Dissertation

SLOW CONVERGENCE OF GRAPHS UNDER MEAN CURVATURE FLOW

zur Erlangung des Grades eines Doktors der Naturwissenschaften

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میری والدہ زاہدہ پروین کے نام

In loving memory of my mother Zahida Perveen

Tag der mündlichen Qualifikation: **27. Januar 2010**
1. Gutachter: **Prof. Dr. Klaus Ecker**
2. Gutachterin: **Dr. Maria Athanassenas**

Erklärung

Ich bestätige hiermit, dass ich diese Arbeit selbständig verfasst und keine anderen als die angegebenen Hilfsmittel und Quellen verwendet habe.

Kashif Rasul

Abstract

In this thesis we study the mean curvature flow of entire graphs in Euclidean space. From the work of Ecker and Huisken, we know that given some initial growth condition at infinity, such graphs become self-similar under the evolution and the convergence is exponentially fast in time.

In this work, we propose an alternative condition at infinity, motivated by looking at the heat equation, and show that under mean curvature flow such a growth condition is preserved for the height and gradient of the graph. For the curvature we propose an analogous result to that of Ecker and Huisken, by proving a spatial decay estimate with slightly stronger condition.

Our main result then says that under mean curvature flow and our condition, the graph also becomes self similar, but slower than in the exponential case.

Zusammenfassung

In dieser Doktorarbeit wird der mittlere Krümmungsfluss von ganzen Graphen im euklidischen Raum betrachtet. Ecker und Huisken zeigen, dass unter gewissen Wachstumsbedingungen im Unendlichen, solche Graphen unter dem Fluss selbst ähnlich werden, wobei die Konvergenz in der Zeit exponentiell schnell ist.

Durch die Wärmeleitungsgleichung motiviert, schlagen wir in dieser Arbeit eine alternative Bedingung im Unendlichen vor und zeigen, dass solch eine Wachstumsbedingung für die Höhe und den Gradienten unter dem mittleren Krümmungsfluss erhalten bleibt.

Unser Hauptresultat besagt, dass unter der alternativen Bedingung, der Graph einer Lösung des mittleren Krümmungsflusses ebenfalls selbstähnlich wird, allerdings mit einer in der Zeit langsameren Konvergenzrate.

Contents

Preface

Mean curvature flow simply stated is the evolution of a hypersurface in its normal direction, with speed equal to the mean curvature at each point. So for some initial surface $F_0\colon M^n \to \mathbb{R}^{n+1}$ we consider

$$\frac{\partial F}{\partial t}(p,t) = \vec{H}(p,t) = -\nu \sum_{i=1}^{n} \kappa_i,$$

where ν is the outward pointing unit normal at each point p of the surface and κ_i are the principle curvatures at this point. This evolution has been studied due to its connection with geometry and physics of interfaces [15], in particular the motion of grain boundaries in an annealing pure metal [2]. This process is the gradient flow of the area functional

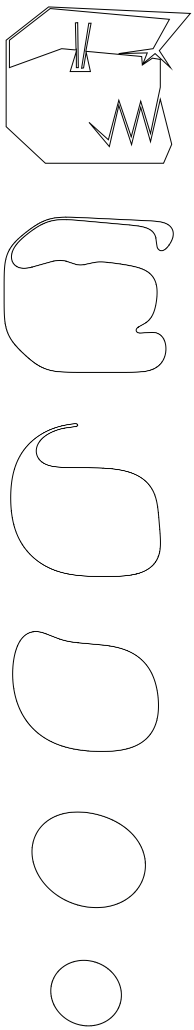
$$\frac{\partial}{\partial t}|M_t^n| = -\int_{M_t^n} |\vec{H}|^2 \,\mathrm{d}\mu_t,$$

and is a quasi-linear (weakly) parabolic partial differential equation evolving the local embedding map of the hypersurface. When stated in the curvature setting:

$$\partial_t A_j^i = \Delta_{M_t} A_j^i + A_j^i |A|^2,$$

where A_{ij} is the second fundamental form, it is described as a reaction-diffusion system, with the reaction part (from a cubic in the curvature term) causing the formation of *singularities*, and the diffusion from a Laplace-Beltrami operator causing the singularities to be *self-similar*.

The investigation of mean curvature flow started around the late '70s by the work of Brakke [2] where he studied such flows on a more general class of surfaces, called *varifolds*, in the geometric measure theory setting. His results when restricted to smooth hypersurfaces which encounter singularities for the first time, and satisfy certain additional assumptions, state that these hypersurfaces at the singular time are still smooth except for a lower dimen-



A closed embedded curve becomes convex and eventually disappears. Image from [3].

sional set. In the class of hypersurfaces with positive mean curvature[1] the beautiful result of Brian White gives the maximum size of the singular set as one less than the dimension of the hypersurface, and this is *optimal* in view of some special solutions.

These non-classical methods, those of geometric measure theory as well as viscosity or level-set methods where developed to deal with the solution of mean curvature flow after the formation of singularities at some time $t = T$, when the maximal curvature

$$A(t) := \max_{M_t^n} \sqrt{\kappa_1^2 + \cdots + \kappa_n^2} \to \infty \text{ as } t \to T,$$

and thus the classical differential geometric and partial differential equation methods fail. In this work we will however not be concerned with the solutions after the first singularity has occurred and thus remain in the classical realm, even though the recent work of Huisken and Sinestrari uses the classical methods with “surgeries” to extend the flow beyond the singularity. Thus we introduce the most simple case, that of curve shortening flow.

For the case of curves, Gage and Hamilton [8] proved that convex curves evolving by mean curvature remain convex and shrink to a circular point. Shortly afterwards, Grayson in his paper [9] completed the result by proving that a nonconvex curve stays embedded and becomes convex in finite time and that no singularity occurs during this process as shown in the snapshots in the margin. Huisken later used a result which classifies all types of singularities of mean curvature flow in his paper [12] to give a more intuitive proof of Grayson’s result [13]. Instead of controlling the shape of the curve at each time, Huisken used a *monotonicity formula* for some isoperimetric ratio to show that the curve becomes a round circle at the singularity, and in fact this is the only embedded limiting case of a singularity.

Around the same time as Gage and Hamilton’s result for curves, Huisken in his now classical paper [11] proved the corresponding result for *surfaces*, namely that compact convex initial surfaces contract smoothly in finite time and become spherical in the process. Oddly enough his proof does not work for curves.[2] The approach of the proof was inspired by Hamilton’s results [10], since the evolution of certain curvature quantities turned out to be similar to those Hamilton had when evolving the metric of a compact three-dimensional manifold with positive Ricci curvature in the direction of the Ricci curvature, i.e. via Ricci flow:

$$\frac{\partial}{\partial t} g_{ij} = -2R_{ij},$$

[1]Positive mean curvature holds by the maximum principle for the duration of the evolution if it does for the initial hypersurface.

[2]Huisken’s proof shows that the asymptotic shape of the solution to mean curvature flow is totally umbilic and the only such hypersurfaces are spheres.

to obtain a metric of constant curvature in the limit.

This connection of mean curvature flow with Hamilton's Ricci flow has a deeper reason. Like the mean curvature flow, the Ricci flow is also a reaction-diffusion system of partial differential equations, albeit in an intrinsic setting, and the reaction terms are quadratic in the Riemann tensor. Thus many properties though different are analogous for Ricci flow and mean curvature flow.

On the other hand we have the simple heat equation:

$$u_t(x,t) = \Delta u(x,t) \text{ with } u(x,0) = u_0(x),$$

and its properties, namely the rescaling property that if $u(x,t)$ is a solution then

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t), \quad \lambda > 0 \quad \text{is also a solution,}$$

which gives us optimal estimates on the regularity of solutions; smoothing property:

$$|D^m u|(x,t) \leq \frac{C_m}{t^{m/2}} \sup|u|,$$

where if we only know the supremum of the solution is bounded, then the derivatives of any order of the solution are bounded; and finally we have Harnack type positivity properties:

$$\partial_t \log u \geq |D \log u|^2 - \frac{n}{2t},$$

saying that the solution cannot fall off too quickly. All these properties carry over in some form to the mean curvature flow setting, giving us a rich source of results.

It turns out that it is crucial to understand the properties of the self-similar solutions which relates the properties of these flows to the geometry of the space. The idea here is to find monotone quantities, non-increasing in time, in these flows, by simply using integration by parts, such that these quantities are constant precisely on the self-similar solutions. In the case of mean curvature flow Huisken's monotonicity formula [12] and its generalisation by Ecker and Huisken [6] as well as a local version by Ecker [4], yields important information about the nature of singularities and other properties of the flow. In the Ricci flow case, the famous Perelman's entropy are such monotone quantities, among others, that Perelman uses for his proof of the Poincaré conjecture.

As we alluded to earlier, the surface forms a singularity at a point when the maximal curvature tends to infinity as we approach some maximal time $t = T$ and we can classify this singularity according to the rate that the curvature blows up. The natural growth rate suggested by the scaling symmetry

$(x,t) \mapsto (\lambda x, \lambda^2 t)$ is

$$\sup_{M^n \times [0,T)} |A(t)| \leq C(T-t)^{-1/2}.$$

If this rate holds for some constant $C < \infty$, then the surface is said to have a "Type 1" singularity, and if not we call the singularity a "Type 2". The monotonicity formula is useful in studying Type 1 singularities of mean curvature flow. If we rescale our surface via the natural scaling we can show that as the surface encounters a Type 1 singularity, it becomes self-similar. Angenent and Velázquez in their paper [1] construct solutions of mean curvature flow that have Type 2 singularities by looking at rotationally symmetric surfaces with non-generic neck pinches. The idea here is to use topological methods in the analysis of singularities and obtain a complete asymptotic understanding of the various kind of blow-ups via the method of matched asymptotic expansions.

For the case of entire graphs, Ecker and Huisken in their paper [6] show that such graphs become self-similar provided they were initially somewhat well behaved at infinity. They show that indeed this behaviour at infinity is necessary for asymptotic convergence, in view of a counter example which does not converge asymptotically. Further they show using this condition that the convergence is exponentially fast in time to an expanding self-similar solution. Stavrou in his paper [16] proves asymptotic convergence using a much weaker condition to that in [6]. In their later paper Ecker and Huisken [7] used the local properties of mean curvature flow to obtain interior estimates, to deduce the fact that if the initial graph was only Lipschitz continuous, then it would have a smooth solution for all time under mean curvature flow, without the need for any assumption on the growth and curvature of the graph at infinity. This is surprising since such a result does not hold for the heat equation, where one needs to give a condition at infinity for existence.

This work also deals with entire graphs as in the Ecker and Huisken setting [6]. We being with Chapter 1, where we introduce the notations and basic differential geometry to prove Huisken's monotonicity formula, which we use to prove a Weak Maximum principle we will need in the subsequent chapters.

We then motivate this work in Chapter 2, by looking at the Ecker and Huisken condition at infinity:

$$\langle x, \nu \rangle^2 \leq c_3 (1 + |x|^2)^{1-\delta}, \qquad c_3 < \infty, \qquad \delta > 0,$$

where x is the position vector, ν the outward unit normal and c_3 some constant, for the simple case of the heat equation. Since we have an explicit solution for the heat equation, we show how for the case of cones as initial data, this condition implies a convergence to an expanding self-similar solution in exponential time. We then propose an alternative condition to that of

Ecker and Huisken, namely:

$$\langle x,\nu\rangle^2 \leq c\frac{1+|x|^2-u^2}{\log^\delta(e+|x|^2)}, \qquad c<\infty, \qquad \delta>0,$$

where u is the height above some hyperplane, and show that in this case, we have *polynomial* in time convergence to the self-similar solution.

Our main result thus states that under mean curvature flow of graphs with such a *logarithmic* condition at infinity, the rescaled solutions $\tilde{M}_s$ converge to a self-similar graph in polynomial time via the following estimate for some $0<\gamma<2$:

$$\sup_{\tilde{M}_s}\frac{(\tilde{H}+\langle\tilde{x},\tilde{\nu}\rangle)^2\tilde{v}^2}{\tilde{\eta}_2^p\log^{-\epsilon}\tilde{\eta}_1} \leq (1+s)^{-\gamma}\sup_{M_0}\frac{(H+\langle x,\nu\rangle)^2v^2}{\eta_2^p\log^{-\epsilon}\eta_1}$$

where v is the gradient function, $0<\epsilon<\delta$, $0<p<1$, and for some choice of test functions η_1 and η_2. Note that this logarithmic condition is much *weaker* than the Ecker and Huisken condition.

To begin with we show in Chapter 3, that the height of the graph if initially in a logarithmic growth class, stays so during the duration of the evolution. Similarly in Chapter 4 we derive a priori logarithmic estimates for the gradient of the graph.

For the shorttime existence of the solution of entire graphs, we use the result of Ecker and Huisken [7], however in Chapter 5, by restricting to the case of linear growth, we show longtime existence for Lipschitz initial data, which was done by Ecker and Huisken in their paper [6]. We however then prove a logarithmic spatial decay estimate for the curvature and all its derivatives and show this behaviour is maintained during the evolution.

In Chapter 6, we study the behaviour of the solution as time tends to infinity and show that under our logarithmic assumption the graphs become self-similar, and in fact this convergence is polynomial in time. Unlike the Angenent and Velázquez method of using spectral analysis to obtain the rate of convergence for self-similarly contracting solutions of mean curvature flow, we use only the Maximum principle together with test functions to obtain our result.

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Chapter 1

Introduction

We fix our notations and present some definitions and then present some properties of mean curvature flow which we will use in the subsequent chapters. Since we study a geometric flow we will start with an introduction to some basic differential geometric facts. Much of this material is covered in detail in the book by Ecker [5].

1.1 Notations

We will study hypersurfaces in $\mathbb{R}^{n+1}$, denoted by M which are smooth and properly embedded, contained in some open set $U \subset \mathbb{R}^{n+1}$. We denote such a surface by a map from an open set $M^n \subset \mathbb{R}^n$ by $F\colon M^n \to \mathbb{R}^{n+1}$ and write $F(M^n) = M$.

M is properly embedded if $F^{-1}(K) \subset M^n$ is compact for a compact set $K \subset U$.

At every point $p \in M^n$, the coordinate tangent vectors $\partial_i F(p) := \partial F / \partial p_i$ for $i = 1, \ldots, n$, form a basis of the tangent space $T_x M$ at $x = F(p)$, The metric on M is then given by $g_{ij} = \partial_i F \cdot \partial_j F$ and then the area element of M is the root of the determinant of the metric:

$$\sqrt{g} = \sqrt{\det[g_{ij}]}.$$

The inverse metric is just the inverse matrix of g_{ij} and is given by $g^{ij} = [g_{ij}]^{-1}$.

1.2 Geometry of surfaces

The *tangential gradient* of a function on our surface $h\colon M \to \mathbb{R}$ is

$$\nabla^M h = g^{ij} \partial_i h \partial_j F$$

and the *covariant derivative* of a smooth tangent vector $X = X^i \partial_i F$ on M is

$$\nabla_i^M X^j = \partial_i X^j + \Gamma_{ik}^j X^k.$$

Here Γ_{ij}^k are the *Christoffel symbols* $\Gamma_{ij}^k = g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij})/2$.

The trace of covariant derivative tensor X on M is called the *tangential divergence* of X and is denoted by

$$\operatorname{div}_M X = \nabla_i^M X^i = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} X_j)$$

and the *Laplace-Beltrami* operator of h on M is then the divergence of the tangential gradient of h and is written as

The divergence also makes sense for a smooth vector field $X\colon M \to \mathbb{R}^{n+1}$ which is not necessarily tangent to M. In this case $\operatorname{div}_M X = g^{ij} \partial_i X \cdot \partial_j F$.

$$\Delta_M h = \operatorname{div}_M \nabla^M h = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j h).$$

Note that ν the choice of unit normal to our hypersurface M satisfies

$$\nu \cdot \partial_i F = 0$$

on M for $1 \leq i \leq n$ and we define the *second fundamental form* of M as

$$A_{ij} = \partial_i \nu \cdot \partial_j F = -\nu \cdot \partial_i \partial_j F.$$

The *Weingarten map*, which maps tangent vectors to tangent vectors, is

$$A_j^i = g^{ik} A_{kj}$$

and its n eigenvalues $\kappa_1, \ldots, \kappa_n$ are called the *principal curvatures* of M. The *mean curvature* is then defined as

$$H = \sum_{i=1}^n \kappa_i = g^{ij} A_{ij} = g^{ij} \partial_i \nu \cdot \partial_j F = \operatorname{div}_M \nu,$$

and the *mean curvature vector* of M is

Note than $\vec{H}$ is now an inward pointing vector.

$$\vec{H} = -H\nu.$$

In the absence of coordinate systems we can still define derivatives of functions and vector fields on hypersurfaces M by projecting them from $\mathbb{R}^{n+1}$ onto the tangent space of M. For a $x \in M$ we thus define the projection operator $\mathrm{p}_{T_x M}\colon \mathbb{R}^{n+1} \to T_x M$ by

$$\mathrm{p}_{T_x M}(w) = w - (\nu(x) \cdot w)\nu(x).$$

Then for a suitably smooth function $h\colon U \to \mathbb{R}$, the tangential gradient of h is

$$\nabla^M h = \mathrm{p}_{T_x M}(Dh(x)) = Dh(x) - \nu(x) \cdot Dh(x)\nu(x)$$

where $Dh(x)$ is the gradient of h in $\mathbb{R}^{n+1}$ and $x \in M$. Similarly for a suitably differentiable vector field $X\colon U \to \mathbb{R}^{n+1}$, the tangential divergence with respect to M is defined as

$$\operatorname{div}_M X(x) = \operatorname{div}_{\mathbb{R}^{n+1}} X(x) - \nu(x) \cdot D_{\nu(x)} X(x)$$

where $D_{\nu(x)} X(x)$ is the derivative in the $\nu(x)$ direction, given by

$$D_{\nu(x)} X(x) = \left[\frac{\partial X_i}{\partial x_j}\right]_{1 \le i,j \le n+1} \nu(x).$$

Finally, the Laplace-Beltrami of a suitable h, given by $\Delta_M h = \operatorname{div}_M \nabla^M h$, is

$$\Delta_M h = \operatorname{div}_M Dh + \vec{H} \cdot Dh = \Delta_{\mathbb{R}^{n+1}} h - D^2 h(\nu, \nu) + \vec{H} \cdot Dh \qquad (1.1)$$

where $D^2 h(\nu, \nu) := \nu \cdot D_\nu Dh$ is the second derivative of h in the normal direction.

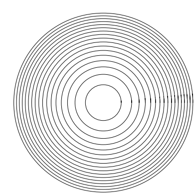
If our hypersurface has no boundary $\partial M = \varnothing$ or the vector field X has compact support the divergence theorem is:

$$\int_M \operatorname{div}_M X = -\int_M \vec{H} \cdot X.$$

Thus for a test function $\phi \in C_0^2(\mathbb{R}^{n+1})$ this implies that

$$0 = \int_M \operatorname{div}_M D\phi + \vec{H} \cdot D\phi = \int_M \Delta_M \phi.$$

1.3 Mean curvature flow



A classical solution to mean curvature flow, where a sphere shrinks to a point in finite time.

We denote a smooth family of embeddings from an open subset M^n of $\mathbb{R}^n$ by $F_t = F(\cdot, t)\colon M^n \to \mathbb{R}^{n+1}$ with $F_t(M^n) = M_t$ for $t \in I$, where I an open interval of $\mathbb{R}$.

We say that this family of hypersurfaces moves by *mean curvature flow* if

$$\frac{\partial F}{\partial t}(p, t) = \vec{H}(F(p, t)) \qquad (1.2)$$

for $p \in M^n$ and $t \in I$.

We also have the fact that

$$-H\nu = -(\operatorname{div}_M \nu)\nu = g^{ij} \partial_i \partial j F = \Delta_M F$$

so that we can write (1.2) as

$$\frac{\partial F}{\partial t}(p,t) - \Delta_M F(p,t) = 0.$$

We will consider hypersurfaces which are entire graphs of the form $M_t = \operatorname{graph} u(\cdot,t)$ for $u(\cdot,t)\colon \mathbb{R}^n \to \mathbb{R}$ and $t \in I$. In other words the last component of the map $F(\cdot,t)$ of M_t can be expressed as a function of the first n components and we can write

$$F(p,t) = (\hat{F}(p,t), u(\hat{F}(p,t),t)).$$

If we denote by Du the derivative of u with respect to $\hat{x} = \hat{F}(p,t)$, then the upward unit normal vector is given by

$$\nu = \frac{(-Du,1)}{\sqrt{1+|Du|^2}}$$

and the mean curvature of the graph is

$$-H = \operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right).$$

Now since

$$\frac{\partial F}{\partial t} = \left(\frac{\partial \hat{F}}{\partial t}, \frac{\partial u}{\partial t} + Du\frac{\partial \hat{F}}{\partial t}\right)$$

we have by (1.2)

$$\frac{\partial F}{\partial t}\cdot \nu = -H$$

which gives us the partial differential equation for mean curvature flow of a graph given by $u(\hat{F}(p,t),t)$:

Actually we have shown that (1.2) is equivalent to (1.3) up to tangential diffeomorphisms.

$$\frac{\partial u}{\partial t} = \sqrt{1+|Du|^2}\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right). \tag{1.3}$$

In [6] the authors have proved long time existence and uniqueness of a solution to (1.3) provided the initial data $u(\cdot,0) = u_0(\cdot)$ grows at most linearly. Of particular interest are the asymptotic properties of solutions to mean curvature flow. Huisken has shown that convex closed hypersurfaces asymptotically converge to spheres [11]. Ecker and Huisken have shown in the case of entire graphs and under some condition on the growth of the initial hypersurface and some condition near infinity, the solution becomes asymptotically self-similar.

1.4 Self-similar expanding solutions

A self-similar or *homothetic* solution to mean curvature flow is one which moves by scaling and is given by the general form

$$M_t = \lambda(t) M_{t_1} \tag{1.4}$$

for a given time t_1 and a positive $\lambda(t)$ which needs to be specified. We can then specify the ansatz, for a family of embeddings $\tilde{F}(\cdot, t) \colon M^n \to \mathbb{R}^{n+1}$, namely:

$$\tilde{F}(q,t) = \lambda(t)\tilde{F}(q,t_1)$$

which satisfies the evolution equation

$$\left(\frac{\partial \tilde{F}}{\partial t}(q,t)\right)^{\perp} = \vec{H}(\tilde{F}(q,t))$$

for $q \in M^n$. This says that up to tangential diffeomorphisms, the motion described by $\tilde{F}$ is equivalent to its normal motion along the mean curvature vector given by F.

Using the fact that the mean curvature scales like $1/\lambda(t)$ we have from the above evolution equation:

$$\lambda'(t)\tilde{F}(q,t_1)^{\perp} = \frac{1}{\lambda(t)}\vec{H}(\tilde{F}(q,t_1))$$

from which we see that

$$\alpha \equiv \frac{d}{dt}\lambda^2(t) = 2\lambda(t)\lambda'(t)$$

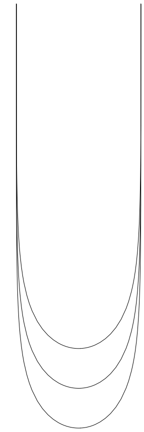
is independent of t. Setting the condition $\lambda(t_1) = 1$ from our ansatz we obtain the following positive solution to our ODE:

$$\lambda(t) = \sqrt{1 + \alpha(t - t_1)} \tag{1.5}$$

for t satisfying $1 + \alpha(t - t_1) > 0$. Thus setting $x = \tilde{F}(q,t)$ for $x \in M^n$, we have from our ansatz $\tilde{F}(q,t_1) = x/\lambda(t)$, which when substituted into the above equation gives:

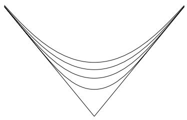
$$\vec{H}(x) = \frac{\alpha x^{\perp}}{2\lambda^2(t)}$$

for $1 + \alpha(t - t_1) > 0$, describing *expanding homothetic solutions* for $\alpha > 0$ and *contracting homothetic solutions* for $\alpha < 0$.



The "grim reaper" or translating soliton solution of mean curvature flow.

1.4.1 Self-similar solutions in the graph setting



The solution of mean curvature flow with a cone as initial data.

In the graph setting expanding homothetic solutions to (1.3) arise quite naturally as solutions "coming out of a cone" i.e. when the initial surface M_0 is a cone. Suppose that if u_0 is a cone with vertex at 0 then we can write:

$$u_0(\lambda p) = \lambda u_0(p)$$

for all $\lambda \geq 0$ and $p \in \mathbb{R}^n$. Now if we define

$$u_0^\lambda(p) := \frac{1}{\lambda} u_0(\lambda p)$$

for $\lambda > 0$, then

$$u^\lambda(p,t) = \frac{1}{\lambda} u(\lambda p, \lambda^2 t)$$

solves (1.3) with the initial condition $u^\lambda(p,0) = u_0^\lambda(\lambda p)/\lambda = u_0(p)$ for $\lambda > 0$. By the uniqueness of solutions at least in the class of functions with at most linear growth, we get that

$$u(p,t) = \frac{1}{\lambda} u(\lambda p, \lambda^2 t)$$

and setting $\lambda = 1/\sqrt{t}$ in the above gives

$$u(p,t) = \sqrt{t} u\left(\frac{p}{\sqrt{t}}, 1\right).$$

Differentiate both sides with respect to t we get:

$$\frac{\partial u}{\partial t} = \frac{1}{2} t^{-1/2} u(\frac{p}{\sqrt{t}}, 1) - \frac{1}{2} t^{-3/2} Du(\frac{p}{\sqrt{t}}, 1) \cdot p.$$

Since we know from (1.3) that $\partial u / \partial t = -\sqrt{1 + |Du|^2} H$, we conclude that:

$$H + \frac{1}{2t} F \cdot \nu = 0,$$

which corresponds to $\alpha = 1$ and $t_1 = 1$ in (1.5). Therefore for all $p \in M_1$ the identity

$$H + \frac{1}{2} F \cdot \nu = 0$$

holds.

This together with the initial condition $\lim_{t \to 0} M_t = M_0$ implies that every M_t is asymptotic to the initial cone as $|p| \to \infty$ for $p \in M_t$ in the sense

$$\lim_{|p| \to \infty} \frac{|u(p,t) - u_0(p)|}{|p|} = 0$$

for every $t > 0$.

1.5 Integral form of mean curvature flow

We can also study the equation (1.2) in the integral setting. This method of using energy estimates provide us with the most important tools in understanding the formation of singularities and their asymptotics. These tools where first established for (1.2) by Huisken and also generalised by Huisken and Ecker in their work.

To begin with we will need to show how the area element evolves as the surface flows by its mean curvature. Recall that:

$$\mathrm{d}\mu_t(p) = \sqrt{\det g_{ij}(p,t)}\,\mathrm{d}\mu_{M^n}(p)$$

where $\mathrm{d}\mu_{M^n}$ is the volume form on the parameter manifold M^n and the metric at (p,t) is

$$g_{ij}(p,t) = \partial_i F(p,t) \cdot \partial_j F(p,t).$$

Then we use the fact that the coordinate derivatives of $F(p,t)$ commute to calculate:

$$\partial_t g_{ij} = 2\partial_t \partial_i F \cdot \partial_j F = 2\partial_i \partial_t F \cdot \partial_j F.$$

Next we use the evolution equation $\partial_t F = -H\nu$, and ν being the normal implies $\nu \cdot \partial_j F = 0$ for $1 \leq j \leq n$ to get

$$\partial_t g_{ij} = 2\partial_i(-H\nu) \cdot \partial_j F = -2H\partial_i \nu \cdot \partial_j F = -2HA_{ij}$$

and also

$$\partial_t g^{ij} = 2HA^{ij}.$$

The derivative of $g := \det[g_{ij}]$, the determinate of a matrix $[g_{ij}]$, is $\partial_t g = \mathrm{trace}([g_{ij}]^{-1}\partial_t[g_{ij}])g$ or $\partial_t g = g g^{ij}\partial_t g_{ij}$ and so the area element evolves by:

$$\partial_t \sqrt{g} = -H^2\sqrt{g} = -|\vec{H}|^2\sqrt{g}$$

which proves the following lemma:

Lemma 1.1 (Evolution of Area Element)**.** *The area element of a solution $(M_t)_{t\in I}$ of (1.2) satisfies the equation*

$$\frac{\partial}{\partial t}\mathrm{d}\mu_t = -|\vec{H}|^2 \mathrm{d}\mu_t$$

for all $t \in I$.

The fact that we consider smooth, properly embedded solutions of mean curvature flow implies that the n-dimensional Hausdorff measure on the family of surfaces satisfies

$$\mathcal{H}^n(M_t \cap K) < \infty$$

for all $K \subset\subset \mathbb{R}^{n+1}$ and for all $t \in I$. This is needed for us to integrate test functions on $\mathbb{R}^{n+1}$ which have compact support over M_t. Another assumption in view of the divergence theorem is that our family of solutions has no boundary inside the set in which we integrate over them. With this in mind and by using the chain-rule and the above lemma, we can state the integral form of mean curvature flow as:

Proposition 1.1 (Mean Curvature Flow in Integral Form)**.** *For any smooth, properly embedded family of solutions $(M_t)_{t\in I}$ of (1.2) in an open subset $U \subset \mathbb{R}^{n+1}$, the following holds*

$$\frac{d}{dt}\int_{M_t}\phi = \int_{M_t}\vec{H}\cdot D\phi - |\vec{H}|^2\phi$$

for all $t \in I$ and $\phi \in C^1_0(U)$.

Similarly for time dependent test functions $\phi \in C^1(U \times I)$, we can specify that $\phi(\cdot, t) \in C^2_0(U)$ and $\phi_t(\cdot, t) \in C^0_0(U)$ for all $t \in I$. This implies then that the time derivative ϕ_t is integrable on M_t and by the divergence theorem we have $\int_{M_t}\Delta_{M_t}\phi = 0$ for all $t \in I$. This together with the above equations then give us the following given the assumptions on our test function:

Proposition 1.2 (Time-Dependent Test Function)**.** *A smooth, properly embedded solution $(M_t)_{t\in I}$ of (1.2) in $U \subset \mathbb{R}^{n+1}$ satisfies*

$$\frac{d}{dt}\int_{M_t}\phi = \int_{M_t}\frac{d\phi}{dt} - |\vec{H}|^2\phi = \int_{M_t}\frac{\partial\phi}{\partial t} + \vec{H}\cdot D\phi - |\vec{H}|^2\phi$$

which due to the Divergence Theorem gives:

$$\frac{d}{dt}\int_{M_t}\phi = \int_{M_t}\frac{\partial\phi}{\partial t} \pm \operatorname{div}_{M_t} D\phi - |\vec{H}|^2\phi = \int_{M_t}\left(\frac{d}{dt} \pm \Delta_{M_t}\right)\phi - |\vec{H}|^2\phi,$$

given test function ϕ defined above.

1.5.1 Monotonicity formula

The *monotonicity formula* which was proved by Huisken describes the behaviour of the integral of the “backward heat-kernel” over our surface M_t. We can define backward heat-kernel centered at the origin as

$$\Phi(x,t) = \frac{1}{(-4\pi t)^{n/2}}\exp\left(\frac{|x|^2}{4t}\right)$$

for $x \in \mathbb{R}^{n+1}$ and $t > 0$ and its translate for some fixed $x_0 \in \mathbb{R}^{n+1}$ and time $t_0 > t$, as:

$$\Phi_{(x_0,t_0)}(x,t) := \Phi(x - x_0, t - t_0) = \frac{1}{(4\pi(t_0 - t))^{n/2}}\exp\left(-\frac{|x - x_0|^2}{4(t_0 - t)}\right).$$

Geometrically we can see that as t approached t_0, the kernel is scaled and concentrates at the point x_0. The monotonicity formula then says that under mean curvature flow the area of the hypersurface near any point in non-increasing on any scale. It is in fact strictly decreasing, unless the hypersurface is homothetically contracting about this point.

Theorem 1.1 (Huisken's Monotonicity Formula). *If M_t is a surface satisfying (1.2) for $t < t_0$, then we have*

$$\frac{d}{dt}\int_{M_t} \Phi_{(x_0,t_0)}\, \mathrm{d}\mu_t = -\int_{M_t} \left| \vec{H} - \frac{\nabla^{\perp}\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}} \right|^2 \Phi_{(x_0,t_0)}\, \mathrm{d}\mu_t.$$

Proof. The proof uses the fact that Φ is an extrinsically defined function of M_t, in which case the total derivative of Φ along M_t equals:

$$\frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + D\Phi \cdot \vec{H}$$

and the Laplace-Beltrami given by (1.1). Thus if we assume we are at the origin $(x_0, t_0) = (0, 0)$ we have the following pointwise result:

$$\left(\frac{d}{dt} + \Delta_{M_t}\right)\Phi = \frac{\partial\Phi}{\partial t} + \mathrm{div}_{M_t} D\Phi + 2\vec{H}\cdot\nabla^{\perp}\Phi,$$

where we use the fact that since $\vec{H}$ is a normal vector to M_t, then

$$\vec{H}\cdot D\Phi = \vec{H}\cdot\nabla^{\perp}\Phi.$$

Completing the square then gives us:

$$\begin{aligned}\left(\frac{d}{dt} + \Delta\right)\Phi &= \frac{\partial\Phi}{\partial t} + \mathrm{div}_{M_t} D\Phi + \frac{|\nabla^{\perp}\Phi|^2}{\Phi} \\ &\quad - \left|\vec{H} - \frac{\nabla^{\perp}\Phi}{\Phi}\right|^2 \Phi + |\vec{H}|^2\Phi.\end{aligned}$$

Next we calculate from the definition of Φ:

$$\begin{aligned}\frac{\partial\Phi}{\partial t} + \mathrm{div}_{M_t} D\Phi + \frac{|\nabla^{\perp}\Phi|^2}{\Phi} &= \left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}^{n+1}}\right)\Phi - D^2\Phi(\nu,\nu) + \frac{|D\Phi\cdot\nu|^2}{\Phi} \\ &= 0,\end{aligned}$$

since

$$\left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}^{n+1}}\right)\Phi = \frac{\Phi}{2t}$$

and the remaining two terms using $D\Phi = x\Phi/(2t)$ similarly gives $-\Phi/(2t)$. Thus

$$\left(\frac{d}{dt} + \Delta_{M_t}\right)\Phi - |\vec{H}|^2\Phi = -\left|\vec{H} - \frac{\nabla^\perp\Phi}{\Phi}\right|^2\Phi, \tag{1.6}$$

and in the case of compact M_t, by Proposition 1.2 we are done. In the general case we use Theorem 1.2 given below, with $f \equiv 1$ to obtain the result. □

Theorem 1.2 (Weighted Monotonicity Formula). *Suppose $(M_t)_{t\in I}$ is a family of solutions to (1.2), and that f is a sufficiently smooth (possibly time-dependent) function defined on $(M_t)_{t\in I}$, such that all integrals are finite and integration by parts is permitted, i.e.*

$$\int_{M_t}(|f| + |\frac{\partial f}{\partial t}| + |Df| + |D^2 f|)\Phi_{(x_0,t_0)} < \infty$$

for all times $t \in I$ with $t < t_0$ and fixed point $(x_0, t_0) \in \mathbb{R}^{n+2}$. Then for these times we have:

$$\frac{d}{dt}\int_{M_t} f\Phi_{(x_0,t_0)} = \int_{M_t}\left(\left(\frac{d}{dt} - \Delta_{M_t}\right)f - \left|\vec{H} - \frac{\nabla^\perp\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2 f\right)\Phi_{(x_0,t_0)}.$$

Proof. We consider a time-dependent compactly supported test function ϕ. From Proposition 1.2 we have

$$\begin{aligned}\frac{d}{dt}\int_{M_t}\Phi\phi &= \int_{M_t}\Phi\frac{d\phi}{dt} + \phi\frac{d\Phi}{dt} - |\vec{H}|^2\Phi\phi \\ &= \int_{M_t}\Phi\left(\frac{d}{dt} - \Delta_{M_t}\right)\phi + \left(\left(\frac{d}{dt} + \Delta_{M_t}\right)\Phi - |\vec{H}|^2\Phi\right)\phi,\end{aligned}$$

where we have used the integration by parts formula

$$\int_M \phi\Delta_M\Phi = -\int_M \nabla^M\phi\cdot\nabla^M\Phi = \int_M \Phi\Delta_M\phi.$$

In view of (1.6) we have:

$$\frac{d}{dt}\int_{M_t}\Phi\phi = \int_{M_t}\Phi\left(\frac{d}{dt} - \Delta_{M_t}\right)\phi - \left|\vec{H} - \frac{\nabla^\perp\Phi_{(x_0,t_0)}}{\Phi_{(x_0,t_0)}}\right|^2\Phi\phi.$$

Now we proceed as in [14] and choose $\phi = \chi_R$ where

$$\chi_{B_R} \leq \chi_R \leq \chi_{B_{2R}},$$

χ_{B_R} being the characteristic function for the ball B_R and

$$R|D\chi_R| + R^2|D^2\chi_R| \leq C_0.$$

We therefore have

$$\begin{aligned} \left| \left(\frac{d}{dt} - \Delta_{M_t} \right) \chi_R \right| &= \left| \left(\frac{\partial}{\partial t} - \Delta_{\mathbb{R}^{n+1}} \right) \chi_R + D^2\chi_R(\nu,\nu) \right| \\ &\leq \frac{C(n, C_0)}{R^2} \chi_{B_{2R} \setminus B_R}. \end{aligned}$$

Since we assumed that $\int_{M_t} \Phi \leq \infty$ we can therefore let $R \to \infty$ and use the standard converge theorems for integrals to conclude the Monotonicity formula when M_t is not compact.

To prove the general case we use $\phi = f\chi_R$ and proceed in the same way as above. □

1.6 Weak Maximum principle

In order to prove estimates for different geometric quantities, we will need to use the following weak maximum principle which we will prove using the monotonicity formula of the previous section. We will use the the maximum principle in cases where our test function is extrinsically defined, in which case we can express a

$$h(p,t) = f(x,t) \text{ where } x = F(p,t).$$

Here $h\colon M^n \times [t_1, t_0) \to \mathbb{R}$ and $f\colon U \times [t_1, t_0) \to \mathbb{R}$ and U is an open set in $\mathbb{R}^{n+1}$ containing the family of surfaces M_t. The maximum principle then states:

Proposition 1.3 (Weak Maximum Principle)**.** *For $(M_t)_{t\in(t_1,t_0)}$, a family of solutions to (1.2), suppose $f\colon U \times [t_1, t_0) \to \mathbb{R}$ is sufficiently smooth for $t > t_1$, continuous on $M^n \times [t_1, t_0]$ and satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta_{M_t} \right) f \leq \vec{a} \cdot \nabla^{M_t} f,$$

for some vector field $\vec{a}\colon M^n \times [t_1, t_0) \to \mathbb{R}^{n+1}$ which is well-defined in the sense $a_0 = \sup_{M\times[t_1,t_0)} |\vec{a}| < \infty$. Then

$$\sup_{M_t} f \leq \sup_{M_{t_1}} f$$

for all $t \in [t_1, t_0]$.

Proof. Let $k = \sup_{M_{t_1}} f$ and define $f_k := \max(f - k, 0)$. Then from our assumption on f, we have

$$\begin{aligned}\left(\frac{d}{dt} - \Delta_{M_t}\right) f_k^2 &= 2f_k \left(\frac{d}{dt} - \Delta_{M_t}\right) f_k - 2|\nabla f_k|^2 \\ &\leq 2f_k \vec{a} \cdot \nabla f_k - 2|\nabla f_k|^2.\end{aligned}$$

By Young's inequality and our assumption on the vector field $\vec{a}$ we obtain

$$\left(\frac{d}{dt} - \Delta_{M_t}\right) f_k^2 \leq \frac{1}{2} a_0^2 f_k^2.$$

We now use Theorem 1.2 with $f = f_k^2$ to obtain the inequality

$$\frac{d}{dt} \int_{M_t} f_k^2 \Phi \, d\mu_t \leq \frac{1}{2} a_0^2 \int_{M_t} f_k^2 \Phi \, d\mu_t$$

which implies that f_k is constant in time t. But initially at $t = t_1$, $f_k = 0$ so we have that $f_k \equiv 0$, which gives us our result. □

Chapter 2

Comparison with the Heat Equation

Since we have an explicit form for the solution of the heat equation, we use it to calculate the rate of convergence of rescaled solutions for different assumptions on the growth rate of conic initial data.

2.1 Simplest case

When $n = 1$, the equation (1.3) is just

$$\frac{\partial u}{\partial t} = \frac{\Delta u}{1 + Du^2}, \tag{2.1}$$

and its linearization is just the heat equation

$$\frac{\partial u}{\partial t} = \Delta u, \tag{2.2}$$

with the initial data given by $u(p, 0) = u_0(p)$. The solution of (2.2) for a $u \in C^2$ is given by:

$$u(p,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(z) e^{-(p-z)^2/4t} \, \mathrm{d}z.$$

If we define $q = (z - p)/\sqrt{4t}$ then we get

$$u(p,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} u_0(p + q\sqrt{4t}) e^{-q^2} \, \mathrm{d}q.$$

Rescaling the solution parabolically, i.e.

$$\Phi(y,\tau) = \frac{u(p,t)}{\sqrt{2t}}, \quad y = \frac{p}{\sqrt{2t}}, \quad \tau = \log\sqrt{2t},$$

then gives

$$\Phi(y,\tau) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\tau} u_0\left(e^{\tau}(y+q\sqrt{2})\right) e^{-q^2}\,\mathrm{d}q.$$

Notice that the solution coming out of the cone $u_0(y) = |y|$ is given by

$$u(y,t) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} |y+q\sqrt{4t}|e^{-q^2}\,\mathrm{d}q,$$

which at time $t = 1/2$ it is independent of t and so is equal to

$$u(y,1/2) = \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} |y+q\sqrt{2}|e^{-q^2}\,\mathrm{d}q = \Phi^{\infty}(y) := \lim_{\tau\to\infty}\Phi(y,\tau).$$

Thus we call $\Phi^{\infty}(\cdot)$ the self-similar solution to the heat equation coming out of this cone $u_0(y) = |y|$.

If we now stipulate how our initial data $u_0(p)$ converges to its tangent cone at infinity, then we can calculate the rate at which the solutions to the heat equation become self-similar as $t \to \infty$.

2.2 Ecker and Huisken condition

In their paper [6], Ecker and Huisken consider an initial surface which grows *linearly* and has bounded curvature. In the simplified case when $n = 1$ we can write our initial surface as $x_0 = (p, u_0(p))$. The unit normal is then given by

$$\nu_0 = \frac{(-u_0', 1)}{\sqrt{1+(u_0')^2}}.$$

The linear growth condition is then

$$v_0 = \langle \nu_0, e_2\rangle^{-1} = \sqrt{1+(u_0')^2} \leq c_1$$

for some fixed constant $c_1 \geq 1$. The bounded curvature then implies that the second derivative of u_0 is bounded too.

In addition Ecker and Huisken require that their initial surface satisfy the following estimate

$$\langle x_0, \nu_0\rangle^2 \leq c_3(1+|x_0|^2)^{1-\delta}$$

for some constant $c_3 < \infty$ and $\delta > 0$. We write this condition in terms of u_0 to get

$$\frac{|pu_0' - u_0|^2}{1+(u_0')^2} \leq \frac{c_3}{(1+p^2+u_0^2)^{\delta-1}}.$$

Note that the linear growth condition implies

$$\frac{|pu_0' - u_0|^2}{c_1^2} \leq \frac{|pu_0' - u_0|^2}{1+(u_0')^2} \leq \frac{c_3}{(1+p^2+u_0^2)^{\delta-1}} \leq \frac{C}{p^{2\delta-2}}$$

which is just the following, after we divide both sides by p^2

$$\frac{|pu_0' - u_0|}{p^2} \leq \frac{C}{p^{\delta+1}}.$$

Note that the left hand side is just the derivative of u_0/p, so we have

$$\left|\frac{d}{dp}\frac{u_0(p)}{p}\right| \leq \frac{C}{p^{\delta+1}},$$

which finally gives us the Ecker and Huisken condition:

$$\left|\lim_{r\to\infty}\frac{u_0(r)}{r} - \frac{u_0(p)}{p}\right| \leq \frac{C}{p^\delta}.$$

2.3 Rate of convergence to self-similar solutions

If the initial data $u_0(p)$ is a cone tangent to the standard cone at infinity, then it satisfies all the assumptions of the previous section if

$$\left|\frac{u_0(p)}{p} - 1\right| \leq \frac{C}{|p|^\delta},$$

i.e. it approaches its tangent cone at infinity at a rate proportional to $|x|^\delta$, for some $\delta > 0$. This then implies that

$$|u_0(p) - |p|| \leq C|p|^{1-\delta}$$

which we use to get:

$$\begin{aligned}
|\Phi(y,\tau) - \Phi^\infty(y)| &= \left|\frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\tau} u_0\left(e^\tau(y+q\sqrt{2})\right) e^{-q^2}\,\mathrm{d}q - \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} |y+q\sqrt{2}| e^{-q^2}\,\mathrm{d}q\right| \\
&\leq \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}} e^{-\tau}\left|u_0\left(e^\tau(y+q\sqrt{2})\right) - |e^\tau(y+q\sqrt{2})|\right| e^{-q^2}\,\mathrm{d}q \\
&\leq \frac{Ce^{-\delta\tau}}{\sqrt{\pi}}\int_{\mathbb{R}} |y+q\sqrt{2}|^{1-\delta} e^{-q^2}\,\mathrm{d}q = Ce^{-\delta\tau}.
\end{aligned}$$

Thus we see that the convergence to the self-similar solution coming out of the cone is exponential—at least in the linear case of the heat equation.

Now if we assume our initial data satisfies for some $\delta > 0$ a condition of the form

$$\left|\frac{u_0(p)}{p} - 1\right| \leq \frac{C}{\log^\delta |p|},$$

which implies that

$$|u_0(p) - |p|| \leq \frac{C|p|}{\log^\delta |p|}$$

and we see that

$$\begin{aligned}
|\Phi(y,\tau) - \Phi^\infty(y)| &\leq \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-\tau} \left| u_0\left(e^\tau(y + q\sqrt{2})\right) - |e^\tau(y+q\sqrt{2})| \right| e^{-q^2}\, \mathrm{d}q \\
&\leq \frac{C}{\sqrt{\pi}} \int_{\mathbb{R}} \frac{|y+q\sqrt{2}|}{\tau^\delta + \log^\delta |y+q\sqrt{2}|} e^{-q^2}\, \mathrm{d}q \\
&\leq C\tau^{-\delta}.
\end{aligned}$$

In other words we have a slower (τ^δ) rather than exponential, rate of convergence to a self-similar solution.

We can thus reverse the above process to recover the condition analogous to that of Ecker and Huisken so that we have a slower convergence to self-similar solutions in this simplified case. Doing this gives that:

$$\langle x_0, \nu_0 \rangle^2 \leq c_3 \frac{|p|^2}{\log^\delta |x_0|^2}$$

for some $\delta > 0$.

Chapter 3

Height Estimate

We wish to show in this chapter that any initial smooth graph with bounded gradient that satisfies the logarithmic convergence to its tangent cone at infinity, stays in such a class under mean curvature flow.

We define the height of M_t with respect to the hyperplane $\mathbb{R}^n \times \{0\}$ by

$$u(p,t) = \langle x(p,t), e_{n+1} \rangle$$

In particular we have the following lemma, since $x(p,t)$ is the solution to mean curvature flow and so

$$\left(\frac{d}{dt} - \Delta \right) u = 0.$$

Lemma 3.1. *The function $\eta(x,t)$ given by*

$$\eta(x,t) = 1 + |x|^2 - u^2 + (2n+m)t$$

satisfies

$$\left(\frac{d}{dt} - \Delta \right) \eta = 2|\nabla u|^2 + m,$$

for some constant m.

Proof. Since

$$\left(\frac{d}{dt} - \Delta \right) (|x|^2 + 2nt) = 0$$

and

$$\left(\frac{d}{dt} - \Delta \right) u^2 = -2|\nabla u|^2,$$

we have the implied result. □

We have, by the Chain-rule for the Heat operator, for two twice differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $\eta\colon \mathbb{R}^n \to \mathbb{R}$, the composite $f(\eta)$ satisfies

$$\left(\frac{d}{dt} - \Delta\right) f(\eta) = f'(\eta)\left(\frac{d}{dt} - \Delta\right)\eta - f''(\eta)|\nabla\eta|^2,$$

where f' denotes the derivative of f. We use this fact together with the Product-rule for the Heat operator:

$$\left(\frac{d}{dt} - \Delta\right) fg = f\left(\frac{d}{dt} - \Delta\right) g + g\left(\frac{d}{dt} - \Delta\right) f - 2\nabla f \cdot \nabla g$$

for a twice differentiable function $g\colon \mathbb{R} \to \mathbb{R}$, to prove the following lemma:

Lemma 3.2. *For any power δ and two twice differentiable functions $f\colon \mathbb{R} \to \mathbb{R}$ and $\eta\colon \mathbb{R}^n \to \mathbb{R}$, the composite $f^\delta(\eta)$ satisfies*

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f^\delta(\eta) &= \delta f^{\delta-1}\left(\frac{d}{dt} - \Delta\right) f(\eta) - \delta(\delta-1) f^{\delta-2}(f')^2|\nabla\eta|^2 \\ &= \delta f^{\delta-1} f'\left(\frac{d}{dt} - \Delta\right)\eta - \delta f^{\delta-1} f''|\nabla\eta|^2 \\ &\quad - \delta(\delta-1) f^{\delta-2}(f')^2|\nabla\eta|^2.\end{aligned}$$

Proof. By using the above product rule repeatedly and direct calculation we have that:

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f^\delta &= f\left(\frac{d}{dt} - \Delta\right) f^{\delta-1} + f^{\delta-1}\left(\frac{d}{dt} - \Delta\right) f - 2\nabla f^{\delta-1}\cdot\nabla f \\ &= f\left(f\left(\frac{d}{dt} - \Delta\right) f^{\delta-2} + f^{\delta-2}\left(\frac{d}{dt} - \Delta\right) f - 2\nabla f^{\delta-2}\cdot\nabla f\right) \\ &\quad + f^{\delta-1}\left(\frac{d}{dt} - \Delta\right) f - 2\nabla f^{\delta-1}\cdot\nabla f \\ &= \cdots \\ &= \delta f^{\delta-1}\left(\frac{d}{dt} - \Delta\right) f \\ &\quad - 2\left(1 + 2 + \cdots + (\delta-1)\right) f^{\delta-2}(f')^2|\nabla\eta|^2\end{aligned}$$

which gives the result since $\sum_{i=1}^{\delta-1} i = \delta(\delta-1)/2$. □

Thus we have for our particular η, and general f,

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) f^\delta(\eta) &= \delta f^{\delta-1} f'(2|\nabla u|^2 + m) - \delta f^{\delta-1} f''|\nabla\eta|^2 \\ &\quad - \delta(\delta-1) f^{\delta-2}(f')^2|\nabla\eta|^2,\end{aligned}$$

and

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)u^2f^\delta(\eta) &= \delta f^{\delta-1}f'u^2(2|\nabla u|^2+m)-\delta f^{\delta-1}f''u^2|\nabla\eta|^2\\ &\quad -\delta(\delta-1)f^{\delta-2}(f')^2u^2|\nabla\eta|^2-2f^\delta|\nabla u|^2\\ &\quad -4\delta f^{\delta-1}f'u\nabla u\cdot\nabla\eta.\end{aligned}$$

We use this to prove the following lemma:

Lemma 3.3. *For any power δ and two twice differentiable functions functions $f\colon\mathbb{R}\to\mathbb{R}$ and $\eta\colon\mathbb{R}^n\to\mathbb{R}$ we have:*

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\left(\frac{u^2f^\delta(\eta)}{\eta}-f^\delta(\eta)\right) &= \left(\frac{u^2}{\eta}-1\right)\delta f^{\delta-1}\left(f'(2|\nabla u|^2+m)-f''|\nabla\eta|^2\right)\\ &\quad -\left(\frac{u^2}{\eta}-1\right)\delta(\delta-1)f^{\delta-2}(f')^2|\nabla\eta|^2\\ &\quad -2\frac{f^\delta}{\eta}|\nabla u|^2-\frac{u^2f^\delta}{\eta^2}(2|\nabla u|^2+m)\\ &\quad -2\frac{u^2f^\delta}{\eta^3}|\nabla\eta|^2+2\delta\frac{u^2f^{\delta-1}}{\eta^2}f'|\nabla\eta|^2\\ &\quad -4\delta\frac{uf^{\delta-1}}{\eta}f'\nabla u\cdot\nabla\eta+4\frac{uf^\delta}{\eta^2}\nabla u\cdot\nabla\eta.\end{aligned}$$

Proof. To begin with we use the previous lemma to calculate:

$$\left(\frac{d}{dt}-\Delta\right)\eta^{-1}=-\eta^{-2}\left(\frac{d}{dt}-\Delta\right)\eta-2\frac{|\nabla\eta|^2}{\eta^3}=-\frac{2|\nabla u|^2+m}{\eta^2}-2\frac{|\nabla\eta|^2}{\eta^3}.$$

Thus we have the following:

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\frac{u^2}{\eta} &= \frac{1}{\eta}\left(\frac{d}{dt}-\Delta\right)u^2+u^2\left(\frac{d}{dt}-\Delta\right)\eta^{-1}\\ &\quad -2\nabla\eta^{-1}\cdot\nabla u^2\end{aligned}$$

and

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\left(\frac{u^2}{\eta}-1\right)f^\delta &= \left(\frac{u^2}{\eta}-1\right)\left(\frac{d}{dt}-\Delta\right)f^\delta + f^\delta\left(\frac{d}{dt}-\Delta\right)\frac{u^2}{\eta} \\ &\quad -2\nabla\frac{u^2}{\eta}\cdot\nabla f^\delta \\ &= \left(\frac{u^2}{\eta}-1\right)\left(\frac{d}{dt}-\Delta\right)f^\delta + \frac{f^\delta}{\eta}\left(\frac{d}{dt}-\Delta\right)u^2 \\ &\quad + u^2 f^\delta\left(\frac{d}{dt}-\Delta\right)\eta^{-1} - 2f^\delta\nabla\eta^{-1}\cdot\nabla u^2 \\ &\quad -2\nabla\frac{u^2}{\eta}\cdot\nabla f^\delta\end{aligned}$$

which when expanded gives us the result. □

3.1 Logarithmic growth

Now we would like to study M_t, a smooth solution to (1.2), which grows logarithmically. We would like to show that that height $u(\cdot,t)$ satisfies the same logarithmic growth estimates as $u(\cdot,0)$. Note in particular that the non-negative function $|x|^2-u^2$ measures distance in the hyperplane orthogonal to e_{n+1}. Our proposition then states:

Proposition 3.1. *If for some negative constant $-\infty < c_0 \le 0$ and positive power $\delta \ge 1$, the inequality*

$$\frac{u^2}{e+|x|^2-u^2}-1 \le \frac{c_0}{\log^\delta(e+|x|^2-u^2)}$$

is satisfied on M_0, then for all $t>0$,

$$\frac{u^2}{e+|x|^2-u^2+(2n+m)t}-1 \le \frac{c_0}{\log^\delta(e+|x|^2-u^2+(2n+m)t)},$$

and a positive constant $m \ge 4(\delta-1)$.

Proof idea. The proof involves calculating the evolution of

$$\left(\frac{u^2}{e+|x|^2-u^2+(2n+m)t}-1\right)\log^\delta(e+|x|^2-u^2+(2n+m)t)$$

and then using the Weak Maximum Principle to obtain the result. In the calculation below, we will abuse our notations and denote η also by the term with the exponential e,

$$\eta(x,t) = e + |x|^2 - u^2 + (2n+m)t,$$

as the constant does not affect the prior calculations.

Thus when we have for our particular case

$$f(\eta) = \log(\eta), \qquad f'(\eta) = \frac{1}{\eta}, \qquad f''(\eta) = -\frac{1}{\eta^2}$$

then the above lemma gives:

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\left(\frac{u^2 f^\delta(\eta)}{\eta} - f^\delta(\eta)\right) \quad = \quad & \left(\frac{u^2}{\eta} - 1\right)\delta f^{\delta-1}\left(\frac{2|\nabla u|^2 + m}{\eta} + \frac{|\nabla\eta|^2}{\eta^2}\right) \\
& - \left(\frac{u^2}{\eta} - 1\right)\delta(\delta-1)\frac{f^{\delta-2}}{\eta^2}|\nabla\eta|^2 \\
& - 2\frac{f^\delta}{\eta}|\nabla u|^2 - \frac{u^2 f^\delta}{\eta^2}(2|\nabla u|^2 + m) \\
& - 2\frac{u^2 f^\delta}{\eta^3}|\nabla\eta|^2 + 2\delta\frac{u^2 f^{\delta-1}}{\eta^3}|\nabla\eta|^2 \\
& - 4\delta\frac{u f^{\delta-1}}{\eta^2}\nabla u\cdot\nabla\eta + 4\frac{u f^\delta}{\eta^2}\nabla u\cdot\nabla\eta.
\end{aligned}$$

Using Young's inequality we obtain:

$$\left|4\frac{u f^\delta}{\eta^2}\nabla u\cdot\nabla\eta\right| \le 2\frac{f^\delta}{\eta}|\nabla u|^2 + 2\frac{u^2 f^\delta}{\eta^3}|\nabla\eta|^2. \tag{3.1}$$

Also note that in terms of a local orthonormal frame $\{e_i\}_{1\le i\le n}$ on M we have

$$\nabla_i u = \nabla_i\langle x, e_{n+1}\rangle = \langle e_i, e_{n+1}\rangle$$

which implies that

$$\nabla_i\eta = 2|x|\frac{\langle x, e_i\rangle}{|x|} - 2\langle x, e_{n+1}\rangle\langle e_i, e_{n+1}\rangle = 2\langle e_i, x - \langle x, e_{n+1}\rangle e_{n+1}\rangle$$

so that:

$$\begin{aligned}
|\nabla\eta|^2 \quad &= \quad 4\sum_i(\langle x, e_i\rangle - \langle x, e_{n+1}\rangle\langle e_i, e_{n+1}\rangle)^2 \\
&= \quad 4\sum_i(\langle x, e_i\rangle^2 - 2\langle x, e_{n+1}\rangle\langle e_i, e_{n+1}\rangle\langle x, e_i\rangle + \langle x, e_{n+1}\rangle^2\langle e_i, e_{n+1}\rangle^2) \\
&= \quad 4\left(|x|^2 - \langle\nu, x\rangle^2 - 2\langle x, e_{n+1}\rangle^2 + u^2(1 - \langle\nu, e_{n+1}\rangle^2)\right) \\
&\le \quad 4\eta.
\end{aligned} \tag{3.2}$$

Also by the first derivative test for an extrema, we have at such a point $\nabla(u^2 f^\delta/\eta - f^\delta) = 0$. This expands out to give:

$$\begin{aligned} 0 &= \nabla\left(\frac{u^2 f^\delta(\eta)}{\eta} - f^\delta(\eta)\right) = 2\frac{u f^\delta \nabla u}{\eta} + \delta\frac{u^2 f^{\delta-1}\nabla\eta}{\eta^2} \\ &\quad - \frac{u^2 f^\delta \nabla\eta}{\eta^2} - \delta\frac{f^{\delta-1}\nabla\eta}{\eta}, \end{aligned}$$

which we rearrange to get:

$$\begin{aligned} 2\delta\frac{u^2 f^{\delta-1}}{\eta^3}|\nabla\eta|^2 - 4\delta\frac{u f^{\delta-1}}{\eta^2}\nabla u\cdot\nabla\eta &= -2\delta^2\frac{f^{\delta-2}|\nabla\eta|^2}{\eta^2} + 2\delta^2\frac{u^2 f^{\delta-2}|\nabla\eta|^2}{\eta^3} \\ &\leq -2\delta^2\frac{f^{\delta-2}|\nabla\eta|^2}{\eta^2} + 2\delta^2\frac{f^{\delta-2}|\nabla\eta|^2}{\eta^2} \\ &= 0. \end{aligned} \tag{3.3}$$

The last inequality above comes if we assume that initially

$$\frac{u^2}{\eta} \leq 1$$

which is preserved during mean curvature flow as shown by the following adaption of Proposition 2.2 of Ecker and Huisken in [6]:

Proposition 3.2 (A priori Height Estimate)**.** *If for some $c_0 < \infty$, the inequality*

$$\frac{u^2}{1+|x|^2-u^2} \leq C_0$$

is satisfied on M_0, then for all $t > 0$, and some constant $m \geq 0$,

$$\frac{u^2}{1+|x|^2-u^2+(2n+m)t} \leq C_0.$$

Proof. We calculate the evolution equation

$$\begin{aligned} \left(\frac{d}{dt}-\Delta\right)\frac{u^2}{\eta} &= -2\frac{|\nabla u|^2}{\eta} - \frac{u^2}{\eta^2}(2|\nabla u|^2+m) - 2u^2\frac{|\nabla\eta|^2}{\eta^3} \\ &\quad + 4u\frac{\nabla\eta\cdot\nabla u}{\eta^2}. \end{aligned}$$

By Young's inequality we have:

$$\left|4u\frac{\nabla\eta\cdot\nabla u}{\eta^2}\right| \leq 2\frac{|\nabla u|^2}{\eta} + 2u^2\frac{|\nabla\eta|^2}{\eta^3}$$

which then implies that

$$\left(\frac{d}{dt}-\Delta\right)\frac{u^2}{\eta}\leq-\frac{u^2}{\eta^2}(2|\nabla u|^2+m)\leq 0.$$

Therefore by the Weak Maximum Principle (Proposition 1.3) the result follows. □

We are now ready to prove our proposition, namely if M_t is a smooth solution of mean curvature flow and if initially M_0 converges to its tangent cone logarithmically, then such a rate is preserved during the evolution. In other words the solutions remain in the same growth class they started in.

Proof of Proposition 3.1. By using the inequalities (3.1) and (3.3) we obtain from the evolution equation for some positive power δ, the following inequality at the maximum point:

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\left(\frac{u^2}{\eta}-1\right)f^\delta \quad\leq\quad & -\delta\left(1-\frac{u^2}{\eta}\right)f^{\delta-1}\left(\frac{2|\nabla u|^2+m}{\eta}+\frac{|\nabla\eta|^2}{\eta^2}\right)\\ & +\delta(\delta-1)\left(1-\frac{u^2}{\eta}\right)\frac{f^{\delta-2}}{\eta^2}|\nabla\eta|^2\\ & -\frac{u^2f^\delta}{\eta^2}(2|\nabla u|^2+m).\end{aligned}$$

Note that since $u^2/\eta-1$ by our initial assumption is negative, the only positive term we have in the above inequality is

$$\delta(\delta-1)\left(1-\frac{u^2}{\eta}\right)\frac{f^{\delta-2}}{\eta^2}|\nabla\eta|^2\leq 4\delta(\delta-1)\left(1-\frac{u^2}{\eta}\right)\frac{f^{\delta-2}}{\eta},$$

since by (3.2) $|\nabla\eta|^2\leq 4\eta$. Now we will choose a positive constant m so that we can control this term. In particular we see that m must be chosen so that

$$\delta\left(1-\frac{u^2}{\eta}\right)\frac{f^{\delta-1}}{\eta}\left(4\frac{\delta-1}{f}-m\right)\leq 0.$$

Thus since $f(\eta)\geq\log e=1$, if we choose $m\geq 4(\delta-1)$, we can drop this remaining negative term to obtain

$$\left(\frac{d}{dt}-\Delta\right)\left(\frac{u^2}{\eta}-1\right)f^\delta\leq 0.$$

Therefore once again by the Weak Maximum Principle in Proposition 1.3, the result we need is obtained. □

Chapter 4

Gradient Function Estimates

In this chapter we wish to show that our family of surfaces M_t remains a graph for all times. We thus need to estimate $\nu \cdot e_{n+1}$ from below or equivalently the term

$$v = \frac{1}{\nu \cdot e_{n+1}} = \sqrt{1 + |Du|^2},$$

defined as the *gradient function*, from above.

We recall from Appendix A that

$$\left(\frac{d}{dt} - \Delta\right) \nu = |A|^2 \nu,$$

which we use to prove the following lemma from [6]:

Lemma 4.1. *The quantity v satisfies the evolution equation*

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2\frac{|\nabla^{M_t} v|^2}{v}.$$

Proof. We have that

$$\left(\frac{d}{dt} - \Delta\right) (\nu \cdot e_{n+1}) = |A|^2 v^{-1},$$

which gives us the result

$$\left(\frac{d}{dt} - \Delta\right) v = -|A|^2 v - 2\frac{|\nabla^{M_t} v|^2}{v}$$

by using Lemma 3.2. □

An immediate corollary of the above lemma in view of the Maximum Principle is that

Corollary 4.1. *If v is bounded at time $t = 0$, it remains bounded by the same constant.*

We have form the work of Ecker and Huisken the following proposition

Proposition 4.1. *If for some constant $c_1 < \infty$, $p \geq 0$,*

$$v \leq c_1(1 + |x|^2 - u^2)^p$$

at time $t = 0$, then for $t > 0$ the inequality holds

$$v(x,t) \leq c_1 \left(1 + |x|^2 - u^2 + (2n+m)t\right)^p$$

holds for some positive constant $m \geq 0$.

Proof. Setting

$$\eta(x,t) = 1 + |x|^2 - u^2 + (2n+m)t,$$

we compute from the previous lemmas:

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) v\eta^{-p} &= v\left(\frac{d}{dt} - \Delta\right)\eta^{-p} + \eta^{-p}\left(\frac{d}{dt} - \Delta\right) v \\
&\quad - 2\nabla v \cdot \eta^{-p} \\
&= -p(p+1)|\nabla\eta|^2 v\eta^{-p-2} - 2p(|\nabla u|^2 + m)v\eta^{-p-1} \\
&\quad - |A|^2 v\eta^{-p} - 2\frac{|\nabla v|^2}{v\eta^p} + 2p\eta^{-p-1}\nabla v \cdot \nabla\eta.
\end{aligned}$$

The last term we estimate via Young's inequality as

$$|2p\eta^{-p-1}\nabla v \cdot \nabla\eta| \leq 2\frac{|\nabla v|^2}{v\eta^p} + \frac{1}{2}p^2|\nabla\eta|^2 v\eta^{-p-2},$$

which gives us for positive p and m

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) v\eta^{-p} &= -p(\frac{p}{2}+1)|\nabla\eta|^2 v\eta^{-p-2} - 2p(|\nabla u|^2 + m)v\eta^{-p-1} \\
&\quad - |A|^2 v\eta^{-p} \\
&\leq 0,
\end{aligned}$$

and the conclusion follows by the Maximum Principle. □

Indeed we can also derive logarithmic estimates for v similar to those derived in the previous section for the height. More precisely we have that

Proposition 4.2. *If for come constant $c_1 < \infty$, and $\delta \geq 0$,*

$$v \leq c_1 \log^{\delta}(e + |x|^2 - u^2)$$

at time $t = 0$, then for $t > 0$ the inequality

$$v \leq c_1 \log^{\delta}(e + |x|^2 - u^2 + 2nt)$$

holds.

Proof. We have from the previous lemma for $m = 0$ that:

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right) v \log^{-\delta} \eta \quad &= \quad \log^{-\delta} \eta \left(\frac{d}{dt} - \Delta\right) v + v \left(\frac{d}{dt} - \Delta\right) \log^{-\delta} \eta \\
&\quad - 2\nabla \log^{-\delta} \eta \cdot \nabla v \\
&= \quad -|A|^2 v \log^{-\delta} \eta - 2\frac{|\nabla v|^2}{v} \log^{-\delta} \eta - 2\delta v \frac{\log^{-\delta-1} \eta}{\eta} |\nabla u|^2 \\
&\quad - \delta v \frac{\log^{-\delta-1} \eta}{\eta^2} |\nabla \eta|^2 - \delta(\delta+1) v \frac{\log^{-\delta-2} \eta}{\eta^2} |\nabla \eta|^2 \\
&\quad + 2\delta \frac{\log^{-\delta-1} \eta}{\eta} \nabla \eta \cdot \nabla v.
\end{aligned}$$

Now we estimate

$$\left| 2\delta \frac{\log^{-\delta-1} \eta}{\eta} \nabla \eta \cdot \nabla v \right| \leq 2\frac{|\nabla v|^2}{v} \log^{-\delta} \eta + \frac{1}{2} \delta^2 v \frac{\log^{-\delta-2} \eta}{\eta^2} |\nabla \eta|^2$$

and the result follows after we drop all the remaining negative terms because of the positive $\delta \geq 0$. ☐

Proposition 4.3. *If for some negative constant $-\infty < c_1 \leq 0$, and $\delta \geq 1$,*

$$\frac{v}{e + |x|^2 - u^2} - 1 \leq \frac{c_1}{\log^{\delta}(e + |x|^2 - u^2)}$$

at time $t = 0$, then for $t > 0$ the inequality

$$\frac{v(x,t)}{e + |x|^2 - u^2 + 2nt} - 1 \leq \frac{c_1}{\log^{\delta}(e + |x|^2 - u^2 + 2nt)}$$

holds.

Proof. From the previous lemmas we have that

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\frac{v}{\eta} &= \frac{1}{\eta}\left(\frac{d}{dt}-\Delta\right)v+v\left(\frac{d}{dt}-\Delta\right)\frac{1}{\eta}\\ &\quad -2\nabla\eta^{-1}\cdot\nabla v\\ &= -\frac{|A|^2 v}{\eta}-2\frac{|\nabla v|^2}{\eta v}-2v\frac{|\nabla u|^2}{\eta^2}-2v\frac{|\nabla\eta|^2}{\eta^3}\\ &\quad +2\eta^{-2}\nabla\eta\cdot\nabla v,\end{aligned}$$

which because of Young's inequality

$$|2\eta^{-2}\nabla\eta\cdot\nabla v|\le 2\frac{|\nabla v|^2}{\eta v}+2v\frac{|\nabla\eta|^2}{\eta^3},$$

gives us

$$\left(\frac{d}{dt}-\Delta\right)\left(\frac{v}{\eta}-1\right)\le -\frac{|A|^2 v}{\eta}-2v\frac{|\nabla u|^2}{\eta^2}\le 0.$$

Similarly

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\log^{\delta}\eta &= 2\delta\frac{\log^{\delta-1}\eta}{\eta}|\nabla u|^2+\delta\frac{\log^{\delta-1}\eta}{\eta^2}|\nabla\eta|^2\\ &\quad -\delta(\delta-1)\frac{\log^{\delta-2}\eta}{\eta^2}|\nabla\eta|^2.\end{aligned}$$

Therefore we have the following evolution equation

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\left(\frac{v}{\eta}-1\right)\log^{\delta}\eta &\le 2\delta\left(\frac{v}{\eta}-1\right)\frac{\log^{\delta-1}\eta}{\eta}|\nabla u|^2\\ &\quad +\delta\left(\frac{v}{\eta}-1\right)\frac{\log^{\delta-1}\eta}{\eta^2}|\nabla\eta|^2\\ &\quad -\delta(\delta-1)\left(\frac{v}{\eta}-1\right)\frac{\log^{\delta-2}\eta}{\eta^2}|\nabla\eta|^2\\ &\quad -2\nabla\log^{\delta}\eta\cdot\nabla\frac{v}{\eta},\end{aligned}$$

and at an extrema we have

$$\nabla\left(\frac{v}{\eta}-1\right)\log^{\delta}\eta=0,$$

which after expanding and re-arranging gives us:

$$2\delta^2\left(\frac{v}{\eta}-1\right)\frac{\log^{\delta-2}\eta}{\eta^2}|\nabla\eta|^2=-2\nabla\log^{\delta}\eta\cdot\nabla\frac{v}{\eta}.$$

Substituting this into our evolution equation above and noting also that by our initial assumption on the constant c_1 begin negative, and the fact that $\log^{\delta}\eta \geq 1$, we have that initially

$$\left(\frac{v}{\eta} - 1\right) \leq 0,$$

which by the above proposition remains so during the course of the evolution, we therefore obtain after dropping the obvious negative terms the following

$$\left(\frac{d}{dt} - \Delta\right)\left(\frac{v}{\eta} - 1\right)\log^{\delta}\eta \leq (\delta^2 + \delta)\left(\frac{v}{\eta} - 1\right)\frac{\log^{\delta-2}\eta}{\eta^2}|\nabla\eta|^2 \leq 0$$

where the last term is also negative for $\delta \geq 0$. Thus using the Maximum principle we obtain our result. □

Chapter 5

Curvature Estimates

Here we present the work of Ecker and Huisken, which guarantee's long-time existence of a solution to mean curvature flow, for which it is crucial to obtain a priori bounds for the second fundamental form on M_t.

We shall only look at the case of *linear growth* by assuming that for some fixed constant $c_1 \geq 1$, the inequality

$$v \leq c_1 \tag{5.1}$$

holds everywhere initially on M_0, and by the result in the previous chapter, hence holds for all time $t \geq 0$.

Lemma 5.1. *The curvature satisfies the inequality*

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 v^2 \leq -2\frac{1}{v} \nabla v \cdot \nabla(|A|^2 v^2).$$

Proof. We have the evolution equation

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 = -2|\nabla A|^2 + 2|A|^4 \leq -2|\nabla|A||^2 + 2|A|^4,$$

where we have used Kato's inequality $|\nabla|A||^2 \leq |\nabla A|^2$. We also have

$$\left(\frac{d}{dt} - \Delta\right) v^2 = -2|A|^2 v^2 - 6|\nabla v|^2,$$

and finally,

$$2\nabla|A|^2 \cdot \nabla v^2 = \nabla|A|^2 \cdot \nabla v^2 + 4v|A|\nabla|A| \cdot \nabla v,$$

which we can re-write in terms of $\nabla(|A|^2v^2)$ since

$$\nabla(|A|^2v^2) = |A|^2\nabla v^2 + v^2\nabla|A|^2$$

which after re-arranging gives

$$\begin{aligned}\nabla|A|^2\cdot\nabla v^2 &= \frac{1}{v^2}\nabla v^2\cdot\nabla(|A|^2v^2) - \frac{1}{v^2}|\nabla v^2|^2|A|^2 \\ &= 2\frac{1}{v}\nabla v\cdot\nabla(|A|^2v^2) - \frac{1}{v^2}|\nabla v^2|^2|A|^2.\end{aligned}$$

Therefore we have by Young's inequality

$$\begin{aligned}2\nabla|A|^2\cdot\nabla v^2 &= 2\frac{1}{v}\nabla v\cdot\nabla(|A|^2v^2) - \frac{1}{v^2}|\nabla v^2|^2|A|^2 + 4v|A|\nabla|A|\cdot\nabla v \\ &= 2\frac{1}{v}\nabla v\cdot\nabla(|A|^2v^2) - 4|\nabla v|^2|A|^2 + 4v|A|\nabla|A|\cdot\nabla v \\ &\geq 2\frac{1}{v}\nabla v\cdot\nabla(|A|^2v^2) - 6|\nabla v|^2|A|^2 - 2|\nabla|A||^2v^2.\end{aligned}$$

This then gives us that

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)|A|^2v^2 &= v^2\left(\frac{d}{dt}-\Delta\right)|A|^2 + |A|^2\left(\frac{d}{dt}-\Delta\right)v^2 \\ &\quad - 2\nabla|A|^2\cdot\nabla v^2 \\ &\leq -2\frac{1}{v}\nabla v\cdot\nabla(|A|^2v^2),\end{aligned}$$

as stated in the lemma. □

An immediate corollary in view of the above lemma is an a priori estimate for solutions of mean curvature flow with bounded gradient and bounded curvature, together with the fact that

$$\frac{1}{v}|\nabla v| \leq |A|v,$$

which implies that the vector given by $\vec{a} = -2v^{-1}\nabla v$ is bounded. The exact statement of the corollary is given by:

Corollary 5.1. *If M_t is a smooth solution of (1.2) with bounded gradient and bounded curvature on each M_t, then there is the a priori estimate*

$$\sup_{M_t}|A|^2v^2 \leq \sup_{M_0}|A|^2v^2.$$

5.1 Higher derivatives of the curvature

Following the work of Huisken, we wish to use the uniform estimates on $|A|^2$ to estimate all derivatives of A in terms of their initial data.

To begin with we have the following theorem of Huisken from [11]:

Theorem 5.1. *For any m we have an equation*

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)|\nabla^m A|^2 &\leq C(n,m)\sum_{i+j+k=m}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A| \\ &\quad -2|\nabla^{m+1}A|^2 \\ &\leq C_m(1+|\nabla^m A|^2)-2|\nabla^{m+1}A|^2,\end{aligned}$$

where C_m depends on n, m, and on upper bounds for $|A|^2,\dots,|\nabla^{m-1}A|^2$.

Note the last inequality comes from the applying Young's inequality to the expression

$$C(n,m)\sum_{i+j+k=m}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^m A|,$$

which shows that the dependency on the lower order derivatives is polynomial.

The proof follows as in Hamilton's paper [10], where we use the notation $S*T$ for any linear combination of tensors formed by contraction on S and T by g. The covariant derivative involves the Christoffel symbols, and we observe that the time derivative of the Christoffel symbols Γ^i_{jk} is given by

$$\begin{aligned}\partial_t\Gamma^i_{jk} &= \frac{1}{2}g^{il}\left(\nabla_j\partial_t g_{kl}+\nabla_k\partial_t g_{jl}-\nabla_l\partial_t g_{jk}\right) \\ &= -g^{il}\left(\nabla_j(HA_{kl})+\nabla_k(Hh_{jl})-\nabla_l(HA_{jk})\right) \\ &= A*\nabla A,\end{aligned}$$

since the evolution equation of g is given by $\partial_t g_{ij}=-2HA_{ij}$.

As in Hamilton, we have the following lemma:

Lemma 5.2. *If S and T are tensors satisfying the evolution equation*

$$\left(\frac{d}{dt}-\Delta\right)S=T,$$

then the covariant derivative ∇S satisfies an equation of the form

$$\left(\frac{d}{dt}-\Delta\right)\nabla S=\nabla T+A*\nabla A*S+A*A*\nabla S.$$

Proof. In view of the time derivative of the Christoffel symbol, we have that

$$\frac{d}{dt}\nabla S = \nabla \frac{d}{dt} S + A * \nabla A * S,$$

since

$$\begin{aligned}\frac{d}{dt}\nabla_i S^i &= \frac{d}{dt}\partial_i S^i + \frac{d}{dt}\Gamma^j_{ik} S^k \\ &= \partial_i \frac{d}{dt} S^i + \Gamma^j_{ik}\frac{d}{dt} S^k + S^k \frac{d}{dt}\Gamma^j_{ik} \\ &= \nabla_i \frac{d}{dt} S^i + A * \nabla A * S.\end{aligned}$$

Substituting the evolution of S then gives us:

$$\frac{d}{dt}\nabla S = \nabla \Delta S + \nabla T + A * \nabla A * S.$$

Finally interchanging the derivatives

$$\nabla \Delta S = \Delta \nabla S + A * A * \nabla S + A * \nabla A * S,$$

and this completes the proof. □

Following exactly like in Hamilton, we thus have the following theorem

Theorem 5.2. *The m-th covariant derivative $\nabla^m |A|$ of the second fundamental form satisfies an evolution equation of the form*

$$\left(\frac{d}{dt} - \Delta\right) \nabla^m A = \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A.$$

Proof. If $m = 0$ we have that

$$\left(\frac{d}{dt} - \Delta\right) A_{ij} = |A|^2 A_{ij}$$

which gives us the explicit form of the cubic term. We proceed by induction on m, using the previous lemma. This gives

$$\begin{aligned}\frac{d}{dt}\nabla^{m+1} A &= \Delta \nabla^{m+1} A + A * A * \nabla^{m+1} A + A * \nabla A * \nabla^m A \\ &\quad + \nabla \left(\sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A \right),\end{aligned}$$

and the result follows by the distributive rule for the covariant derivative. □

As a corollary we thus have the following result also from Hamilton:

Corollary 5.2. *For any m we have an evolution equation*

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)|\nabla^m A|^2 &= -2|\nabla^{m+1}A|^2 \\ &\quad + \sum_{i+j+k=m} \nabla^i A * \nabla^j A * \nabla^k A * \nabla^m A.\end{aligned}$$

Proof. From the previous theorem we have that

$$\frac{d}{dt}|\nabla^m A|^2 = 2\langle \nabla^m A, \frac{d}{dt}\nabla^m A\rangle + A * \nabla^m A * \nabla^m A,$$

where the extra term comes from the variation of the g^{ij} defining the norm $|\cdot|^2$. The Laplace-Beltrami operator also gives

$$\Delta|\nabla^m A|^2 = 2\langle \nabla^m A, \Delta\nabla^m A\rangle + 2|\nabla^{m+1}A|^2.$$

which then gives:

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)|\nabla^m A|^2 &= 2\langle \nabla^m A, \left(\frac{d}{dt}-\Delta\right)\nabla^m A\rangle - 2|\nabla^{m+1}A|^2 \\ &\quad + A * \nabla^m A * \nabla^m A,\end{aligned}$$

and the result follows. □

We can now follow exactly as in Huisken's paper [12], to show that all higher derivatives of the curvature on M_t are bounded, by using the uniform estimate on $|A|^2$ and Theorem 5.1, we have the following proposition:

Proposition 5.1. *If M_t is a smooth solution to (1.2) such that v, $|A|^2$, $|\nabla A|^2$, ..., $|\nabla^m A|^2$ are bounded on each M_t, then we have for all $t \geq 0$ the a priori estimate:*

$$\sup_{M_t}|\nabla^m A| \leq C(m)$$

where $C(m)$ depends on m, n, c_1 the gradient bound, and $\sup_{M_0}|\nabla^j A|$ for $0 \leq j \leq m$.

Proof. We know by Corollary 5.1 that result holds for $m = 0$. So we proceed by induction on m. Suppose we have the result for $m-1$. Then there is a constant B, depending on m, n, C_0, and M_0 such that

$$\left(\frac{d}{dt}-\Delta\right)|\nabla^m A|^2 \leq B(1+|\nabla^m A|^2).$$

Now we add enough of the evolution equation of $|\nabla^{m-1}A|^2$ to control the right hand side. By Theorem 5.1 we have that

$$\left(\frac{d}{dt}-\Delta\right)(|\nabla^m A|^2+B|\nabla^{m-1}A|^2)\leq -B|\nabla^m A|^2+B_1,$$

where B_1 depends on B and $C(j)$ for $0\leq j\leq m-1$. Since $|\nabla^{m-1}A|^2$ is already bounded, this inequality implies that $|\nabla^m A|^2$ can be estimated uniformly in t by a constant depending on its initial data and on B and B_1. □

5.2 Longtime existence

To derive existence of a longtime solution for Lipschitz initial data, we will first need the following estimates interior in time for the curvature and all its derivatives, as done by the work of Ecker and Huisken.

Proposition 5.2. *Let M_t be a smooth solution of (1.2) with bounded gradient, $v\leq c_1$. Then for each $m\geq 0$ there is a constant $C(m)$ depending on c_1, n, and m such that*

$$t^{m+1}|\nabla^m A|^2\leq C(m)$$

holds uniformly on M_t.

Proof. For the case $m=0$ we compute from the results above, the evolution equation

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)(2t|A|^2v^2+v^2) &\leq -\frac{2}{v}\nabla v\cdot\nabla(2t|A|^2v^2)-6|\nabla v|^2\\ &\leq -\frac{2}{v}\nabla v\cdot\nabla(2t|A|^2v^2)-4|\nabla v|^2\\ &= -\frac{2}{v}\nabla v\cdot\nabla(2t|A|^2v^2+v^2).\end{aligned}$$

By the maximum principle we thus have that the estimate

$$2t|A|^2v^2+v^2\leq c_1^2$$

holds uniformly on M_t. We now proceed by induction on m. We have that for arbitrary power $l\geq 0$ the estimate:

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)(t^{l+1}|\nabla^l A|^2) \leq\ & (l+1)t^l|\nabla^l A|^2-2t^{l+1}|\nabla^{l+1}A|^2\\ &+C(n,l)t^{l+1}\sum_{i+j+k=l}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^l A|.\end{aligned}$$

Suppose the result holds for $m-1$, then we can estimate the last term as

$$\begin{aligned} t^{l+1}\sum_{i+j+k=l}|\nabla^i A||\nabla^j A||\nabla^k A||\nabla^l A| &\leq Ct^{l+1}\sum_{i+j+k=l} t^{-i/2-j/2-1}|\nabla^k A||\nabla^l A| \\ &\leq Ct^{l/2}\sum_{k\leq l} t^{k/2}|\nabla^k A||\nabla^l A| \\ &\leq C\sum_{k\leq l} t^k|\nabla^k A|^2, \end{aligned}$$

where the constant $C = C(n, l, c_1)$. Then we obtain for $l \leq m$ the estimate

$$\left(\frac{d}{dt}-\Delta\right)(t^{l+1}|\nabla^l A|^2) \leq -2t^{l+1}|\nabla^{l+1}A|^2 + C\sum_{k\leq l} t^k|\nabla^k A|^2.$$

We can therefore choose a constant k_1 big enough so that

$$\begin{aligned} \left(\frac{d}{dt}-\Delta\right)(t^{m+1}|\nabla^m A|^2 + k_1 t^m|\nabla^{m-1}A|^2) &\leq C\sum_{k\leq m} t^k|\nabla^k A|^2 - 2k_1 t^m|\nabla^m A|^2 \\ &\quad + C\sum_{k\leq m-1} t^k|\nabla^k A|^2 \\ &\leq C\sum_{k\leq m-1} t^k|\nabla^k A|^2. \end{aligned}$$

We then proceed similarly choosing constants $k_2, k_3, \ldots, k_m$ such that

$$\left(\frac{d}{dt}-\Delta\right)\left(t^{m+1}|\nabla^m A|^2 + \sum_{i=1}^m k_i t^{m+1-i}|\nabla^{m-i}A|^2\right) \leq C|A|^2.$$

The last term we can control by using the evolution equation of v^2 and selecting a constant k_{m+1} big enough such that:

$$\left(\frac{d}{dt}-\Delta\right)\left(t^{m+1}|\nabla^m A|^2 + \sum_{i=1}^m k_i t^{m+1-i}|\nabla^{m-i}A|^2 + k_{m+1}v^2\right) \leq 0.$$

The result then follows from the Maximum principle. □

Using the above proposition, Ecker and Huisken obtain the existence of a longtime solution for Lipschitz initial data:

Theorem 5.3. *If the initial hypersurface M_0 is Lipschitz continuous and satisfies*

$$\sup_{M_0} v \leq c_1,$$

then the mean curvature flow problem (1.2) has a longtime solution for all $t > 0$ and satisfies a priori estimates in Corollary 4.1 and Proposition 5.2.

5.3 Spatial decay

After obtaining decay estimates in time we are now able to show also that initial spatial decay behaviour is preserved during the course of the evolution. If we consider scaled solutions $(M_s^\rho)_{s\in(0,1)}$, where

$$M_s^\rho = \frac{1}{\rho} M_{\rho^2 s}$$

where we may choose $\rho = 1$, then the second fundamental form of (M_s^ρ) given by A_ρ satisfies a scaling property which we obtain by setting $x = \rho y$ and $t = \rho^2 s$. We then have for $x \in M_t$ and $y \in M_s^\rho$ that

$$|\nabla^m A_\rho(y)|^2 = \rho^{2(m+1)} |\nabla^m A(x)|^2,$$

so that the statements

$$|\nabla^m A_\rho(y)|^2 \leq c_2(m)$$

for $y \in M_s^\rho \cap B_{1/2}$, $s \in (3/4, 1)$ and

$$|\nabla^m A(x)|^2 \leq \frac{c_2(m)}{\rho^{2(m+1)}}$$

for $x \in M_t \cap B_{\rho/2}$, $t \in (3/4\rho^2, \rho^2)$ are equivalent. In view of this we propose the following proposition which satisfies the correct scaling of the second fundamental form:

Proposition 5.3. *Let M_t be a smooth solution of (1.2), satisfying $v \leq c_1$, and the additional assumption*

$$|\nabla^m A|^2 \leq c_2(m) \frac{\log^{\delta(m+1)}(e + |x|^2)}{(e + |x|^2)^{m+1}}$$

at time $t = 0$, $m \geq 0$ and $\delta \geq 0$. Then for all $t > 0$

$$|\nabla^m A|^2 \leq C_m \frac{\log^{\delta(m+1)}\left(e + \left(\sqrt{|x|^2 + 2nt} - \sqrt{\beta t}\right)^2\right)}{\left(e + \left(\sqrt{|x|^2 + 2nt} - \sqrt{\beta t}\right)^2\right)^{m+1}}$$

where $\beta = \beta(c_1) > 0$ and $C_m = C_m(n, m, c_1, c_2(0), \ldots, c_2(m))$.

Proof idea. Let $g = |A|^2 v^2 f(\eta) + Lv^2$ where $f(\eta)$ is an arbitrary non-negative function and $L > 0$ to be determined later. For the case $m = 0$ we thus have:

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g &= f(\eta)\left(\frac{d}{dt} - \Delta\right)|A|^2 v^2 + |A|^2 v^2 \left(\frac{d}{dt} - \Delta\right) f(\eta) \\ &\quad - 2\nabla f(\eta) \cdot \nabla(|A|^2 v^2) + L\left(\frac{d}{dt} - \Delta\right) v^2 \\ &\leq -2\frac{f(\eta)}{v}\nabla v \cdot \nabla(|A|^2 v^2) + |A|^2 v^2 \left(\frac{d}{dt} - \Delta\right) f(\eta) \\ &\quad - 2\nabla f(\eta) \cdot \nabla(|A|^2 v^2) - 2L(|A|^2 v^2 - 3|\nabla v|^2).\end{aligned}$$

Note that we have

$$\nabla g = f(\eta)\nabla(|A|^2 v^2) + |A|^2 v^2 \nabla f(\eta) + 2vL\nabla v,$$

so that we if we multiply both sides by $-2(\nabla v)/v$ we end up with:

$$-\frac{2}{v}\nabla v \cdot \nabla g = -2\frac{f}{v}\nabla v \cdot \nabla(|A|^2 v^2) - 2|A|^2 v \nabla v \cdot \nabla f - 4L|\nabla v|^2.$$

Similarly multiplying both sides by $-2(\nabla f)/f$ gives:

$$-\frac{2}{f}\nabla f \cdot \nabla g = -2\nabla f \cdot \nabla(|A|^2 v^2) - 2\frac{|\nabla f|^2}{f}|A|^2 v^2 - 4L\frac{v}{f}\nabla v \cdot \nabla f.$$

Substituting the above equations into the estimate thus gives

$$\begin{aligned}\left(\frac{d}{dt} - \Delta\right) g &\leq -2\left(\frac{\nabla v}{v} + \frac{\nabla f}{f}\right) \cdot \nabla g + |A|^2 v^2 \left(\frac{2}{v}\nabla f \cdot \nabla v + \frac{2}{f}|\nabla f|^2\right. \\ &\quad \left. + \left(\frac{d}{dt} - \Delta\right) f - 2L\right) + 4L\frac{v}{f}\nabla v \cdot \nabla f - 2L|\nabla v|^2.\end{aligned}$$

By Young's inequality we have that

$$\left|4L\frac{v}{f}\nabla v \cdot \nabla f\right| \leq 2L\frac{v^2}{f^2}|\nabla f|^2 + 2L|\nabla v|^2,$$

and we estimate the vector $2v^{-1}\nabla v$ using the inequality $v^{-1}|\nabla v| \leq |A|v$, and from the Proposition for the long-time existence for the case $m = 0$ and the fact $C(0) = c_1^2/2$. This together then implies that

$$\frac{2}{v}\nabla v \cdot \nabla f \leq c_1\sqrt{\frac{2}{t}}|\nabla f|.$$

Thus we finally have the estimate

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g \quad &\le \quad -2\left(\frac{\nabla v}{v}+\frac{\nabla f}{f}\right)\cdot\nabla g+|A|^2v^2\left(c_1\sqrt{\frac{2}{t}}|\nabla f|+\frac{2}{f}|\nabla f|^2\right.\\ &\quad \left.+\left(\frac{d}{dt}-\Delta\right)f-2L\right)+2L\frac{v^2}{f^2}|\nabla f|^2. \end{aligned} \tag{5.2}$$

Now we define

$$\eta(x,t)=e+\left(\sqrt{|x|^2+2nt}-\sqrt{\beta t}\right)^2$$

where $\beta>0$ will be chosen later. Recall that we have the inequality (3.2)

$$|\nabla\eta|^2\le 4\eta$$

and also in view of the fact that $(d/dt-\Delta)(|x|^2+2nt)=0$ we have

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\eta &= 2\left(\sqrt{|x|^2+2nt}-\sqrt{\beta t}\right)\left(\frac{d}{dt}-\Delta\right)\sqrt{|x|^2+2nt}\\ &\quad -2\left|\nabla\sqrt{|x|^2+2nt}\right|^2+\beta-\sqrt{\frac{\beta}{t}(|x|^2+2nt)}\\ &= \left(\sqrt{|x|^2+2nt}-\sqrt{\beta t}\right)\frac{\left|\nabla|x|^2\right|^2}{2(|x|^2+2nt)^{3/2}}\\ &\quad -\frac{\left|\nabla|x|^2\right|^2}{2(|x|^2+2nt)}+\beta-\sqrt{\frac{\beta}{t}(|x|^2+2nt)}\\ &= \frac{\left|\nabla|x|^2\right|^2}{2(|x|^2+2nt)}-\sqrt{\beta t}\frac{\left|\nabla|x|^2\right|^2}{2(|x|^2+2nt)^{3/2}}\\ &\quad -\frac{\left|\nabla|x|^2\right|^2}{2(|x|^2+2nt)}+\beta-\sqrt{\frac{\beta}{t}(|x|^2+2nt)}\\ &\le \beta-\sqrt{\frac{\beta}{t}(|x|^2+2nt)}. \end{aligned} \tag{5.3}$$

Proof of Proposition 5.3. As stated above, for the case $m=0$ if $g=|A|^2v^2f(\eta)+Lv^2$, where $f(\eta)$ is an arbitrary non-negative function and $L>0$ to be determined later, we have the evolution equation of g given by (5.2). Now we define

$$f(\eta(x,t))=\frac{\eta(x,t)}{\log^{\delta}\eta(x,t)},\qquad f'(\eta)=\frac{\log\eta-\delta}{\log^{\delta+1}\eta},\qquad f''(\eta)=\frac{\delta(1+\delta-\log\eta)}{\eta\log^{\delta+2}\eta},$$

and for some $\beta > 0$ to be chosen later and $\eta(x,t)$ whose evolution equation is given by (5.3).

Thus we begin by estimating the terms in (5.2) by first calculating the evolution of $f(\eta)$:

$$\begin{aligned}
\left(\frac{d}{dt}-\Delta\right) f(\eta) &= \eta\left(\frac{d}{dt}-\Delta\right)\log^{-\delta}\eta + \log^{-\delta}\eta\left(\frac{d}{dt}-\Delta\right)\eta \\
&\quad - 2\nabla\eta\cdot\nabla\log^{-\delta}\eta \\
&= -\delta\log^{-\delta-1}\eta\left(\frac{d}{dt}-\Delta\right)\eta - \frac{\delta}{\eta}|\nabla\eta|^2\log^{-\delta-1}\eta \\
&\quad - \frac{\delta(\delta+1)}{\eta}|\nabla\eta|^2\log^{-\delta-2}\eta + \log^{-\delta}\eta\left(\frac{d}{dt}-\Delta\right)\eta \\
&\quad + 2\frac{\delta}{\eta}|\nabla\eta|^2\log^{-\delta-1}\eta \\
&= \left(1-\frac{\delta}{\log\eta}\right)\log^{-\delta}\eta\left(\frac{d}{dt}-\Delta\right)\eta \\
&\quad + \delta\left(1-\frac{\delta+1}{\log\eta}\right)\frac{|\nabla\eta|^2}{\eta}\log^{-\delta-1}\eta \\
&\le \left(\frac{d}{dt}-\Delta\right)\eta + 4\delta \le 4\delta + \beta - \sqrt{\frac{\beta}{t}(|x|^2+2nt)},
\end{aligned}$$

where we use the inequality (3.2) $|\nabla\eta|^2 \le 4\eta$ and the fact that $\log\eta \ge 1$. The next term we estimate is:

$$\begin{aligned}
\frac{2}{f}|\nabla f|^2 &= 2\frac{f}{\eta^2}|\nabla\eta|^2 + 2\delta^2\frac{f}{\eta^4\log^2\eta}|\nabla\eta|^2 \\
&\quad - 4\delta\frac{f}{\eta^3\log\eta}|\nabla\eta|^2 \\
&\le 8\frac{f}{\eta} + 8\delta^2\frac{f}{\eta^3\log^2\eta} \le 8(1+\delta^2/e^2),
\end{aligned}$$

which also gives us that

$$2L\frac{v^2}{f^2}|\nabla f|^2 \le 8(1+\delta^2/e^2)L\frac{v^2}{f}.$$

Finally we estimate

$$|\nabla f| \le |\nabla\eta|\log^{-\delta}\eta - \frac{\delta}{\eta}|\nabla\eta|\log^{-\delta-1}\eta \le |\nabla\eta| \le 2\sqrt{|x|^2+2nt} + 2\sqrt{\beta t},$$

which we use to get

$$c_1\sqrt{\frac{2}{t}}|\nabla f| \le c_1\sqrt{\frac{2}{t}}|\nabla \eta| \le 2c_1\sqrt{\frac{2}{t}(|x|^2+2nt)}+2c_1\sqrt{2\beta}.$$

We therefore have the final estimate for g by substituting the above estimates into (5.2):

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)g &\le -2\left(\frac{\nabla v}{v}+\frac{\nabla f}{f}\right)\cdot\nabla g+|A|^2v^2\Big(2c_1\sqrt{2\beta}+\beta+4\delta \\ &\quad +8(1+\delta^2/e^2)-2L\Big)-\sqrt{\frac{|x|^2+2nt}{t}}\left(\sqrt{\beta}-2\sqrt{2}c_1\right)|A|^2v^2 \\ &\quad +8(1+\delta^2/e^2)L\frac{v^2}{f(\eta)} \\ &\le \vec{b}\cdot\nabla g+|A|^2v^2\left(2c_1\sqrt{2\beta}+\beta+4\delta+8(1+\delta^2/e^2)-2L\right) \\ &\quad +8(1+\delta^2/e^2)L\frac{v^2}{f(\eta)},\end{aligned}$$

for some large enough $\beta=\beta(c_1)$, where we define

$$\vec{b}=-2\left(\frac{\nabla v}{v}+\frac{\nabla f}{f}\right).$$

If we now choose L large depending on β, c_1 and δ, and define $k=\sup_{M_0}g+9(1+\delta^2/e^2)Lc_1^2$, we obtain

$$\left(\frac{d}{dt}-\Delta\right)g\le\vec{b}\cdot\nabla g-\frac{g-k}{f(\eta)}$$

where we have used the estimate $v(x,t)\le c_1$ once again. Now let $g_k=\max(g-k,0)$, and since $g_k\cdot(g-k)=g_k^2$, we obtain the result using the Maximum Principle with g_k^2.

For the case $m=1$, we compute as in the previous proposition the evolution

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)|\nabla A|^2f^2(\eta) &= f^2\left(\frac{d}{dt}-\Delta\right)|\nabla A|^2+|\nabla A|^2\left(\frac{d}{dt}-\Delta\right)f^2(\eta) \\ &\quad -2\nabla f^2\cdot\nabla|\nabla A|^2 \\ &\le c(n)|A|^2|\nabla A|^2f^2(\eta)-2|\nabla^2A|^2f^2(\eta) \\ &\quad +8\left(\frac{\log\eta-\delta}{\log^{\delta+1}\eta}\right)^2|\nabla\eta|^2|\nabla A|^2+2|\nabla^2A|^2f^2.\end{aligned}$$

Since by (3.2) $|\nabla\eta|^2 \leq 4\eta$ and $|A|^2 f(\eta) \leq C_0$ (recall that $v \leq 1$) we estimate

$$\left(\frac{d}{dt} - \Delta\right) |\nabla A|^2 f^2(\eta) \leq c(n, \delta, C_0)|\nabla A|^2 f(\eta).$$

Similarly we derive

$$\left(\frac{d}{dt} - \Delta\right) |A|^2 f(\eta) \leq -|\nabla A|^2 f(\eta) + c(n, \delta, C_0)|A|^2.$$

Also recall that for $v \geq 1$ one has

$$\left(\frac{d}{dt} - \Delta\right) v^2 \leq -2|A|^2$$

so that if we choose large enough positive constants K and L depending on n, δ and C_0 we have that

$$\left(\frac{d}{dt} - \Delta\right) (|\nabla A|^2 f^2(\eta) + K|A|^2 f(\eta) + Lv^2) \leq 0.$$

The proposition for $m = 1$ then follows from the Maximum Principle. We iterate over m similarly to prove the general statement. □

Chapter 6

Asymptotic behaviour

In this chapter we study the behaviour of our surfaces as the time t goes to infinity. Through out this chapter we will assume certain conditions on our initial hypersurface. We will state these assumptions before we use them below.

As our initial graph evolves under mean curvature flow, it will move off to infinity with speed proportional to $1/\sqrt{t}$, so studying their global shape as time goes to infinity will give us no insight, unless we rescale the surfaces back and prevent them from diverging to infinity.

We therefore define the following rescaling:

$$\tilde{F}(s) = \frac{F(t)}{\sqrt{2t+1}}$$

where the new time variable is given by

$$s = \frac{1}{2}\log(2t+1)$$

for $0 \leq s < \infty$. The rescaled mean curvature flow then becomes

$$\frac{\partial \tilde{F}}{\partial s} = \vec{\tilde{H}} - \tilde{F} \tag{6.1}$$

with the same initial condition

$$\tilde{F}(\cdot, 0) = F(\cdot, 0).$$

Now for the rescaled surfaces denoted by $\tilde{M}_s = \tilde{F}(\cdot, s)(M)$ we have the following result:

Theorem 6.1. *Suppose M_0 satisfies the linear growth condition (5.1) and has bounded curvature. If in addition it satisfies*

$$\langle x, \nu\rangle^2 \le c\frac{1+|x|^2-u^2}{\log^\delta(e+|x|^2)} \tag{6.2}$$

for some constant $c<\infty$ and some power $\delta>0$ then the solutions $\tilde{M}_s$ of the rescaled mean curvature flow (6.1) converge as $s\to\infty$ to a limiting surface $\tilde{M}_\infty$ which is self-similar, i.e. it satisfies

$$F^\perp = \vec{H}.$$

Before we prove this we show that the up to a time dependent factor, the condition (6.2) is preserved for all time.

Lemma 6.1. *Suppose our initial graph M_0 has bounded gradient and curvature and we have*

$$\langle x, \nu\rangle^2 \le c\frac{1+|x|^2-u^2}{\log^\delta(e+|x|^2)}$$

for some constant $c<\infty$ and positive $\delta\ge 0$, then for all $t>0$, M_t also satisfies

$$\langle x, \nu\rangle^2 \le c(t)\frac{1+|x|^2-u^2+2nt}{\log^\delta(e+|x|^2+2nt)}.$$

Proof. Let $f=\langle x,\nu\rangle$, then we have

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)f^2 &= 2f^2|A|^2-4Hf-2|\nabla f|^2\\ &\le C(f^2+1)-2|\nabla f|^2.\end{aligned}$$

Also if we define

$$\eta_1 = e+|x|^2+2nt \quad \text{and} \quad \eta_2 = 1+|x|^2-u^2+2nt$$

then by the product-rule for the Heat operator,

$$\begin{aligned}\left(\frac{d}{dt}-\Delta\right)\log^\delta\eta_1 &= \delta\frac{|\nabla\eta_1|^2}{\eta_1^2}\log^{\delta-1}\eta_1-\delta(\delta-1)\frac{|\nabla\eta_1|^2}{\eta_1^2}\log^{\delta-2}\eta_1\\ &= \delta\left(\frac{1}{\log\eta_1}-\frac{\delta-1}{\log^2\eta_1}\right)\frac{|\nabla\eta_1|^2}{\eta_1^2}\log^\delta\eta_1\\ &\le 4\delta\left(\frac{1}{\log\eta_1}-\frac{\delta-1}{\log^2\eta_1}\right)\frac{1}{\eta_1}\log^\delta\eta_1 \le C\log^\delta\eta_1,\end{aligned}$$

where we have used the inequality $|\nabla\eta_1|^2 \leq 4\eta_1$ and denoted all constants which depend on the curvature bound and t by C. Similarly we have

$$\left(\frac{d}{dt} - \Delta\right)\eta_2^{-1} = -2\frac{|\nabla\eta_2|^2}{\eta_2^3} - 2\frac{|\nabla u|^2}{\eta_2^2},$$

so that

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\frac{f^2}{\eta_2} &= \frac{1}{\eta_2}\left(\frac{d}{dt} - \Delta\right)f^2 + f^2\left(\frac{d}{dt} - \Delta\right)\frac{1}{\eta_2} - 2\nabla f^2 \cdot \nabla\frac{1}{\eta_2} \\
&\leq \frac{C}{\eta_2}(f^2+1) - 2\frac{|\nabla f|^2}{\eta_2} - 2f^2\frac{|\nabla\eta_2|^2}{\eta_2^3} - 2f^2\frac{|\nabla u|^2}{\eta_2^2} \\
&\quad + 4\frac{f}{\eta_2^2}\nabla f \cdot \nabla\eta_2 \\
&\leq \frac{C}{\eta_2}(f^2+1) - 2f^2\frac{|\nabla u|^2}{\eta_2^2} \leq \frac{C}{\eta_2}(f^2+1),
\end{aligned}$$

where we have used Young's inequality

$$\left|4\frac{f}{\eta_2^2}\nabla f \cdot \nabla\eta_2\right| \leq 2\frac{|\nabla f|^2}{\eta_2} + 2f^2\frac{|\nabla\eta_2|^2}{\eta_2^3}.$$

Therefore since $\eta_2^{-1}\log^\delta\eta_1 \leq c$, some constant $c > 0$ we have:

$$\begin{aligned}
\left(\frac{d}{dt} - \Delta\right)\frac{f^2}{\eta_2}\log^\delta\eta_1 &= \frac{f^2}{\eta_2}\left(\frac{d}{dt} - \Delta\right)\log^\delta\eta_1 + \log^\delta\eta_1\left(\frac{d}{dt} - \Delta\right)\frac{f^2}{\eta_2} \\
&\quad - 2\nabla\frac{f^2}{\eta_2} \cdot \nabla\log^\delta\eta_1 \\
&\leq C\left(\frac{f^2}{\eta_2}\log^\delta\eta_1 + 1\right) - 2\nabla\frac{f^2}{\eta_2} \cdot \nabla\log^\delta\eta_1.
\end{aligned}$$

Now we calculate

$$-2\nabla\frac{f^2}{\eta_2} \cdot \nabla\log^\delta\eta_1 = 2\delta\frac{f^2\log^{\delta-1}\eta_1}{\eta_1\eta_2^2}\nabla\eta_1 \cdot \nabla\eta_2 - 4\delta\frac{f\log^{\delta-1}\eta_1}{\eta_1\eta_2}\nabla f \cdot \nabla\eta_1,$$

and once again by Young's inequality we can estimate the first term by

$$\begin{aligned}
\left|2\delta\frac{f^2\log^{\delta-1}\eta_1}{\eta_1\eta_2^2}\nabla\eta_1 \cdot \nabla\eta_2\right| &\leq \frac{f^2}{\eta_2}\log^\delta\eta_1\left(4\delta^2\frac{|\nabla\eta_1|^2}{\eta_1^2\log^2\eta_1} + \frac{|\nabla\eta_2|^2}{\eta_2^2}\right) \\
&\leq C\frac{f^2}{\eta_2}\log^\delta\eta_1,
\end{aligned}$$

since $|\nabla \eta_1|^2 \leq 4\eta_1$ and similarly $|\nabla \eta_2|^2 \leq 4\eta_2$. Thus dropping the negative term gives us the following estimate:

$$\left(\frac{d}{dt} - \Delta\right) \frac{f^2}{\eta_2} \log^\delta \eta_1 \leq C \left(\frac{f^2}{\eta_2} \log^\delta \eta_1 + 1\right),$$

which by the Maximum Principle implies the result that we require. □

In order to prove the theorem we will need to know the evolution equation of a test-function in the rescaled case. The test function we use will have the following general form

$$\tilde{\rho}(\tilde{x}, s) = g(\tilde{x}, s) h(s)$$

where

$$g(\tilde{x}, s) = \frac{\log^\delta(\tilde{\eta}_1)}{\tilde{\eta}_2^p},$$

for some positive powers $\delta > 0$ and $p > 0$ to be specified later.

The heat operator of $\tilde{\rho}$ is

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) \tilde{\rho}(\tilde{x}, s) = h(s) \left(\frac{d}{ds} - \tilde{\Delta}\right) g(\tilde{x}, s) + h'(s) g(\tilde{x}, s)$$

and since

$$\begin{aligned}
\left(\frac{d}{ds} - \tilde{\Delta}\right) g(\tilde{x}, s) &= \frac{1}{\tilde{\eta}_2^p} \left(\frac{d}{ds} - \tilde{\Delta}\right) \log^\delta \tilde{\eta}_1 + \log^\delta \tilde{\eta}_1 \left(\frac{d}{ds} - \tilde{\Delta}\right) \frac{1}{\tilde{\eta}_2^p} \\
&\quad - 2\nabla \log^\delta \tilde{\eta}_1(\tilde{\eta}_1) \cdot \nabla \frac{1}{\tilde{\eta}_2^p} \\
&= \delta \frac{\log^{\delta-1} \tilde{\eta}_1}{\tilde{\eta}_2^p} \left(\frac{d}{ds} - \tilde{\Delta}\right) \tilde{\eta}_1 - \delta(\delta - 1) \frac{|\nabla \tilde{\eta}_1|^2}{\tilde{\eta}_1^2 \tilde{\eta}_2^p} \log^{\delta-2} \tilde{\eta}_1 \\
&\quad - p \frac{\log^\delta \tilde{\eta}_1}{\tilde{\eta}_2^{p+1}} \left(\frac{d}{ds} - \tilde{\Delta}\right) \tilde{\eta}_2 - p(p+1) \frac{|\nabla \tilde{\eta}_2|^2}{\tilde{\eta}_2^{p+2}} \log^\delta \tilde{\eta}_1 \\
&\quad - 2\nabla \log^\delta \tilde{\eta}_1 \cdot \nabla \tilde{\eta}_2^{-p}
\end{aligned}$$

we get the following:

Proposition 6.1. *For twice differentiable functions $\tilde{\eta}_1(\tilde{x}, s)$ and $\tilde{\eta}_2(\tilde{x}, s)$ and continuous $h(s)$, such that*

$$\tilde{\rho}(\tilde{x}, s) = g(\tilde{x}, s) h(s) = \frac{\log^\delta \tilde{\eta}_1}{\tilde{\eta}_2^p} h(s)$$

for some positive powers $\delta > 0$ and $p > 0$, we have

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\rho}(\tilde{x},s) \quad \leq \quad & \delta\frac{\tilde{\rho}}{\log\tilde{\eta}_1}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\eta}_1 - \delta(\frac{\delta}{2} - 1)\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho} \\ & - p\frac{\tilde{\rho}}{\tilde{\eta}_2}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\eta}_2 + p(p-1)\frac{|\nabla\tilde{\eta}_2|^2}{\tilde{\eta}_2^2}\tilde{\rho} \\ & + h'(s)g(\tilde{x},s).\end{aligned}$$

Proof. The crucial ingredient of the proof comes from expanding the term

$$-2h(s)\nabla\log^\delta\tilde{\eta}_1\cdot\nabla\tilde{\eta}_2^{-p} = 2\delta p\frac{\tilde{\rho}}{\tilde{\eta}_1\tilde{\eta}_2\log\tilde{\eta}_1}\nabla\tilde{\eta}_1\cdot\nabla\tilde{\eta}_2,$$

which by Peter-Paul's inequality gives:

$$\left|2\delta p\frac{\tilde{\rho}}{\tilde{\eta}_1\tilde{\eta}_2\log\tilde{\eta}_1}\nabla\tilde{\eta}_1\cdot\nabla\tilde{\eta}_2\right| \leq 2p^2\frac{|\nabla\tilde{\eta}_2|^2}{\tilde{\eta}_2^2}\tilde{\rho} + \frac{1}{2}\delta^2\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho}.$$

Therefore since

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\rho}(\tilde{x},s) \quad = \quad & \delta\frac{\tilde{\rho}}{\log\tilde{\eta}_1}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\eta}_1 - \delta(\delta - 1)\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho} \\ & - p\frac{\tilde{\rho}}{\tilde{\eta}_2}\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\eta}_2 - p(p+1)\frac{|\nabla\tilde{\eta}_2|^2}{\tilde{\eta}_2^2}\tilde{\rho} + h'(s)g(\tilde{x},s) \\ & - 2\nabla\log^\delta\tilde{\eta}_1\cdot\nabla\frac{h(s)}{\tilde{\eta}_2^p},\end{aligned}$$

we obtain our result by using the above estimate. □

6.1 Proof of Theorem 6.1

The result of the Theorem will follow from the following estimate for some $0 < \gamma < 2$:

$$\sup_{\tilde{M}_s}\frac{(\tilde{H} + \langle\tilde{x},\tilde{\nu}\rangle)^2\tilde{v}^2}{\tilde{\eta}_2^p\log^{-\epsilon}\tilde{\eta}_1} \leq (1+s)^{-\gamma}\sup_{M_0}\frac{(H + \langle x,\nu\rangle)^2 v^2}{\eta_2^p\log^{-\epsilon}\eta_1}$$

where $0 < \epsilon < \delta$, $0 < p < 1$, and for some choice of test functions η_1 and η_2. Note that this implies *polynomial* convergence on compact subsets, instead of exponentially fast convergence, obtained in regard to the corresponding estimate of Ecker and Huisken.

We will make use of the following lemma from Ecker and Huisken [6]:

Lemma 6.2. *The normalised quantity $\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle$ satisfies the evolution equation*

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle) = (|\tilde{A}|^2 - 1)(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle).$$

Proof. To begin with note that $\tilde{H} = \psi^{-1}(t)H$ and $\langle \tilde{x}, \tilde{v} \rangle = \psi(t)\langle x, \nu \rangle$ where $\psi(t) = 1/\sqrt{2t+1}$ is the rescaling factor. As in Appendix B, we therefore say that H is of "degree" -1 and $\langle x, \nu \rangle$ is of "degree" 1. This together with the evolution equations

$$\left(\frac{d}{dt} - \Delta\right) H = |A|^2 H \qquad \text{and} \qquad \left(\frac{d}{dt} - \Delta\right)\langle x, \nu \rangle = |A|^2 \langle x, \nu \rangle - 2H,$$

and Lemma B.1 for calculating rescaled evolution equations gives us

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right)(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle) &= |\tilde{A}|^2(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle) - 2\tilde{H} + \tilde{H} - \langle \tilde{x}, \tilde{v} \rangle \\ &= (|\tilde{A}|^2 - 1)(\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle),\end{aligned}$$

which is the result we want. □

Similarly we have

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{v}^2 = -2|\tilde{A}|^2\tilde{v}^2 - 6|\nabla \tilde{v}|^2$$

which gives us the following inequality for $f^2 = (\tilde{H} + \langle \tilde{x}, \tilde{v} \rangle)^2 \tilde{v}^2$

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) f^2 \leq -2f^2 - 2\frac{1}{\tilde{v}}\nabla \tilde{v} \cdot \nabla f^2.$$

Multiplying this with a test function $\tilde{\rho}(\tilde{x}, s)$ we derive

$$\begin{aligned}\left(\frac{d}{ds} - \tilde{\Delta}\right) f^2 \tilde{\rho}(\tilde{x}, s) \leq\ & -2f^2\tilde{\rho} - 2\frac{\tilde{\rho}}{\tilde{v}}\nabla \tilde{v} \cdot \nabla f^2 + f^2\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\rho} \\ & - 2\nabla\tilde{\rho} \cdot \nabla f^2.\end{aligned}$$

Note that since $\nabla(f^2\tilde{\rho}) = \tilde{\rho}\nabla f^2 + f^2\nabla\tilde{\rho}$, we can write

$$\begin{aligned}-2\frac{\tilde{\rho}}{\tilde{v}}\nabla \tilde{v} \cdot \nabla f^2 - 2\nabla\tilde{\rho} \cdot \nabla f^2 =\ & -2\left(\frac{\nabla \tilde{v}}{\tilde{v}} + \frac{\nabla \tilde{\rho}}{\tilde{\rho}}\right)\nabla(f^2\tilde{\rho}) + 2\frac{f^2}{\tilde{v}}\nabla\tilde{\rho} \cdot \nabla \tilde{v} \\ & + 2\frac{f^2}{\tilde{\rho}}|\nabla\tilde{\rho}|^2,\end{aligned}$$

so that we end up with

$$\begin{aligned}\left(\frac{d}{ds}-\tilde{\Delta}\right)f^2\tilde{\rho}(\tilde{x},s) &\leq -2f^2\tilde{\rho}+f^2\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\rho}+2\frac{f^2}{\tilde{\rho}}|\nabla\tilde{\rho}|^2\\ &\quad -2\left(\frac{\nabla\tilde{v}}{\tilde{v}}+\frac{\nabla\tilde{\rho}}{\tilde{\rho}}\right)\nabla(f^2\tilde{\rho})+2\frac{f^2}{\tilde{v}}\nabla\tilde{\rho}\cdot\nabla\tilde{v}. \quad (6.3)\end{aligned}$$

Proof of Theorem 6.1. When we set $h(s)=(1+s)^\gamma$ for some positive γ, then we have that:

$$h'(s)=\frac{\gamma}{1+s}h(s),$$

thus giving us the following estimate for the evolution of $\tilde{\rho}$ via Proposition 6.1

$$\begin{aligned}\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\rho}(\tilde{x},s) &\leq \epsilon\frac{\tilde{\rho}}{\log\tilde{\eta}_1}\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\eta}_1+\epsilon(1-\frac{\epsilon}{2})\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho}\\ &\quad -p\frac{\tilde{\rho}}{\tilde{\eta}_2}\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\eta}_2+p(p-1)\frac{|\nabla\tilde{\eta}_2|^2}{\tilde{\eta}_2^2}\tilde{\rho}\\ &\quad +\frac{\gamma}{1+s}\tilde{\rho}.\end{aligned}$$

Now define $\tilde{\eta}_1$ and $\tilde{\eta}_2$ as

$$\tilde{\eta}_1=e+\alpha|\tilde{x}|^2 \qquad \text{and} \qquad \tilde{\eta}_2=1+\beta|\tilde{x}|^2-\beta\tilde{u}^2,$$

for some positive constants α and β to be determined later. We have that since both $|x|^2$ and u^2 are of "degree" 2, together with the fact that

$$\left(\frac{d}{dt}-\Delta\right)|x|^2=-2n \qquad \text{and} \qquad \left(\frac{d}{dt}-\Delta\right)u^2=-2|\nabla u|^2,$$

the heat operator of $\tilde{\eta}_1$ is given by:

$$\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\eta}_1=-2\alpha(|\tilde{x}|^2+n),$$

and that of $\tilde{\eta}_2$ by:

$$\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\eta}_2=-2\beta(|\tilde{x}|^2+n)+2\beta(|\nabla\tilde{u}|^2+\tilde{u}^2).$$

This then implies the following estimate for $0<p<1$, $0<\epsilon<\delta$, and $s>0$:

$$\begin{aligned}\left(\frac{d}{ds}-\tilde{\Delta}\right)\tilde{\rho} &\leq 2\left(p\beta-\frac{\epsilon\alpha}{\log\tilde{\eta}_1}\right)(|\tilde{x}|^2+n)\tilde{\rho}\\ &\quad +\frac{1}{2}\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho}+\frac{\gamma}{1+s}\tilde{\rho}\\ &\leq \frac{1}{2}\frac{|\nabla\tilde{\eta}_1|^2}{\tilde{\eta}_1^2\log^2\tilde{\eta}_1}\tilde{\rho}+\gamma\tilde{\rho},\end{aligned}$$

where we have chosen β such that $\beta \leq \epsilon\alpha/p \log \eta_1$ so that we can drop first term in the above inequality and that $\epsilon - \epsilon^2/2 \leq 1/2$ for $\epsilon > 0$.

Moreover we obtain from the estimates

$$|\nabla \tilde{\eta}_1|^2 \leq 4\alpha\tilde{\eta}_1 \qquad \text{and} \qquad |\nabla \tilde{\eta}_2|^2 \leq 4\beta\tilde{\eta}_2,$$

the following estimates:

$$\left(\frac{d}{ds} - \tilde{\Delta}\right)\tilde{\rho} \leq (2\alpha + \gamma)\tilde{\rho},$$

and

$$\begin{aligned} |\nabla \tilde{\rho}| &\leq \left(\epsilon \frac{|\nabla \tilde{\eta}_1|}{\tilde{\eta}_1 \log \tilde{\eta}_1} + p\frac{|\nabla \tilde{\eta}_2|}{\tilde{\eta}_2}\right)\tilde{\rho} \\ &\leq 2(\epsilon\sqrt{\alpha} + p\sqrt{\beta})\tilde{\rho} \leq 2\sqrt{\epsilon\alpha}(\sqrt{\epsilon} + \sqrt{p})\tilde{\rho}. \end{aligned}$$

which then gives

$$2\frac{f^2}{\tilde{\rho}}|\nabla\tilde{\rho}|^2 \leq 8\epsilon\alpha(\sqrt{\epsilon} + \sqrt{p})^2 f^2\tilde{\rho},$$

and together with the estimate $|\nabla \tilde{v}|/\tilde{v} \leq |\tilde{A}|\tilde{v} \leq c_1 c(0)$ also gives:

$$2\frac{f^2}{\tilde{v}}\nabla\tilde{\rho}\cdot\nabla\tilde{v} \leq c(c_1, c(0), n)\sqrt{\epsilon\alpha}(\sqrt{\epsilon} + \sqrt{p})f^2\tilde{\rho}.$$

Thus we finally have after substituting the above estimate into (6.3):

$$\begin{aligned} \left(\frac{d}{ds} - \tilde{\Delta}\right) f^2\tilde{\rho} &\leq \vec{a}\cdot\nabla(f^2\tilde{\rho}) + \big(2\alpha + \gamma + c\sqrt{\epsilon\alpha}(\sqrt{\epsilon} + \sqrt{p}) \\ &\quad + 8\epsilon\alpha(\sqrt{\epsilon} + \sqrt{p})^2 - 2\big) f^2\tilde{\rho}, \end{aligned}$$

where

$$\vec{a} = -2\left(\frac{\nabla\tilde{v}}{\tilde{v}} + \frac{\nabla\tilde{\rho}}{\tilde{\rho}}\right).$$

Choosing α, β, and γ suitably small depending on ϵ, p and c, we see that:

$$\left(\frac{d}{ds} - \tilde{\Delta}\right) f^2\tilde{\rho} \leq \vec{a}\cdot\nabla(f^2\tilde{\rho})$$

for all $s > 0$. Lemma 6.1 ensures that $f^2\tilde{\rho}$ vanishes at infinity which enables us to apply the parabolic maximum principle to conclude that $f^2\tilde{\rho}$ is uniformly bounded by its initial data.

Finally we use the result of Stavrou [16] to conclude uniform convergence to self-similar solutions, since our assumption is stronger than his. □

Appendix A

Derivation of Equations

In this chapter we wish to investigate the equation for mean curvature flow and derive some results which we will need.

Recall that for a properly embedded map $F\colon \Omega \to \mathbb{R}^{n+1}$, where $\Omega \subset \mathbb{R}^n$, the coordinate tangent vectors are denoted by $\partial_i F(p)$ and they form a basis of the tangent space $T_x M$ at $x = F(p)$ at every $p \in \Omega$. The metric on M is then defined by g_{ij}.

The *Riemann curvature tensor* of M is defined by

$$\nabla_i^M \nabla_j^M X_k - \nabla_j^M \nabla_i^M X_k = R_{ijkl}^M X^l$$

where X is a tangent vector field on M and the Hessian operator $\nabla_i^M \nabla_j^M$ is defined as

$$\nabla_i^M \nabla_j^M := \nabla_{\tau_i}^M \nabla_{\tau_j}^M - \nabla_{\nabla_{\tau_i}^M \tau_j}^M$$

where $\tau_1, \ldots, \tau_n$ are the basis of a local orthonormal frame. The Riemann tensor has the property that $R_{ijkl}^M = -R_{jikl}^M$ and $R_{ijkl}^M = -R_{klij}^M$, and the *Gauss equations* express this tensor in terms of the second fundamental form of M by:

$$R_{ijkl}^M = A_{ik} A_{jl} - A_{il} A_{jk}.$$

The *Codazzi equations* then says that the 3-tensor of covariant derivatives of the second fundamental form given by $\nabla^M A = \{\nabla_i^M A_{jk}\}$ is totally symmetric.

In the case of tensor fields, we denote the covariant derivatives, Hessian, and Laplacian operator analogously to the vector fields case. For instance in an orthonormal frame $\tau_1, \ldots, \tau_n$, we denote the component of $\nabla_i^M \nabla_j^M A$ with respect to τ_i by $\nabla_i^M \nabla_j^M A_{kl}$.

A.1 Preliminary Results

To begin with, we wish to calculate the Laplacian of the mean curvature, Laplacian of the second fundamental form and its squared norm and finally the Laplacian of the unit normal vector field. Recall that the mean curvature and the square of the second fundamental form are given by

$$H = A^i_i = g^{ij}A_{ij} \quad \text{and} \quad |A|^2 = A^i_j A^j_i = g^{ij}g^{kl}A_{ik}A_{jl}.$$

By working in a local orthonormal frame we can simply use the lower indices only, and we also use subscripts to denote derivatives where there use will be clear from the context of the computation. For example the Riemann curvature tensor can then be written as

$$X_{ijk} - X_{ikj} := \nabla^M_k \nabla^M_j X_i - \nabla^M_j \nabla^M_k X_i = R^M_{kjil} X_l = X_l R_{lijk}$$

by the symmetry of the Riemann tensor. Similarly

$$A_{ijkl} - A_{ijlk} := \nabla^M_l \nabla^M_k A_{ij} - \nabla^M_k \nabla^M_l A_{ij} = A_{im} R^M_{mjkl} + A_{mj} R^M_{imkl}.$$

Now by the Codazzi equations we have

$$\Delta_M A_{ij} = A_{ijkk} = A_{ikjk} = A_{kijk},$$

which by the above identity for the second fundamental form gives

$$\Delta_M A_{ij} = A_{kikj} + A_{km} R^M_{mijk} + A_{mi} R_{mkjk}.$$

Once again by the Codazzi equations and Gauss equation we get

$$\begin{aligned} \Delta_M A_{ij} &= A_{kkij} + A_{km}(A_{mj}A_{ik} - A_{mk}A_{ij}) + A_{mi}(A_{mj}A_{kk} - A_{mk}A_{kj}) \\ &= H_{ij} - |A|^2 A_{ij} + H A_{ik} A_{kj}, \end{aligned}$$

which is referred to as *Simons' identity* in its standard notation

$$\Delta_M A_{ij} = \nabla^M_i \nabla^M_j H - |A|^2 A_{ij} + H A_{ik} A^k_j.$$

Contracting the Simons' identity with $A^{ij} = g^{ik}g^{jl}A_{kl}$ then gives us the formula

$$\frac{1}{2}\Delta_M |A|^2 = A^{ij}\nabla^M_i \nabla^M_j H + |\nabla^M A|^2 + H A_{ij} A^i_k A^k_j - |A|^4,$$

where $|\nabla^M A|^2$ is the squared norm of the tensor $\{\nabla^M_k A_{ij}\}$.

In order to calculate the Laplacian of the vector field ν, it is more convenient to work with geodesic normal coordinates on M, i.e. the metric is

$g_{ij} = \delta_{ij}$ and the tangential component of $\partial_i \partial_j F$, at a point $x = F(p) \in M$ where we compute, is $(\partial_i \partial_j F)^T = 0$. Recall we have from the definition of the second fundamental form, $A_{ij} = -\partial_i \partial_j F \cdot \nu$, that $\partial_i \nu = \partial_i \nu \partial_j F \cdot \partial_j F = A_{ij} \partial_j F$. The Codazzi equations are simply $\partial_i A_{ij} = \partial_j A_{ii}$. Thus pointwise we have

$$\partial_i \partial_j \nu = \partial_i (A_{ij} \partial_j F) = \partial_i A_{ij} \partial_j F + A_{ij} \partial_i \partial_j F = \partial_j A_{ii} \partial_j F - A_{ij} A_{ij} \nu$$

from the Codazzi identity and definition. Thus we get:

$$\Delta_M \nu = -|A|^2 \nu + \nabla^M H,$$

which is referred to as the *Jacobi field equation*.

A.2 Metric and curvature

We will need to derive the time derivatives of the geometric quantities whose Laplacian we calculated in the previous section. This together with the Laplacian will then all us to calculate the Heat operator of various expressions, which we will need in our study.

Recall that if our manifold satisfies (1.2) then the metric evolves as

$$\partial_t g_{ij} = -2HA_{ij},$$

the inverse metric by contracting the above equation with g^{ij} gives

$$\partial_t g^{ij} = 2HA^{ij},$$

and the area element $\sqrt{g}$, then satisfies

$$\partial_t \sqrt{g} = -H^2 \sqrt{g} = -|\vec{H}|^2 \sqrt{g}.$$

To calculate the derivative of the second fundamental form A_{ij}, we once again do the calculation in geodesic normal coordinates, i.e. $g_{ij} = \delta_{ij}$ and $(\partial_i \partial_j F)^T = 0$ at the point $x = F(p,t) \in M$. Since $A_{ij} = -\partial_i \partial_j F \cdot \nu$ we get

$$\partial_t A_{ij} = -\partial_t (\partial_i \partial_j F \cdot \nu) = -\partial_i \partial_j \partial_t F \cdot \nu - \partial_i \partial_j F \cdot \partial_t \nu,$$

but since by $\partial_t \nu$ is tangential, the last term drops out. We use the evolution equation (1.2) to get:

$$\partial_t A_{ij} = \partial_i \partial_j H + H \partial_i \partial_j \nu \cdot \nu = \partial_i \partial_j H - H \partial_i \nu \cdot \partial_j \nu.$$

Since at the point $x = F(p,t)$ in normal coordinates, $\nabla_i^{M_t} \nabla_j^{M_t} H = \partial_i \partial_j H$, we have:

$$\partial_t A_{ij} = \nabla_i^{M_t} \nabla_j^{M_t} H - H A_{ik} A_j^k.$$

To Calculate the deviate of the normal field ν we use the fact that $\partial_t \nu$ is a tangential field in view of the fact that ν is a unit vector. We can then write $\partial_t \nu$ in terms of the the tangent vectors $\partial_j F$ to obtain

$$\partial_t \nu = g^{ij} \partial_t \nu \cdot \partial_j F \partial_i F = -g^{ij} \nu \cdot \partial_j (-H\nu) \partial_i F = g^{ij} \partial_j H \partial_i F,$$

where we have used the identity $\nu \cdot \partial_i F = 0$, the product rule and the evolution equation (1.2). By definition the last term is the tangential gradient of H denoted by

$$\partial_t \nu = \nabla^{M_t} H.$$

A.3 Heat operator of terms

We are now ready to calculate the Heat operator for the geometric terms we are interested in. By combining the results of the previous two sections we have in particular for $H = g^{ij} A_{ij}$

$$(\partial_t - \Delta_{M_t}) H = H|A|^2.$$

Contracting the evolution equation of A_{ij} with A^{ij}, we obtain

$$\partial_t |A|^2 = 2A^{ij} \nabla_i^{M_t} \nabla_j^{M_t} H + 2HA_{ik} A_j^k A^{ij},$$

which combined with the Laplacian of $|A|^2$ gives us

$$(\partial_t - \Delta_{M_t}) |A|^2 = 2|A|^4 - 2|\nabla^{M_t} A|^2.$$

The evolution of the unit normal ν is thus

$$(\partial_t - \Delta_{M_t}) \nu = |A|^2 \nu.$$

Appendix B

Rescaled mean curvature flow

In order to study the asymptotic behaviour of M_t as t gets larger, we will need to rescale the surface by keeping some geometric quantity fixed, for example the total area of the surfaces or the total enclosed volume. We do this by multiplying the solution F of (1.2) for each time $0 \leq t < \infty$ with a positive constant

$$\psi(t) = \frac{1}{\sqrt{2t+1}}$$

such that

$$\tilde{F}(s) = \psi(t)F(t)$$

where we have introduce a new time variable $0 \leq s < \infty$ given by

$$s(t) = \frac{1}{2}\log(2t+1),$$

such that

$$\frac{ds}{dt} = \psi^2.$$

The various geometric quantities then scale like

$$\tilde{g}_{ij} = \partial_i \tilde{F} \cdot \partial_j \tilde{F} = \psi^2 g_{ij}, \quad \tilde{A}_{ij} = \psi A_{ij}, \quad \tilde{H} = \psi^{-1} H, \quad |\tilde{A}|^2 = \psi^{-2}|A|^2,$$

etc. and the rescaled evolution equation for F is given by

$$\begin{aligned} \frac{d\tilde{F}}{ds} &= \psi^{-2}\frac{d\tilde{F}}{dt} = \psi^{-2}\frac{d\psi}{dt}F + \psi^{-1}\frac{dF}{dt} \\ &= -\tilde{H}\tilde{\nu} - \tilde{F}. \end{aligned} \tag{B.1}$$

To calculate the evolution of the rescaled area element we note that

$$\partial_s \tilde{g} = \tilde{g}\tilde{g}^{ij}\partial_s \tilde{g}_{ij},$$

thus giving us

$$\partial_s \sqrt{\tilde{g}} = \sqrt{\tilde{g}} \tilde{g}^{ij} \partial_i \partial_s \tilde{F} \cdot \partial_j \tilde{F} = \sqrt{\tilde{g}}\, \mathrm{div}_{\tilde{M}_s} \frac{\partial \tilde{F}}{\partial s} = -\sqrt{\tilde{g}}(\tilde{H}^2 + n).$$

For other evolution equations we can use the following lemma to compute the rescaled evolution.

Lemma B.1. *Suppose the expressions P and Q, formed from g and A, satisfy*

$$\frac{dP}{dt} = \pm \Delta P + Q,$$

and P has "degree" α, i.e. $\tilde{P} = \psi^{\alpha} P$. Then Q has degree $\alpha - 2$ and

$$\frac{d\tilde{P}}{ds} = \pm \tilde{\Delta} \tilde{P} + \tilde{Q} - \alpha \tilde{P}.$$

Proof. The proof follows from calculating

$$\begin{aligned} \frac{d\tilde{P}}{ds} &= \psi^{\alpha-2} \frac{dP}{dt} + \alpha \psi^{\alpha-3} \psi'(t) P \\ &= \pm \psi^{\alpha-2} \Delta P + \psi^{\alpha-2} Q - \alpha \psi^{\alpha} P \\ &= \pm \tilde{\Delta} \tilde{P} + \tilde{Q} - \alpha \tilde{P}, \end{aligned}$$

where we have used the fact that $\psi'(t) = -\psi^3(t)$. □

Using this lemma, we can then convert unchanged many results to the rescaled setting. For example a corollary to the above lemma and the Monotonicity Theorem of Huisken is that:

Corollary B.1. *If the surfaces $\tilde{M}_s$ satisfy the rescaled evolution equation (B.1), then we have*

$$\frac{d}{ds} \int_{\tilde{M}_s} \tilde{\Phi}\, d\tilde{\mu}_s = - \int_{\tilde{M}_s} \left| -\tilde{H}\tilde{\nu} - \frac{\nabla^{\perp} \tilde{\Phi}}{\tilde{\Phi}} \right|^2 \tilde{\Phi}\, d\tilde{\mu}_s,$$

for $\tilde{\Phi} = \psi^{-n} \Phi$, the rescaled backward heat kernel, given by

$$\tilde{\Phi}(\tilde{x} - \tilde{x}_0) = \exp\left(-\frac{|\tilde{x} - \tilde{x}_0|^2}{2} \right).$$

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