cones are the connected components of $(\mathbb{R}^3)^* \setminus \{(a,b,c) : a = \pm b\}$. (Observe that these are non-pointed cones). The first one goes from B to D based on the equality

$$D = B + (-2,-2,0) + (1+\sqrt{2})(2,-2,0).$$

Its length is $1 + (1+\sqrt{2}) = 2 + \sqrt{2}$ and its open polar cone is

$$\{(a,b,c) : a + b < 0, a - b > 0\}.$$

Again, symmetry of Δ gives paths for the other three connected components of $(\mathbb{R}^3)^* \setminus \{(a,b,c) : a = \pm b\}$.

Together, these two sets of paths show width $\geq 2+\sqrt{2}$ for all lattice functionals except for the integer multiples of $\frac{1}{2}(0,0,1)$, $\frac{1}{2}(1,1,0)$ and $\frac{1}{2}(1,-1,0)$. These three give widths $2+\sqrt{2}$, $2+2\sqrt{2}$ and $2+2\sqrt{2}$ to Δ, respectively. $\square$

Remark 3.8. The family of tetrahedra $\Delta(x,y,z)$ also contains

$$\Delta(3,3,3) = \operatorname{conv}\{(3,3,3),(3,-3,-3),(-3,3,-3),(-3,3,-3)\},$$

which is the third dilation of a unimodular simplex. In this sense, $\Delta(x,y,z)$ is a common generalization of two of the three existing lattice tetrahedra of maximal width [AKW17]. This is further motivation for Conjecture 3.2.

3.3 Local maximality of the tetrahedron Δ

As further evidence for Conjecture 3.2, we can prove a local version of it, namely:

Theorem 3.9. *Δ is a strict local maximizer for width among hollow tetrahedra. That is, every small perturbation of Δ is either non-hollow or has width strictly smaller than $2+\sqrt{2}$.*

Corollary 3.10. *Δ is a strict local maximizer for width among hollow convex 3-bodies. That is, every convex body K in a neighborhood of Δ is either non-hollow or has width strictly smaller than $2+\sqrt{2}$.*

Proof. Let $p_1, \dots, p_4$ denote lattice points lying respectively in the relative interior of the four facets of Δ; their specific coordinates are given in Section 3.3.1, and are not needed here. Let K be a hollow convex body in a neighborhood of Δ. Let $H_1, \dots, H_4$ be planes weakly separating K from $p_1, \dots, p_4$, respectively. The fact that K is close to Δ and each p_i is in the relative interior of a different facet of Δ implies that $H_1, \dots, H_4$ are close to the facet planes of Δ. The tetrahedron Δ' defined by $H_1, \dots, H_4$ contains K and by Theorem 3.9 has width bounded by $2 + \sqrt{2}$. □

To prove Theorem 3.9, in Section 3.3.1 we transform it into the more explicit Theorem 3.11. Then, in Section 3.3.2 and Section 3.3.3 we give two proofs of the latter; the first one uses the KKT criterion, and the second one is more direct and elementary, although it amounts to the same computations.

3.3.1 Setting the problem

To prove Theorem 3.9 we find more convenient to look at perturbations of the lattice, keeping Δ fixed, rather than the other way around. That is to say, we fix Δ to have the vertex coordinates shown in (3.2) and let $\Lambda(\mathbf{t})$ be the affine lattice generated by:

$$\begin{aligned} p_1 &= (-1,-1,-1) + (t_{11}, t_{12}, t_{13}), \qquad & p_2 &= (1,-1,1) + (t_{21}, t_{22}, t_{23}) \\ p_3 &= (1,1,-1) + (t_{31}, t_{32}, t_{33}), \qquad & p_4 &= (-1,1,1) + (t_{41}, t_{42}, t_{43}) \end{aligned}$$

where the t_{ij}'s are variables. Observe that $\Lambda(\mathbf{0}) = \Lambda$. Our task is to study the width of Δ with respect to $\Lambda(\mathbf{t})$ as a function of $\mathbf{t}$ and show that $\mathbf{0}$ is a strict local maximizer of it, under the constraint that Δ is hollow.

Since a tetrahedron of maximal width necessarily has at least one lattice point on (the relative interior of) every facet, and since the facets of Δ contain each a single point of Λ, there is no loss of generality in constraining the variables t_{ij} to values where we have the coplanarities $a_1a_2a_3p_4$, $a_1a_2p_3a_4$, $a_1p_2a_3a_4$ and $p_1a_2a_3a_4$. In practice this means we can express the t_{*3}'s in terms of the t_{*1}'s and t_{*2}'s as follows:

$$\begin{aligned} t_{13} &= -\frac{(2+\sqrt{2})t_{11} + \sqrt{2}t_{12}}{2}, \qquad & t_{23} &= \frac{-\sqrt{2}t_{21} + (2+\sqrt{2})t_{22}}{2}, \\ t_{33} &= \frac{(2+\sqrt{2})t_{31} + \sqrt{2}t_{32}}{2}, \qquad & t_{43} &= \frac{\sqrt{2}t_{41} - (2+\sqrt{2})t_{42}}{2}. \end{aligned}$$

Thus, in what follows we denote

$$\mathbf{t} := (t_{11}, t_{12}, t_{21}, t_{22}, t_{31}, t_{32}, t_{41}, t_{42})$$

our vector of only eight variables.

In this setting the seven functionals that attain the maximum width of Δ are no longer linear in $\mathbf{t}$. To derive their exact form, consider the 3×3 matrix

$$M(\mathbf{t}) = \begin{pmatrix} p_4(\mathbf{t}) - p_1(\mathbf{t}) \\ p_2(\mathbf{t}) - p_1(\mathbf{t}) \\ p_3(\mathbf{t}) - p_1(\mathbf{t}) \end{pmatrix}$$

as a function of $\mathbf{t}$. The rows of M are a basis for the linear lattice $\overrightarrow{\Lambda}(\mathbf{t})$, so the columns of its inverse $N(\mathbf{t}) := M(\mathbf{t})^{-1}$ form the corresponding dual basis in $\overrightarrow{\Lambda}(\mathbf{t})^*$. That is, the columns of $N(\mathbf{0})$ are the functionals

$$\frac{1}{4}(-1,1,1),\ \frac{1}{4}(1,-1,1),\ \frac{1}{4}(1,1,-1) \in \overrightarrow{\Lambda}^*$$

and the columns of $N(\mathbf{t})$ are their respective perturbations in $\overrightarrow{\Lambda}(\mathbf{t})^*$. Hence, denoting $N^i(\mathbf{t})$ the i-th column of $N(\mathbf{t})$, the seven lattice functionals that attain the maximum width of Δ at $\mathbf{t} = \mathbf{0}$ are

$$c_{+++}(\mathbf{t}) := N^1(\mathbf{t}) + N^2(\mathbf{t}) + N^3(\mathbf{t}), \qquad c_{-++}(\mathbf{t}) := N^1(\mathbf{t}),$$
$$c_{+-+}(\mathbf{t}) := N^2(\mathbf{t}), \qquad c_{++-}(\mathbf{t}) := N^3(\mathbf{t}),$$

$$\begin{aligned} c_x(\mathbf{t}) &:= N^2(\mathbf{t}) + N^3(\mathbf{t}), \\ c_y(\mathbf{t}) &:= N^1(\mathbf{t}) + N^3(\mathbf{t}), \\ c_z(\mathbf{t}) &:= N^1(\mathbf{t}) + N^2(\mathbf{t}). \end{aligned}$$

In Theorem 3.11 below, we will show that for every $\mathbf{t}$ close enough to $\mathbf{0}$ at least one of these functionals gives width less than $2 + \sqrt{2}$ to Δ, which implies Theorem 3.9.

The width of Δ with respect to $c_z(\mathbf{t})$ is difficult to express because $c_z(\mathbf{0})$ attains its maximum at two of the vertices of Δ (a_1 and a_3) and its minimum at the other two (a_2 and a_4). But for each of the other six functionals, at $\mathbf{t} = \mathbf{0}$ we have a unique maximizing and minimizing vertex of Δ. Hence, for $\mathbf{t}$ close to $\mathbf{0}$ those vertices still maximize and minimize, and we get closed expressions for the width of Δ with respect to each functional:

$$\begin{aligned}
f_1(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{+++}(\mathbf{t})) = \langle c_{+++}(\mathbf{t}), a_1 - a_4\rangle,\\
f_2(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{-++}(\mathbf{t})) = \langle c_{-++}(\mathbf{t}), a_3 - a_4\rangle,\\
f_3(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{++-}(\mathbf{t})) = \langle c_{++-}(\mathbf{t}), a_2 - a_3\rangle,\\
f_4(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{+-+}(\mathbf{t})) = \langle c_{+-+}(\mathbf{t}), a_1 - a_2\rangle,\\
f_5(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{x}(\mathbf{t})) = \langle c_{x}(\mathbf{t}), a_1 - a_3\rangle,\\
f_6(\mathbf{t}) &:= \mathrm{width}(\Delta, c_{y}(\mathbf{t})) = \langle c_{y}(\mathbf{t}), a_2 - a_4\rangle.
\end{aligned}$$

Then, Theorem 3.9 is a direct consequence of the following statement, of which we give two proofs in the rest of the paper:

Theorem 3.11. *The system of* 6 *inequalities in eight variables*

$$f_i(\mathbf{t}) \geq 2 + \sqrt{2}, \qquad i \in \{1, \dots, 6\}$$

has an isolated solution at $\mathbf{t} = \mathbf{0}$.

3.3.2 A proof via the KKT theorem

We use the following version of the Karush-Kuhn-Tucker (KKT) conditions for optimality. Suppose we have the following problem on n variables $\mathbf{x} \in \mathbb{R}^n$:

$$\begin{array}{rl}
\text{maximize} & f(\mathbf{x})\\
\text{subject to} & g_i(\mathbf{x}) \geq 0, i = 1, \dots, m.
\end{array}$$

Assume all the functions are twice continuously differentiable.

Define the associated *Lagrangian function* as

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}),$$

where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m) \in \mathbb{R}^m$ is a vector of Lagrange multipliers.

In this setting, the following is Theorem 14.19 in [GNS09]:

Theorem 3.12. *Let* $R = \{\mathbf{x} : g_i(\mathbf{x}) \geq 0 \text{ for all } i\}$ *and let* $\mathbf{x}_*$ *be a point in* R. *Suppose there exists a vector* $\boldsymbol{\lambda}_* \in \mathbb{R}^m$ *such that*

1. $\nabla_{\mathbf{x}} \mathcal{L}(\mathbf{x}_*, \boldsymbol{\lambda}_*) = \mathbf{0}$,

2. $\boldsymbol{\lambda}_* \geq \mathbf{0}$,

3. $\langle \boldsymbol{\lambda}_*, g_i(\mathbf{x}_*) \rangle = 0$ *for every* i,

4. $\nabla^2_{\mathbf{xx}} \mathcal{L}(\mathbf{x}_*, \boldsymbol{\lambda}_*)$ *is negative definite in the subspace* $\{\nabla_{\mathbf{x}} g_i(\mathbf{x}_*) = \mathbf{0}\ \forall i\}$.

Then $\mathbf{x}_*$ *is a strict local maximizer of* f *on* R.

First proof of Theorem 3.11. Let us define $g_i(\mathbf{t}) = f_i(\mathbf{t}) - (2+\sqrt{2})$ for $i = 2, \dots, 6$ and $f(\mathbf{t}) = f_1(\mathbf{t})$. Then, our statement is equivalent to the following: *The origin is a strict local maximizer of* f *on the region* $R := \{\mathbf{t} : g_i(\mathbf{t}) \geq 0 \quad \forall i \in \{2, \dots, 6\}\}$. We prove this by applying Theorem 3.12 to the functions $f(\mathbf{t})$ and $g_i(\mathbf{t})$. All computations are done in `Sagemath` [SageMath].

We first compute the gradients of the functions $f, g_2, \dots, g_6$ and evaluate them at $\mathbf{0}$, obtaining:

$$\begin{aligned}
\nabla f(\mathbf{0}) &= \frac{1}{4}(-1,1,-1,-2,0,0,-2,1) && + \frac{\sqrt{2}}{8}(-2,0,1,-1,0,0,-3,1)\\
\nabla g_2(\mathbf{0}) &= \frac{1}{4}(-2,1,0,0,1,2,1,1) && + \frac{\sqrt{2}}{8}(-1,-1,0,0,1,3,0,2)\\
\nabla g_3(\mathbf{0}) &= \frac{1}{4}(0,0,2,-1,1,-1,1,2) && + \frac{\sqrt{2}}{8}(0,0,3,-1,2,0,-1,1)\\
\nabla g_4(\mathbf{0}) &= \frac{1}{4}(-1,-2,-1,-2,2,-1,0,0) && + \frac{\sqrt{2}}{8}(-1,-3,0,-2,1,1,0,0)\\
\nabla g_5(\mathbf{0}) &= \frac{1}{2}(1,0,-1,0,-1,0,1,0) && + \frac{\sqrt{2}}{2}(1,0,0,0,-1,0,0,0)\\
\nabla g_6(\mathbf{0}) &= \frac{1}{2}(0,1,0,1,0,-1,0,-1) && + \frac{\sqrt{2}}{2}(0,0,0,1,0,0,0,-1)
\end{aligned}$$

This set of six vectors happens to have rank five. The following is the unique dependence among them:

$$\nabla f(\mathbf{0}) + \nabla g_2(\mathbf{0}) + \nabla g_3(\mathbf{0}) + \nabla g_4(\mathbf{0}) + \sqrt{2}\big(\nabla g_5(\mathbf{0}) + \nabla g_6(\mathbf{0})\big) = 0.$$

Observe that the coefficients of the dependence are all positive. We define $\boldsymbol{\lambda}_*$ as the vector of these coefficients (forgetting the coefficient of ∇f):

$$\boldsymbol{\lambda}_* := (1,1,1,\sqrt{2},\sqrt{2}).$$

The Lagrangian function at $\boldsymbol{\lambda}_*$ is thus:

$$\mathcal{L}(\mathbf{t}, \boldsymbol{\lambda}_*) = f(\mathbf{t}) + g_2(\mathbf{t}) + g_3(\mathbf{t}) + g_4(\mathbf{t}) + \sqrt{2}\big(g_5(\mathbf{t}) + g_6(\mathbf{t})\big).$$

The linear dependence among the gradients at $\mathbf{0}$ implies the first condition in Theorem 3.12, namely

$$\nabla\mathcal{L}(\mathbf{0}, \boldsymbol{\lambda}_*) = \mathbf{0}.$$

Condition (2) is true by construction and condition (3) is obvious since $g_i(\mathbf{0}) = \mathbf{0}$.

Thus, only condition (4) is still to be verified. For this we need to compute the Hessian of $\mathcal{L}(\mathbf{0}, \boldsymbol{\lambda}_*)$ in the 3-dimensional vector subspace $\{\nabla g_i(\mathbf{0}) = 0 : i = 2, \dots, 6\}$. This subspace admits the parametric form $\{\mathbf{v}(s_1, s_2, s_3) : s_1, s_2, s_3 \in \mathbb{R}\}$, where

$$\begin{aligned}\mathbf{v}(s_1, s_2, s_3) :=& \left(1, 0, 0, 0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}-2}{2}\right) s_1 \\ &+ \left(0, 1, 0, -\sqrt{2}, \frac{2-\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{2-3\sqrt{2}}{2}\right) s_2 \\ &+ \left(0, 0, 1, -1, 1-\sqrt{2}, 1, 0, -\sqrt{2}\right) s_3.\end{aligned}$$

The Hessian of $\mathcal{L}(\mathbf{v}(s_1, s_2, s_3), \boldsymbol{\lambda}_*)$ at $s_1 = s_2 = s_3 = 0$ is

$$\nabla^2_{\mathbf{tt}}\mathcal{L}(\mathbf{v}(0,0,0), \boldsymbol{\lambda}_*) = \begin{bmatrix} -\frac{19\sqrt{2}}{2} - 13 & -\frac{5\sqrt{2}}{2} - 4 & -\sqrt{2} - 2 \\ -\frac{5\sqrt{2}}{2} - 4 & -\frac{39\sqrt{2}}{2} - 27 & -16\sqrt{2} - 22 \\ -\sqrt{2} - 2 & -16\sqrt{2} - 22 & -19\sqrt{2} - 26 \end{bmatrix},$$

which is indeed negative definite. □

3.3.3 A proof using linear functions as multipliers

We keep the notation of the previous section, except we now define $g_i(\mathbf{t}) = f_i(\mathbf{t}) - (2+\sqrt{2})$ for $i = 1, \dots, 6$ (instead of only $i \geq 2$) and have no functional f. We have seen that there is a unique positive dependence $\sum_{i=1}^{6} \lambda_i \nabla g_i(\mathbf{0}) = \mathbf{0}$ between $\nabla g_i(\mathbf{0})$, namely

$$\boldsymbol{\lambda}_* = (1, 1, 1, 1, \sqrt{2}, \sqrt{2}).$$

Using Taylor expansion we consider the functions $\lambda_i g_i$ decomposed into a linear term (gradient), a quadratic term (Hessian) and higher order terms:

$$\lambda_i g_i(\mathbf{t}) = \underbrace{l_i(\mathbf{t})}_{\text{linear}} + \underbrace{q_i(\mathbf{t})}_{\text{quadratic}} + \underbrace{r_i(\mathbf{t})}_{\text{rest}}.$$

The condition $\sum_{i=1}^{6} \lambda_i \nabla g_i(\mathbf{0}) = \mathbf{0}$ just means that $\sum_{i=1}^{6} l_i$ is identically zero. We now consider a positive constant $c \in \mathbb{R}_{\geq 0}$ (to be specified later) and define the function

$$\begin{aligned} g &= \sum_{i=1}^{6} (c - l_i)(l_i + q_i + r_i) \\ &= \underbrace{c \sum_{i=1}^{6} l_i}_{=0} + \underbrace{\sum_{i=1}^{6} (cq_i - l_i^2)}_{=:q \text{ (quadratic)}} + \underbrace{\sum_{i=1}^{6} cr_i - l_i(q_i + r_i)}_{=:r \text{ rest}}. \end{aligned}$$

Lemma 3.13. *If the Hessian of g is negative definite at $\mathbf{t} = \mathbf{0}$ then Theorem 3.11 holds.*

Proof. Since $g = 0$ and $\nabla g = 0$ at the origin, the Hessian being negative definite, there is a neighborhood U_1 of the origin such that g is strictly negative in $U_1 \setminus \{\mathbf{0}\}$. On the other hand, there is another neighborhood U_2 in which all the multipliers $c - l_i$ are positive, since $c > 0$ and $l_i(\mathbf{0}) = 0$.

Thus, for any $\mathbf{t} \in U_1 \cap U_2 \setminus \{\mathbf{0}\}$ there is an i such that $\lambda_i g_i(\mathbf{t}) < 0$; that is, $f_i(\mathbf{t}) \leq 2 + \sqrt{2}$. $\square$

Lemma 3.14. *The Hessian of g is negative definite at $\mathbf{t} = \mathbf{0}$ for any sufficiently small $c > 0$.*

Proof. The Hessian of g is

$$\sum_{i=1}^{6} (cq_i - l_i^2) = c \sum_{i=1}^{6} q_i - \sum_{i=1}^{6} l_i^2.$$

At $c = 0$ this equals $-\sum_{i=1}^{6} l_i^2$, which is negative semi-definite with null-space equal to

$$V = \{\nabla g_i(\mathbf{0}) = \mathbf{0} : i = 2, \ldots, 6\}.$$

This is the same 3-dimensional subspace as in the first proof (we now have an extra gradient $\nabla g_1(\mathbf{0})$ but it is a linear combination of the other five). On the other hand, the other summand $\sum_{i=1}^{6} q_i$ is nothing but the Hessian of the Lagrangian $\mathcal{L}(\mathbf{0}, \boldsymbol{\lambda}_*)$ that we defined in that proof, which we showed to be negative definite on V. Thus, for a sufficiently small c the sum of the two is negative definite. $\square$

Remark 3.15. One advantage of the second proof over the first one is that it gives explicit sufficient conditions for a neighborhood $U = U_1 \cap U_2$ to have $\mathbf{0}$ as the unique solution of the system. As a first step towards constructing an explicit neighborhood U we have checked that any $c \in (0, 0.4)$ is valid for Theorem 3.14.

Chapter 4

Hollow polytopes of large width

In this chapter, we investigate lower bounds on the flatness constants. Recall that in Section 1.4 we introduced the flatness theorem, which states that in a fixed dimension d, the width of hollow convex bodies is bounded above by a constant; we denoted this constant by $w_c(d)$. Here,we are also interested in the following specializations of the constant:

The *flatness constant* is

$$w_c(d) := \sup_{\substack{C \text{ hollow } d\text{-dim} \\ \text{convex body}}} \operatorname{width}(C, \Lambda),$$

while $w_p(d)$, $w_s(d)$ and $w_e(d)$ are the supremums of the same, respectively over all hollow lattice d-polytopes, hollow lattice d-simplices, and empty lattice d-simplices.

Observe that the specializations w_p, w_s and w_e take integer values. Clearly we have

$$w_e(d) \leq w_s(d) \leq w_p(d) \leq w_c(d),$$

and by the flatness theorem (theorem 1.14) we know they are all finite. A first, easy lower bound for $w_s(d)$ (and thus for $w_p(d)$ and $w_c(d)$) is $d \leq w_s(d)$, since $d\Delta_d$, the d-th dilation of the unimodular d-simplex, is a hollow d-dimensional lattice simplex, and its width is exactly d (see Figure 4.1).

To the best of our knowledge, what is known about lower bounds for the flatness constants in low dimension can be summarized as follows.

- Since the d-th dilation of a unimodular d-simplex is hollow and has width d, $w_s(d) \geq d$.

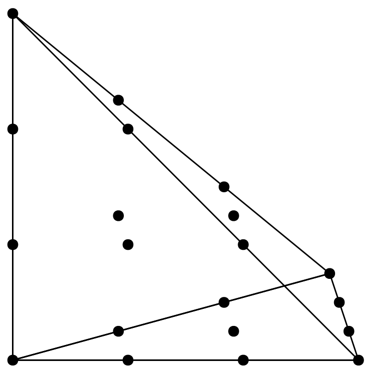


FIGURE 4.1: The third dilation of the standard unimodular tetrahedron; it is hollow and has width 3.

- Sebő [Seb99] showed $w_e(d) \geq d-2$.[1]
- Conway and Thompson (see [MH73, Theorem I.9.5]) showed a lower bound of $\Omega(d)$ for the maximum width of hollow ellipsoids.
- Dash et al. [DDG+14] (Theorem 3.2 and the paragraphs before it) show that

$$3.1547\ldots = 2 + \frac{2}{\sqrt{3}} \leq w_s(3) \leq w_c(3) \leq 1 + \frac{2}{\sqrt{3}} + \left(\frac{90}{\pi}\right)^{1/3} = 4.2439\ldots$$

- The following exact values are known for small d:

d	$w_e(d)$	$w_s(d)$	$w_p(d)$	$w_c(d)$
1	1	1	1	1
2	1	2	2	$1+\frac{2}{\sqrt{3}}$ [Hur90]
3	1 [Whi64]	3 [AKW17]	3 [AKW17]	
4	4 [IVnS19]			

In this paper we establish some new lower bounds, both for specific dimensions (Sections 4.2 and 4.3) and in the asymptotic sense (Section 4.4).

In Sections 4.2 and 4.3, we show $w_p(14) \geq 15$ and $w_s(404) \geq 408$:

Theorem 4.1. *There is a hollow lattice* 14*-polytope of width* 15 *and a hollow lattice* 404*-simplex of width* 408*.*

We do not know of any hollow lattice d-polytope of width larger than d in previous literature, although Francisco Santos has since made us aware in personal communication of a 10-dimensional empty simplex of width 11.

[1] As Sebő points out, for even d the bound can be increased to $d-1$.

Our main technical tool, both in Theorem 4.1 and for the asymptotic results, is to use dilated *direct sums* of polytopes and convex bodies. Let $C_i \subset \mathbb{R}^{d_i}$, $i = 1, \dots, m$, be convex bodies containing the origin. Their *direct sum* [HRGZ97] (sometimes called *free sum* [AB15]) is defined as

$$C_1 \oplus \cdots \oplus C_m := \left\{ (\lambda_1 x_1, \dots, \lambda_m x_m) : x_i \in C_i, \lambda_i \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

For a constant $k \in \mathbb{R}_{\geq 0}$, kC denotes the dilation of C by a factor of k. For a given lattice polytope or convex body C containing the origin (not necessarily in the interior) let us denote $C^{\oplus m} = \bigoplus_{i=1}^{m} mC$, the m-fold direct sum of mC with itself. The following proposition is a particular case of Theorem 4.8 in Section 4.1.

Proposition 4.2. *1.* $\mathrm{width}(C^{\oplus m}) = m\ \mathrm{width}(C)$.

2. If C is hollow then $C^{\oplus m}$ is hollow.

A consequence of this is the following statement, proved in Section 4.4.

Theorem 4.3. *Let $w_* : \mathbb{N} \to \mathbb{R}$ denote any of the functions w_s, w_p, or w_c. Then*

$$\lim_{d\to\infty} \frac{w_*(d)}{d} = \sup_{d\in\mathbb{N}} \frac{w_*(d)}{d}.$$

This, in turn, implies our main asymptotic result:

Theorem 4.4.

$$\lim_{d\to\infty} \frac{w_p(d)}{d} = \lim_{d\to\infty} \frac{w_c(d)}{d} \geq \frac{2+\sqrt{2}}{3} = 1.138\dots$$
$$\lim_{d\to\infty} \frac{w_s(d)}{d} \geq \frac{102}{101} = 1.1\dots$$

Proof. From Theorem 4.3, together with the explicit lower bounds $w_c(3) \geq 2 + \sqrt{2}$ (Theorem 3.1) and $w_s(10) \geq 11$ (personal communication from Francisco Santos), we obtain

$$\lim_{d\to\infty} \frac{w_c(d)}{d} \geq \frac{2}{3} + \frac{\sqrt{2}}{3}, \qquad \lim_{d\to\infty} \frac{w_s(d)}{d} \geq \frac{102}{101}.$$

Thus, we only need to show the equality

$$\sup_{d\in\mathbb{N}} \frac{w_c(d)}{d} = \sup_{d\in\mathbb{N}} \frac{w_p(d)}{d}.$$

The "$\geq$" is obvious. For the "$\leq$", let C be a hollow convex body such that $\mathrm{width}(C)/\dim(C)$ is very close to $\sup_d w_c(d)/d$. We can approximate C arbitrarily by a hollow rational polytope P, and choose an integer m such that mP is a lattice polytope. By Proposition 4.2 we have that $P^{\oplus m}$ is a hollow lattice polytope of dimension $m\dim(C)$ and

$$\frac{\mathrm{width}(P^{\oplus m})}{\dim(P^{\oplus m})} = \frac{\mathrm{width}(P)}{\dim(P)} \simeq \frac{\mathrm{width}(C)}{\dim(C)}. \qquad \square$$

Another implication of our analysis of direct sums together with the values $w_c(2) = 2.1547\ldots$ and $w_c(3) \geq 3.4142\ldots$ (proved in Section 3.2) is

Proposition 4.5 (Section 4.4)**.** *For every d we have*

$$\begin{aligned} w_c(d+1) &\geq w_c(d) + 1, \\ w_c(d+2) &\geq w_c(d) + 2.1547\ldots, \\ w_c(d+3) &\geq w_c(d) + 3.4142\ldots, \end{aligned}$$

As a consequence,

$$\frac{w_c(d)}{d} \geq \frac{1}{2} 2.1547\cdots = 1.0773\ldots \qquad \forall d \geq 2.$$

In Section 4.5 we study the width of empty simplices. We do not know whether $\lim_{d\to\infty} \frac{w_e(d)}{d}$ exists. However, we can prove the following slightly weaker result, using as a base case the empty 10-dimensional simplex of width 11 found by Francisco Santos.

Theorem 4.6 (Section 4.5)**.** *For every $d, m \in \mathbb{N}$ we have*

$$w_e(dm) \geq (m-3)\, w_e(d).$$

In particular,

$$\limsup_{d\to\infty} \frac{w_e(d)}{d} = \sup_{d\in\mathbb{N}} \frac{w_e(d)}{d} \geq 1.1$$

Theorem 4.6 disproves the following guess from [Seb99, p. 403]: "it seems to be reasonable to think that the maximum width of an empty integer simplex in $\mathbb{R}^n$ is n + constant" (unless the constant is zero).

We believe our results are a first step towards the main goal concerning flatness lower bounds, which would be to show that $\sup_d w_*(d)/d = \infty$, at least for w_c.

4.1 Hollow direct sums

Since we will often be using direct sums of polytopes, let us remind the reader of their combinatorial structure.

Lemma 4.7. *Let $P = P_1 \oplus \cdots \oplus P_m$ be a direct sum of polytopes. Then:*

1. *If F_i is a face of P_i that does not contain the origin for each i then the join $F_1 * \cdots * F_m$ of them is a face of P that does not contain the origin of dimension $\sum_i \dim(F_i) + m - 1$. All faces of P that do not contain the origin arise in this way.*

2. *If F_i is a face of P_i that contains the origin for each i then the direct sum $F_1 \oplus \cdots \oplus F_m$ of them is a face of P that contains the origin of dimension $\sum_i \dim(F_i)$. All faces of P that contain the origin arise in this way.*

In particular, the non-zero vertices of P are the points of the form $(0, \dots, 0, v, 0, \dots, 0)$, with v a non-zero vertex of the corresponding P_i, and 0 is a vertex of P if and only if it is a vertex of every P_i. □

Our main technical result is the following theorem. Proposition 4.2 is the case $C_1 = \cdots = C_m$ and $k_i = m$ of it. Part (4) of Theorem 4.8 is equivalent to Corollary 5.5(a) in [AB15].

Theorem 4.8. *Let $C_1, \dots, C_m$ be convex bodies containing the origin and let $k_1, \dots, k_m > 0$ be dilation factors. Let*

$$C := \bigoplus_i k_i C_i = k_1 C_1 \oplus \cdots \oplus k_m C_m.$$

Then:

1. *If $k_i C_i$ is a lattice polytope for every i then C is a lattice polytope.*

2. *If C_i is a simplex with a vertex at the origin for every i then C is a simplex with a vertex at the origin.*

3. *The width of C equals $\min_i\{k_i \operatorname{width}(C_i)\}$.*

4. *If C_i is hollow for every i and $\sum_i \frac{1}{k_i} \geq 1$ then C is hollow.*

Proof. Part (1) is obvious, from the description of the vertices of direct sums in Lemma 4.7. For part (2) let d_i be the dimension of C_i. Each C_i has d_i non-zero vertices plus the origin so, by the same Lemma, C has $d_1 + \cdots + d_m$ vertices plus the origin. Since C lives in dimension $d_1 + \cdots + d_m$, it must be a simplex.

To prove (3), first note that $\mathrm{width}(k_i C_i) = k_i \,\mathrm{width}(C_i)$, so we can assume w.l.o.g. $k_i = 1$ for all i. Let $f_i \in \Lambda_i^*$ be a lattice direction for which $\mathrm{width}(C_i)$ is obtained. Then

$$\begin{aligned}\mathrm{width}(P) &\leq \mathrm{width}(C, (0, \ldots, 0, f_i, 0, \ldots, 0)) \\ &= \mathrm{width}(C_i, f_i) = \mathrm{width}(C_i).\end{aligned}$$

This proves that $\mathrm{width}(C) \leq \min_i\{\mathrm{width}(C_i)\}$. For the other inequality, given any lattice functional $g = (g_1, \ldots, g_m) \in \Lambda^* \setminus \{0\} = \oplus_i \Lambda_i^* \setminus \{0\}$, we want to show that $\mathrm{width}(C, g) \geq \mathrm{width}(C_i)$ for some i. For this, let us choose any i with $g_i \neq 0$. Then:

$$\begin{aligned}\mathrm{width}(C, g) &= \max_{c,c' \in C} |g^\intercal c - g^\intercal c'| \\ &\geq \max_{c_i, c_i' \in C_i} |g^\intercal(0, \ldots, 0, c_i, 0, \ldots, 0) - g^\intercal(0, \ldots, 0, c_i', 0, \ldots, 0)| \\ &= \mathrm{width}(C_i, g_i) \geq \mathrm{width}(C_i) \geq \min_j \mathrm{width}(C_j).\end{aligned}$$

Finally, to prove part (4), suppose by contradiction that C is not hollow, and let $c \in \operatorname{int} C \cap \Lambda$. Since $c \in \operatorname{int} C$, we can write $c = (\lambda_1 k_1 c_1, \ldots, \lambda_m k_m c_m)$ with $c_i \in \operatorname{int} C_i$ and $\lambda_i > 0$ with $\sum \lambda_i = 1$. On the other hand, since $c \in \Lambda$, we know that each $\lambda_i k_i c_i \in \Lambda_i$. Since C_i is hollow and $c_i \in \operatorname{int} C_i$, we have that $\lambda_i k_i > 1$. This implies $\sum \frac{1}{k_i} < \sum \lambda_i = 1$, contradicting our assumption. $\square$

Observe that the assumption that the C_is contain the origin is no loss of generality: lattice polytopes can be translated to have the origin as a vertex; convex bodies can first be enlarged so that they have lattice points in the boundary, then translated. In both cases, the direct sum C of Theorem 4.8 can be constructed using these modified C_is.

4.2 A hollow lattice 14-polytope of width 15

Let A, B and C be the vertices of an equilateral triangle Δ in the plane; without loss of generality, $A = (0, 0)$, $B = (1, 0)$, $C = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Let Λ be the lattice they generate:

$$\Lambda := \left\{(a, b\sqrt{3}) \in \mathbb{R}^2 : 2a, 2b, a + b \in \mathbb{Z}\right\}.$$

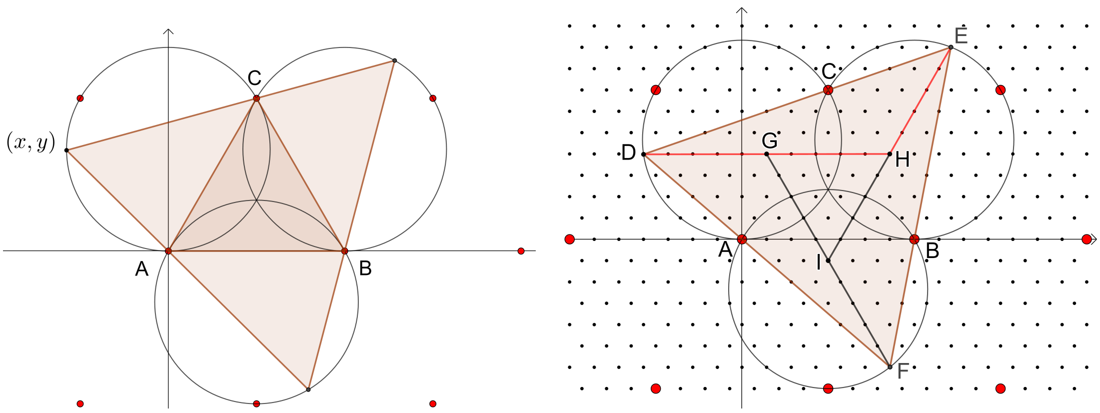


FIGURE 4.2: Left: the hollow equilateral triangle $T(x, y)$ circumscribed to the triangle Δ, depending on the position of (x, y) along the circle S_1 (Hurkens position, maximizing width, is shown in the picture). Right: the refinement of the lattice by a factor of seven creates a lattice point in the circle and close to Hurkens position.

We consider the family $\{T(x, y)\}$ of equilateral triangles circumscribed around Δ, where (x, y) denotes, by convention, the vertex lying between A and C. A point (x, y) defines such a triangle if and only if it lies outside Δ and along the circle

$$S_1 := \left\{ (x, y) : x^2 + \left(y - \frac{\sqrt{3}}{3} \right)^2 = \frac{1}{3} \right\}.$$

It is easy to see that every triangle in the family is hollow. For example, $T(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ is a hollow lattice triangle of width two, unimodularly equivalent to the second dilation of Δ. The triangle $T\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$, pictured in Figure 4.2 (left) maximizes the width of the family and was shown by Hurkens [Hur90] to maximize width among all hollow convex 2-bodies (see also Averkov and Wagner [AW12]).

We now consider the seventh refinement $\Lambda' := \frac{1}{7}\Lambda$ of Λ. The circle S_1 contains, apart from the points A, B, C, additional points of Λ'. In particular, if we fix $T := T(D)$ for the point $D = \left(-\frac{4}{7}, \frac{2\sqrt{3}}{7}\right) \in \Lambda' \cap S_1$ we get a triangle with vertices in Λ' and of width close to the maximum, since D is close to Hurken's point $\left(-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}\right)$ (See Figure 4.2, again). Specifically:

$$T := T(D) = \operatorname{conv}\left(\left(-\frac{4}{7}, \frac{2\sqrt{3}}{7} \right), \left(\frac{17}{14}, \frac{9\sqrt{3}}{14} \right), \left(\frac{6}{7}, -\frac{3\sqrt{3}}{7} \right) \right).$$

Lemma 4.9. *The triangle T defined above is hollow and of width $2 + 1/7 = 2.1419$ with respect to Λ. It is also rational, with its seventh dilation being a lattice triangle.*

Proof. It is clear by our construction that T is hollow with respect to Λ, and since it has its vertices in Λ', its seventh dilation is a lattice triangle of Λ.

We now claim that the width of T in Λ' is 15. It is easy to check that it has width 15 with respect to the three functionals f_0, f_1 and f_2 that define edges of Δ. We call E and F the vertices of T in clockwise order from D. To show that the width of T is at least 15, we apply Lemma 3.4 to each of the three paths DHE (drawn in red in Figure 4.2), EIF and FGD. These paths have lattice length equal to 15. It is easy to see that the polar cones of the paths are $\operatorname{cone}(f_0, f_1)$, $\operatorname{cone}(f_0, f_2)$ and $\operatorname{cone}(f_1, f_2)$, so by Lemma 3.4, functionals in the interior of any of these cones give width at least 15 to T. The only (primitive) functionals not in the open cones are precisely f_0, f_1 and f_2 which, as said above, yield width 15. $\square$

We can now prove the first half of Theorem 4.1:

Theorem 4.10. *$T^{\oplus 7}$ is a 14-dimensional hollow lattice polytope of width 15. It has 21 vertices and 2^7+7 facets (2^7 simplices and seven combinatorially of the form segment$\oplus$triangle$^{\oplus 6}$).*

Proof. The first claim follows from Proposition 4.2 and Lemma 4.9. $T^{\oplus 7}$ has 21 vertices by the description of vertices of direct sums in Lemma 4.7. The same lemma implies the following description of the facets:

1. Facets of $T^{\oplus 7}$ that do not contain the origin are the joins of edges of T that do not contain the origin. Since there are two such edges to chose from in T and joins of simplices are simplices, we obtain the 2^7 stated simplices.

2. Facets of $T^{\oplus 7}$ that contain the origin are of the form

$$T \oplus \cdots \oplus T \oplus F \oplus T \oplus \cdots \oplus T$$

   where F is the edge of T that contains the origin. Since F can be placed anywhere in the sum, we have seven such facets. $\square$

4.3 A hollow lattice 404-simplex of width 408

We now turn back to the question of lower bounds for the specializations of the flatness constant; here we focus on w_s. In particular, we construct a lattice *simplex* of width larger than its dimension. To do this via Theorem 4.8, we need a rational hollow simplex with the origin as a vertex and of width larger than its dimension, which can be found in dimension four. We do not know whether one exists in dimension three.

Lemma 4.11. *There is a rational hollow 4-simplex S of width $4 + 4/101$ and with a lattice vertex whose 101-th dilation is a lattice simplex.*

Proof. It is known that the following lattice 4-simplex is *empty*, that is, it has no lattice points other than its vertices, and it has width four ([HZ00, IVnS19]):

$$S_0 := \operatorname{conv}\{(0,0,0,0),(1,0,0,0),(0,1,0,0),(0,0,1,0),(6,14,17,101)\}.$$

Observe that the facet of S_0 opposite the origin lies in the hyperplane $101x_1 + 101x_2 + 101x_3 - 36x_4 = 101$. Since 101 is coprime with 36, dilating S_0 by a factor of $102/101$ gives a hollow simplex S: apart of the five vertices of S_0 (which lie in the boundary of S) all other lattice points must be in the facet-defining hyperplane $101x_1 + 101x_2 + 101x_3 - 36x_4 = 102$. □

Applying Proposition 4.2 to the hollow simplex S, we obtain that $S^{\oplus 101}$ is a 404-dimensional lattice simplex of width 408. This proves the second half of Theorem 4.1.

Remark 4.12. Any dilation of S_0 by a factor strictly greater than $102/101$ is not hollow anymore, since the point

$$(1,2,3,14) = \frac{17}{101}(1,0,0,0) + \frac{6}{101}(0,1,0,0) + \frac{65}{101}(0,0,1,0) + \frac{14}{101}(6,14,17,101)$$

lies in the relative interior of the facet of S opposite the origin.

4.4 General lower bounds

In this section, we apply Theorem 4.8 to the explicit examples from Sections 4.2–4.3 to obtain lower bounds for $w_c(d)$, $w_p(d)$ and $w_s(d)$ in general dimension d. In particular, we prove Theorem 4.3 and Proposition 4.5.

Corollary 4.13. *For $w_* = w_c$, w_p or w_s we have that*

$$w_*(md) \geq m\, w_*(d), \quad \forall m \in \mathbb{N}.$$

For w_c we have the more general result

$$w_c(d_1 + \cdots + d_m) \geq w_c(d_1) + \cdots + w_c(d_m), \quad \forall d_1, \ldots, d_m \in \mathbb{N}. \tag{4.1}$$

Proof. For the first inequality, let K be a hollow convex d-body (resp., a lattice d-polytope, a lattice d-simplex) achieving $w_c(d)$ (resp. $w_p(d)$, $w_s(d)$ and, in the case of a lattice polytope, assume without loss of generality that the origin is a vertex of K). Then, apply Theorem 4.8 with $C_i = K$ and $k_i = m$ for all i. This gives a dm-dimensional hollow convex body (resp., a lattice polytope, a lattice simplex) of width $m\, w_*(d)$.

For the case of w_c we have more freedom, since we do not need the k_is to be integers. Thus, if for each $i = 1, \ldots, m$ we let the C_is in Theorem 4.8 be hollow convex bodies of width $w_c(d_i)$ and we take $k_i = (\sum_j w_c(d_j))/w_c(d_i)$ for each i, we obtain a hollow convex body C of width $\sum_j w_c(d_j)$ and dimension $\sum_j d_j$. $\square$

Inequality (4.1) implies that w_c is strictly increasing. For w_p and w_s we can only prove weak monotonicity:

Corollary 4.14.

$$w_p(d+1) \geq w_p(d), \quad w_s(d+1) \geq w_s(d).$$

Proof. Let $C_1 = [0,1]$ and let C_2 be a lattice polytope (resp. a hollow simplex) of dimension d and with $\mathrm{width}(C) = w_*(d)$. Apply Theorem 4.8 with $k_1 > w_*(d)$ and $k_2 = 1$. $\square$

Question 4.15. Are w_p, w_s or w_e strictly increasing? Since these w_* take only integer values, strict monotonicity is equivalent to the inequality

$$w_*(d+1) \geq w_*(d) + 1.$$

(For w_e even non-strict monotonicity is unclear, due to its more arithmetic nature).

We can now prove Theorem 4.3 and Proposition 4.5:

Proof of Theorem 4.3. By Corollaries 4.13 and 4.14, the three sequences $w_c(d)$, $w_p(d)$ and $w_s(d)$ satisfy the conditions of the following elementary statement:

If a sequence $(a_d)_{d\in\mathbb{N}}$ satisfies $a_{d+1} \geq a_d$ and $a_{md} \geq m\, a_d$ $\forall d, m \in \mathbb{N}$, then

$$\lim_{d\to\infty} \frac{a_d}{d} = \sup_{d\in\mathbb{N}} \frac{a_d}{d}. \qquad \square$$

Proof of Proposition 4.5. The inequalities

$$\begin{aligned} w_c(d+1) &\geq w_c(d) + 1, \\ w_c(d+2) &\geq w_c(d) + 1 + \tfrac{2}{\sqrt{3}}, \\ w_c(d+3) &\geq w_c(d) + 2 + \sqrt{2}, \end{aligned}$$

follow from applying Equation (4.1) of Corollary 4.13 with $w_c(1) = 1$ $w_c(2) = 1 + \frac{2}{\sqrt{3}}$ [Hur90] and $w_c(3) \geq 2 + \sqrt{2}$ (Section 3.2).

Any integer $d \geq 2$ can be written as $d = 2a + 3b$ for some nonnegative integers a, b. Then for all $d \geq 2$, the inequalities above yield

$$\begin{aligned} w_c(d) \geq w_c(2)a + w_c(3)b &\geq \left(1 + \frac{2}{\sqrt{3}}\right) a + \left(2 + \sqrt{2}\right) b \\ &\geq \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)(2a + 3b) = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right) d. \end{aligned}$$ □

4.5 Lower bound for empty simplices

To prove the asymptotic lower bound of Theorem 4.6, we use the following lemma:

Lemma 4.16. *Let $P = \operatorname{conv}(0, v_1, \ldots, v_d) \subset \mathbb{R}^d$ be an empty d-simplex of width w and let $m \geq 2$ be an integer. For each $i \in [d]$ and $j \in [m]$, let*

$$v_i^{(j)} := 0 \oplus \cdots \oplus v_i \oplus \cdots \oplus 0 \in \mathbb{R}^{md}$$

with v_i in the j-th summand, and define

$$w_i^{(j)} := (m-2)v_i^{(j)} + v_{i+1}^{(j+1)} \in \mathbb{R}^{md},$$

with i taken modulo d and j modulo m. Let

$$P_m := \operatorname{conv}\left(\{0\} \cup \left\{w_i^{(j)} : (i,j) \in [d] \times [m]\right\}\right).$$

Then: (1) $\operatorname{width}(P_m) \geq (m-3)w$*; and* (2) P_m *is empty.*

Proof. Observe that P_m is contained in $P^{\oplus m} = \bigoplus_{j=1}^m mP$ and tries to approximate it: the vertices of $P^{\oplus m}$ are 0 and $\{mv_i^{(j)} : (i,j) \in [d] \times [m]\}$, and the vertices $w_i^{(j)}$ of P_m are close to them.

To prove (1), let $f = f_1 \oplus \cdots \oplus f_m$ be an integer functional. Assume without loss of generality that

$$\max_j \{\operatorname{width}(P, f_j)\} = \operatorname{width}(P, f_1).$$

Let us denote $v_0 = 0$ and let $i^+, i^- \in \{0, \ldots, d\}$ be indices such that $f_1(v_{i^+})$ and $f_1(v_{i^-})$ are the maximum and minimum values of f_1 on P, respectively. Then,

$$\begin{aligned} |f(w_{i^+}^{(1)} - w_{i^-}^{(1)})| &\geq |f_1((m-2)v_{i^+} - (m-2)v_{i^-})| - |f_2(v_{i^++1} - v_{i^-+1})| \\ &\geq (m-2)|f_1(v_{i^+} - v_{i^-})| - |f_1(v_{i^+} - v_{i^-})| \\ &= (m-3)|f_1(v_{i^+} - v_{i^-})| \geq (m-3)w. \end{aligned}$$

For part (2), to search for a contradiction assume P_m is not empty. Let $z \in P_m \cap \mathbb{Z}^{dm}$ be an integer point different from 0 and from the $w_i^{(j)}$s. We can then write z as a convex combination of the vertices of P_m. That is:

$$z = \sum_{j=1}^{m} \sum_{i=1}^{d} \lambda_i^{(j)} w_i^{(j)} = \sum_{j=1}^{m} \sum_{i=1}^{d} ((m-2)\lambda_i^{(j)} + \lambda_{i-1}^{(j-1)}) v_i^{(j)}, \tag{4.2}$$

with $\lambda_i^{(j)} \geq 0$ and $\sum_{i,j} \lambda_i^{(j)} \leq 1$.

But since $P_m \subset P^{\oplus m}$, we can also write

$$z = \mu_1 z_1 \oplus \cdots \oplus \mu_m z_m, \tag{4.3}$$

with each $z_j \in P$, $\mu_j z_j \in \mathbb{Z}^d$, $\mu_j \geq 0$ and $\sum_j \mu_j \leq m$. Comparing Equations (4.2) and (4.3) we obtain

$$\mu_j z_j = \sum_{i=1}^{d} ((m-2)\lambda_i^{(j)} + \lambda_{i-1}^{(j-1)}) v_i. \tag{4.4}$$

Claim: $\sum_i \lambda_i^{(j)} \neq 0$ *for every* j. Indeed, if there is a j where this sum is zero, assume without loss of generality that $\sum_i \lambda_i^{(j-1)} \neq 0$. Then Equation (4.4) gives

$$\mu_j z_j = \sum_{i=1}^{d} \lambda_{i-1}^{(j-1)} v_i,$$

which is a nonzero point in P. Since P is empty and $\mu_j z_j \in \mathbb{Z}^d$, we conclude that one of the $\lambda_i^{(j-1)}$s equals 1, so that $z = w_i^{(j-1)}$, a contradiction because z was assumed not to be a vertex of P_m.

From the claim and Equation (4.4) it follows that $\mu_j z_j \neq 0$ for all j. In order for $\mu_j z_j$ to be a lattice point we need $\mu_j \geq 1$ (because $0 < \mu_j < 1$ implies $\mu_j z_j$ to be a lattice point in P but not a vertex of P, which is not possible). Since on the other hand $\sum_j \mu_j \leq m$, we conclude that $\mu_j = 1$ for every j. This implies that every z_j is a non-zero lattice point of P; that is, for each j there is an i_j such that $z_j = v_{i_j}$. Equation (4.4) now becomes

$$v_{i_j} = \sum_{i=1}^{d} ((m-2)\lambda_i^{(j)} + \lambda_{i-1}^{(j-1)}) v_i.$$

Since the v_is are independent, we have

$$1 = (m-2)\lambda_{i_j}^{(j)} + \lambda_{i_j-1}^{(j-1)}, \qquad \forall j.$$

Summing over j we get the contradiction

$$m = \sum_j (m-2)\lambda_{i_j}^{(j)} + \sum_j \lambda_{i_j - 1}^{(j-1)} \leq (m-2) + 1 = m - 1. \qquad \square$$

Remark 4.17. Lemma 4.16 and its proof generalize Sebő's construction of empty m-simplices of width $m-2$ [Seb99]. Indeed, letting $P = [0,1]$, our lemma gives an empty m-simplex of width (at least) $m-3$. Sebő's $m-2$ is obtained with an additional argument that works for $[0,1]$ but not (as far as we can see) for an arbitrary P.

Proof of Theorem 4.6. Let P be an empty d-simplex of maximum width; that is, with $\mathrm{width}(P) = w_e(d)$. Applying Lemma 4.16 to P we obtain a sequence $(P_m)_{m \in \mathbb{N}}$ of empty md-simplices of width $(m-3)w_e(d)$, which implies $w_e(dm) \geq (m-3)\, w_e(d)$.

From this fact, combined with $w_e(1) = 1$, we obtain

$$\limsup_{d \to \infty} \frac{w_e(d)}{d} = \sup_{d \in \mathbb{N}} \frac{w_e(d)}{d} \geq 1. \qquad \square$$

Chapter 5

Covering radius and a discrete analogue of surface area

Recall from Section 1.4 the definition of the *covering radius* of a convex body K in $\mathbb{R}^d$ with respect to a lattice Λ:

$$\mu_n(K, \Lambda) = \min \left\{ \mu \geq 0 : \mu K + \Lambda = \mathbb{R}^d \right\}.$$

Throughout this chapter, we may leave out the subscript n whenever it is clear that we are dealing with the covering radius, and not lower covering minima. Unless stated otherwise, we consider $\Lambda = \mathbb{Z}^d$ and drop it from the notation. The covering radius will therefore often be simply denoted by $\mu(K)$.

In the chapter, we are interested in upper bounds on the covering radius of non-hollow *lattice polytopes*, that is, polytopes all of whose vertices are lattice points. If we drop the non-hollow condition it is easy to show that the maximum covering radius of a lattice d-polytope equals d, with equality if and only if the polytope is a *unimodular simplex*; that is, one of the form $\operatorname{conv}(\{\mathbf{0}, b_1, \ldots, b_d\})$ where $\{b_1, \ldots, b_d\}$ is a lattice basis for Λ, or a lattice translate of that. (See corollary 5.45 for a proof of a more general statement).

The existence of interior lattice points makes the problem more difficult and interesting. The natural candidate to play the role of the unimodular simplex is

$$S(\mathbf{1}_{d+1}) := \operatorname{conv}(\{-\mathbf{1}_d, e_1, \ldots, e_d\}),$$

since it is the unique non-hollow d-polytope of minimum volume (see [BHW07, Thm. 1.2]). Here $\mathbf{1}_d = (1, \ldots, 1)$ denotes the all-one vector in dimension d, and e_i denotes the ith coordinate unit vector.[1]

The covering radius of $S(\mathbf{1}_{d+1})$ was computed in [GMS17, Prop. 4.9]:

$$\mu(S(\mathbf{1}_{d+1}), \mathbb{Z}^d) = \frac{d}{2}. \tag{5.1}$$

Since the covering radius is additive with respect to direct sums (see Section 5.1), direct sums of simplices of the form $S(\mathbf{1}_l)$ or lattice translates thereof also have covering radius equal to $d/2$. We conjecture that this procedure gives *all* the non-hollow lattice polytopes of maximum covering radius in a given dimension:

Conjecture A. Let $P \subseteq \mathbb{R}^d$ be a non-hollow lattice d-polytope. Then

$$\mu(P) \leq \frac{d}{2},$$

with equality if and only if P is obtained by direct sums and/or translations of simplices of the form $S(\mathbf{1}_l)$.

Example 5.1. In dimension two, $S(\mathbf{1}_3)$ has covering radius 1, and so do the following triangle and square:

$$\begin{aligned} S(\mathbf{1}_2) \oplus ((1 + S(\mathbf{1}_2)) &= \operatorname{conv}(\{(1,0), (-1,0), (0,2)\}), \\ S(\mathbf{1}_2) \oplus S(\mathbf{1}_2) &= \operatorname{conv}(\{(1,0), (-1.0), (0,1), (0,-1)\}). \end{aligned}$$

In dimension three, translations and/or direct sums of the $S(\mathbf{1}_l)$s produce nine pairwise non-equivalent non-hollow 3-polytopes of covering radius 3/2, that we describe in Lemma 5.24.

One motivation for conjecture A comes from a seemingly simple question about covering minima, whose definition we now recall from Section 1.4. The *d-th covering minimum* of a convex body $K \subseteq \mathbb{R}^n$ with respect to a lattice $\Lambda \subseteq \mathbb{R}^n$ (Definition 1.15) is

$$\mu_d(K, \Lambda) := \max_{\pi} \mu(\pi(K), \pi(\Lambda)),$$

where π runs over all linear projections $\pi : \mathbb{R}^n \to \mathbb{R}^d$ such that $\pi(\Lambda)$ is a lattice.

Since $S(\mathbf{1}_{n+1})$ projects to $S(\mathbf{1}_{d+1})$ for every $d < n$, we use (5.1) and get

$$\mu_d(S(\mathbf{1}_{n+1})) \geq \mu_d(S(\mathbf{1}_{d+1})) = \frac{d}{2}. \tag{5.2}$$

[1] The notation $S(\mathbf{1}_{d+1})$ comes from the fact that this is a particular case of the simplices $S(\omega)$, $\omega \in \mathbb{R}^{d+1}_{>0}$ introduced below. We call $S(\mathbf{1}_{d+1})$ the *standard terminal simplex* since *terminal* is used in the literature for lattice simplices with the origin in the interior and no lattice points other than the origin and the vertices.

The converse inequality was conjectured in [GMS17]:

Conjecture B ([GMS17, Rem. 4.10])**.** For every $n \in \mathbb{N}$ and $d \leq n$,

$$\mu_d(S(\mathbf{1}_{n+1})) = \frac{d}{2}. \tag{5.3}$$

In section 5.2 we prove:

Theorem 5.2 (Equivalence of Conjectures A and B)**.** *For each $d \in \mathbb{N}$, the following are equivalent:*

(i) $\mu(P) \leq \frac{\ell}{2}$ for every non-hollow lattice ℓ-polytope P and for every $\ell \leq d$.

(ii) conjecture B holds for every $\ell \leq d$. That is, $\mu_\ell(S(\mathbf{1}_{n+1})) = \frac{\ell}{2}$, for every $\ell, n \in \mathbb{N}$ with $\ell \leq d \leq n$.

Theorem 5.3 (Corollary 5.21 and Theorem 5.29)**.** *Conjecture A, hence also conjecture B, holds in dimension up to three.*

The computation of the covering radius for $S(\mathbf{1}_{d+1})$ can be generalized to the following class of simplices: For each $\omega = (\omega_0, \ldots, \omega_d) \in \mathbb{R}^{d+1}_{>0}$, we define

$$S(\omega) := \operatorname{conv}(\{-\omega_0 \mathbf{1}_d, \omega_1 e_1, \ldots, \omega_d e_d\}).$$

In section 5.4 we derive the following closed formula for $\mu(S(\omega))$. Therein and in the rest of the paper we denote by $\mathrm{Vol}_\Lambda(K)$ the *normalized volume* of a convex body K with respect to a lattice Λ, which equals the Euclidean volume $\mathrm{vol}(K)$ of K normalized such that a unimodular simplex of Λ has volume one.

Theorem 5.4. *For every $\omega \in \mathbb{R}^{d+1}_{>0}$, we have*

$$\mu(S(\omega)) = \frac{\sum_{0 \leq i < j \leq d} \frac{1}{\omega_i \omega_j}}{\sum_{i=0}^{d} \frac{1}{\omega_i}} = \frac{1}{2} \frac{\sum_{i=0}^{d} \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S(\omega)))}{\mathrm{Vol}_{\mathbb{Z}^d}(S(\omega))},$$

where $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the linear projection vanishing at the ith vertex of $S(\omega)$.

In [GMS17], the authors conjecture an optimal lower bound on the *covering product* $\mu_1(K) \cdot \ldots \cdot \mu_d(K) \cdot \mathrm{Vol}_{\mathbb{Z}^d}(K)$ for any convex body $K \subseteq \mathbb{R}^d$. As a consequence of the explicit formula for $\mu(S(\omega))$, we confirm this conjecture for the simplices $S(\omega)$ (see corollary 5.51).

Observe that the volume expression on the right in theorem 5.4 can be defined for every simplex with the origin in its interior as follows:

Definition 5.5. Let $S = \operatorname{conv}(\{v_0, \dots, v_d\})$ be a d-simplex with the origin in its interior. We say that S has *rational vertex directions* if the line through the origin and the vertex v_i has rational direction, for every $0 \le i \le d$.

Writing $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ for the linear projection vanishing at v_i, we define the *discrete surface area* of such a simplex S as

$$\operatorname{Surf}_{\mathbb{Z}^d}(S) := \sum_{i=0}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)).$$

Note that $\operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)) = \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(F_i))$, with F_i being the facet of S opposite to the vertex v_i. In this sense, the sum of these numbers is indeed a version of the "surface area" of S, except that the volume of each facet is computed with respect to its projection from the opposite vertex.

Motivated by this definition and theorem 5.4 we propose the following conjecture, which is the main object of study in this chapter:

Conjecture C. Let S be a d-simplex with the origin in its interior and with rational vertex directions. Then

$$\mu(S) \le \frac{1}{2} \frac{\operatorname{Surf}_{\mathbb{Z}^d}(S)}{\operatorname{Vol}_{\mathbb{Z}^d}(S)}. \tag{5.4}$$

In Section 5.3 we give additional motivation for this conjecture. We show that it implies conjecture A (corollary 5.35), that it holds in dimension two (theorem 5.41), and that in arbitrary dimension it holds up to a factor of two (proposition 5.36).

Covering criteria such as the one in Conjecture C are rare in the literature, but very useful as they reduce the question of covering to computing less complex geometric functionals such as volume or (variants of the) surface area (cf. [Gru07, Sect. 31]). A classical inequality of this type is the following result of Hadwiger. We regard conjecture C as a discrete analog thereof.

Theorem 5.6 (Hadwiger [Had70])**.** *For every convex body K in $\mathbb{R}^d$*

$$\mu(K) \le \frac{1}{2} \frac{\operatorname{surf}(K)}{\operatorname{vol}(K)},$$

where $\operatorname{vol}(K)$ *and* $\operatorname{surf}(K)$ *are the Euclidean volume and surface area of K.*

Observe that the statement of conjecture C is more intrinsic than Hadwiger's inequality. This is because the Euclidean surface area is not invariant under unimodular transformations, so that the bound in theorem 5.6 depends on the particular representative of

K in its unimodular class. Moreover the inequality only holds for the standard lattice $\mathbb{Z}^d$ and cannot easily be transfered to other lattices (cf. [Sch92] for partial results for arbitrary lattices). In constrast, our proposed relation in conjecture C *is* unimodularly invariant and there is no loss of generality in restricting to the standard lattice as we do (see lemma 5.34 for details on these claims). Moreover, our proposed inequality in conjecture C is tight for the large class of simplices $S(\omega)$.

At the end of Section 5.3, we complement our investigations on conjecture C by extending it to the case where the origin lies in the boundary of the simplex S, rather than in the interior.

Another way to extend conjecture A is to ask for the maximal covering radius among lattice polytopes with at least $k \geq 1$ interior lattice points. The natural conjecture is:

Conjecture D. Let $k, d \in \mathbb{N}$ be nonnegative integers. Then, for every lattice d-polytope P with k interior lattice points we have

$$\mu(P) \leq \frac{d-1}{2} + \frac{1}{k+1}.$$

Equality holds for $k = 1$ if and only if P is obtained by direct sums and/or translations of simplices of the form $S(\mathbf{1}_l)$, and for $k \geq 2$, if and only if P is obtained by direct sums and/or translations of the segment $[0, k+1]$ and simplices $S(\mathbf{1}_l)$.

In section 5.5, we prove this conjecture in dimension two (see theorem 5.55). Observe that no analog of conjecture D makes sense for other covering minima. Indeed, the maximum dth covering minimum μ_d among non-hollow lattice n-polytopes with k interior lattice points does not depend on k or n, for $d < n$: It equals the maximum covering radius among non-hollow lattice d-polytopes, since every non-hollow lattice d-polytope can be obtained as the projection of a $(d+1)$-polytope with arbitrarily many interior lattice points. In fact, assuming conjecture A this maximum is given by

$$\mu_d(S(k, 1, \ldots, 1)) = \mu_d(S(\mathbf{1}_{d+1})) = \frac{d}{2}, \quad \text{for all } n > d \text{ and } k \in \mathbb{N}.$$

In summary, we prove the following relationships between our conjectures:

$$\begin{array}{ccl}
 & & \textit{conjecture C} \\
 & & \Downarrow \\
\textit{conjecture D} & \Rightarrow & \textit{conjecture A} \\
 & & \Downarrow \\
 & & \textit{conjecture A} \ \text{ without equality case} \\
 & & \Updownarrow \\
 & & \textit{conjecture B}
\end{array}$$

We further prove that all these conjectures hold in dimension two, that conjecture A holds in dimension three and that conjecture C holds for the simplices of the form $S(\omega)$.

5.1 Preliminaries

This section develops some tools that will be essential for our analyses. We first describe how the covering radius behaves with respect to projections, and more importantly, that it is an additive functional on direct sums of convex bodies and lattices. Afterwards we introduce and study the concept of *tight covering* that facilitates our equality characterizations, for example the one in Theorem 5.3.

Projection and direct sum

Lemma 5.7. *Let $K \subseteq \mathbb{R}^d$ be a convex body containing the origin, and let $\pi : \mathbb{R}^d \to \mathbb{R}^l$ be a rational linear projection, so that $\pi(\mathbb{Z}^d)$ is a lattice. Let $Q = K \cap \pi^{-1}(\mathbf{0})$ and let $L = \pi^{-1}(\mathbf{0})$ be the linear subspace spanned by Q. Then, we have*

$$\mu(K, \mathbb{Z}^d) \leq \mu(Q, \mathbb{Z}^d \cap L) + \mu(\pi(K), \pi(\mathbb{Z}^d)).$$

Proof. Let us abbreviate $\mu_Q = \mu(Q, \mathbb{Z}^d \cap L)$ and $\mu_\pi = \mu(\pi(K), \pi(\mathbb{Z}^d))$. Let $x \in \mathbb{R}^d$ be arbitrary. Then, $\pi(x)$ is covered by $\mu_\pi \cdot \pi(K) + \pi(\mathbb{Z}^d) = \pi\left(\mu_\pi K + \mathbb{Z}^d\right)$. Hence, there exists a point $x' \in \mathbb{R}^d$ such that the segment $[x, x']$ is parallel to L and such that x' is covered by $\mu_\pi K + \mathbb{Z}^d$. On the other hand, $y = x - x' \in L$ is covered by $\mu_Q Q + (\mathbb{Z}^d \cap L)$. Since $Q \subseteq K$, this implies that $x = y + x'$ is covered by $(\mu_Q + \mu_\pi)K + \mathbb{Z}^d$, as claimed. $\square$

A particularly interesting case of the above result is when K decomposes as a direct sum. Let $\mathbb{R}^d = V \oplus W$ be a decomposition into complementary linear subspaces with $\dim(V) = \ell$ and $\dim(W) = d - \ell$. The *direct sum* of two convex bodies $K \subseteq V, L \subseteq W$ both containing the origin is defined as

$$K \oplus L := \{\lambda x + (1 - \lambda)y : x \in K, y \in L, \lambda \in [0, 1]\} \subseteq \mathbb{R}^d.$$

The direct sum of two lattices $\Lambda \subseteq V$, $\Gamma \subseteq W$ is defined as

$$\Lambda \oplus \Gamma := \{x + y : x \in \Lambda, y \in \Gamma\} \subseteq \mathbb{R}^d.$$

With these definitions we can now formulate:

Corollary 5.8. *Let $\mathbb{R}^d = V \oplus W$ be a decomposition as above, let $K \subseteq V$, $L \subseteq W$ be convex bodies containing the origin, and let $\Lambda \subseteq V$, $\Gamma \subseteq W$ be lattices. Then,*

$$\mu_d(K \oplus L, \Lambda \oplus \Gamma) = \mu_\ell(K, \Lambda) + \mu_{d-\ell}(L, \Gamma).$$

Proof. The inequality $\mu_d(K \oplus L, \Lambda \oplus \Gamma) \leq \mu_\ell(K, \Lambda) + \mu_{d-\ell}(L, \Gamma)$ is a special case of Lemma 5.7, via the natural projection $\mathbb{R}^d = V \oplus W \to V$.

For the other inequality, let $x \in V$ be a point not covered by $cK + \Lambda$ for some $c < \mu_\ell(K, \Lambda)$ and let $y \in W$ be a point not covered by $\bar{c}L + \Gamma$ for some $\bar{c} < \mu_{d-\ell}(L, \Gamma)$. We claim that $x + y \in V \oplus W = \mathbb{R}^d$ is not covered by $(c + \bar{c})(K \oplus L) + \Lambda \oplus \Gamma$, and thus $c + \bar{c} \leq \mu_d(K \oplus L, \Lambda \oplus \Gamma)$. Since, c and $\bar{c}$ were taken arbitrarily, this implies $\mu_\ell(K, \Lambda) + \mu_{d-\ell}(L, \Gamma) \leq \mu_d(K \oplus L, \Lambda \oplus \Gamma)$.

Assume to the contrary, that $x + y \in (c + \bar{c})(K \oplus L) + \Lambda \oplus \Gamma$, that is, $x + y = (c + \bar{c})(\lambda p + (1 - \lambda)q) + w + z$, for some $\lambda \in [0, 1]$, $p \in K$, $q \in L$, $w \in \Lambda$, and $z \in \Gamma$. Since the sums are direct, we get $x = (c + \bar{c})\lambda p + w$ and $y = (c + \bar{c})(1 - \lambda)q + z$, which by assumption implies $(c + \bar{c})\lambda > c$ and $(c + \bar{c})(1 - \lambda) > \bar{c}$. These two inequalities cannot hold at the same time, and we arrive at a contradiction. $\square$

Tight covering

Definition 5.9. Let $K \subseteq \mathbb{R}^d$ be a convex body and let Λ be a lattice. Then, K is called *tight for* Λ if for every convex body $K' \supsetneq K$, we have

$$\mu(K', \Lambda) < \mu(K, \Lambda).$$

Definition 5.10. Let $K \subseteq \mathbb{R}^d$ be a convex body of covering radius μ with respect to a lattice Λ. A point $p \in \mathbb{R}^d$ is *last covered by* K if

$$p \notin \operatorname{int}(\mu \cdot K) + \Lambda.$$

Let P be a d-polytope, let F be a facet of P, and let p be a point that is last covered by P. We say that p *needs* F if $p \in \operatorname{relint}(\mu \cdot F) + \Lambda$.

Lemma 5.11. *Let $K \subseteq \mathbb{R}^d$ be a convex body of covering radius μ with respect to a lattice Λ. Then, the following properties are equivalent:*

i) K is tight for Λ.

ii) K is a polytope and for every facet F of K and for every last covered point p, p needs F.

iii) K is a polytope and every facet of every hollow translate of $\mu \cdot K$ is non-hollow.

iv) Every hollow translate of $\mu \cdot K$ is a maximal hollow convex body with respect to inclusion.

Proof. The equivalence of iii) and iv) is the characterization of maximal hollow convex bodies by Lovász [Lov89]. For the equivalence of i) and iv) observe that, by definition, μ is the largest constant such that (a) $\mu \cdot K$ has a hollow lattice translate and (b) the inequality $\mu(K', \Lambda) < \mu(K, \Lambda)$ in the definition of tightness is nothing but maximality of all such hollow translates.

We now show the equivalence of i) and ii). Suppose there is a facet F of K that is not needed by some last covered point p. Let $K' = \operatorname{conv}(K \cup \{x\})$, where $x \notin K$ is a point *beyond* F, meaning that x violates the inequality that defines F, but satisfies all other facet-inducing inequalities of K. Then

$$\mu(K', \Lambda) = \mu(K, \Lambda),$$

because p is still a last covered point of K' (for the same dilate μ).

Conversely, if K is not tight let K' be a convex body strictly containing K and that has the same covering radius. Let F be a facet of K with $\operatorname{relint}(F) \subseteq \operatorname{int}(K')$. Let p be a point that is last covered by K'. Since the covering radii are equal and $K \subsetneq K'$, p must also be last covered by K. Since we chose F so that $\operatorname{relint}(F)$ is in the interior of K', p does not need F. $\square$

Example 5.12. It is not sufficient for tightness that "every facet is needed by *some* last covered point." An example showing this is the hexagon $P = \operatorname{conv}(\{\pm(1,0), \pm(0,1), \pm(1,1)\})$ with respect to the integer lattice. P has covering radius $2/3$, the same as the triangle $\operatorname{conv}(\{(-1,1), (2,1), (-1,-2)\})$ that properly contains it, so it is not tight. It has two orbits of last covered points, with representatives $\pm(2/3, 1/3)$, each of which needs three of the six edges of P.

Lemma 5.13. *Every simplex is tight for every lattice.*

Proof. We use Lemma 5.11. Let Δ be a simplex of covering radius μ with respect to a lattice Λ, and let p be a point last covered by Δ. That is, $p \notin \operatorname{int}(\mu\Delta) + \Lambda$. Let $F_0, F_1, \ldots, F_d$ be the facets of Δ, with interior facet normals $v_0, \ldots, v_d$.

Every neighborhood of p is covered by $\mu\Delta + \Lambda$, and p can only lie in lattice translates of the boundary of $\mu\Delta$. Suppose, in order to get a contradiction, that a certain facet F_i

is not needed by p. This implies that for every $\mu\Delta + z$ ($z \in \Lambda$) containing p there is a facet $F_j \neq F_i$ such that $\mu\Delta + z \subset H_j^p$, where

$$H_j^p := \{x \in \mathbb{R}^d : v_j^\intercal x \leq v_j^\intercal p\}$$

is the translation to p of the j-th facet-defining half-space of Δ. This implies that we have a neighborhood of p covered by the d affine half-spaces with p in the boundary corresponding to the indices $j \neq i$. This is impossible since the corresponding d normals are linearly independent. □

Lemma 5.14. *Let K_1 and K_2 be convex bodies containing the origin and let Λ_1 and Λ_2 be lattices. Then, K_1 and K_2 are tight for Λ_1 and Λ_2, respectively, if and only if $K_1 \oplus K_2$ is tight for $\Lambda_1 \oplus \Lambda_2$.*

Proof. First of all, let $K' \supsetneq K_1 \oplus K_2$ be a convex body and let K_1' and K_2' be the projection of K' onto the linear span of K_1 and K_2, respectively. Clearly, either $K_1' \supsetneq K_1$ or $K_2' \supsetneq K_2$, so that by Corollary 5.8 and the tightness of K_1 and K_2, we have

$$\begin{aligned}\mu(K_1 \oplus K_2, \Lambda_1 \oplus \Lambda_2) &= \mu(K_1, \Lambda_1) + \mu(K_2, \Lambda_2) > \mu(K_1', \Lambda_1) + \mu(K_2', \Lambda_2)\\ &= \mu(K_1' \oplus K_2', \Lambda_1 \oplus \Lambda_2) \geq \mu(K', \Lambda_1 \oplus \Lambda_2),\end{aligned}$$

since $K_1' \oplus K_2' \subseteq K'$. Therefore, $K_1 \oplus K_2$ is tight for $\Lambda_1 \oplus \Lambda_2$.

Conversely, if say K_1 is not tight for Λ_1, then there exists $K_1' \supsetneq K_1$ such that $\mu(K_1, \Lambda_1) = \mu(K_1', \Lambda_1)$. Then, $K_1' \oplus K_2 \supsetneq K_1 \oplus K_2$ and by Corollary 5.8

$$\begin{aligned}\mu(K_1' \oplus K_2, \Lambda_1 \oplus \Lambda_2) &= \mu(K_1', \Lambda_1) + \mu(K_2, \Lambda_2) = \mu(K_1, \Lambda_1) + \mu(K_2, \Lambda_2)\\ &= \mu(K_1 \oplus K_2, \Lambda_1 \oplus \Lambda_2),\end{aligned}$$

so $K_1 \oplus K_2$ is not tight for $\Lambda_1 \oplus \Lambda_2$. □

Lemma 5.15. *Let $\Lambda' \subsetneq \Lambda$ be two lattices in $\mathbb{R}^d$, and let $K \subseteq \mathbb{R}^d$ be a convex body. Then,*

$$\mu(K, \Lambda) \leq \mu(K, \Lambda').$$

Proof. Let $\mu = \mu(K, \Lambda)$ and $\mu' = \mu(K, \Lambda')$. Then, $\mu' K + \Lambda' \subseteq \mu' K + \Lambda$, so $\mu \leq \mu'$. An example where equality holds is the following: Let $K = [-1, 1]^d$ and let Λ be an arbitrary refinement of $\mathbb{Z}^d$ contained in $\mathbb{R}^{d-1} \times \mathbb{Z}$. Then, $\mu(K, \mathbb{Z}^d) = \mu(K, \Lambda) = 1/2$. □

Remark 5.16. The inequality in Lemma 5.15 may not be strict, even for simplices. An example is the simplex $(I \oplus I')' \oplus I$ of Lemma 5.24 below. It has the same covering radius as $S(\mathbf{1}_4)$ (equal to $3/2$), yet it is isomorphic to $S(\mathbf{1}_4)$ when regarded with respect to the sublattice of index two generated by its vertices and its interior lattice point. This can easily be derived from its depiction in the bottom-center of Figure 5.2, or from its coordinates in Table 5.1 (in these coordinates the sublattice is $\{(x, y, z) \in \mathbb{Z}^3 : x \in 2\mathbb{Z}\}$).

5.2 Conjectures A and B: Equivalence and small dimensions

Equivalence of Conjectures A and B

As an auxiliary result we first reduce conjecture A to lattice simplices.

Lemma 5.17. *Every non-hollow lattice polytope contains a non-hollow lattice simplex of possibly smaller dimension.*

Proof. Consider a triangulation T of the given lattice polytope P whose only vertices are the vertices of P. Since P is non-hollow it contains an interior lattice point, say $p \in \mathrm{int}(P) \cap \mathbb{Z}^d$. Let S be the unique, possibly lower-dimensional, simplex in T that contains p in its relative interior. By definition, S is non-hollow and contained in P. $\square$

Corollary 5.18. *Conjecture A reduces to lattice simplices. More precisely, conjecture A holds in every dimension $\leq d$, if and only if it holds for lattice simplices in every dimension $\leq d$.*

Proof. One direction is trivially true. We prove the other one by induction on d. Let $P \subseteq \mathbb{R}^d$ be a non-hollow lattice polytope. In view of lemma 5.17, we find an ℓ-dimensional non-hollow lattice simplex $S \subseteq P$. If $\ell = d$, then we simply have $\mu(P) \leq \mu(S)$. So, let us assume that $\ell < d$ and assume that conjecture A is proven for any dimension $< d$. Assume also that S contains the origin in its interior and write L_S for the linear hull of S. We now apply lemma 5.7 to the projection π onto $L_S^\perp$. Observe that $S \subseteq P \cap \pi^{-1}(\mathbf{0}) = P \cap L_S$, and that S is non-hollow with respect to $\mathbb{Z}^d \cap L_S$ and $\pi(P)$ is non-hollow with respect to the lattice $\pi(\mathbb{Z}^d)$. We get that

$$\mu(P) \leq \mu(S, \mathbb{Z}^d \cap L_S) + \mu(\pi(P), \pi(\mathbb{Z}^d)) \leq \frac{\ell}{2} + \frac{d-\ell}{2} = \frac{d}{2}. \qquad \square$$

Proof of theorem 5.2. Suppose first that for $\ell \leq d$ every lattice ℓ-polytope P has $\mu(P) \leq \ell/2$. Since $S(\mathbf{1}_{n+1})$ projects to $S(\mathbf{1}_{\ell+1})$, we have by (5.1)

$$\mu_\ell(S(\mathbf{1}_{n+1}), \mathbb{Z}^n) \geq \mu_\ell(S(\mathbf{1}_{\ell+1}), \mathbb{Z}^\ell) = \frac{\ell}{2}.$$

For the converse inequality, let $\pi : \mathbb{R}^n \to \mathbb{R}^\ell$ be an integer projection along which the value of $\mu_\ell(S(\mathbf{1}_{n+1}))$ is attained. Then, $\pi(S(\mathbf{1}_{n+1}))$ is non-hollow with respect to the lattice $\pi(\mathbb{Z}^n)$, and thus

$$\mu_\ell(S(\mathbf{1}_{n+1}), \mathbb{Z}^n) = \mu_\ell(\pi(S(\mathbf{1}_{n+1})), \pi(\mathbb{Z}^n)) \leq \frac{\ell}{2}.$$

For the reverse implication (ii) $\Rightarrow$ (i), suppose conjecture B holds in every dimension $\ell \leq d$. Let P be a lattice ℓ-polytope with at least one interior lattice point, which without loss of generality we assume to be the origin $\mathbf{0}$. By corollary 5.18 we can assume P to be a simplex, and we let $v_0, \ldots, v_\ell$ be its vertices. Let $(b_0, \ldots, b_\ell) \in \mathbb{N}^{\ell+1}$ be a multiple of the barycentric coordinates of $\mathbf{0}$ in P; that is, assume that

$$\mathbf{0} = \frac{1}{N} \sum_{i=0}^{\ell} b_i v_i, \tag{5.5}$$

where $N = \sum_{i=0}^{\ell} b_i \geq \ell + 1$. Consider the $(N-1)$-dimensional simplex $S(\mathbf{1}_N)$, and the affine projection $\pi : \mathbb{R}^{N-1} \to \mathbb{R}^\ell$ that sends exactly b_i vertices of $S(\mathbf{1}_N)$ to v_i, $i = 0, \ldots, \ell$. Expression (5.5) implies that π sends the origin to the origin, which in turn implies π to be an integer projection. In particular,

$$\mu(P, \mathbb{Z}^\ell) \leq \mu_\ell(\pi(S(\mathbf{1}_N)), \pi(\mathbb{Z}^{N-1})) \leq \mu_\ell(S(\mathbf{1}_N), \mathbb{Z}^{N-1}) = \frac{\ell}{2},$$

since $\pi(\mathbb{Z}^{N-1}) \subseteq \mathbb{Z}^\ell$. □

Conjecture A in dimensions 2 and 3

We here prove conjecture A in dimensions two and three, including the case of equality. We bigin in dimension two.

Let $I = [-1, 1]$ and $I' = [0, 2]$ be intervals of length two centered at 0 and 1, respectively.

Lemma 5.19. *The three polygons $S(\mathbf{1}_3)$, $I \oplus I$, and $I \oplus I'$ have covering radius equal to one.*

Proof. For $S(\mathbf{1}_3)$ this is just eq. (5.1). For the other two polygons it follows from Corollary 5.8, since they are unimodularly equivalent to direct sums of segments of length two. □

We now show that every other non-hollow lattice polygon contains a (unimodularly equivalent) copy of one of these three, which implies Conjecture A. For this let us

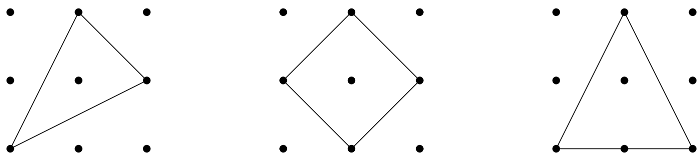


FIGURE 5.1: The non-hollow lattice polygons $S(\mathbf{1}_3)$, $I \oplus I$ and $I \oplus I'$ of covering radius equal to one.

consider the following auxiliary family of lattice triangles with k interior lattice points: For each $k \in \mathbb{N}$, and $\alpha \in \{0,1\}$ let

$$M_k(\alpha) = \operatorname{conv}(\{(-1,0),(1,\alpha),(0,k+1)\}). \tag{5.6}$$

Observe that

$$M_1(0) = I \oplus I', \quad M_1(1) \cong S(\mathbf{1}_3), \quad \text{and} \quad \forall k \geq 2,\ M_{k-1}(\alpha) \subsetneq M_k(\alpha).$$

Lemma 5.20. *Every non-hollow lattice polygon P contains a unimodular copy of either $M_1(0) = I \oplus I'$, $M_1(1) \cong S(\mathbf{1}_3)$ or $I \oplus I$.*

Proof. Without loss of generality, assume the origin is in the interior of P. Consider the complete fan whose rays go through all non-zero lattice points in P. We call this the *lattice fan* associated to P, and it is a complete unimodular fan. Since a 2-dimensional fan is uniquely determined by its rays, we denote $\mathbb{F}\{v_1,\dots,v_m\}$ the fan with rays through $v_1,\dots,v_m \in \mathbb{R}^2$. In particular, the lattice fan of P is denoted $\mathbb{F}\{P \cap \mathbb{Z}^2\}$.

By the classification of complete unimodular fans, see [Ewa96, Thm. V.6.6], $\mathbb{F}\{P \cap \mathbb{Z}^2\}$ can be (modulo unimodular equivalence) obtained by successively refining the lattice fan of either $S(\mathbf{1}_3)$ or

$$\mathbb{F}_l := \mathbb{F}\{(0,-1),(0,1),(-1,0),(1,l)\},$$

for some $l \in \mathbb{Z}_{\geq 0}$. Observe that $\mathbb{F}_0$ is the lattice fan of $I \oplus I$, $\mathbb{F}_1$ refines the lattice fan of $S(\mathbf{1}_3) \cong M_1(1)$ and, for every $l \geq 2$ we have that $\mathbb{F}_l$ is unimodularly equivalent to the fan of

$$\begin{cases} M_k(0) & \text{if } l = 2k \text{ is even, and} \\ M_k(1) & \text{if } l = 2k-1 \text{ is odd.} \end{cases}$$

This, together with the fact that $M_1(\alpha) \subseteq M_k(\alpha)$ for every $k \geq 1$, implies that P contains one of $M_1(0)$, $M_1(1)$ or $I \oplus I$. $\square$

Corollary 5.21. *Let P be a non-hollow lattice polygon. Then*

$$\mu(P) \leq 1,$$

with equality if and only if P is unimodularly equivalent to one of $S(\mathbf{1}_3)$, $I \oplus I$, or $I \oplus I'$.

Proof. By Lemma 5.20, unless P is one of $S(\mathbf{1}_3)$, $I \oplus I$ or $I \oplus I'$ it strictly contains one of them. If the latter happens then its covering radius is strictly smaller than 1, since the three of them are tight by Lemma 5.13 and Lemma 5.14. $\square$

Remark 5.22. The covering radius of $M_k(\alpha)$ can be computed explicitly via

$$M_k(0) \cong I \oplus [0, k+1], \quad \text{and} \quad M_k(1) \cong S(k, 1, 1).$$

Indeed, this implies

$$\mu(M_k(0)) = \frac{1}{2} + \frac{1}{k+1} = \frac{k+3}{2k+2}, \quad \text{and} \quad \mu(M_k(1)) = \frac{1+\frac{2}{k}}{2+\frac{1}{k}} = \frac{k+2}{2k+1},$$

by Corollary 5.8 and Theorem 5.4, respectively. We see that, indeed, their covering radius equals 1 for $k = 1$ and is strictly smaller for greater k.

We now proceed to the three-dimensional case of Conjecture A. We first need to introduce the following concept:

Definition 5.23. A *minimal d-polytope* is a non-hollow lattice d-polytope not properly containing any other non-hollow lattice d-polytope.

In this language, our results in dimension 2 can be restated as: There are exactly three minimal 2-polytopes, they have covering radius 1, and every other non-hollow lattice 2-polytope has strictly smaller covering radius.

In dimension three things are a bit more complicated. To start with, instead of three direct sums of (perhaps translated) simplices of the form $S(\mathbf{1}_i)$ there are nine, that we now describe. As in the previous section, let $I = [-1, 1] = S(\mathbf{1}_2)$ and $I' = [0, 2]$. In a similar way we define:

$$\begin{aligned}
S'(\mathbf{1}_3) &= (1,1) + S(\mathbf{1}_3) = \operatorname{conv}(\{(0,0),(2,1),(1,2)\}),\\
(I \oplus I')^\circ &= (0,-1) + (I \oplus I') = \operatorname{conv}(\{(0,1),(\pm 1,-1)\}),\\
(I \oplus I')' &= (0,-2) + (I \oplus I') = \operatorname{conv}(\{(0,0),(\pm 1,-2)\}).
\end{aligned}$$

Put differently, $S'(\mathbf{1}_3)$ is $S(\mathbf{1}_3)$ translated to have the origin as a vertex; the other two are $I \oplus I'$ translated to have the origin in the interior and at the "apex", respectively.

Lemma 5.24. *There are the following nine non-equivalent lattice* 3*-polytopes of covering radius* 3/2*, obtained as direct sums of (perhaps translated) simplices of the form* $S(\mathbf{1}_d)$*:*

$$\begin{gathered} S(\mathbf{1}_4), \\ S(\mathbf{1}_3) \oplus I, \qquad S'(\mathbf{1}_3) \oplus I, \qquad S(\mathbf{1}_3) \oplus I', \\ I \oplus I \oplus I, \qquad I \oplus I \oplus I', \\ (I \oplus I')^\circ \oplus I, \qquad (I \oplus I')' \oplus I, \qquad (I \oplus I')^\circ \oplus I'. \end{gathered}$$

The last five polytopes are illustrated in Figure 5.2, which is borrowed from [BS19, p. 123]. Observe that the last three can equally be written as

$$I \oplus (I \oplus I)', \qquad I \oplus (I' \oplus I)', \qquad I \oplus (I \oplus I')'',$$

where $(I \oplus I)'$ denotes $I \oplus I$ translated to have the origin as a vertex and $(I \oplus I')''$ is $I \oplus I'$ translated to have the origin at an endpoint of its edge of length two.

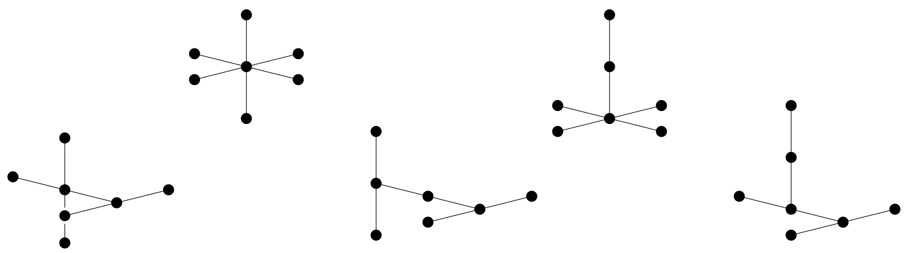


FIGURE 5.2: The five non-hollow lattice 3-polytopes that can be obtained by translations and direct sums of $I = [-1, 1]$.

Proof. That all the described direct sums are non-hollow follows from the following more general fact: The direct sum of two or more non-hollow lattice polytopes containing the origin is non-hollow if (and only if) all but at most one of the summands has the origin in its interior. Indeed, if the summand exists then its interior point(s) are interior in the sum; if it doesn't then the origin is an interior point in the sum.

With this in mind, we only need to check that the nine described polytopes are pairwise unimodularly non-equivalent, which is left to the reader. □

A second difference with dimension two is that these nine non-hollow lattice 3-polytopes are no longer the only minimal ones. Minimal non-hollow 3-polytopes have been classified and there are 26 with a single interior lattice point (see [Kas10, Thm. 3.1] and Tables 2 & 4 therein) plus the infinite family described in Theorem 5.26 below.

To prove conjecture A in dimension three we show that, on the one hand, the covering radii of the 26 with a single interior lattice point can be explicitly computed and/or bounded, giving the following result, the proof of which we postpone to Section 5.6.1.

Theorem 5.25. *Among the 26 minimal non-hollow* 3*-polytopes with a single interior lattice point, all except the nine in Lemma 5.24 have covering radius strictly smaller than* $3/2$*.*

On the other hand, all the (infinitely many) minimal non-hollow 3-polytopes with more than one interior lattice point have covering radius strictly smaller than $3/2$, as we now prove. For any $k \in \mathbb{N}$ and $\alpha, \beta \in \{0, 1\}$, we define $M_k(\alpha, \beta)$ to be the following lattice tetrahedron:

$$M_k(\alpha, \beta) = \operatorname{conv}(\{(1, 0, 0), (-1, 0, \alpha), (0, 1, k + 1), (0, -1, k + 1 - \beta)\}). \tag{5.7}$$

Theorem 5.26 ([BK16, Prop. 4.2])**.** *Every minimal* 3*-polytope with* $k \geq 2$ *interior lattice points is equivalent by unimodular equivalence or refinement of the lattice to* $M_k(\alpha, \beta)$ *for some* $\alpha, \beta \in \{0, 1\}$*.*

Theorem 5.26 is a version of [BK16, Prop. 4.2], although more explicit than the original one. An example where refinement is needed in the statement is $M_k(0, 0)$ considered with respect to the lattice Λ generated by $\mathbb{Z}^3$ and $(1/q, 1 - 1/q, 0)$, with q and $k + 1$ coprime. $M_k(0, 0)$ is still minimal with respect to Λ because it contains no point of $\Lambda \setminus \mathbb{Z}^3$.

Proof. Let P be a minimal lattice 3-polytope with more than one interior lattice point, and let L be a line containing two of them. Without loss of generality we assume that $L = \{(0, 0, z) : z \in \mathbb{R}\}$ and $L \cap P$ is the segment between $(0, 0, z_1)$ and $(0, 0, z_2)$, with $z_1 \in [0, 1)$ and $z_2 \in (r, r + 1]$ for some $r \in \{2, \ldots, k\}$, so that L contains r interior lattice points of P.

Claim 1: *The minimal faces of* P *containing respectively* $(0, 0, z_1)$ *and* $(0, 0, z_2)$ *are non-coplanar edges.* Let F_1 and F_2 be those faces. If one of them, say F_1, had dimension two, then $\operatorname{conv}(F_1 \cup \{(0, 0, r)\})$ would be a non-hollow lattice polytope strictly contained in P. If one of them, say F_1, had dimension zero then necessarily $F_1 = \{(0, 0, z_1)\} = \{(0, 0, 0)\}$. This would imply $\operatorname{conv}(P \cap \mathbb{Z}^3 \setminus \{\mathbf{0}\})$ to be a non-hollow lattice polytope strictly contained in P. Thus, F_1 and F_2 are both edges of P. They cannot be coplanar, since otherwise there would be vertices p and q of P, one on either side of the hyperplane $\operatorname{aff}(F_1 \cup F_2)$, and the polytope $\operatorname{conv}(F_1 \cup \{(0, 0, r), p, q\})$ would be non-hollow and strictly contained in P.

Hence, $\operatorname{conv}(F_1 \cup F_2)$ is a non-hollow lattice tetrahedron and by minimality, $P = \operatorname{conv}(F_1 \cup F_2)$. We denote v_i and w_i the vertices of F_i, for $i = 1, 2$.

Claim 2: *All the lattice points in the tetrahedron P other than the four vertices are on the line L.* Let H_i be the plane containing the line L and the edge F_i, for $i = 1, 2$. The polytope $Q = \operatorname{conv}(L \cap P \cup \{v_1, w_1, v_2\}) \subset P$ is contained in H_1^+, one of the two halfspaces defined by H_1; furthermore, the facet of Q lying on H_1 is non-hollow, since $(0, 0, 1)$ is in its relative interior. Therefore, if P contained any lattice point u other than the vertex w_2 in the open halfspace $(H_1^-)^o$ then $\operatorname{conv}(Q \cup \{u\})$ would be a non-hollow lattice polytope strictly contained in P. Thus there are no lattice points in the open halfspace $(H_1^-)^o$. Since the same can be said for the other halfspaces, H_1^+ and $H_2^{\pm}$, all lattice points of P except its four vertices must lie on L.

In particular, we have $r = k$.

Claim 3: *The endpoint $(0, 0, z_i)$ equals the mid-point of the edge $F_i = \operatorname{conv}(\{v_i, w_i\})$.* Let us only look at $i = 1$, the other case being symmetric. Let $u_1 = (0, 0, 1)$ and $u_2 = (0, 0, 2)$ be the first two interior lattice points of P along L. The triangles $\operatorname{conv}(\{u_1, u_2, v_1\})$ and $\operatorname{conv}(\{u_1, u_2, w_1\})$ are empty lattice triangles in the plane H_1, hence they have the same area. Thus, v_1 and w_1 are at the same distance from (and on opposite sides of) the line L, which implies the statement.

In particular, $z_1 \in [0, 1)$ and $z_2 \in (k, k + 1]$ are either integers or half-integers, so they can be written as $z_1 = \alpha/2$ and $z_2 = k + 1 - \beta/2$ for some $\alpha, \beta \in \{0, 1\}$. It is now clear that the affine transformation that fixes L and sends $v_1 \mapsto (1, 0, 0)$ and $v_2 \mapsto (0, 1, k+1)$, sends P to $M_k(\alpha, \beta)$. The map may send $\mathbb{Z}^3$ to a different lattice Λ, but Λ refines $\mathbb{Z}^3$ since $(1, 0, 0)$, $(0, 1, k + 1)$, $(0, 0, 1)$ and $(0, 0, 2)$ are in Λ and they generate $\mathbb{Z}^3$. □

Corollary 5.27. *Every minimal 3-polytope with $k \geq 2$ interior lattice points has covering radius strictly smaller than $3/2$.*

Proof. The projection of $M_k(\alpha, \beta)$ along the z direction is $I \oplus I$ and the fiber over the origin is the segment $\{0\} \times \{0\} \times [\alpha/2, k + 1 - \beta/2]$, of length $k + 1 - (\alpha + \beta)/2$. Thus, by Lemma 5.7,

$$\mu(M_k(\alpha, \beta)) \leq \mu(I \oplus I) + \mu([\alpha/2, k + 1 - \beta/2]) = 1 + \frac{1}{k + 1 - \frac{\alpha+\beta}{2}} \leq \frac{3}{2}.$$

Moreover, the last inequality is met with equality only in the case $k = 2$, $\alpha = \beta = 1$. But for $M_2(1, 1)$ we can consider the projection $(x, y, z) \mapsto x$, whose image is I and whose fiber is

$$\operatorname{conv}(\{(0, 1/2), (1, 3), (-1, 2)\}) \cong S(3/2, 1, 1).$$

Thus, by Lemma 5.7 and Theorem 5.4, we have

$$\mu(M_2(1,1)) \le \mu(I) + \mu(S(3/2,1,1)) = \frac{1}{2} + \frac{7/3}{8/3} = \frac{11}{8} < \frac{3}{2}. \qquad \square$$

In fact we can be more explicit:

Remark 5.28. The covering radius of $M_k(\alpha, \beta)$ admits a closed expression:

$$\mu(M_k(0,0)) = \mu(I \oplus [0,k+1] \oplus I) = 1 + \frac{1}{k+1}.$$
$$\mu(M_k(1,0)) = \mu(M_k(0,1)) = \mu(I \oplus M_k(1)) = 1 + \frac{3}{4k+2},$$
$$\mu(M_k(1,1)) = 1 + \frac{1}{2k}.$$

The first formula directly follows from lemma 5.7. The second one also does, using Remark 5.22. For the third one, see lemma 5.65. For $k = 1$ the three expressions reduce to $3/2$, which is consistent with the descriptions $M_1(0,0) \cong I \oplus (I \oplus I')'$, $M_1(0,1) \cong I \oplus S(\mathbf{1}_3)$, and $M_1(1,1) \cong S(\mathbf{1}_4)$.

We are now ready to prove conjecture A in dimension three:

Theorem 5.29. *Let P be a non-hollow lattice 3-polytope. Then*

$$\mu(P) \le \frac{3}{2},$$

with equality if and only if P is unimodularly equivalent to one of the nine polytopes in Lemma 5.24.

Proof. Let P be a non-hollow lattice 3-polytope, and let T be a minimal one contained in it. If T is not one of the nine in Lemma 5.24 then T, and hence P, has covering radius strictly smaller than $3/2$ by either Corollary 5.27 or Theorem 5.25. If T is one of the nine and $P \neq T$ then

$$\mu(P) < \mu(T) = \frac{3}{2},$$

since these nine are tight by Lemma 5.13 and Lemma 5.14. $\square$

Another proof of conjecture A in dimension two

Let $v \in \mathbb{R}^d_{\ge 1}$ and let $\Delta_v := \operatorname{conv}(\{-v, e_1, \ldots, e_d\})$. The following result says that bounds for the covering radii of this class of simplices translate to bounds for all non-hollow lattice polytopes (cf. corollary 5.18).

Lemma 5.30. *Let Δ be a non-hollow lattice d-simplex. Then, there is a vector $v \in \mathbb{Q}^d_{\geq 1}$ such that*

$$\mu(\Delta) \leq \mu(\Delta_v).$$

Proof. Write $\Delta = \operatorname{conv}(\{w_0, w_1, \ldots, w_d\})$ and assume without loss of generality that $\mathbf{0} \in \operatorname{int}(\Delta) \cap \mathbb{Z}^d$. Let $(\beta_0, \beta_1, \ldots, \beta_d)$ be the barycentric coordinates of the origin with respect to Δ. That is, $\beta_i \geq 0$, $\sum_{i=0}^d \beta_i = 1$, and $\mathbf{0} = \sum_{i=0}^d \beta_i w_i$. We may assume without loss of generality that $0 < \beta_0 \leq \beta_i$, for all $i \in \{1, \ldots, d\}$. Now, let $W = (w_1, \ldots, w_d) \in \mathbb{Z}^{d \times d}$ and let $v \in \mathbb{R}^d$ be such that $W\Delta_v = \Delta$. Clearly, $\beta_0 v = W^{-1}\left(\sum_{i=1}^d \beta_i w_i\right) = \sum_{i=1}^d \beta_i e_i$, and hence $v = (\frac{\beta_1}{\beta_0}, \ldots, \frac{\beta_d}{\beta_0}) \in \mathbb{Q}^d_{\geq 1}$.

With these observations, we get

$$\mu(\Delta) \leq \mu(\Delta, W\mathbb{Z}^d) = \mu(W\Delta_v, W\mathbb{Z}^d) = \mu(\Delta_v),$$

as desired. □

In dimension two, this approach leads to another proof of conjecture A:

Proposition 5.31. *For every $v \in \mathbb{R}^2_{\geq 1}$, we have*

$$\mu(\Delta_v) \leq 1.$$

Equality holds if and only if $v \in \{(a, 1), (1, a)\}$, for some $1 \leq a \leq 2$.

Proof. Due to symmetry, we can assume $v = (v_1, v_2)$ with $v_1 \geq v_2$. If $v_2 > 1$, then Δ_v strictly contains the triangle Δ_w, for some $w = (w_1, 1) \in \mathbb{R}^2_{\geq 1}$. By lemma 5.13 triangles are tight for every lattice, so that $\mu(\Delta_v) < \mu(\Delta_w)$ and it thus suffices to consider $v = (a, 1)$, for $a \geq 1$.

Let F_0 be the edge of Δ_v not containing v, and let F_1 and F_2 be the edges of Δ_v not containing e_1 and e_2, respectively. Further, let $\ell = \{(x, y) : x + y = 1\}$ be the line containing F_0. An elementary calculation provides us with the following intersection points:

$$\ell \cap (F_1 + e_1) = \left\{\left(\tfrac{2}{a+2}, \tfrac{a}{a+2}\right)\right\}, \qquad \ell \cap (F_2 + e_2) = \left\{\left(\tfrac{1}{a+2}, \tfrac{a+1}{a+2}\right)\right\},$$
$$\ell \cap (F_1 + (1,1)) = \left\{\left(\tfrac{2-a}{a+2}, \tfrac{2a}{a+2}\right)\right\}, \qquad \ell \cap (F_2 + (1,1)) = \left\{\left(\tfrac{2}{a+2}, \tfrac{a}{a+2}\right)\right\}.$$

This already shows that the translates $\{0,1\}^2 + \Delta_v$ cover the unit cube $[0,1]^2$, for every $a \geq 1$, so that $\mu(\Delta_v) \leq 1$ as claimed.

In order to decide the equality case, observe that in the covering of $[0,1]^2$ by these four translates, the point $\left(\frac{2}{a+2}, \frac{a}{a+2}\right)$ is covered last, and is not contained in the interior of any of the four triangles. However, the translate $(2,1)+\Delta_v$ may contain this point in the interior. Noting that

$$\ell \cap (F_1 + (2,1)) = \left\{\left(\tfrac{4-a}{a+2}, \tfrac{2a-2}{a+2}\right)\right\},$$

this happens if and only if $4-a<2$, that is, $a>2$. $\square$

Unfortunately, the analogous result fails in higher dimensions:

Example 5.32. The method described in section 5.6.2 can be used to compute that

$$\mu(\Delta_{(3/2,1,1)}) = \frac{14}{9} > \frac{3}{2}.$$

Counterexamples in higher dimensions can be constructed from this example as follows: Let $v \in \mathbb{R}^d_{\geq 1}$ be such that $\mu(\Delta_v) > \frac{d}{2}$, and let S be the non-hollow lattice $(d+1)$-simplex arising as the direct sum of Δ_v and $I' = [0,2]$. In view of lemma 5.30, there exists $w \in \mathbb{R}^{d+1}_{\geq 1}$ such that

$$\mu(\Delta_w) \geq \mu(S) = \mu(\Delta_v) + \mu(I') > \frac{d}{2} + \frac{1}{2} = \frac{d+1}{2},$$

where we also used corollary 5.8.

5.3 Conjecture C

We here focus on conjecture C. We show that it implies conjecture A, we prove it up to a factor of two in arbitrary dimension, and we prove it in dimension two. In the last paragraph, we investigate how the proposed bound changes if we allow the origin to be contained in the boundary of the given simplex.

As a preparation, let us first reinterpret conjecture C in terms of (reciprocals of) certain lengths. To this end, let $S = \operatorname{conv}(\{v_0, \ldots, v_d\})$ be a d-simplex with the origin in its interior, and assume that it has *rational vertex directions*, that is, the line through the origin and the vertex v_i has rational direction, for every $0 \leq i \leq d$.

As in conjecture C, let π_i be the linear projection to dimension $d-1$ vanishing at v_i. Finally, let ℓ_i be the lattice length of $S \cap \pi_i^{-1}(\mathbf{0})$. Put differently, let u_i be the point where the ray from v_i through $\mathbf{0}$ hits the opposite facet of S and let ℓ_i be the ratio between the length of $[u_i, v_i]$ and the length of the primitive lattice vector in the same direction. In formula:

$$\ell_i := \mathrm{Vol}_{\mathbb{Z}^d \cap \mathbb{R}v_i}([u_i, v_i]).$$

Lemma 5.33. *For every $i \in \{0, 1, \ldots, d\}$, we have*

$$\frac{1}{\ell_i} = \frac{\mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}{\mathrm{Vol}_{\mathbb{Z}^d}(S)}.$$

In particular, Conjecture C is equivalent to the inequality

$$\mu(S) \leq \frac{1}{2} \sum_{i=0}^{d} \frac{1}{\ell_i}. \tag{5.8}$$

Proof. By construction, we have $\pi_i(S) = \pi_i(F_i)$, where F_i is the facet of S opposite to the vertex v_i. Therefore, $\mathrm{vol}(S) = \frac{1}{d} \mathrm{vol}(\pi_i(S)) \mathrm{vol}([u_i, v_i])$. The determinants of the involved lattices are related by $1 = \det(\mathbb{Z}^d) = \det(\pi_i(\mathbb{Z}^d)) \det(\mathbb{Z}^d \cap \mathbb{R}v_i)$ (cf. [Mar03, Prop. 1.2.9]). Hence,

$$\begin{aligned}
\mathrm{Vol}_{\mathbb{Z}^d}(S) = \frac{d! \, \mathrm{vol}(S)}{\det(\mathbb{Z}^d)} &= \frac{(d-1)! \, \mathrm{vol}(\pi_i(S))}{\det(\pi_i(\mathbb{Z}^d))} \frac{\mathrm{vol}([u_i, v_i])}{\det(\mathbb{Z}^d \cap \mathbb{R}v_i)} \\
&= \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)) \, \mathrm{Vol}_{\mathbb{Z}^d \cap \mathbb{R}v_i}([u_i, v_i]),
\end{aligned}$$

as desired. □

We now also detail the claim in the introduction, that the discrete surface area defined in definition 5.5 is invariant under unimodular transformations.

Lemma 5.34. *Let S be a d-simplex with the origin in its interior and with rational vertex directions. Let A be an invertible linear transformation. Then*

$$\mathrm{Surf}_{A\mathbb{Z}^d}(AS) = \mathrm{Surf}_{\mathbb{Z}^d}(S).$$

In particular, if A is unimodular, we have $\mathrm{Surf}_{\mathbb{Z}^d}(AS) = \mathrm{Surf}_{\mathbb{Z}^d}(S)$.

Proof. As before we write $S = \mathrm{conv}(\{v_0, \ldots, v_d\})$ and we let π_i be the projection vanishing at v_i, for $0 \leq i \leq d$. Clearly, $AS = \mathrm{conv}(\{Av_0, \ldots, Av_d\})$ and the corresponding projection $\bar{\pi}_i$ vanishing at Av_i can be written as $\bar{\pi}_i = \pi_i A^{-1}$. Therefore, we get

$$\mathrm{Surf}_{A\mathbb{Z}^d}(AS) = \sum_{i=0}^{d} \mathrm{Vol}_{\bar{\pi}_i(A\mathbb{Z}^d)}(\bar{\pi}_i(AS)) = \sum_{i=0}^{d} \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)) = \mathrm{Surf}_{\mathbb{Z}^d}(S),$$

as claimed. □

Conjecture C implies Conjecture A

Corollary 5.35. *conjecture C $\implies$ conjecture A.*

Proof. In view of Corollary 5.18, it suffices to consider lattice simplices. Therefore, let $S = \operatorname{conv}(\{v_0, \dots, v_d\})$ be a lattice d-simplex containing the origin in its interior. Furthermore, let ω_i be the lattice length of the segment $[\mathbf{0}, v_i]$. Then, $1 - \omega_i/\ell_i$ is the i-th barycentric coordinate of the origin with respect to the vertices of S, so that

$$\sum_{i=0}^{d} \left(1 - \frac{\omega_i}{\ell_i}\right) = 1$$

and, hence, $\sum_{i=0}^{d} \omega_i/\ell_i = d$. On the other hand, for a lattice simplex we have $\omega_i \geq 1$. Thus, assuming Conjecture C holds for S, we have

$$\mu(S) \leq \frac{1}{2} \sum_{i=0}^{d} \frac{1}{\ell_i} \leq \frac{1}{2} \sum_{i=0}^{d} \frac{\omega_i}{\ell_i} = \frac{d}{2}.$$ □

Conjecture C holds up to a factor of two

In the formulation of lemma 5.33, conjecture C is easily proved inductively up to a factor of two.

Proposition 5.36. *Let $S = \operatorname{conv}(\{v_0, \dots, v_d\})$ be a d-simplex with the origin in its interior and with rational vertex directions. Then*

$$\mu(S) \leq \sum_{i=0}^{d} \frac{1}{\ell_i},$$

with the lattice lengths ℓ_i defined as above.

Proof. As above, let u_i be the intersection of the line $\mathbb{R}v_i$ with the facet F of S opposite to v_i, so that ℓ_i is the lattice length of $Q := [u_i, v_i] \subseteq S$. Note, that u_i lies in the relative interior of F. Also, let π_i be the linear projection vanishing at v_i. By the assumptions on S, the projection π_i is rational and thus $\pi_i(S)$ is a $(d-1)$-dimensional simplex having the origin in its interior and with rational vertex directions with respect to $\pi_i(\mathbb{Z}^d)$.

Using lemma 5.7 and the induction hypothesis for $\pi_i(S)$, we get

$$\mu(S, \mathbb{Z}^d) \leq \mu(Q, \mathbb{Z}^d \cap L_Q) + \mu(\pi_i(S), \pi_i(\mathbb{Z}^d)) \leq \frac{1}{\ell_i} + \sum_{j \neq i} \frac{1}{\ell'_j}, \tag{5.9}$$

where the ℓ'_j are the corresponding lattice-lengths in $\pi_i(S)$. Thus, to prove the proposition we only need to show that $\ell'_j \geq \ell_j$, for all $j \neq i$. In fact, since the one-dimensional lattice $\pi_i(\mathbb{Z}^d) \cap \pi_i(\mathbb{R}v_j)$ refines $\pi_i(\mathbb{Z}^d \cap \mathbb{R}v_j)$, we have

$$\begin{aligned}\ell_j = \mathrm{Vol}_{\mathbb{Z}^d\cap\mathbb{R}v_j}([u_j, v_j]) &= \mathrm{Vol}_{\pi_i(\mathbb{Z}^d\cap\mathbb{R}v_j)}([\pi_i(u_j), \pi_i(v_j)]) \\ &\leq \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)\cap\pi_i(\mathbb{R}v_j)}([\pi_i(u_j), \pi_i(v_j)]) \leq \ell'_j.\end{aligned}$$

Here, the last inequality comes from the fact that $[\pi_i(u_j), \pi_i(v_j)] \subseteq \pi_i(S)$ is contained in the ray from the vertex $\pi_i(v_j)$ of $\pi_i(S)$ through the origin. $\square$

Remark 5.37. Corollary 5.43 in the next section proves Conjecture C in the plane. So we can base the inductive proof above on the stronger assumption that $\mu(S') \leq c_{d-1}\sum_{i=0}^{d-1}\frac{1}{\ell'_i}$, where S' is a $(d-1)$-dimensional simplex and c_{d-1} is a suitable constant with $c_2 = 1/2$. Summing the thus modified inequality (5.9) for all indices $0 \leq i \leq d$, yields the recursion $(d+1)c_d = 1 + dc_{d-1}$. Solving it shows that

$$\mu(S) \leq \frac{2d-1}{2d+2}\sum_{i=0}^{d}\frac{1}{\ell_i},$$

for all d-simplices S with the origin in its interior and with rational vertex directions. This is a good bound in $\mathbb{R}^3$ since $c_3 = 5/8$.

Conjecture C in dimension two

We will now prove Conjecture C in dimension two. Our first remarks however are valid in arbitrary dimension.

Throughout this paragraph, let $S = \mathrm{conv}(\{v_0, \ldots, v_d\})$ be a simplex with the origin in its interior and with rational vertex directions. For each $i = 0, \ldots, d$, let p_i be the primitive positive multiple of v_i. Let $\alpha = (\alpha_0, \ldots, \alpha_d) \in \mathbb{N}^{d+1}$ be the primitive integer linear dependence among the p_i's. That is,

$$\sum_{i=0}^{d}\alpha_i p_i = \mathbf{0} \qquad \text{and} \qquad \gcd(\alpha_0, \ldots, \alpha_d) = 1.$$

Also, for each i, let $\beta_i = \alpha_i\|p_i\|/\|v_i\| \in \mathbb{R}_{>0}$, so that

$$\sum_{i=0}^{d}\beta_i v_i = \sum_{i=0}^{d}\alpha_i p_i = \mathbf{0}.$$

Remark 5.38. The fact that the p_i's are primitive imposes some condition on the vector $\alpha \in \mathbb{N}^{d+1}$. Namely, for each $i \in \{0, \dots, d\}$, we have

$$\gcd(\alpha_j : j \neq i) = 1.$$

Indeed, let Λ be the lattice generated by $\{p_0, p_1, \dots, p_d\}$, and let Λ_i be the sublattice generated by $\{p_j : j \neq i\}$. Then, the primitive vector of Λ_i in the direction of p_i is

$$\frac{\sum_{j \neq i} \alpha_j p_j}{\gcd(\alpha_j : j \neq i)} = \frac{-\alpha_i p_i}{\gcd(\alpha_j : j \neq i)},$$

which is an integer multiple of p_i if, and only if, $\gcd(\alpha_j : j \neq i) = 1$.

As in the previous sections, for each i let ℓ_i be the lattice length of $S \cap \mathbb{R}v_i$. The following lemma says that the vectors α and $\beta = (\beta_0, \beta_1, \dots, \beta_d)$ contain all the information about S needed to compute the right-hand side in (5.8).

Lemma 5.39. *The lattice length of $S \cap \mathbb{R}v_i$ equals*

$$\ell_i = \frac{\alpha_i}{\beta_i} + \frac{\alpha_i}{\sum_{j \neq i} \beta_j} = \frac{\alpha_i}{\beta_i} \cdot \frac{\sum_{j=0}^{d} \beta_j}{\sum_{j \neq i} \beta_j}.$$

Proof. To slightly simplify notation, we do the computations for $i = 0$. For this, let us use the vectors $p_1, \dots, p_d$ as the basis for a linear coordinate system in $\mathbb{R}^d$. In these coordinates, p_0 becomes

$$p_0 = -\frac{1}{\alpha_0} (\alpha_1, \dots, \alpha_d).$$

On the other hand, the equation of the facet of S opposite to v_0 is

$$\sum_{j=1}^{d} \frac{\beta_j}{\alpha_j} x_j = 1,$$

so that this facet intersects the line spanned by p_0 in the point

$$\frac{(\alpha_1, \dots, \alpha_d)}{\sum_{j=1}^{d} \beta_j} = \frac{-\alpha_0}{\sum_{j=1}^{d} \beta_j} p_0.$$

Thus, the segment $S \cap \mathbb{R}v_0$ has endpoints $\frac{\alpha_0}{\beta_0} p_0$ and $\frac{-\alpha_0}{\sum_{j=1}^{d} \beta_j} p_0$, which implies the statement. $\square$

Remark 5.40. Observe that the quantity ω_i in the proof of corollary 5.35 equals α_i/β_i. With this in mind, one easily recovers the equality $\sum_i \frac{\omega_i}{\ell_i} = d$ used in that proof, from lemma 5.39.

Our proof of Conjecture C in two dimensions is based on applying Lemma 5.7 to the projection $\pi : \mathbb{R}^2 \to \mathbb{R}$ along the direction of v_i, for some fixed $i \in \{0, 1, 2\}$. Then, with the notation above,

(i) α_0, α_1 and α_2 are pairwise coprime, by Remark 5.38.

(ii) The lattice length of $S \cap \pi^{-1}(\mathbf{0})$ is ℓ_i.

(iii) The lattice length of $\pi(S)$ equals

$$\frac{\alpha_j\alpha_k}{\beta_j} + \frac{\alpha_j\alpha_k}{\beta_k} = \frac{\alpha_j\alpha_k}{\beta_j\beta_k}(\beta_j + \beta_k),$$

where $\{j, k\} = \{0, 1, 2\} \setminus \{i\}$. Here we use that the projection of the segment $[\mathbf{0}, v_j] = \frac{\alpha_j}{\beta_j}[\mathbf{0}, p_j]$ has length $\alpha_k\frac{\alpha_j}{\beta_j}$, since $\gcd(\alpha_j, \alpha_k) = 1$ implies that $\pi(\frac{p_j}{\alpha_k})$ is a primitive lattice point in the projection.

Writing $L = \pi^{-1}(\mathbf{0})$, Lemma 5.7 gives us

$$\mu(S) \leq \mu(S \cap L, \mathbb{Z}^2 \cap L) + \mu(\pi(S), \pi(\mathbb{Z}^2)).$$

Hence, the inequality (5.8) would follow from:

$$\frac{1}{\ell_j} + \frac{1}{\ell_k} - \frac{1}{\ell_i} \geq \frac{2\beta_j\beta_k}{\alpha_j\alpha_k(\beta_j + \beta_k)}. \tag{5.10}$$

We prove this inequality under mild assumptions.

Theorem 5.41. *Let $S = \operatorname{conv}(\{v_0, v_1, v_2\}) \subseteq \mathbb{R}^2$ be a triangle with the origin in its interior and with rational vertex directions. Let the vectors α and β, and the lengths ℓ_i be defined as above, and let p_0, p_1 and p_2 be primitive in the directions of v_0, v_1 and v_2. Assume that $(\alpha_0, \alpha_1, \alpha_2) \neq (1, 1, 1)$. Then, the inequality* (5.10) *holds for some choice of $i \in \{0, 1, 2\}$.*

Moreover, the inequality is strict unless $(\alpha_0, \alpha_1, \alpha_2) = (2, 1, 1)$ and $\beta_1 = \beta_2$, up to reordering the indices.

Example 5.42.

(i) The necessity of $(\alpha_0, \alpha_1, \alpha_2) \neq (1, 1, 1)$ is shown by the following example. If $S = S(1, 1, 1)$ (so that $\alpha_i = \beta_i = 1$ for all i), then

$$\frac{1}{\ell_j} + \frac{1}{\ell_k} - \frac{1}{\ell_i} = \frac{2}{3} \qquad \text{and} \qquad \frac{2\beta_j\beta_k}{\alpha_j\alpha_k(\beta_k + \beta_k)} = 1,$$

so the inequality fails.

(ii) Even if $(\alpha_0, \alpha_1, \alpha_2) \neq (1,1,1)$, it is not true that (5.10) holds *for every* $i \in \{0,1,2\}$. For $\omega > 0$, consider the simplex

$$S = \operatorname{conv}(\{(0,\omega), (-1,-1), (1,-1)\}).$$

It has parameters $(\alpha_0, \alpha_1, \alpha_2) = (2,1,1)$, $(\beta_0, \beta_1, \beta_2) = \left(\frac{2}{\omega}, 1, 1\right)$, $\ell_0 = \omega + 1$, and $\ell_1 = \ell_2 = \frac{2\omega+2}{\omega+2}$. For $i = 0$, we indeed have

$$\frac{1}{\ell_1} + \frac{1}{\ell_2} - \frac{1}{\ell_0} = 1 = \frac{2\beta_1\beta_2}{\alpha_1\alpha_2(\beta_1+\beta_2)}.$$

But for $i \in \{1,2\}$, we get

$$\frac{1}{\ell_j} + \frac{1}{\ell_k} - \frac{1}{\ell_i} = \frac{1}{\ell_0} = \frac{1}{\omega+1} < \frac{2}{\omega+2} = \frac{2\beta_j\beta_k}{\alpha_j\alpha_k(\beta_j+\beta_k)}.$$

Proof of Theorem 5.41. Case 1: At most one of the α_is equals 1. Say $\alpha_1 \neq 1 \neq \alpha_2$. With no loss of generality assume $\ell_2 \geq \ell_1$. Then, by Lemma 5.39,

$$\frac{1}{\ell_0} + \frac{1}{\ell_1} - \frac{1}{\ell_2} \geq \frac{1}{\ell_0} = \frac{\beta_0}{\alpha_0} \cdot \frac{\beta_1+\beta_2}{\beta_0+\beta_1+\beta_2} > \frac{\beta_0}{\alpha_0} \cdot \frac{\beta_1}{\beta_0+\beta_1} \geq \frac{2\beta_0\beta_1}{\alpha_0\alpha_1(\beta_0+\beta_1)}.$$

Case 2: Two of the α_is equal 1. Assume that $\alpha_1 = \alpha_2 = 1$. The condition $(\alpha_0, \alpha_1, \alpha_2) \neq (1,1,1)$ then implies $\alpha_0 \geq 2$, so that Lemma 5.39 gives

$$\begin{aligned}\frac{1}{\ell_1} + \frac{1}{\ell_2} - \frac{1}{\ell_0} &= \frac{\beta_1(\beta_0+\beta_2)}{\beta_0+\beta_1+\beta_2} + \frac{\beta_2(\beta_0+\beta_1)}{\beta_0+\beta_1+\beta_2} - \frac{\beta_0}{\alpha_0} \cdot \frac{\beta_1+\beta_2}{\beta_0+\beta_1+\beta_2} \\ &= \frac{2\beta_1\beta_2 + \left(1 - \frac{1}{\alpha_0}\right)\beta_0(\beta_1+\beta_2)}{\beta_0+\beta_1+\beta_2} \overset{*}{\geq} \frac{2\beta_1\beta_2 + \frac{1}{2}\beta_0(\beta_1+\beta_2)}{\beta_0+\beta_1+\beta_2}.\end{aligned}$$

Thus, the inequality we want to prove is

$$\frac{2\beta_1\beta_2 + \frac{1}{2}\beta_0(\beta_1+\beta_2)}{\beta_0+\beta_1+\beta_2} \geq \frac{2\beta_1\beta_2}{\beta_1+\beta_2}$$

or, equivalently,

$$2\beta_1\beta_2(\beta_1+\beta_2) + \frac{1}{2}\beta_0(\beta_1+\beta_2)^2 \geq 2\beta_1\beta_2(\beta_0+\beta_1+\beta_2).$$

This is equivalent to $(\beta_1+\beta_2)^2 \overset{*}{\geq} 4\beta_1\beta_2$, which clearly holds.

The two inequalities we used, marked with "$\overset{*}{\geq}$", are equalities if and only if $\alpha_0 = 2$ and $\beta_1 = \beta_2$, respectively. ☐

We now prove Conjecture C for $d = 2$, which together with corollary 5.21 and proposition 5.31, gives the third proof of conjecture A in the plane.

Corollary 5.43. *Conjecture C holds in dimension two.*

Proof. Let $S = \operatorname{conv}(\{v_0, v_1, v_2\}) \subseteq \mathbb{R}^2$ be a triangle with the origin in its interior and with rational vertex directions. Let the vectors α and β, and the lengths ℓ_i be defined as above, taking p_0, p_1 and p_2 primitive. In view of Lemma 5.33 we need to show that

$$\mu(S) \leq \frac{1}{2}\left(\frac{1}{\ell_0} + \frac{1}{\ell_1} + \frac{1}{\ell_2}\right).$$

If $(\alpha_0, \alpha_1, \alpha_2) = (1, 1, 1)$, then consider the lattice Λ generated by p_0, p_1, p_2. Let A be the linear transformation sending e_i to p_i, for $i = 1, 2$. Then, $\Lambda = A\mathbb{Z}^2$ and $S = AS(\omega)$ for a suitable $\omega \in \mathbb{R}^3_{>0}$. Moreover, since the p_is are primitive, the lattice lengths ℓ_i are the same for every pair $(S, \mathbb{Z}^2)$, (S, Λ), and $(S(\omega), \mathbb{Z}^2)$. Observing that $\Lambda \subseteq \mathbb{Z}^2$ is a sublattice, we may therefore apply Theorem 5.4 and get

$$\mu(S) \leq \mu(S, \Lambda) = \mu(S(\omega), \mathbb{Z}^2) = \frac{1}{2}\left(\frac{1}{\ell_0} + \frac{1}{\ell_1} + \frac{1}{\ell_2}\right).$$

So, we assume that $(\alpha_0, \alpha_1, \alpha_2) \neq (1, 1, 1)$ and thus we can apply Theorem 5.41, which provides us with an index $i \in \{0, 1, 2\}$ such that the inequality (5.10) holds. As we saw above, this implies the desired bound. □

Analogs to conjecture C with the origin in the boundary

As we said in the introduction, the question analogous to Conjecture A for general lattice polytopes has an easy answer: the maximum covering radius among all d-dimensional lattice polytopes equals d and is attained by, and only by, unimodular simplices. This phenomenon generalizes to analogs of theorem 5.4 and conjecture C, which admit easy proofs. The generalization concerns the simplices $S(\omega)$, except we now allow one of the entries of ω (typically the first one) to be zero so that the origin becomes a vertex:

Proposition 5.44. *For an $\omega \in \mathbb{R}^d_{>0}$ let*

$$S(0, \omega) := \operatorname{conv}(\{\mathbf{0}, \omega_1 e_1, \ldots, \omega_d e_d\}).$$

Then

$$\mu(S(0, \omega)) = \sum_{i=1}^{d} \frac{1}{\omega_i} = \frac{\sum_{i=1}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S(\omega)))}{\operatorname{Vol}_{\mathbb{Z}^d}(S(\omega))},$$

where $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ is the linear projection that forgets the i-th coordinate.

Proof. $S(0, \omega)$ can be redescribed as

$$\left\{ x \in \mathbb{R}^d_{\geq 0} : \sum_{i=1}^{d} \frac{x_i}{\omega_i} \leq 1 \right\}.$$

In this form it is clear that $\mu(S(0,\omega))$ equals the unique $\mu \in [0, \infty)$ such that $\mathbf{1}_d$ lies in the boundary of $\mu \cdot S(0, \omega)$, which equals $\sum_i \frac{1}{\omega_i}$, as stated. $\square$

Corollary 5.45. *Let $S = \operatorname{conv}(\{\mathbf{0}, v_1, \ldots, v_d\}) \subseteq \mathbb{R}^d$ be a d-simplex with rational vertex directions. For each $i = 1, \ldots, d$, let $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the linear projection vanishing at v_i. Then,*

$$\mu(S) \leq \frac{\sum_{i=1}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}{\operatorname{Vol}_{\mathbb{Z}^d}(S)},$$

with equality if and only if S is unimodularly equivalent (by a transformation fixing the origin) to $S(0, \omega)$ for some $\omega \in \mathbb{R}^d_{>0}$.

Proof. Let $p_1, \ldots, p_d \in \mathbb{Z}^d$ be the primitive vertex directions of S, so that $v_i = \omega_i p_i$, where ω_i is the lattice length of the segment $[\mathbf{0}, v_i]$, for each $i = 1, \ldots, d$. Then, the linear map sending $p_i \mapsto e_i$, $i = 1, \ldots, d$, sends S to $S(0, \omega)$ and $\mathbb{Z}^d$ to a lattice Λ containing $\mathbb{Z}^d$. This implies

$$\mu(S, \mathbb{Z}^d) = \mu(S(0, \omega), \Lambda) \leq \mu(S(0, \omega), \mathbb{Z}^d) = \frac{\sum_{i=1}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}{\operatorname{Vol}_{\mathbb{Z}^d}(S)},$$

by Proposition 5.44.

The 'if' in the equality case is obvious: in this case $\Lambda = \mathbb{Z}^d$. For the 'only if' suppose that Λ is a proper superlattice of $\mathbb{Z}^d$ and let $p \in \Lambda \cap [0, 1)^d \setminus \{\mathbf{0}\}$ be a non-zero lattice point in the half-open unit cube. Let $\mu = \mu(S(0, \omega), \mathbb{Z}^d) = \frac{\sum_{i=1}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}{\operatorname{Vol}_{\mathbb{Z}^d}(S)}$. Then, the point $\mathbf{1}$ is the only point in the unit cube $[0, 1]^d$ that is last covered by $\mathbb{Z}^d + \mu \cdot S(0, \omega)$. Since $\mathbf{1}$ lies in the interior of $p + \mu \cdot S(0, \omega)$, the covering radius of $S(0, \omega)$ is strictly smaller with respect to Λ than it is with respect to $\mathbb{Z}^d$. $\square$

Our next results say that proposition 5.44 and corollary 5.45 are not only analogs (without the factor of two) of theorem 5.4 and conjecture C, but also a limit of them when we make one of the vertices tend to zero. We consider this as additional evidence for Conjecture C. Formally:

Theorem 5.46. *Let $S = \operatorname{conv}(\{v_0, \ldots, v_d\})$ be a d-simplex with the origin in its interior and with rational vertex directions. For each $i \in \{0, \ldots, d\}$ consider the one-parameter family of simplices*

$$S_t^{(i)} := \operatorname{conv}(\{v_0, \ldots, t v_i, \ldots, v_d\}), \quad t \in [0, 1],$$

so that $S_1^{(i)} = S$ *and* $S_0^{(i)} = \operatorname{conv}(\{v_1, \dots, \mathbf{0}, \dots, v_d\})$. *For each* $i = 0, \dots, d$ *let* $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ *be the linear projection vanishing at* v_i.

Then, there is an index $j \in \{0, \dots, d\}$ *such that*

$$\lim_{t\to 0} \frac{1}{2} \frac{\sum_{i=0}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_t^{(j)}))}{\operatorname{Vol}_{\mathbb{Z}^d}(S_t^{(j)})} \geq \frac{\sum_{i=0, i\neq j}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0^{(j)}))}{\operatorname{Vol}_{\mathbb{Z}^d}(S_0^{(j)})}, \tag{5.11}$$

with equality if and only if the primitive lattice vectors parallel to $v_0, \dots, v_d$ *add up to zero.*

Observe that the condition for equality includes, but is more general than, the case when S is of the form $S(\omega)$.

Proof. For each i, let u_i be the primitive lattice vector parallel to v_i, and let $U = \{u_0, \dots, u_d\}$. We choose j to be an index minimizing the (absolute value of the) determinant of $U \setminus \{u_i\}$ among all i. Observe that S is of the form $S(\omega)$ if and only if all those determinants are equal to 1.

To simplify notation, in the rest of the proof we assume $j = 0$ and we drop the superindex from the notation $S_t^{(j)}$.

Since the volume functional is continuous, we have

$$\lim_{t\to 0} \operatorname{Vol}_{\mathbb{Z}^d}(S_t) = \operatorname{Vol}_{\mathbb{Z}^d}(S_0),$$

and, for each $i = 1, \dots, d$,

$$\lim_{t\to 0} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_t)) = \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0)).$$

Thus, the only thing to prove is that

$$\lim_{t\to 0} \operatorname{Vol}_{\pi_0(\mathbb{Z}^d)}(\pi_0(S_t)) \geq \sum_{i=1}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0)).$$

The volume on the left-hand side does not depend on t because the vertex of S_t that depends on t is projected out by π_0. Moreover, this volume equals $\sum_{i=1}^{d} \operatorname{Vol}_{\pi_0(\mathbb{Z}^d)}(\pi_0(F_i))$, where F_i is the facet of S_0 opposite to v_i. Similarly, $\operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0)) = \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(F_i))$. Hence, the inequality follows from

$$\operatorname{Vol}_{\pi_0(\mathbb{Z}^d)}(\pi_0(F_i)) \geq \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(F_i)). \tag{5.12}$$

Both sides of eq. (5.12) are integer multiples of $\mathrm{Vol}_{\mathbb{Z}^d\cap \mathrm{aff}\,(F_i)}(F_i)$, with the proportionality factors being the lattice distances from F_i to u_0 and to u_i, respectively. These distances are proportional to the determinants of $U \setminus \{u_i\}$ and $U \setminus \{u_0\}$, so our assumption on u_0 minimizing this implies the statement. Moreover, we have equality if, and only if, all the determinants of $U \setminus \{u_i\}$ are equal to that of $U \setminus \{u_0\}$. This in turn is equivalent to $\sum_{i=0}^d u_i = \mathbf{0}$. □

Corollary 5.47. *In the conditions of theorem 5.46 and for the index j mentioned therein, we have*

$$\lim_{t\to 0} \mu(S_t^{(j)}) \leq \lim_{t\to 0} \frac{1}{2} \frac{\sum_{i=0}^d \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_t^{(j)}))}{\mathrm{Vol}_{\mathbb{Z}^d}(S_t^{(j)})},$$

with equality if and only if the primitive lattice vectors parallel to $v_0, \dots, v_d$ add up to zero.

Proof. This follows from theorem 5.46 since

$$\lim_{t\to 0} \mu(S_t^{(j)}) = \mu(S_0^{(j)}) \leq \frac{\sum_{i=0, i\neq j}^d \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0^{(j)}))}{\mathrm{Vol}_{\mathbb{Z}^d}(S_0^{(j)})},$$

where the last inequality is corollary 5.45. □

Remark 5.48. eq. (5.11) is not true for all choices of j. Without any assumption on j the proof of theorem 5.46 carries through up to the point where we say that eq. (5.11) would follow from eq. (5.12), but the latter inequality is not true in general. For a specific example, let $S = \mathrm{conv}\,(\{(0,-1),(1,1),(-1,1)\})$ and consider $j = 0$. Then, for $i = 1, 2$,

$$\mathrm{Vol}_{\pi_0(\mathbb{Z}^d)}(\pi_0(F_i)) = 1 < 2 = \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(F_i)).$$

This gives

$$\lim_{t\to 0} \frac{1}{2} \frac{\sum_{i=0}^d \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_t^{(0)}))}{\mathrm{Vol}_{\mathbb{Z}^d}(S_t^{(0)})} = \frac{1}{2} \cdot \frac{2+2+2}{2} = \frac{3}{2},$$

and

$$\frac{\sum_{i=1}^d \mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S_0^{(0)}))}{\mathrm{Vol}_{\mathbb{Z}^d}(S_0^{(0)})} = \frac{2+2}{2} = 2.$$

We finally look at the intermediate case where $\mathbf{0}$ is in the boundary of $S = \mathrm{conv}\,(\{v_0, \dots, v_d\})$ but not a vertex. We can generalize conjecture C to

Conjecture E. Let $S = \mathrm{conv}\,(\{v_0, \dots, v_d\})$ be a d-simplex with $\mathbf{0} \in S \setminus \{v_0, \dots, v_d\}$, and with rational vertex directions. Let $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the linear projection vanishing at v_i. Let $I \subset \{0,\dots,d\}$ be the set of labels of facets of S containing $\mathbf{0}$. Then

$$\mu(S) \le \frac{1}{2}\frac{\sum_{i=0}^{d} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S)) + \sum_{i\in I} \operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}{\operatorname{Vol}_{\mathbb{Z}^d}(S)}. \tag{5.13}$$

Proposition 5.49. *conjecture E $\iff$ conjecture C.*

Proof. The implication conjecture E $\implies$ conjecture C is obvious, since the latter is the case $I = \emptyset$ of the former.

For the other implication, for each $i = 0,\dots,d$, let

$$\ell_i = \frac{\operatorname{Vol}_{\mathbb{Z}^d}(S)}{\operatorname{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))},$$

which equals the lattice length of the segment $S \cap \operatorname{lin}(\{v_i\})$. The inequality in conjecture E we want to prove becomes

$$\mu(S) \le \frac{1}{2}\sum_{i\notin I}\frac{1}{\ell_i} + \sum_{i\in I}\frac{1}{\ell_i}.$$

Let $S_I = \operatorname{conv}(\{v_i : i \notin I\})$, and $S_{\overline{I}} = \operatorname{conv}(\{\mathbf{0}\} \cup \{v_i : i \in I\})$. Observe that S_I equals the intersection of the facets of S containing $\mathbf{0}$, hence it is a $(d-|I|)$-simplex with $\mathbf{0}$ in its relative interior. $S_{\overline{I}}$ is an $|I|$-simplex with $\mathbf{0}$ as a vertex. Hence, conjecture C and proposition 5.44 respectively say:

$$\mu(S_I) \le \frac{1}{2}\sum_{i\notin I}\frac{1}{\ell_i} \qquad \text{and} \qquad \mu(S_{\overline{I}}) \le \sum_{i\in I}\frac{1}{\ell_i}.$$

Consider the linear projection $\pi_I : \mathbb{R}^d \to \mathbb{R}^I$ vanishing on S_I. By lemma 5.7

$$\mu(S) \le \mu(S_I) + \mu(\pi_I(S)),$$

so it only remains to show that

$$\mu(\pi_I(S)) \le \mu(S_{\overline{I}}).$$

This holds because π_I is an affine bijection from $S_{\overline{I}}$ to $\pi_I(S)$, so that $\pi_I(S)$ can be considered to be the same as $S_{\overline{I}}$ except regarded with respect to a (perhaps) finer lattice. $\square$

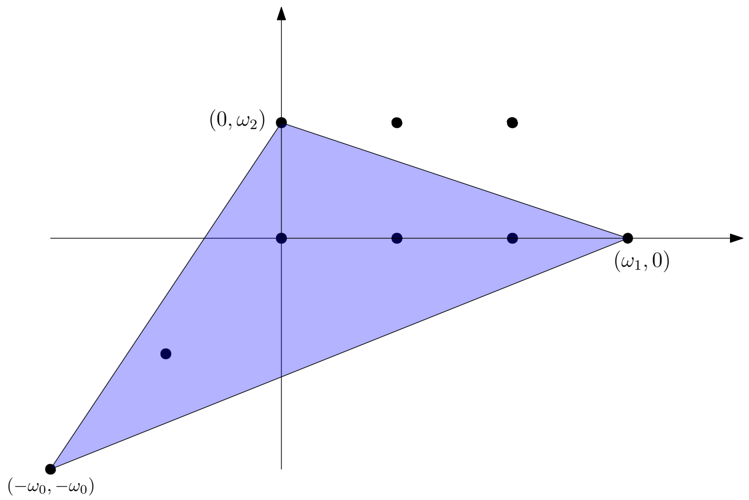


FIGURE 5.3: An example of a simplex $S(\omega)$ in dimension two, with $\omega = (2, 4, 1)$

5.4 Covering minima of the simplex $S(\omega)$

The covering radius of $S(\omega)$

We here prove theorem 5.4. That is, we compute the covering radius of simplices of the form

$$S(\omega) = \operatorname{conv}(\{-\omega_0 \mathbf{1}_d, \omega_1 e_1, \ldots, \omega_d e_d\}),$$

see Figure 5.3.

Proof of theorem 5.4. The simplex $S(\omega)$ can be triangulated into the $d+1$ simplices

$$S_i = \operatorname{conv}(\{\mathbf{0}, \omega_0 e_0, \omega_1 e_1, \ldots, \omega_d e_d\} \setminus \{\omega_i e_i\}), \quad 0 \leq i \leq d,$$

where $e_0 = -\mathbf{1}_d$. Writing $[d]_0 := \{0, 1, \ldots, d\}$, we define

$$\mathring{P}_i = \left\{ \sum_{j \in [d]_0 \setminus \{i\}} \alpha_j e_j : 0 \leq \alpha_j < 1 \right\}$$

the half-open parallelotope spanned by the primitive edge directions of S_i incident to the origin. Let $i \in [d]_0$ be fixed. Then, for any $x \in \mathbb{R}^d$ there is a lattice point $v_i \in \mathbb{Z}^d$ such that $x \in v_i + \lambda S_i$ and the dilation factor $\lambda \geq 0$ is the smallest possible. Let $L_i(x)$ be the set of all such lattice points v_i. For a fixed $v \in \mathbb{Z}^d$, we define

$$R_i(v) = \left\{ x \in \mathbb{R}^d : v \in L_i(x) \right\}$$

to be the region of points that are associated to v in this way.

Explicitly these regions are translates of the $\mathring{P}_i$, more precisely we claim that $R_i(v) = v + \mathring{P}_i$, for all $i \in [d_0]$.

Indeed, let $x \in R_i(v)$, and let $\lambda \geq 0$ be smallest possible such that $x \in v + \lambda S_i$. By the definition of S_i, we can write $x - v = \sum_{j\in[d]_0\setminus\{i\}} \alpha_j e_j$, for some $\alpha_j \geq 0$. If there would be an index j such that $\alpha_j \geq 1$, then $x \in v + e_j + \lambda S_i$ and the intersection of this simplex and $v + \lambda S_i$ is a smaller homothetic copy of S_i containing x. Thus, λ is not minimal and this contradiction implies that $x \in v + \mathring{P}_i$. Conversely, if $x - v = \sum_{j\in[d]_0\setminus\{i\}} \alpha_j e_j \in \mathring{P}_i$, and $\lambda \geq 0$ is minimal such that $x \in v + \lambda S_i$, then $x - v$ lies in the facet of λS_i not containing the origin. Since $0 \leq \alpha_j < 1$, for all $j \in [d]_0 \setminus \{i\}$, the scalar λ is not only minimal for v, but for any lattice point. Hence, $v \in L_i(x)$.

With this observation, the regions $R_i(v)$ are seen to be induced by the arrangement of the hyperplanes $\{x_i = a\}, \{x_i - x_j = a\}$ for all $j \in [d]_0 \setminus \{i\}$ and $a \in \mathbb{Z}$, where we define $x_0 = 0$. We call this arrangement A^i_d. Moreover, for a point x in the interior of $R_i(v)$, the associated lattice point is unique, and we call it $v_i(x)$.

The smallest common refinement A_d of the arrangements $\mathrm{A}^0_d, \ldots, \mathrm{A}^d_d$ is known as the *alcoved arrangement* (see [BS18, Ch. 7] for a detailed description). The full-dimensional cells of A_d, also called its *chambers*, are lattice translations of the simplices

$$C_\pi = \operatorname{conv}\left(\{\mathbf{0}, e_{\pi(1)}, e_{\pi(1)} + e_{\pi(2)}, \ldots, e_{\pi(1)} + \ldots + e_{\pi(d)}\}\right),$$

where π is a permutation of $\{1, \ldots, d\}$.

Each chamber of A_d is the intersection of regions $R_i(v)$. More precisely,

$$\begin{aligned}\operatorname{int}(C_\pi) &= R_0(\mathbf{0}) \cap R_{\pi(1)}(e_{\pi(1)}) \cap \ldots \cap R_{\pi(d)}(e_{\pi(1)} + \ldots + e_{\pi(d)}) \\ &= \mathring{P}_0 \cap (e_{\pi(1)} + \mathring{P}_{\pi(1)}) \cap \ldots \cap (e_{\pi(1)} + \ldots + e_{\pi(d)} + \mathring{P}_{\pi(d)}).\end{aligned}$$

Therefore, the chambers C_π are exactly those regions of points in $\mathbb{R}^d$ that, for each $i \in [d]_0$, are associated to the same lattice point, that is, $v_i(x) = v_i(y)$ for all $x, y \in \operatorname{int}(C_\pi)$.

After these preparations, we are ready to compute the covering radius of $S(\omega)$. Note that, since $[0,1]^d$ is a fundamental cell of $\mathbb{Z}^d$, we only need to find the smallest dilation factor μ so that the lattice translates of $\mu S(\omega)$ cover the unit cube. Moreover, we may focus on what happens within one chamber C_π, and by symmetry we assume that $\pi = \mathrm{Id}$. Among all points in $C_{\mathrm{Id}} = \operatorname{conv}(\{\mathbf{0}, e_1, e_1 + e_2, \ldots, e_1 + \ldots + e_d\})$, we are looking for a point y which is last covered by dilations of $S_i + e_{[i]}$, for some $i \in [d]_0$, and the factor of dilation needed. Here, we write $e_{[i]} = e_1 + \ldots + e_i$. If we let $\ell_i : \mathbb{R}^d \to \mathbb{R}$ be the linear functional which takes value 1 on the facet F_i of $S(\omega)$ that is opposite to $\omega_i e_i$, this is equivalent to

$$y = \operatorname*{argmax}_{x \in C_{\mathrm{Id}}} \min_{i \in [d]_0} |\ell_i(x - e_{[i]})|.$$

The key observation is that y is the point where all the values $|\ell_i(y - e_{[i]})|$, $0 \le i \le d$, are equal. This is because $\ell_i(x - e_{[i]})$ is nonnegative for $x \in C_{\mathrm{Id}}$ and because there is a positive linear dependence among the functionals ℓ_i, so there cannot be a point y' where they all achieve a larger value than at a point where they all achieve the same value. Therefore, y satisfies the conditions

$$\ell_0(y) = \ell_i(y - e_{[i]}), \quad \text{for every } 1 \le i \le d.$$

The explicit expression of the functionals ℓ_i is

$$\ell_0(x) = \sum_{j=1}^{d} \omega_j^{-1} x_j \quad \text{and} \quad \ell_i(x) = \sum_{j \in [d] \setminus \{i\}} \omega_j^{-1} x_j - \left(\sum_{j \in [d]_0 \setminus \{i\}} \omega_j^{-1} \right) x_i.$$

Thus we need to solve the system of the following equations:

$$\sum_{j=1}^{d} \omega_j^{-1} y_j = \sum_{j \in [d] \setminus \{i\}} \omega_j^{-1} y_j - \left(\sum_{j \in [d]_0 \setminus \{i\}} \omega_j^{-1} \right) y_i + \omega_0^{-1} + \sum_{j > i} \omega_j^{-1}, \quad 1 \le i \le d.$$

This system is solved by $y = (y_1, \ldots, y_d)$ with

$$y_i = \frac{\omega_0^{-1} + \omega_{i+1}^{-1} + \ldots + \omega_d^{-1}}{\omega_0^{-1} + \omega_1^{-1} + \ldots + \omega_d^{-1}}.$$

The value that the functionals take at y is by what we said above the covering radius of $S(\omega)$, and it is given by

$$\mu(S(\omega)) = \ell_0(y) = \frac{\sum_{0 \le i < j \le d} \omega_i^{-1} \omega_j^{-1}}{\sum_{i=0}^{d} \omega_i^{-1}},$$

as desired. □

Corollary 5.50. *Let $S \subseteq \mathbb{R}^d$ be a simplex with the origin it its interior and with rational vertex directions. If the primitive vertex directions $p_0, p_1, \ldots, p_d$ of S satisfy $p_0 + p_1 + \ldots + p_d = \mathbf{0}$, then Conjecture C holds for S.*

Proof. The proof is basically given already in Corollary 5.43. Consider the lattice Λ generated by $p_0, p_1, \ldots, p_d$, and let A be the linear transformation sending e_i to p_i, for $i = 1, \ldots, d$. Then, $\Lambda = A\mathbb{Z}^d$ and $S = AS(\omega)$ for a suitable $\omega \in \mathbb{R}^{d+1}_{>0}$. Since the p_is are primitive, the lattice lengths $\ell_i = \frac{\mathrm{Vol}_{\mathbb{Z}^d}(S)}{\mathrm{Vol}_{\pi_i(\mathbb{Z}^d)}(\pi_i(S))}$ are the same for every pair $(S, \mathbb{Z}^d)$,

(S,Λ), and $(S(\omega),\mathbb{Z}^d)$. Using that $\Lambda \subseteq \mathbb{Z}^d$ is a sublattice, we therefore apply Theorem 5.4 and get

$$\mu(S) \leq \mu(S,\Lambda) = \mu(S(\omega),\mathbb{Z}^d) = \frac{1}{2}\sum_{i=0}^{d}\frac{1}{\ell_i}. \qquad \square$$

Observe that theorem 5.4 says that eq. (5.4) in conjecture C is an equality for simplices of the form $S(\omega)$. Other simplices may also produce an equality, as the triangle $T = S(\mathbf{1}_2) \oplus S'(\mathbf{1}_2)$ shows:

$$\frac{1}{2}\frac{\sum_{i=0}^{2}\mathrm{Vol}_{\pi_i(\mathbb{Z}^2)}(\pi_i(T))}{\mathrm{Vol}_{\mathbb{Z}^2}(T)} = \frac{1}{2}\cdot\frac{3+3+2}{4} = 1 = \mu(T).$$

The covering product conjecture

The following conjecture was proposed in [GMS17], which was the initial motivation to compute the covering minima of the simplex $S(\mathbf{1}_{d+1})$.

Conjecture F ([GMS17, Conj. 4.8])**.** For every convex body $K \subseteq \mathbb{R}^d$,

$$\mu_1(K)\cdot\ldots\cdot\mu_d(K)\cdot\mathrm{vol}(K) \geq \frac{d+1}{2^d}.$$

Equality is attained for the simplex $S(\mathbf{1}_{d+1})$.

conjecture F is known to hold for $d = 2$ [Sch95]. We show it in arbitrary dimension for the simplices $S(\omega)$.

Corollary 5.51. *For every $\omega \in \mathbb{R}^{d+1}_{>0}$, we have*

$$\mu_1(S(\omega))\cdot\ldots\cdot\mu_d(S(\omega))\cdot\mathrm{Vol}_{\mathbb{Z}^d}(S(\omega)) \geq \frac{(d+1)!}{2^d}.$$

Equality can hold only if $\omega_0 = \omega_1 = \ldots = \omega_d$.

Proof. Since every permutation of the vertices of $S(\mathbf{1})$ is a unimodular transformation, and since the considered product functional is invariant under unimodular transformations, we can assume that $\omega_0 \leq \omega_1 \leq \ldots \leq \omega_d$. By Theorem 5.4, the covering radius of $S(\omega)$ is given by

$$\mu(S(\omega)) = \frac{\sigma_{d-1}(\omega_0,\omega_1,\ldots,\omega_d)}{\sigma_d(\omega_0,\omega_1,\ldots,\omega_d)},$$

where $\sigma_j(\omega_0,\omega_1,\ldots,\omega_d) = \sum_{0\leq i_1<\ldots<i_j\leq d}\prod_{\ell=1}^{j}\omega_{i_\ell}$ is the *j-th elementary symmetric function* in the ω_i's. Writing $\omega_I = (\omega_0,\omega_{i_1},\ldots,\omega_{i_j})$, for every index set $I = \{i_1,\ldots,i_j\} \subseteq$

$\{1, \ldots, d\}$, $|I| = j$, we project onto the j-dimensional coordinate plane indexed by I and obtain $\mu_j(S(\omega)) \geq \mu_j(S(\omega_I))$. In particular, choosing $I = \{1, \ldots, j\}$, we have

$$\mu_j(S(\omega)) \geq \frac{\sigma_{j-1}(\omega_0, \omega_1, \ldots, \omega_j)}{\sigma_j(\omega_0, \omega_1, \ldots, \omega_j)}. \tag{5.14}$$

Next, in view of $\omega_j \geq \omega_{j-1} \geq \ldots \geq \omega_0$, we get

$$\begin{aligned}\frac{\sigma_{j-1}(\omega_0, \ldots, \omega_j)}{\sigma_{j-1}(\omega_0, \ldots, \omega_{j-1})} &= \frac{\sigma_{j-1}(\omega_0, \ldots, \omega_{j-1}) + \omega_j\, \sigma_{j-2}(\omega_0, \ldots, \omega_{j-1})}{\sigma_{j-1}(\omega_0, \ldots, \omega_{j-1})} \\ &= 1 + \frac{\omega_j\, \sigma_{j-2}(\omega_0, \ldots, \omega_{j-1})}{\sigma_{j-1}(\omega_0, \ldots, \omega_{j-1})} \geq 1 + \frac{\binom{j}{2}}{j} = \frac{j+1}{2},\end{aligned} \tag{5.15}$$

with strict inequality unless $\omega_j = \omega_{j-1} = \ldots = \omega_0$.

Finally, computing the volumes of the pyramids over the $d+1$ facets of $S(\omega)$ with apex at the origin, we obtain $\mathrm{Vol}_{\mathbb{Z}^d}(S(\omega)) = \sigma_d(\omega_0, \omega_1, \ldots, \omega_d)$. Combining this with (5.14) and (5.15) yields

$$\begin{aligned}\mu_1(S(\omega)) \cdot \ldots \cdot \mu_d(S(\omega)) \cdot \mathrm{Vol}_{\mathbb{Z}^d}(S(\omega)) &\geq \prod_{j=1}^{d} \frac{\sigma_{j-1}(\omega_0, \ldots, \omega_j)}{\sigma_j(\omega_0, \ldots, \omega_j)} \sigma_d(\omega_0, \ldots, \omega_d) \\ &= \prod_{j=1}^{d} \frac{\sigma_{j-1}(\omega_0, \ldots, \omega_j)}{\sigma_{j-1}(\omega_0, \ldots, \omega_{j-1})} \geq \frac{(d+1)!}{2^d}.\end{aligned}$$

Furthermore, equality can only hold if $\omega_0 = \omega_1 = \ldots = \omega_d$ as otherwise (5.15) would be strict for $j = d$. □

Note that if Conjecture B holds, then the simplex $S(\mathbf{1}_{d+1})$ attains equality in Corollary 5.51 (this was the original motivation in [GMS17] to state Conjecture B).

With the notation of the proof above, for each $I \subseteq \{1, \ldots, d\}$, $|I| = j$, we have $\mu_j(S(\omega_I)) \leq \mu_j(S(\omega_0, \omega_1, \ldots, \omega_j))$, just because $S(\omega) \subseteq S(\bar{\omega})$, whenever $\omega_i \leq \bar{\omega}_i$, for all i. Therefore, the bound in (5.14) is maximal among coordinate projections of $S(\omega)$. This suggests the following common generalization of Conjecture B and Theorem 5.4.

Conjecture 5.52. *For every $\omega \in \mathbb{R}^{d+1}_{>0}$ with $\omega_0 \leq \omega_1 \leq \ldots \leq \omega_d$, and every $j \in \{1, \ldots, d\}$, the j-th covering minimum of the simplex $S(\omega)$ is attained by the projection to the first j coordinates. That is:*

$$\mu_j(S(\omega)) = \mu_j(S(\omega_0, \ldots, \omega_j)) = \frac{\sigma_{j-1}(\omega_0, \omega_1, \ldots, \omega_j)}{\sigma_j(\omega_0, \omega_1, \ldots, \omega_j)}.$$

Besides the case $j = d$ (Theorem 5.4) also the case $j = 1$ of conjecture 5.52 holds. Assuming that $\omega_0 \leq \omega_1 \leq \ldots \leq \omega_d$, it states that $\mu_1(S(\omega)) = \frac{1}{\omega_0 + \omega_1}$. Since (5.14) provides the lower bound, this is equivalent to

$$\det(\mathbb{Z}^d|L_z) \leq \frac{\|S(\omega)|L_z\|}{\omega_0+\omega_1},$$

for all primitive $z \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, where $L_z = \mathrm{lin}\{z\}$. In view of $\det(\mathbb{Z}^d|L_z) = \|z\|^{-1}$ and $e_i|L_z = \frac{z_i}{\|z\|^2}z$, it follows from an elementary computation.

5.5 Conjecture D: Lattice polytopes with k interior lattice points

We now look at Conjecture D, that is, we investigate the maximum covering radius among lattice d-polytopes with at least $k \geq 2$ interior lattice points. The conjectured maximum covering radius $\frac{d-1}{2} + \frac{1}{k+1}$ is attained by the polytopes of the form

$$[0, k+1] \oplus T_1 \oplus \cdots \oplus T_m,$$

where each T_i is a non-hollow lattice d_i-polytope of covering radius $d_i/2$, with $\sum_{i=1}^m d_i = d-1$. The different T_i can be translated to have their (unique) interior lattice point at different positions along the segment $[0, k+1]$ in much the same way as in the examples of Lemma 5.24. In the following we analyze the possibilities in dimensions two and three:

Example 5.53. In dimension two we have a single T_i, the segment $[-1, 1]$, but we can place it at different heights with respect to $[0, k+1]$. For each k we can construct $\lfloor (k+3)/2 \rfloor$ non-isomorphic lattice polygons with k interior lattice points and of covering radius $\frac{1}{2} + \frac{1}{k+1}$, namely:

$$\mathrm{conv}\left(\{(0,0), (0,k), (-1,i), (1,i)\}\right), \quad i = 0, \ldots, \lfloor (k+1)/2 \rfloor.$$

The case $i = 0$ coincides with the triangle $M_k(0)$; the cases $i > 0$ produce kite-shaped quadrilaterals.

Observe that the triangle $M_k(1) \cong S(k, 1, 1)$ is very similar to but has smaller area than $M_k(0)$. One could expect it to achieve a larger covering radius but it does not, as computed in remark 5.22:

$$\mu(M_k(1)) = \frac{k+2}{2k+1} = \frac{1}{2} + \frac{3}{4k+2} < \frac{1}{2} + \frac{1}{k+1}, \quad \text{if } k \geq 2.$$

Example 5.54. In dimension three we can have $[0, k+1] \oplus T$ with $\dim(T) = 2$ or $[0, k+1] \oplus T_1 \oplus T_2$ with $\dim(T_1) = \dim(T_2) = 1$.

If the latter happens then $T_1 = T_2 = [-1,1] = I$ and, again, they can be placed at different heights along the segment $[0,k+1]$. This gives quadratically many non-isomorphic octahedra (when both T_1 and T_2 intersect $[0,k+1]$ in the interior), linearly many triangular bipyramids (when one intersects in the interior and the other at an end-point), plus the square pyramid $[0,k+1] \oplus I \oplus I$ and the tetrahedron $M_k(0,0)$ (when both intersect at end-points).

In the case $[0,k+1] \oplus T$, T can be either $S(\mathbf{1}_3)$ or $I \oplus I'$; the case $T = I \oplus I$ being already covered above. This produces two tetrahedra $[0,k+1] \oplus S(\mathbf{1}_3)$ and $[0,k+1] \oplus I \oplus I'$, plus linearly many triangular bipyramids.

As happened in dimension two, the computations of remark 5.28 show that $M_k(1,0)$ and $M_k(1,1)$ have covering radius strictly smaller than $1 + \frac{1}{k+1}$, even if their volume is smaller than that of $M_k(0,0)$.

The rest of this section is devoted to prove conjecture D in dimension two. Since we proved conjecture A in the plane (corollary 5.21), it suffices to consider lattice polygons with at least two interior lattice points. More precisely, we show:

Theorem 5.55. *Let P be a non-hollow lattice polygon with $k \geq 2$ interior lattice points. Then $\mu(P) \leq \frac{1}{2} + \frac{1}{k+1}$, with equality if and only if P is the direct sum of two lattice segments of lengths 2 and $k+1$.*

Our proof is split up into five steps distinguishing cases with respect to the following parameters: A lattice polytope P has *(lattice) width* $\omega \in \mathbb{N}$ if there is an affine integer projection from P to the segment $[0,\omega]$ but not to $[0,\omega-1]$. Remember that the width is the reciprocal of the first covering minimum. The numbers m, m', and k will denote the maximum number of collinear lattice points, maximum number of collinear *interior* lattice points, and the number of interior lattice points of P, respectively. We proceed as follows:

Step 1: $(\omega = 2)$ in lemma 5.56

Step 2: $(\omega \geq 3, m \geq 4)$ except for $(\omega = 3, m = 4, k \geq 5)$ in lemma 5.57

Step 3: $(\omega = 3, m = 4, k \geq 5)$ in lemma 5.59

Step 4: $(\omega \geq 3, m' \geq 3)$ in lemma 5.60

Step 5: $(\omega \geq 3, m' \leq 2)$

It will turn out that equality in the bound of theorem 5.55 can only occur in the first case, that is, when P has width two.

Lemma 5.56. *Let P be a lattice polygon with $k \geq 2$ interior lattice points. Then, the following are equivalent:*

(i) P has width equal to two.

(ii) All the interior lattice points of P are collinear.

Moreover, if this happens then P satisfies conjecture D, with equality if and only if P is the direct sum of two lattice segments of lengths 2 and $k+1$.

Proof. The fact that width two implies that all interior lattice points are collinear is obvious. For the converse, without loss of generality assume that the k interior lattice points of P are $(0,1), \ldots, (0,k)$. We claim that $P \subset [-1,1] \times \mathbb{R}$. Suppose to the contrary that P has a lattice point (x,y) with $|x| \geq 2$. Then the triangle T with vertices $(0,1)$, $(0,2)$ and (x,y) is fully contained in $\operatorname{int}(P)$ except perhaps for the vertex (x,y) which may be in the boundary of P. Now, since T is a non-unimodular triangle it contains at least one lattice point other than its vertices. That point is in $\operatorname{int}(P)$, which contradicts the collinearity assumption.

This finishes the proof of the stated equivalence. Let us now show conjecture D for a lattice polygon P satisfying (i) and (ii). We keep the convention that the interior lattice points in P are $(0,1), \ldots, (0,k)$. Let S be the segment $P \cap (\{0\} \times \mathbb{R})$. We distinguish three cases, depending on whether none, one, or both of the end-points of S are lattice points:

- If exactly one is a lattice point, then P contains a copy of $M_k(1)$, whose covering radius is strictly smaller than $\frac{1}{2} + \frac{1}{k+1}$ (see example 5.53).
- If none is a lattice point then $S = \{0\} \times [1/2, k+1/2]$. Without loss of generality we have
$$P = \operatorname{conv}(\{(-1,0), (-1,a), (1,1), (1,1+b)\}),$$
where a and b are nonnegative integers with $a+b = 2k$. There are two possibilities: If $a = b = k$ then P is a parallelogram of covering radius at most $1/2$, because $\frac{1}{2}P$ contains a fundamental domain of $\mathbb{Z}^2$. If $a \neq b$ then one of them, say a, is at least $k+1$. In this case, P contains the triangle $\operatorname{conv}(\{(-1,0), (-1,a), (1,1)\})$ whose covering radius is bounded by $1/2 + 1/a \leq 1/2 + 1/(k+1)$. Since triangles are tight, equality can only hold when P coincides with this triangle, implying $b = 0$. But in that case $a = 2k$ and $1/2 + 1/a < 1/2 + 1/(k+1)$, since $k \geq 2$.
- If both end-points of S are lattice points, then they are given by $(0,0)$ and $(0,k+1)$. Applying lemma 5.7 to the projection that forgets the second coordinate gives the

upper bound: the fiber S has length $k+1$ and the projection has length 2. For the case of equality, observe that if P has lattice points $u \in \{-1\} \times \mathbb{R}$ and $v \in \{1\} \times \mathbb{R}$ such that the mid-point of uv is integral then P contains (an affine image of) the direct sum of $[-1,1]$ and a segment of length $k+1$. Since that direct sum is tight (lemma 5.14), P either equals the direct sum or it has strictly smaller covering radius.

Thus, we can assume that P does not have such points u and v. Put differently, P has a single lattice point on each side of S and the height of these points have different parity. Without loss of generality we can assume

$$P = \operatorname{conv}(\{(0,0),(0,k+1),(-1,0),(1,a)\}),$$

for an odd $a \in [1, 2k+1]$. We claim that the *proof* of lemma 5.7 implies that $\mu(P)$ is *strictly smaller* than $\lambda := 1/2 + 1/(k+1)$. Indeed, that proof is based on the fact that λP contains the following parallelogram Q, which is a fundamental domain for $\mathbb{Z}^2$:

$$Q = \operatorname{conv}\left(\left\{\left(-\frac{1}{2},0\right),\left(-\frac{1}{2},1\right),\left(\frac{1}{2},\frac{a}{2}\right),\left(\frac{1}{2},1+\frac{a}{2}\right)\right\}\right).$$

But we can argue that, moreover, the vertices of Q are its only points not contained in the interior of λP, and that each of these vertices is in the interior of some lattice translation of λP because the vertical offset of the left and right edges of Q is not an integer. This implies λ to be strictly larger than $\mu(P)$. □

For the rest of the proof of theorem 5.55, we can now assume that the width ω of P is at least three. Let L be the line containing the maximum number m of collinear lattice points in P. We will frequently use the following upper bound, obtained from lemma 5.7 applied to the projection along the direction of L:

$$\mu(P) \le \frac{1}{\omega} + \frac{1}{m-1}. \tag{5.16}$$

Lemma 5.57. *If P is a non-hollow lattice polygon with $(\omega \ge 3, m \ge 4, k \ge 2)$, except for $(\omega = 3, m = 4, k \ge 5)$, then P satisfies conjecture D with strict inequality.*

Proof. We look separately at the possibilitites for maximum number m of collinear interior lattice points of P:

If $m \ge 7$ then eq. (5.16) gives

$$\mu(P) \le \frac{1}{\omega} + \frac{1}{m-1} \le \frac{1}{3} + \frac{1}{6} = \frac{1}{2},$$

and there is nothing to prove.

If $m \in \{5,6\}$ then the same argument works as long as $\omega \geq 4$. If $\omega = 3$, then we have $k \leq 2m$, because all interior lattice points lie in two parallel lines orthogonal to the direction in which the width is attained. Thus, we get the following, depending on the value of m:

$$\begin{aligned}
\mu(P) &\leq \frac{1}{3} + \frac{1}{5} = \frac{8}{15} \quad < \quad \frac{15}{26} = \frac{1}{2} + \frac{1}{13} \leq \frac{1}{2} + \frac{1}{k+1}, \qquad \text{if } m = 6. \\
\mu(P) &\leq \frac{1}{3} + \frac{1}{4} = \frac{7}{12} \quad < \quad \frac{13}{22} = \frac{1}{2} + \frac{1}{11} \leq \frac{1}{2} + \frac{1}{k+1}, \qquad \text{if } m = 5.
\end{aligned}$$

So, for the rest of the proof we assume $m = 4$. If $\omega \geq 6$ then eq. (5.16) again gives $\mu(P) \leq \frac{1}{6} + \frac{1}{3}$. Thus, assume $\omega \in \{3,4,5\}$ and suppose without loss of generality that $P \subset [0,\omega] \times \mathbb{R}$. The observation that $k \leq 4(\omega - 1)$ (because there are $\omega - 1$ intermediate lines $\{i\} \times \mathbb{R}$, each with at most $m = 4$ lattice points) discards the case $\omega = 5$:

$$\mu(P) \leq \frac{1}{5} + \frac{1}{3} = \frac{8}{15} \quad < \quad \frac{19}{34} \leq \frac{1}{2} + \frac{1}{k+1}.$$

In the case $\omega = 4$ we could a priori have up to $3 \times 4 = 12$ interior lattice points. But more than 10 would imply at least three in each of the three lines $\{i\} \times \mathbb{R}$, $i = 1,2,3$. This would make P contain a parallelogram Q with vertical edge of length two and horizontal width two. Such a Q has $\mu(Q) \leq \frac{1}{2}$, since $\frac{1}{2}Q$ contains a fundamental domain of $\mathbb{Z}^2$. Thus, we can assume $k \leq 10$ and we get

$$\mu(P) \leq \frac{1}{4} + \frac{1}{3} = \frac{7}{12} \quad < \quad \frac{1}{2} + \frac{1}{11} \leq \frac{1}{2} + \frac{1}{k+1}.$$

In the final case, $m = 4$, $\omega = 3$, and $k \leq 4$, we get

$$\mu(P) \leq \frac{1}{3} + \frac{1}{3} = \frac{2}{3} \quad < \quad \frac{1}{2} + \frac{1}{5} \leq \frac{1}{2} + \frac{1}{k+1}. \qquad \square$$

Remark 5.58. Lattice polytopes with $m \leq 3$ contain at most nine lattice points in total, since they cannot have two points in the same residue class modulo $(3\mathbb{Z})^2$. In particular, they have $k \leq 6$. On the other hand, the polytopes with $(\omega = 3, m = 4)$ have $k \leq 8$ because they have at most four points in each of the two intermediate lines along the direction where $\omega = 3$ is attained. Thus, the cases not covered by lemma 5.56 and lemma 5.57 have between 3 and 8 interior lattice points. Castryck [Cas12] enumerated all lattice polygons with $k \leq 30$ up to unimodular equivalence, and showed that there are $120+211+403+714+1023+1830$ of them with k equal to 3, 4, 5, 6, 7, and 8. The rest of the section could be replaced by a computer-aided computation of the covering radius of these 4301 polygons.

The missing case ($\omega = 3$, $m = 4$, $k \geq 5$) in lemma 5.57 is dealt with separately, since it needs some ad-hoc arguments.

Lemma 5.59. *Suppose P has width 3 (assume it is contained in $[0,3] \times \mathbb{R}$), its maximum number of collinear points is 4, and it has $k \geq 5$ interior lattice points. Then at least one of the following conditions holds:*

(i) P has four collinear lattice points along one of the intermediate vertical lines $\{1\} \times \mathbb{R}$ or $\{2\} \times \mathbb{R}$ and (at least) three of them are interior to P,

(ii) P contains a parallelogram with one vertical edge of length two and horizontal width two.

In both cases we have

$$\mu(P) < \frac{1}{2} + \frac{1}{k+1}.$$

Proof. We first prove the conclusion. If P contains a parallelogram Q as stated in (ii) then $\mu(P) \leq \mu(Q) = \frac{1}{2}$ and we are done. Suppose, then, that P contains the four collinear points $(1, i)$, $i = 1, 2, 3, 4$ and that the first three are interior. Then the segment $P \cap \{x = 1\}$ has length at least $3 + \frac{1}{3} = \frac{10}{3}$, because its bottom end-point cannot be above $(1, \frac{2}{3})$. Thus

$$\mu(P) \leq \frac{1}{3} + \frac{3}{10} = \frac{19}{30} \quad < \quad \frac{9}{14} = \frac{1}{2} + \frac{1}{7} \leq \frac{1}{2} + \frac{1}{k+1},$$

if $k \in \{5, 6\}$. In the case $k \geq 7$, we can assume the four collinear lattice points $(1, i)$, $i = 1, 2, 3, 4$ are actually interior to P. Therefore, the segment $P \cap \{x = 1\}$ has length at least $3 + \frac{2}{3} = \frac{11}{3}$, because its bottom end-point cannot be above $(1, \frac{2}{3})$ and its top end-point cannot be below $(1, 4 + \frac{1}{3})$. Thus

$$\mu(P) \leq \frac{1}{3} + \frac{3}{11} = \frac{20}{33} \quad < \quad \frac{11}{18} = \frac{1}{2} + \frac{1}{9} \leq \frac{1}{2} + \frac{1}{k+1}.$$

Let us now assume that P is neither in the conditions of (i) or (ii) and let us derive a contradiction. By (the negation of) (i), P has at most three interior lattice points along each of the two vertical lines. Since it has at least five in total, we assume without loss of generality that

$$(1,1), (1,2), (1,3), (2,1), (2,2) \in \operatorname{int}(P).$$

The proof is based on arguing that certain additional points must or cannot be in P. This is illustrated in fig. 5.4 where the points that *must* be in P are drawn as black dots and the ones that *cannot* as crosses. The initial points that we assume in $\operatorname{int}(P)$ are drawn as white dots. The labels of the points indicate the order in which they appear in the proof:

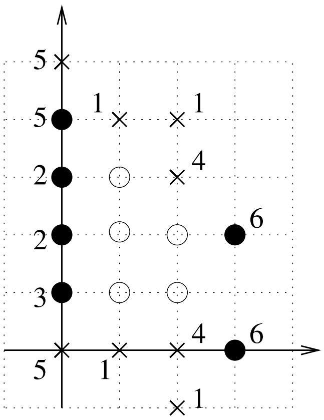


FIGURE 5.4: Illustration of the proof of lemma 5.59.

1) None of the points $(1, 0)$, $(1, 4)$, $(2, -1)$, or $(2, 4)$ can be in P, since their presence would give condition (i).

2) The left end-point of the top (respectively, bottom) edge of P meeting the line $\{x = 1\}$ must then be of the form $(0, a)$ with $a \geq 3$ (respectively, with $a \leq 2$). Hence, $(0, 2)$ and $(0, 3)$ are in P.

3) One of $(0, 1)$ or $(0, 4)$ must be in P, for otherwise the edges going from $(0, 2)$ and $(0, 3)$ to the right must go strictly below and above $(2, 0)$ and $(2, 3)$ respectively, giving four interior points along $\{x = 2\}$. Assume without loss of generality that $(0, 1) \in P$.

4) Since we already have an intersection of length two with $\{x = 0\}$, the intersection with $\{x = 2\}$ must have length strictly smaller than 2, in order for P not to be in the conditions of part (ii). Thus, $(2, 0), (2, 3) \notin P$.

5) Now the top edge of P crossing $\{x = 1\}$ must have its left end-point above $(0, 3)$, because $(2, 3) \notin P$, which implies $(0, 4) \in P$. Since we already have four collinear points in $\{x = 0\}$, neither $(0, 0)$ nor $(0, 5)$ is in P.

6) Now the only possibility for the right end-points of the top and bottom edges of P are $(3, 0)$ and $(3, 2)$ (remember that the white dots in the figure are meant to be in the interior of P).

This gives a contradiction, since P is now as described in part (ii). $\square$

Lemma 5.60. *Suppose P has width at least three and three interior collinear lattice points. Then, P has four collinear lattice points.*

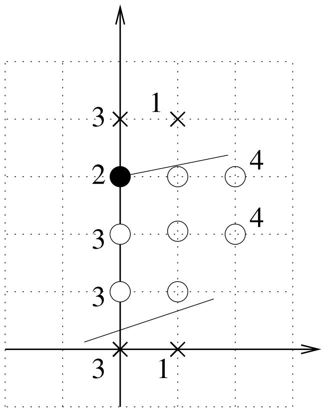


FIGURE 5.5: Illustration of the proof of lemma 5.60.

Proof. Suppose P contains $(1,1)$, $(1,2)$ and $(1,3)$ in its interior and moreover that P does not contain four collinear lattice points. We are going to arrive to a contradiction. Similarly to the proof of lemma 5.59, we illustrate our reasoning in fig. 5.5:

1) $(1,0)$ and $(1,4)$ are exterior to P, and the length of $P \cap \{x=1\}$ is greater than two.

2) Since P does not have a vertex in $\{x=1\}$, one of the intersections $P \cap \{x=0\}$ or $P \cap \{x=2\}$ has at least the same length as $P \cap \{x=1\}$. Suppose it is $P \cap \{x=0\}$. If P does not have a vertex in $\{x=0\}$ then it has at least three lattice points in $\{x=-1\}$ and at least one in $\{x \geq 2\}$. That would make at least ten lattice points in total, which would imply $m \geq 4$ as we observed in Remark 5.58. So, without loss of generality we assume that the top edge crossing $\{x=1\}$ has a vertex at $(0,3)$.

3) Then $(0,0)$ and $(0,4)$ are exterior to P, in order not to have four collinear points, and $(0,2)$ and $(0,1)$ are interior to P, since $P \cap \{x=0\}$ has length larger than two.

4) The edges crossing $\{x=1\}$ must cross $\{x=2\}$ above $(2,3)$ and below $(2,2)$ respectively, so these two points are also in the interior of P.

So, we have identified eight lattice points in P. But none of them can be an end-point of the bottom edge of P crossing $\{x=1\}$. Thus, P has at least ten lattice points, which implies $m \geq 4$. $\square$

Proof of theorem 5.55. After Lemmas 5.56–5.60, the only case left to consider is when P has at least three interior lattice points (since otherwise it has width two), but no three of them collinear. Moreover, we can assume P does not contain four collinear points.

Let Q be the convex hull of all the interior lattice points in P. Q has at most four lattice points, because if there were five then two of them would be in the same residue class modulo $(2\mathbb{Z})^2$, giving three collinear ones. This gives only three possibilities for Q: it is equivalent to either a unimodular triangle, a unit parallelogram, or $S(\mathbf{1}_3)$.

No boundary lattice point p of P can be at lattice distance more than one from Q, because otherwise the triangle with vertices p and two of the lattice points in Q would contain additional lattice points, which would necessarily be in the interior of P. Thus, P is fully contained in one of the three polygons drawn in fig. 5.6. In each case, let R denote the polygon in the figure; we want to show that every subpolygon of R containing all the white dots (the polygon Q) in its interior has covering radius strictly smaller than $\frac{1}{2} + \frac{1}{k+1}$, where $k = 3$ in cases (A) and (B), and $k = 4$ in case (C):

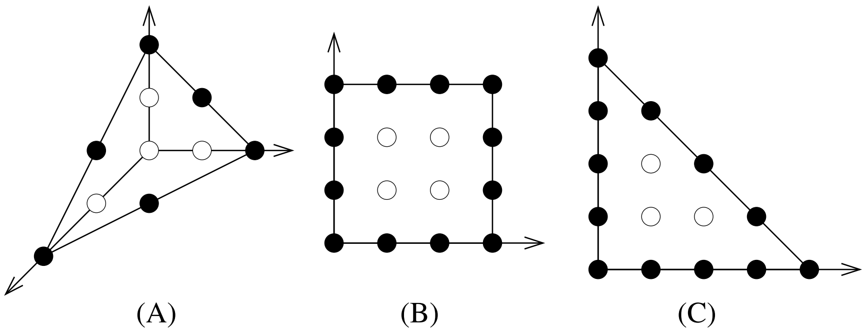


FIGURE 5.6: The three cases in the proof of theorem 5.55.

(A) If $Q = S(\mathbf{1}_3)$, then $R = 2Q = S(2,2,2)$. The only lattice subpolygon of R containing Q in its interior is R itself, whose covering radius is $1/2$ by theorem 5.4 (or by the fact that it coincides with $2S(\mathbf{1}_3)$).

(B) If Q is a unit parallelogram, without loss of generality we assume that $Q = [1,2]^2$ and $R = [0,3]^2$. We distinguish cases:

1) P contains at least one lattice point from the relative interior of each edge of R. The only possibility for R not to contain four collinear points is that P equals $S := \operatorname{conv}(\{(1,0),(3,1),(2,3),(0,2)\})$ (or its mirror reflection). It is easy to calculate that $\mu(S) = \frac{3}{5} < \frac{1}{2} + \frac{1}{5}$, since Q, which is a fundamental domain, is inscribed in the dilation of S of factor $\frac{3}{5}$ centered at $(\frac{3}{2}, \frac{3}{2})$.

2) Along some edge, P does not contain any relative interior point of R. Say P contains neither $(1,0)$ nor $(2,0)$. Then, it must contain the edge from $(0,0)$ to $(3,1)$ (or its mirror reflection, which gives an analogous case). If P contains $(0,1)$ then we have four collinear points. If it does not, then it contains the edge from $(0,0)$ to $(1,3)$. In particular, P contains the triangle with vertices $(0,0)$, $(3,1)$ and $(1,3)$. This triangle is a translate of $S(1,2,2)$, hence its covering radius equals $\frac{5}{8} < \frac{1}{2} + \frac{1}{5}$ by theorem 5.4.

(C) If Q is a unimodular triangle, then without loss of generality we can assume that $Q = \operatorname{conv}(\{(1,1),(1,2),(2,1)\})$, so that P is contained in $R = \operatorname{conv}(\{(0,0),(4,0),(0,4)\})$. There are two possibilities:

1) P contains a vertex of R, say $(0,0)$. It must also contain (at least) one lattice point on the opposite edge $\{x+y=4\}$. But:
   - i) If $(2,2) \in P$ then $\mu(P) \leq \mu([0,2]^2) = \frac{1}{2}$.
   - ii) If $(4,0)$ or $(0,4)$ is in P then P contains five collinear points.
   - iii) If $(3,1)$ is in P then $P \cap \{y=1\}$ has length at least $8/3$. Lemma 5.7 for the projection along this line gives $\mu(P) \leq \frac{1}{3} + \frac{3}{8} = \frac{17}{24} < \frac{3}{4}$. The case $(1,3) \in P$ is symmetric to this one.
2) P does not contain a vertex of R. Then, in order for $(1,1)$ to be in the interior of P, P must contain (at least) one of the points $(1,0)$ or $(0,1)$. The same reasoning for the other two interior points gives that P contains one of $(3,0)$ and $(3,1)$, and one of $(0,3)$ and $(1,3)$. Out of the eight combinations of one point from each pair the only ones that do not produce four collinear points in P are the triangle $\operatorname{conv}(\{(0,1),(3,0),(1,3)\})$ and its reflection along the diagonal $\{x=y\}$. In lemma 5.66 we compute the covering radius of this triangle to be $5/7$, which is smaller than $3/4$. $\square$

5.6 The 26 minimal 1-point lattice 3-polytopes

The 26 minimal non-hollow lattice 3-polytopes with a single interior lattice point were classified by Kasprzyk [Kas10]. We describe them in Tables 5.1 and 5.2, in the same order as they appear in Kasprzyk's Tables 2 and 4. Table 5.1 contains the 16 that are tetrahedra and Table 5.2 the 10 that are not. For each of them we list its vertices as the columns of a matrix and include a description that is explained in section 5.6.1. For the tetrahedral examples in Table 5.1 we also include the *volume vector* (a,b,c,d) consisting of the normalized volumes of the pyramids from the origin over the facets. The given descriptions allow us to bound the covering radius away from $3/2$ for each of the 17 polytopes that are not equivalent to one in lemma 5.24, thus obtaining a first proof of Theorem 5.25. A second proof is by explicitly computing their covering radius via solving a suitable mixed-integer linear program as explained in section 5.6.2. The covering radius obtained for each is also shown in the tables, and is highlighted in bold-face for the nine of Lemma 5.24, which are the ones with $\mu = 3/2$.

$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,1,1)$	$\begin{pmatrix} -2 & 2 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$ $(2,2,2,2)$	$\begin{pmatrix} -5 & 5 & 0 & 0 \\ -3 & 2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{pmatrix}$ $(5,5,5,5)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,1,2)$
$S(\mathbf{1}_4) = T(1,1,1,1)$	$(I \oplus I')' \oplus I$	$T(5,5,5,5)$	$T(1,1,1,2)$
$\boldsymbol{\mu = 3/2}$	$\boldsymbol{\mu = 3/2}$	$\mu = 9/10$	$\mu = 7/5$
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,1,3)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,2,2)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,2,3)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,2,4)$
$S(\mathbf{1}_3) \oplus I'$	$S(\mathbf{1}_3)' \oplus I$	$T(1,1,2,3)$	$(I \oplus I')^\circ \oplus I'$
$\boldsymbol{\mu = 3/2}$	$\boldsymbol{\mu = 3/2}$	$\mu = 9/7$	$\boldsymbol{\mu = 3/2}$
$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,3,4)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,3,5)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -6 & 0 & 0 & 1 \end{pmatrix}$ $(1,1,4,6)$	$\begin{pmatrix} -2 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{pmatrix}$ $(1,2,3,5)$
$\mathrm{Pyr}_3(S(\mathbf{1}_3))$	$I' \oplus M_2(1)$	$I' \oplus M_2(0)$	$T(1,2,3,5)$
$\mu = 11/9$	$\mu = 13/10$	$\mu = 4/3$	$\mu = 12/11$
$\begin{pmatrix} -3 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{pmatrix}$ $(1,3,4,5)$	$\begin{pmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 2 & 0 \\ -4 & 0 & 1 & 1 \end{pmatrix}$ $(2,2,3,5)$	$\begin{pmatrix} -3 & 2 & 0 & 0 \\ -4 & 1 & 1 & 0 \\ -5 & 1 & 0 & 1 \end{pmatrix}$ $(2,3,5,7)$	$\begin{pmatrix} -4 & 3 & 0 & 0 \\ -3 & 1 & 1 & 0 \\ -5 & 2 & 0 & 1 \end{pmatrix}$ $(3,4,5,7)$
$T(1,3,4,5)$	$\mathrm{Pyr}_4(S(\mathbf{1}_3))$	$T(2,3,5,7)$	$T(3,4,5,7)$
$\mu = 14/13$	$\mu = 7/6$	$\mu = 1$	$\mu = 18/19$

TABLE 5.1: The covering radius of the minimal non-hollow tetrahedra with exactly one interior lattice point.

5.6.1 First proof of Theorem 5.25: theoretical bounds

The 26 polytopes of Table 5.1 and Table 5.2 are as follows. In some cases the description allows us to compute the covering radius exactly, and in other cases to show that it is strictly smaller than 3/2:

- The nine from Lemma 5.24, of covering radius 3/2.
- There are three more in Table 5.1 that decompose as direct sums, namely (translations of) $I' \oplus M_2(1)$, $I' \oplus M_2(0)$, and $I \oplus Q_4$. Here $M_k(\alpha) = M_k(2,1;\alpha,0)$ is the triangle defined in Equation (5.6) and

$$Q_4 = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}.$$

$\begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$ $S(\mathbf{1}_3) \oplus I$	$\begin{pmatrix} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$ $I \oplus Q_4$
$\boldsymbol{\mu = 3/2}$	$\mu = 4/3$
$\begin{pmatrix} 1 & 0 & -1 & 1 & -1 \\ 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 3 & -3 \end{pmatrix}$ $\mathrm{Bipyr}_3(S(\mathbf{1}_3) \oplus I)$	$\begin{pmatrix} 1 & 0 & 0 & -2 & -2 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$ $I \oplus I \oplus I'$
$\mu = 17/18$	$\boldsymbol{\mu = 3/2}$
$\begin{pmatrix} 1 & 0 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$ $(I \oplus I')^\circ \oplus I$	$\begin{pmatrix} 1 & 0 & -2 & 1 & -3 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix}$ $\mathrm{Bipyr}_2(I \oplus I \oplus I')$
$\boldsymbol{\mu = 3/2}$	$\mu = 7/8$
$\begin{pmatrix} 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 2 & -2 \end{pmatrix}$ $\mathrm{Bipyr}_2((I \oplus I')^\circ \oplus I)$	$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$ $I \oplus I \oplus I$
$\mu = 1$	$\boldsymbol{\mu = 3/2}$
$\begin{pmatrix} 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$ $\mathrm{Pyr}_3([0,1]^2)$	$\begin{pmatrix} 1 & 0 & -1 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & -2 \end{pmatrix}$ $\mathrm{Bipyr}_2(I \oplus I \oplus I)$
$\mu = 4/3$	$\mu = 3/4$

TABLE 5.2: The covering radius of the minimal non-hollow non-tetrahedra with exactly one interior lattice point.

For the first two the decomposition is enough to compute their covering radius, via Remark 5.22:

$$\mu(I' \oplus M_2(1)) = \mu(I') + \mu(M_2(1)) = \frac{1}{2} + \frac{4}{5} = \frac{13}{10},$$
$$\mu(I' \oplus M_2(0)) = \mu(I') + \mu(M_2(0)) = \frac{1}{2} + \frac{5}{6} = \frac{4}{3}.$$

For the third one we use that Q_4 strictly contains $S(\mathbf{1}_3)$ and that $S(\mathbf{1}_3)$ is tight (lemma 5.13) to obtain:

$$\mu(I \oplus Q_4) = \mu(I) + \mu(Q_4) < \mu(I) + \mu(S(\mathbf{1}_3)) = \frac{3}{2}.$$

- There are four in Table 5.2 that are the same as the four of covering radius 3/2 except considered with respect to a finer lattice. Since they are also (skew) bipyramids, they are marked as $\mathrm{Bipyr}_i(-)$, where i is the index of the superlattice. Let P be one of them. In each case the intersection of P with the plane $z = 0$ is a non-hollow lattice polygon (in fact, it is one of the three of covering radius equal to one). Applying Lemma 5.7 to the projection π onto the z-coordinate we get:

$$\mu(P) \le \mu(P \cap \{z = 0\}) + \mu(\pi(P)) \le 1 + \frac{1}{4} = \frac{5}{4},$$

since $\pi(P)$ has length at least four in all cases.

- The three marked as $\mathrm{Pyr}_i(Q)$ are (skew) pyramids of height i over a polygon Q. More precisely: $\mathrm{Pyr}_3(S(\mathbf{1}_3))$ and $\mathrm{Pyr}_4(S(\mathbf{1}_3))$ are both tetrahedra with a facet isomorphic to $S(\mathbf{1}_3)$ in the plane $x - 2y + z = 2$ and the opposite vertex at distance three and four, respectively, from that facet. $\mathrm{Pyr}_3([0,1]^2)$ is a pyramid with base a unimodular parallelogram in the plane $x + y + z = 1$ and the apex at distance three. In the three cases, $\mu(Q) = 1$ so that Lemma 5.7 applied to the projection that has the base of the pyramid as fiber gives

$$\mu(\mathrm{Pyr}_i(Q)) \le \mu(Q) + \frac{1}{i} = 1 + \frac{1}{i} \le \frac{4}{3}.$$

- The eight described as $T(a, b, c, d)$ in Table 5.1 are the *terminal tetrahedra*, that is, the lattice tetrahedra with only five lattice points, their four vertices plus the origin. These have previously been classified by Kasprzyk [Kas06] and Reznick [Rez06, Thm. 7], and appear also as the last eight rows in [BS16, Table 1]. The parameters a, b, c, and d are the normalized volumes of the pyramids from the origin over the facets. One of them coincides with $S(\mathbf{1}_4) = T(1, 1, 1, 1)$. The rest of this paragraph describes a way to bound their covering radius when $a+b+c+d$ is relatively big. In the next paragraph we compute it exactly when it is small.

Let A, B, C, D be the vertices of $T(a, b, c, d)$ labeled in the natural way (so that a is the determinant of BCD, b is the determinant of ACD, etc.). Since $T(a, b, c, d)$ is terminal, the triangle formed by the origin and any two vertices is unimodular, and so there is no loss of generality in taking C and D to be the points $(1, 0, 0)$ and $(0, 1, 0)$, respectively. Once this is done, A and B must have z coordinate equal to b and $-a$ (or vice versa), in order for the determinants of BCD and ACD to be a and b, respectively. Then, in order for the determinants of ABD and ACD to be c and d, the segment AB must intersect the plane $z = 0$ at $(-c/(a + b), -d/(a + b))$. That is, $T(a, b, c, d) \cap \{z = 0\}$ is the triangle $\Delta_{(c/(a+b),d/(a+b))}$ seen in Lemma 5.30. Then, Lemma 5.7 applied to the z coordinate gives:

$$\mu(T(a, b, c, d)) \le \mu\left(\Delta_{\left(\frac{c}{a+b}, \frac{d}{a+b}\right)}\right) + \frac{1}{a + b}.$$

Now, Proposition 5.31 says that whenever $c, d \ge a + b$ the first summand is ≤ 1. Thus:

$$\mu(T(1, 2, 3, 5)) \le \frac{4}{3}, \quad \mu(T(1, 3, 4, 5)) \le \frac{5}{4}, \quad \mu(T(2, 3, 5, 7)) \le \frac{6}{5}.$$

For the simplex $T(5,5,5,5)$ we can use that the triangle $\Delta_{(1/2,1/2)}$ coincides with $S(1,1,1/2)$, so Theorem 5.4 gives us its exact covering radius $5/4$. Thus:

$$\mu(T(5,5,5,5)) \leq \frac{5}{4} + \frac{1}{10} < \frac{3}{2}.$$

The same applies to $T(3,4,5,7)$ since $\Delta_{(5/7,1)}$ contains the point $(1/2,1/2)$ and hence the triangle $\Delta_{(1/2,1/2)}$:

$$\mu(T(3,4,5,7)) \leq \mu\left(\Delta_{\left(\frac{5}{7},1\right)}\right) + \frac{1}{7} \leq \mu\left(\Delta_{\left(\frac{1}{2},\frac{1}{2}\right)}\right) + \frac{1}{7} = \frac{5}{4} + \frac{1}{7} < \frac{3}{2}.$$

This shows the desired inequality for all the tetrahedra $T(a,b,c,d)$ except the two smallest ones, $T(1,1,1,2)$ and $T(1,1,2,3)$. For these two, corollary 5.63 below shows $\mu(T(1,1,1,2)) = \frac{7}{5}$ and $\mu(T(1,1,2,3)) = \frac{9}{7}$.

Computing the covering radius of a lattice simplex

Let $T = \operatorname{conv}(\{v_0, v_1, \dots, v_d\})$ be a lattice simplex of normalized volume $V = \operatorname{Vol}_\Lambda(T)$ with respect to a certain lattice Λ. The affine map defined by $v_0 \mapsto \mathbf{0}$ and $v_i \mapsto V \cdot e_i$, $i = 1, \dots, d$, sends T to the dilated standard simplex $V \cdot \operatorname{conv}(\{\mathbf{0}, e_1, \dots, e_d\})$ and Λ to an intermediate lattice between $\mathbb{Z}^d$ and $V\mathbb{Z}^d$, which we still denote by Λ. Observe that $\Lambda_T := \Lambda/V\mathbb{Z}^d$ is a subgroup of $\mathbb{Z}^d/V\mathbb{Z}^d = (\mathbb{Z}_V)^d$ of order V and that

$$\mathbb{Z}^d/\Lambda = (\mathbb{Z}_V)^d/(\Lambda/V\mathbb{Z}^d)$$

is, hence, a finite abelian group of order V^{d-1}. The *Cayley graph* G associated with the quotient group $\mathbb{Z}^d/\Lambda$ is the directed graph with vertex set $\mathbb{Z}^d/\Lambda$ and edges $(x+\Lambda, x+e_i+\Lambda)$, for $x \in \mathbb{Z}^d$ and $1 \leq i \leq d$. The following is a particular case of [MS13, Lem. 3 & 4] (cf. also [GMS17, Thm. 4.11]):

Lemma 5.61. *In the above conditions, let G be the Cayley graph of the quotient group $\mathbb{Z}^d/\Lambda$ and let $\delta = \delta(G)$ be the (directed) diameter of G. Equivalently, δ is the maximum distance from $\mathbf{0}$ to any other node of G. Then,*

$$\mu(T) = \frac{\delta + d}{V}.$$

Proof. The covering radius of the standard d-simplex $\operatorname{conv}(\{\mathbf{0}, e_1, \dots, e_d\})$ with respect to a sublattice Λ of $\mathbb{Z}^d$ equals $\delta + d$. (This is the case $v = (1, \dots, 1)$ of [GMS17, Thm. 4.11]). We divide this by V since we are looking at the Vth dilation of the standard simplex. $\square$

In all cases of interest for us the group Λ_T is cyclic; that is, there is a lattice point $p \in \Lambda$ such that Λ equals the lattice generated by the vertices of T together with p. We say that T is a *cyclic simplex* when this happens. In these conditions, let $\frac{1}{V}(a_0, \ldots, a_d)$ be the barycentric coordinates of p, so that $a_0, \ldots, a_d$ are integers which add up to V. Then,

$$\mathbb{Z}^d/\Lambda \cong (\mathbb{Z}_V)^d/\langle(a_1, \ldots, a_d)\rangle.$$

In particular:

Corollary 5.62. *Let T be a cyclic simplex of normalized volume V with generator $\frac{1}{V}(a_0, \ldots, a_d)$ and let $G(V; a_1, \ldots, a_d)$ be the Cayley graph of $\mathbb{Z}^d/\Lambda \cong (\mathbb{Z}_V)^d/\langle(a_1, \ldots, a_d)\rangle$ with respect to the standard generators. Then*

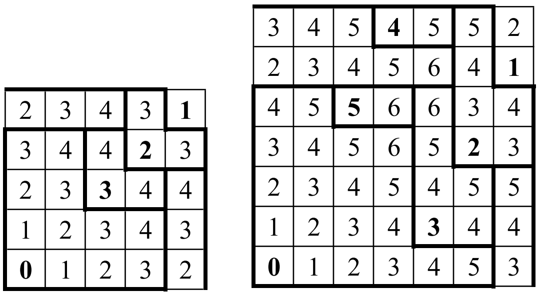
$$\mu(T) = \frac{\delta + d}{V},$$

where δ is the diameter of $G(V; a_1, \ldots, a_d)$.

Let us now look at an arbitrary lattice tetrahedron with an interior lattice point p and let $a, b, c, d \in \mathbb{N}$ be the normalized volumes of the pyramids with apex at p over the facets of T. When $\gcd(a, b, c, d) = 1$ we have that Λ_T is cyclic of order $V = a+b+c+d$ and with generator $\frac{1}{V}(a, b, c)$. Thus, corollary 5.62 gives an easy way to compute the covering radius of T. The case $a = 1$ is particularly simple, since then $G(V; a, b, c)$ coincides with the Cayley graph of $(\mathbb{Z}_V)^2$ with respect to the generators $(1, 0)$, $(0, 1)$ and $(-b, -c)$. That is, $G(V; a, b, c)$ has $(\mathbb{Z}_V)^2$ as vertex set and from each vertex (i, j) we have the following three arcs:

$$(i, j) \to (i, j+1), \quad (i, j) \to (i+1, j), \quad (i, j) \to (i-b, j-c).$$

With this in mind, fig. 5.7 shows the computation of $\delta(G(5; 1, 1, 1))$ and $\delta(G(7; 1, 1, 2))$ in the following way: a grid with V^2 cells represents the nodes of $G(V; 1, b, c)$ with the origin at the south-west corner. In each cell we have written its distance from the origin. The grid has to be regarded as a torus, so that every cell has an east, west, north and south neighbor. Moving to the north or east increases the distance from the origin by at most one unit, and when it does not increase the corresponding wall is highlighted in bold to signify that the corresponding arc of $G(V; a, b, c)$ is not used in any shortest path from the origin. Observe that, by commutativity, most of the arcs of the form $(i, j) \to (i-b, j-c)$ are irrelevant for the diameter: only those arriving to cells with bold south and west are needed in order to verify that the diagrams are correct. Such cells have their distances also in bold. For example, the cell $(V-b, V-c)$ is the one labeled **1** and with south and west edges in bold.



$G(5;1,1,1)$ $G(7;1,1,2)$

FIGURE 5.7: Graphical computation of $\delta(G(5;1,1,1)) = 4$ and $\delta(G(7;1,1,2)) = 6$, which imply $\mu(T(1,1,1,2)) = \frac{7}{5}$ and $\mu(T(1,1,2,3)) = \frac{9}{7}$.

Corollary 5.63. $\mu(T(1,1,1,2)) = \frac{7}{5}$ *and* $\mu(T(1,1,2,3)) = \frac{9}{7}$.

Remark 5.64. The condition $a = 1$ used in the computations can be weakened to $\gcd(a,V) = 1$, since in this case $G(V;a,b,c) = G(V;1,ba^{-1},ca^{-1})$, where a^{-1} denotes the inverse of a modulo V. Although we only show the computations for $T(1,1,1,2)$ and $T(1,1,2,3)$, fourteen of the sixteen simplices in Table 5.1 satisfy this; all but $(I \oplus I')' \oplus I$ and $T(5,5,5,5)$.

This method can also be applied to the minimal tetrahedra $M_k(1,1)$ of Equation (5.7):

Lemma 5.65. *For every $k \in \mathbb{N}$ we have*

$$\mu(M_k(1,1)) = 1 + \frac{1}{2k}.$$

Proof. $M_k(1,1)$ has normalized volume $4k$ and the point $p = (0,0,1)$ has barycentric coordinates $\frac{1}{4k}(1,1,2k-1,2k-1)$. Thus, $\mu(M_k(1,1))$ equals $(\delta+3)/4k$ where δ is the diameter of the graph $G(4k;1,2k-1,2k-1)$. This diameter equals $4k-1$, as derived from fig. 5.8. □

Also, we can easily compute the covering radius of the triangle needed at the end of the proof of theorem 5.55.

Lemma 5.66.

$$\mu(\mathrm{conv}(\{(0,1),(3,0),(1,3)\})) = \frac{5}{7}.$$

Proof. The triangle has normalized area 7 and the point $(1,1)$ has barycentric coordinates $\frac{1}{7}(1,2,4)$. Thus,

$$\mu(\mathrm{conv}(\{(0,1),(3,0),(1,3)\})) = \frac{\delta(G(7;1,2)) + 2}{7} = \frac{3+2}{7}.$$

The (easy) computation $\delta(G(7;1,2)) = 3$ is left to the reader. □

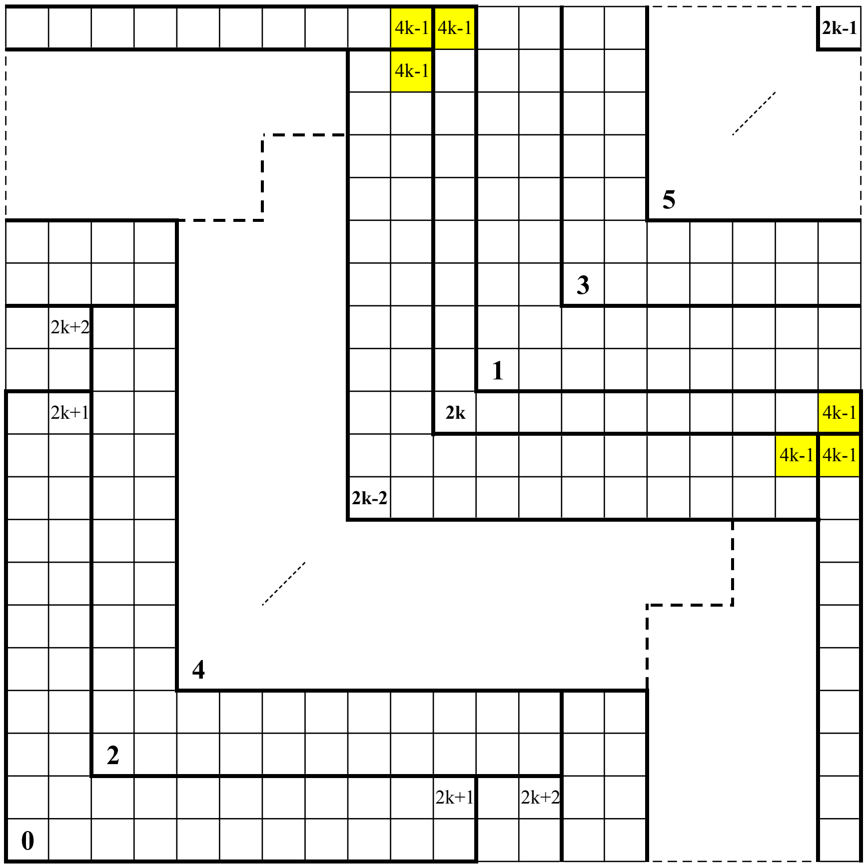


FIGURE 5.8: Computation of $\mu(M_k(1,1)) = 1 + 1/2k$ via $\delta(G(4k; 1, 2k-1, 2k-1)) = 4k - 1$. Only the distance to some cells is shown. The ones achieving the diameter are highlighted in yellow.

5.6.2 Second proof of Theorem 5.25: computer calculations

Here we describe an algorithmic computation of covering radii based on a formulation of $\mu(P)$ as the optimal value of a mixed-integer program. This formulation is already implicit in Kannan's paper [Kan92, Sect. 5].

Let $P = \{x \in \mathbb{R}^d : a_i^\intercal x \le b_i, 1 \le i \le m\}$ be a polytope with outer facet normals $a_i \in \mathbb{R}^d$ and right hand sides $b_i \in \mathbb{R}$. Without loss of generality, we assume that $b_i > 0$, that is, P contains the origin in its interior. Since P is bounded, there exists a finite subset $N_P \subseteq \mathbb{Z}^d$ such that $\mu(P)P + N_P$ contains the unit cube $[0,1]^d$.

Proposition 5.67. *The covering radius $\mu(P)$ is equal to the optimal value of the following linear mixed-integer program:*

$$\begin{array}{rrcll} \text{maximize} & \mu & & & \\ \text{s.t.} & a_i^\intercal x & \geq & \mu b_i + a_i^\intercal \ell - M(1-y_i^\ell), & \forall i = 1,\dots,m,\ \forall \ell \in N_P \\ & \sum_{i=1}^m y_i^\ell & \geq & 1, & \forall \ell \in N_P \\ & y_i^\ell & \in & \{0,1\}, & \forall i = 1,\dots,m,\ \forall \ell \in N_P \\ & x & \in & [0,1]^d. & \end{array}$$

The constant $M > 0$ is chosen large enough such that every non-active inequality involving M is redundant.

Proof. By the periodicity of the arrangement $\mu P + \mathbb{Z}^d$, we get that

$$\mu(P) = \min\{\mu \geq 0 : [0,1]^d \subseteq \mu P + N_P\}.$$

Hence, the covering radius equals the minimal $\mu \geq 0$ such that for all $x \in [0,1]^d$ there exists an $\ell \in N_P$ such that $x \in \mu P + \ell$. This gives a mixed-integer program with infinitely many constraints. In order to turn it into a finite program, we may also interpret the covering radius as the supremum among $\mu \geq 0$ such that there exists an $x \in [0,1]^d$ such that $x \notin \mu P + N_P$.

Modeling this non-containment condition can be done as follows: For a fixed $\ell \in N_P$, we have $x \notin \mu P + \ell$ if and only if there exists a defining inequality of P that is violated, that is, there exists an $i \in \{1,\dots,m\}$ such that $a_i^\intercal x > \mu b_i + a_i^\intercal \ell$. Introducing the binary variable y_i^ℓ for each $1 \leq i \leq m$ and each $\ell \in N_P$, and using a large enough constant $M > 0$, this is modeled by the first two lines in the program, as the condition $\sum_{i=1}^m y_i^\ell \geq 1$ ensures that at least one inequality is violated for ℓ.

We can replace the supremum by a maximum and the strict inequality $a_i^\intercal x > \mu b_i + a_i^\intercal \ell$ by a non-strict one, since P is compact and the covering radius is in fact an attained maximum. $\square$

In order to make this formulation effective, we need to find a suitable finite subset $N_P \subseteq \mathbb{Z}^d$: A point $x \in [0,1]^d$ is contained in $z + \mu(P)P$, for some $z \in \mathbb{Z}^d$, if and only if $z \in [0,1]^d - \mu(P)P$. Hence, for any theoretically proven upper bound $\mu(P) \leq \mu$, we can solve the mixed-integer program in Proposition 5.67 with respect to $N_P = ([0,1]^d - \mu P) \cap \mathbb{Z}^d$ and obtain the covering radius of P.

Based on these considerations, we employed the SCIP solver in exact solving mode [CKSW13] and computed the covering radius of the 26 minimal lattice 3-polytopes with the results given in Tables 5.1 and 5.2.

Declaration of Authorship

I, Giulia Codenotti, declare that this thesis titled, “Covering properties of lattice polytopes” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.
- This dissertation has not yet been submitted in the same or similar form in any previous doctoral procedure.

Signed: ______________________________

Date: ______________________________

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