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A fermionic de Finetti theorem

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Quantum versions of de Finetti's theorem are powerful tools, yielding conceptually important insights into the security of key distribution protocols or tomography schemes and allowing one to bound the error made by mean-field approaches. Such theorems link the symmetry of a quantum state under the exchange of subsystems to negligible quantum correlations and are well understood and established in the context of distinguishable particles. In this work, we derive a de Finetti theorem for finite sized Majorana fermionic systems. It is shown, much reflecting the spirit of other quantum de Finetti theorems, that a state which is invariant under certain permutations of modes loses most of its anti-symmetric character and is locally well described by a mode separable state. We discuss the structure of the resulting mode separable states and establish in specific instances a quantitative link to the quality of the Hartree-Fock approximation of quantum systems. We hint at a link to generalized Pauli principles for one-body reduced density operators. Finally, building upon the obtained de Finetti theorem, we generalize and extend the applicability of Hudson's fermionic central limit theorem. *Published by AIP Publishing.* https://doi.org/10.1063/1.4998944

I. INTRODUCTION

Being first formulated for infinite systems of distinguishable particles,[1–3] a body of finite sized instances of *quantum de Finetti theorems* has been developed and improved in recent years.[4–14] Their essential and common feature is that they allow one to bound the suppression of quantum correlations in reduced states of quantum states that exhibit a *permutation invariance*. In their basic readings for finite systems,[4,5] they state that local reductions of a quantum state which is invariant under the exchange of parts of the system are in trace-distance well approximated by convex combinations of independent identically distributed (i.i.d.) product states. Triggered by these initial results, different ramifications have been explored. Relaxing, for instance, the assumption of obtaining i.i.d. product states allows one to consider large subsystems,[6,7] while changing the distance measure to the operational distinction using only local operations and classical communication (LOCC) norms alters the sensitivity of the resulting bounds to the dimension of the local Hilbert spaces from linear to logarithmic and can therefore be applied in more general settings.[11,13,14]

These results have gained a considerable attention in recent years specifically in the context of *quantum information theory*. Here, they are important as they yield insights into tomography problems,[3,6] are used to prove the general security of quantum key distribution protocols,[7,8] or allow one to analyze more general settings in hypothesis testing schemes.[15] At the same time, they give rise to quasipolynomial time algorithms for entanglement testing.[14]

In addition to these important uses for problems arising in quantum information theory, de Finetti theorems have key implications to problems in *quantum many-body physics*. They immediately yield bounds on the accuracy of *mean field approximations* employed on permutation invariant systems for distinguishable particles. In this context, it is even possible to lift the rather restrictive assumption of permutation invariance and one can derive bounds based on the connectivity of the systems' interaction graph[16] while maintaining much of the spirit of the original statement. In bosonic systems, naturally featuring a permutation invariance of particles, de Finetti theorems control the use of the well established mean field description based on the Gross-Pitaevskii equation (see, for instance, Refs.

17–22). What is more, bosonic Gaussian de Finetti theorems have been considered that resemble the results obtained here.[23]

For fermionic systems, the above de Finetti theorems when literally applied to first and second quantized readings are only of limited use due to the intrinsic anti-symmetry constraints of the fermionic states. As such, they also do not allow us to control and bound mean field solutions. This seems a particularly grave omission in the light of the fact that mean field approaches are key to our understanding of interacting fermionic systems. They constitute an essential tool to understand the fundamental properties of fermionic systems arising in the context of condensed matter theory and quantum chemistry. Most prominently, the *Hartree-Fock approximation*, the fermionic mean field approximation on the level of particles, is often able to capture properties of interacting systems surprisingly accurately and provides a starting point of more involved numerical and analytical approaches.[24–26] Next to the Hartree-Fock approximation, other mean field approaches based on generalized Gutzwiller wave functions[27] or on product states on the level of single particle modes[28,29] can be introduced. Understanding and bounding on a rigorous level the validity of these mean field approaches and revealing, just as in the case of distinguishable particles in Ref. 16, the underlying structures necessary for their success are therefore highly desirable. In fact, what seems urgently missing in many situations are performance guarantees for Hartree-Fock approaches.

As for bosonic or distinguishable particles, de Finetti type theorems promise a way forward here. In Refs. 29 and 30, de Finetti type theorems are provided for fermionic systems. These theorems and investigations characterize the set of states which are invariant under an arbitrary permutation of the fermionic modes in the thermodynamic limit. In both cases, a full permutation invariance in the state is assumed which combined with the canonical anti-symmetric structure of fermionic systems leads to cancellations in expectation values, as we will argue. What is more, precise bounds for finite-sized systems in trace norm are so far not in the focus of attention.

Extending and complementing the results, in this work, we derive a *fermionic mode de Finetti theorem for finite system sizes*, much in the spirit of Refs. 4 and 5. In addition, we show that we can derive our result without interfering by assumption with the anti-symmetry of fermionic states. By contrast, we find that given a relaxed version of permutation invariance of the fermionic state defined in detail below, the anti-symmetric character of the state vanishes in the same way as the quantum correlations. In addition, we discuss the structure of the obtained product states and relate them in special cases to fermionic Gaussian states. With this, we provide a stepping stone towards understanding and bounding the mean field approximation such as the Hartree-Fock approach in finite fermionic systems.

Further, we argue that our theorem naturally enables us to extend results that are originally formulated for i.i.d. product states, just as in the case of distinguishable particles. We make this notion explicit by discussing fermionic central limit theorems which in combination with our theorem yield the structural insight that permutation invariant states appear to be convex combinations of Gaussian states when probed on large scales.[31] By making this step, we provide instances in which central limit type arguments hold away from the case of i.i.d. product states or states with clustering correlations.[32,33]

This work is organized as follows. We start by introducing our setting and fixing the necessary notation and definitions. We then prove our main result, a mode de Finetti theorem for finite fermionic systems stated in Theorem 2. We conclude by discussing the structure of the obtained product states and implications for the approximation of permutation invariant ground states. In doing so, we reconsider Hudson's fermionic central limit theorem[31] for finite sized systems in the Appendix and show that a permutation invariant state is approximately a convex mixture of Gaussian states in Fourier space.

II. SETTING AND PREPARATION

A. Definitions

In the following, we consider a finite *fermionic lattice-system* with V sites and p fermionic modes per side. To each of the $K = Vp$ modes, we associate the creation and annihilation operators

$f_j^{\alpha\dagger}$ and f_j^{α} with $\alpha \in [p] = \{1, \ldots, p\}$ and $j \in [V]$, which fulfill the canonical *anti-commutation relation*

$$\{f_j^{\alpha\dagger}, f_k^{\beta}\} = \delta_{j,k}\delta_{\alpha,\beta}\mathbb{1}, \qquad \{f_j^{\alpha}, f_k^{\beta}\} = 0, \quad \forall j,k \in [V], \, \forall \alpha,\beta \in [p]. \tag{1}$$

It is convenient to introduce the *Majorana operators*

$$m_j^{2\alpha-1} = f_j^{\alpha\dagger} + f_j^{\alpha}, \tag{2}$$

$$m_j^{2\alpha} = i(f_j^{\alpha\dagger} - f_j^{\alpha}), \tag{3}$$

which satisfy the Majorana anti-commutation relation

$$\{m_x^{\alpha}, m_y^{\beta}\} = 2\delta_{x,y}\delta_{\alpha,\beta}\mathbb{1}. \tag{4}$$

We define the *parity operators* of a site $j \in [V]$ as

$$P_j = \prod_{\alpha\in[p]} \left(\mathbb{1} - 2f_j^{\alpha\dagger}f_j^{\alpha}\right) = (-i)^p \prod_{\alpha=1}^{p} m_j^{2\alpha-1}m_j^{2\alpha}. \tag{5}$$

We denote by $\mathcal{F}_K$ the *fermionic Fock space* of K modes and by $\mathcal{D}(\mathcal{F}_K)$ the set of fermionic states ρ on K modes respecting the *parity superselection rule*, i.e., $[\rho, \prod_{j\in[V]}P_j] = 0$.[28,34] With this, every state $\rho \in \mathcal{D}(\mathcal{F}_K)$ will be an even operator and have a vanishing expectation value with all $m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r}$ for odd r and hence any odd operator. For a permutation $\pi \in S_V$ and a given product of Majorana operators $m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r}$, we introduce the notation

$$\pi(m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r}) = m_{\pi(j_1)}^{\alpha_1} \ldots m_{\pi(j_r)}^{\alpha_r}, \tag{6}$$

which extends linearly to general fermionic operators as they can be uniquely expanded in the Majorana operator basis. We are now in the position to define a key concept, the *permutationally invariant fermionic states*.

Definition 1 (Permutation invariant fermionic state). *Given a fermionic system of V sites with p modes per sites and Majorana operators $\{m_x^{\alpha}\}_{(x,\alpha)\in[V]\times[p]}$, a fermionic state ρ respecting the fermionic superselection rule is called permutation invariant if it fulfills the following conditions*:

(1) *for all $(j_1, \alpha_1) < \ldots < (j_r, \alpha_r) \in ([V] \times [p])^{\times r}$ and $\pi \in S_V$ with $(\pi(j_1), \alpha_1) < \ldots < (\pi(j_r), \alpha_r)$ preserving that order, we have*

$$\mathrm{tr}\left(\rho\, m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r}\right) = \mathrm{tr}\left(\rho\, \pi(m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r})\right), \tag{7}$$

(2) *for all $(j_1, \alpha_1) < \ldots < (j_r, \alpha_r) \in ([V] \times [p])^{\times r}$ with $|\{\alpha_k | j_k = j\}|$ even for all $j \in [V]$ and all $\pi \in S_V$, we have*

$$\mathrm{tr}\left(\rho\, m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r}\right) = \mathrm{tr}\left(\rho\, \pi(m_{j_1}^{\alpha_1} \ldots m_{j_r}^{\alpha_r})\right), \tag{8}$$

where, for tuples, $<$ indicates a lexicographical ordering.

B. Preliminaries

Note that once we picked an arbitrary ordering of sites, the state is only permutation invariant with respect to general forward permutations which especially do not exchange odd operators [condition (1)] and the general permutation of even operators [condition (2)] which commute if supported on different sites. The finite sized version of the fully permutation invariant states considered in Refs. 29 and 30 fulfill this definition. Further, however, we allow for natural signs appearing for fermionic states which are lost for fully permutation invariant states. Considering in the simplest case $\mathrm{tr}(\rho\, m_1^1 m_2^1)$ and a permutation π that would exchange sites 1 and 2, we obtain for fully permutation invariant states and from the Majorana anti-commutation relations $\mathrm{tr}(\rho\, m_1^1 m_2^1) = \mathrm{tr}(\rho\, m_2^1 m_1^1) = -\mathrm{tr}(\rho\, m_1^1 m_2^1)$ which of course leads to $\mathrm{tr}(\rho\, m_1^1 m_2^1) = 0$. By not restricting these permutations explicitly in Definition 1, we implicitly allow for these natural signs and do not assume the corresponding expectation values to vanish trivially. By this, Definition 1 is more general than full permutation invariance as, for instance, the states

$$\rho = \frac{1}{(2^p)^V}\left(\mathbb{1} + i\tan\left(\frac{\pi}{2V}\right)\mu \sum_{\substack{j,l\in[V]:\\ j<l}} m_j^1 m_l^1\right), \tag{9}$$

with $\mu \in [-1, 1]$, have the non-trivial expectation values for $a \neq b$,

$$\mathrm{tr}(\rho\, m_a^1 m_b^1) = \begin{cases} i\tan\left(\frac{\pi}{2V}\right)\mu & \text{if } b < a \\ -i\tan\left(\frac{\pi}{2V}\right)\mu & \text{if } a < b \end{cases}. \tag{10}$$

Hence, the state is permutation invariant according to Definition 1 and our main result applies to it but fails to be fully permutation invariant for $\mu \neq 0$.

Note that replacing the full permutation invariance of a fermionic state by Definition 1 lifts further constraints next to allowing for non-trivial expectation values for $m_a^\alpha m_b^\alpha$ terms. The two expectation values $\mathrm{tr}(\rho\, m_a^\alpha m_b^\beta)$ and $\mathrm{tr}(\rho\, m_a^\beta m_b^\alpha)$ for $\alpha \neq \beta$ and $a < b$ are equal for fully permutation invariant states, while Definition 1 leaves them uncorrelated. This also illustrates that Definition 1 is more general than assuming a full permutation invariance up to a potential sign when swapping two odd operators while Definition 1 again includes these cases.

It is convenient to define for a generic operator P the maps C_P^σ with $\sigma = \pm$ by their action on an arbitrary operator X,

$$C_P^\sigma(X) = \frac{1}{2}\Big(X + \sigma PXP\Big). \tag{11}$$

For any two operators P and X, we can bound the operator norm of $C_P^\sigma(X)$ via

$$\|C_P^\sigma(X)\| \leq \frac{1 + \|P\|^2}{2}\|X\|, \tag{12}$$

and thus in the special case of $\|P\| = 1$, we obtain $\|C_P^\sigma(X)\| \leq \|X\|$.

The map $C_{P_j}^\sigma$, for $\sigma = +/-$ and P_j being the parity operators defined above, erases all terms from X which involve an odd/even number of Majorana operators on site j which can be verified by noting that

$$C_{P_j}^{+/-}(m_j^{\alpha_1}\dots m_j^{\alpha_r}) = \begin{cases} m_j^{\alpha_1}\dots m_j^{\alpha_r} & \text{for even/odd } r \\ 0 & \text{for odd/even } r \end{cases}. \tag{13}$$

We will use the notation that $C_{P_j}^+(X)$ is called even on site j and $C_{P_j}^-(X)$ is called odd on site j. The map

$$C := C_{P_V}^+ \circ \dots \circ C_{P_1}^+ \tag{14}$$

restricted to the its action on states constitutes a *quantum channel* with $\rho \mapsto C(\rho) \in \mathcal{D}(\mathcal{F}_K)$ is locally even on all sites. The expectation values of ρ and $C(\rho)$ are closely related. For any Majorana word $A = m_{j_1}^{\alpha_1}\dots m_{j_r}^{\alpha_r}$ by using the cyclicity of the trace, we have

$$\mathrm{tr}\Big[C(\rho)A\Big] = \mathrm{tr}\Big[\rho\, C(A)\Big] = \begin{cases} \mathrm{tr}(\rho A) & \text{if } A \text{ is even on all sites} \\ 0 & \text{else} \end{cases}. \tag{15}$$

In view of analyzing the structure of fermionic mode product states using Hudson's central limit theorem,[31] we further define, for a fermionic system with K modes, creation and annihilation operators $f_j^\dagger$ and f_j for $j \in [K]$ and state ρ the cumulants K_w^ρ with $w = 2, 4, \dots, 2K$ via

$$\mathrm{tr}(\rho f_{j_1}^{c_1}\dots f_{j_w}^{c_w}) = \sum_{P\in\mathcal{P}_{[w]}} \sigma_P \prod_{p\in P} K_{|p|}^\rho((f_{j_k}^{c_k})_{k\in p}), \tag{16}$$

where $c_j = -1, 1$ and $f_j^1 = f_j$ and $f_j^{-1} = f_j^\dagger$; $\mathcal{P}_{[w]}$ denotes the *set of all partitions* of the set $[w]$ into increasingly ordered parts of even size and σ_P denotes the sign of the permutation π which orders the sequence $(k)_{k\in p, p\in P}$.

III. A FERMIONIC DE FINETTI THEOREM FOR FINITE SYSTEMS

We now proceed to prove our main result stated in Theorem 2. In Lemma 1, we will first show that in permutation invariant fermionic states, terms sensitive to the fermionic anti-symmetry are suppressed in the system size such that essentially the fermionic character of the system is lost. More concretely, if ρ is a permutation invariant state, then ρ and $C(\rho)$ turn out to be approximately locally indistinguishable. We then proceed in Theorem 2 to exploit this fact by approximating a permutation invariant fermionic state with a permutation invariant state of qubits using the Jordan-Wigner transformation which allows us to employ standard quantum de Finetti theorems for finite systems in order to obtain the final result.

A. Suppression of the anti-symmetric character

We start by discussing the suppression of the anti-symmetric character of permutation invariant fermionic states in trace norm.

Lemma 1 (Suppression of the anti-symmetric character). *Let ρ be a permutation invariant fermionic state on a system of $V \geq 6$ sites with p modes per site. Then for any $k < V$, we have that*

$$\|\mathrm{tr}_{[V]\setminus[k]}(\rho) - \mathrm{tr}_{[V]\setminus[k]}[C(\rho)]\|_1 \leq \frac{2}{\sqrt{3}} \frac{2^{2p}\sqrt{k-1}^3}{V}, \tag{17}$$

where C is the quantum channel introduced above and $\mathrm{tr}_{[V]\setminus[k]}(\omega)$ *denotes the reduced state of* $\omega \in \{\rho, C(\rho)\}$ *to the first k sites.*

Proof. In order to prove the theorem, we first rewrite the one-norm distance of two states using expectation values via

$$\begin{aligned}\|\mathrm{tr}_{[V]\setminus[k]}(\rho) - \mathrm{tr}_{[V]\setminus[k]}[C(\rho)]\|_1 &= \sup_{\substack{A:\|A\|=1, A^\dagger=A\\ \mathrm{supp}(A)\subset[k]}} |\mathrm{tr}(\rho A - C(\rho)A)| \\ &= \sup_{\substack{A:\|A\|=1, A^\dagger=A\\ \mathrm{supp}(A)\subset[k]}} |\mathrm{tr}(\rho[A - C(A)])|. \end{aligned}\tag{18}$$

For $k = 1$, the bound is trivially fulfilled as then $C(A) = A$ by the overall evenness of A. We assume therefore $1 < k < V/2$ for the following.

We bound the expectation value by decomposing a general observable A into different contributions using the maps C^+ and C^- for different operators P. Using the local parity operator P_1, we define the two operators $C^-_{P_1}(A) = A_1$ and $C^+_{P_1}(A)$ which are both bounded in operator norm by the norm of A. As discussed above, A_1 will contain all terms of A which are odd on site 1, whereas in $C^+_{P_1}(A)$, all terms which are even on site 1 are collected. We continue by decomposing $C^+_{P_1}(A)$ into $C^-_{P_2}[C^+_{P_1}(A)] = A_2$ and $C^+_{P_2}[C^+_{P_1}(A)]$. The operator A_2 contains now all terms of A which are even on site 1 but odd on site 2. We can iterate this process and define for $l \in [k-1]$ the operators

$$A_l = C^-_{P_l} \circ C^+_{P_{l-1}} \circ C^+_{P_{l-2}} \circ \ldots \circ C^+_{P_1}(A) \tag{19}$$

and $A_k = C^+_{P_k} \circ \ldots \circ C^+_{P_1}(A)$ which fulfill

$$A = \sum_{l=1}^{k} A_l \tag{20}$$

and $\|A_l\| \leq \|A\|$ for all $l \in [k]$ as $\|P_j\| = 1$ for all $j \in [K]$.

Next, we decompose the operators A_l for $l \in [k-1]$. Given $l \in [k-1]$, we define the two operators $C^{-}_{m_l^1}(A_l)$ and $C^{+}_{m_l^1}(A_l)$ (here it is important to note that m_j^α is a Hermitian operator with eigenvalues ± 1). As each A_l is overall even, the operator $C^{-}_{m_l^1}(A_l)$ contains all terms of A_l which involve the operator m_l^1 and $C^{+}_{m_l^1}(A_l)$ collects all terms without m_l^1. We can iterate this with all m_l^α operators and obtain a decomposition

$$A_l = \sum_{r=1}^{p} \sum_{1\leq \alpha_1 < \ldots < \alpha_{2r-1} \leq 2p} m_l^{\alpha_1} \ldots m_l^{\alpha_{2r-1}} B_{l,(\alpha_j)_{j\in[2r-1]}} \tag{21}$$

for any $l \in [k-1]$ with

$$\|m_l^{\alpha_1} \ldots m_l^{\alpha_{2r-1}} B_{l,(\alpha_j)_{j\in[2r-1]}}\| \leq \|A\|. \tag{22}$$

The operators $B_{l,(\alpha_j)_{j\in[2r-1]}}$ are overall odd, even on the sites $1, \ldots, l-1$, and act trivially on site l.

Next, we introduce a set of permutations of the V sites in order to exploit the permutation invariance of the state. For this, we decompose for a given $l \in [k-1]$ the set $[k]$ into a left part $[l-1]$, a right part $[k]\backslash[l]$ and the site l. The permutations are then supposed to permute the site l on one of about $V/2$ many sites in the middle of the system. The left and right parts are then permuted independently from the permutation of the site l in a block to the left and to the right of the middle part on which l is permuted, where the block structure of the left and right blocks is preserved (consecutive sites stay consecutive) and the position of the left and right blocks is correlated. To make this concrete, let $\tau_i^j \in S_V$ denote the transposition of the sites i and j and define $n_k = \lfloor V/2(k-1)\rfloor$. Further, we introduce for $x \in [n_k]$ and $l \in [k-1]$ the abbreviations

$$b_x^l := V - (x-1)(k-l), \tag{23}$$

$$c_x^l := n_k(l-1) - (x-1)(l-1), \tag{24}$$

and the set

$$V_l^1 := \{j \in [V] : n_k(l-1) < j \leq V - n_k(k-l)\}. \tag{25}$$

We then define for any $l \in [k-1]$, $a \in V_l^1$, and $x \in [n_k]$ the permutations $\pi_{a,x}^{(l)}$ and $\pi_x^{(l)}$ by

$$\pi_{a,x}^{(l)} := \tau_a^l \, \tau_{b_x^l}^k \tau_{b_x^l-1}^{k-1} \ldots \tau_{b_x^l-k+l+1}^{l+1} \; \tau_{c_x^l}^{l-1} \tau_{c_x^l-1}^{l-2} \ldots \tau_{c_x^l-l+2}^{1} = \tau_a^l \pi_x^{(l)}, \tag{26}$$

which are visualized in Fig. 1. By definition, the permutations $\pi_{a,x}^{(l)}$ do not change the relative order of the sites $[k]$ or any $l \in [k-1], x \in [n_k]$ and $a \in V_l^1$. We obtain, therefore, for any $l \in [k-1]$, for a permutation invariant state ρ and operator $m_l^{\alpha_1} \ldots m_l^{\alpha_{2r-1}} B_{l,(\alpha_j)_{j\in[2r-1]}}$ in the above decomposition,

$$\begin{aligned} \operatorname{tr}(\rho m_l^{\alpha_1} \ldots m_l^{\alpha_{2r-1}} B_{l,(\alpha_j)_{j\in[2r-1]}}) &= \operatorname{tr}\left[\rho \frac{1}{|V_l^1| n_k} \sum_{a\in V_l^1} \sum_{x\in[n_k]} \pi_{a,x}^{(l)}\left(m_l^{\alpha_1} \ldots m_l^{\alpha_{2r-1}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)\right] \\ &= \operatorname{tr}\left[\rho \frac{1}{|V_l^1| n_k} \sum_{a\in V_l^1} \sum_{x\in[n_k]} \pi_{a,x}^{(l)}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)\right], \end{aligned} \tag{27}$$

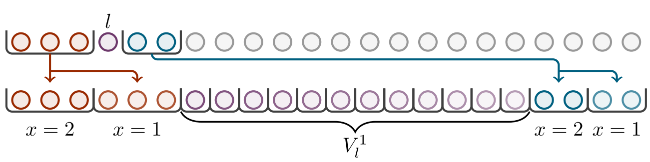

FIG. 1. Illustration of the permutation constructed in Eq. (26) for $l = 4$, $k = 6$, and $V = 22$. The permutations $\tau_a^l \pi_x^{(l)}$ permute site l into the central part (highlighted in purple), and the left and right parts (red and blue) are permuted into the bins to the left and right of the central part. The position of the left and right parts is correlated and fixed by the choice of x. The final position of l in the central part is specified by a.

where in the last line, we introduced the abbreviation $m_l^{\vec{\alpha}} := m_l^{\alpha_1} \dots m_l^{\alpha_{2r-1}}$ in order to simplify the notation. Exploiting Cauchy-Schwarz's inequality we obtain for any state ρ and operator A

$$\|\mathrm{tr}(\rho A)\|^2 = \|\mathrm{tr}(\sqrt{\rho}\sqrt{\rho}A)\|^2 \leq \mathrm{tr}(\rho A A^\dagger). \tag{28}$$

Applying this to Eq. (27) yields

$$\begin{aligned} &|\mathrm{tr}\left(\rho m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)|^2 \\ &\leq \mathrm{tr}\left[\rho \frac{1}{|V_l^1|^2 n_k^2} \sum_{a\in V_l^1} \sum_{x\in[n_k]} \sum_{b\in V_l^1} \sum_{y\in[n_k]} \pi_{a,x}^{(l)}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right) \pi_{b,y}^{(l)}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)^\dagger\right] \\ &= \mathrm{tr}\left[\rho \frac{1}{|V_l^1|^2 n_k^2} \sum_{a,b\in V_l^1} \sum_{x,y\in[n_k]} m_a^{\vec{\alpha}} \pi_x^{(l)}\left(B_{l,(\alpha_j)_{j\in[2r-1]}}\right) \pi_y^{(l)}\left(B_{l,(\alpha_j)_{j\in[2r-1]}}\right)^\dagger m_b^{\vec{\alpha}\dagger}\right]. \end{aligned} \tag{29}$$

The sum over x and y is intrinsically symmetric in x and y, whereas the operators $\pi_x^{(l)}(B_{l,(\alpha_j)_{j\in[2r-1]}})$ and $\pi_y^{(l)}(B_{l,(\alpha_j)_{j\in[2r-1]}})$ for $x \neq y$ anti-commute due to the overall oddness of $B_{l,(\alpha_j)_{j\in[2r-1]}}$. Therefore, all terms with $x \neq y$ vanish in the sum. The same arguments for the operators $m_a^{\vec{\alpha}}$ and $m_b^{\vec{\alpha}}$ yield that only terms with $a = b$ and $x = y$ contribute to the sum, and hence

$$|\mathrm{tr}(\rho m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}})|^2 \leq \mathrm{tr}\left[\rho \frac{1}{|V_l^1|^2 n_k^2} \sum_{a\in V_l^1} \sum_{x\in[n_k]} \pi_{a,x}^{(l)}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)^\dagger\right)\right]. \tag{30}$$

As ρ is permutation invariant, we obtain

$$\begin{aligned} |\mathrm{tr}(\rho m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}})|^2 &\leq \frac{|V_l^1| n_k}{|V_l^1|^2 n_k^2} \mathrm{tr}\left[\rho\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\left(m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\right)^\dagger\right)\right] \\ &\leq \frac{\|m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}}\|^2}{|V_l^1| n_k} \leq \frac{\|A\|^2}{|V_l^1| n_k}. \end{aligned} \tag{31}$$

The assumption $\|A\| = 1$ yields

$$|\mathrm{tr}(\rho m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}})| \leq \left(\frac{1}{|V_l^1| n_k}\right)^{1/2}. \tag{32}$$

By construction, $C(\rho)$ is even on all sites, i.e., for $l \in [k-1]$ $\mathrm{tr}(C(\rho)A_l) = 0$ and further $\mathrm{tr}(\rho A_k) = \mathrm{tr}(C(\rho)A_k)$. We then obtain from the above decomposition of A,

$$|\mathrm{tr}(\rho A) - \mathrm{tr}(C(\rho)A)| \leq \sum_{l=1}^{k} |\mathrm{tr}(\rho A_l) - \mathrm{tr}(C(\rho)A_l)| = \sum_{l=1}^{k-1} |\mathrm{tr}(\rho A_l)| \tag{33}$$

$$\leq \sum_{l=1}^{k-1} \sum_{r=1}^{p} \sum_{1\leq\alpha_1<\ldots<\alpha_r\leq 2p} |\mathrm{tr}(\rho m_l^{\vec{\alpha}} B_{l,(\alpha_j)_{j\in[2r-1]}})| \leq \frac{2^{2p}(k-1)}{2\sqrt{|V_l^1| n_k}}, \tag{34}$$

as there are $2^{2p}/2$ different $m_l^{\vec{\alpha}}$ involved per site (due to the oddness constraint by construction). Using that

$$\max_{w\in[n]} \frac{\frac{n}{w} - \lfloor \frac{n}{w} \rfloor}{\frac{n}{w}} \leq \frac{1}{2}, \tag{35}$$

we can simplify the bound to

$$|\mathrm{tr}(\rho A) - \mathrm{tr}(C(\rho)A)| \leq \frac{2}{\sqrt{3}} \frac{2^{2p}\sqrt{k-1}^3}{V}. \tag{36}$$

If we assume $V \geq 6$, the bound yields a value greater than 2 for all $k \geq V/2$ such that the bound applies to all k. □

B. A fermionic de Finetti theorem

The state $C(\rho)$ is locally even on all sites and therefore fully permutation invariant, i.e., $\mathrm{tr}(C(\rho)A) = \mathrm{tr}(C(\rho)\pi(A))$ for all $\pi \in S_V$ by Definition 1. By virtue of the Jordan-Wigner transformation, we can then map $C(\rho)$ to a permutation invariant state on V 2^p dimensional spins which fulfill a de Finetti theorem. In order to allow for a more compact notation, we define for a state $\xi \in \mathcal{D}(\mathcal{F}_p)$ and $k \in \mathbb{N}$ the state $\xi^{\otimes k} \in \mathcal{D}(\mathcal{F}_{kp})$ as the k-fold copy of ξ. The individual copies are hereby completely uncorrelated meaning that $\mathrm{tr}(m_1^{\vec{\alpha}^{(1)}} \ldots m_k^{\vec{\alpha}^{(k)}} \xi^{\otimes k}) = \mathrm{tr}(m_1^{\vec{\alpha}^{(1)}} \xi) \cdot \ldots \cdot \mathrm{tr}(m_1^{\vec{\alpha}^{(k)}} \xi)$ for any sets of indices $\vec{\alpha}^{(l)} \in [p]^{\times r_l}$, with $r_l \leq 2p$ for all $l = 1, \ldots, k$. Note that the notation is motivated by the fact that under the Jordan Wigner transformation, the state $\xi^{\otimes k}$ is indeed the standard tensor product of the state ξ in the proper sense if the sites are ordered appropriately. We extend this notation in the obvious sense to products of different states, i.e., for $\rho_1 \in \mathcal{D}(\mathcal{F}_{p_1})$ and $\rho_2 \in \mathcal{D}(\mathcal{F}_{p_2})$ then $\rho_1 \otimes \rho_2 \in \mathcal{D}(\mathcal{F}_{p_1+p_2})$ denotes the state on the joint system. With this, we arrive at the following main statement:

Theorem 2 (A fermionic de Finetti theorem). *Let ρ be a permutation invariant fermionic state on a system of $V \geq 6$ sites with p modes per site. Then there exist for $r < \infty$ and $l \in [r]$ states $\xi_l \in \mathcal{D}(\mathcal{F}_p)$ with $\mathcal{F}_p$ being the fermionic Fock space of p modes and $a_l \in [0, 1]$ with $\sum_l a_l = 1$ such that*

$$\|\mathrm{tr}_{[V]\setminus[k]}(\rho) - \sum_{l=1}^{r} a_l \xi_l^{\otimes k}\|_1 \leq \frac{2}{\sqrt{3}} \frac{2^{2p}\sqrt{k-1}^3}{V} + 2\frac{2^{2p}k}{V}, \tag{37}$$

where $\xi_l^{\otimes k}$ denotes the k-fold copy of ξ_l on $\mathcal{F}_{kp}$.

Proof. By Lemma 1, we can bound

$$\|\mathrm{tr}_{[V]\setminus[k]}(\rho) - \mathrm{tr}_{[V]\setminus[k]}[C(\rho)]\|_1 \leq \frac{2}{\sqrt{3}} \frac{2^{2p}\sqrt{k-1}^3}{V}, \tag{38}$$

where $C(\rho)$ is locally even. Note that $C(\rho)$ is the fully permutation invariant as

$$\mathrm{tr}(C(\rho)\pi(A)) = \mathrm{tr}(\rho C(\pi(A))) = \mathrm{tr}(\rho\pi(C(A))) = \mathrm{tr}(\rho C(A)) = \mathrm{tr}(C(\rho)A) \tag{39}$$

for ρ being permutationally invariant according to Definition 1. By virtue of the Jordan-Wigner transformation, $C(\rho)$ can therefore be viewed as a permutation invariant state on $\otimes^V \mathbb{C}^{2^p}$. From the de Finetti theorem for mixed states on finite spin systems,[5] we then obtain that there exist states $\chi_l \in \mathcal{B}(\mathbb{C}^{2^p})$ and weights $a_l \in [0, 1]$ for $l = 1, \ldots, r > \infty$ such that

$$\|\mathrm{tr}_{[V]\setminus[k]}[C(\rho)] - \sum_{l=1}^{r} a_l \chi_j^{\otimes k}\|_1 \leq 2\frac{2^{2p}k}{V}. \tag{40}$$

Defining the states $\xi_l = C^+_{Z^{\otimes p}}(\chi_l)$ with Z being the Pauli z operator and $Z^{\otimes p}$ corresponding to the Jordan Wigner transformed local parity operator, we find that for $C^{(k)} = C^+_{(Z^{\otimes p})_k} \circ \cdots \circ C^+_{(Z^{\otimes p})_1}$, under the Jordan-Wigner identification,

$$C^{(k)}(\mathrm{tr}_{[V]\setminus[k]}[C(\rho)]) = \mathrm{tr}_{[V]\setminus[k]}[C(\rho)] \tag{41}$$

and

$$C^{(k)}(\chi_l^{\otimes k}) = \xi_l^{\otimes k}. \tag{42}$$

Note that by construction, ξ_l are even operators and therefore states on the Fock space $\mathcal{F}_p$. By the contractiveness of a channel, we find

$$\|C^{(k)}\Big(\mathrm{tr}_{[V]\setminus[k]}[C(\rho)] - \sum_{l=1}^{r} a_l \chi_j^{\otimes k}\Big)\|_1 \leq \|\mathrm{tr}_{[V]\setminus[k]}[C(\rho)] - \sum_{l=1}^{r} a_l \chi_j^{\otimes k}\|_1 \tag{43}$$

such that

$$\|\mathrm{tr}_{[V]\setminus[k]}(\rho) - \sum_{l=1}^{r} a_l \xi_l^{\otimes k}\|_1 \leq \frac{2}{\sqrt{3}} \frac{2^{2p}\sqrt{k-1}^3}{V} + 2\frac{2^{2p}k}{V}, \tag{44}$$

which proves the main statement of the theorem. □

IV. STRUCTURE OF FERMIONIC MODE PRODUCT STATES AND GROUND STATE APPROXIMATION

The states appearing in the above de Finetti theorem, fermionic mode product states, may be somewhat uncommon for fermionic systems on the first sight. We therefore would like to elaborate on their structure in the following as they can be connected to more natural fermionic states in certain limiting cases. Further, we wish to highlight two applications of the above theorem.

A. Implications for mean field approximations

First, we explain how it helps us to bound mean field approximations to fermionic systems in special cases. Second, we explain how it leads to generalizations of established results for fermionic systems using the example of fermionic central limit theorems. For this, given a fermionic permutation invariant state ρ with r, a_l, and ξ_l being the coefficients and states corresponding to ρ according to Theorem 2, we define the abbreviation $\sigma_k = \sum_{l=1}^{r} a_l \xi_l^{\otimes k}$, omitting any reference to ρ as it will be clear which ρ is considered from the context.

In the case of a single mode per site, i.e., $p = 1$, $\xi_l \in \mathcal{D}(\mathcal{F}_1)$ are of the form $\xi_l = \alpha_l |0\rangle\langle 0| + (1-\alpha_l)|1\rangle\langle 1|$ with $\alpha_l \in [0, 1]$. We then obtain that

$$\sigma_k = \sum_{i_1,\ldots,i_k=0}^{1} b_{i_1,\ldots,i_k} |i_1,\ldots,i_k\rangle\langle i_1,\ldots,i_k|, \tag{45}$$

with $b_{i_1,\ldots,i_k} \geq 0$ and $\sum b_{i_1,\ldots,i_k} = 1$, meaning that σ_k is diagonal and therefore the convex combination of Fock basis states, i.e., pure Gaussian states. The same holds for all $p \leq 3$ where it can also be shown that every pure state $\xi \in \mathcal{D}(\mathcal{F}_p)$ is a Gaussian state.[35] Any mixed state ξ_l can be decomposed into the convex combination of pure states and hence a convex combination of pure Gaussian states for $p \leq 3$. As for two pure Gaussian states ξ and ξ', we have that $\xi \otimes \xi'$ is pure Gaussian as well this has the obvious yet important implication that the full σ_k is then again a convex combination of pure Gaussian states as well.

We can relate this to approximating a permutation invariant ground state of a given physical model. Let $\mathcal{S}_V$ denote a collection of subsets of size k of $[V]$. Consider a fermionic system of size V with a permutation invariant ground state ρ_{GS} and Hamiltonian

$$H = \frac{1}{|\mathcal{S}_V|} \sum_{S \in \mathcal{S}_V} H_S, \tag{46}$$

where we assume the H_S terms to be normalized $\|H_S\| \leq 1$ and to be supported on the modes of the sites $S \subset [V]$ only. Under the assumption of a permutation invariant ground state, we then obtain

$$\begin{aligned} 4\frac{2^{2p}k^{3/2}}{V} &\geq \frac{1}{|\mathcal{S}_V|} \sum_{S \subset \mathcal{S}_V} \Big|\mathrm{tr}\Big(H_S\Big[\sum_{l=1}^{r} a_l \xi_l^{\otimes k} - \mathrm{tr}_{[V-k]}(\rho_{\mathrm{GS}})\Big]\Big)\Big| \\ &\geq \mathrm{tr}\Big(H\Big[\sum_{l=1}^{r} a_l \xi_l^{\otimes V} - \rho_{\mathrm{GS}}\Big]\Big) \\ &\geq \min_{\xi \in \mathcal{D}(\mathcal{F}_p)} \mathrm{tr}(H\xi^{\otimes V}) - E_{\mathrm{GS}}, \end{aligned} \tag{47}$$

where a_l and ξ_l are the coefficients and states from Theorem 2 corresponding to ρ_{GS}, and the last step follows from the linearity of the energy expectation value.

For $p \leq 3$, in particular for the important case $p = 2$ reflecting fermions with a spin, we therefore directly obtain by convexity that Theorem 2 allows us to bound for the above defined models the best energy obtained from a pure Gaussian ground state approximation. In other words, these models are instances in which Theorem 2 allows us to bound the error made by using a Hartree-Fock approximation, hence giving a performance certificate.

However, let us also note that the assumptions made on the system are rather strict. The normalization of the Hamiltonian and more importantly the property of having a permutation invariant ground state in the first place are very restrictive. The above argumentation therefore does not yield bounds on common systems encountered most naturally but serves as an illustration for how mode de Finetti theorems are in principle capable of providing insights into particle product state approximation like the Hartree-Fock method.

B. A central limit theorem for correlated fermionic states

Next to understanding the structure of σ_k for low p, the two limiting cases for the size of the subsystem k can be understood. If we consider on-site observables only, i.e., $k = 1$, by Theorem 2, σ_1 agrees up to an error decreasing with the system size with the single site reduction of the initial states ρ where $\mathrm{tr}_{[V]\setminus\{1\}}(\rho)$ can be any state of $\mathcal{D}(\mathcal{F}_p)$, for instance, also far away from any Gaussian state. In short, σ_1 can obviously be any state in $\mathcal{D}(\mathcal{F}_p)$. However, in the case of a large subsystem $k \gg 1$, the products $\xi_l^{\otimes k}$ acquire an additional structure which is captured by a fermionic central limit theorem. In the Appendix, we show that all Fourier moments of $\xi_l^{\otimes k}$ are the moments of a Gaussian state up to an error scaling as k^{-1}. By this, σ_k can be thought of as a convex combination of Gaussian states in the limit of large k for observables which are smeared over the whole subsystem of size k.

To be precise, we introduce the Fourier modes

$$a_q^\alpha = \frac{1}{\sqrt{V}} \sum_{j=1}^{V} e^{2\pi i \frac{jq}{V}} f_j^\alpha, \tag{48}$$

with $q = -\lfloor (V-1)/2 \rfloor, \ldots, \lfloor V/2 \rfloor$. Then the extension of Hudson's central limit theorem[31] presented in the Appendix in Lemma 3 implies now that all cumulants of order $w > 2$ are suppressed in the number of copies V by

$$|K_w^{\rho^{\otimes V}}(a_{q_1}^{c_1,\alpha_1}, \ldots, a_{q_w}^{c_w,\alpha_w})| \leq \frac{1}{\sqrt{V}^{2-w}} |K_w^\rho(f_1^{c_1,\alpha_1}, \ldots, f_1^{c_w,\alpha_w})|. \tag{49}$$

In addition, we see that the second cumulants decouple into contributions from the modes $q = 0$, if V is even $q = V/2$ and of q and $-q$, and all these contributions are closely related to the second cumulants of the copied state ρ as

$$|K_2^{\rho^{\otimes V}}(a_{q_1}^{c_1,\alpha_1}, a_{q_2}^{c_2,\alpha_2})| = K_2^\rho(f_1^{c_1,\alpha_1}, f_1^{c_2,\alpha_2}) \delta_{(c_1 q_1 + c_2 q_2) \bmod V, 0}. \tag{50}$$

We therefore obtain that the cumulants of $\rho^{\otimes V}$ are approximated by the cumulants of $\rho_0 \otimes \rho_{V/2} \otimes_{q=1}^{\lfloor (V-1)/2 \rfloor} \rho_{q,-q}$ where the individual states are Gaussian and $\rho_0 = \rho_{V/2}$ and $\rho_{q,-q} = \rho_{q',-q'}$ for all admissible q, q'. On the one hand, this observation reveals the structure of i.i.d. mode product states when probed on large subsystems. On the other hand, our mode de Finetti theorem in combination with the extended Hudson central limit theorem in Lemma 3 yields immediately the following corollary:

Corollary 1 (A central limit theorem for correlated states). *Let ρ be a permutation invariant state according to Definition 1 on $V \geq 6$ sites, then for any $k \leq V$, the Fourier moments of the reductions* $\mathrm{tr}_{\setminus[k]}(\rho)$ *converge to the one of a convex combination of Gaussian states with an error decreasing as* $k^{-1} + k^{3/2}/V$.

This exemplifies how insights about i.i.d. product states immediately extend to the more general structure of permutation invariance and we obtain a fermionic central limit type theorem for states with long range correlations. In mindset, this is reminiscent of the dynamical central limit theorems allowing for initial correlations as presented in Ref. 36 and building upon the bosonic Ref. 37.

C. Comments on one-particle reduced density operators

In this final comment, we hint at a link to consequences of permutation invariance of fermionic states to spectral properties of *one-particle reduced density operators* (1-RDM). It is known that spectra of 1-RDM arising from pure fermionic states give rise to a *convex polytope*,[38,39] giving rise to *generalized Pauli constraints*. General mixed fermionic states do not have to fulfill such constraints.[40] However, for permutation invariant fermionic states, again new constraints emerge for the 1-RDM. The object in the focus of attention here is the 1-RDM, for K modes defined as the *correlation matrix* $\mathbb{1}\geq\Gamma\geq 0$ with entries

$$\Gamma_{j,k} = \langle f_j^\dagger f_k\rangle, \tag{51}$$

for $p = 1$ and $j, k = 1, \ldots, K$. For fixed particle number N, one has $\mathrm{tr}(\Gamma) = N$. In the symmetric setting considered here, one finds $\langle f_j^\dagger f_j\rangle = a$ and $\langle f_j^\dagger f_k\rangle = b$ for $j > k$ and $\langle f_j^\dagger f_k\rangle = b^*$ for $j < k$, with $a = N/K$ and $|b| \leq 8/(3^{1/2}K)$, by our theorem. Further, one can show for $b = |b|e^{i\phi}$ with $\phi\in\mathbb{R}$ that the 1-RDM has eigenvalues

$$\lambda_k = \begin{cases} \dfrac{N}{K} + |b|\dfrac{\cos\left(\frac{2\pi}{K}k + \frac{(K-2)}{K}\phi\right) - \cos(\phi)}{1-\cos\left(\frac{2\pi}{K}k - \frac{2}{K}\phi\right)} & \text{if } b\notin\mathbb{R} \\ \dfrac{N}{K} - b + bK\delta_{k,0} & \text{if } b\in\mathbb{R} \end{cases}, \tag{52}$$

for $k = 0, \ldots, K-1$. What is more, $\mathbb{1} \geq \Gamma \geq 0$ implies further constraints to b. Hence, we find that from permutation invariance and the fermionic character alone, one can identify constraints, beyond the *standard Pauli constraints* that $\lambda_k \in [0, 1]$ for all $k = 1, \ldots, K$. This statement only takes the case $p = 1$ into account. For $p > 1$, a richer structure emerges, as here the correlation matrix Γ takes the form

$$\Gamma = \begin{pmatrix} A & B & \ldots & B \\ B^\dagger & A & \ddots & \vdots \\ \vdots & \ddots & \ddots & B \\ B^\dagger & \ldots & B^\dagger & A \end{pmatrix}, \tag{53}$$

with A being Hermitian with its trace fixed by the particle number, and the entries of B again being suppressed in the system size.

V. OUTLOOK

In this work, we have presented a fermionic mode de Finetti theorem for finite sized systems, stated precisely in Theorem 2. We have shown that we can derive this theorem without assuming a full permutation invariance of the state, which in combination with the canonical anti-commutation relations would lead to forcing specific correlators to vanish and imposes therefore additional constraints on the system. We instead provide an operational definition of permutation invariance, restricting ourselves to a more natural setting for fermionic states which does not interfere with the intrinsic anti-commutation of such systems. Interestingly, by virtue of the Jordan Wigner transformation, this of course immediately also provides an extension to de Finetti theorems of distinguishable particles which we have not discussed so far, namely, in cases where the state of the system can be mapped to a permutation invariant fermionic state. Investigating the potential of such a generalization will be the subject of future research. Further, we have discussed the structure of the resulting mode product states and connected them in different limiting cases to Gaussian states. In doing so, we in addition illustrated how it allows us to generalize established results for i.i.d. product states naturally to permutation invariant states by considering an extension to Hudson's central limit theorem discussed in the Appendix. Our theorem provides a further step into understanding the structure of fermionic states and provides a mathematical underpinning of mean field approaches, complementing previous results formulated or primarily investigated in the thermodynamic limit.[29,30] Similar to the rich structures present in permutation invariant systems of distinguishable particles, we expect that further generalization and insights can be obtained in the near future. It remains an interesting and important question whether fermionic mean field approaches can be bound in non-permutation invariant settings along the lines of Ref. 16, to give rise to quality certificates of Hartree-Fock approaches based on interaction graphs alone.

Let us also note that in bosonic systems, particle de Finetti theorems are easily available as the states are intrinsically symmetric under the exchange of particles and in addition the number of relevant single particle modes, which controls the local dimension of each particle, can be much smaller than the total particle number, e.g., in the setting of Bose-Einstein condensation. Both features are absent in fermionic systems such that it remains open and subject of future research if a non-trivial fermionic particle de Finetti theorem can be formulated which would allow us to bound the Hartree-Fock approach on more general grounds and might yield deeper and important insights into the structure of fermionic systems. It is the hope that the present work stimulates such further approaches.

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APPENDIX: EXTENSION OF HUDSON'S CENTRAL LIMIT THEOREM

Given a fermionic state ρ on p modes, it is known that a certain reduction of $\rho^{\otimes V}$ converges to the Gaussian state with the same second moments as ρ for $V \to \infty$ by the central limit theorem formulated by Hudson.[31] In its precise formulation, the theorem states that for any $V \in \mathbb{N}$, we define the modes

$$a_0^\alpha = \frac{1}{\sqrt{V}} \sum_{j=1}^{V} f_j^\alpha. \tag{A1}$$

Then for any observable A that can be written with the modes a_0^α, $a_0^{\alpha\dagger}$ only, we obtain

$$\lim_{V\to\infty} \mathrm{tr}(\rho^{\otimes V} A) = \mathrm{tr}(\rho_G \tilde{A}), \tag{A2}$$

where $\tilde{A}$ is constructed from A by replacing all a_0^α and $a_0^{\alpha\dagger}$ by f_1^α and $f_1^{\alpha\dagger}$, respectively, and ρ_G denotes the Gaussian state on p modes with the same second moments as ρ. We can take this result a step further and can investigate $\rho^{\otimes V}$ globally. For this, consider the Fourier modes

$$a_q^\alpha = \frac{1}{\sqrt{V}} \sum_{j=1}^{V} e^{2\pi i \frac{jq}{V}} f_j^\alpha, \tag{A3}$$

with $q = -\lfloor (V-1)/2 \rfloor, \ldots, \lfloor V/2 \rfloor$. We then obtain the following lemma:

Lemma 3 (Fermionic central limit theorem). *Given a fermionic state ρ on $V \in \mathbb{N}$ sites and p modes per site, we then obtain for any $w = 2, 4, \ldots, 2pV$, sequences $c_1, \ldots, c_w$, $\alpha_1, \ldots, \alpha_w$ and $q_1, \ldots, q_w$ with $c_j \in \{\pm 1\}$, $\alpha_j \in [p]$, and q_j as above such that all triples (c_j, α_j, q_j) are distinct that cumulants are bounded as*

$$K_w^{\rho^{\otimes V}}(a_{q_1}^{c_1,\alpha_1}, \ldots, a_{q_w}^{c_w,\alpha_w}) = \frac{1}{\sqrt{V}^w} K_w^\rho(f_1^{c_1,\alpha_1}, \ldots, f_1^{c_w,\alpha_w}) \sum_{j=1}^{V} e^{\frac{2\pi i}{V} \sum_{l=1}^{w} c_l q_l j}. \tag{A4}$$

Note that we could completely decouple the state $\rho^{\otimes V}$ into a V-fold copy of the same Gaussian state if we would not have used the Fourier modes but the modes created from a tensor product of Hadamard gates as transformation which essentially follows from the considerations in Ref. 41, building up upon Ref. 42. Using such central limit theorems, the *extremality* of fermionic Gaussian states for a number of interesting properties can be derived,[41,42] beyond the observation that the maximum von-Neumann entropy $\rho \mapsto S(\rho)$ for given second moments is attained by Gaussian states, and the minimum of the *coherent information* $\rho \mapsto S(\rho_A) - S(\rho)$, for given second moments, A reflecting the modes of a subsystem, is again assumed for fermionic Gaussian states.[43]

Proof. We prove this lemma by induction. Let $w = 2$. We then find

$$K_2^{\rho^{\otimes V}}(a_{q_1}^{c_1,\alpha_1}, a_{q_2}^{c_2,\alpha_2}) = \frac{1}{V}\sum_{j,l=1}^{V} e^{2\pi i \frac{c_1 q_1 j + c_2 q_2 l}{V}} \operatorname{tr}(\rho^{\otimes V} f_j^{c_1,\alpha_1} f_l^{c_2,\alpha_2}). \tag{A5}$$

As ρ is an even operator, the terms of the above sum are non-zero only for $l = j$ such that we obtain

$$K_2^{\rho^{\otimes V}}(a_{q_1}^{c_1,\alpha_1}, a_{q_2}^{c_2,\alpha_2}) = \frac{1}{V}\sum_{j=1}^{V} e^{2\pi i \frac{(c_1 q_1 + c_2 q_2) j}{V}} \operatorname{tr}(\rho^{\otimes V} f_j^{c_1,\alpha_1} f_j^{c_2,\alpha_2}) \tag{A6}$$

$$= \frac{1}{\sqrt{V}^2} K_2^{\rho}(f_1^{c_1,\alpha_1} f_1^{c_2,\alpha_2}) \sum_{j=1}^{V} e^{\frac{2\pi i}{V}(c_1 q_1 + c_2 q_2) j}, \tag{A7}$$

as the expectation value is independent of j. In order to access higher cumulants for $w > 2$, consider

$$\sum_{P \in \mathcal{P}_{[w]}} \sigma_P \prod_{p \in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) = \frac{1}{\sqrt{V}^w} \operatorname{tr}\Big(\rho^{\otimes V} \prod_{l=1}^{w} \sum_{j=1}^{V} e^{\frac{2\pi i}{V} c_l q_l j} f_j^{c_l,\alpha_l}\Big). \tag{A8}$$

We define $\mathcal{P}_{[w]}^V$ to be the set of all partitions of $[w]$ into V increasingly ordered sets of even size including empty sets. The idea is now that every such partition labels one configuration in the product of the sums on the right-hand side of Eq. (A8) in the sense that for $(z_1, \ldots, z_V) = Z \in \mathcal{P}_{[w]}^V$ and the indices contained in z_j are associated with site j (with no index associated in the case of z_j being empty). We can then write

$$\sum_{P \in \mathcal{P}_{[w]}} \sigma_P \prod_{p \in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) = \frac{1}{\sqrt{V}^w} \sum_{Z \in P_{[w]}^V} \sigma_Z \prod_{j \in [w] : |z_j| > 0} \operatorname{tr}\Big(\rho \prod_{l \in z_j} e^{\frac{2\pi i}{V} c_l q_l j} f_1^{c_l,\alpha_l}\Big). \tag{A9}$$

Inserting the definition of the cumulants then results in

$$\begin{aligned}&\sum_{P \in \mathcal{P}_{[w]}} \sigma_P \prod_{p \in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) \\ &= \frac{1}{\sqrt{V}^w} \sum_{Z \in P_{[w]}^V} \sigma_Z \prod_{j \in [w] : |z_j| > 0} \sum_{P \in \mathcal{P}_{z_j}} \sigma_P \prod_{p \in P} K_{|p|}^{\rho}((f_1^{c_l,\alpha_l})_{l\in p}) e^{\frac{2\pi i}{V} \sum_{l \in p} c_l q_l j}.\end{aligned} \tag{A10}$$

The expression above looks rather convoluted. We can simplify it significantly by realizing that if we expand all sums and products, the collection of all P in one term forms a partition of $[w]$ while the partition Z determines which index appears on which site. Summing over Z will then yield that every partition appears on every site such that we can write

$$\sum_{P \in \mathcal{P}_{[w]}} \sigma_P \prod_{p \in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) = \frac{1}{\sqrt{V}^w} \sum_{P \in \mathcal{P}_{[w]}} \sigma_P \prod_{p \in P} \Big(\sum_{j=1}^{V} K_{|p|}^{\rho}((f_1^{c_l,\alpha_l})_{l\in p}) e^{\frac{2\pi i}{V} \sum_{l \in p} c_l q_l j}\Big), \tag{A11}$$

where one can check that the sign σ_P results from the product of σ_Z and all σ_P's in Eq. (A10). Inserting the induction hypothesis for all cases in which partitions into sets smaller than w appear yields

$$
\begin{aligned}
&\sum_{P\in\mathcal{P}_{[w]}} \sigma_P \prod_{p\in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) \\
&= \sum_{P\in\mathcal{P}_{[w]}:|P|>1} \sigma_P \prod_{p\in P} K_{|p|}^{\rho^{\otimes V}}((a_{q_l}^{c_l,\alpha_l})_{l\in p}) + \frac{1}{\sqrt{V}^w} K_w^{\rho}((f_1^{c_l,\alpha_l})_{l\in[w]}) \sum_{j=1}^{V} e^{\frac{2\pi i}{V}\sum_{l=1}^{w} c_l q_l j}. \qquad (A12)
\end{aligned}
$$

Eliminating the common terms on both sides of the equation yields the result. □

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