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Random partial differential equations on evolving hypersurfaces

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1. Introduction

Partial differential equations (PDEs) appear in the mathematical modelling of a great variety of processes. Most of these equations contain various parameters that describe some physical properties, for example permeability or thermal conductivity. Usually it is presumed that these parameters are precisely given and a PDE is considered in a deterministic manner. However, often this is not the case, but there is a degree of uncertainty regarding the given data. Clearly, one would like to quantify the effect of uncertain parameters..

First, let us comment on various causes of uncertainty in model inputs. Generally, we can separate those causes into two main categories. The first category consists of uncertainty due to incomplete knowledge. This means that, in principle, it could be removed by performing additional measurements or having complete information. However, those measurements are typically very costly or impractical. This type of uncertainty is known as epismetic uncertainty. The second type is the so-called aleatoric uncertainty and it refers to the uncertainty of a phenomenon that comes from its own nature. It appears due to some unexpected or uncontrolled circumstances and cannot be reduced or removed by additional measurements. Thus, it relates to those quantities that are different every time we run the experiment due to information that cannot be controlled or measured, such as wind vibration.

For a more detailed discussion about the types and causes of uncertainty we refer the reader to [36, 69]. We will mainly concentrate on the epismetic type of uncertainty. Thus, we will think about uncertainty in the way it is interpreted in [36]: "uncertainty may be thought of as a measure of the incompleteness of one's knowledge or information about an unknown quantity whose true value could be established if a perfect measuring device were available."

On this background, uncertainty quantification (UQ) has developed to a flourishing and very active mathematical field. We would refer to Sullivan [118] for the underlying mathematical concepts, typical UQ objectives and numerous examples. Concerning other basic references on UQ, we would point out [13, 69, 94, 96].

The overall goal is to identify and quantify uncertainty. In particular, given some information about the uncertainty of input data, we want to study the uncertainty of the system output, which is a solution function in the PDE setting. There are several approaches to the quantification of uncertainty, what is meant exactly by UQ and the corresponding mathematical framework. Some common approaches are: the worst case scenario, the probabilistic approach, Bayesian interface, the measure-theoretic approach, etc. For more details on these approaches see [69] and the references therein.

This thesis will concentrate on the probabilistic approach which characterizes uncertainty by statistical information, such as probability density function, mean value, variance etc. Thus, we interpret the input data of a PDE as random fields. This results in a PDE with random coefficients, also known as a random partial differential equation (RPDE). Hence, the solution is also a random field and the aim is to determine its statistics or the statistics of some functional

applied to it. Furthermore, we would like to analyse the impact of a given uncertainty of random input data on the solution. There is a growing interest in RPDEs as these equations occur in many applications, such as hydrogeology, material science, fluid dynamics, biological fluids etc. This in turn leads especially to the growing development of numerical analysis and numerical methods for solving RPDEs, [10, 11, 14, 15, 27, 29, 32, 33, 69, 77]. Note that most of those papers deal with elliptic RPDEs. Parabolic PDEs with random coefficients, specifically, have so far been studied in the following papers , [10, 28, 77, 88].

All these papers have considered equations on a bounded *flat* fixed domain in $\mathbb{R}^d$. However, it is known and well studied that, in a variety of applications, these models can be better formulated on both stationary and evolving *curved* domains, cf., e.g. [114]. Thus, one would prefer to study a PDE whose domain is an evolving n-dimensional curved surface embedded in $\mathbb{R}^{n+1}$. Such PDEs are called *evolving surface partial differential equations*. The extension of the Boussinesq equations to any interface geometry has already been studied in [114]. Over the past years, deterministic surface PDEs have gained increasing interest due to a variety of applications including biological modelling [89] and engineering [99]. For this thesis specifically, the motivating example is modelling the transport of a surface active agent (surfactant) on the interface between two fluids [81, 117].

Since domains on which those equations are posed are curved, the framework will be geometric. This means that regular Cartesian derivatives in this setting are replaced with tangential gradients. Furthermore, this leads to notions like Laplace-Beltrami operator, tangential gradient et cetera. Additionally, since the domain is changing in time, a natural time derivative to consider is the so-called material derivative, which is the derivative on a space-time domain that computes the time rate of change of any quantity along the flow of the surface. Note that we assume the surface evolution to be prescribed. One could also consider the evolving hypersurface that is a solution of a given geometric PDE, such as motion by mean curvature or the Willmore flow. For a general overview on geometric PDEs, we refer the reader to [41].

Concerning the computational methods for surface PDEs, a broad review on this topic is presented in [59]. In general, there exist two main categories of numerical methods for surface PDEs. One idea is to use an explicit representation of the surface and approximate it using a triangulated surface on which calculations are then performed. This approach can be traced back to the pioneering paper of Dziuk [54] on the surface finite element method to compute the solution of the Laplace-Beltrami equation on a curved domain. Dziuk and Elliott [56] later extended this work first to parabolic equation on stationary surfaces and then to the evolving surface finite element method (ESFEM) for PDEs on moving hypersurfaces ([55, 58]). The other approach uses implicit representation of the surface and embeds the surface into Cartesian space. A typical example of these kind of methods is a level-set method [115].

With this we have motivated and introduced two natural ways of approaching PDEs from different points of view: random PDEs and surface PDEs. Both uncertainty quantification and the geometric framework are well-developed fields and of great interest. However, to the best of our knowledge there is no mathematical theory that merges these two fields. The aim of this work is exactly that, to bring these two areas together and to consider random PDEs on moving hypersurfaces. First, we develop an appropriate setting and formulation of the random equation on evolving hypersurfaces, prove its well-posedness and consider different types of uncertainty of the initial data: uniformly bounded and log-normal coefficients. Then, we will derive and

analyse a numerical method of solving surface RPDE, which will result in the so-called evolving surface finite element - Monte-Carlo (ESFEM-MC) method. We derive discretization error estimates for the ESFEM-MC method and present some numerical examples that confirm our predicted order of convergence.

Let us make the previous statements more precise and comprehensible. We wish to analyse the following advection-diffusion equation with random coefficients on evolving hypersurface $\{\Gamma(t)\}_{t\in[0,T]}$

$$\begin{aligned} \partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + u \nabla_\Gamma \cdot \mathbf{w} &= f \\ u(0) &= u_0 \end{aligned} \tag{1.0.1}$$

where ∇_Γ is the tangential surface gradient, $\nabla_\Gamma \cdot$ is the tangential divergence, $\partial^\bullet$ is the material derivative and $\mathbf{w}$ is the velocity field of the evolution. In contrast to the deterministic case, the diffusion coefficient α, the source function f and the initial value u_0 are random. Hence the solution u will also be a random field. The equation (1.0.1) models the transport of a scalar quantity, e.g. a surfactant, along a moving two-dimensional interface [117]. The surfactant is transported by advection via the tangential fluid velocity and by diffusion within the surface.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and elements $\omega \in \Omega$ be samples. For the analysis part, we will consider two cases: when the random coefficient α is uniformly bounded from above and below and when α has the log-normal distribution. First of all, we need to define an appropriate framework for solving the equation (1.0.1). This part is based on the work of Alphonse et al. presented in [4], where the abstract setting of the PDE on an evolving Hilbert space has been considered. The main idea is to overcome the difficulty that our domain changes over time, by connecting the space at arbitrary time t with the fixed initial space and incorporate this pull-back into the definition of the solution space. We will prove that this general framework can be adapted for our setting. It consists of defining a Bochner-Sobolev type of spaces, defining the material derivative and a solution space and selecting an appropriate Gelfand triple. Many of these results are based on the tensor product structure of the spaces that we consider, such as $L^2(\Omega, H^1(\Gamma)) \cong L^2(\Omega) \otimes H^1(\Gamma)$. Thus we will utilize results from tensor spaces, presented in Section 2.5 where we will clarify the notion of tensor spaces and present some results that will be used in our analysis.

In analogy to the elliptic case [94], for the parabolic PDE with random coefficients there exist two weak formulations: path-wise (for a fixed sample ω) and "mean" (which also includes integration over Ω). A more direct way (as in [10]) of proving the integrability of the solution with respect to $\mathbb{P}$ is when we integrate the equation over the spatial domain and in addition also take expectations, which allows us to apply the Banach–Nečas–Babuška [BNB] theorem directly to the whole solution space. We will call this approach the "mean-weak" formulation. This result guarantees the measurability and the existence of the first and second moments of the solution and bounds of their norms, which motivates us to adopt this approach in the uniform case when the bilinear forms are uniformly bounded.

In many practical applications in the geosciences and biology [31], flow and transfer in porous media are processes that are usually analysed, and log-normally distributed random coefficients play an important role. As explained for example in [65], if the diffusion coefficient varies drastically within a layer, it is appropriate to expand its logarithm in an affine series of independent identically distributed normal random variables. The log-normal random parameter has already

been analysed for the elliptic equations in many papers, for example in [28, 29, 65, 111] and in the parabolic case in [88, 106]. However, in this case the bilinear forms are not uniformly distributed any more, so that we cannot consider the "mean-weak" formulation, since the direct integration over Ω would not lead to a well-posed problem. Instead, we consider the path-wise formulation and the Karhunen-Loève (KL) expansion of the coefficient α. Using this approach for each realisation we obtain parametrized deterministic problem. Thus, we get a family of deterministic weak formulations over the spatial domain that can be solved $\mathbb{P}$-almost surely. Since we are considering a PDE with random coefficients, we are interested in the statistics of the solution, i.e. we want to prove that the solution is in $L^2(\Omega)$. In order to achieve that via the path-wise approach, we need to prove the measurability of the solution with respect to $\mathbb{P}$ and a uniform bound for the $L^2(\Omega)$-norm (or a higher order norm). The proof that we present is similar in spirit to the proof done by Gittelson in [65]. It is based on defining an auxiliary Gaussian measure and controlling the inf-sup constant from the existence theory for the solution of the deterministic PDE. In order to better understand those results and the necessary conditions that ensure the existence of the KL expansion of a random variable, in Section 2.6 we present a discussion on the representation of a random field. In particular, in Section 2.7 we consider the representation of a Gaussian random filed. These results on well-posedness are published in [47].

Concerning the numerical analysis we will restrict the discussion to the case when the random coefficient α is uniformly bounded from both above and below. We derive and analyse the ESFEM-MC method. Following Dziuk and Elliott [55], the space discretization is performed by random piecewise linear finite element functions on simplicial approximations $\Gamma_h(t)$ of the surface $\Gamma(t)$, $t \in [0, T]$. To deal with uncertainty we apply the standard Monte-Carlo approach. First we prove the well-posedness of the semi-discrete problem. The main difficulty is that the solution space in this case is not Hilbert. Hence, we can not directly apply the BNB theorem as we did in the continuous case. Instead, we consider a path-wise approach and prove the measurability of the solution by proving the continuous dependence of the semi-discrete solution of the equation on the initial data, which is by assumption measurable. Moreover, the next step is to define a path-wise Ritz projection and prove its regularity and measurability properties. We present optimal error estimates for the resulting semi-discrete scheme which then provide corresponding error estimates for expectation values and Monte-Carlo approximations. Application of efficient solution techniques, such as adaptivity [45], multigrid methods [86], and Multilevel Monte-Carlo techniques [15, 29, 32] is very promising, but beyond the scope of this thesis. In our numerical experiments we investigate a corresponding fully discrete scheme based on an implicit Euler method and observe optimal convergence rates. These results are presented in Chapters 6, 7 and 8 and are going to appear in [48].

We postponed the numerical analysis for the case when the coefficient α has log-normal distribution for the following reasons. The main reason is that the general results concerning the representation of a Gaussian random field on an evolving curved domain are missing. These results, particularly sample regularity of the random field, such as continuity, or differentiability (if we want to consider a higher order approximation) are needed in order to apply the standard FEM results. Moreover, for computations we need to use the truncated KL expansion for which we need to calculate the basis functions of the function space on the evolving hypersurface. Some work in this direction has been done by Schwab and Lang [75, 87] in the case of a sphere

$\mathbb{S}^2$. Furthermore, this work has been generalized to the domain $\mathbb{S}^2 \times [0, T]$, cf. [40]. However, more general results about log-normal fields on evolving hypersurfaces will be part of a future research and will not be presented in this thesis. Still, we present a brief summary of results obtained so far, main difficulties and challenges of this field in Section 2.7. Thus, some results in this section might not always be presented in their generality and with full precision, but instead appropriate references for more details are proposed.

We conclude this thesis with a chapter on the outlook for further development. In this chapter we start to analyse a question that naturally comes up: what happens if the given velocity is a random field? First we clarify that this assumption leads to a PDE posed on a so-called random tube or a random non-cylindrical domain. PDEs on random domains have been already studied by many authors [25, 26, 73, 124]. In particular, Harbrecht suggests different approaches in various papers [73, 74], and comments on their advantages and disadvantages. One of standard approaches in dealing with random domains is to consider the domain mapping method introduced in [124]. The main idea is to pull back the PDE onto a fixed domain. In this way, we reformulate a PDE on a random domain into a random PDE on a fixed domain. However, in the existing work, to the best of our knowledge, no one has considered a parabolic PDE posed on a random domain that changes in time. This specifically is the last problem that we will consider. We will prove the well-posedness of the heat equation posed on a flat evolving random domain i.e. a random tube. This formulation brings us to the framework of PDEs on a non-cylindrical domain. This field is well-established ([23, 35, 84, 90]) and we give a brief overview of existing results, mainly focusing on the work of Zolésio, [38, 44, 52, 53], that will be exploited in our calculations. Notably, we give a detailed analysis of necessary regularity assumptions on the initial data, particularly velocity and its associated flow, that will ensure the well-posedness of the considered equation. The well-posedness is proved utilizing the standard results for parabolic PDEs. This work can be generalized to elliptic PDEs on a curved random domain. This is a work in progress together with Church, Elliott and Kornhuber and will not be developed further in this thesis.

Let us finish by commenting on the possible directions of research motivated by this thesis. We have already mentioned analysis of Gaussian random fields on evolving curved domains, which would lead to a numerical analysis and computations for the case when the coefficient has a log-normal distribution. Moreover, it is natural to investigate other numerical methods that would lead to faster and more efficient computations, as well as numerical methods for the case when the velocity is random. Furthermore, one may ask if it is possible to have a rough evolution of a hypersurface. This would lead to a random PDE with rough coefficients. These topics exceed the scope of this thesis and are left for future research.

2. Preliminaries

We will only consider a fixed finite time interval $[0, T]$, where $T \in (0, \infty)$. Furthermore, we will denote by $D((0, T))$ the space of $\mathbb{R}$-valued C^∞-smooth functions with compact support in $(0, T)$. We will reuse the same constants C in calculations multiple times if their exact value is not important. Moreover, integrals will be usually written without measure, unless it is not clear which terms are integrated.

2.1. Probability spaces

A mathematical model for uncertainty and randomness is a random variable. As illustrated by Bertrand's paradox in [63, Sec 2.1.1], one has to be careful in defining the term random. The precise way to do it is by introducing the probability space setting. In this subsection we recall some basic concepts from measure and probability theory. We will mainly follow [37] and for more information we refer also to e.g., [19, 102].

We start with non-empty set Ω, called *sample space*, with points $\omega \in \Omega$ that are called *samples*. Next, we define a subsets of Ω which we can measure and that will be referred to as events.

Definition 2.1.1. A *σ-algebra* $\mathcal{F}$ on Ω is a family $\mathcal{F}$ of subsets of Ω with these properties

(i) $\emptyset \in \mathcal{F}$

(ii) $F \in \mathcal{F} \Rightarrow F^C \in \mathcal{F}$, where $F^C = \Omega \setminus F$

(iii) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a *measurable space*.

The subsets of Ω that belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets and in the probability theory context they are called *events*.

Definition 2.1.2. If $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ are two measurable spaces, then a function $X : \Omega \to E$ such that

$$X^{-1}(A) := \{\omega \in \Omega : X(\omega) \in A\} = \{X \in A\} \in \mathcal{F} \tag{2.1.1}$$

for all $A \in \mathcal{E}$, is called *$\mathcal{F}$-measurable* or *a random variable*.

Note that the measurability of a function depends only on the $\sigma-$algebra, we do not need to define measures on these spaces.

To avoid atypical cases and technical difficulties, we assume that the range space E is a separable Banach space (for example if E is non-separable, the sum of two random variables

doesn't have to be a random variable). Furthermore, the separability assumption enables us to define a Bochner integral, see Section 2.2. Recall that the Borel σ-algebra on E is the smallest algebra containing all open (closed) subsets of E and it will be denoted as $\mathcal{B}(E)$. The elements $B \in \mathcal{B}(E)$ are called Borel. If both spaces Ω and E are separable Banach spaces, the measurable function $X : (\Omega, \mathcal{B}(\Omega)) \to (E, \mathcal{B}(E))$ is also called *a Borel function*. In this case in Definition 2.1.2 it is enough to consider just open sets A. The following lemma describes the Borel σ-algebra on a separable Banach space.

Lemma 2.1.3. [37, Proposition 1.3] Let E be a separable Banach space. Then $\mathcal{B}(E)$ is the smallest σ-field of subsets of E containing all sets of the form

$$\{x \in X : \varphi(x) \leq \alpha\}, \quad \varphi \in E^*, \alpha \in \mathbb{R}.$$

By the previous lemma we can characterize E-valued random variables on a separable Banach space E: a mapping $X : \Omega \to E$ is an E-valued random variable iff, for arbitrary $\varphi \in E^*$, $\varphi(X) : \Omega \to \mathbb{R}$ is an R-valued random variable.

Definition 2.1.4. A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a σ-additive function $\mathbb{P} : \mathcal{F} \mapsto [0, 1]$ such that $\mathbb{P}(\Omega) = 1$. The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called *a probability space*. It is called a *complete* probability space if any subset A of any $B \in \mathcal{F}$ with $\mathbb{P}(B) = 0$ is also in $\mathcal{F}$.

Note that any measure space can be completed by adding to its σ-algebra all subsets of sets of zero measure [49, Sec. 3]. Therefore, assuming that the probability space is complete is not a significant restriction.

Remark 2.1.5. Considering the complete probability space helps us to avoid some pathological behaviours that are hard to control. For instance, if we were to consider the non-complete probability space, it could happen that if we change a random variable on a zero measure set we can obtain a function that is no longer a random variable! To see this, as in [80], let us consider $\Omega = [0, 1]$ and the Borel σ-algebra on Ω, but with the Lebesgue measure. Then we can create a subset A of a Cantor set (thus it has Lebesgue measure zero) that is not Borel measurable. Changing a constant 1 random variable on that set A, for example taking it to be χ_{A^C}, we obtain a function that is not any more $\mathcal{B}([0, 1])$-measurable, since the inverse of $\{0\}$ is $A \notin \mathcal{B}([0, 1])$.

Next, we would like to discuss the separability of the space $L^2(\Omega, \mathcal{F}, \mathbb{P})$. An example of non-separable L^2-space is if we consider $\Omega = \mathbb{R}$ with the counting measure. The question is what kind of properties of the measure space would provide the separability of the $L^2(\Omega, \mathcal{F}, \mathbb{P})$. For that purpose we define the notion of separable measure space. The following results are taken from [22, 119].

Definition 2.1.6. The measure space $(\Omega, \mathcal{F}, \mathbb{P})$ is called *separable* if there exists a countable family $\{E_n\}_{n\in\mathbb{N}}$ of subsets of $\mathcal{F}$ such that the σ-algebra which the family $\{E_n\}_{n\in\mathbb{N}}$ generates coincides with $\mathcal{F}$, i.e. $\mathcal{F}$ is generated by a countable collection of subsets.

An example of a separable measure space is when Ω is a separable metric space and $\mathcal{F} = \mathcal{B}(\Omega)$. The separability of the space is necessary in order to have the isomorphism of tensor spaces, which will be used later in the expansion of the random field. For that purpose, we need the following result.

Theorem 2.1.7. [22, Theorem 4.13] Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a separable measure space. Then $L^p(\Omega)$ is separable for any $p, 1 < p < \infty$.

The alternative approach to get the criteria for separability of $L^2(\Omega)$ is to define a new metric space, since it is clear what is meant by the separability of metric spaces. Namely, we can define a metric on the equivalent classes of $\mathcal{F}$ by

$$d(A, B) := \mathbb{P}(A \triangle B).$$

Then by [119, Theorem 13.8] we conclude that the space $L^p(\Omega, \mathcal{F}, \mathbb{P}), 1 \leq p < \infty$ is separable iff the metric space $(\mathcal{F}, d)$ is separable.

Remark 2.1.8. Note that having the σ-finite measure μ is not enough to ensure the separability of the space $L^p(\Omega, \mathcal{F}, \mu)$, i.e. there exists a finite measure space $(\Omega, \mathcal{F}, \mu)$ such that $L^p(\Omega, \mathcal{F}, \mu)$ is not separable, cf. [78].

To this end, we will always suppose to have a complete separable measure space. We make this assumption precise for our underlying probability space.

Assumption 2.1.9. $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete separable probability space.

We finish this section with a discussion how one can prove the measurability of the function, defined in Definition 2.1.2

Lemma 2.1.10. Every continuous function from one metric space into another is a Borel function.

Proof. Since we are considering Borel σ-algebra , it is enough to check (2.1.1) for open sets. According to the definition of a continuous function, the inverse image of every open set is an open set, which proves the claim. □

Remark 2.1.11. The previous statement is not true if we consider some σ-algebra other than Borel. The counterexample of a continuous function on $\mathbb{R}$ that is not Lebesgue measurable can be found for example in [49, Proposition 4.2.1]. The comment about this result is that the Lebesgue σ-algebra on $\mathbb{R}$ may be too large.

The following result shows that the composition of a continuous and measurable function is a measurable function (note that for this result we need the range space to be a complete Banach space).

Lemma 2.1.12. Let $(X, \mathcal{M})$ be a measurable space and $(Y, \mathcal{B}(Y)), (Z, \mathcal{B}(Z))$ be two metric spaces. If $f : X \to Y$ is $\mathcal{M}$-measurable function and $g : Y \to Z$ is a continuous function, then the composition $g \circ f : X \to Z$ is $\mathcal{M}$-measurable.

Proof. Let $V \in \mathcal{B}(Z)$ be open. Then by continuity of g, it follows that $g^{-1}(V) \in \mathcal{B}(Y)$ is also open. In the end, measurability of f implies $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \in \mathcal{M}$, which completes the proof. □

Let $(X, \mathcal{M})$ be a measurable space and $(Y, \mathcal{B}(Y))$ be a separable Banach space. One can easily prove, cf. [49], that $\max(f, g), \min(f, g)$ and arithmetic operations (such as $f \pm g$, fg) over measurable functions $f, g : X \to Y$, give measurable functions. Furthermore, if f_n is a sequence of measurable functions from X into Y, then $\sup_n f_n, \inf_n f_n, \limsup_n f_n, \inf_m \sup_{m \geq n} f_n$ define measurable functions.

The important property of measurability is that it prevails through limit processes, i.e. the point-wise limit of measurable functions is measurable:

Theorem 2.1.13. [49, Theorem 4.2.2] Let $(X, \mathcal{M})$ be a measurable space and $(Y, \mathcal{B}(Y))$ be a metric space. Furthermore, let f_n be measurable functions from X into Y such that for all $x \in X$, $f_n(x) \to f(x)$ in Y, then f is measurable.

The previous result can be generalized to the case when we have the μ-a.e. point-wise convergence, where $(X, \mathcal{M}, \mu)$ is a complete measure space. More precisely:

Lemma 2.1.14. Let $(X, \mathcal{M}, \mu)$ be a complete measurable space and $(Y, \mathcal{B}(Y))$ be a metric space. Let f_n be measurable functions from X into Y such that for $\mu -$ a.e. $x \in X$ $f_n(x) \to f(x)$, in Y . Then f is measurable.

Proof. The main idea, as presented in [79], is to use the fact that if f is measurable on a complete measure space and $f = g$, μ-a.e., then g is also measurable. Let us define

$$A := \{x | f_n(x) \to f(x)\}.$$

Then by definition of μ-a.e. point-wise convergence, $\mu(A^c) = 0$, we obtain that A is measurable. Hence, defining $g_n(x) := 1_A(x) f_n(x)$ gives us a sequence of measurable functions and $g_n(x)$ converges for every x to function $g(x) := 1_A(x) f(x)$. It follows from Theorem 2.1.13 that function g is measurable. Additionally, $f = g$, μ-a.e. which by completeness of measure space, implies the measurability of function f.

Remark 2.1.15. Note that the previous result doesn't hold when the underlying measure space is not complete. The counterexample is similar to one discussed in Remark 2.1.5. Namely, if we consider again the set A which is a subset of a Cantor set that is not Borel measurable and take the sequence $f_n \equiv 1$, for every n. Then $f_n \to 1_A$, Lebesgue a.e., but 1_A is not Borel measurable.

□

2.2. Bochner spaces

The Bochner space is a straightforward generalization of the Lebesgue space to a Banach space valued functions. We want to define the integral of an E-valued random variable $X : \Omega \to E$. For this purpose, let $(E, \mathcal{B}(E))$ be a separable Banach space and we will need the following result.

Lemma 2.2.1. [37, Lemma 1.1] Let E be a separable metric space with metric d and let X be an E-valued RV. Then there exists a sequence $(X_m)_{m \in \mathbb{N}}$ of simple E-valued RVs such that, for arbitrary $\omega \in \Omega$, the sequence $(d(X(\omega), X_m(\omega)))_{m \in \mathbb{N}}$ is monotonically decreasing to zero.

For a simple random variable, i.e. the one that takes only a finite number of values

$$X = \sum_{i=1}^{N} x_i \chi_{A_i}, \quad A_i \in \mathcal{F},\, x_i \in E$$

Lemma 2.2.1 allows us to define

$$\int_B X(\omega) d\mathbb{P} := \sum_{i=1}^{N} x_i \mathbb{P} A_i \cap B, \quad \forall B \in \mathcal{F}.$$

One can show that the previous definition does not depend on the representation of X and that standard properties of integrals hold.

In order to define an integral of a general E-valued integral we need the following lemma.

Lemma 2.2.2. [37, Lemma 1.5] Let E be a separable Banach space and let X be an E-valued random variable defined on $(\Omega, \mathcal{F})$. Then the real valued function $\|X\| : \Omega \to \mathbb{R}$ is measurable.

The previous Lemma ensures that the following definition is proper. We say that the random variable X is *Bochner integrable* if

$$\int_\Omega \|X\| d\mathbb{P} < \infty.$$

Let X be integrable. Then by [37, Lemma 1.1] there exists a sequence $\{X_m\}$ of simple random variables such that $\{\|X(\omega) - X_m(\omega)\|\}_m \downarrow 0$. Now we can define the *Bochner's integral* of X by

$$\mathbb{E}[X] := \int_\Omega X d\mathbb{P} := \lim_{n\to\infty} \int_\Omega X_n(\omega) d\mathbb{P}.$$

and $\mathbb{E}[X]$ is called *the expectation of X* (w.r.t. $\mathbb{P}$). Bochner's integral has many of the standard properties of the *Lebesgue integral*. The one that we will specifically use is Fubini's theorem. In order to state Fubini's theorem, we first need to define the product of probability spaces. Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i), i = 1, 2$ be two measurable spaces. Then the product σ-algebra $\mathcal{F}_1 \times \mathcal{F}_2$ is defined as the smallest σ-algebra containing all the sets of the form $A_1 \times A_2, A_i \in \mathcal{F}_i, i = 1, 2$. Furthermore, the measure $\mathbb{P}_1 \otimes \mathbb{P}_2$ on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ is defined by

$$\mathbb{P}_1 \otimes \mathbb{P}_2(A_1 \times A_2) := \mathbb{P}_1(A_1)\mathbb{P}_2(A_2), \quad A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2.$$

For integration with respect to a product measure, we have the following important result on iterated integrals. The proof can be found for example in [119, Theorem 6.6].

Theorem 2.2.3. (Fubini-Tonelli) Let $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, $i = 1, 2$, be probability spaces and let $f : \Omega_1 \times \Omega_2 \to E$ be measurable. Then if one of the integrals exists

$$\int_{\Omega_1\times\Omega_2} \|f\|_E d(\mathbb{P}_1 \otimes \mathbb{P}_2), \quad \int_{\Omega_1} \left(\int_{\Omega_2} \|f\|_E d\mathbb{P}_2 \right) d\mathbb{P}_1, \quad \int_{\Omega_2} \left(\int_{\Omega_1} \|f\|_E d\mathbb{P}_1 \right) d\mathbb{P}_2$$

then all three exist and are equal:

$$\int_{\Omega_1\times\Omega_2} \|f\|_E d(\mathbb{P}_1 \otimes \mathbb{P}_2) = \int_{\Omega_1} \left(\int_{\Omega_2} \|f\|_E d\mathbb{P}_2 \right) d\mathbb{P}_1 = \int_{\Omega_2} \left(\int_{\Omega_1} \|f\|_E d\mathbb{P}_1 \right) d\mathbb{P}_2.$$

We finish this section by defining the space $L^p(\Omega, \mathcal{F}, \mathbb{P}; E)$, for $p \geq 1$ as the set of all equivalence classes of $E - valued$ random variables (w.r.t. the equivalence relation $X \sim Y \iff X = Y$ a.s.) with the norm

$$\|X\|_p = (E(\|X\|^p))^{\frac{1}{p}}, \quad p \in [1, \infty)$$

and

$$\|X\|_\infty = \text{ess sup}_{\omega \in \Omega} \|X(\omega)\|.$$

Let $V \underset{i}{\hookrightarrow} H \cong H^* \underset{i'}{\hookrightarrow} V^*$ be a Gelfand triple (we recall the notion of a Gelfand triple in Section 3.1). Our goal is to define the standard Soboloev-Bochner solution space for parabolic PDEs. Let us first recall the existing results on vector-valued distributions. Every $u \in L^2(0, T; V)$ defines a vector-valued distribution $\mathrm{T}_u : D((0, T)) \to V$ through the H-valued integral

$$\varphi \mapsto \int_0^T y(t)\varphi(t)dt.$$

We will identify T_u and u. Now we can define its distributional derivative. We say that $u \in L^2(0, T; V)$ has a *weak derivative* $u' \in L^2(0, T; V^*)$ if there exists $w \in L^2(0, T; V^*)$ such that

$$\mathrm{T}'_u(\xi) = \int_0^T \xi'(t)(u(t), v)_H = -\int_0^T \xi(t) \langle w(t), v \rangle_{V^*,V}, \quad \forall \xi \in D(0, T), \quad \forall v \in V \tag{2.2.1}$$

and we write $w = u'$. Further we can define the standard Sobolev-Bochner space as

$$\mathcal{W}(V_0, V_0^*) = \{u \in L^2(0, T; L^2(\Omega, H^1(\Gamma_0))) \mid u' \in L^2(0, T; L^2(\Omega, H^{-1}(\Gamma_0)))\}. \tag{2.2.2}$$

The space $\mathcal{W}(V_0, V_0^*)$ is a Hilbert space with the inner product defined via:

$$(u, v)_{\mathcal{W}(V_0, V_0^*)} := \int_0^T \int_\Omega (u(t, \omega), v(t, \omega))_{H^1(\Gamma_0)} + \int_0^T \int_\Omega (u'(t, \omega), v'(t, \omega))_{H^{-1}(\Gamma_0)}.$$

The next theorem states the main properties of the space $\mathcal{W}(V_0, V_0^*)$.

Theorem 2.2.4. The following statements hold

i) The embedding $\mathcal{W}(V_0, V_0^*) \subset C([0, T], H)$ is continuous.

ii) The embedding $\mathcal{W}(V_0, V_0^*) \subset D([0, T], V)$ is dense.

iii) Let $u, v \in \mathcal{W}(V_0, V_0^*)$, then the mapping

$$t \mapsto (u(t), v(t))_H$$

is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(u(t), v(t))_H = \left\langle u'(t), v(t) \right\rangle_{V^*,V} + \left\langle u(t), v'(t) \right\rangle_{V,V^*}$$

holds for almost every $t \in [0, T]$. The last expression implies the integration by parts formula

$$(u(T), v(T))_H - (u(0), v(0))_H = \int_0^T \left\langle u'(t), v(t) \right\rangle_{V^*,V} + \int_0^T \left\langle u(t), v'(t) \right\rangle_{V,V^*}.$$

Proof. For the density result see [92, Theorem 2.1] and for the result see [116]. □

The weak derivative can be characterized in terms of vector-valued test-functions. Since we will use this result in our definition of a weak material derivative, we state it also here for completeness. Let us denote by $D([0,T],V)$ the space of all C^∞-smooth V-valued test functions with compact support in $[0,T]$.

Theorem 2.2.5. The weak derivative condition (2.2.1) is equivalent to

$$\int_0^T (u(t),\psi'(t))_H = -\int_0^T \langle u'(t),\psi(t)\rangle_{V^*,V} \quad \forall \psi \in D((0,T),V). \tag{2.2.3}$$

Proof. The direct implication follows from Theorem 2.2.4, iii). To see that (2.2.3) implies (2.2.1), test (2.2.3) with $\xi v \in D((0,T),V)$, where $\xi \in D((0,T))$ and $v \in V$. □

2.3. Hypersurfaces

Let us first recall some basic theory about hypersurfaces and Sobolev spaces on hypersurfaces that we will need to treat surface PDEs. For more details we refer to [41, 59, 120].

Definition 2.3.1. Let $k \in \mathbb{N} \cup \{\infty\}$. $\Gamma \subset \mathbb{R}^{n+1}$ is called *C^k-hypersurface* if for every point $x_0 \in \Gamma$, there exists an open set $U \subset \mathbb{R}^{n+1}$ containing x_0 and a function $\varphi \in C^k(U)$ such that

$$U \cap \Gamma = \{x \in U : \varphi(x) = 0\} \quad \text{and} \quad \nabla\varphi \neq 0 \text{ on } \Gamma \cap U. \tag{2.3.1}$$

The linear space

$$T_x\Gamma = \{\tau \in \mathbb{R}^{n+1} : \exists \gamma : (-\epsilon,\epsilon) \to \mathbb{R}^{n+1} \text{ differentiable }, \gamma((-\epsilon,\epsilon)) \subset \Gamma, \gamma(0) = x,\ \gamma'(0) = \tau\}$$

is the *tangent space* to Γ at $x \in \Gamma$. From the definition it directly follows that $T_x\Gamma = [\nabla\varphi(x)]^\perp$, where φ is the notion from (2.3.1). Hence, $T_x\Gamma$ is an n-dimensional vector subspace of $\mathbb{R}^{n+1}$.

A vector $\nu(x) \in \mathbb{R}^{n+1}$ is called a *unit normal vector* at $x \in \Gamma$ if $\nu(x) \perp T_x\Gamma$ and $|\nu(x)| = 1$. From the previous characterization of the tangent space, we have

$$\nu(x) = \pm\frac{\nabla\varphi(x)}{|\nabla\varphi(x)|}.$$

A C^1-hypersurface is called *orientable* if there exists a continuous vector field $\nu : \Gamma \to \mathbb{R}^{n+1}$ such that $\nu(x)$ is a unit normal vector to Γ for all $x \in \Gamma$.

Remark 2.3.2. One can also define a *parametrized C^k-surface* $\Gamma \subset \mathbb{R}^{n+1}$ by: for every point $x_0 \in \Gamma$ there exists a local parametrization $X : V \to U \cap \Gamma$, where $V \subset \mathbb{R}^n$ is an open and connected set and $x_0 \in U \subset \mathbb{R}^{n+1}$ is an open set, such that $X \in C^k(V,\mathbb{R}^{n+1})$, X is a bijection and rank of ∇X is n on V. The map X^{-1} is called a local chart. The connection between the parametrized surfaces and hypersurfaces is presented in [120] and it can be shown that locally parametrised hypersurfaces and hypersurfaces are the same.

In view of the previous remark, we say that a function $f : \Gamma \to \mathbb{R}$ is k-times differentiable if all $f \circ X_i : V_i \to \mathbb{R}$ are k times differentiable, for all local parametrizations from the atlas $(X_i)_{i\in I}$, $\cup_i X_i(V_i) = \Gamma$.

Assumption 2.3.3. Assume that Γ is a C^2, compact, connected, orientable, without a boundary, n-dimensional hypersurface, embedded in $\mathbb{R}^{n+1}$ for $n = 1, 2$, or 3.

For the definition of a C^k hypersurface with a boundary and more details on this topic, we refer the reader to [120, Ch. 19].

Definition 2.3.4. Let $\Gamma \subset \mathbb{R}^{n+1}$ be a C^1-hypersurface. For a function $f : \Gamma \to \mathbb{R}$, which is differentiable in an open neighbourhood of Γ, we define the *tangential gradient* of f at $x \in \Gamma$ by

$$\nabla_\Gamma f(x) := \nabla \tilde{f}(x) - \nabla \tilde{f}(x) \cdot \nu(x)\nu(x),$$

where $\tilde{f}$ is a smooth extension of f to a neighbourhood of the Γ and ∇ is the usual gradient in $\mathbb{R}^{n+1}$.

For the construction of the extension $\tilde{f}$ see the proof of [59, Theorem 2.10]. Note that $\nabla_\Gamma f(x)$ is the orthogonal projection of $\nabla \tilde{f}(x)$ onto $T_x\Gamma$

$$\nabla_\Gamma f(x) = P(x)\nabla \tilde{f}(x),$$

where $P(x)_{ij} = \delta_{ij} - \nu_i(x)\nu_j(x), i, j = 1, \ldots, n+1$. Hence, it is a tangential vector. Moreover, one can show that a tangential gradient depends only on the values of f on $\Gamma \cap U$ [59, Lemma 2.4], which makes the previous definition of the tangential gradient independent of the extension $\tilde{f}$.

The tangential gradient is a vector-valued quantity and for its components we will use the notation

$$\nabla_\Gamma f(x) = (\underline{D}_1 f(x), \ldots, \underline{D}_{n+1} f(x)).$$

From the definition of the tangential gradient, one directly gets

$$\nabla_\Gamma(\alpha f + \beta g) = \alpha \nabla_\Gamma f + \beta \nabla_\Gamma g, \qquad \nabla_\Gamma(fg) = g\nabla_\Gamma f + f\nabla_\Gamma g$$

for differentiable functions $f, g : \Gamma \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$. Note that in general $\underline{D}_i$ and $\underline{D}_j$ do not commute, but it holds

$$\underline{D}_i \underline{D}_j f - \underline{D}_j \underline{D}_i f = \{(\underline{D}_k \nu_j)\nu_i - (\underline{D}_k \nu_i)\nu_j\}\underline{D}_k f \quad i, j = 1, \ldots n+1.$$

We define the surface divergence of a vector field $v : \Gamma \to \mathbb{R}^{n+1}$ by

$$\nabla_\Gamma \cdot v := \sum_{i=1}^{n+1} \underline{D}_i v_i,$$

which yields a natural definition of a surface Laplacian for a function $f \in C^2(\Gamma)$, known as the *Laplace-Beltrami operator*

$$\Delta_\Gamma f(x) = \nabla_\Gamma \cdot \nabla_\Gamma f(x) = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f(x) \quad x \in \Gamma.$$

Let us define

$$C^1(\Gamma) := \{f : \Gamma \to \mathbb{R} : f(x) \text{ is differentiable for every } x \in \Gamma \text{ and } \underline{D}_j f : \Gamma \to \mathbb{R},\ j = 1, \dots, n+1 \text{ are continuous } \}.$$

Similarly we can define $C^j(\Gamma)(j \in \mathbb{N})$ provided that Γ is a C^k-hypersurface with $k \geq j$.

Let $\Gamma \in C^2$. Then $\nu \in C^1(\Gamma)$ and we can define

$$\mathcal{H}_{ij} := \underline{D}_i \nu_j, \quad i, j = 1, \dots, n+1.$$

Since $\underline{D}_j \nu_k = \underline{D}_k \nu_j$, it follows that the matrix $\mathcal{H}$ is symmetric. Moreover, one can show that zero is an eigenvalue of $\mathcal{H}$ i.e. $\mathcal{H}\nu = 0$. The matrix $\mathcal{H}$ is called the *extended Weingarten map* and it maps tangent space into itself. Its restriction to the tangent space is called *Weingarten map* and its eigenvalues $k_i, i = 1, \dots, n$ (except the trivial eigenvalue in the normal direction) are called *principal curvatures* of Γ. Now for any $x \in \Gamma$ we can define the *mean curvature* of Γ at x by

$$H(x) := \text{trace}\mathcal{H}(x) = \sum_{1}^{n+1} k_i.$$

The mean curvature appears in the basic calculations, such as partial integration.

Remark 2.3.5. If Γ is defined as a zero level-set of a function φ in $\mathbb{R}^2$, i.e., $\Gamma = \{x \in \mathbb{R}^2 : \varphi(x) = 0\}$, then the mean curvature can be directly calculated from the formula

$$H = \nabla \cdot \frac{\nabla \varphi}{|\nabla \varphi|} = \frac{1}{|\nabla \varphi|} \sum_{j,k=1}^{3} (\delta_{jk} - \frac{\varphi_{x_j} \varphi_{x_k}}{|\nabla \varphi|^2}) \varphi_{x_j x_k}.$$

Since working with charts and atlases is not convenient in terms of numerical analysis, and using global Fermi coordinates is a better choice, we will introduce the sign distance function for Γ. Namely, Assumption 2.3.3 enables us to use Jordan-Brouwer theorem, which implies that Γ is a boundary of an open, bounded set $G \subset \mathbb{R}^{n+1}$. In this case a useful level set representation can be obtained with the help of the signed distance function, where the signed distance function is defined by

$$d(x) = \begin{cases} \inf_{y \in \Gamma} |x - y|, & x \in \mathbb{R}^{n+1} \backslash \overline{G} \\ -\inf_{y \in \Gamma} |x - y|, & x \in G \end{cases}.$$

It directly follows that d is globally Lipschitz continuous with the Lipschitz constant 1. Utilizing C^2 regularity of Γ, we get that it satisfies both interior and exterior conditions, enabling us to prove the following lemma which introduces the global coordinates.

Lemma 2.3.6. We define

$$U_\delta := \{x \in \mathbb{R}^{n+1} || d(x)| < \delta\}.$$

Then $d \in C^k(U_\delta)$ and for every $x \in U_\delta$ there exists a unique $a(x) \in \Gamma$ such that

$$x = a(x) + d(x)\nu(a(x)). \tag{2.3.2}$$

In addition

$$\nabla d(x) = \nu(a(x)), \quad |\nabla d(x)| = 1 \quad \forall x \in U_\delta.$$

For every point $x \in U_\delta$ we can extend the normal in the normal direction $\nu(x) = \nu(a(x))$. Hence, from previous lemma we conclude that every point $x \in U_\delta$ can be described by its Fermi coordinates $a(x)$ and $d(x)$ using (2.3.2)

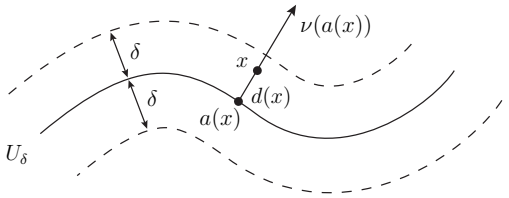


Figure 2.1.: Fermi coordinates $x = a(x) + d(x)\nu(a(x))$.

Exploiting Fermi coordinates, we can prove the co-area formula, cf. [62], which enables us to prove the formula for integration by parts on surfaces Γ. We state here the result in the general form.

Theorem 2.3.7. Assume that Γ is a hypersurface in $\mathbb{R}^{n+1}$ with a smooth boundary $\partial\Gamma$ and that $f \in C^1(\overline{\Gamma})$. Then

$$\int_\Gamma \nabla_\Gamma f dA = \int_\Gamma f H \nu dA + \int_{\partial\Gamma} f \mu dA. \tag{2.3.3}$$

where μ denotes the co-normal vector that is normal to $\partial\Gamma$ and tangent to Γ.

It is important to note that in our case, under Assumption 2.3.3, Γ doesn't have a boundary, the last term in (2.3.3) vanishes. Moreover, dA in the first two integrals over Γ in (2.3.3) denotes the n-dimensional surface measure, while in the last integral over $\partial\Gamma$ denotes the $n-1$ surface measure. Combining (2.3.3) with the product rule, we derive *Green's formula*

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g dA = -\int_\Gamma f \Delta_\Gamma g dA + \int_{\partial\Gamma} f \nabla_\Gamma g \cdot \mu dA, \tag{2.3.4}$$

where the last term vanishes if Γ doesn't have a boundary, which will be the case of our interest.

We will consider a weak formulation of PDEs on Γ, which leads to the concept of Sobolev spaces on surfaces as natural solution space. Thus, we need first to introduce the notion of a weak derivative, which will be induced by the formula for integration by parts on Γ. Let $\Gamma \in C^2$, which implies the existence of the mean curvature and allows us to use the integration by parts.

We define $L^p(\Gamma), p \in [1, \infty]$ as usual, i.e. as a set of all functions $f : \Gamma \to \mathbb{R}$ that are measurable with respect to the surface measure dA such that

$$\|f\|_{L^p(\Gamma)} := \left(\int_\Gamma |f(x)|^p \right)^{1/p} < \infty, \quad p < \infty$$

and for $p = \infty$ we take the essential supremum norm. The standard results hold, namely, $L^p(\Gamma)$ is a Banach space and $L^2(\Gamma)$ is a Hilbert space. Moreover, the spaces $C^0(\Gamma)$ and $C^1(\Gamma)$ are dense in $L^p(\Gamma)$, for $p < \infty$.

Definition 2.3.8. We say that a function $f \in L^1(\Gamma)$ has *a weak derivative* $g_i = D_i f \in L^1(\Gamma)$, $i \in \{1, \dots, n+1\}$ if for every function $\phi \in \mathcal{C}^1(\Gamma)$ with compact support and every i it holds

$$\int_\Gamma f \underline{D}_i \phi dA = -\int_\Gamma \phi g_i dA + \int_\Gamma f \phi H \nu_i dA.$$

The Sobolev space on Γ is defined by

$$H^{1,p}(\Gamma) = \{f \in L^p(\Gamma) \mid D_i f \in L^p(\Gamma), i = 1, \dots, n+1\}$$

with the norm

$$\|f\|_{H^{1,p}(\Gamma)} = \left(\|f\|_{L^p(\Gamma)} + \|\nabla_\Gamma f\|_{L^p(\Gamma)}\right)^{\frac{1}{p}}.$$

In particular, for $p = 2$ we will use the notation $H^1(\Gamma) = H^{1,2}(\Gamma)$.

Exploiting global coordinates and the standard Poincaré inequality in $\mathbb{R}^{n+1}$, we can deduce the Poincaré inequality on surfaces.

Theorem 2.3.9. Assume that Γ is C^3 and $1 \leq p < \infty$. Then, there exists a constant C_P such that for every function $f \in H^{1,p}(\Gamma)$ with $\int_\Gamma f dA = 0$ one has the inequality

$$\|f\|_{L^p(\Gamma)} \leq C_P \|\nabla_\Gamma f\|_{L^p(\Gamma)}. \tag{2.3.5}$$

Proof. We refer to [59, Theorem 2.12]. □

2.4. Moving surfaces

Let us define the family of evolving surfaces $\{\Gamma(t)\}$ for $t \in [0, T]$ that we will consider. For each $t \in [0, T]$ we assume that $\Gamma(t)$ satisfies the same properties as Γ and we set $\Gamma_0 := \Gamma(0)$. Furthermore, we assume the existence of a flow $\overline{\Phi} : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0, T]$ its restriction $\Phi_t^0 := \Phi(t, \cdot) : \Gamma_0 \to \Gamma(t), \Phi \in \mathcal{C}^1([0, T], \mathcal{C}^2(\Gamma_0))$ is a diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt}\Phi_t^0(\cdot) &= \mathbf{v}(t, \Phi_t^0(\cdot)) \\ \Phi_0^0(\cdot) &= \mathrm{Id}(\cdot), \end{aligned}$$

where $\mathbf{v} : [0, T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a velocity field. We assume that $\mathbf{v}(t, \cdot) \in \mathcal{C}^2(\Gamma(t))$ and that it has uniformly bounded divergence

$$|\nabla_{\Gamma(t)} \cdot \mathbf{v}(t)| \leq C_{\mathbf{v}} \quad \text{for all } t \in [0, T]. \tag{2.4.1}$$

In the following we will write ∇_Γ instead of $\nabla_{\Gamma(t)}$, whenever it is clear which surface $\Gamma(t)$ the gradient relates to.

Remark 2.4.1. Besides the normal velocity $\mathbf{v}_\nu = \mathbf{v} \cdot \nu \nu$, which is enough to define the evolution of the surface, we assume that the surface also has an advective tangential velocity $\mathbf{v}_\tau$ that describes the motion of points along the surface. Hence, we assume that we are given a global velocity field $\mathbf{v}$ that can be decomposed as $\mathbf{v} = \mathbf{v}_\nu + \mathbf{v}_\tau$. In addition, we assume that the physical velocity agrees with the velocity of the parametrisation. For remark about the different notions of velocities for an evolving hypersurface see for example [5, Remark 2.6].

Let us define the space-time domain by

$$\mathcal{G}_T := \bigcup_{t\in[0,T]} \Gamma(t) \times \{t\}. \tag{2.4.2}$$

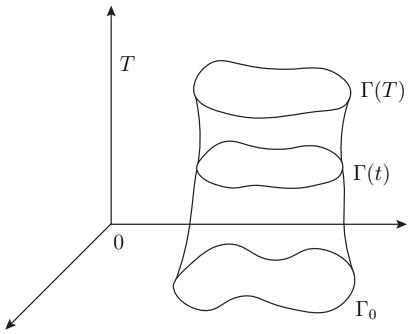


Figure 2.2.: Example of a space-time domain $\mathcal{G}_T$.

Furthermore, applying Lemma 2.3.6 for every $\Gamma(t)$ we infer that $\Gamma(t)$ can be represented as the zero level set

$$\Gamma(t) = \{x \in \mathcal{N}(t) \,|\, d(x,t) = 0\}, \quad t \in [0,T],$$

of a signed distance function $d = d(x,t)$ defined on an open neighbourhood $\mathcal{N}(t)$ of $\Gamma(t)$ such that $|\nabla d| \neq 0$ for $t \in [0,T]$. Note that d, d_t, d_{x_i}, $d_{x_i x_j} \in \mathcal{C}^1(\mathcal{N}_T)$ with $i, j = 1, \ldots, n+1$ holds for

$$\mathcal{N}_T := \bigcup_{t\in[0,T]} \mathcal{N}(t) \times \{t\}. \tag{2.4.3}$$

We also choose $\mathcal{N}(t)$ such that for every $x \in \mathcal{N}(t)$ and $t \in [0,T]$ there exists a unique $p(x,t) \in \Gamma(t)$ such that

$$x = p(x,t) + d(x,t)\nu(p(x,t),t), \tag{2.4.4}$$

and fix the orientation of $\Gamma(t)$ by choosing the normal vector field $\nu(x,t) := \nabla d(x,t)$. Note that the constant extension of a function $\eta(\cdot,t)\colon \Gamma(t) \to \mathbb{R}$ to $\mathcal{N}(t)$ in the normal direction is given by $\eta^{-l}(x,t) = \eta(p(x,t),t)$, $p \in \mathcal{N}(t)$. Later on, we will use (2.4.4) to define the lift of functions on approximate hypersurfaces.

We finish this section with stating the so-called Leibniz (or Transport) formula or the time derivative of integrals over moving surfaces. Thus, the Leibniz formula is a generalization of the classical Reynolds' Transport Formula for curved domains. The proof can be found in [55, Lemma 2.1].

Theorem 2.4.2. Let $\{\Gamma(t)\}_{t\in[0,T]}$ be an evolving surface defined as above. Furthermore, assume that f is a function defined on $\mathcal{N}_T$ such that all the following quantities exist. Then

$$\frac{d}{dt}\int_{\Gamma(t)} f = \int_{\Gamma(t)} \partial^\bullet f + f\nabla_\Gamma \cdot \mathbf{v}. \tag{2.4.5}$$

Moreover,

$$\frac{1}{2}\frac{d}{dt}\int_{\Gamma(t)}|\nabla_\Gamma f|^2=\int_\Gamma \nabla_\Gamma f\cdot\nabla_\Gamma(\partial^\bullet f)+\frac{1}{2}\int_{\Gamma(t)}|\nabla_\Gamma f|^2\nabla_\Gamma\cdot\mathbf{v}-\int_{\Gamma(t)}D(\mathbf{v})\nabla_\Gamma f\cdot\nabla_\Gamma f, \quad (2.4.6)$$

where the deformation tensor is given by $D(\mathbf{v})_{ij}=\frac{1}{2}(\underline{D}_i\mathbf{v}_j+\underline{D}_j\mathbf{v}_i)$, $i,j=1,\dots n$.

2.5. Tensor products

The function spaces which will be used later have tensor product structure. We will mainly use the connection between tensor structure of functions and separation of variables. In this section we summarize without proofs the relevant material on tensor spaces. We will mainly focus on what tensor space is and how elementary tensors can be defined. For more details we refer to [71, 109].

We start with the most common examples of tensors. Then we will generalize these structures for general vector spaces and try to explain difficulties that appear when one defines tensor spaces in functional analysis. One usually starts with introducing tensors as vectors and matrices. Namely, the example of tensors that appears almost everywhere are vectors. Furthermore, matrices can be identified with tensors of order 2 and they correspond to linear mappings. These objects can be naturally generalized to tensors of order $d\geq 3$ which are studied in multi-linear algebra. Particularly, the set of tensors can be expressed by

$$\mathbb{R}^I:=\{v=(v_i)_{i\in I}:v_i\in\mathbb{R}\}$$

where $I=I_1\times\cdots\times I_d$ and $I_j=\{1,\dots,n_j\}$ for every $j\in\{1,\dots,d\}$. Defining the tensor product

$$v:=v^{(1)}\otimes\ldots\otimes v^{(d)}=\bigotimes_{j=1}^{d}v^{(j)}\in\mathbb{R}^I$$

via its entries

$$v_i=v[i_1\dots i_d]=v^{(1)}_{i_1}\cdot\ldots\cdot v^{(d)}_{i_d}\quad i\in I$$

we obtain the relation between $\mathbb{R}^{I_j}$ and $\mathbb{R}^I$, because the tensor space $\mathbb{R}^I$ can be written as

$$\bigotimes_{j=1}^{d}\mathbb{R}^{I_j}=\text{span}\{v^{(1)}\otimes\cdots\otimes v^{(d)}:v^{(j)}\in\mathbb{R}^{I_j},1\leq j\leq d\}. \quad (2.5.1)$$

In the infinite dimensional case, if we want to obtain a complete space (Banach or Hilbert), we need to modify the definition (2.5.1) and consider the completion of the span of elementary tensors with respect to a suitable norm. This is where the complexity appears since this norm is not fixed by the normed spaces that generate the tensor space. We will comment later on the choice of this norm in the Banach case and the Hilbert case.

In order to better understand the definition of topological tensor space, we start by introducing the algebraic foundation of tensor spaces, i.e. defining a tensor product of two vector spaces, which will again be a vector space.

Let U, V and M be vector spaces over the same field $\mathbb{K}$. The case $\mathbb{K} = M = \mathbb{R}$ will be the one of interest to us, but here we will keep a more general setting. We are interested in the bilinear mappings

$$\Phi : U \times V \to M.$$

The set of all these mappings $\mathcal{B}(U, V; M)$ is a vector space with respect to addition and scalar multiplication. It is clear that although $\mathcal{B}(U, V; M)$ and the vector space $\mathcal{L}(U \times V; M)$ of all linear mappings $L : U \times V \to M$ are closely related, they behave differently in many aspects. Since there are many drawbacks of bilinear mappings (e.g. there is no open mapping theorem for bilinear surjective maps, there is no Hahn-Banach theorem for bilinear continuous form etc.), one would like to try to reduce the study of bilinear maps to the study of linear maps. This is the motivation of introducing an algebraic tensor space. More precisely the question is

Is it possible to construct a new vector space

$$\mathcal{T} = U \otimes_a V$$

using just the vector spaces U and V, such that for every vector space M there exists some natural isomorphism $\Phi \mapsto \Phi'$ of $\mathcal{B}(U, V; M)$ onto a corresponding vector space $\mathcal{L}(T; M)$?

Instead of considering a bilinear map Φ of vector space $U \times V$ into any M, answering the previous question would enable us to consider a linear map Φ' from the new space $\mathcal{T}$ to M, i.e. we would get the natural connection

$$\mathcal{B}(U, V; M) \rightleftarrows \mathcal{L}(\mathcal{T}, M).$$

This result is known as a universality property and its precise statement and proof can be found in [43]. Here we will just state the result. Note that a similar construction is used in the construction of the ring of polynomials $\mathbb{K}[X]$.

Theorem 2.5.1. For any vector spaces U and V over the same filed $\mathbb{K}$, there exists at least one pair $(\mathcal{T}, \varphi)$ of vector space $\mathcal{T}$ and a bilinear map $\varphi : U \times V \to \mathcal{T}$ such that

a) the vector space $\mathcal{T}$ is generated by vectors $\varphi(u, v)$ from $\varphi(U \times V)$

b) for every vector space M over $\mathbb{K}$, the mapping

$$L \mapsto L \circ \varphi, \quad \mathcal{L}(\mathcal{T}, M) \to \mathcal{B}(U \times V, M)$$

is one isomorphism of a vector space $\mathcal{L}(\mathcal{T}, M)$ onto a vector space $\mathcal{B}(U \times V, M)$.

Moreover, this pair is unique up to the isomorphism, i.e. if there is some other pair $(\mathcal{T}', \varphi')$ that satisfies a) and b) then there exists a unique isomorphism $F : \mathcal{T} \to \mathcal{T}'$ for which it holds $\varphi' = F \circ \varphi$.

The vector space from the previous theorem is called the *algebraic tensor space* and it is denoted by

$$\mathcal{T} = U \otimes_a V$$

and $\varphi(u, v) = u \otimes v$ are called *elementary tensors*. Any element from T is called a *tensor*.

In other words, the mapping

$$\begin{aligned} \varphi : U \times V &\to U \otimes_a V && (2.5.2) \\ (u, v) &\mapsto u \otimes v && (2.5.3) \end{aligned}$$

defines one bilinear map and the whole algebraic tensor space $U \otimes_a V$ is generated by elementary tensors $u \otimes v$. Moreover, for every bilinear map $\Phi : U \times V \to M$ there exists a unique linear map $L : U \otimes_a V \to M$ for which the diagram

$$\begin{array}{ccc} U \times V & \xrightarrow{\Phi} & M \\ & \searrow^{\varphi} & \uparrow L \\ & & U \otimes_a V \end{array}$$

commutes for every vector space M over the same field $\mathbb{K}$ and this mapping defines an isomorphism

$$\mathcal{B}(U \times V; M) \cong \mathcal{L}(U \otimes V, M).$$

In particular, for $M = \mathbb{K}$ the previous relation becomes

$$\mathcal{B}(U \times V; M) \cong (U \otimes V)^* \cong \mathcal{L}(U, V^*).$$

Although not every tensor from $U \otimes_a V$ has the form $u \otimes v$, it is a finite linear combination of these elementary tensors, but this representation is not unique. However, since

$$\alpha \cdot (u \otimes v) = \alpha \cdot \varphi(u, v) = \varphi(\alpha u, v) = (\alpha u) \otimes v,$$

it follows that every tensor from $U \otimes_a V$ is a finite sum of elementary tensors. Moreover, if $\{e_i\}_{i=1}^m$ and $\{f_j\}_{j=1}^n$ are basis of U and V respectively, then $\{e_i \otimes f_j\}$ forms a basis of $U \otimes_a V$.

Remark 2.5.2. The previous result just states the existence of a pair $(\mathcal{T}, \varphi)$. For the construction one uses the quotient space form of the free vector space over $U \times V$. More precisely, the algebraic tensor space is defined as $U \otimes_a V := \mathcal{V}_{\text{free}}(U \times V)/_N$, where $\mathcal{V}_{\text{free}}$ is a free vector space and $N := \text{span}\{\sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j (u_i, v_i) - (\sum_{i=1}^m \alpha_i u_i, \sum_{j=1}^n \beta_j v_j)\}$ and an elementary tensor is $u \otimes v := c_{(u,v)}$. For more details see for example [71].

In algebraic constructions, span is always a finite combination and infinite sums, as well as the limit of sequences, can't be defined without topology. Since we are interested in function spaces, the next step is to define a topological tensor space, i.e. to make it complete and normed. For that purpose let U and V now be Banach spaces. The analysis of topological tensor space has been started by Schatten and Grothendick. The motivation to consider $X = U \times V$ of two Banach spaces, is that they are related to linear operator spaces and the tensor product structure allows us to transfer some properties of U and V to X, which are easier to consider. Thus the main goal is to define topology on tensor product space and this will be done considering the completion of $U \otimes_a V$ w.r.t. a given norm $\|\cdot\|$. This is where the difficulty appears, since it is not clear how to define this norm. Note that in the finite dimensional case the algebraic tensor space $U \otimes_a V$ is already complete. Furthermore, it is enough that just one of the spaces is finite dimensional (see [71, Corollary 4.61]).

The completion of the algebraic tensor space is called *Banach tensor space* and is denoted by

$$U \otimes_{\|\cdot\|} V := \overline{U \otimes_a V}^{\|\cdot\|}.$$

It is important to notice that the previous definition strongly depends on the norm $\|\cdot\|$. The reasonable question that appears is: how is the norm on $U \otimes_{\|\cdot\|} V$ connected to the norms on U and V? More precisely, do $\|\cdot\|_U$ and $\|\cdot\|_V$ determine the norm $\|\cdot\|$ on the algebraic tensor product in a canonical way? In general one would like to find a *crossnorm* i.e. a norm on $U \otimes_a V$ that satisfies

$$\|u \otimes v\| := \|u\|_U \|v\|_V, \quad \forall u \in U, v \in V. \tag{2.5.4}$$

Unfortunately, contrary to linear mappings, defining the norm on the elementary tensors doesn't determine the norm on the whole space $U \otimes_a V$. Norms that are crossnorm always exist, but they are not unique. Hence, the topological tensor space is not uniquely determined by $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$, but depends on the choice of the norm. The necessary condition which ensures that $\|u \otimes v\|$ is finite is the continuity of the tensor which is equivalent to

$$\exists C : \|u \otimes v\| \leq C \|u\|_U \|v\|_V. \tag{2.5.5}$$

In particular, note that every crossnorm is continuous. It turns out that the strongest possible norm that ensures continuity is a projective norm defined by

$$\|x\|_\wedge := \inf\{\sum_{i=1}^{n} \|u_i\|_U \|v_i\|_V : x = \sum_{i=1}^{n} u_i \otimes v_i,\ n \in \mathbb{N}\} \quad \text{for } x \in U \otimes_a V.$$

This means that if there is some other norm $\|\cdot\|$ that satisfies (2.5.5) with a constant C, then it also holds $\|\cdot\| \leq C\|\cdot\|_\wedge$. Moreover, any other continuous norm leads to a bigger topological tensor space i.e. $U \otimes_\wedge V$ is the smallest Banach tensor space that contains $U \otimes_a V$. For more details see [71].

Now we want to construct a concrete example of tensor space $\mathcal{T}$ and elementary tensor $\varphi(u, v) = u \otimes v$, whose existence were stated by the abstract result in Theorem 2.5.1. Sometimes Banach tensor space is defined in this concrete way and we want to show how that fits into the general picture. For every $u \in U$ and $v \in V$ we define a map $\Phi_{u,v} : \mathcal{B}(U, V) \to \mathbb{R}$ by

$$\Phi_{u,v}(B) := B(u, v) \quad \forall B \in \mathcal{B}(U \times V, \mathbb{R}).$$

Such a defined map $\Phi_{u,v}$ is a linear functional on $\mathcal{B}(U \times V; \mathbb{R})$ and we will show that it will be an elementary tensor. Now we can define $\mathcal{T}$, which will have a role of algebraic tensor space, as a span of all such $\Phi_{u,v}, u \in U, v \in V$. This, as commented before, coincides with all finite sums of these functionals. Hence, we can define a Banach tensor space as a space of all finite sums of $\Phi_{u,v}$:

$$\mathcal{T} := \left\{ \sum_{i=1}^{n} \Phi_{u,v} : u \in U, v \in V, n \in \mathbb{N} \right\}. \tag{2.5.6}$$

Furthermore, let us define a mapping $\varphi : U \times V \to \mathcal{T}$ via

$$\varphi : (u, v) \mapsto \Phi_{u,v},$$

which directly implies the assumption a) in Theorem 2.5.1 is fulfilled. We want to show that $u \otimes v := \varphi(u, v) = \Phi_{u,v}$ is an elementary tensor. Since $\Phi_{u,v} \in \mathcal{B}(U \times V)^*$, we have

$$\Phi_{u+u',v}(B) = B(u + u'; v) = B(u, v) + B(u', v) = \Phi_{u,v}(B) + \Phi_{u',v}(B),$$

which implies that the map φ is bilinear. It is left to show that such a defined pair $(\mathcal{T}, \varphi)$ satisfies the condition $b)$ from Theorem 2.5.1, i.e. that for every $\Phi : U \times V \to \mathbb{R}$ there exists a unique linear map $L : \mathcal{T} \to \mathbb{R}$ such that $\Phi = L \circ \varphi$. It is enough to define L on a basis of $\mathcal{T}$. For that reason let us consider $e = (e)_{r \in R}$ and $f = (f)_{s \in S}$ basis of U and V, respectively. Then one can directly check that

$$F := \{\Phi_{e_r, f_s} : r \in R, s \in S\}$$

is the basis of $\mathcal{T}$. Hence it must be

$$L(\Phi_{e_r, f_s}) = \Phi(e_r, f_s),$$

which uniquely defines the linear map L.

From Theorem 2.5.1 we conclude that

$$u \otimes v := \varphi(u, v) = \Phi_{u,v} : B \to B(u, v) \quad \forall B \in \mathcal{B}(U \times V, \mathbb{R})$$

defines one *concrete example* of elementary tensor. The corresponding algebraic tensor space $\mathcal{T} = U \otimes_a V$ is a vector space of all finite sums of such bilinear forms.

The important special case when the crossnorm $\|\cdot\|$ on the algebraic tensor space is unique is when we consider the algebraic tensor space of two *Hilbert* spaces and we want that the obtained tensor space itself is a Hilbert space. In this case we define an inner product on the algebraic tensor space that is compatible w.r.t. to the original inner products, and in this way we obtain a pre-Hilbert space.

Let us make this construction more precise. For that purpose let U and V be Hilbert spaces. As already announced, this case is simpler than the case when we consider just Banach spaces, because there is no issue about the choice of the norm, because of the linearity of the inner product. Namely, according to [71, Lemma 4.124] defining the scalar product on the elementary tensor

$$\langle u \otimes v, \tilde{u} \otimes \tilde{v} \rangle := \langle u, \tilde{u} \rangle_U \langle v, \tilde{v} \rangle_V \tag{2.5.7}$$

and extending it by linearity, determines a unique scalar product on the whole algebraic tensor space $U \otimes_a V$. One can prove that the norm induced by this scalar product satisfies all necessary conditions for a norm on a Banach topological space, i.e. it is a crossnorm (for more details see [71, Sect 4.4]). Hence, we can define the *(Hilbert) tensor space* $U \otimes V$ as a completion of $U \otimes_a V$ w.r.t. a unique norm derived from the scalar product defined by (2.5.7).

This general construction of a Hilbert tensor space as a closure of the algebraic tensor space w.r.t. a norm induced by the inner product is often made more concrete in the literature (for example in [109]) by introducing a concrete elementary tensor. Now we will define this particular elementary tensor and show that it is just a specific way of how one can construct the Hilbert tensor space. Before the definition, let us note that it is enough to consider these spaces just on the level of algebra, i.e. to construct just the algebraic tensor space. This follows from Theorem 2.5.1, which gives us the isomorphism between algebraic tensor spaces, and then completion of these spaces will provide us the isomorphism between Hilbert tensor spaces.

Let $u \in U$ and $v \in V$. We define $\Phi_{u,v} \equiv u \otimes v : U \times V \to \mathbb{R}$ by

$$\Phi_{u,v}(x, y) \equiv u \otimes v(x, y) := \langle u, x \rangle_U \langle v, y \rangle_V, \quad \forall x \in U, \forall y \in V,$$

which will have the role of a "concrete" elementary tensor. It is clear that $\Phi_{u,v} \in \mathcal{B}(U,V)$. The space $\mathcal{T}$ is now the span of all $\Phi_{u,v}$ and it coincides with all finite sums of these bilinear maps $\Phi_{u,v}$. The map $\varphi : U \times V \to \mathcal{T}$ must be defined as

$$\varphi(u,v) \mapsto \Phi_{u,v}$$

and it is clearly bilinear. It is left to show that such a defined pair $(\mathcal{T}, \varphi)$ satisfies the assumption b) from Theorem 2.5.1, i.e. that for every $\Phi : U \times V \to M$, there exists a unique linear map $L : \mathcal{T} \to M$ such that

$$\Phi = L \circ \varphi.$$

Since L is linear, it is enough to define it on a basis of $\mathcal{T}$. Hence, the next task is to determine the basis of $\mathcal{T}$. Since, as already explained, it is enough to consider U and V on the algebraic level, let

$$e = \{e_r\}_{r\in R} \text{ and } f = \{f_s\}_{s\in S}$$

be any two vector basis of U and V, respectively. Explicitly, this means that any $u \in U$ and $v \in V$ can be represented as a *finite* linear combination of basis vectors

$$u = \sum_{r=1}^{N_r} \alpha_r e_r \text{ and } v = \sum_{s=1}^{N_s} \beta_s f_s.$$

Thus we have

$$\Phi_{u,v} = \varphi(u,v) = \sum_{r=1,s=1}^{N_r,N_s} \alpha_r\beta_s \cdot \varphi(e_r, f_s) = \sum_{r=1,s=1}^{N_r,N_s} \lambda_{r,s} \cdot \Phi_{e_r,f_s},$$

which implies that

$$\mathbb{F} := \{\Phi_{e_r,f_s} \,:\, r \in R, s \in S\}$$

spans the whole space $\mathcal{T}$. It is left to show its algebraic linear independence. Let

$$\sum_{r=1,s=1}^{N_r,N_s} \lambda_{r,s} \cdot \Phi_{e_r,f_s} = 0$$

which yields

$$\sum_{r=1,s=1}^{N_r,N_s} \lambda_{r,s} \cdot \Phi_{e_r,f_s}(x,y) = \sum_{r=1,s=1}^{N_r,N_s} \lambda_{r,s} \cdot \langle e_r, x\rangle_U \langle f_s, y\rangle_V, \quad \forall x \in U, \forall y \in V. \tag{2.5.8}$$

Since these sums are finite, there exists some element $x_0 \in U$ which is orthogonal on every e_r except on one of them, for example e_i. We can define this $x_0 := w + e_i$, $w \in (\text{span}\{e_1, \ldots, e_{n_r}\})^{\perp}$. Similarly, we can choose y_0 such that $\langle y_0, f_s\rangle = \delta_{js}$. For such x_0 and y_0 relation (2.5.8) becomes $\lambda_{ij} = 0$. Hence $\mathbb{F}$ is a basis of $\mathcal{T}$. Now the linear map L is defined in a natural way by

$$L(\Phi_{e_r,f_s}) := \Phi(e_r, f_s).$$

Remark 2.5.3. Another special case when the crossnorm is uniquely determined is when we consider the tensor product of C^*-algebra and a matrix, and we want that obtained ensor space is C^*-algebra itself (for details see [43]).

We finish this section by stating some of the main results concerning the Hilbert tensor spaces that we will use later.

Theorem 2.5.4. The tensor space $U \otimes V$ is a Hilbert space. If $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ are basis of Hilbert spaces U and V, then $\{e_j \otimes f_k\}_{j,k\in\mathbb{N}}$ constitute a basis of $U \otimes V$.

Proof. The proof can be found for example in [109]. □

Theorem 2.5.5. Let (X, μ) and (Y, ν) be measure spaces such that $L^2(X, \mu)$ and $L^2(Y, \nu)$ are separable. Then, the following holds:

a) There is a unique isometric isomorphism

$$L^2(X, \mu) \otimes L^2(Y, \nu) \cong L^2(X \times Y, \mu \times \nu)$$

so that $f \otimes g \mapsto fg$.

b) If H is a separable Hilbert space then there is a unique isometric isomorphism

$$L^2(X, \mu) \otimes H \cong L^2(X, \mu; H)$$

so that $f(x) \otimes \varphi \mapsto f(x)\varphi$.

Proof. The proof can be found for example in [109]. □

2.6. Karhunen-Loève expansion

There are two types of problems concerning the expansion of a random variable one could discuss. The first one is how to construct a random variable on a function space X using randomized series. More precisely, we can define the random variable $u : \Omega \times D \to \mathbb{R}$, $D \subset \mathbb{R}^d$

$$u := m_0 + \sum_{j=1}^{\infty} u_j \phi_j \tag{2.6.1}$$

where $\{\phi_j\}_{j=1}^{\infty}$ is a sequence in the Banach space X, $m_0 \in X$ and $\{u_j\}_{j=1}^{\infty}$ is randomized sequence of the form $u_j := \gamma_j \xi_j$ where $\{\gamma_j\}_{j=1}^{\infty}$ is a deterministic sequence and $\{\xi_j\}_{j=1}^{\infty}$ is an i.i.d. random sequence. Choosing the specific space X and distribution of ξ_1, (2.6.1) defines different random variables. The typical examples of a constructed random variable u would be: uniform, Besov or Gaussian random variable. These type of constructions are discussed in [39, Sec 2] and additionally they analyse how the regularity of the resulting random variable u depends on the sequence $\{\gamma_j\}_{j=1}^{\infty}$. We will discuss briefly the case when ξ_1 is uniformly distributed in the Chapter 4 and when ξ_1 is a Gaussian random variable.

The second problem would be: given a random variable u, which assumptions about u ensure the randomized series representation of the form (2.6.1)? Furthermore, what can we say about

the functions that appear in this expansion, more specifically, about the distribution of a random sequence $\{\xi_j\}_{j=1}^{\infty}$? These type of questions will be discussed in this section. In particular, we will give a brief overview of the Karhunen-Loève expansion.

The Karhunen-Loève (KL) expansion is one of the possible representations of a random variable and it is well-studied in the field of uncertainty quantification (see [96, 111, 118]). In the 1940s, Karhunen [82, 83] introduced the use of spectral theoretic methods in the analysis of stochastic processes. His approach has been developed by the papers and books by Loève in the 1950s [93] and by Ash [7] in 1965.

The KL expansion is based on the singular value decomposition of the covariance operator of a random variable. Its advantage is that it characterizes the random variable by its covariance and mean value. Since the Gaussian random variable is uniquely determined by its mean value and covariance, it will always have a KL expansion. Furthermore, the uncorrelated Gaussian random variables are also independent and linear combinations of Gaussian random variables is a Gaussian random variable, which makes the KL expansion especially useful and important for the representation of Gaussian random variables. However, this is not true for other distributions and we do not have much information about the distribution of random variables that appear in its expansion. In addition, the eigenvalue problem that needs to be solved to obtain the functions from the representation is not explicitly solvable in most of the cases. Thus, the KL expansion is primarily used for the representation of the Gaussian random variables.

One of the reasons to use the KL expansion is that it makes computations feasible. Furthermore, among other possible decompositions it is optimal in a sense that it minimizes the total mean-square error. Besides the practical importance, KL expansion also enables us to reformulate the PDE with random coefficients as a parametric family of deterministic problems i.e., into a deterministic PDE posed on an infinitely dimensional parameter space (see Chapter 5). This transition is achieved by the mapping that arises from the KL expansion and maps the probability space into an appropriate parametric space with a proper Gaussian measure. This reformulation will turn out to be very useful in the proof of well-posedness of the RPDE with a log-normal coefficient (see Section 5.3).

Before we present the proof of the Karhunen-Loève theorem, we will state the assumptions about a random variable u that we want to expand, discuss the properties of its covariance operator and recall Mercer's theorem that will be used in the proof. For this part we mainly follow Chapter 11.1 in [3, 13, 118].

Let $D \subset \mathbb{R}^d$ be a *compact* domain. Note that the assumption about compactness is not necessary, we can instead consider a first-countable topological space equipped with a complete Borel measure, cf. [118]. However, the compactness assumption simplifies the rest of the assumptions and since this assumption is fulfilled in our setting, we will assume it also for this more general presentation. Recall that for our problem, D will be the space-time domain $\mathcal{G}_T$ defined by (2.4.2), which is compact as a continuous image of a compact set $[0, T] \times \Gamma_0$.

We consider a stochastic process $u : \Omega \times D \to \mathbb{R}$ for which we assume that it is *mean-square integrable* i.e. $u \in L^2(\Omega \times D)$. Without loss of generality we assume that u is centred i.e. $\forall x \in D$,
$\mathbb{E}[u(x)] = 0$, otherwise we can consider the random variable $v := u - \mathbb{E}[u]$. Furthermore, let $C_u : D \times D \to \mathbb{R}$ be the covariance function, sometimes called the auto-covariance function or the two-point correlation. It describes the spatial (or temporal) covariance of a random variable

and it is defined as a covariance of the values of a random variable at two locations x and y:

$$C_u(x,y) := \text{cov}(u(x),u(y)) = \mathbb{E}[u(\cdot,x)u(\cdot,y)] = \int_\Omega u(\omega,x)u(\omega,y)d\mathbb{P}.$$

We assume that C_u is *continuous*. Since D is compact, the previous assumption implies that C_u is bounded and hence it is square-integrable i.e. $C_u \in L^2(D\times D)$. The mapping C_u is often called a Hilbert-Schmidt kernel, because it will produce a Hilbert-Schmidt integral operator.

To see this, let us associate to it a linear integral operator $\mathcal{C}_u : L^2(D) \to L^2(D)$ by

$$[\mathcal{C}_u v](x) := \int_D C_u(x,y)v(y)dy. \tag{2.6.2}$$

In this setting, $\mathcal{C}_u$ is called a covariance operator.

Remark 2.6.1. Sometimes $\mathcal{C}_u$ is defined by

$$(\mathcal{C}_u v, w) := \int_D\int_D C_u(x,y)v(x)w(y)dxdy,$$

which is consistent with the definition (2.6.2) by Riesz's theorem.

Remark 2.6.2. If D is not compact, but just a first-countable topological space, then for the proof we will present we need to assume that C_u is both continuous and square-integrable.

The following lemma summarizes the properties of the covariance operator.

Lemma 2.6.3. Let C_u be continuous and $\mathcal{C}_u$ defined by (2.6.2). Then the following holds:

a) $\mathcal{C}_u$ is self-adjoint,

b) $\mathcal{C}_u$ is bounded,

c) $\mathcal{C}_u$ is compact,

d) $\mathcal{C}_u$ is positive.

Proof. a) Utilizing the symmetry of $C_u(x,y)$ and Fubini's theorem 2.2.3, we obtain that $\mathcal{C}_u$ is self-adjoint:

$$\langle \mathcal{C}_u p, q\rangle_{L^2(D)} = \int_D \mathcal{C}_u p(y)q(y)dy = \int_D \left(\int_D C_u(x,y)q(y)dy\right)p(x)dx = \langle p, \mathcal{C}_u q\rangle_{L^2(D)}.$$

b) The Cauchy-Schwarz inequality implies

$$\|\mathcal{C}_u v\|^2_{L^2(D)} = \int_D \left|\int_D C_u(x,y)v(y)dy\right|^2 dx \le \|C_u\|_{L^2(D\times D)}\|v\|_{L^2(D)}$$

which entails the boundedness of the operator $\mathcal{C}_u$.

c) Since

$$\|\mathcal{C}_u\|_{HS} = \text{trace }(\mathcal{C}_u) = \|C_u\|^2_{L^2(D\times D)},$$

it follows that $\mathcal{C}_u$ is a Hilbert-Schmidt operator on $L^2(D)$, and thus it is compact.

d) From Fubini's theorem 2.2.3 we compute

$$\begin{aligned}\langle \mathcal{C}_u v, v\rangle &= \int_D \left(\int_D \mathbb{E}[u(x)u(y)]v(y)dy\right) v(x)dx \\ &= \mathbb{E}\left[\left(\int_D u(x)v(x)dx\right)\left(\int_D u(y)v(y)dy\right)\right] = \mathbb{E}\left[\left(\int_D u(x)v(x)dx\right)^2\right] \geq 0,\end{aligned}$$

thus $\mathcal{C}_u$ is positive. □

Definition 2.6.4. Let D be a metric space. A function $K : D \times D \to \mathbb{R}$ is called a *Mercer kernel* if

a) K is continuous

b) K is symmetric: $K(x, y) = K(y, x) \quad \forall x, y \in D$

c) K is positive-semi definite: for any finite combination of points $x_1, \ldots, x_n$ the Gram matrix

$$G := \begin{bmatrix} K(x_1, x_1) & \ldots & K(x_1, x_n) \\ \vdots & \ddots & \vdots \\ K(x_n, x_1) & \ldots & K(x_n, x_n) \end{bmatrix}$$

is positive semi-definite i.e. $\xi \cdot G\xi \geq 0, \forall \xi \in \mathbb{R}^n$.

The previous definition can be generalized to any first-countable topological space D. Note that under the continuity assumption, C_u is a Mercer kernel. The following Mercer's theorem provides a series representation for the Mercer kernel based on the spectral representation of the associated integral operator. This result will be utilized in the proof of the KL theorem. For the proof see [67].

Theorem 2.6.5 (Mercer)**.** Let D be a first-countable topological space equipped with a complete Borel measure μ and let $K : D \times D \to \mathbb{R}$ be a Mercer kernel. If $x \mapsto K(x, x)$ lies in $L^1(X, \mu)$, then there exists an orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ of $L^2(D, \mu)$ consisting of eigenfunctions of the integral operator

$$v \mapsto \int_D K(\cdot, y)v(y)d\mu(y)$$

with a non-negative eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$. Furthermore, the eigenfunctions corresponding to non-zero eigenvalues are continuous, and

$$K(x, y) = \sum_{n\in\mathbb{N}} \lambda_n \psi_n(x)\psi_n(y)$$

where this series converges absolutely, uniformly over compact subsets of D.

Now we can prove the main theorem of this section.

Theorem 2.6.6 (Karhunen-Loève)**.** Let $D \subset \mathbb{R}^n$ be compact and $u : \Omega \times D \to \mathbb{R}$ be a centred square-integrable, stochastic process i.e., $u \in L^2(\Omega \times D)$ and $\mathbb{E}u = 0$. Furthermore, we assume that u has a continuous covariance function. Then

$$u(\omega, x) = \sum_{n\in\mathbb{N}} Z_n(\omega)\psi_n(x) \tag{2.6.3}$$

in the mean-square sense, where the $\{\psi_n\}_{n=1}^{\infty}$ are orthonormal eigenfunctions of the covariance operator $\mathcal{C}_u$, the corresponding eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ are non-negative and the convergence of the series is in $L^2(\Omega, L^\infty(D))$. The coefficients are given by

$$Z_n(\omega) = \int_D u(\omega, x)\psi_n(x)dx.$$

Furthermore, the random variables Z_n are centred, uncorrelated and have variance λ_n:

$$\mathbb{E}[Z_n] = 0 \quad \text{and} \quad \mathbb{E}[Z_m Z_n] = \lambda_n \delta_{mn}.$$

Proof. As already discussed, the covariance function C_u is a Mercer kernel and by the continuity assumption it is integrable on the diagonal. Hence, we can apply Mercer's theorem. Or alternatively (as it is done in [96]), according to Lemma 2.6.3, $\mathcal{C}_u$ is a compact, self-adjoint, positive operator, which enables us to directly apply the spectral theorem (see [12]). At any rate, we obtain the orthonormal basis $\{\psi_n\}_{n=1}^{\infty}$ of $L^2(D)$ consisting of eigenfunctions of the covariance operator $\mathcal{C}_u$ and the corresponding non-negative eigenvalues $(\lambda_n)_{n=1}^{\infty}$. Moreover, the eigenvalues and eigenvectors are connected as the solutions of the Fredholm equation of the second kind

$$\int_D C_u(x, y)\psi_n(y)dy = \lambda_n\psi_n(x) \tag{2.6.4}$$

and the eigenfunctions corresponding to a non-zero eigenvalues are continuous on D. In this basis the covariance function has the representation

$$C_u(x, y) = \sum_n \lambda_n\psi_n(x)\psi_n(y),$$

where the convergence is absolute and uniform on $D \times D$. Since by Theorem 2.5.5, $u \in L^2(\Omega \times D) \cong L^2(\Omega, L^2(D))$, we can use the basis $\{\psi_n\}_{n=1}^{\infty}$ of $L^2(D)$ to present $u(\omega)$

$$u(\omega, x) = \sum_{n\in\mathbb{N}} Z_n(\omega)\psi_n(x), \tag{2.6.5}$$

with the random coefficients given by the orthogonal projection

$$Z_n(\omega) = \int_D u(\omega, x)\psi_n(x)dx$$

where the equality should be understood in the mean-square sense i.e. with convergence in $L^2(\Omega \times D)$. To be more precise, we have that for every fixed sample $\omega \in \Omega$, the realization $\tilde{u} := u(\omega, \cdot) : D \to \mathbb{R}$ has the expansion

$$\tilde{u} = u(\omega) = \sum_{n\in\mathbb{N}} Z_n(\omega)\psi_n.$$

An even stronger result holds as a consequence of Mercer's theorem, that is, we even have convergence in $L^2(\Omega, L^\infty(D))$ i.e.

$$\lim_{N\to\infty} \mathbb{E}[|u(x) - u_N(x)|^2] = 0 \quad x \in D$$

uniformly, and hence point-wise, in D, where

$$u_N(\omega, x) = \sum_{i=1}^{N} Z_i(\omega)\psi_i(x)$$

and (2.6.5) holds for every $x \in D$. To prove this statement, let us calculate the following

$$\begin{aligned}
s_N(x) := \mathbb{E}[|u(x) - u_N(x)|^2] &= \mathbb{E}[u(x)^2] - 2\mathbb{E}[u(x)\sum_{i=1}^{N} Z_i\psi_i(x)] + \mathbb{E}[\sum_{i,j=1}^{N} Z_iZ_j\psi_i(x)\psi_j(x)] \\
&= C_u(x,x) - 2\sum_{i=1}^{N}\left(\int_D \mathbb{E}[u(x)u(y)]\psi_i(y)dy\right)\psi_i(x) + \sum_{i=1}^{N}\lambda_i\psi_i(x)^2 \\
&= C_u(x,x) - 2\sum_{i=1}^{N}\left(\int_D C_u(x,y)\psi_i(y)dy\right)\psi_i(x) + \sum_{i=1}^{N}\lambda_i\psi_i(x)^2 \\
&= C_u(x,x) - 2\sum_{i=1}^{N}[\mathcal{C}_u\psi_i](x)\psi_i(x) + \sum_{i=1}^{N}\lambda_i\psi_i(x)^2 \\
&= C_u(x,x) - 2\sum_{i=1}^{N}\lambda_i\psi_i(x)^2 + \sum_{i=1}^{N}\lambda_i\psi_i(x)^2 \\
&= C_u(x,x) - \sum_{i=1}^{N}\lambda_i\psi_i(x)^2.
\end{aligned}$$

Invoking Mercer's theorem we have

$$\lim_{N\to\infty} s_N = 0$$

uniformly in D. It is left to prove properties of the coefficients Z_n. Observe that although the law of random variables Z_n is unknown, we can still directly compute the following

$$\mathbb{E}[Z_n] = \int_D \mathbb{E}[u(x)]\psi_n(x)dx = 0$$

and

$$\begin{aligned}\mathbb{E}[Z_n Z_m] &= \mathbb{E}\left[\int_D u(x)\psi_n(x)\int_D u(y)\psi_m(y)dxdy\right] \\ &= \mathbb{E}\left[\int_D\int_D \psi_n(x)u(x)u(y)\psi_m(y)dxdy\right] \\ &= \int_D\int_D \mathbb{E}[u(x)u(y)]\psi_n(x)\psi_m(y)dxdy \\ &= \int_D \psi_n(x)\left(\int_D C_u(x,y)\psi_m(y)dy\right)dx \\ &= \int_D \psi_n(x)\int_D \lambda_m\psi_m(x)dxdy = \lambda_m\delta_{nm}.\end{aligned}$$

Now we directly get

$$\text{Var}[Z_n] = \mathbb{E}[(Z_n - \mathbb{E}[Z_n])^2] = \mathbb{E}[Z_n^2] = \lambda_n,$$

which implies

$$\text{Var}[u] = \sum_{n\in\mathbb{N}} \psi_n(x)^2 \text{Var}[Z_n] = \sum_{n\in\mathbb{N}} \lambda_n\psi_n(x)^2.$$

Hence, $\{Z_n\}_{n=1}^{\infty}$ are centered, orthogonal and with unit variance. Since $\mathbb{E}[Z_n] = 0$, the orthogonality implies that they are also uncorrelated. However, in general, Z_n are not independent. □

Since the random coefficients Z_n are orthogonal in the probability space and deterministic functions ψ_n are orthogonal in $L^2(D)$, sometimes the expansion (2.6.3) is called *bi-orthogonal*.

Let us comment on the assumptions of Theorem 2.6.6, since there are different versions of this theorem in the literature. For instance in [96], instead of continuity of the covariance function C_u the authors assume the mean-square continuity of the random variable u i.e.

$$\lim_{x\to x'} \|u(\cdot,x) - u(\cdot,x')\|^2_{L^2(\Omega)} = 0, \quad \forall x \in D.$$

However, the following Lemma states that these two assumptions are equivalent.

Lemma 2.6.7. A stochastic process is mean-square continuous if and only if its auto-correlation function C_u is continuous on $D \times D$.

Proof. For the proof see for example [3, Lemma 4.2]. □

Furthermore, we did not have to use Mercer's theorem to obtain the representation (2.6.3). Instead, we could apply the singular value decomposition to the operator $\mathcal{C}_u$ that results in the existence of the orthogonal system $\{\psi_n\}_{n=1}^{\infty}$ and the expansion (2.6.3). The difference is that without the continuity assumption of the covariance C_u , we just get the convergence in $L^2(\Omega, L^2(D))$, cf. [13]. Thus, stronger assumptions about the covariance function, particularly continuity, allow us to apply Mercer's theorem, which results in a stronger convergence result.

Let $\lambda_m = 0$, for some m and consider the corresponding coefficient Z_m in (2.6.3). Since $\mathbb{E}[Z_m] = 0$ and $\text{Var}[Z_m] = \lambda_m = 0$, we conclude that $Z_m = 0$. Thus, in the KL expansion, we

can consider just those eigenvectors that correspond to a non-zero eigenvalue. However, note that in this case we also exclude those eigenvectors ψ_m that correspond to zero eigenvalues and then the system $\{\psi_n\}_{n=1}^{\infty}$ is then just orthonormal but not necessarily the basis of $L^2(D)$. This conclusion enables us to normalize the coefficients Z_n in the KL expansion and define

$$\eta_n := \frac{1}{\sqrt{\lambda_n}} Z_n,$$

which leads to a more familiar version of expansion (2.6.3) given by

$$u(\omega, x) = \sum_{n\in\mathbb{N}} \sqrt{\lambda_n}\eta_n(\omega)\psi_n(x), \tag{2.6.6}$$

where now

$$\eta_n(\omega) = \frac{1}{\sqrt{\lambda_n}} \int_D u(\omega, x)\psi_n(x)dx.$$

Note that infinitely many processes have the same covariance kernel C_u, and hence will lead to the same basis $\{\psi_n\}_{n=1}^{\infty}$. What distinguishes these processes is the joint probability of the random variables η_n , nevertheless, in general we don't know what this distribution is. This is what makes the Gaussian random variable special and in particular convenient, since in this case we know that the distribution of η_n is also Gaussian.

Remark 2.6.8. The KL expansion is the only decomposition of the random variable based on the eigenvalues of $\mathcal{C}_u$ that results in orthogonal random variables η_n (see [64]).

Let us mention two basic examples concerning the KL expansion. Namely, we can find explicit KL expansion of a Brownian motion and Brownian bridge. These and other examples can be found in the book [127].

Example 2.6.9. Let $W(t)_{t\in[0,T]}$ be a Brownian motion. Then $C_W(t, s) = \min\{t, s\}$. One can show that for $t, s \in [0, 1]$, the eigenvectors of the covariance function $\min\{t, s\}$ are

$$\Psi_n(t) = \sqrt{2}\sin\left(\left(n - \frac{1}{2}\right)\pi t\right)$$

and the corresponding eigenvalues are

$$\lambda_n = \frac{1}{\left(n - \frac{1}{2}\right)^2 \pi^2}.$$

Then $W(t)$ can be written in the KL expansion in the following way

$$W(t) = \frac{2\sqrt{2}}{\pi} \sum_{n\in\mathbb{N}} \eta_n \frac{\sin\left(\left(n - \frac{1}{2}\right)\pi t\right)}{2n - 1},$$

where η_n are mutually independent standard Gaussian random variables. △

Example 2.6.10. Another example of an explicit KL expansion can be obtained if we consider the Brownian bridge. The Brownian bridge $\{B_t\}_{t\in[0,1]}$ can be derived from the Brownian motion $\{W_t\}_{t\in[0,1]}$ by conditioning $W_1 = 0$. Thus, $B_t = W_t - tW_1$. One can show that the Brownian bridge is a centred Gaussian process and its covariance is given by $C_B(s,t) = \min\{s,t\} - st$ and that the eigenvectors of the covariance function are given by

$$\psi_n(t) = \sqrt{2}\sin(n\pi t)$$

with corresponding eigenvalues

$$\lambda_n = \frac{1}{n^2\pi^2}.$$

Thus the KL expansion of the Brownian bridge on $[0,1]$ is given by

$$B(t) = \frac{\sqrt{2}}{\pi}\sum_{n\in\mathbb{N}}\frac{1}{n}\eta_n\sin(n\pi t),$$

where $\eta_n \sim N(0,1)$ are i.i.d. and the convergence in mean is almost sure (see Section 2.7). $\triangle$

In practice the infinite dimensional KL expansion is of little use and one usually considers the finite dimensional sum

$$\hat{u}(\omega,x) = \sum_{i=1}^{N_{KL}}\sqrt{\lambda_i}u_i(x)\eta_i(\omega).$$

To see how we should choose N_{KL} such that the truncation error is satisfied and that the KL expansion is an appropriate way to approximate a random variable we consider the mean-square error:

$$\epsilon^2_{N_{KL}} := \mathbb{E}[\|u(x,\cdot) - \hat{u}(x,\cdot)\|^2_{L^2(D)}].$$

It turns out that $\epsilon^2_{N_{KL}}$ is the smallest when $\hat{u}$ is defined as a truncation of N_{KL} elements of the KL expansion. Namely, the KL expansion is optimal in the sense that from all the approximations of N_{KL} elements, KL is the one that minimizes the total mean square error. In addition, the direct calculations imply

$$\begin{aligned}\epsilon^2_{N_{KL}} &= \sum_{i,j>N_{KL}}\sqrt{\lambda_i\lambda_j}(u_i,u_j)\mathbb{E}[\eta_i,\eta_j]\\ &= \sum_{i,j>N_{KL}}\sqrt{\lambda_i\lambda_j}\delta_{ij}\delta_{ij} = \sum_{i>N_{KL}}\lambda_i.\end{aligned}$$

Hence, how fast the mean square error goes to zero depends on the decay of eigenvalues of $\mathcal{C}_u$, which decrease to zero as i goes to infinity. Thus, the number N_{KL} that we need to achieve some specific error threshold depends on the correlation function of the process and the more correlated the process is, the smaller N_{KL} we need.

Remark 2.6.11. The decay of eigenvalues in the KL expansion depends on the smoothness of the covariance functions, and different notions of regularity of a covariance function provide some special bounds for the eigenvalues in the KL expansion. For more details see [113].

Remark 2.6.12. When we are considering a PDE with random coefficients, to get the finite representation of a random solution u, it is often assumed that a random coefficient α in the equation depends only on the finite number of random variables $\alpha(\omega, x) = \alpha(Y_1(\omega), \dots, Y_N(\omega), x)$, where $\mathbb{E}Y_i = 0$ and $\mathbb{E}[Y_i Y_j] = \delta_{ij}$ and in addition it is assumed that Y_i are independent. Thus, one immediately considers the truncated KL expansion. Applying the Doob-Dynkin Lemma A.3.1, together with the finite dimensional noise assumption, shows that the solution of a random PDE can also be expressed by a finite number of random variables

$$u(\omega, x) = u(Y_1(\omega), \dots, Y_N(\omega), x).$$

The natural question that appears is: when can we calculate the eigenvalues λ_n and eigenvectors ψ_n of the integral operator $\mathcal{C}_u$, that we need in order to get the KL expansion? TThe answer to this question depends on when we are able to solve the Fredholm equation (2.6.4). It has been found (see for instance [64, 96]) that in the case when the random variable u has a known rational spectral density, the analytical solutions of (2.6.4) are available. Furthermore, as already mentioned, if the random variable is Gaussian, then the explicit representation of its eigenvalues and eigenvectors of the covariance kernel are available, and thus we can write down the exact KL representation. Other than that, just in some specific cases (see [64]) the analytical solutions are available, but in general we need to apply the numerical methods for solving the eigenvalue problems.

One can also consider the KL expansion of a *vector-valued* random variable. This situation naturally appears if we want to consider a PDE on a random domain $D(\omega) \subset \mathbb{R}$ and we assume that there exists a random mapping V that connects the random domain $D(\omega)$ with some fixed domain D, i.e. $V(\omega) : D \to D(\omega)$. Hence V is an example of a vector-valued random variable. This setting will appear in Chapter 9. For this reason we briefly comment on what happens with the KL expansion of a vector-valued random variable. More on this topic can be found, for example, in [68, 73, 93].

Let $D \subset \mathbb{R}^d$ be a fixed domain and

$$V(x, \omega) = [v_1(x, \omega), \dots, v_d(x, \omega)]^T \in L^2(\Omega, L^2(D, \mathbb{R}^d)).$$

Then the mean of V is a vector given by

$$\mathbb{E}[V](x) = [\mathbb{E}[v_1](x), \dots, \mathbb{E}[v_d](x)]^T \in L^2(D, \mathbb{R}^d),$$

where $\mathbb{E}[v_i](x) \in L^2(D)$ is the mean of a scalar random variable v_i. Now the covariance function $\mathrm{Cov}[V]$ of V is matrix-valued and it is given by

$$\mathrm{Cov}[V](x, y) = [\mathrm{Cov}_{i,j}(x, y)]_{i,j=1}^d \in L^2(D \times D, \mathbb{R}^{d \times d}),$$

with

$$\mathrm{Cov}_{i,j}(x, y) = \mathbb{E}\left[(v_i(x, \omega) - \mathbb{E}[v_i](x))\,(v_j(y, \omega) - \mathbb{E}[v_j](y))\right] \in L^2(D \times D).$$

The covariance operator $\mathcal{C}_V : L^2(D, \mathbb{R}^d) \to L^2(D, \mathbb{R}^d)$ is defined by

$$[\mathcal{C}_V v](x) := \int_D \mathrm{Cov}[V](x, y) v(y) dy.$$

Note that $\mathcal{C}_V$ is the analogue of the operator $\mathcal{C}_u$ defined by (2.6.2) in the scalar case.

As in the scalar case, it can be proved that the covariance operator $\mathcal{C}_V$ is a non-negative, symmetric, trace class operator with the trace $\|V - \mathbb{E}[V]\|_{L^2(\Omega, L^2(D,\mathbb{R}^d))}$ (for the proof see [73, Lemma 3.1]). Hence, $\mathcal{C}_V$ is a compact operator and has a spectral decomposition

$$\mathcal{C}_V u = \sum_{k\in\mathbb{N}} \lambda_k (u, \varphi_k)_{L^2(D,\mathbb{R}^d)} \varphi_k, \quad \forall u \in L^2(D, \mathbb{R}^d),$$

where $\{\varphi_k\}_{k=1}^{\infty}$ is an orthonormal set of eigenvectors in $L^2(D, \mathbb{R}^d)$ and $\{\lambda_k\}_{k=1}^{\infty}$ are corresponding eigenvalues. As a result, we obtain that the KL expansion of the vector field V is given by

$$V(x, \omega) = \mathbb{E}[V](x) + \sum_{k\in\mathbb{N}} \sqrt{\lambda_k} \varphi_k(x) X_k(\omega)$$

and

$$X_k(\omega) = \frac{1}{\sqrt{\lambda_k}} \int_D (V(x, \omega) - \mathbb{E}[V](x))^T \varphi_k(x) dx.$$

Note that these formulae are analogue to formulae in the scalar case, the only difference is that we did not assume that V is centred, hence we have an additional mean value term and instead of $L^2(D)$ space we have the corresponding $L^2(D, \mathbb{R}^d)$ space.

We will finish this section with a remark that comments on the limitations of the KL expansion and possible alternative approaches.

Remark 2.6.13. Let us note that an obvious issue that appears in calculation of the KL expansion is that we need to know the correlation function of the random variable that we represent, but in general we don't have this information a priori for the variables that we want to calculate, i.e. we do not know the joint probability of random variables η_i a priori. Moreover, the correlation function of the unknown random variable is usually exactly the one of statistic we want to calculate. Furthermore, as already mentioned, in most of the cases we can not solve the eigenvalue problem (2.6.4). Hence, some other representations of the random variables are needed, such as polynomial chaos expansion, where one prescribes a priori the form of random coefficients as polynomials of independent random variables with prescribed distribution. In the case of Gaussian and uniform measures, the polynomial chaos representations use Hermite and Legendre polynomials, respectively and other probability measures give rise to other polynomial systems. For more details on this topic see for example [96].

Another approach is to consider the expansion w.r.t. the basis of the Cameron-Martin space. More on this topic can be found for example in [39, 111] and some brief remarks are made is Section 2.7.

Remark 2.6.14. Let us finish this section by mentioning the so-called *hierarchic discrete spectral expansion* introduced in [66]. This is an alternative representation of Gaussian random fields. In contrast to KL expansion, it can be constructed directly from the covariance kernel, thus one does not need to calculate eigenvalues of the covariance operator. Furthermore, it does not assume any particular structure of the covariance kernel and underlying domain.

2.7. Gaussian and log-normal fields

In this section we want to analyse a log-normal random field on the space-time domain defined by (2.4.2). Hence, instead of considering the expansion of the Gaussian filed, we consider the expansion of its logarithm. Thus, if we want to talk about the log-normal random field on $\mathcal{G}_T$, we first have to investigate the Gaussian random field on $\mathcal{G}_T$. One of the standard approaches when one wants to approximate and simulate a Gaussian random field is to write its KL expansion. In particular, in the previous section we discussed the KL expansion of any random field on a domain D. For the special case of Gaussian random field the KL expansion has additional properties, i.e. we know more about the distribution of random variables that appear in the expansion. Furthermore, as discussed in the conclusion of the previous section, there are other possible ways to represent the Gaussian random field, such as spectral decomposition or expansion w.r.t. the basis of its Cameron-Martin space. Natural questions arise: when do these type of expansions exist and what are their properties, how does the regularity of the covariance function influence the sample regularity of the Gaussian field etc. Although these types of questions, as well as more general approaches to the representation of the Gaussian field on $\mathcal{G}_T$, exceed the scope of this thesis, we will give a brief overview of the existing results, with the goal to describe challenges and possibilities for their application or generalization. In particular, these problems have been considered for the log-normal diffusion on the sphere [75, 87] and generalized for the sphere cross time in [40]. In order to explain what kind of sample regularity results we would need to have if we want to consider the numerical analysis of log-normal diffusions on the $\mathcal{G}_T$, we will briefly present the existing results on the sphere and outline the difficulties of their generalization to the evolving hypersurface.

In this section we will state the results about Gaussian random fields mainly following the book [2]. There exists various literature that covers the theory of Gaussian random fields on general parameter space, such as [1, 20, 111] etc.

We have already defined the random variable on any measurable space $(\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Since we want to consider $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}$, we need a notion of random fields, in particular random function defined over a Euclidean space. We start with recalling some basic general definitions and especially, properties of Gaussian random fields (GRF). Let $\mathbb{R}^{\mathbb{T}}$ denote the space of all real functions on a non-empty set $\mathbb{T}$ equipped with the topology of point-wise convergence. Usually $\mathbb{T}$ is a subset of a Euclidean space or, more generally, from a Riemannian manifold. Especially, often $\mathbb{T} = [0, T]$ or $\mathbb{T} = [0, \infty)$ represents time, and we talk about stochastic (random) processes. The general definition of a random space is stated in [2, Def 1.1.2]:

Definition 2.7.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{T}$ a topological space. Then a measurable mapping $f : \Omega \to \mathbb{R}^{\mathbb{T}}$ is called a *real valued random field*. Measurable mappings $f : \Omega \to (\mathbb{R}^{\mathbb{T}})^d, d > 1$, are called *vector valued random fields*.

Thus, a random field $\{X(t) : t \in \mathbb{T}\}$, opposed to a random process, denotes a collection of RVs indexed by a set that is not necessarily a subset of $\mathbb{R}$. Hence, it is a mapping $f : \Omega \times \mathbb{T} \to \mathbb{R}$ that is $\mathcal{F} \otimes \mathcal{T}$-measurable, where $\mathcal{T}$ is a Borel σ-field generated by the topology on $\mathbb{T}$. Note that we will not use the notation of a random process for $\mathbb{T} \subset \mathbb{R}$, but expressions random processes and random fields will be used equally.

Remark 2.7.2. To avoid differences in sample path behaviour of equivalent processes, we assume that all random fields are separable throughout in the sense of Doob, i.e. sample functions are determined by their values on a dense subset. A precise definition of a separable random field is stated in ([1, Def. 1.1.3]). Furthermore, in the same book the authors present an example of different properties of processes in a non-separable setting.

Let us now consider a special Gaussian distribution. What makes Gaussian RVs so tempting is the analytical form of their density which makes it possible to get many explicit results, which is not possible in most of the other cases.

A real valued random variable $X : \Omega \to \mathbb{R}$ is said to be a *Gaussian random variable* (GRV) if its distribution has the density function with respect to Lebesgue measure given by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}},$$

for some $m \in \mathbb{R}$ and $\sigma > 0$. The number m is called the mean and σ^2 the variance of X and we write $X \sim \mathcal{N}(m, \sigma^2)$. If $m = 0$, then the RV is called centred. A Gaussian distribution is completely determined by its first and second moment.

An $\mathbb{R}^d$-valued random variable $X : \Omega \to \mathbb{R}^d$ is called *a multivariate Gaussian* (vector) if for every $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$ the $\mathbb{R}$-valued variable $\sum_{i=1}^d \alpha_i X_i$ is Gaussian. By the Cramér-Wold theorem, the distribution of a random vector $(X_1, \ldots, X_n)$ is uniquely determined by the laws of all linear combinations $\sum_{i=1}^d \alpha_i X_i$. Thus, we can equivalently define a multivariate Gaussian distribution via the the joint density of a random vector $X = (X_1, X_2, \ldots, X_d)$ given with

$$\varphi(x) = \frac{1}{(2\pi)^{d/2}|C|^{1/2}} e^{-\frac{1}{2}(x-m)'C^{-1}(x-m)},$$

where $m \in \mathbb{R}^d$ is a mean vector, with $m_i := \mathbb{E}[X_j]$, C is a non-negative definite $d \times d$ covariance matrix with elements $c_{ij} = \mathbb{E}[(X_i - m_i)(X_j - m_j)]$ and $|C| := \det C$.

From the previous specific analytical form of probability density, it follows that a Gaussian random vector is completely determined by its mean and covariance function. Furthermore, for Gaussian RVs, notions uncorrelated and independent are equivalent. By direct calculations it follows that linear transformations preserve the Gaussian distribution of random vectors. GRVs are also important for theoretical reasons, because in contrast to Lebesgue measure, we can define an infinite dimensional Gaussian measure. These properties make Gaussian processes preferable to work with and there exists a very rich and well-understood general theory about of Gaussian random fields.

A Gaussian random field is a collection of random variables such that any finite number of RVs has a joint Gaussian distribution. More precisely:

Definition 2.7.3. A random field $X : \Omega \times \mathbb{T} \to \mathbb{R}$ is called $\mathbb{R}$-*valued Gaussian random field* (GRF) if for each $1 \leq n < \infty$ and every $(t_1, ..., t_n) \in \mathbb{T}^n$, the finite dimensional distributions of $(X_{t_1}, ..., X_{t_n})$ are multivariate Gaussian. Moreover, a $\mathbb{R}$-valued random filed $\alpha : \Omega \times \mathbb{T} \to \mathbb{R}_+$ is *log-normal RF* if $\log \alpha$ is an Gaussian RF on $\mathbb{T}$.

Remark 2.7.4. Analogously, one can define a multivariate Gaussian fields $X : \Omega \to (\mathbb{R}^{\mathbb{T}})^d$ taking values in $\mathbb{R}^d$ as fields for which the linear combination $\sum_{i=1}^d \alpha_i X_{t,i}$ is a real valued Gaussian field for every $\alpha \in \mathbb{R}^d$.

As emphasized in the book [2], what makes a Gaussian random field special, is that its definition and some of its basic properties have very little to do with the parameter space $\mathbb{T}$ on which the Gaussian random variable $f : \Omega \to \mathbb{R}^{\mathbb{T}}$ is defined. This fact ensures a substantial generality of spaces where we can define Gaussian random fields, especially important cases being when $\mathbb{T}$ is either Euclidean space or Riemann manifold.

As expected, a Gaussian process is characterized by its mean function $m : \mathbb{T} \to \mathbb{R}$ and covariance function $C : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, that is symmetric and positive definite. Note that if finite number of RVs are jointly Gaussian, each of them is Gaussian. However, the opposite is not true, if X_1 and X_2 are Gaussian, then (X_1, X_2) doesn't have to be jointly Gaussian.

Remark 2.7.5. As discussed in Adler's book [1], there are two main approaches in defining a random field:

• measure theoretic approach, via measurable mappings, which leads to a probabilistic setting,

• probabilistic approach, defining a random field as a collection of random variables and measures.

The natural question is how these two approaches are connected. More precisely, if we use the second approach, can we always find a random field according to the first approach that has these measures as finite-dimensional distributions? Not every family of measures corresponds to a random field. However, we will always assume that our family does correspond to one. Kolmogorov gave necessary and sufficient conditions when this is the case: we need the given family of measures to satisfy the so-called properties of symmetry and consistency. For more details see [1, 20]. Now, let us comment on this matter in our specific case of GRF. One can define the GRF in a probabilistic approach as in Definition 2.7.3. Another approach would be to start with a Gaussian measure [39, 118]. In $\mathbb{R}^n$, as we saw, Gaussian measures are defined via the Radon-Nikodym derivative w.r.t. the Lebesgue measure. However, since there is no infinite dimensional Lebesgue measure, in order to generalize the definition, it is better to start with the characterization of the Gaussian measures via push-forward. Namely, it holds that a push-forward of a Gaussian measure by any linear functional is a Gaussian measure on $\mathbb{R}$. Thus, we say that a Borel measure μ on a normed vector space V is a Gaussian measure if for every linear functional $l \in V^*$ the push-forward $l_*\mu$ is Gaussian on $\mathbb{R}$. Then one can define (generalized) Gaussian RP as measurable mapping which induces a Gaussian measure. However, if we define a GRP as in Definition 2.7.3, then two legitimate questions emerge. Namely, given a Gaussian process with paths in a linear functional space, is there a Gaussian measure on the function space that is induced by a given process? And conversely, given a Gaussian measure on a linear functional space, is there a Gaussian process with paths in the function space, which induces the given measure? These questions were analysed in the paper [107] and positively answered for the function spaces $C(I), C^n(I), AC(I)$, for I being an appropriate interval, and for the space $L^2(\mathbb{T}, \mathcal{A}, \nu)$, where $(\mathbb{T}, \mathcal{A}, \nu)$ is a σ-finite measure space. For more details on this topic see [20].

According to [20, Prop 2.3.9], the GRP as a family of RVs such that all their finite linear combinations are Gaussian induces a Gaussian measure μ on the path space $\mathbb{R}^{\mathbb{T}}$ with the topology of point-wise convergence. Let us state some of the properties of GRP that we will exploit. By Fernique's theorem (see [118, Th. 2.47]), Gaussian process has all finite moments. Sazonov's theorem answers the question which operators can be covariance operators. Namely, they have to be positive, self-adjoint and trace class. As discussed in [118], the translation of a Gaussian

measure doesn't have to have a density w.r.t. original measure, this gives rise to Cameron-Martin spaces. For a Gaussian measure on a Banach space it is possible to associate to it a Hilbert space that is a subset of the given Banach space, this subset is called a Cameron-Martin space. It characterizes which Gaussian measures are singular and which are equivalent, these are the only two options that can happen in the infinite dimensional setting. As explained in [72], the significance of the Cameron-Martin space is that it tells us in which directions translations of the Gaussian measure are 'quasi-invariant' i.e. the translated measure has the same null sets as the original measure. Note that a Cameron-Martin space is very small, in the sense that in the infinite dimensional setting it even has Gaussian measure zero. The Gaussian measure is determined by its Cameron-Martin space. In the finite dimensional case $\mathbb{R}^n$, the Cameron-Martin space is given by the range of the covariance matrix. Later we will exploit the basis of Cameron-Martin space to expand the Gaussian random field. For more details about Cameron-Martin space see [39, 72, 118].

The next interesting problem concerns the sample regularity of GRF. The spatial smoothness of samples is particularly important for numerical analysis since it can be exploited to improve the order of convergence. Sample regularity of random fields and processes has been widely analysed. The review of existing results can be found in [6, 104]. These results are mainly based on the Kolmogorov-Chentsov theorem (see [37]) that enables the existence of continuous modifications of stochastic processes and derives bounds on the exponent of a sample Hölder continuity (see Apendix A.4). Furthermore, in [6] they prove Kolmogorov-Chentsov theorem in a more general sense and show sample differentiability of random fields on domains of cone type and on manifolds.

Thus, the regularity of random fields can be expressed in terms of regularity of its covariance function. If we create GRF by random series according to (2.6.1), as discussed in the previous section, similar regularity results can be obtained in terms of the deterministic sequence γ and ξ. Namely, let H be a Hilbert space of $\mathbb{R}$-valued functions on domain $D \subset \mathbb{R}^d$, $\{\phi_j\}_{j=1}^{\infty}$ be an ONB of H and we set $\xi \sim N(0,1)$. Then (2.6.1) defines a Gaussian measure $\mathcal{N}(m_0, C)$ on H where C depends on γ. Assuming that $\gamma_j \asymp j^{-s/d}$, $s > 0$, the series exists as an L^2-limit in appropriate Hilbert space. For more details see [39, Sec. 2.4,2.5].

The next step would be to consider the representation of Gaussian random fields. One possibility is to consider a spectral representation, which is a general way of generating all stationary fields and which helps us to better understand sample path properties of stationary fields. More about stationary RFs can be found in [2]. The main property of a strictly stationary random field over $\mathbb{T}$, with respect to the group operation $+$, is that its finite dimensional distributions are invariant under this operation. In this work we will not consider the stochastic representation, but rather the KL expansion type of representation. However, we shall be trying to explain what the difference is between these two. The general spectral representation of a mean square continuous, centred, (Gaussian) stationary random field on $\mathbb{R}^N$ is given by [2, Theorem 1.4.4]. A special case of isotropic fields simplifies a general spectral representation. First of all, one characterizes isotropic fields in a stationary case by the property that the covariance function depends only on the Euclidean length $|t|$ of the vector t i.e. $C(t) = C(|t|)$. This assumption leads to many consequences, such as a special form of the covariance function given by [2, Theorem 1.4.6], limiting effects of the spectrum and simplifications of the spectral measure. Moreover,

the spectral representation of isotropic fields on $\mathbb{R}^{\mathbb{T}}$, given by

$$f(t) = f(r, \theta) = \sum_{m,l=0}^{\infty} f_{ml}(r) h_{ml}^{(N-1)}(\theta), \tag{2.7.1}$$

is based on the so-called spherical harmonics $h_{ml}^{(N-1)}$ on the $(N-1)$ sphere $\mathbb{S}^{N-1}$, which form the orthonormal basis of square integrable function on $\mathbb{S}^{N-1}$ and the family of mutually uncorrelated, stationary, one dimensional process $\{f_{ml}\}$. For more details see [2, Th. 1.4.7]. It is important to note that in the spectral representation (2.7.1) there is a hidden stochastic process entering via f_{ml} and spectral noise, and this makes a significant difference between (2.7.1) and the similar looking KL expansion!

As already mentioned, we will rather consider the KL expansion of a centred Gaussian process, which is a special case of general orthogonal expansion with respect to the orthonormal basis of its Cameron-Martin space. That is, every centred Gaussian process with a continuous covariance function has an expansion of the form

$$f(t) = \sum_{n=1}^{\infty} \xi_n \varphi_n(t), \tag{2.7.2}$$

where $\{\varphi_n\}_{n=1}^{\infty}$ is an orthonormal basis for the Cameron Martin space and $\{\xi_n\}_{n=1}^{\infty}$ is an i.i.d. sequence of Gaussian variables. For more details see [2, Th. 2.5.1]. In addition, if $\mathbb{T}$ is a "nice" subset of $\mathbb{R}^N$, then (2.7.2) leads to the KL expansion.

Remark 2.7.6. Besides the previously given comments, assumptions that ensure the continuity of the covariance function of a Gaussian process are discussed in [2, Sect. 2.1]. Moreover, as a consequence of the representation (2.7.2), one can show that Gaussian processes are either continuous with probability one or discontinuous with probability one.

The representation (2.7.2) has many practical implications in simulations of stationary processes on Euclidean space, since it provides a simple method for sampling Gaussian measures. In principle, one would need to truncate the sum and determine $\{\varphi_n\}$ which form the orthonormal basis of the Cameron-Martin space and are solutions of the eigenvalue problem that involves covariance. We will not explain the details of how to derive the orthogonal representation (2.7.2), but we refer the interested reader to [2, 20, 39, 111]. Instead, we will consider the earlier analysed KL expansion.

As already announced, when $\mathbb{T}$ is a nice subset of $\mathbb{R}^N$, i.e. more precisely, when $\mathbb{T}$ *is a compact subset of* $\mathbb{R}^N$, there is a simplification of finding the orthonormal functions φ_n and orthogonal representation leads to the KL expansion. Hence, the KL expansion is a special case of an expansion of a Gaussian random field w.r.t. the orthonormal basis of its Cameron-Martin space that has independent standard coefficients. To see this, one has to prove that $\{\sqrt{\lambda_n}\psi_n\}_{n=1}^{\infty}$ forms a complete orthonormal system in the Cameron-Martin space (where (λ_n, ψ_n) is the eigenpair of the covariance operator). For the proof see [2, Lemma 2.5.6]. Hence, by setting $\varphi_n = \sqrt{\lambda_n}\psi_n$ in (2.7.2), we obtain the KL expansion (2.6.6) of f:

$$f = \sum_{n} \sqrt{\lambda_n} \xi_n \psi_n. \tag{2.7.3}$$

Considering the KL expansion of GRF leads to certain simplifications. In this case, ξ_n are jointly Gaussian, as linear functional of a Gaussian random field f. Note that by Fernique's theorem, every GRF has a KL expansion. Moreover, in Gaussian case, uncorrelated implies independence. Hence, ξ_n are i.i.d. $N(0,1)$ random variables. The independence of ξ_n implies the almost sure convergence of the KL expansion. Namely, the mean-square convergence of the KL expansion implies the convergence in probability, which together with the independence of ξ_n and utilizing Elemadi's inequality, entail the almost sure convergence. For more details see [19, Th. 5.10].

Having setting up this general framework with abstract $\mathbb{T}$, let us now state some facts about our specific case of a Gaussian random process α that is defined on $\mathcal{G}_T$, defined by (2.4.2). Hence, $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}$ is a Gaussian random field parametrized over $\mathbb{T} = \mathcal{G}_T$. Since $\mathbb{T} = \mathcal{G}_T$ is compact, as a continuous image of a compact set $\Gamma_0 \times [0,T]$, the representation (2.7.3) holds.

Furthermore, there are other results concerning more specific properties that can be deduced from the expansion, such as sample regularity. These results are important for the convergence and error analysis. In order for these results to hold, one needs more structure on the space $\mathbb{T}$, such as being a Riemannian manifold or for it to have a group structure, which enables us to use the group representation theory as in [97].

We are interested in the case when diffusion coefficient $\alpha = \exp\beta$, where β is a GRF on $\mathcal{G}_T$. In Chapter 5 we will prove the well-posedness of this problem and give an a priori bound of the solution. However, if we want to do numerical analysis, in order to use results concerning the FEM error, we need to obtain results concerning the sample regularity and integrability of the solution u in terms of the RF α.

Exploiting the smoothness of the exponential function and compactness of the domain, we conclude that the sample regularity of $\exp\beta$ is the same as the sample regularity of β. These results are well-studied in the case when the space of the realizations is a flat domain D (see [28] and references therein) and in the case when $D = \mathbb{S}^2$, see [75].

If we consider the Poisson equation on the flat domain, the idea is to use the following FEM result [70]. Let $D \subset \mathbb{R}^d$ be convex or with C^2 boundary, $f \in L^2(D)$ and $\alpha \in C^{0,t}(\overline{D})$. Then for any $0 < t < 1$ and any $0 < s < t$, $s \neq \frac{1}{2}$, there exists $C = C(d, D, s, t)$ such that

$$\|u\|_{H^{1+s}} \leq C\|\alpha^{-1}\|^2_{L^\infty}\|\alpha\|_{L^\infty}\|\alpha\|^2_{C^{0,t}}\|f\|_{L^2}, \tag{2.7.4}$$

where u is the weak solution of the Poisson equation $-\nabla\cdot(\alpha\nabla u) = f$. For $t = 1$, we get the same estimate for $\|u\|_{H^2}$. To provide the assumption $\alpha \in C^{0,t}(\overline{D})$ from the previous result, we utilize A.4.2, that is based on Kolmogorov's theorem A.4.1. Namely, if the covariance function of a GRF β satisfies $C \in C^{0,2a}, a > 0$, then $\beta \in C^{0,b}(\overline{D})$ a.s., for any $b < a$ and hence $\alpha = \exp\beta \in C^{0,b}(\overline{D})$. Since we would like to estimate $\mathbb{E}\|u - u_h\|_{H^1_0(D)}$, where u_h is the FEM approximation, we need the estimate for the expression $\mathbb{E}(\exp(p\|\beta\|_{C(\overline{D})})), p > 1$, that we can obtain from Fernique's theorem. In addition, we also estimate the term $\mathbb{E}|\alpha|_{C^{0,b}(\overline{D})}$, i.e. $|\alpha|_{C^{0,b}(\overline{D})}$, independently of the sample.

The natural question that arises is: can we generalize these results to an evolving hypersurface? Concerning the required ESFEM results, to the best of our knowledge, the higher order estimates of the type (2.7.4) are not considered in the general case (some results are available in [45]). Furthermore, necessary sample regularity results of a GRF on general evolving hypersurface are still not established. These questions are the topic of the current research and will not be answered in this thesis.

However, results concerning Gaussian and log-normal random fields on the sphere are presented in [75, 87]. Particularly, they utilize the general Kolmogorov's theorem proved in [6] for the special case of the sphere. The general existing results about GRF on subsets of Euclidean space [125] require a group structure on the space of realizations, which is not the case for the sphere. Instead, they exploit results from [97] to present isotropic GRF on the sphere with respect to spherical harmonic functions. Considering the GRF over a sphere is of particular interest in many applications, such as cell-biology, cosmology, meteorology [91, 97]. In our setting, this means that we don't have an evolution, but we consider just the static case, i.e. $\Gamma(t) \equiv \Gamma, \forall t$.

Let us first recall the definition of *isotropic* GRF over a Riemann manifold (see [30]).

Definition 2.7.7. Let $\{X(p) : p \in M\}$ be a smooth centred GRF parametrized over an N-dimensional Riemann manifold $\mathbb{T} = M$. We say that X is isotropic over M if its covariance function can be written as

$$C(p, q) = \rho(d_M^2(p, q)), \quad \text{for any } p, q \in M, \tag{2.7.5}$$

where ρ is a real function on $[0, \infty)$ and d_M is a geodesic distance.

Relation (2.7.5) can be interpreted in the sense that $C(p, q)$ behaves isotropically over M, since it depends only on $d_M(p, q)$. For general Riemann manifolds, isotropic GF can be constructed via spectral representations.

The approach presented in [97] is based on the group representation theory. We sketch the idea of this approach and comment on the difficulties in its generalization. Let G be a topological compact group with the Haar measure dg and random field X on G i.e. $X = \{X(g) : g \in G\}$, where the Borel σ-algebra on G is generated by its topology. The authors are particularly interested in the connection between isotropy, the representation of compact groups and spectral analysis of random fields. Many ideas are based on the group representation theory and the Peter-Weyl theorem from harmonic analysis, which allows the generalized Fourier expansion to be applied for functions defined on the arbitrary compact group. The spectral representation results, the so-called Stochastic Peter-Weyl theorem, can be proved using the group theoretic and harmonic analysis point of view (see [97, Th. 5.5]). These results mainly concern random fields that have invariance properties: isotropy (invariance w.r.t. rotations) and stationarity (invariance w.r.t. translations).

The special case of the general group setting approach is the group of rotations $G = SO(3)$. Unfortunately, the sphere $\mathbb{S}^2$ does not have a group structure, but has the quotient space structure: $\mathbb{S}^2 \cong SO(3)/SO(2)$. The previous isomorphism enables us to use the general results that hold for the group G in order to characterize the isotropic spherical random fields, i.e. random fields indexed by the sphere $X = \{X(t) : t \in \mathbb{S}^2\}$.

Thus, the results concerning the existence of the KL expansion of isotropic GRF proved in [97] are based on the special structure of the sphere, i.e. that it has quotient space structure, and in particular it relies on the spherical harmonic functions. For the definition of the spherical harmonic function and different notions of isotropy see [87], it is based on the Legendre polynomials. The significance of the spherical harmonic functions follows from the Peter-Weyl's theorem [97, Th 3.29] which implies that spherical harmonic functions form an orthonormal basis of $L^2(\mathbb{S}^2, \mathbb{C})$ and they are eigenfunctions of $\Delta_{\mathbb{S}^2}$. Hence, every 2-weakly isotropic random

field on the sphere can be represented by its KL expansion w.r.t. spherical harmonic functions Y_{lm}. We state these results for completeness.

Theorem 2.7.8. [87, Theorem 2.3] Let X be a 2-weakly isotropic random field on $\mathbb{S}^2$. Then it holds

- X satisfies $\mathbb{P}$-almost surely
$$\int_{\mathbb{S}^2} X(x)^2 d\sigma(x) < \infty.$$
- X admits a Karhunen–Loève expansion
$$X = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm} \tag{2.7.6}$$
with
$$a_{lm} = \int_{S^2} X(y) \overline{Y_{lm}}(y) d\sigma(y)$$
for $l \in \mathbb{N}_0$ and $m \in \{-l, \dots, l\}$.
- The series expansion (2.7.6) converges in $L^2(\Omega \times S^2; \mathbb{R})$ i.e.
$$\lim_{L\to\infty} \mathbb{E}\left(\int_{\mathbb{S}^2} \left(X(y) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(y) \right)^2 d\sigma(y) \right) = 0.$$
- The series expansion (2.7.6) converges in $L^2(\Omega; \mathbb{R})$ for all $x \in \mathbb{S}^2$, i.e. for all $x \in \mathbb{S}^2$,
$$\lim_{L\to\infty} \mathbb{E}\left(\left(X(x) - \sum_{l=0}^{L} \sum_{m=-l}^{l} a_{lm} Y_{lm}(x) \right)^2 \right) = 0.$$

The previous theorem especially implies that $X \in L^2(\Omega, L^2(\mathbb{S}^2))$. Furthermore, in the special case, when X is isotropic Gaussian random field, coefficients $(a_{lm} : l \in \mathbb{N}_0, m = 0, \dots l)$ are independent Gaussian random variables.

To show converge rates and to develop efficient simulations, one exploits the special properties of random coefficients $(a_{lm} : l \in \mathbb{N}_0, m = -l, \dots, l)$ (for details see [87]). Moreover, they characterize the smoothness of the covariance function via decay of angular power spectrum. Namely, the regularity of the kernel $C(\beta(x)\beta(y))$ is equivalent to the weighted 2-summability of the angular power spectrum. For this reason, they make the assumption about the summability of the angular power spectrum.

Utilizing Kolmogorov's theorem, they characterize sample Hölder's continuity and sample differentiability [87, Th 4.5, Th 4.6]. As we already saw in the flat case, regularity properties of the samples are important for derivation of convergence rates. The special structure based on the spherical harmonic functions of the KL expansion leads to convergence rates of truncated expansion. Based on the previous results about the GRF on $\mathbb{S}^2$, in [75] they consider numerical

analysis of the elliptic equation on the sphere with isotropic log-normal coefficient. They deduce the error estimates for MLMCFEM and MLMC Spectral Element discretization and show how the particular geometry of the sphere allows more precise convergence results. Furthermore, the authors derive higher order approximation of the solution. These results can be generalized to d-dimensional sphere $\mathbb{S}^{d-1}$, $d \geq 3$.

Furthermore, the results on the sphere are generalized in [40] for the case of sphere cross time. The authors consider expansion, sample regularity and approximation of the GRF on $\mathbb{S}^2 \times [0, T]$. In particular, they are interested in how the regularity properties of the field can evolve over time. For the representation of the field on the sphere cross time they use both KL expansion and Hermite polynomials for space, while for time they use Schoeneberg's functions. Note that their space domain $\mathbb{S}^2$ does not evolve in time.

Observe that generalizing these results, first for any stationary compact connected C^2 hypersurface, and then for the evolving family $\Gamma(t)$, is not straight forward. The analysis in [87] strongly relies on the special expression of the KL expansion w.r.t. spherical harmonic functions. Furthermore, the existence of the KL expansion strongly depends on the quotient space structure of the sphere. One idea is to try to identify when it is possible to characterize $\mathcal{G}_T$ as a quotient space and, using the general results from [98] for the underlying group. This is a work in progress.

3. Function spaces

In this chapter we will define the function spaces that we will mainly consider in the case when the diffusion coefficient has uniform distribution.

3.1. Gelfand triple

In this section, we will introduce the basic Gelfand triple that will be used to define the solution space for (1.0.1). Note that sometimes instead of Gelfand triple, the term rigged Hilbert space is used, especially in the context of quantum mechanics. Gelfand triples were introduced by the Gelfand school around 1955. We begin by recalling the notion of a Gelfand triple. In general, a Gelfand triple is a scheme that equips a Hilbert space with a dense topological vector space that will be "good" test functions and its dual space enlarges the starting Hilbert space with corresponding distributions. In order to build this structure we will use the notion of (anti)dual space and conjugate operator (definitions of these objects are stated in the Appendix). The following definition and properties are adopted from [123, Section 17.1 Gelfand triple].

Let V be a reflexive Banach space and H a Hilbert space. Suppose that V is embedded in H: $V \underset{i}{\hookrightarrow} H$, where i is continuous, injective embedding. Furthermore, we assume that $\operatorname{Im} i$ is dense in H. Applying (A.1.1) once to i and once to i', we obtain that $i' : H^* \to V^*$ is injective and $\operatorname{Im} i'$ is dense in V', where i' is the conjugate operator of i and we identify H and H^* by Riesz' theorem A.1.1. Moreover, since $\|i\| = \|i'\|$, the continuity of i' follows. Altogether, we have

$$V \underset{i}{\hookrightarrow} H \cong H^* \underset{i'}{\hookrightarrow} V^*, \tag{3.1.1}$$

where both embeddings i and i' are continuous, injective and have dense images in H and V^*. A structure of this kind is called a *Gelfand triple* and H is referred as a pivot space.

Remark 3.1.1. As stated in [22, Ch. 5.2], the canonical isomorphism i' from H^* to V^* is simply the restriction to V of continuous linear functionals φ on H:

$$\langle i'\varphi, v\rangle_{V^*,V} = \langle \varphi, iv\rangle_{H^*,H} = \langle \varphi, v\rangle_{H^*,H} \quad \forall \varphi \in H^*,\, v \in V.$$

A simple example of a Gefand triple is

$$l^1 \hookrightarrow l^2 \hookrightarrow l^\infty.$$

When we consider a Gelfand triple, one can see the continuous extension of $(\cdot,\cdot)_H$ on $V^* \times V$ as a new representation formula for the functionals from V^*. That is, for $v' \in V^*$, we have

$$v'(v) = \langle v', v\rangle_{V^*,V} = \lim_{n\to\infty} (h_n, v)_H, \tag{3.1.2}$$

where $h_n \in H$ and $h_n \to v'$ in V^*. More details about the previous formula are given in the Appendix. Note that if in addition $v' \equiv h \in H$, we have

$$\langle h, v\rangle_{V^*,V} = (h, v)_H \quad v \in V,$$

i.e. one says that the duality pairing between V and V^* is compatible with the inner product on H.

Let us also mention one delicate situation, when V is also a Hilbert space. In this case, utilizing the Riesz isomorphism, we could also identify V with its dual space V^*, but then (3.1.1) becomes absurd. The example that we can not simultaneously identify V and H with their dual spaces is given in [22, Ch 5.2]. The idea is to consider $H = l^2$ and its subspace $V := \{u = (u_n)_n : \sum_n n^2 u_n^2 < \infty\}$. Then the dual space $V^* = \{f = (f_n)_n : \sum_n \frac{1}{n^2} u_n^2 < \infty\}$ is strictly bigger than H, and hence equality does not hold in (3.1.1). Hence, one should identify the pivot space H and H^* and not V. This is also explained by (3.1.2) where we identify functionals from V^* by $(\cdot, \cdot)_H$, and not by scalar product in V.

Now we want to define the Gelfand triple for our problem. In the discrete case the standard choice for the Gelfand triple at the fixed time t is

$$H^1(\Gamma(t)) \hookrightarrow L^2(\Gamma(t)) \hookrightarrow H^{-1}(\Gamma(t)),$$

where $H^{-1}(\Gamma(t))$ is the dual space of $H^1(\Gamma(t))$. That this is indeed a Gelfand triple, follows from the fact that $C_c^\infty(\Gamma(t))$ is dense in both spaces: $L^2(\Gamma(t)$ and $H^1(\Gamma(t))$, cf. [22, Corollary 4.23]. The continuity follows from Poincaré inequality.

To extend this scheme to the probabilistic setting, let us define

$$V(t) := L^2(\Omega, H^1(\Gamma(t))) \quad \text{and} \quad H(t) := L^2(\Omega, L^2(\Gamma(t))), \tag{3.1.3}$$

for each $t \in [0, T]$. By [46, Ch IV, Th. 1] the dual space of $L^p(\Omega, \mu, X)$ is $L^q(\Omega, \mu, X^*)$ iff the Banach space X has the Radon-Nikodym property, where μ is a finite measure, $1 \le p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Since every Hilbert space has the Radon-Nikodym property, it follows that the dual space of $V(t)$ is the space $V^*(t) = L^2(\Omega, H^{-1}(\Gamma(t)))$.

Remark 3.1.2. The Radon-Nikodym property is related to the vector valued extension of the Radon-Nikodym theorem. Namely, the Radon-Nikodym theorem fails to hold in general for the Bochner integral. If X is a Banach space and the generalization of the Radon-Nikodym theorem holds with values in X, then we say that X has the Radon-Nikodym property. Every Hilbert space has a Radon-Nikodym property. Even more generally, all separable dual spaces and reflexive spaces have the Radon-Nikodym property. Examples of spaces that don't have the Radon-Nikodym property are c_0, $L^\infty(D)$ and $L^1(D)$, where D is open and bounded subset of $\mathbb{R}^n$. More on this topic can be found for example in [46, Ch. III].

Since all spaces $L^2(\Omega)$, $L^2(\Gamma(t))$ and $H^1(\Gamma(t))$ are separable Hilbert spaces, Theorem 2.5.5 implies

$$L^2(\Omega, H^1(\Gamma(t))) \cong L^2(\Omega) \otimes H^1(\Gamma(t)) \tag{3.1.4}$$

$$L^2(\Omega, L^2(\Gamma(t))) \cong L^2(\Omega) \otimes L^2(\Gamma(t)). \tag{3.1.5}$$

Remark 3.1.3. For convenience we will often (but not always) write $u(\omega, x)$ instead of $u(\omega)(x)$, which is justified by the aforementioned isomorphisms.

Lemma 3.1.4. Let L, X, Y be separable Hilbert spaces such that the inclusion

$$X \underset{i'}{\hookrightarrow} Y \tag{3.1.6}$$

is continuous and dense. Then there exists a dense inclusion

$$L \otimes X \underset{i}{\hookrightarrow} L \otimes Y. \tag{3.1.7}$$

Proof. First, we need to prove the existence of the inclusion i. This can be proved utilizing the quotient space structure of tensor spaces commented in Remark 2.5.2. Namely, let U and W be some vector spaces such that $U \hookrightarrow W$ and U' and W' are their subspaces respectively, $U' \subset U$, $W' \subset W$. Then,

$$x - y \in U' \Rightarrow i(x - y) \in W', \tag{3.1.8}$$

implies $U/_{U'} \hookrightarrow W/_{W'}$. Take $U := \mathcal{V}_{\text{free}}(L \times X)$, $U' = N_U$ and $W := \mathcal{V}_{\text{free}}(L \times Y)$, $W' = N_W$, where $\mathcal{V}_{\text{free}}$ and N are introduced in Remark 2.5.2. Then, (3.1.8) is obviously satisfied. Hence, the inclusion i exists.

In order to prove that i is dense, we first prove the statement on the algebraic tensor level, i.e., that $L \otimes_a X$ is dense in $L \otimes_a Y$. For that purpose, let $\varepsilon > 0$ be arbitrary and let $l \otimes y \in L \otimes_a Y$, $l \in L$, $y \in Y$ be an elementary tensor. Then the density assumption implies that there exists $x \in X$ such that $\|y - x\|_Y < \frac{\varepsilon}{\|l\|_L}$. Thus, $l \otimes x \in L \otimes_a X$ and it holds

$$\|l \otimes y - l \otimes x\|_{L \otimes_a X} = \|l\|_l \|y \otimes x\|_Y < \varepsilon.$$

In a similar way we can prove the same inequality for any element from $L \otimes_a Y$. Specifically, an arbitrary element from $L \otimes_a Y$ is a finite sum of elementary tensors, i.e. it is $\sum_{i=1}^n l_i \otimes y_i$, where $l_i \in L, y_i \in Y, i = 1, \ldots, n$, $n \in \mathbb{N}$. Then, for every $y_i \in Y$ there exists an $x_i \in X$ such that $\|y_i - x_i\|_Y < \frac{\varepsilon}{nC}$, where $C := \max_{1 \le i \le n} \|l_i\|_L$. Consequently, for $\sum_{i=1}^n l_i \otimes x_i \in L \otimes_a X$ we have

$$\|\sum_{i=1}^n l_i \otimes y_i - \sum_{i=1}^n l_i \otimes x_i\|_{L \otimes_a Y} \le \sum_{i=1}^n \|l_i\|_L \|y_i - x_i\|_Y < \varepsilon.$$

The last inequality implies that

$$L \otimes_a X \underset{i}{\hookrightarrow} L \otimes_a Y \tag{3.1.9}$$

is dense. In order to show that the same holds for the Hilbert tensor spaces, i.e., for the completion of (3.1.9) w.r.t. the appropriate norm, we prove that

$$L \otimes_a X \underset{i}{\hookrightarrow} L \otimes Y \tag{3.1.10}$$

is dense. Accordingly, let $\varepsilon > 0$ and $y \in L \otimes Y$ be arbitrary. Then, since $L \otimes_a Y$ is dense in $L \otimes Y$, there exists an element $\tilde{y} \in L \otimes_a Y$ such that

$$\|y - \tilde{y}\|_{L \otimes Y} < \frac{\varepsilon}{2}. \tag{3.1.11}$$

Since $\tilde{y}$ is an element of the algebraic tensor product, it has the form $\tilde{y} = \sum_{j=1}^{N} l_j \otimes y_j$, for some $l_j \in L, y_j \in Y, N \in \mathbb{N}$. Moreover, since X is dense in Y, then for every j, there exists $x_j \in X$ such that $\|y_j - x_j\|_Y < \frac{\varepsilon}{2N\|l_j\|_L}$. Hence, after summation over j, we obtain

$$\|\tilde{y} - \sum_{j=1}^{N} l_j \otimes x_j\|_{L \otimes_a Y} \leq \sum_{j=1}^{N} \|l_j\|_L \|y_j - x_j\|_Y < \frac{\varepsilon}{2}. \tag{3.1.12}$$

Utilizing the triangle inequality, expressions (3.1.11) and (3.1.12), imply that the inclusion (3.1.10) is dense. To deduce the final step in the proof, note that since the inclusion i' in (3.1.7) is dense and Lipschitz, it follows that the inclusion in (3.1.10) is dense and Lipschitz. Hence it has a Lipschitz extension on the completion of $L \otimes_a X$, which is exactly $L \otimes X$. This proofs the lemma. □

Theorem 3.1.5. $V(t) \hookrightarrow H(t) \hookrightarrow V^*(t)$ is a Gelfand triple for every $t \in [0, T]$.

Proof. First note that the continuity of the inclusion follows directly from Poincaré inequality and the fact that the Poincaré constant depends just on the domain, but not on the sample ω. Moreover, we would like to exploit the fact that $H^1(\Gamma(t))$ is dense and continuous in $L^2(\Gamma(t))$. Thus, it is natural to consider tensor structure of spaces $V(t)$ and $H(t)$, stated in (3.1.4) and (3.1.5). Then, according to Lemma 3.1.4, there exists a dense inclusion $V(t) \hookrightarrow H(t)$, which completes the proof. □

3.2. Compatibility of spaces

In order to treat the evolving spaces, we need to define special Bochner-type function spaces such that for every $t \in [0, T]$ we have $u(t) \in V(t)$. In general, if we have an evolving family of Hilbert spaces $X = (X(t))_{t \in [0,T]}$, the idea is to connect the space $X(t)$ at any time $t \in [0, T]$ with some fixed space, for example with the initial space $X(0)$. Thus we construct the family of maps $\phi_t : X(0) \to X(t)$, which we call the pushforward map. We denote the inverse of ϕ_t by $\phi_{-t} : X(t) \to X(0)$ and call it the pullback map. The following definition is adapted from [4].

Remark 3.2.1. This approach is similar to the Arbitrary Lagrangian Eulerian [ALE] framework. Furthermore, the following setting can be generalized for the family X of Banach spaces, as presented in [4], however, this case is of no interest to us at this moment.

Definition 3.2.2. The pair $\{X, (\phi_t)_{t \in [0,T]}\}$ is *compatible* if the following conditions hold:

- for every $t \in [0, T]$, ϕ_t is a linear homeomorphism such that ϕ_0 is the identity map
- there exists a constant C_X which is independent of t such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X(0)} \quad \text{for every } u \in X(0) \\ \|\phi_{-t} u\|_{X(0)} &\leq C_X \|u\|_{X(t)} \quad \text{for every } u \in X(t) \end{aligned}$$

- the map $t \mapsto \|\phi_t u\|_{X(t)}$ is continuous for every $u \in X(0)$.

Note that for the given family $\{X(t)\}$ there are usually many different mappings ϕ_t such that the pair $\{X, (\phi_t)_{t\in[0,T]}\}$ is compatible.

We will denote the dual operator of ϕ_t by $\phi_t^* : X^*(t) \to X^*(0)$. As a consequence of the previous conditions, we obtain that ϕ_t^* and its inverse are also linear homeomorphisms which satisfy the following conditions

$$\begin{aligned} \|\phi_t^* f\|_{X^*(0)} &\leq C_X \|f\|_{X^*(t)} \quad \text{for every } f \in X^*(t) \\ \|\phi_{-t}^* f\|_{X^*(t)} &\leq C_X \|f\|_{X^*(0)} \quad \text{for every } f \in X^*(0). \end{aligned}$$

For the Gelfand triple $L^2(\Omega, H^1(\Gamma(t))) \subset L^2(\Omega, L^2(\Gamma(t))) \subset L^2(\Omega, H^{-1}(\Gamma(t)))$ we define the pull-back operator $\phi_{-t} : L^2(\Omega, L^2(\Gamma(t))) \to L^2(\Omega, L^2(\Gamma_0))$ in the following way

$$(\phi_{-t}u)(\omega)(x) := u(\omega)(\Phi_t^0(x)) \quad \text{for every } x \in \Gamma(0),\ \omega \in \Omega.$$

Remark 3.2.3. Since we are interested only in the dual operator of $\phi_t\big|_V$, we will denote it by $\phi_t^* : V^*(t) \to V_0^*$.

The next step is to prove that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}\big|_{V_0})$ are compatible pairs. The proof is similar to the proof of [122, Lemma 3.2].

Let $J_t^0(\cdot) := \det D_{\Gamma_0}\Phi_t^0(\cdot)$ denote the Jacobian determinant of the matrix representation of $D_{\Gamma_0}\Phi_t^0(\cdot)$ w.r.t. orthonormal basis of the respective tangent space, (where $(D_{\Gamma_0}\Phi_t^0)_{ij} := \underline{D}_j(\Phi_t^0)_i$). Thus, J_t^0 presents the change area of the element when transformed from Γ_0 to $\Gamma(t)$. The assumptions for the flow Φ_t^0 imply $J_t^0 \in \mathcal{C}^1([0,T]\times\Gamma_0)$ and the uniform bound for the field J_t^0:

$$\frac{1}{C_J} \leq J_t^0(x) \leq C_J \quad \text{for every } x \in \Gamma_0 \text{ and for all } t \in [0,T], \tag{3.2.1}$$

where C_J is a positive constant.

The substitution formula for integrable functions $\zeta : \Gamma(t) \to \mathbb{R}$ reads

$$\int_{\Gamma(t)} \zeta = \int_{\Gamma_0} (\zeta \circ \Phi_t^0) J_t^0 = \int_{\Gamma_0} \phi_{-t}\zeta J_t^0.$$

Using the Leibniz formula stated in Theorem 2.4.2 for differentiation of a parameter dependent surface integral, it can be shown [122, Lemma 3.2] that

$$\frac{d}{dt} J_t^0 = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t)) J_t^0. \tag{3.2.2}$$

Lemma 3.2.4. The pairs $\big(H, (\phi_t)_{t\in[0,T]}\big)$ and $\Big(V, (\phi_t\big|_{V_0})_{t\in[0,T]}\Big)$ are compatible.

Proof. The proof is similar to the proof of [122, Lemma 3.3]. However, we will state the proof in order to show that constants that appear are independent of the sample ω.

We first prove the statement for the pair $\big(H, (\phi_t)_{t\in[0,T]}\big)$. Let u be from $L^2(\Omega, L^2(\Gamma(t)))$. Then we have

$$\|\phi_{-t}u\|^2_{L^2(\Omega,L^2(\Gamma_0))} = \int_\Omega \int_{\Gamma(t)} |u(\omega)(y)|^2 \frac{1}{J_t^0((\Phi_t^0)^{-1}(y))} \leq C_J \|u\|^2_{L^2(\Omega,L^2(\Gamma(t)))}, \tag{3.2.3}$$

where we have used the substitution formula and the boundedness of J_t^0. It is clear that ϕ_{-t} is linear and that its continuity follows immediately from the previous estimate

$$\|\phi_{-t}u - \phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} = \|\phi_{-t}(u-v)\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C_J^{1/2}\|u-v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Since Φ_t^0 is $\mathcal{C}^2$-diffeomorphism, it follows that ϕ_{-t} is bijective and its inverse (the pushforward) is defined by

$$\phi_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma(t))), \quad (\phi_t v)(\omega, x) = v(\omega) \circ (\Phi_t^0)^{-1}(x).$$

Similarly as for ϕ_{-t}, we can prove that ϕ_t is well defined, that it satisfies the norm boundedness relation and it is continuous. Thus, ϕ_t is a linear homeomorphism.
Since the probability space does not depend on time, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega,L^2(\Gamma(t)))}$ follows directly from [122, Lemma 3.3.] and the triangle inequality.

In order to prove the compatibility of the family $(V, (\phi_t\big|_{V_0})_{t\in[0,T]})$, let $v \in L^2(\Omega, H^1(\Gamma(t)))$ and $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$. Utilizing the substitution formula and integration by parts on $\Gamma(t)$ we obtain

$$\begin{aligned}
&\int_\Omega \int_{\Gamma_0} \phi_{-t} v(\omega,x) \nabla_\Gamma \varphi(\omega,x) = \int_\Omega \int_{\Gamma(t)} v(\omega,x)(D\bar{\Phi}_t(x))^T \nabla_\Gamma(\phi_t\varphi(\omega,x)) J_{-t}^0(x) \\
&= -\int_\Omega \int_{\Gamma(t)} \phi_t\varphi(\omega,x)(\nabla_\Gamma(v(\omega,x)(D\bar{\Phi}_t(x))^T) J_{-t}^0(x) \qquad (3.2.4)\\
&+ v(\omega,x)(D\bar{\Phi}_t(x))^T \nabla_\Gamma J_{-t}^0(x) + v(\omega,x)(D\bar{\Phi}_t(x))^T J_{-t}^0(x) H_0\nu_0 \qquad (3.2.5)\\
&= -\int_\Omega \int_{\Gamma(t)} \phi_t\varphi(\omega,x) s(\omega,x) J_{-t}^0(x) \\
&= -\int_\Omega \int_{\Gamma_0} [\phi_{-t}s(\omega,x) - H_0\nu_0\phi_{-t}v(\omega,x)]\varphi(\omega,x) + H_0\nu_0\phi_{-t}v(\omega,x)\varphi(\omega,x), \qquad (3.2.6)
\end{aligned}$$

where s is the function that we get from the partial integration. Note that s depends only on the mean curvature and derivative of $\bar{\Phi}_t$ which can be bounded independently of time and ω. Thus, $\|s(\omega)\|_{L^2(\Gamma(t))^{(n+1)}} \leq C\|v(\omega)\|_{H^1(\Gamma(t))}$, where C does not depend on ω and t. Furthermore, we get

$$\|s\|_{L^2(\Omega,L^2(\Gamma(t))^{n+1})} \leq C\|v\|_{L^2(\Omega,H^1(\Gamma(t)))}.$$

Hence, the estimate from the first part of the proof implies

$$\phi_{-t}v \in L^2(\Omega, L^2(\Gamma_0)) \quad \text{and} \quad \|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C'\|v\|_{L^2(\Omega,H^1(\Gamma(t)))}. \qquad (3.2.7)$$

On the other hand, from the partial integration on the hypersurface we have

$$\int_\Omega \int_{\Gamma_0} \phi_{-t}v(\omega,x)\nabla_\Gamma\varphi(\omega,x) = -\int_\Omega \int_{\Gamma_0} \varphi(\omega,x)(\nabla_\Gamma(\phi_{-t}v)(\omega,x) + \phi_{-t}v(\omega,x)H_0\nu_0).$$

From the last relation and (3.2.6), since they hold for every $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$, we get

$$\nabla_\Gamma(\phi_{-t}v)(\omega,x) = \phi_{-t}s(\omega,x) - H_0\nu_0(\phi_{-t}v)(\omega,x). \qquad (3.2.8)$$

For $v \in L^2(\Omega, L^2(\Gamma(t)))$, we have already proved in (3.2.3) that $\|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}$. Therefore, the following estimate follows

$$\|H_0\nu_0(\phi_{-t}v)(\omega,x)\|_{L^2(\Omega,L^2(\Gamma_0))} \leq |H_0|C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

The last inequality, (3.2.7) and (3.2.8), imply

$$\|\phi_{-t}v\|_{L^2(\Omega,H^1(\Gamma_0))} \leq C_V\|v\|_{V(t)},$$

where C_V depends on the global bound on $|H_t|$, $\|\partial\bar{\Phi}_t\|$ and $\|\partial_{ij}\bar{\Phi}_t\|$ with $1 \leq i,j \leq n+1$, $t \in [0,T]$ and these bounds are deterministic and independent of time.

Similarly to the previous case, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega,H^1(\Gamma(t)))}$ follows from [122, Lemma 3.3] and the probability space's independence of time, which completes the proof. □

3.3. Bochner-type spaces

In this section, we want to define Bochner-type spaces of random functions that are defined on evolving spaces. The Bochner integral was introduced in Section 2.2. In order to strictly define these spaces we will require that the pull-back of u belongs to the fixed initial space $V(0)$. These spaces are a special case of general function spaces defined in [4]:

Definition 3.3.1. For a compatible pair $(X,(\phi_t)_t)$ we define spaces:

$$L^2_X := \left\{ u : [0,T] \ni t \mapsto (\bar{u}(t),t) \in \bigcup_{s\in[0,T]} X(s)\times\{s\} \mid \phi_{-(\cdot)}\bar{u}(\cdot) \in L^2(0,T;X(0)) \right\},$$
$$L^2_{X^*} := \left\{ f : [0,T] \ni t \mapsto (\bar{f}(t),t) \in \bigcup_{s\in[0,T]} X^*(s)\times\{s\} \mid \phi_{-(\cdot)}\bar{f}(\cdot) \in L^2(0,T;X^*(0)) \right\}.$$

Like the standard Bochner spaces, these spaces consist of equivalence classes of functions agreeing almost everywhere in $[0,T]$. Note that previous spaces strongly depend on the map ϕ_t.

Remark 3.3.2. In the following we will identify $u(t) = (\overline{u}(t),t)$ with $\overline{u}(t)$, for brevity of notation.

In order to understand these spaces better, we will state their most important properties. More details and proofs of the following statements can be found in [4].

Lemma 3.3.3. (The isomorphism with standard Bochner spaces and the equivalence of norms) The maps

$$L^2(0,T;X_0) \ni u \mapsto \phi_{(\cdot)}u(\cdot) \in L^2_X$$
$$L^2(0,T;X_0^*) \ni f \mapsto \phi_{-(\cdot)}f(\cdot) \in L^2_{X^*}$$

are isomorphisms. Furthermore, the equivalence of norms holds

$$\frac{1}{C_X}\|u\|_{L^2_X} \leq \|\phi_{(\cdot)}u(\cdot)\|_{L^2(0,T;X_0)} \leq C_X\|u\|_{L^2_X} \quad \forall u \in L^2_X$$
$$\frac{1}{C_X}\|f\|_{L^2_{X^*}} \leq \|\phi_{-(\cdot)}f(\cdot)\|_{L^2(0,T;X_0^*)} \leq C_X\|f\|_{L^2_X} \quad \forall f \in L^2_{X^*}.$$

Proof. See [4, Lemma 2.10, Lemma 2.11]. □

The spaces L^2_X and $L^2_{X^*}$ are separable Hilbert spaces ([4, Corollary 2.11]) with the inner product defined as

$$(u,v)_{L^2_X} = \int_0^T (u(t),v(t))_{X(t)}dt$$
$$(f,g)_{L^2_{X^*}} = \int_0^T (f(t),g(t))_{X^*(t)}dt.$$

For $f \in L^2_{X^*}$ and $u \in L^2_X$ the map

$$t \mapsto \langle f(t), u(t)\rangle_{X^*(t),X(t)}$$

is integrable on $[0,T]$, see [4, Lemma 2.13]. Utilizing the integrability of this map and Fubini's theorem 2.2.3, in [4, Lemma 2.15] the authors prove that the spaces $L^2_{X^*}$ and $(L^2_X)^*$ are isometrically isomorphic. Furthermore, the duality pairing of $f \in L^2_{X^*}$ with $u \in L^2_X$ is given by

$$\langle f,u\rangle_{L^2_{X^*},L^2_X} = \int_0^T \langle f,u\rangle_{X^*(t),X(t)}\,dt.$$

Let us now consider the specific family of evolving spaces, namely the one defined by (3.1.3). By Lemma 3.2.4, the spaces L^2_V, $L^2_{V^*}$ and L^2_H are well-defined. Moreover, identifying $L^2_{V^*}$ and $(L^2_V)^*$ and exploiting the density of the space $L^2(0,T;V_0)$ in $L^2(0,T;H_0)$, Lemma 3.3.3 and Theorem 3.1.5, we obtain the following result.

Lemma 3.3.4.

$$L^2_{L^2(\Omega,H^1(\Gamma(t)))} \hookrightarrow L^2_{L^2(\Omega,L^2(\Gamma(t)))} \hookrightarrow L^2_{L^2(\Omega,H^{-1}(\Gamma(t)))}$$

is a Gelfand triple.

3.4. Material derivative

This section is motivated by the abstract framework from Chapter 2.4 in [4]. We plan to define a time derivative that will also take into account the spatial movement, i.e. the material derivative for random functions. As a first step, let us consider the spaces of pushed-forward continuously differentiable functions

$$\mathcal{C}^j_V := \{u \in L^2_V \mid \phi_{-(\cdot)}u(\cdot) \in \mathcal{C}^j([0,T],L^2(\Omega,H^1(\Gamma_0)))\} \text{ for } j \in \{0,1,\dots\}.$$

Definition 3.4.1. For $u \in \mathcal{C}_V^1$ *the strong material derivative* $\dot{u} \in \mathcal{C}_V^0$ is defined by

$$\dot{u}(t) = \phi_t \left(\frac{d}{dt} \phi_{-t} u(t) \right) \tag{3.4.1}$$

for every $t \in [0, T]$.

Using the smoothness of $\Gamma(t)$ and Φ_t^0, for every $\omega \in \Omega$ each function $u(t, \omega) : \Gamma(t) \to \mathbb{R}$ can be extended to a neighbourhood of $\bigcup_{t \in [0,T]} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+2}$ such that $\nabla u(\omega)$ and $u_t(\omega)$ of the extension in the neighbourhood are well-defined for every ω (for the construction of the extension see [59]). Utilizing the chain rule, for $u \in \mathcal{C}_V^1$ and $y \in \Gamma_0$, we get

$$\begin{aligned} \frac{d}{dt} \phi_{-t} u(t) &= \frac{d}{dt} (u(t, \omega, \Phi_t^0(y))) \\ &= u_t(t, \omega, \Phi_t^0(y)) + \nabla u \,|_{(t, \omega, \Phi_t^0(y))} \cdot \mathbf{v}(t, \Phi_t^0(y)) \\ &= \phi_{-t} u_t(t, \omega, y) + \phi_{-t} \nabla u(t, \omega, y) \cdot \phi_{-t}(\mathbf{v}(t, y)). \end{aligned}$$

Thus, we get the following explicit formula for the *strong material derivative*

$$\dot{u}(t, \omega, x) = u_t(t, \omega, x) + \nabla u(t, \omega, x) \cdot \mathbf{v}(t, x), \tag{3.4.2}$$

for every $x \in \Gamma(t)$ and $\omega \in \Omega$.

Remark 3.4.2. Note that the right-hand side of (3.4.2) does not depend on extension, so dependence of every extension (i.e. neighbourhood) on ω is irrelevant.

Just as in the deterministic case, it might happen that the equation does not have a solution if requesting $u \in \mathcal{C}_V^1$. Hence, we aim to define a weak material derivative that needs less regularity. In addition to the case when we consider a fixed domain, we will have an extra term that will take into account the movement of the domain. As usual in this setting (see for example [4]), the idea is to pull-back the inner product on $L^2(\Omega, L^2(\Gamma(t)))$ onto the fixed space $L^2(\Omega, L^2(\Gamma_0))$. Then, on the product space of $L^2(\Omega, L^2(\Gamma_0))$ we can define a bilinear form $\hat{b}$ as inner-product of push-forward mappings. Furthermore, defining $\hat{c}$ as a regular time derivative of this bilinear form, the extra term c in the weak material derivative will be the push-forward of $\hat{c}$ onto $H(t) \times H(t)$. Let us make this construction more precise.

First we define the bounded bilinear form $\hat{b}(t, \cdot, \cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ for every $t \in [0, T]$ by:

$$\begin{aligned} \hat{b}(t, u_0, v_0) &:= (\phi_t u_0, \phi_t v_0)_{L^2(\Omega, L^2(\Gamma(t)))} \\ &= \int_\Omega \int_{\Gamma(0)} u_0(\omega, x) v_0(\omega, x) J_t^0(x). \end{aligned}$$

Note that there exists an operator $T_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ s.t. the bilinear form $\hat{b}(t; \cdot, \cdot)$ can be represented as a scalar product on $L^2(\Omega, L^2(\Gamma_0))$ i.e.:

$$\hat{b}(t; u_0, v_0) = \langle T_t u_0, v_0 \rangle_{L^2(\Omega, L^2(\Gamma_0))} = \int_\Omega \int_{\Gamma_0} T_t u_0(\omega, x) v_0(\omega, x).$$

From the definition of the bilinear form $\hat{b}(t,\cdot,\cdot)$ we get the precise definition of the operator T_t:

$$T_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0)) \text{ defined by } T_t u_0(\omega, x) = u_0(\omega, x) J_t^0(x), \tag{3.4.3}$$

for every $x \in \Gamma_0$.

Moreover, we define the map $\theta : [0,T] \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ as the classical time derivative of the norm on $L^2(\Omega, L^2(\Gamma(t)))$:

$$\theta(t, u_0) := \frac{d}{dt} \|\phi_t u_0\|^2_{L^2(\Omega, L^2(\Gamma(t)))} \quad \forall u_0 \in L^2(\Omega, L^2(\Gamma_0)),$$

in the classical sense.

Lemma 3.4.3. a) The map θ is well-defined and for each $t \in [0,T]$ the map

$$u_0 \mapsto \theta(t, u_0) \quad u_0 \in L^2(\Omega, L^2(\Gamma_0)) \tag{3.4.4}$$

is continuous.

b) For every $t \in [0,T]$ there exists a deterministic constant C that is independent of time such that

$$|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| \leq C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}.$$

Proof. a) Using the substitution formula, the formula (3.2.2) and the assumption (2.4.1) we get:

$$\begin{aligned} \theta(t, u_0) &= \frac{d}{dt} \int_\Omega \int_{\Gamma(0)} |u_0(\omega, x)|^2 J_t^0(x) \\ &= \int_\Omega \int_{\Gamma(0)} |u_0(\omega, x)|^2 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t)) J_t^0(x) \\ &= \int_\Omega \int_{\Gamma(0)} |u_0(\omega, x)|^2 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t,x)) J_t^0(x) \leq C \|u_0\|^2_{L^2(\Omega, L^2(\Gamma_0))}. \end{aligned} \tag{3.4.5}$$

Utilizing the uniform in t boundedness of integrand and the Dominant Convergence theorem, we can interchange the derivative and the integral in (3.4.5). Hence, θ is well-defined. To prove the continuity of (3.4.4), note that $u \in L^2(\Omega, L^2(\Gamma_0))$, which implies $u^2 \in L^1(\Omega, L^1(\Gamma_0))$. Consequently, we obtain that if $u_n \to u$ in $L^2(\Omega, L^2(\Gamma_0))$, then $u_n^2 \to u^2$ in $L^1(\Omega, L^1(\Gamma_0))$. Now the continuity follows from:

$$\begin{aligned} |\theta(t, u_n) - \theta(t, u)| &\leq \int_\Omega \int_{\Gamma_0} |u_n^2(\omega, x) - u^2(\omega, x)| |\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t,x)) J_t^0(x)| \\ &\leq C \|u_n^2 - u^2\|_{L^1(\Omega, L^1(\Gamma_0))} \to 0. \end{aligned}$$

b) Using the Cauchy-Schwarz inequality, (3.2.1) and (3.2.2) we get the estimate:

$$\begin{aligned} |\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| &= \|4 \frac{d}{dt} \hat{b}(t; u_0, v_0)\| \\ &= 4 | \int_\Omega \int_{\Gamma_0} u_0(\omega, x) v_0(\omega, x) \frac{d}{dt} J_t^0(x)| \\ &\leq C | \, (u_0, v_0) \, |_{L^2(\Omega, L^2(\Gamma_0))} \\ &\leq C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}. \quad \square \end{aligned}$$

We define the bilinear form $\hat{c}(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ as a partial time derivative of $\hat{b}$

$$\begin{aligned}\hat{c}(t;u_0,v_0) &:= \frac{\partial}{\partial t}\hat{b}(t;u_0,v_0) = \frac{1}{4}(\theta(t,u_0+v_0)-\theta(t,u_0-v_0)) \\ &= \int_\Omega \int_{\Gamma_0} u_0(\omega,x)v_0(\omega,x)\phi_{-t}(\nabla_{\Gamma(t)}\cdot \mathbf{v}(t,x))J_t^0(x).\end{aligned}$$

From [4, Lemma 2.27] it follows that for every $u, v \in \mathcal{C}^1([0,T]; L^2(\Omega, L^2(\Gamma_0)))$ the map

$$t \mapsto \hat{b}(t;u(t),v(t))$$

is differentiable in the classical sense and the formula for differentiation of the scalar product on $L^2(\Omega, L^2(\Gamma(t)))$ is

$$\frac{d}{dt}\hat{b}(t;u(t),v(t)) = \hat{b}(t;u'(t),v(t)) + \hat{b}(t;u(t),v'(t)) + \hat{c}(t;u(t),v(t)).$$

We will generalise this result in Section 3.5 for less regular functions u and v.

The next step is to define the extra term that appears in the definition of the weak material derivative. As we have already announced, we pull the functions back to $\Gamma(0)$ and apply bilinear form $\hat{c}$ to them. More precisely, we define the bilinear form $c(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma(t))) \times L^2(\Omega, L^2(\Gamma(t))) \to \mathbb{R}$ by

$$c(t;u,v) := \hat{c}(t;\phi_{-t}u, \phi_{-t}v) = \int_\Omega \int_{\Gamma(t)} u(\omega,z)v(\omega,z)(\nabla_{\Gamma(t)}\cdot \mathbf{v}(t,x)).$$

Lemma 3.4.4. For every $u, v \in L^2_V$, the map

$$t \mapsto c(t;u(t),v(t))$$

is measurable. Furthermore, c is bounded independently of t by a deterministic constant:

$$|c(t;u,v)| \le C\|u\|_{L^2(\Omega,L^2(\Gamma(t)))}\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Proof. From Lemma 3.4.3 it follows that we can apply the Corollary of [4, Lemma 2.26], which proves the Lemma. □

Utilizing the previous result, we can define the weak material derivative.

Definition 3.4.5. We say that $\partial^\bullet u \in L^2_{V^*}$ is *a weak material derivative* of $u \in L^2_V$ if and only if

$$\begin{aligned}\int_0^T \langle \partial^\bullet u(t), \eta(t)\rangle_{V^*(t),V(t)} &= -\int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t;u(t),\eta(t)) \\ = \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\dot{\eta}(t,\omega,x) &- \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\eta(t,\omega,x)\nabla_{\Gamma(t)}\cdot \mathbf{v}(t,x),\end{aligned} \tag{3.4.6}$$

holds for all $\eta \in \mathcal{D}_V(0,T) = \{\eta \in L^2_V \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0,T); L^2(\Omega, H^1(\Gamma_0)))\}$.

Note that it can be directly shown that if it exists, the weak material derivative is unique and every strong material derivative is also a weak material derivative.

3.5. Solution space

In this section we define the solution space, based on the general framework presented in [4]. We will require the solution of the equation (1.0.1) to belong to the space L^2_V and to have a weak material derivative in its dual $L^2_{V^*}$. Hence, we define the solution space as:

$$W(V, V^*) := \{u \in L^2_V \mid \partial^\bullet u \in L^2_{V^*}\}.$$

In order to prove that the solution space is a Hilbert space and that it has some additional properties, we will connect $W(V, V^*)$ with the standard Sobolev-Bochner space $\mathcal{W}(V_0, V_0^*)$ defined by (2.2.2) for which these properties are known. We will show that the previous two types of spaces are connected in a natural way, i.e. that the pull-back of the functions from the solution space belongs to the Sobolev-Bochner space and vice versa. In addition, we also have the equivalence of the norms. First we will prove the technical result which is similar to [122, Lemma 3.6].

Lemma 3.5.1. Let $w \in \mathcal{W}(V_0, V_0^*)$ and $f \in \mathcal{C}^1([0,T] \times \Gamma_0)$. Then $fw \in \mathcal{W}(V_0, V_0^*)$ and

$$(fw)' = \partial_t f w + f w', \tag{3.5.1}$$

where $\langle fw', \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma_0)), L^2(\Omega, H^1(\Gamma_0))} = \langle w', f\varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma_0)), L^2(\Omega, H^1(\Gamma_0))}$.

Proof. We will first prove the Lemma for $\varphi \in \mathcal{D}([0,T], L^2(\Omega, H^1(\Gamma_0)))$. By [122, Lemma 3.6], $f \in \mathcal{C}^1([0,T] \times \Gamma_0)$, which implies

$$f \in \mathcal{C}([0,T], \mathcal{C}^1(\Gamma_0)) \text{ and } f \in \mathcal{C}^1([0,T], \mathcal{C}(\Gamma_0)). \tag{3.5.2}$$

In order to prove that $f\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$ we can treat deterministic function f as a random function that is constant in ω. More precisely, if we define the function $\tilde{f}(t, \omega, x) := f(t, x)$, it follows from (3.5.2) $\tilde{f} \in \mathcal{C}([0,T], L^2(\Omega, \mathcal{C}^1(\Gamma_0))$. This can be strictly shown by defining the function $g : \mathcal{C}(\Gamma_0) \to L^2(\Omega, \mathcal{C}(\Gamma_0))$, $g(f)(\omega, x) := f(x)$. Note that since g is linear, it is a $\mathcal{C}^\infty$- function and for every t we have $g(f(t)) = \tilde{f}(t)$. Hence, we have

$$\tilde{f}\varphi \in \mathcal{C}([0,T], L^2(\Omega, H^1(\Gamma_0))) \cap \mathcal{C}^1([0,T], L^2(\Omega, L^2(\Gamma_0)))$$

which implies $\tilde{f}\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$ and thus, $f\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$.

It is left to prove that formula (3.5.1) is valid. We will prove this utilizing the characterization of the weak derivative (2.2.3) and partial integration [4, Lemma 2.1(3)]:

$$\int_0^T \langle fw', \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))} = -\int_0^T \langle w, (f\varphi)' \rangle_{L^2(\Omega, H^1(\Gamma(t))), L^2(\Omega, H^{-1}(\Gamma(t)))}$$
$$= -\int_0^T \langle \partial_t f w, \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))} - \int_0^T \langle fw, \varphi' \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))}.$$

Thus,

$$\int_0^T \langle fw, \varphi' \rangle_{L^2(\Omega, L^2(\Gamma_0))} = \int_0^T \langle \partial_t f w + f w', \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))},$$

i.e. $(fw)' = \partial_t f w + f w'$. The density result Theorem 2.2.4 ii) implies that we can approximate every function fw by continuous $L^2(\Omega, H^1(\Gamma_0))$-valued functions and we conclude that $fw \in L^2(\Omega, H^1(\Gamma_0))$. The similar argument implies $(fw)' \in L^2(\Omega, H^{-1}(\Gamma_0))$. □

Corollary 3.5.2. Letting $T_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ be the operator defined by (3.4.3), it holds

$$u \in \mathcal{W}(V_0, V_0^*) \text{ if and only if } T_{(\cdot)}u(\cdot) \in \mathcal{W}(V_0, V_0^*). \tag{3.5.3}$$

Proof. It is enough to apply Lemma 3.5.1 to functions $f = J^0_{(\cdot)}$ and $f = \frac{1}{J^0_{(\cdot)}}$, which are both from the space $\mathcal{C}^1([0,T] \times \Gamma_0)$. □

Theorem 3.5.3. The following equivalence holds

$$v \in W(V, V^*) \text{ if and only if } \phi_{-(\cdot)}v(\cdot) \in \mathcal{W}(V_0, V_0^*), \tag{3.5.4}$$

and the norms are equivalent

$$C_1\|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0,V_0^*)} \leq \|v\|_{W(V,V^*)} \leq C_2\|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0,V_0^*)}. \tag{3.5.5}$$

Remark 3.5.4. Following the notation from [4], we say that there exists an *evolving space equivalence* between the spaces $W(V, V^*)$ and $\mathcal{W}(V_0, V_0^*)$ if and only if they satisfy (3.5.4) and (3.5.5). The previous theorem enables us to transfer the properties of the space $\mathcal{W}(V_0, V_0^*)$ to the space $W(V, V^*)$.

Proof. Let $u \in \mathcal{W}(V_0, V_0^*)$. For every $t \in [0,T]$ we define a map $\hat{S}(t) : V_0^* \to V_0^*$ by

$$\hat{S}(t)u'(t) := J_t^0 u'(t). \tag{3.5.6}$$

Note that since J_t^0 is bounded independently of t and has an inverse, this implies that $\hat{S}(t)$ has an inverse, and both $\hat{S}(t)$ and $\hat{S}^{-1}(t)$ are bounded independently of t. Furthermore, from the uniform bound on J_t^0 we have $\hat{S}(\cdot)u'(\cdot) \in L^2(0,T;V_0^*)$. In the end, utilizing the product rule (3.5.1), we get

$$(T_t u(t))' = (J_t^0 u(t))' = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t)) J_t^0 u(t) + J_t^0 u'(t) = \hat{S}(t)u'(t) + \hat{C}(t)u(t),$$

where T_t is defined by (3.4.3) and $\hat{C}(t) : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ is defined as

$$\hat{C}(t, \omega, x) = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{v}(t)) J_t^0(x),$$

i.e. $\left\langle \hat{C}(t)u_0, v_0 \right\rangle := \hat{c}(t; u_0, v_0)$. Thus, using in addition Corollary 3.5.2, we can apply [4, Theorem 2.32.], which yields that there exists an evolving space equivalence between $W(V, V^*)$ and $\mathcal{W}(V_0, V_0^*)$. □

Corollary 3.5.5. The solution space $W(V, V^*)$ is a Hilbert space with the inner product defined via

$$(u, v)_{W(V,V^*)} = \int_0^T \int_\Omega (u(t), v(t))_{H^1(\Gamma(t))} + \int_0^T \int_\Omega (\partial^\bullet u(t), \partial^\bullet v(t))_{H^{-1}(\Gamma(t))}.$$

More properties of the space $W(V, V^*)$ can be derived as a consequence of the evolving space equivalence with the space $\mathcal{W}(V_0, V_0^*)$ and its features stated in Theorem 2.2.4. We state some of them in the following lemma.

Lemma 3.5.6. The following statements hold:

1. Space $W(V, V^*)$ is embedded into C_H^0.
2. The embedding $D_V([0,T]) \subset W(V, V^*)$ is dense.
3. For every $u \in W(V, V^*)$ the following inequality is valid

$$\max_{t\in[0,T]} \|u(t)\|_{H(t)} \leq C\|u\|_{W(V,V^*)}.$$

Proof. See [4, Lemma 2.35, Lemma 2.36]. □

As a consequence of the previous Lemma, the evaluation $t \mapsto u(t)$ is well-defined. As a result, we will be able to specify initial conditions for the PDE. Furthermore, we can define the subspace

$$W_0(V, V^*) := \{u \in W(V, V^*) | u(0) = 0\}, \tag{3.5.7}$$

that we will use in the proof of the well-posedness of a considered problem, in order to restrict ourself to the zero initial value. Note that $W_0(V, V^*)$ is a Hilbert space as a closed linear subspace of $W(V, V^*)$.

Our next result states how to generalize the result about the differentiation of the inner product on $H(t) = L^2(\Omega, L^2(\Gamma(t)))$ of functions from $\mathcal{C}_H^1$, to functions from the solution space.

Theorem 3.5.7. (Transport theorem.) For all $u, v \in W(V, V^*)$, the map

$$t \mapsto (u(t), v(t))_{L^2(\Omega, L^2(\Gamma(t)))}$$

is absolutely continuous on $[0, T]$ and

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \partial^\bullet u(t), v(t)\rangle_{V^*(t),V(t)} + \langle \partial^\bullet v(t), u(t)\rangle_{V^*(t),V(t)} + c(t; u(t), v(t)), \tag{3.5.8}$$

holds for almost all $t \in [0, T]$.

Proof. The proof is based on the density of the space $D_V[0, T]$ in the space $W(V, V^*)$ and the Transport formula for the functions from $\mathcal{C}_H^1$. For a detailed proof, we refer the reader to [4, Theorem 2.38.]. □

The previous theorem gives us the formula for integration by parts.

Corollary 3.5.8. For all $u, v \in W(V, V^*)$, the integration by parts holds

$$(u(T), v(T))_{H(T)} - (u(0), v(0))_{H_0} =$$
$$\int_0^T \left(\langle \partial^\bullet u(t), v(t)\rangle_{V^*(t),V(t)} + \langle \partial^\bullet v(t), u(t)\rangle_{V^*(t),V(t)} + c(t; u(t), v(t)) \right).$$

To discuss the regularity results, we define a new space in which the weak derivative of a function has more regularity.

Definition 3.5.9. Let

$$W(V,H) := \{u \in L^2_V \,|\, \partial^\bullet u \in L^2_H\}.$$

In order to prove the properties of the previous space, similarly as we did for $W(V,V^*)$, we connect $W(V,H)$ with the standard Sobolev-Bochner space $\mathcal{W}(V_0,H_0)$.

Lemma 3.5.10. There is an evolving space equivalence between $W(V,H)$ and $\mathcal{W}(V_0,H_0) \equiv \{v \in L^2(0,T;L^2(\Omega,H^1(\Gamma_0))) \,|\, v' \in L^2(0,T;L^2(\Omega,L^2(\Gamma_0)))\}$.

Proof. Since The Jacobian J_t^0 is uniformly bounded, both in time and space (see (3.2.1)), applying [4, Theorem 2.33] to the restriction $\hat{S}(t)\colon H_0 \to H_0$ of the map defined by (3.5.6), completes the proof. □

Corollary 3.5.11. $W(V,H)$ is a Hilbert space.

4. Uniformly bounded random diffusion coefficient

In this chapter we will consider the case when the diffusion coefficient is uniformly bounded away from zero and from above, which allows us to consider the "mean-weak" formulation and directly apply the Banach-Nečas-Babuška [BNB] theorem about the existence and uniqueness of the solution. By the "mean-weak" formulation, we meant the variational formulation that also includes the integration over the probability space Ω. The BNB theorem is stated in Theorem 4.2.1.

The setting when the coefficient is uniformly bounded from above and away from zero is often considered by many others in the field of uncertainty quantification (see [10, 18, 33, 77]). The main advantage of the assumption concerning the existence of an uniform bound is that it enables us to use known deterministic bounds.

Before we formulate the problem we want to consider, let us first point out a few facts about uniformly bounded random variables and give some concrete examples. Let X be a random variable on some domain D, $X : \Omega \times D \to \mathbb{R}$. Assuming that X is uniformly bounded away from zero and from above ensures the existence of *constants* $X_{\min}, X_{\max} \in (0, \infty)$ such

$$\mathbb{P}\left(\omega \in \Omega : X(\omega, x) \in [X_{\min}, X_{\max}],\ \forall x \in \overline{D}\right) = 1. \tag{4.0.1}$$

One of the obvious characterizations of uniformly bounded random variables is that it is necessary and sufficient that it has compact support. Hence, normal random variables are not uniformly bounded. Moreover, a uniformly bonded random variable has all moments:

$$\mathbb{E}[|X^n|] = \mathbb{E}[|X|^n] \leq \mathbb{E}[C^n] = C^n < \infty \quad \forall n.$$

A typical example of a continuous uniformly bounded distribution is a *(continuous) uniform distribution* or rectangular distribution. In one dimensional case, it is defined by two parameters: minimum a and maximum b. We write $X \sim \mathcal{U}(a, b)$. Its probability density function is constant between these two parameters (see Figure 4.1) and is given with

$$f(x, y) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$$

The name uniform comes from the property that the probability that X falls within any interval $[x, x + d] \subset [a, b]$ of the fixed size d is independent of the interval itself, but depends just on the interval size. The previous conclusion follows from the direct calculations:

$$\mathbb{P}(X \in [x, x + d]) = \int_x^{x+d} \frac{1}{b-a} dy = \frac{d}{b-a}.$$

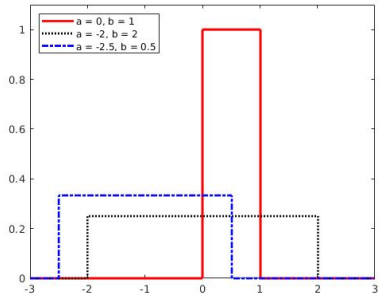


Figure 4.1.: Probability density functions of the uniform distributions.

For $a = 0$ and $b = 1$, the distribution $\mathcal{U}(0,1)$ is called standard uniform distribution. Note that the sum of two independent uniform distributions is not uniform, but the so-called triangular distribution. To see this, consider $X, Y \sim \mathcal{U}(0,1)$ with density functions f_X and f_Y. Then the sum $Z = X + Y$ is a random variable with density function f_Z, where f_Z is the convolution of f_X and f_Y:

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy = \begin{cases} z & 0 \leq z \leq 1 \\ 2-z & 1 < z \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

Hence, Z is not uniformly distributed. The definition of the uniform distribution can be generalized to the n-dimensional case. Let $D \subset \mathbb{R}^n$ be a Borel set with a positive and finite measure. Then uniform probability distribution on D is defined to be zero outside D and $1/\lambda^d(S)$ in D. Furthermore, for any measurable set A it holds

$$\mathcal{U}(D)(A) = \frac{\lambda^d(A \cap D)}{\lambda^d(D)}.$$

Since the sum of uniformly distributed random variables is not uniformly distributed, in particular, $X_0 + \sum_{i=1}^{N} X_i(\omega)\phi(x)$ does not define a uniform distribution, but it is uniformly bounded. This type of sum is a typical example of a uniformly bounded random variable. For example

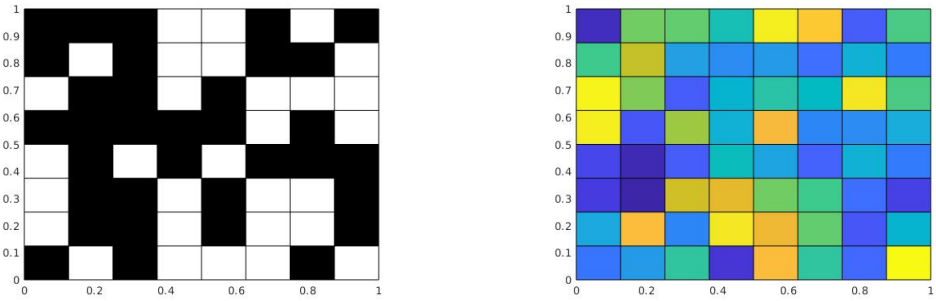
$$a(x,\omega) = 5 + \cos(x)\, Y_1(\omega) + \sin(x)\, Y_2(\omega),$$

obviously satisfy (4.0.1), where $Y_1, Y_2 \sim \mathcal{U}[-1,1]$ and in addition, they are independent.

Our next example is known as the "random checkerboard model". Let $Q_k := k + [0,1)^d$, $k \in \mathbb{Z}^d$ denote the unit cube with a corner at k. Furthermore, let $\{a_k\}_{k\in\mathbb{Z}^d}$ be a collection of i.i.d. random variables such that $0 < \lambda \leq a_k \leq \Lambda < \infty$ holds with probability one. Then define

$$a(\omega, x) := \sum_{k\in\mathbb{Z}^d} a_k(\omega)\chi_{Q_k}(x) \tag{4.0.2}$$

where χ is a characteristic function. That is, $a(x) = a_k$ if $x \in Q_k$, i.e., the random function $a(x)$ is piecewise constant, taking random variables on the cubes Q_k. Thus, a is uniformly bounded



(a) Black and white checkerboard

(b) $a_k := 1 + Y_k$, $Y_k \sim \mathcal{U}(0,1)$

Figure 4.2.: Realizations of the random checkerboard model

form below and above by λ and Λ, respectively. Note that the sequence $\{a_k\}_{k\in\mathbb{Z}^d}$ completely determines the random field $a(x)$. Although a_k might take a continuum of values, the name "random checkerboard model" is inspired by the case when a_k takes only two values, which correspond to colours white and black. Two realizations of $a(x)$ are presented in Figure 4.2, where the different colours represent different values of a_k. The left figure represents the case when a_k takes only two values and this is the standard random black and white checkerboard model. The right figure is a realization of $a(x)$ for $a_k := 1 + Y_k$, $Y_k \sim \mathcal{U}(0,1)$.

Note that the random field a does not change if we shift the domain. Thus, a is stationary w.r.t. shifts in $x \in \mathbb{R}^d$:

for any integer $m > 0$ and a.e. $(x_1, \dots, x_m)$, $x_i \in \mathbb{R}^d, i = 1, \dots, m$ it holds

$$(a(x_1), \dots, a(x_m)) \sim (a(x_1 + h), \dots, a(x_m + h)) \qquad \forall h \in \mathbb{Z}^d.$$

This property makes the random field $a(x)$ a typical example in the homogenization theory.

As we already announced in Section 2.6, the general idea how to construct uniformly bounded random variable is presented in [39, Section 2.2]. The authors create random functions by randomizing the coefficients of a series expansion of a function. We will explain how to create uniformly bounded random variable in this manner. Let X denote a Banach space, $\{\psi_j\}_{j=1}^{\infty}$ be a normalized sequence in X and $D \subset \mathbb{R}^d$ be a domain. We define the random variable by

$$u := m_0 + \sum_{j=1}^{\infty} u_j \psi_j \tag{4.0.3}$$

where $m_0 \in X$ and we randomize u_j by setting $u_j = \xi_j \gamma_j$. In order to construct a uniformly bonded random variable, we take $X = L^{\infty}(D)$ and $\{\xi_j\}_j$ is i.i.d. sequence with $\xi_1 \sim \mathcal{U}(-1,1)$ and $\{\gamma_j\}_{j=1}^{\infty} \in l^1$ is a deterministic sequence. We consider $\{\xi_j\}_{j=1}^{\infty}$ as a random element in the probability space $(\mathbb{R}^{\infty}, \mathcal{B}(\mathbb{R}^{\infty}), \mathbb{P})$.

Since we assumed that $\{\xi_j\}_{j=1}^{\infty}$ are independent, we obtain the product structure of the joint probability i.e. the product measure of $\xi = (\xi_j)_{j=1}^{\infty}$ on $[-1,1]^{\mathbb{N}}$ on appropriate σ-algebra is given by

$$\mu_0(d\xi) := \otimes_j \frac{1}{2} \lambda(d\xi_j).$$

Assume furthermore that there exist positive constants m_{min} and m_{max} and δ such that

$$\begin{aligned}\operatorname*{ess\,inf}_{x\in D} m_0(x) &\geq m_{\min}\\ \operatorname*{ess\,sup}_{x\in D} m_0(x) &\leq m_{\max}\\ \|\gamma\|_{l^1} &= \frac{\delta}{1+\delta} m_{\min}.\end{aligned}$$

Since X is not a separable space, it can happen that ψ_j or u live in a subspace X' of X, i.e. we will have the convergence of (4.0.3) in X'. In the considered case of uniformly distributed ξ_1 , X' is a closer of the linear span of the functions $(m_0, \{\psi_j\}_{j=1}^{\infty})$ with respect to the infinity norm on X. Then, $(X', \|\cdot\|_{\infty})$ is separable. The following theorem states that under previous assumptions, (4.0.3) defines the uniformly bounded random variable on X'. For the proof see [39, Theorem 2.1].

Theorem 4.0.1. The following holds $\mathbb{P}$-almost surely: the sequence of functions $\{u^N\}_{N=1}^{\infty}$ given by

$$u_N := m_0 + \sum_{j=1}^{N} u_j \psi_j$$

is Cauchy in X' and the limiting function u given by (4.0.3) satisfies

$$\frac{1}{1+\delta} m_{\min} \leq u(x) \leq m_{\max} + \frac{\delta}{1+\delta} m_{\min} \quad \text{a.e.} \quad x \in D.$$

In addition, if we assume that m_0 is Hölder continuous and $\{\psi_j\}_{j=1}^{\infty}$ have uniform Hölder exponent, plus assuming suitable decay of $\{\gamma_j\}_{j=1}^{\infty}$, utilizing Kolmogorov's continuity theorem A.4.1, we obtain Hölder continuity of u with an appropriate exponent. For more details see [39, Th. 2.3].

A typical example of this kind of random coefficient would be in a case of material with inclusion of random conductivity

$$a(\omega, x) = a_0 + \sum_{i=1}^{N} 1_{D_n}(x)\eta_n(\omega).$$

4.1. Formulation of the problem

We consider an initial value problem for an advection-diffusion equation on the evolving surface $\Gamma(t), t \in [0, T]$, with a uniformly bounded random coefficient α, which is given by

$$\begin{aligned}\partial^{\bullet} u - \nabla_{\Gamma} \cdot (\alpha \nabla_{\Gamma} u) + u \nabla_{\Gamma} \cdot \mathbf{v} &= f \quad \text{in } L^2_{V^*}\\ u(0) &= u_0 \quad \text{in } L^2_{H_0}.\end{aligned} \tag{4.1.1}$$

Here the initial function u_0 and source term f are also random functions.

Remark 4.1.1. The initial condition is meaningful thanks to the embedding $W(V, V^*) \subset \mathcal{C}_H^0$ stated in Lemma 3.5.6.

Existence and uniqueness can be stated on the following assumption.

Assumption 4.1.2. The diffusion coefficient α satisfies the following conditions

a) $\alpha \colon \Omega \times \mathcal{G}_T \to \mathbb{R}$ is a $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T)$-measurable, where $\mathcal{G}_T$ is the space-time surface $\mathcal{G}_T := \bigcup_t \Gamma(t) \times \{t\}$.

b) α is uniformly bounded from above and below in the sense that there exist positive constants $\alpha_{\min}$ and $\alpha_{\max}$ such that

$$0 < \alpha_{\min} \leq \alpha(\omega, x, t) \leq \alpha_{\max} < \infty \quad \forall (x,t) \in \mathcal{G}_T \tag{4.1.2}$$

holds for $\mathbb{P}$-a.e. $\omega \in \Omega$

and the initial function satisfies $u_0 \in L^2(\Omega, L^2(\Gamma_0))$ and the source term $f \in L^2_{V^*}$.

A *"mean-weak" solution* of (4.1.1) is a solution in the following sense.

Problem 4.1.3. [Mean-weak form of the random advection-diffusion equation on $\{\Gamma(t)\}_{t\in[0,T]}$] Find $u \in W(V, V^*)$ that point-wise satisfies the initial condition $u(0) = u_0 \in H(0)$ and

$$\begin{aligned} \langle \partial^\bullet u(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))} + \int_\Omega \int_{\Gamma(t)} \alpha(t) \nabla_\Gamma u(t) \cdot \nabla_\Gamma v \\ + \int_\Omega \int_{\Gamma(t)} u(t) v \nabla_\Gamma \cdot \mathbf{v} = \langle f(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))}, \end{aligned} \tag{4.1.3}$$

for every $v \in L^2(\Omega, H^1(\Gamma(t)))$ and a.e. $t \in [0, T]$.

Remark 4.1.4. As observed in [4], formulations (4.1.1) and (4.1.3) are equivalent, where the first one includes the integration over time and the second one is asked to hold a.e. in $[0, T]$. The direct implication is based on forming the duality pairing of (4.1.1) and separability of V_0. The reversed implication follows from the density of simple functions in L^2_V, cf. [4, Lemma 2.9]. In the proof of well-posedness of (4.1.3) we will use the bilinear form that includes integration over time. This is justified by the previous arguments.

In order to simplify the notation we introduce the notation

$$V(t) := L^2(\Omega, H^1(\Gamma(t))) \qquad H(t) := L^2(\Omega, L^2(\Gamma(t)))$$

and define the bilinear form $a(t; \cdot, \cdot) : V(t) \times V(t) \to \mathbb{R}$ by

$$a(t; u, v) := \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u(\omega, x) \cdot \nabla_\Gamma v(\omega, x). \tag{4.1.4}$$

Let us state some of the properties of the bilinear form a.

Lemma 4.1.5. The map

$$t \mapsto a(t; u(t), v(t)) \tag{4.1.5}$$

is measurable for all $u, v \in L^2_V$. Furthermore, there exist positive deterministic constants C_1, C_2 and C_3 that are independent of t such that the following holds for almost every $t \in [0, T]$

$$\begin{aligned} a(t; v, v) &\geq C_1 \|v\|^2_{L^2(\Omega, H^1(\Gamma(t)))} - C_2 \|v\|^2_{L^2(\Omega, L^2(\Gamma(t)))} \quad \forall v \in V(t) && (4.1.6) \\ |a(t; u, v)| &\leq C_3 \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))} \quad \forall u, v \in V(t). && (4.1.7) \end{aligned}$$

Proof. The measurability of (4.1.5) follows directly from the Fubini-Tonelli theorem 2.2.3. Moreover, the assumption (4.1.2) directly implies that

$$a(t; v, v) \geq \alpha_{min} \|\nabla_\Gamma v\|^2_{L^2(\Omega, L^2(\Gamma))},$$

thus we can take $C_1 = C_2 = \alpha_{\min}$. Using again (4.1.2) and the Cauchy-Schwarz inequality we get that $C_3 = \alpha_{\max}$

$$\begin{aligned} \left| \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u \cdot \nabla_\Gamma v \right| &\leq \alpha_{\max} | \langle \nabla_\Gamma u, \nabla_\Gamma v \rangle_{L^2(\Omega, L^2(\Gamma(t)))} | \\ &\leq \alpha_{\max} \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))}. \quad \square \end{aligned}$$

Remark 4.1.6. In [4] the authors remarked that formulation (4.1.1) implicitly claims that $\nabla_\Gamma \cdot (\alpha \nabla_\Gamma u)$ and $u \nabla_\Gamma \cdot \mathbf{v}$ belong to $L^2_{V^*}$. This holds as a corrolarry of Lemma 4.1.5 and Assumption 2.4.1 on the velocity.

4.2. Existence and uniqueness

The Banach-Nečas-Babuška [BNB] theorem is the main tool for proving the well-posedness of the abstract problem of the form

$$\text{find } u \in \mathcal{V} \text{ such that } \mathcal{A}(u, w) = \mathcal{L}(w) \quad \forall w \in \mathcal{W}$$

where in the general setting $\mathcal{V}, \mathcal{W}$ are Banach spaces and $\mathcal{W}$ is reflexive, $\mathcal{A} : \mathcal{V} \times \mathcal{W} \to \mathbb{R}$ is a bilinear form and $\mathcal{L} : \mathcal{W} \to \mathbb{R}$ is a linear form. We will state the BNB theorem and give some remarks, and the proof can be found in [61].

Theorem 4.2.1. Let X be a Banach space and Y be a reflexive Banach space. Consider $d(\cdot, \cdot) : X \times Y \to \mathbb{R}$ a bounded bilinear form and $f \in Y^*$. Then the following are equivalent

(i) There is a unique solution $x \in X$ to the problem

$$d(x, y) = \langle f, y \rangle_{Y^*, Y} \qquad \text{for all } y \in Y$$

satisfying

$$\|x\|_X \leq C \|f\|_{Y^*}. \tag{4.2.1}$$

(ii) a) There exists $\beta > 0$ such that

$$\inf_{x\in X}\sup_{y\in Y}\frac{d(x,y)}{\|x\|_X\|y\|_Y} \geq \beta \qquad \text{inf-sup condition} \tag{4.2.2}$$

b) For arbitrary $y \in Y$ if

$$d(x,y) = 0 \quad \text{holds for all } x \in X, \tag{4.2.3}$$

then $y = 0$.

Furthermore, the estimate (4.2.1) holds with the constant $C = \frac{1}{\beta}$.

The condition 2(b) can be expressed equivalently as

$$\sup_{x\in X}|d(x,y)| > 0 \quad \forall y \in Y, y \neq 0,$$

and the inf-sup condition (5.2.9) can be written equivalently as

$$\exists\beta > 0, \quad \sup_{y\in Y}\frac{d(x,y)}{\|x\|_X\|y\|_Y} \geq \beta\|x\|_X \quad \forall x \in X.$$

The inf-sup condition is also known as the Babuška-Brezis condition. Moreover, there exists also the third equivalent condition to the inf-sup condition given by:

$$\exists\beta > 0, \quad \inf_{x\in X}\sup_{y\in Y}\frac{d(x,y)}{\|x\|_X\|y\|_Y} = \inf_{y\in Y}\sup_{x\in X}\frac{d(x,y)}{\|x\|_X\|y\|_Y} \geq \beta. \tag{4.2.4}$$

The BNB theorem is often referred to as a generalization of the Lax-Milgram theorem. Namely, if d is defined on $X \times X$, i.e. $X = Y$, and d is coercive in X:

$$\exists\beta_0 > 0, \quad d(x,x) \geq \beta_0\|x\|_X^2, \quad \forall x \in X,$$

then it satisfies the conditions of the BNB theorem. To see this, take $x = y$ to obtain

$$\sup_{y\in X} \in X\frac{d(x,y)}{\|y\|_X} \geq \frac{d(x,x)}{\|x\|_X} \leq \beta_0\|x\|_X.$$

Furthermore, if for arbitrary $y \in Y = X$, $d(x,y) = 0$ holds for all $x \in X$, choosing again $x = y \in X$ we obtain

$$0 = d(x,x) \geq \beta_0\|x\|_X^2 \geq 0,$$

thus $\|x\| = 0$ i.e. $y = x = 0$.

Remark 4.2.2. Let us give a brief history of comments concerning the BNB theorem. The following remarks and more details can be found in [110]. In 1962, Nečas [100] proved the first version of the theorem which consists of the implication $(4.2.4 \Rightarrow (i))$ in a Hilbert setting, as a direct consequence of the Lax-Milgram theorem. Later in 1971, Babuška [8] stated the theorem in the context of finite element methods. In 2002 Ern and Guermond presented the part

$(i) \Leftrightarrow (ii)$ and named it the Bancah-Nečas-Babuška theorem in [61] since its proof was based on Banach's results and they outline the following:

The BNB Theorem plays a fundamental role in this book. Although it is by no means standard, we have adopted the terminology "BNB Theorem" because the result is presented in the form below was first stated by Nečas in 1962 [100] and popularized by Babuška in 1972 in the context of finite element methods. From a functional analysis perspective, this theorem is a rephrasing of two fundamental results by Banach: the Closed Range Theorem and the Open Mapping Theorem. [61, page 84]

The goal is to apply the BNB theorem to our problem $(4.1.1)$. This has been done in the general setting in [4, Theorem 3.6] and in the special case of diffusion problems on evolving surfaces in [103, Ch. 4]. However, for completeness we state the proof adjusted to our particular problem. Before we define the spaces X and Y and bilinear form $d(\cdot,\cdot)$, we first show how we can transform the problem to the initial value problem with zero initial value. To see this, first consider the appropriate initial value problem on the fixed domain. Then there exists a solution $z \in \mathcal{W}(V_0, V_0^*)$ with $z(0) = u_0$ such that $\|z\|_{\mathcal{W}(V_0,V_0^*)} \leq C\|u_0\|_{H_0}$. Defining the function $\tilde{z} := \phi z \in W(V, V^*)$ which satisfies $\tilde{z}(0) = z_0$ we transform $(4.1.1)$ into a PDE with zero initial condition by setting $y := u - \tilde{z}$

$$\begin{aligned} \partial^\bullet y - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma y) + y \nabla_\Gamma \cdot \mathbf{v} &= \tilde{f} \\ y(0) &= 0, \end{aligned} \tag{4.2.5}$$

where $\tilde{f} := f - \partial^\bullet \tilde{z} + \nabla_\Gamma \cdot \alpha \nabla_\Gamma \tilde{z} - \tilde{z} \nabla_\Gamma \cdot \mathbf{v} \in L^2_{V^*}$. Now it is enough to prove the well-posedness of the suitable "mean-weak" formulation of (4.2.5).

We define X as the solution space that additionally satisfies the zero initial value condition i.e. it is the Hilbert space defined by (3.5.7)

$$X := W_0(V, V^*) = \{u \in W(V, V^*) | u(0) = 0\}$$

and $Y = L^2_V$ and the bilinear form

$$d(u, v) := \langle \partial^\bullet u(t), v \rangle_{L^2_{V^*}, L^2_V} + \int_0^T a(t; u(t), v(t)) + \int_0^T c(t; u(t), v(t)).$$

The linearity and boundedness of $d : X \times Y \to \mathbb{R}$ is straightforward:

$$\begin{aligned} |d(u, v)| &\leq |\langle \partial^\bullet u, v \rangle_{L^2_{V^*}, L^2_V}| + \int_0^T C_3 \|u\|_{V(t)} \|v\|_{V(t)} + C_{\mathbf{v}} \|u\|_{L^2_H} \|v\|_{L^2_H} \\ &\leq \|\partial^\bullet u\|_{L^2_{V^*}} \|v\|_{L^2_V} + C\|u\|_{L^2_V} \|v\|_{L^2_V} \\ &\leq C\|v\|_Y (\|\partial^\bullet u\|_{L^2_{V^*}} + \|u\|_{L^2_V}) \leq C\sqrt{2}\|v\|_Y \|u\|_X, \end{aligned}$$

where in the first inequality we used the boundedness of $a(t; u, v)$ proved in (4.1.7) and Cauchy-Schwarz inequality.

Remark 4.2.3. The Cauchy-Schwarz inequality for duality pairing follows directly from the Riesz theorem A.1.1: for any functional $l \in V^*$ there exists a unique element $z \in V$ such that $\langle l, v \rangle_{V^*,V} = (z, v)_V$ and $\|l\|_{V^*} = \|z\|_V$, hence according to the standard Cauchy-Schwarz inequality for the inner product we have

The next two lemmas are essential for proving the assumptions of the BNB theorem 4.2.1 and the proofs are similar to general deterministic proofs presented in [4].

Lemma 4.2.4. For all $u \in W_0(V, V^*)$, there exists a function $v_u \in L^2_V$ such that

$$d(u, v_u) \geq C\|u\|_{W(V,V^*)}\|v_u\|_{L^2_V}. \tag{4.2.6}$$

Proof. Let $u \in W_0(V, V^*)$ and take $u_\gamma(t) := e^{-\gamma t}u(t)$. Then

$$\partial^\bullet u_\gamma(t) = e^{-\gamma t}\partial^\bullet u(t) - \gamma u_\gamma(t) \in V^*(t),$$

hence $u_\gamma \in W_0(V, V^*)$ and

$$\langle \partial^\bullet u(t), u_\gamma(t)\rangle_{V^*(t),V(t)} = \langle \partial^\bullet u_\gamma(t), u(t)\rangle_{V^*(t),V(t)} + \gamma \langle u(t), u_\gamma(t)\rangle_{V^*(t),V(t)}.$$

Regrouping the previous equality and integrating over $[0, T]$ we infer

$$\int_0^T \langle \partial^\bullet u(t), u_\gamma(t)\rangle_{V^*(t),V(t)} = \frac{1}{2}\int_0^T \Big(\langle \partial^\bullet u_\gamma(t), u(t)\rangle_{V^*(t),V(t)} + \langle \partial^\bullet u(t), u_\gamma(t)\rangle_{V^*(t),V(t)}\Big) + \frac{1}{2}\gamma\int_0^T \gamma \langle u(t), u_\gamma(t)\rangle_{V^*(t),V(t)}.$$

Using the Transport theorem 3.5.7 for functions u and u_γ, from the previous equation we obtain

$$\begin{aligned}
&\int_0^T \langle \partial^\bullet u(t), u_\gamma(t)\rangle_{V^*(t),V(t)} = \\
&\frac{1}{2}\int_0^T \frac{d}{dt}(u(t), u_\gamma(t))_{H(t)} - \frac{1}{2}\int_0^T c(t; u(t), u_\gamma(t)) + \frac{1}{2}\gamma\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)} = \\
&\frac{1}{2}e^{-\gamma T}\|u(T)\|^2_{H(T)} - \frac{1}{2}\int_0^T e^{-\gamma t}c(t; u(t), u(t)) + \frac{1}{2}\gamma\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)},
\end{aligned}$$

where in the last inequality we used $u(0) = 0$.

Utilizing the bound (2.4.1) and the coercivity (4.1.6) of $a(t; \cdot, \cdot)$ we get

$$\begin{aligned}
&d(u, u_\gamma) = \\
&\int_0^T \langle \partial^\bullet u(t), u_\gamma(t)\rangle_{V^*(t),V(t)} + \int_0^T a(t; u(t), u_\gamma(t)) + \int_0^T e^{-\gamma t}c(t; u(t), u(t)) \geq \\
&\frac{1}{2}\int_0^T e^{-\gamma t}c(t; u(t), u(t)) + \frac{1}{2}\gamma\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)} + \int_0^T e^{-\gamma t}a(t; u(t), u(t)) \geq \\
&-\frac{C_{\mathbf{v}}}{2}\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)} + \frac{1}{2}\gamma\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)} + \int_0^T e^{-\gamma t}(C_1\|u\|^2_{V(t)} - C_2\|u\|^2_{H(t)}) = \\
&C_1\int_0^T e^{-\gamma t}\|u(t)\|^2_{V(t)} + \frac{\gamma - C_{\mathbf{v}} - 2C_2}{2}\int_0^T e^{-\gamma t}\|u(t)\|^2_{H(t)} \geq C_1 e^{-\gamma T}\|u\|^2_{L^2_V},
\end{aligned} \tag{4.2.7}$$

for $\gamma \geq C_{\mathbf{v}} + 2C_2$. The last estimate establishes the control of $\|u\|_V$ on the right hand side of (4.2.6). In order to bound the full norm $\|u\|_{W(V,V^*)}$, we need to also control $\|\partial^\bullet u\|_{V^*}$.This

is accomplished by using the Riesz' representation theorem A.1.1 that states that there exists a unique $z \in L^2_V$ such that

$$\langle \partial^\bullet u, v\rangle_{L^2_{V^*},L^2_V} = (z,v)_{L^2_V} \quad \text{for all } v \in L^2_V \quad \text{and} \quad \|z\|_{L^2_V} = \|\partial^\bullet u\|_{L^2_{V^*}}.$$

Thus taking $v = z \in L^2_V$ we obtain

$$\langle \partial^\bullet u, z\rangle_{L^2_{V^*},L^2_V} = (z,z)_{L^2_V} = \|\partial^\bullet u\|^2_{L^2_{V^*}}.$$

Therefore utilizing (4.1.7) and Young's inequality, we infer

$$\begin{aligned} d(u,z) &\geq \|\partial^\bullet u\|^2_{L^2_{V^*}} - \int_0^T C_3\|u\|_{V(t)}\|z(t)\|_{V(t)} \\ &\geq \|\partial^\bullet u\|^2_{L^2_{V^*}} - (C_3^2\|u\|^2_{L^2_V} + \|z\|^2_{L^2_V}) = \frac{1}{2}\|\partial^\bullet u\|^2_{L^2_{V^*}} - C\|u\|^2_{L^2_V}. \end{aligned} \tag{4.2.8}$$

The estimate (4.2.8) provides the control of $\|\partial^\bullet u\|_{L^2_{V^*}}$ at the expanse of $\|u\|_{L^2_V}$, which is controlled by the estimate (4.2.7). Accordingly we define the ansatz $v_u = z + \mu u_\gamma \in L^2_V$, where $\mu > 0$ is a sufficiently large and we obtain

$$\begin{aligned} \|v_u\|_{L^2_V} &\leq \|z\|_{L^2_V} + \mu\|u_\gamma\|_{L^2_V} \\ &= \|\partial^\bullet u\|_{L^2_{V^*}} + \mu\left(\int_0^T |e^{-\gamma t}|^2\|u(t)\|^2_{V(t)}\right)^{\frac{1}{2}} \\ &\leq \|\partial^\bullet u\|_{L^2_{V^*}} + \mu\|u\|_{L^2_V} \\ &\leq \mu\sqrt{2}\|u\|_{W(V,V^*)} \end{aligned} \tag{4.2.9}$$

From (4.2.7), (4.2.8), (4.2.9) and setting $\mu e^{-\gamma T}C_1 - C = 1/2$, we conclude

$$\begin{aligned} d(u,v_u) &\geq \frac{1}{2}\|\partial^\bullet u\|^2_{L^2_{V^*}} - C\|u\|^2_{L^2_V} + \mu e^{-\gamma T}C_1 \\ &\geq \frac{1}{2}\|u\|^2_{W(V,V^*)} \geq \frac{\sqrt{2}}{4\mu}\|u\|_{W(V,V^*)}\|v_u\|_{L^2_V}, \end{aligned}$$

which completes the proof. □

Note that the inf-sup condition (5.2.9) is a direct consequence of the previous Lemma.

Lemma 4.2.5. If for any given $v \in L^2_V$, the equality $d(u,v) = 0$ holds for all $u \in W(V,V^*)$, then $v = 0$.

Proof. Setting $u = \eta \in \mathcal{D}_V$ in $d(\eta, v) = 0$, we infer

$$\begin{aligned} (\dot\eta, v)_{L^2_H} = (v,\dot\eta)_{L^2_H} = -\int_0^T a(t;\eta(t),v(t)) - \int_0^T c(t;\eta(t),v(t)) \\ = -\int_0^T a(t;v(t),\eta(t)) - \int_0^T c(t;v(t),\eta(t)), \end{aligned}$$

where in the last step we used the symmetry of the bilinear forms $a(t;\cdot,\cdot)$ and $c(t;\cdot,\cdot)$. From the definition of the material derivative (3.4.6) and previous equality, we conclude

$$\langle \partial^\bullet v, \eta \rangle_{L^2_{V^*}, L^2_V} = a(t; v(t), \eta(t)), \quad \forall \eta \in \mathcal{D}_V. \tag{4.2.10}$$

Hence, $\partial^\bullet v \in L^2_{V^*}$, and thus $v \in W(V, V^*)$. Since $\mathcal{D}((0,T), V_0) \subset L^2((0,T), V_0)$ is dense, it follows that $\mathcal{D}_V \subset L^2_V$ is dense, which infers that (4.2.10) holds for all $u \in L^2_V$. In particular, if we choose $u \in W_0(V, V^*)$, by assumption $d(u, v) = 0$ and (4.2.10) we get

$$\langle \partial^\bullet u, v \rangle_{L^2_{V^*}, L^2_V} + a(t; v, u) + \int_0^T c(t; u(t), v(t)) = 0.$$

Utilizing Cor. 3.5.8, we obtain

$$\int_0^T \frac{d}{dt}(u(t), v(t))_{H(t)} = (u(T), v(T))_{H(T)} = 0 \quad \forall u \in W_0(V, V^*),$$

consequently $v(T) = 0$. We proceed as in the first step of the proof of Lemma 4.2.4, by setting $u(t) = v_\gamma(t) = e^{-\gamma t} v(t) \in L^2_V$ in (4.2.10) and calculating

$$\begin{aligned}
0 &= \langle \partial^\bullet v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} - \int_0^T a(t; v(t), v_\gamma(t)) \\
&= \frac{1}{2}(\langle \partial^\bullet v, v_\gamma \rangle_{L^2_{V^*}, L^2_V} + \langle \partial^\bullet v_\gamma, v \rangle_{L^2_{V^*}, L^2_V}) + \frac{1}{2}\gamma(v, v_\gamma)_{L^2_H} - \int_0^T a(t; v(t), v_\gamma(t)) \\
&= \frac{1}{2}\int_0^T \frac{d}{dt}(v(t), v_\gamma(t))_{H(t)} - \frac{1}{2}\int_0^T c(t; v, v_\gamma) + \frac{1}{2}\gamma(v, v_\gamma)_{L^2_H} - \int_0^T a(t; v(t), v_\gamma(t)) \\
&\leq -\frac{1}{2}\int_0^T c(t; v, v_\gamma) + \frac{1}{2}\gamma(v, v_\gamma)_{L^2_H} - \int_0^T a(t; v(t), v_\gamma(t)),
\end{aligned}$$

where in the last step we used that $v(T) = 0$. Thanks to (2.4.1) and (4.1.6), we end up with

$$0 \leq (C_{\mathbf{v}} + \gamma + 2C_2) \int_0^T e^{-\gamma t} \|v(t)\|^2_{H(t)} - 2C_1 \int_0^T e^{-\gamma t} \|v(t)\|^2_{V(t)}.$$

Choosing $\gamma = -C_{\mathbf{v}} - 2C_2$, we conclude $v = 0$ in L^2_V, which finishes the proof. □

After developing all the necessary results, we can now formulate the theorem about the existence and uniqueness of a "mean-weak" solution of the equation (4.1.3).

Theorem 4.2.6. Under the Assumption 4.1.2 for given $f \in L^2_{V^*}$ and $u_0 \in H_0$, there exists a unique "mean-weak" solution $u \in W(V, V^*)$ satisfying (4.1.3) such that

$$\|u\|_{W(V,V^*)} \leq C(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}}) \tag{4.2.11}$$

where $V = (V(t))_{t\in[0,T]}$ is the family of spaces $V(t) = L^2(\Omega, H^1(\Gamma(t)))$, V^* is the family of corresponding dual spaces and $H_0 = L^2(\Omega, L^2(\Gamma_0))$.

Proof. As a direct consequence of proceeding two lemmas we obtain that the assumptions of the BNB Theorem 4.2.1 are fulfilled, which yields the existence and uniqueness of mean-weak solution $y \in W_0(V, V^*)$ to (4.2.5), with the estimate

$$\|y\|_{W(V,V^*)} \leq C\|\tilde{f}\|_{L^2_{V^*}}.$$

Setting $u = y + \tilde{z}$ (note that y depends on $\tilde{z}$), we obtain the unique mean-weak solution of (4.1.3) that satisfies the a priori bound (4.2.11). □

4.3. Regularity

Let us now assume more regularity of the input data. More precisely, let $f \in L^2_H$ and $u_0 \in V_0$. We will prove that in this case we also have more regularity for the solution, i.e. its material derivative. Before we state this result, we will prove some technical results.

If $u_0 \in V_0$ and $f \in L^2_H$, the Problem 5.2.1 of the "mean-weak" solution transforms to:

Problem 4.3.1 (Weak form of the random advection-diffusion equation on $\{\Gamma(t)\}$)**.** Find $u \in W(V, H)$ that point-wise satisfies the initial condition $u(0) = u_0 \in V(0)$ and

$$\int_\Omega\int_{\Gamma(t)} \partial^\bullet u(t)\varphi + \int_\Omega\int_{\Gamma(t)} \alpha(t)\nabla_\Gamma u(t)\cdot\nabla_\Gamma\varphi + \int_\Omega\int_{\Gamma(t)} u(t)\varphi\nabla_\Gamma\cdot\mathbf{w}(t) = \int_\Omega\int_{\Gamma(t)} fv,, \quad (4.3.1)$$

for every $\varphi \in L^2(\Omega, H^1(\Gamma(t)))$ and a.e. $t \in [0, T]$.

Lemma 4.3.2. There exists a basis $\{\chi^0_j\}_{j\in\mathbb{N}}$ of $V_0 \equiv L^2(\Omega, H^1(\Gamma_0))$ and for every $u_0 \in V_0$ there exists a sequence $\{u_{0k}\}_{k\in\mathbb{N}}$ with $u_{0k} \in \text{span}\{\chi^0_1, \ldots, \chi^0_k\}$ for every k, such that

$$\begin{aligned} u_{0k} &\to u_0 \text{ in } V_0 \\ \|u_{0k}\|_{H_0} &\leq \|u_0\|_{H_0} \\ \|u_{0k}\|_{V_0} &\leq \|u_0\|_{V_0}. \end{aligned} \quad (4.3.2)$$

Proof. Since $H^1(\Gamma_0)$ is compactly embedded in $L^2(\Gamma_0)$, there exists an orthonormal basis $\{w_m\}_{m\in\mathbb{N}}$ in $L^2(\Gamma_0)$ such that

$$(u, w_m)_{L^2(\Gamma_0)} = \lambda_m^{-1}(u, w_m)_{H^1(\Gamma_0)} \quad \forall u \in H^1(\Gamma_0) \quad (4.3.3)$$

and in addition, $\{\lambda_m^{-1/2} w_m\}_{m\in\mathbb{N}}$ is an orthonormal basis of $H^1(\Gamma_0)$ (see for instance [108, Theorem 6.2-1]). On the other hand, since $L^2(\Omega)$ is separable, it has an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. It follows according to Theorem 2.5.4 that $\{w_m e_n\}_{m,n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega, L^2(\Gamma_0))$ and $\{\lambda^{-1/2} w_m e_n\}_{m,n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\Omega, H^1(\Gamma_0))$. Let $u_0 \in L^2(\Omega, H^1(\Gamma_0))$ be arbitrary. Then, (4.3.3) implies

$$(u_0, e_n w_m)_{L^2(\Omega,L^2(\Gamma_0))} = \lambda_m^{-1}(u_0, e_n w_m)_{L^2(\Omega,H^1(\Gamma_0))}. \quad (4.3.4)$$

Thus we have

$$u_0 = \sum_{m,n}(u_0, e_n w_m)_{L^2(\Omega,L^2(\Gamma_0))} e_n w_m = \sum_{m,n}(u_0, e_n w_m)_{L^2(\Omega,H^1(\Gamma_0))}\lambda_m^{-1} e_n w_m.$$

Now we can define

$$u_{0k} := \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega,L^2(\Gamma_0))} e_n w_m = \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega,H^1(\Gamma_0))} \lambda_m^{-1} e_n w_m,$$

where the last equality follows from (4.3.4). We choose M_k and N_k such that they both converge to ∞, as $k \to \infty$. Defined like this, u_{0k} satisfies conditions (4.3.2) from the Lemma. □

If we write $\chi_j^t := \phi_t(\chi_j^0)$, where $\{\chi_j^0\}_{j\in\mathbb{N}}$ is a basis of V_0, then by [4, Lemma 5.1] it follows that $\{\chi_j^t\}_{j\in\mathbb{N}}$ is a countable basis of $V(t)$. Now we define the space

$$\tilde{C}_V^1 := \{u \,|\, u(t) = \sum_{j=1}^{m} \alpha_j(t)\chi_j^t, m \in \mathbb{N}, \alpha_j \in AC([0,T]) \text{ and } \alpha_j' \in L^2(0,T)\},$$

where $AC([0,T])$ is the space of absolutely continuous functions from $[0,T]$.

For improved regularity of the solution, we will also need the following assumption on the material derivative of the random coefficient α.

Assumption 4.3.3. The diffusion coefficient α fulfils $\alpha(\omega,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{G}_T)$ for $\mathbb{P}$-a.e $\omega \in \Omega$, which implies boundedness of $|\dot{\alpha}(\omega)|$ on $\mathcal{G}_T$, and we assume that this bound is uniform in $\omega \in \Omega$.

Lemma 4.3.4. a) The map

$$t \mapsto a(t; y(t), y(t))$$

is an absolutely continuous function on $[0,T]$ for all $y \in \tilde{C}_V^1$.

b) $a(t; v, v) \geq 0$ for all $v \in V(t)$.

c)

$$\frac{d}{dt} a(t; y(t), y(t)) = 2a(t; y(t), \partial^\bullet y(t)) + r(t; y(t)) \quad \forall y \in \tilde{C}_V^1,$$

where the derivative is taken in the classical sense and $r(t;\cdot) : V(t) \to \mathbb{R}$ satisfies

$$|r(t; v)| \leq C_3 \|v\|_{V(t)}^2 \quad \forall v \in V(t).$$

Proof. Part b) follows immediately from the assumption (4.1.2). In order to prove parts a) and c), let us first take $\eta \in C_V^\infty$. Since the probability space Ω does not depend on time, it does not have any influence in taking a time derivative, thus the analogue Transport formulae from the deterministic case (that can be found in [58, Lemma 2.1]) still hold in our setting. By applying this formula to the bilinear form $a(t;\cdot,\cdot)$ we get

$$\frac{d}{dt} a(t; \eta(t), \eta(t)) = 2a(t; \eta(t), \partial^\bullet \eta(t)) + r(t; \eta(t)), \tag{4.3.5}$$

where the function $r(t; \eta(t))$ is defined by

$$r(t; \eta(t)) := \int_\Omega \int_{\Gamma(t)} \dot{\alpha} |\nabla_\Gamma \eta|^2 + \alpha |\nabla_\Gamma \eta|^2 \nabla_\Gamma \cdot \mathbf{v} - 2\nabla_\Gamma \eta (D_\Gamma(\mathbf{v})) \nabla_\Gamma \eta$$

with the deformation tensor $(D_\Gamma \mathbf{v}(t))_{ij} := \underline{D}_j \mathbf{v}^i(t)$.

In accordance with similar arguments as in [5, Ch. 5.1], which are based on the density result of space C_V^∞ in $\tilde{C}_V^1$, we can conclude that the previous formula is also true for every function $\eta \in \tilde{C}_V^1$. Furthermore, the boundedness of $r(t;\cdot)$ follows directly from the assumptions about the velocity (2.4.1) and assumption (4.3.3). This proves c). It remains to prove part a). This claim follows directly from the previous calculation, which implies that both the function $a(t;\eta(t),\eta(t))$ and its time derivative (i.e. the right hand side of (4.3.5)) belong to $L^1(0,T)$, from which it follows that $t \mapsto a(t;\eta(t),\eta(t))$ has an absolutely continuous representative. □

Theorem 4.3.5. Let Assumption 4.1.2 hold and additionally assume that Assumption 4.3.3 is fulfilled. Then for given $f \in L_H^2$ and $u_0 \in V_0$, there exists a unique "mean-weak" solution $u \in W(V,H)$ satisfying (4.3.1) and the following a priori estimate holds

$$\|u\|_{W(V,H)} \leq C(\|u_0\|_{V_0} + \|f\|_{L_H^2}).$$

Proof. From Lemma 4.1.5, Lemma 4.3.2, and Lemma 4.3.4, it follows that we can apply the general theorem [4, Theorem 3.13] about the regularity of the solution of parabolic PDEs on evolving space, which implies the theorem. Here we just give the main idea of the proof, referring the interested reader to [4, Ch. 5]. The basis of the proof is the abstract pushed-forward Galerkin method, which is the generalization of the Galerkin approximation of the advection-diffusion equation on evolving hypersurfaces analysed in [59]. First we construct the countable pushed forward basis $\chi_j^t := \phi_t(\chi_j^0)$ of $V(t)$ and prove its transport property $\dot{\chi}_j^t = 0$. Now we can define the approximation spaces

$$V_N(t) := \text{span}\{\chi_1^t, \dots, \chi_N^t\} \subset V(t)$$

and

$$L_{V_N}^2 := \left\{ u \in L_V^2 |\, u(t) = \sum_{j=1}^N \alpha_j(t)\chi_j^t,\ \alpha_j : [0,T] \to \mathbb{R} \right\},$$

and note that $\cup_j L_{V_j}^2$ is dense in L_V^2. Furthermore, construct the finite dimensional solutions u_N and prove the well-posedness of the finite dimensional problem together with an a priori bound for the solution and its material derivative

$$\|u_N\|_{L_V^2} \leq C\left(\|u_0\|_{H_0} + \|f\|_{L_{V^*}^2}\right) \quad \text{and } \|\dot{u}_N\|_{L_V^2} \leq C\left(\|u_0\|_{V_0} + \|f\|_{L_H^2}\right).$$

This yields weak convergence results

$$u_N \rightharpoonup u \quad \text{and } \dot{u}_N \rightharpoonup w.$$

The last step is to prove that $w = \dot{u}$, and that $u \in W(V,H)$ is the solution to (4.3.1). □

Remark 4.3.6. One could also prove the well-posedness of (4.1.3) using the Galerkin approximation technique. This proof is presented in the abstract setting in [4, Sec 5.4].

5. Log-normal random diffusion coefficient

In this chapter we will consider the case when the diffusion coefficient has a log-normal distribution introduced by Definition 2.7.3 and satisfies the assumption concerning its series representation. We will use results and definitions from Sections 2.6 and 2.7, and especially our sample space Θ which will be defined by (5.1.3) with measure γ defined by (5.1.1). Since, in this case, the random coefficient is not uniformly bounded with respect to the parameter $y \in \Theta$, the integration of the path-wise formulation over Θ with respect to γ does not lead to a well-posed "mean-weak" formulation. Thus, we can not apply the BNB Theorem 4.2.1, as we did in the uniform case in the Chapter 4. Instead, for each realization y, we will consider a path-wise formulation for which we know, from the deterministic setting, that it has a unique solution $u(y)$. Since we are interested in the statistics of the solution, especially expectation and variance, we want to prove p-integrability of the solution with respect to γ. This consists of two steps, the first of which is proving the measurability of the map $y \mapsto u(y)$ and the second of which is proving the bound for the L^p-norm.

Hence, in this chapter the diffusion coefficient $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}_+$ is a log-normal random field. The definition of GRF and its properties were discussed in Section 2.7. We will consider a series expansion of its logarithm.

Assumption 5.0.1. There exists a sequence $(Y_k)_{k\in\mathbb{N}}$ of i.i.d. standard Gaussian random variables on Ω and functions $\alpha_k \in L^\infty(\mathcal{G}_T)$ for $k \in \mathbb{N}$ with $b := (\|\alpha_k\|_{L^\infty(\mathcal{G}_T)})_{k\in\mathbb{N}} \in l^1(\mathbb{N})$, i.e. $\sum_k b_k < \infty$, where $b_k := \|\alpha_k\|_{L^\infty(\mathcal{G}_T)}$, such that the diffusion coefficient has the form

$$\alpha(\omega; x, t) = \exp\left(\sum_{k\geq 1} \alpha_k(x,t) Y_k(\omega)\right). \tag{5.0.1}$$

Remark 5.0.2. Without loss of generality, we have assumed that the logarithm of α is a centered Gaussian random field. Otherwise, one would get the additional factor in the expansion

$$\alpha(\omega; x, t) = \alpha_0(x) \exp\left(\sum_{k\geq 1} \alpha_k(x,t) Y_k(\omega)\right)$$

and would need to assume that $\alpha_0(x) \geq \alpha_0 > 0, \forall x$. For us, $\alpha_0(x) = 1$.

Remark 5.0.3. We discussed in Section 2.7 the conditions when Assumption 5.0.1 is satisfied. In particular, necessary conditions concerning Assumption 5.0.1 are discussed e.g. in [97] and the references given therein. It is shown that the standard measurability conditions (more precisely: measurability, finite-variance and isotropy) imply the mean-square continuity of a random field. It turns out that this is necessary for representation (5.0.1) to hold.

This type of Assumption 5.0.1 is often made in the literature (see for example [111]) and it holds if $\log \alpha$ is Gaussian and we consider its KL expansion. More generally, we can take $(\alpha_k)_k$ to be orthogonal in the Cameron Martin space of $\log \alpha$. For more details we direct the reader to [65, 111].

5.1. Transformation to parametrized discrete formulation and auxiliary measures

In this section we will consider the random PDE as a parametrized discrete problem. For more details we refer to [65, 106, 111].

One of the advantages of using the KL expansion is that it enables us to transform the random PDE into a parametric deterministic problem on a parameter space that is a subset of $\mathbb{R}^{\mathbb{N}}$. Here $\mathbb{R}^{\mathbb{N}}$ is the infinite dimensional parameter space that can be seen as an infinite product space of all real-sequences. When equipped with its product topology, this is a topological space. Moreover, it is also a measure space when we equip it with its Borel σ-algebra $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$, which is the same as the product of Borel σ-algebras.

We will be interested in the mapping

$$Y : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$$

which is defined by $Y(i) := Y_i(\omega)$, for a given sequence of random variables $\{Y_i\}_{i\in\mathbb{N}}$ from the KL expansion. The mapping Y is measurable (see [65, Lemma 2.8]). Following [111], we recall what kind of a product measure is induced on $\mathbb{R}^{\mathbb{N}}$. In general, the pushforward of the measure $\mathbb{P}$ under Y induces a Borel probability measure γ on $\mathbb{R}^{\mathbb{N}}$

$$\gamma := Y_\sharp \mathbb{P} : \mathcal{B}(\mathbb{R}^{\mathbb{N}}) \to [0, \infty) \qquad (Y_\sharp \mathbb{P})(B) := \mathbb{P}(Y^{-1}(B))) \quad \forall B \in \mathcal{B}(\mathbb{R}^{\mathbb{N}}).$$

In the special case, which will be the subject of our interest, when Y_i are i.i.d. with the standard Gaussian distribution, then γ is an infinite product of standard Gaussian measures N_1 on $\mathbb{R}$

$$\gamma := \bigotimes_{k\geq 1} N_1. \tag{5.1.1}$$

Equivalently, γ is a distribution of an i.i.d. sequence of standard Gaussian RVs. Furthermore, γ is itself Gaussian. More details about Gaussian measures on infinite dimensional spaces are presented for example in [20, 39].

Motivated by the analysis in [65] and [111], for the log-normal case under Assumption 5.0.1, we will reformulate the problem with the parameter domain $\mathbb{R}^{\mathbb{N}}$ instead of Ω . Thus, our probability space is $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \gamma)$, with γ defined by (5.1.1). We underline this change by switching from the notation ω to y and from $Y_k(\omega)$ to y_k. Therefore, the diffusion coefficient now has the form

$$\alpha(y; x, t) = \exp\left(\sum_{k\geq 1} \alpha_k(x, t) y_k\right) \tag{5.1.2}$$

for $y = (y_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and we assume that y_k are i.i.d. standard Gaussian random variables on $\mathbb{R}$.

In order to have the convergence of the series (5.1.2) we consider

$$\Theta_b := \left\{ y \in \mathbb{R}^{\mathbb{N}} \mid \sum_{k\geq 1} b_k |y_k| < \infty \right\}. \tag{5.1.3}$$

With Assumption 5.0.1, using [65, Lemma 2.2] the series (5.1.2) converges in $L^\infty(\mathcal{G}_T)$ in the parameter space Θ_b.

Lemma 5.1.1. For any $b \in l^1(\mathbb{N})$ it holds $\Theta_b \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ and $\gamma(\Theta_b) = 1$.

Proof. This proof is presented in [65, Lemma 2.3]. The measurability of Θ_b follows directly from its representation

$$\Theta_b = \bigcup_{N=1}^{\infty} \bigcap_{M=1}^{\infty} \left\{ y \in \mathbb{R}^{\infty} | \sum_{m=1}^{M} \alpha_m |y_m| \leq N \right\}.$$

Since for the random variable $y = (y_m)_m$ on $(\mathbb{R}^\infty, \gamma)$, where y_m are i.i.d. standard Gaussian random variables, we have

$$\int_{\mathbb{R}^\infty} |y_m| d\gamma = \frac{2}{\sqrt{2\pi}} \int_0^\infty \xi \exp\left(-\frac{\xi^2}{2}\right) d\xi = \sqrt{\frac{2}{\pi}},$$

and by applying the monotone convergence theorem we deduce

$$\mathbb{E}\left(\sum_{m=1}^{M} b_m |y_m|\right) = \int_{\mathbb{R}^\infty} \sum_{m=1}^{M} b_m |y_m| d\gamma = \sum_{m=1}^{M} \int_0^\infty b_m |y_m| = \sqrt{\frac{2}{\pi}} \sum_{m=1}^{M} b_m < \infty.$$

Hence the sum converges γ-a.e. on $\mathbb{R}^\infty$. Therefore $\gamma(\Theta_b) = 1$.

□

Instead of the whole space $\mathbb{R}^{\mathbb{N}}$, due to Lemma 5.1.1, we will consider $\Theta = \Theta_b$ as the parameter space with a measure that is a restriction of γ on Θ. From Assumption 5.0.1 it follows that the diffusion coefficient is bounded from above and has a positive lower bound for every $y \in \Theta$.

Lemma 5.1.2. For all $y \in \Theta$, the diffusion coefficient $\alpha(y)$ given by (5.1.2) is well-defined and satisfies

$$0 < \alpha_{\min}(y) := \operatorname*{ess\,inf}_{(x,t)\in\mathcal{G}_T} \alpha(y;x,t) \leq \operatorname*{ess\,sup}_{(x,t)\in\mathcal{G}_T} \alpha(y;x,t) =: \alpha_{\max}(y) < \infty \tag{5.1.4}$$

with

$$\alpha_{\max}(y) \leq \exp\left(\sum_{k\geq 1} b_k |y_k|\right)$$

$$\alpha_{\min}(y) \geq \exp\left(-\sum_{k\geq 1} b_k |y_k|\right).$$

Proof. The proof can be found in [111, Lemma 2.29], as a direct consequence of Assumption 5.0.1. For completeness, we present the same proof here. We will first prove the convergence in $L^\infty(\mathcal{G}_T)$, for every y, of the series (5.1.2). Letting $y \in \Theta$, $x \in \mathcal{G}_T$ and defining $b = (b_m)_{m\in\mathbb{N}} := (\|\alpha_m\|_{L^\infty(\mathcal{G}_T)})_{m\in\mathbb{N}}$, we conclude

$$\sum_m |\alpha_m(x,t)||y_m| \leq \sum_m b_m|y_m| < \infty.$$

The continuity and the positivity of the exponential function yield to

$$\exp(\sum_m \alpha_m(x,t)y_m) = \prod_m \exp(\alpha_m(x,t)y_m) \in (0,\infty),$$

which implies that α is well-defined. The bounds for $\alpha_{\max}(y)$ and $\alpha_{\min}(y)$ follow directly from the KL expansion (5.1.2). □

Remark 5.1.3. Note that the proofs of Lemma 5.1.3 and Lemma 5.1.2 do not depend on the parametric setting $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \gamma)$. Hence, using the same proofs we can show those results for the general probability setting $(\Omega, \mathcal{F}, \mathbb{P})$, i.e. for the corresponding space

$$\Omega_b :=:= \left\{ \omega \in \Omega \,|\, \sum_{k\geq 1} b_k|y_k(\omega)| < \infty \right\}$$

it holds $\mathbb{P}(\Omega_b) = 1$. In order to change to parametric formulation one needs to prove $y(\Omega_b) \subset \Theta_b$, for the proof see [65, Lemma 2.12].

In the previous Lemma 5.1.2 one needs to justify that $\alpha_{\min}(y)$ and $\alpha_{\max}(y)$ exist, i.e. that we can talk about the samples of the Gaussian random field $g(y) = \exp(\alpha(y))$. The idea is to prove that the realizations $\alpha(y)$ are continuous, for every y and then using the compactness argument on $\mathcal{G}_T$, conclude the existence of $\alpha_{\min}(y)$ and $\alpha_{\max}(y)$. To ensure continuity of realizations, we need to have some regularity assumptions about the kernel of the Gaussian random variable which will imply the assumption of Kolmogorov's theorem in a separable Banach space E (see Apendix A.4)

$$\mathbb{E}[\|g(t) - g(s)\|_E]^\delta \leq C|t-s|^{1+\varepsilon}. \tag{5.1.5}$$

These assumptions will not be discussed in this work, but we assume that (5.1.5) holds. However, for some special cases there are papers that analyze when this is achieved. For the flat case in [27, Prop 2.1], it is showed that the following assumption on the form of the kernel

$$\mathrm{cov}[g](x,y) = k(\|x-y\|)$$

for $k \in C^{0,1}(\mathbb{R}^+, \mathbb{R})$, implies that the Gaussian random field admits a version whose trajectories belong to $C^{0,\alpha}(\overline{D}), \alpha < 1/2$. In addition, one can show that $\alpha_{\min}, \alpha_{\max} \in L^p(\Omega)$. The sample regularity of the Gaussian random field on the sphere is analysed in [87] and it is characterized by the decay of the angular power spectrum. More details on this topic have been presented in Section 2.7.

Now we introduce an auxiliary Gaussian measure that we will need in order to prove the integrability of the solution. These results are presented in [65, 106]. For the convenience of the reader, we state and prove some of them.

Definition 5.1.4. For any $\sigma = (\sigma_k)_{k\in\mathbb{N}} \in \exp(l^1(\mathbb{N}))$, i.e. $\sigma_k = \exp(s_k)$ with $(s_k)_{k\in\mathbb{N}} \in l^1(\mathbb{N})$, we define the product measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ by

$$\gamma_\sigma := \bigotimes_{k\geq 1} N_{\sigma_k^2}$$

where $N_{\sigma_k^2}$ is a centered Gaussian measure on $\mathbb{R}$ with a standard deviation σ_k. Note that $\gamma_1 = \bigotimes_{k\geq 1} N_1 = \gamma$ is the standard Gaussian measure on $\mathbb{R}^{\mathbb{N}}$.

The next theorem states the strong relation between measures γ and γ_σ and gives the explicit formula for the density. The proof is based on Kakutani's theorem. Namely, Kakutani's theorem gives us the criteria to determine when two measures are equivalent, based on the sign of the Hellinger integral and in addition it gives the expression for the density function. Let us recall the definition of the Hellinger integral. The Hellinger integral for two probability measures μ and ν on a measure space $(\Omega, \mathcal{F})$ is defined by

$$H(\mu,\nu) := \int_\Omega \sqrt{\frac{d\mu}{d\xi}\frac{d\xi}{d\nu}} d\xi,$$

where $\xi = (\mu+\nu)/2$, hence both μ and ν are absolutely continuous w.r.t. ξ. By Hölder inequality it holds $0 \leq H(\mu,\nu) \leq 1$. Let now $(\mu_k)_k$ and $(\nu_k)_k$ be two sequences of probability measures on $(\Omega, \mathcal{F})$ such that $\mu_k \sim \nu_k$ for every k. We define $\mu := \otimes_k \mu_k$ and $\nu := \otimes_k \nu_k$. Kakutani's theorem states that if $H(\mu,\nu) > 0$ then μ and ν are equivalent and the density function is given by

$$\frac{d\nu}{d\mu}(x) = \lim_{k\to\infty} \prod_k \frac{d\nu_k}{d\mu_k}(x_k) \quad \text{in } L^1(\mathbb{R}^\infty, \mu). \tag{5.1.6}$$

Moreover, if $H(\mu,\nu) = 0$, then μ and ν are singular. Hence, if $\Omega = \mathbb{R}^{\mathbb{N}}$ and μ and ν are product measures on Ω, then there are just two possibilities: μ and ν are either equivalent or singular.

Theorem 5.1.5. For all $\sigma \in \exp(l^1(\mathbb{N}))$, the measure γ_σ is equivalent to γ and the density of γ_σ with respect to γ is given by

$$\frac{d\gamma_\sigma}{d\gamma} = \zeta_\sigma(y) = \left(\prod_{k\geq 1} \frac{1}{\sigma_k}\right) \exp\left(-\frac{1}{2}\sum_{k\geq 1}(\sigma_k^{-2} - 1)y_k^2\right). \tag{5.1.7}$$

Proof. The following proof is presented in [65, Proposition 2.11]. The main idea is to use Kakutani's theorem for $\nu_k = N_{\sigma_m^2}$ and $\mu_k = N_1$, implying $\nu = \gamma_\sigma$ and $\mu = \gamma$. From the density expression

$$\frac{dN_{\sigma_m^2}}{dN_1} = \xi_{\sigma,m}(y_m) = \frac{1}{\sigma_m}\exp(-\frac{1}{2}(\sigma_m^{-1} - 1)y_m^2)$$

we infer

$$H(\nu_k, \mu_k) = \int_{\mathbb{R}} \sqrt{\xi_{\sigma,m}(y_m)} dN_1(y_m) = \sqrt{2}\sigma_m + \sigma_m^{-1} = \exp\left(\frac{1}{2}\beta_m\right)$$

for some β_m with $|\beta_m| \leq \log \sigma_m$. Hence,

$$H(\gamma_\sigma, \gamma) = \prod_m \int_{\mathbb{R}} \sqrt{\xi_{\sigma,m}(y_m)} dN_1(y_m) = \exp\left(\frac{1}{2}\sum_m \beta_m\right) \in (0, \infty),$$

since $|\beta_m| \leq \sigma_m$ and $(\log \sigma_m)_m \in l^1(\mathbb{N})$. Consequently, according to Kakutani's theorem, γ_σ and γ are equivalent and the density is

$$\frac{d\gamma_\sigma}{d\gamma}(y) = \lim_{m\to\infty} \xi_{\sigma,m}(y_m)$$

which is equal to (5.1.7). □

Since from the previous theorem we conclude $\gamma_\sigma(\Theta) = 1$ for every $\sigma \in \exp(l^1(\mathbb{N}))$, it follows that the restriction of γ_σ on Θ is a probability measure. We will be interested in a particular form of the exponential sequence σ that depends exponentially on $b = (b_k)_{k\in\mathbb{N}}$, for $b_k := \|\alpha_k\|_{L^\infty(\mathcal{G}_T)}$ defined in Assumption 5.0.1. Thus, we will consider the class

$$\sigma_k := \exp(\chi b_k) \qquad \chi \in \mathbb{R}, \quad k \in \mathbb{N}$$

and we will use the following notation $\gamma_\chi := \gamma_{\sigma(\chi)}$ and $\zeta_\chi := \zeta_{\sigma(\chi)}$. Specifically, for $\chi = 0$, we get $\sigma_\chi = 1$ and $\gamma_1 = \gamma = \gamma_0$.

Lemma 5.1.6. Let $\eta < \chi$ and $m \geq 0$. Then, for every $y \in \Theta$ it holds

$$\frac{\zeta_\eta(y)}{\zeta_\chi(y)} \exp\left(m \sum_{k\geq 1} b_k |y_k|\right) \leq \exp\left(\left(\frac{m^2 \exp(2\chi\|b\|_{l^\infty})}{4(\chi-\eta)} + \chi - \eta\right)\|b\|_{l^1}\right).$$

Proof. The proof can be found in [111, Lemma 2.32] and it relies on the analytical expression for the density ξ given by (5.1.7) and the standard inequalities.

□

We will need the special case from Lemma 5.1.6, when $\eta = 0$, which gives us the bound for $1/\zeta_\chi(y) \exp\left(m \sum_{k\geq 1} b_k |y_k|\right)$.

The main difficulty which appears is that since the coefficient α is not uniformly bounded in y, the integration of the path-wise formulation over the parameter space Θ would lead to an ill-posed problem on $L^2(\Theta, \gamma; V)$. However, we will still prove that the solution of the path-wise formulation belongs to the space $L^2(\Theta, \gamma; V)$, by proving its γ-measurability and an appropriate bound for the norm. For this we will need the previously defined auxiliary Gaussian measures.

Remark 5.1.7. The idea how to overcome this difficulty and still consider the variational formulation over the parameter space Θ is presented for the elliptic case in [65]. One first considers the stronger measure γ_χ, $\chi > 0$, then $L^p(\Theta, \gamma_\chi) \subset L^p(\Theta, \gamma)$ and $u \in L^2(\Theta, \gamma_\chi; V)$. To overcome the difficulty of ill-posed variational formulation w.r.t. γ, one should consider a variational formulation w.r.t. a measure that is stronger than γ but weaker than γ_χ. For more details we refer the reader to [65].

5.2. Path-wise formulation of the problem

For the path-wise formulation we will consider the Gelfand triple $H^1(\Gamma(t)) \subset L^2(\Gamma(t)) \subset H^{-1}(\Gamma(t))$. Let us define

$$\mathcal{V}(t) := H^1(\Gamma(t)) \quad \text{and} \quad \mathcal{H}(t) := L^2(\Gamma(t)).$$

For simplicity, we will assume that the source term $f \in L^2_{\mathcal{V}^*}$ and the initial data $u_0 \in L^2_{\mathcal{H}}$ are deterministic. Furthermore, let us remark that we can transform the problem (1.0.1) into a PDE with zero initial condition, as already discussed in Section 4.2. Thus, in the following we will assume that $u_0 = 0$.

The solution space for the path-wise formulation will be

$$W_0(\mathcal{V}, \mathcal{V}^*) = \{u \in L^2_{\mathcal{V}} \,|\, \partial^\bullet u \in L^2_{\mathcal{V}^*}, u(0) = 0\},$$

which is a Hilbert space, as a closed linear subspace of $W(\mathcal{V}, \mathcal{V}^*)$.
Let us now state the path-wise weak formulation of (1.0.1):

Problem 5.2.1. [Path-wise weak form of the random advection-diffusion equation on $\{\Gamma(t)\}$] For every $y \in \Theta$ find $u(y) \in W_0(\mathcal{V}, \mathcal{V}^*)$ such that almost everywhere in $[0, T]$ it holds

$$\langle \partial^\bullet u(y), v\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \int_{\Gamma(t)} \alpha(y)\nabla_\Gamma u(y) \cdot \nabla_\Gamma v + \int_{\Gamma(t)} u(y) v \nabla_\Gamma \cdot \mathbf{v} \quad = \langle f, v\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}\,, \tag{5.2.1}$$

for every $v \in \mathcal{V}(t)$.

In order to get a coercive bilinear form, we write (1.0.1) as

$$\partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + (\lambda + \nabla_\Gamma \cdot \mathbf{v})u - \lambda u = f \tag{5.2.2}$$

for any $\lambda \in \mathbb{R}$. Introducing

$$\hat{u}(y) := e^{-\lambda t} u(y) \quad \text{and} \quad \hat{f}(y) := e^{-\lambda t} f(y)$$

and using the product rule, we can rewrite (5.2.2) as

$$\partial^\bullet \hat{u} - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma \hat{u}) + (\lambda + \nabla_\Gamma \cdot \mathbf{v})\hat{u} = \hat{f}. \tag{5.2.3}$$

Furthermore, the path-wise weak form of (5.2.3) is given by:
for every $y \in \Theta$ find $\hat{u}(y) \in W_0(\mathcal{V}, \mathcal{V}^*)$ such that almost everywhere in $[0, T]$ it holds

$$\langle \partial^\bullet \hat{u}(y), \hat{v}\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \hat{a}(y, t; \hat{u}, \hat{v}) = \left\langle \hat{f}, \hat{v} \right\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} \qquad \forall \hat{v} \in \mathcal{V}(t), \tag{5.2.4}$$

where the parametric bilinear form $\hat{a}(y, t; \cdot, \cdot) : \mathcal{V}(t) \times \mathcal{V}(t) \to \mathbb{R}$ is defined by

$$\hat{a}(y, t; \xi, \eta) := \int_{\Gamma(t)} \alpha(y)\nabla_\Gamma \xi \cdot \nabla_\Gamma \eta + (\lambda + \nabla_\Gamma \cdot \mathbf{v})\xi\eta.$$

The advantage of writing the equation in this form is that now the induced bilinear form $\hat{a}(y,t;\cdot,\cdot)$ is coercive and bounded, for sufficiently large λ. Namely for $\lambda > C_{\mathbf{v}}$ and $C_\lambda := \lambda - C_{\mathbf{v}}$ we have

$$\hat{a}(y,t;\eta,\eta) \geq m(y)\|\eta\|^2_{\mathcal{V}(t)} \tag{5.2.5}$$

$$|\hat{a}(y,t;\eta,\xi)| \leq M(y)\|\eta\|_{\mathcal{V}(t)}\|\xi\|_{\mathcal{V}(t)} \tag{5.2.6}$$

where $m(y) := \min(\alpha_{\min}(y), C_\lambda)$ and $M(y) := \max(\alpha_{\max}(y), \lambda + C_{\mathbf{v}})$.

Furthermore, we will also use the following estimate

$$\hat{a}(y,t;\eta,\eta) \geq \min\left(\alpha_{\min}(y), \frac{C_\lambda}{2}\right)\|\eta\|^2_{\mathcal{V}(t)} + \frac{C_\lambda}{2}\|\eta\|^2_{\mathcal{H}(t)}. \tag{5.2.7}$$

Defining the bilinear form $d(y) : W_0(\mathcal{V},\mathcal{V}^*) \times L^2_{\mathcal{V}} \to \mathbb{R}$ by

$$d(y;\xi,\eta) := \int_0^T \langle \partial^\bullet \xi, \eta\rangle_{\mathcal{V}^*,\mathcal{V}} + \hat{a}(y,t;\xi,\eta),$$

the inf-sup constant is given by

$$\beta(y) := \inf_{\eta \in W_0(\mathcal{V},\mathcal{V}^*)} \sup_{\xi \in L^2_{\mathcal{V}}} \frac{|d(y;\eta,\xi)|}{\|\eta\|_{W_0(\mathcal{V},\mathcal{V}^*)}\|\xi\|_{L^2_{\mathcal{V}}}}.$$

Lemma 5.2.2. Let Assumption 5.0.1 hold and additionally assume $\lambda \geq 3C_{\mathbf{v}}$ and (2.4.1). Then for every $y \in \Theta$, there exists a unique solution $\hat{u}(y) \in W_0(\mathcal{V},\mathcal{V}^*)$ to the problem (5.2.4). Moreover, the following estimate holds

$$\|\hat{u}(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)} \leq \frac{1}{\beta(y)}\|\hat{f}\|_{L^2_{\mathcal{V}^*}} \tag{5.2.8}$$

where the inf-sup constant is bounded from below by

$$\beta(y) \geq \frac{\min\left(\frac{m(y)}{M(y)^2}, \alpha_{\min}(y), \frac{C_\lambda}{2}\right)}{\sqrt{2\max(m(y)^{-2},1)}}. \tag{5.2.9}$$

Proof. Under Assumption 5.0.1, the existence and uniqueness of the solution, as well as the estimate (5.2.8) follow from the deterministic result for $\lambda \geq 3C_{\mathbf{v}}$, which can be found in [5] and [55]. In order to prove the bound (5.2.9) we will follow the idea from [112]. The main difference in the proof is that our domain is curved and changing in time, therefore we can not use the standard partial integration formula, but instead we will use partial integration that follows from the Transport theorem and has the additional term that reflects the spatial change in time.

Let $y \in \Theta$ be arbitrary. We start with defining the linear operator $A(y,t) : \mathcal{V}(t) \to \mathcal{V}^*(t)$ induced by

$$\langle A(y,t)\eta, \xi\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} := \hat{a}(y,t;\eta,\xi).$$

Given an arbitrary $0 \neq w(y) \in W_0(\mathcal{V},\mathcal{V}^*)$, we define

$$z_w(y,t) := A^{-1}(y,t)\partial^\bullet w(y,t) \in \mathcal{V}(t)$$

and select the test function

$$v_w(y,t) := z_w(y,t) + w(y,t) \in \mathcal{V}(t).$$

Using (5.2.5) and (5.2.6) we obtain

$$\langle \partial^\bullet w, z_w \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} \geq \frac{m(y)}{M(y)^2} \|\partial^\bullet w\|^2_{\mathcal{V}^*(t)}. \tag{5.2.10}$$

The definition of z_w directly implies

$$\hat{a}(y,t;w,z_w) = \left\langle Aw, A^{-1}\partial^\bullet w \right\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} = \langle w, \partial^\bullet w \rangle_{\mathcal{V}(t),\mathcal{V}^*(t)}. \tag{5.2.11}$$

Analogous to Theorem 3.5.7, the Transport formula for the scalar product in $\mathcal{H}(t)$ holds with

$$c(t;u,v) := \int_{\Gamma(t)} uv \nabla_\Gamma \cdot \mathbf{v}.$$

As a consequence, we obtain the following integration by parts formula (see [4, Corollary 2.41])

$$(u(T),v(T)))_{\mathcal{H}(t)} - (u(0),v(0))_{\mathcal{H}(t)} = \int_0^T \langle \partial^\bullet u, v \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \langle \partial^\bullet v, u \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + c(t;u,v). \tag{5.2.12}$$

Using (5.2.10) and (5.2.11) we arrive at

$$\begin{aligned} d(y;w,v_w) &\geq \int_0^T \frac{m(y)}{M(y)^2} \|\partial^\bullet w\|^2_{\mathcal{V}^*(t)} + \langle \partial^\bullet w, w \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \langle w, \partial^\bullet w \rangle_{\mathcal{V}(t),\mathcal{V}^*(t)} + \hat{a}(y,t;w,w) \\ &\geq \int_0^T \frac{m(y)}{M(y)^2} \|\partial^\bullet w\|^2_{\mathcal{V}^*(t)} - C_{\mathbf{v}} \|w\|^2_{\mathcal{H}(t)} + \frac{C_\lambda}{2} \|w\|^2_{\mathcal{H}(t)} + \min\left(\alpha_{\min}(y), \frac{C_\lambda}{2}\right) \|w\|^2_{\mathcal{V}(t)} \end{aligned}$$

where for the last inequality we used (5.2.7), (5.2.12) and (2.4.1). Taking $\lambda \geq 3C_{\mathbf{v}}$ gives $C_\lambda \geq 2C_{\mathbf{v}}$ and we get

$$d(y;w,v_w) \geq \min\left(\frac{m(y)}{M(y)^2}, \alpha_{\min}(y), \frac{C_\lambda}{2}\right) \|w\|^2_{W_0(\mathcal{V},\mathcal{V}^*)}. \tag{5.2.13}$$

It is left to estimate the norm $\|v_w\|_{L^2_\mathcal{V}}$, which follows directly from (5.2.5)

$$\begin{aligned} \|v_w\|^2_{L^2_\mathcal{V}} &\leq 2\left(\|A^{-1}\partial^\bullet w\|^2_{L^2_\mathcal{V}} + \|w\|^2_{L^2_\mathcal{V}}\right) \\ &\leq 2\max(m(y)^{-2},1)\|w\|^2_{W_0(\mathcal{V},\mathcal{V}^*)}. \end{aligned}$$

Since w is arbitrary, the last estimate, together with (5.2.13), implies the bound (5.2.9). □

Utilizing Lemma 5.2.2 we can prove the bound for the path-wise solution.

Theorem 5.2.3. Let Assumption 5.0.1 hold and additionally assume (2.4.1). Then problem (5.2.1) has a unique solution $u(y) \in W_0(\mathcal{V},\mathcal{V}^*)$ for every $y \in \Theta$ and it satisfies

$$\|u(y)\|_{W(\mathcal{V},\mathcal{V}^*)} \leq \frac{\hat{C}}{\beta(y)} \|f\|_{L^2_{\mathcal{V}^*}}$$

where $\hat{C}$ is independent of y and the inf-sup constant $\beta(y)$ is bounded from below by (5.2.9).

Proof. Similarly as in the previous Lemma 5.2.2, the existence and uniqueness of the path-wise solution follow from the deterministic results (see [5, 55]). In order to get the estimate of the solution norm, we compare the norms $\|u(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)}$ and $\|\hat{u}(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)}$. Since

$$\|\partial^\bullet u(y)\|^2_{L^2_{\mathcal{V}^*}} \leq 2e^{2\lambda T}\left(C\lambda\|\hat{u}(y)\|^2_{L^2_{\mathcal{V}}} + \|\partial^\bullet \hat{u}(y)\|^2_{L^2_{\mathcal{V}^*}}\right)$$

where C is the embedding constant of $L^2_{\mathcal{V}}$ into $L^2_{\mathcal{V}^*}$, using Lemma 5.2.2 we obtain

$$\begin{aligned}\|u(y)\|^2_{W_0(\mathcal{V},\mathcal{V}^*)} &\leq e^{2\lambda T}\left(\|\hat{u}(y)\|^2_{L^2_{\mathcal{V}}} + 2C\lambda\|\hat{u}(y)\|^2_{L^2_{\mathcal{V}}} + 2\|\partial^\bullet \hat{u}(y)\|^2_{L^2_{\mathcal{V}^*}}\right)\\ &\leq e^{2\lambda T}\max(2, 1+2C\lambda)\frac{1}{\beta(y)^2}\|\hat{f}\|^2_{L^2_{\mathcal{V}^*}} \leq \hat{C}^2\frac{1}{\beta(y)^2}\|f\|^2_{L^2_{\mathcal{V}^*}}\end{aligned}$$

where $\hat{C}^2 = e^{\lambda T}\max(2, 1+2C\lambda)$ is independent of y, which completes the proof. □

Remark 5.2.4. Without loss of generality we can assume

$$\alpha_{\min}(y) \leq C_{\mathbf{v}} \leq \frac{\alpha_{\max}(y)}{4}$$

for almost every y. Furthermore, without loss of generality we can assume that $\alpha_{\min}(y) \leq 1$ and $\alpha_{\max}(y) \geq 1$ for almost every y. Previous assumptions just make the calculations less technical, since it simplifies the bound of the inf-sup constant.

Under Assumption 5.2.4, by taking $\lambda = 3C_{\mathbf{v}}$, the bound (5.2.9) becomes

$$\beta(y) \geq \frac{1}{\sqrt{2}}\frac{\alpha_{\min}(y)^2}{\alpha_{\max}(y)^2} \quad \text{for a.e. } y.$$

The previous inequality together and Lemma 5.1.2 imply

$$\|u(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)} \leq \frac{\sqrt{2}}{\hat{C}}\frac{\alpha_{\min}(y)^2}{\alpha_{\max}(y)^2} \leq \frac{\sqrt{2}}{\hat{C}}\left(4\sum_{k\geq 1} b_k|y_k|\right) \tag{5.2.14}$$

for almost every y.

5.3. Integrability of the solution

In this section we will prove the p-integrability of the solution u with respect to γ. The first step is to show the measurability of the map $y \mapsto u(y)$, $\Theta \to W_0(\mathcal{V}, \mathcal{V}^*)$. The main idea of the proof is adopted from [65, Lemma 3.4]. It consists of proving that the solution u is almost surely the limit of measurable functions u_n that are the "mean-weak" solutions of (1.0.1) in the uniform case.

Remark 5.3.1. Let us note that since the sample space Θ is independent of time, it holds

$$L^2(\Theta, L^2_{\mathcal{V}}) \cong L^2(\Theta) \otimes L^2_{\mathcal{V}} \cong L^2_{L^2(\Theta,\mathcal{V})}.$$

From this we deduce

$$W(V, V^*) \cong L^2(\Theta) \otimes W(\mathcal{V}, \mathcal{V}^*) \cong L^2(\Theta, W(\mathcal{V}, \mathcal{V}^*)).$$

We will exploit this isomorphism in the proof of the p-integrability of the solution u with respect to γ, where we will consider the problem in a path-wise sense.

Theorem 5.3.2. The solution $u : \Theta \to W(\mathcal{V}, \mathcal{V}^*)$, $y \mapsto u(y)$ of (5.2.1) is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})$-measurable.

Proof. Since we have proved the well-posedness of the "mean-weak" formulation in the uniform case, the proof of the measurability can be adopted from [65, Lemma 3.4]. Here we just sketch its main idea. We start with defining a subspace Θ_n of Θ, for every $n \in \mathbb{N}$, where the diffusion coefficient is uniformly bounded

$$\Theta_n := \{y \in \Theta \,|\, \alpha_{\max}(y) < n, \alpha_{\min}(y) > \frac{1}{n}\} \subset \Theta.$$

Note that Θ_n is increasing and $\Theta = \cup_n \Theta_n$. Then we consider the "mean-weak" formulation on the parameter space Θ_n. In the uniform case, from Theorem 4.2.6 it follows that there exists a unique solution $u_n \in L^2(\Theta_n, \gamma; W_0(\mathcal{V}, \mathcal{V}^*))$. In particular, u_n is a measurable function on Θ_n. The last step is to prove that u is a.s. limit of u_n, thus it is measurable. This follows since u_n also solves the path-wise equation (5.2.1) for a.e. $y \in \Theta_n$. □

Remark 5.3.3. A natural approach to show that u_n solves the path-wise equation (5.2.1) is to use the Lebesgue differentiation theorem for the measure γ. However, the Lebesgue differentiation theorem does not hold in general for infinite dimensional separable Hilbert space and Gaussian measure on it. These results can be found in [105] where the counter example is given and in [121], where it is shown that under additional assumptions on the spectral representation of the covariance operator of measure, the differentiation theorem is valid in infinitely dimensional Hilbert space with this Gaussian measure. Unfortunately, the considered measure γ does not fulfil these additional assumptions.

Now we can state the result about the p-integrability of the solution.

Theorem 5.3.4. Let $0 < p < \infty$, $\chi > 0$ and $f \in L^2_{\mathcal{V}^*}$. If Assumption 5.0.1 holds and additionally we assume (2.4.1), then the solution u of (5.2.1) belongs to $L^p(\Theta, \gamma; W_0(\mathcal{V}, \mathcal{V}^*))$ and satisfies

$$\|u\|_{L^p(\Theta,\gamma;W_0(\mathcal{V},\mathcal{V}^*))} \leq \overline{c}_{p,\chi} \|f\|_{L^2_{\mathcal{V}^*}}$$

with

$$\overline{c}_{p,\chi} = \frac{\sqrt{2}}{\hat{C}} \exp\left(\frac{4p \exp(2\chi\|b\|_{l^\infty})}{\chi} + \frac{\chi}{p}\right) \|b\|_{l^1}.$$

Proof. With previous results in mind, the proof is similar to the proof stated in [106, Prop. 3.3.2]. However, since the bound for the inf-sup constant β is a bit different in our case, we give the main ideas of the proof. From Theorem 5.2.3 and Theorem 5.3.2 we obtain

$$\begin{aligned}\int_\Theta \|u(y)\|^p_{W_0(\mathcal{V},\mathcal{V}^*)} d\gamma &\leq \int_\Theta \frac{1}{\beta(y)^p} \|f\|^p_{L^2_{\mathcal{V}^*}} d\gamma \\ &= \int_\Theta \zeta_\chi(y)^{-1} \frac{1}{\beta(y)^p} \|f\|^p_{L^2_{\mathcal{V}^*}} d\gamma_\chi \leq \operatorname*{ess\,sup}_y \left(\frac{1}{\zeta_\chi(y)\beta(y)^p} \right) \|f\|^p_{L^2_{\mathcal{V}^*}},\end{aligned}$$

where ξ_χ and γ_χ are defined in Section 5.1. In order to bound $\frac{1}{\zeta_\chi(y)\beta(y)^p}$, we use Lemma 5.1.6 and the bound (5.2.14), which completes the proof. □

6. Evolving surface finite element methods

In this chapter we present evolving surface finite element discretization (ESFEM) for the homogeneous advection-diffusion equation (4.1.1)

$$\partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + u \nabla_\Gamma \cdot \mathbf{w} = 0$$

for the case when α is uniformly bounded in ω from above and below, i.e. Assumption 4.1.2 is satisfied. Following Dziuk & Elliott [55], we introduce space discretization that is performed by random piecewise linear finite element functions on simplicial approximations $\Gamma_h(t)$ of the surface $\Gamma(t), t \in [0,T]$.

For the numerical analysis we will assume more regularity of the input data as in Section 4.3, i.e. $f \in L^2_H$ and $u_0 \in V_0$. Furthermore, we assume that Assumption 4.3.3 is satisfied. Then, according to Theorem 4.3.5, there is a unique solution $u \in W(V,H)$ of Problem 4.3.1.

The following assumption of the diffusion coefficient will ensure the H^2-regularity of the solution.

Assumption 6.0.1. Assume that there exists a constant C independent of $\omega \in \Omega$ such that

$$|\nabla_\Gamma \alpha(\omega, x, t)| \leq C \quad \forall (x,t) \in \mathcal{G}_T$$

holds for $\mathbb{P}$-almost all $\omega \in \Omega$.

Note that (4.1.2) and Assumption 6.0.1 imply that $\|\alpha(\omega,t)\|_{C^1(\Gamma(t))}$ is uniformly bounded in $\omega \in \Omega$. This will be used later to prove an $H^2(\Gamma(t))$ bound. In the subsequent error analysis, we will assume further that u has a path-wise strong material derivative, i.e. that $u(\omega) \in C^1_V$ holds for all $\omega \in \Omega$.

To summarize, from now on, we will assume that Assumptions 4.1.2, 4.3.3 and 6.0.1 are satisfied and, additionally, that u has a path-wise strong material derivative $\dot{u}$, i.e. that $u(\omega) \in C^1_V$ holds for all $\omega \in \Omega$. Thus, in numerical analysis sections, notation $\partial^\bullet$ will be used for the strong material derivative defined by (3.4.1), that coincides with the weak material derivative for sufficiently smooth functions.

Remark 6.0.2. The uniformity condition (4.1.2) is not valid for log-normal random fields. Well-posedness for problems with this kind of random coefficients is stated in [15] assuming the existence of a suitable KL expansion. Sample regularity and differentiability, as typically needed for discretization error estimates, is still open, except in the special case of a sphere [29]. Here, the arguments highly rely on spherical harmonic functions that allow for an explicit representation of the Gaussian random field which in turn provides suitable control of the truncation error of KL expansions and regularity of samples. More general approaches to log-normal random fields are a subject of current studies but would exceed the scope of this thesis.

In order to derive a more convenient formulation of Problem 4.3.1 with an identical solution and test space, we introduce the time dependent bilinear forms

$$\begin{aligned} m(u,\varphi) &:= \int_\Omega \int_{\Gamma(t)} u\varphi, \quad g(\mathbf{w};u,\varphi) := \int_\Omega \int_{\Gamma(t)} u\varphi \nabla_\Gamma \cdot \mathbf{w}, \\ a(u,\varphi) &:= \int_\Omega \int_{\Gamma(t)} \alpha \nabla_\Gamma u \cdot \nabla_\Gamma \varphi, \quad b(\mathbf{w};u,\varphi) := \int_\Omega \int_{\Gamma(t)} B(\omega,\mathbf{w}) \nabla_\Gamma u \cdot \nabla_\Gamma \varphi \end{aligned} \tag{6.0.1}$$

for $u, \varphi \in L^2(\Omega, H^1(\Gamma(t)))$ and each $t \in [0,T]$. The tensor B in the definition of $b(\mathbf{w};u,\varphi)$ takes the form

$$B(\omega,\mathbf{w}) = (\partial^\bullet \alpha + \alpha \nabla_\Gamma \cdot \mathbf{w})\mathrm{Id} - 2\alpha D_\Gamma(\mathbf{w})$$

with Id denoting the identity in $(n+1)\times(n+1)$ and $(D_\Gamma \mathbf{w})_{ij} = \underline{D}_j \mathbf{w}^i$. Note that (2.4.1) and the uniform boundedness of $\partial^\bullet \alpha$ on $\mathcal{G}_T$ imply that $|B(\omega,\mathbf{w})| \le C$ holds $\mathbb{P}$-a.e. $\omega \in \Omega$ with some $C \in \mathbb{R}$.

Transport formula 3.5.7 for the differentiation of the time dependent surface integral then reads

$$\frac{d}{dt} m(u,\varphi) = m(\partial^\bullet u,\varphi) + m(u,\partial^\bullet \varphi) + g(\mathbf{w};u,\varphi), \tag{6.0.2}$$

where the equality holds a.e. in $[0,T]$. As a consequence of (6.0.2), Problem 4.3.1 is equivalent to the following formulation with an identical solution and test space.

Problem 6.0.3 (Weak form of the random advection-diffusion equation on $\{\Gamma(t)\}$)**.** Find $u \in W(V,H)$ that point-wise satisfies the initial condition $u(0) = u_0 \in V(0)$ and

$$\frac{d}{dt} m(u,\varphi) + a(u,\varphi) = m(u,\partial^\bullet \varphi) \qquad \forall \varphi \in W(V,H). \tag{6.0.3}$$

This formulation will be used in the continuation.

6.1. Evolving simplicial surfaces

As a first step towards discretization of the weak formulation (6.0.3) we now consider simplicial approximations of the evolving surface $\Gamma(t)$, $t \in [0,T]$. Let $\Gamma_{h,0}$ be an approximation of Γ_0 consisting of non-degenerate simplices $\{E_{j,0}\}_{j=1}^N =: \mathcal{T}_{h,0}$ with vertices $\{X_{j,0}\}_{j=1}^J \subset \Gamma_0$ such that the intersection of two different simplices is a common lower dimensional simplex or empty. For $t \in [0,T]$, we let the vertices $X_j(0) = X_{j,0}$ evolve with the smooth surface velocity $X_j'(t) = \mathbf{w}(X_j(t),t)$, $j = 1,\dots,J$, and consider the approximation $\Gamma_h(t)$ of $\Gamma(t)$ consisting of the corresponding simplices $\{E_j(t)\}_{j=1}^M =: \mathcal{T}_h(t)$. We assume that shape regularity of $\mathcal{T}_h(t)$ holds uniformly in $t \in [0,T]$ and that $\mathcal{T}_h(t)$ is quasi-uniform, uniformly in time, in the sense that

$$h := \sup_{t\in(0,T)} \max_{E(t)\in\mathcal{T}_h(t)} \operatorname{diam} E(t) \ge \inf_{t\in(0,T)} \min_{E(t)\in\mathcal{T}_h(t)} \operatorname{diam} E(t) \ge ch$$

holds with some $c \in \mathbb{R}$. We also assume that $\Gamma_h(t) \subset \mathcal{N}(t)$ for $t \in [0,T]$ and, in addition to (2.4.4), that for every $p \in \Gamma(t)$ there is a unique $x(p,t) \in \Gamma_h(t)$ such that

$$p = x(p,t) + d(x(p,t),t)\nu(p,t). \tag{6.1.1}$$

Note that $\Gamma_h(t)$ can be considered as an interpolation of $\Gamma(t)$ in $\{X_j(t)\}_{j=1}^J$ and a discrete analogue of the space time domain $\mathcal{G}_T$ is given by

$$\mathcal{G}_T^h := \bigcup_t \Gamma_h(t) \times \{t\}.$$

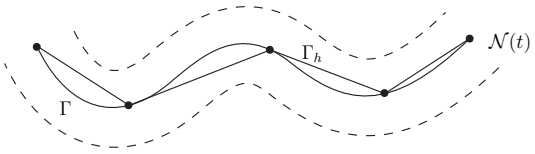


Figure 6.1.: Example of an approximation of a smooth curve Γ by a polygonal curveΓ_h.

We define the tangential gradient of a sufficiently smooth function $\eta_h \colon \Gamma_h(t) \to \mathbb{R}$ in an element-wise sense, i.e., we set

$$\nabla_{\Gamma_h} \eta_h|_E = \nabla \eta_h - \nabla \eta_h \cdot \nu_h \nu_h, \qquad E \in \mathcal{T}_h(t).$$

Here ν_h stands for the element-wise outward unit normal to $E \subset \Gamma_h(t)$. We use the notation $\nabla_{\Gamma_h} \eta_h = (\underline{D}_{h,1}\eta_h, \ldots, \underline{D}_{h,n+1}\eta_h)$.

We define the discrete velocity V_h of $\Gamma_h(t)$ by interpolation of the given velocity $\mathbf{w}$, i.e. we set

$$V_h(X(t),t) := \tilde{I}_h \mathbf{w}(X(t),t), \qquad X(t) \in \Gamma_h(t),$$

with $\tilde{I}_h$ denoting piecewise linear interpolation in $\{X_j(t)\}_{j=1}^J$.

We consider the Gelfand triple on $\Gamma_h(t)$

$$L^2(\Omega, H^1(\Gamma_h(t))) \subset L^2(\Omega, L^2(\Gamma_h(t))) \subset L^2(\Omega, H^{-1}(\Gamma_h(t))) \tag{6.1.2}$$

and denote

$$\mathcal{V}_h(t) := L^2(\Omega, H^1(\Gamma_h(t))) \quad \text{and} \quad \mathcal{H}_h(t) := L^2(\Omega, L^2(\Gamma_h(t))).$$

As in the continuous case, this leads to the following Gelfand triple of evolving Bochner-Sobolev spaces

$$L^2_{\mathcal{V}_h(t)} \subset L^2_{\mathcal{H}_h(t)} \subset L^2_{\mathcal{V}_h^*(t)}. \tag{6.1.3}$$

The discrete velocity V_h induces a discrete strong material derivative in terms of an element-wise version of (3.4.1), i.e., for sufficiently smooth functions $\phi_h \in L^2_{\mathcal{V}_h}$ and any $E(t) \in \Gamma_h(t)$ we set

$$\partial_h^\bullet \phi_h|_{E(t)} := (\phi_{h,t} + V_h \cdot \nabla \phi_h)|_{E(t)}. \tag{6.1.4}$$

We define discrete analogues to the bilinear forms introduced in (6.0.1) on $\mathcal{V}_h(t) \times \mathcal{V}_h(t)$ according to

$$m_h(u_h, \varphi_h) := \int_\Omega \int_{\Gamma_h(t)} u_h \varphi_h, \qquad g_h(V_h; u_h, \varphi_h) := \int_\Omega \int_{\Gamma_h(t)} u_h \varphi_h \nabla_{\Gamma_h} \cdot V_h,$$
$$a_h(u_h, \varphi_h) := \int_\Omega \int_{\Gamma_h(t)} \alpha^{-l} \nabla_{\Gamma_h} u_h \cdot \nabla_{\Gamma_h} \varphi_h,$$
$$b_h(V_h; \phi, U_h) := \sum_{E(t) \in \mathcal{T}_h(t)} \int_\Omega \int_{E(t)} B_h(\omega, V_h) \nabla_{\Gamma_h} \phi \cdot \nabla_{\Gamma_h} U_h$$

involving the tensor

$$B_h(\omega, V_h) = (\partial_h^\bullet \alpha^{-l} + \alpha^{-l} \nabla_{\Gamma_h} \cdot V_h) \mathrm{I}d - 2\alpha^{-l} D_h(V_h)$$

with Id denoting the identity in $(n+1) \times (n+1)$ and $(D_h(V_h))_{ij} = \underline{D}_{h,j} V_h^i$. Here, we denote

$$\alpha^{-l}(\omega, x, t) := \alpha(\omega, p(x,t), t) \quad \omega \in \Omega, \ (x,t) \in \mathcal{G}_T^h \tag{6.1.5}$$

exploiting $\{\Gamma_h(t)\} \subset \mathcal{N}(t)$ and (2.4.4). Later α^{-l} will be called the inverse lift of α.

Note that α^{-l} satisfies a discrete version of Assumptions 4.1.2, 4.3.3 and 6.0.1. In particular, α^{-l} is an $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T^h)$-measurable function, $\alpha^{-l}(\omega, \cdot, \cdot)|_{E_T} \in \mathcal{C}^1(E_T)$ for all space-time elements $E_T := \bigcup_t E(t) \times \{t\}$, and $\alpha_{\min} \leq \alpha^{-l}(\omega, x, t) \leq \alpha_{\max}$ for all $\omega \in \Omega$, $(x,t) \in \mathcal{G}_T^h$.

The next lemma provides a uniform bound for the divergence of V_h and the norm of the tensor B_h that follows from the geometric properties of $\Gamma_h(t)$ in analogy to [60, Lemma 3.3].

Lemma 6.1.1. Under the above assumptions on $\{\Gamma_h(t)\}$, it holds

$$\sup_{t \in [0,T]} \left(\|\nabla_{\Gamma_h} \cdot V_h\|_{L^\infty(\Gamma_h(t))} + \|B_h\|_{L^2(\Omega, L^\infty(\Gamma_h(t)))} \right) \leq c \sup_{t \in [0,T]} \|\mathbf{w}(t)\|_{\mathcal{C}^2(\mathcal{N}_T)}$$

with a constant c depending only on the initial hypersurface Γ_0 and the uniform shape regularity and quasi-uniformity of $\mathcal{T}_h(t)$.

Since the probability space does not depend on time, the discrete analogue of the corresponding transport formulae hold, where the discrete material velocity and discrete tangential gradients are understood in an element-wise sense. The resulting discrete result is stated for example in [58, Lemma 4.2]. The following lemma follows by integration over Ω.

Lemma 6.1.2 (Transport lemma for triangulated surfaces)**.** Let $\{\Gamma_h(t)\}$ be a family of triangulated surfaces evolving with discrete velocity V_h. Let ϕ_h, η_h be time dependent functions such that the following quantities exist. Then

$$\frac{d}{dt} \int_\Omega \int_{\Gamma_h(t)} \phi_h = \int_\Omega \int_{\Gamma_h(t)} \partial_h^\bullet \phi_h + \phi_h \nabla_{\Gamma_h} \cdot V_h.$$

In particular,

$$\frac{d}{dt} m_h(\phi_h, \eta_h) = m(\partial_h^\bullet \phi_h, \eta_h) + m(\phi_h, \partial_h^\bullet \eta_h) + g_h(V_h; \phi_h, \eta_h). \tag{6.1.6}$$

6.2. Finite elements on simplicial surfaces

Following [55], we now introduce an evolving surface finite element discretization (ESFEM) of problem 6.0.3.

For each $t \in [0,T]$ we define the *evolving finite element space*

$$S_h(t) := \{\eta \in \mathcal{C}(\Gamma_h(t)) \mid \eta_E \text{ is affine } \forall E \in \mathcal{T}_h(t)\}. \tag{6.2.1}$$

We denote by $\{\chi_j(t)\}_{j=1,\dots,J}$ the nodal basis of $S_h(t)$, i.e. $\chi_j(X_i(t),t) = \delta_{ij}$ (Kronecker-δ). These basis functions satisfy the transport property [58, Lemma 4.1]

$$\partial_h^\bullet \chi_j = 0. \tag{6.2.2}$$

We consider the following Gelfand triple

$$S_h(t) \subset L_h(t) \subset S_h^*(t), \tag{6.2.3}$$

where all three spaces algebraically coincide but are equipped with different norms inherited from the corresponding continuous counterparts, i.e.,

$$S_h(t) := (S_h(t), \|\cdot\|_{H^1(\Gamma_h(t))}) \quad \text{and} \quad L_h(t) := (S_h(t), \|\cdot\|_{L^2(\Gamma_h(t))}).$$

The dual space $S_h^*(t)$ consists of all continuous linear functionals on $S_h(t)$ and is equipped with the standard dual norm

$$\|\psi\|_{S_h^*(t)} := \sup_{\{\eta \in S_h(t) \mid \|\eta\|_{H^1(\Gamma_h(t))}=1\}} |\psi(\eta)|.$$

Note that all three norms are equivalent as norms on finite dimensional spaces, which implies that (6.2.3) is a Gelfand triple. As a discrete counterpart of (6.1.2), we introduce the Gelfand triple

$$L^2(\Omega, S_h(t)) \subset L^2(\Omega, L_h(t)) \subset L^2(\Omega, S_h^*(t)). \tag{6.2.4}$$

Setting

$$V_h(t) := L^2(\Omega, S_h(t)) \quad H_h(t) := L^2(\Omega, L_h(t)) \quad V_h^*(t) := L^2(\Omega, S_h^*(t))$$

we obtain the finite element analogue

$$L^2_{V_h(t)} \subset L^2_{H_h(t)} \subset L^2_{V_h^*(t)} \tag{6.2.5}$$

of the Gelfand triple (6.1.3) of evolving Bochner-Sobolev spaces. Let us note that since the sample space Ω is independent of time, it holds

$$L^2(\Omega, L^2_X) \cong L^2(\Omega) \otimes L^2_X \cong L^2_{L^2(\Omega,X)} \tag{6.2.6}$$

for any evolving family of separable Hilbert spaces X (see, e.g., Section 2.5). We will exploit this isomorphism for $X = S_h$ in the following definition of the solution space for the semi-discrete problem, where we will rather consider the problem in a path-wise sense.

We define the solution space for the semi-discrete problem as the space of functions that are smooth for each path in the sense that $\phi_h(\omega) \in \mathcal{C}^1_{S_h}$ holds for all $\omega \in \Omega$. Hence, $\partial_h^\bullet \phi_h$ is defined path-wise for path-wise smooth functions. In addition, we require $\partial_h^\bullet \phi_h(t) \in H_h(t)$ to define the semi-discrete solution space

$$W_h(V_h, H_h) := L^2(\Omega, \mathcal{C}^1_{S_h}).$$

The scalar product of this space is defined by

$$(U_h, \phi_h)_{W_h(V_h,H_h)} := \int_0^T \int_\Omega (U_h, \phi_h)_{H^1(\Gamma_h(t))} + \int_0^T \int_\Omega (\partial_h^\bullet U_h, \partial_h^\bullet \phi_h)_{L^2(\Gamma_h(t))}$$

with the associated norm $\|\cdot\|_{W_h(V_h,H_h)}$.

The semi-discrete approximation of Problem 6.0.3, on $\{\Gamma_h(t)\}$ now reads as follows.

Problem 6.2.1 (ESFEM discretization in space)**.** Find $U_h \in W_h(V_h, H_h)$ that point-wise satisfies the initial condition $U_h(0) = U_{h,0} \in V_h(0)$ and

$$\frac{d}{dt} m_h(U_h, \varphi) + a_h(U_h, \varphi) = m_h(U_h, \partial_h^\bullet \varphi) \qquad \forall \varphi \in W_h(V_h, H_h). \tag{6.2.7}$$

In contrast to $W(V, H)$, the semi-discrete space $W_h(V_h, H_h)$ is not complete since the space $C^1_{S_h}$ with the norm

$$\|u\|^2 := \int_0^T \|u\|^2_{H^1(\Gamma_h(t))} + \|\partial_h^\bullet u\|^2_{L^2(\Gamma_h(t))}$$

is not complete, because $C^1_{S_h}$ is isomorphic to $C^1([0,T], \mathbb{R}^N)$ which is not complete with respect to the L^2-norm $\|v\|_0 = (\int_0^T |v(t)|^2 + |v'(t)|^2 \, dt)^{1/2}$. Thus, the proof of the following existence and stability result can not be done in an analogue way as in the continuous case and requires a different kind of argument.

Theorem 6.2.2. The semi-discrete Problem 6.2.1 has a unique solution $U_h \in W_h(V_h, H_h)$ which satisfies the stability property

$$\|U_h\|_{W(V_h,H_h)} \leq C \|U_{h,0}\|_{V_h(0)} \tag{6.2.8}$$

with a mesh-independent constant C depending only on T, $\alpha_{\min}$, and the bound for $\|\nabla_{\Gamma_h} \cdot V_h\|_\infty$ from Lemma 6.1.1.

Proof. In analogy to the continuous case, the Transport theorem, i.e. formulae (6.0.2) and (6.0.3), imply that Problem 6.2.1 is equivalent to find $U_h \in W_h(V_h, H_h)$ that point-wise satisfies the initial condition $U_h(0) = U_{h,0} \in V_h(0)$ and

$$m_h(\partial_h^\bullet U_h, \varphi) + a_h(U_h, \varphi) + g_h(V_h; U_h, \varphi) = 0 \tag{6.2.9}$$

for every $\varphi \in L^2(\Omega, S_h(t))$ and a.e. $t \in [0, T]$.

Let $\omega \in \Omega$ be arbitrary but fixed. We start with considering the deterministic path-wise problem to find $U_h(\omega) \in \mathcal{C}^1_{S_h}$ such that $U_h(\omega;0) = U_{h,0}(\omega)$ and

$$\int_{\Gamma_h(t)} \partial_h^\bullet U_h(\omega)\varphi + \int_{\Gamma_h(t)} \alpha^{-l}(\omega)\nabla_{\Gamma_h} U_h(\omega)\cdot\nabla_{\Gamma_h}\varphi + \int_{\Gamma_h(t)} U_h(\omega)\varphi\nabla_{\Gamma_h}\cdot V_h = 0 \quad (6.2.10)$$

holds for all $\varphi \in S_h(t)$ and a.e. $t \in [0,T]$. Following Dziuk & Elliott [58, Section 4.6], we insert the nodal basis representation

$$U_h(\omega,t,x) = \sum_{j=1}^{J} U_j(\omega,t)\chi_j(x,t) \quad (6.2.11)$$

into (6.2.10) and take $\varphi = \chi_i(t) \in S_h(t)$, $i = 1,\dots,J$, as test functions. Now the transport property (6.2.2) implies

$$\begin{aligned}\sum_{j=1}^{J} \frac{\partial}{\partial t} U_j(\omega) \int_{\Gamma_h(t)} \chi_j\chi_i \quad &+ \quad \sum_{j=1}^{J} U_j(\omega) \int_{\Gamma_h(t)} \alpha^{-l}(\omega)\nabla_{\Gamma_h}\chi_j\cdot\nabla_{\Gamma_h}\chi_i \\ &+ \quad \sum_{j=1}^{J} U_j(\omega) \int_{\Gamma_h(t)} \chi_j\chi_i\nabla_{\Gamma_h}\cdot V_h = 0.\end{aligned} \quad (6.2.12)$$

We introduce the evolving mass matrix $M(t)$ with coefficients

$$M(t)_{ij} := \int_{\Gamma_h(t)} \chi_i(t)\chi_j(t),$$

and the evolving stiffness matrix $S(\omega,t)$ with coefficients

$$S(\omega,t)_{ij} := \int_{\Gamma_h(t)} \alpha^{-l}(\omega,t)\nabla_{\Gamma_h}\chi_j(t)\nabla_{\Gamma_h}\chi_i(t).$$

From [58, Proposition 5.2] it follows

$$\frac{dM}{dt} = M'$$

where

$$M'(t)_{ij} := \int_{\Gamma_h(t)} \chi_j(t)\chi_i(t)\nabla_{\Gamma_h}\cdot V_h(t).$$

Therefore, we can write (6.2.12) as the following linear initial value problem

$$\frac{\partial}{\partial t}(M(t)U(\omega,t)) + S(\omega,t)U(\omega,t) = 0, \qquad U(\omega,0) = U_0(\omega), \quad (6.2.13)$$

for the unknown vector $U(\omega,t) = (U_j(\omega,t))_{i=1}^J$ of coefficient functions. As in [58], there exists a unique path-wise semi-discrete solution $U_h(\omega) \in \mathcal{C}^1_{S_h}$, since the matrix $M(t)$ is uniformly positive definite on $[0,T]$ and the stiffness matrix $S(\omega,t)$ is positive semi-definite for every

$\omega \in \Omega$. Note that the time regularity of $U_h(\omega)$ follows from M, $S(\omega) \in C^1(0,T)$ which in turn is a consequence of our assumptions on the time regularity of the evolution of $\Gamma_h(t)$.

The next step is to prove the measurability of the map $\Omega \ni \omega \mapsto U_h(\omega) \in \mathcal{C}^1_{S_h}$. On $\mathcal{C}^1_{S_h}$ we consider the Borel σ-algebra induced by the norm

$$\|U_h\|^2_{\mathcal{C}^1_{S_h}} := \int_0^T \|U_h(t)\|^2_{H^1(\Gamma_h(t))} + \|\partial_h^\bullet U_h(t)\|^2_{L^2(\Gamma_h(t))}. \tag{6.2.14}$$

We write (6.2.13) in the following form

$$\frac{\partial}{\partial t} U(\omega,t) + A(\omega,t)U(\omega,t) = 0, \qquad U(\omega,0) = U_0(\omega),$$

where

$$A(\omega,t) := M^{-1}(t)\left(M'(t) + S(\omega,t)\right).$$

As $U_{h,0} \in V_h(0)$, the function $\omega \mapsto U_{h,0}(\omega)$ is measurable and since α^{-l} is a $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T^h)$-measurable function, it follows from Fubini's Theorem 2.2.3 that

$$\Omega \ni \omega \mapsto (U_0(\omega), A(\omega)) \in \mathbb{R}^J \times \left(C^1\left([0,T], \mathbb{R}^{J\times J}\right), \|\cdot\|_\infty\right)$$

is a measurable function. Let us show the continuity of the mapping

$$\mathbb{R}^J \times \left(C^1\left([0,T], \mathbb{R}^{J\times J}\right), \|\cdot\|_\infty\right) \ni (U_0, A) \mapsto U \in \left(C^1\left([0,T], \mathbb{R}^{J}\right), \|\cdot\|_\infty\right). \tag{6.2.15}$$

For that purpose let $\varepsilon > 0$ be arbitrary and let U and $\tilde{U}$ solve

$$U' = AU, \quad U(0) = 0 \quad \text{and} \quad \tilde{U}' = \tilde{A}\tilde{U}, \quad \tilde{U}(0) = 0,$$

where $\delta(t) := A(t) - \tilde{A}(t) \in \left(C^1\left([0,T], \mathbb{R}^{J\times J}\right), \|\cdot\|_\infty\right)$ and $\delta_0(t) := U_0 - \tilde{U}_0 \in \mathbb{R}^J$ satisfy

$$\|\delta\|_\infty = \max_{t\in[0,T]} \|\delta(t)\|_{\mathbb{R}^{J\times J}} \le \varepsilon \quad \text{and} \quad \|\delta_0\|_{\mathbb{R}^J} \le \varepsilon. \tag{6.2.16}$$

Then for $w := U - \tilde{U}$ we have

$$\begin{aligned} w' &= AU - \tilde{A}\tilde{U} \\ &= Au - \tilde{A}u + \tilde{A}u - \tilde{A}\tilde{U} \\ &= (A - \tilde{A})U + \tilde{A}(U - \tilde{U}) \\ &= \delta U + (A - \delta)w. \end{aligned}$$

The last equality implies

$$\max_{t\in[0,T]} \|w'(t)\| \le \max_{t\in[0,T]} \|u\|_\infty + \left((\|A\|_\infty + \|\delta\|_\infty)\|w\|_\infty\right). \tag{6.2.17}$$

Moreover, from (6.2.17) we conclude

$$w(t) = \delta_0 + \int_0^t (\delta U + (A - \delta)w)ds.$$

Utilizing the triangulation inequality and (6.2.16), we infer

$$\begin{aligned}\|w(t)\|_{\mathbb{R}^J} &\leq \|\delta_0\|_{\mathbb{R}^J} + \int_0^t \left(\|\delta(s)\|_{\mathbb{R}^{J\times J}}\|U(s)\|_{\mathbb{R}^J} + \|(A-\delta)(s)\|_{\mathbb{R}^{J\times J}}\|w(s)\|_{\mathbb{R}^J}\right) ds \\ &\leq \|\delta_0\|_{\mathbb{R}^J} + t\|\delta\|_\infty\|U\|_\infty + \|A-\delta\|_\infty \int_0^t \|w(s)\|_{\mathbb{R}^J} ds \\ &\leq (1+t\|U\|_\infty\varepsilon) + (\|A\|_\infty+\varepsilon)\int_0^t \|w(s)\|_{\mathbb{R}^J} ds.\end{aligned}$$

By Gronwall's lemma, the last inequality gives the bound

$$\|w(t)\|_{\mathbb{R}^J} \leq (1+t\|U\|_\infty)\,\varepsilon\, e^{(\|A\|_\infty+\varepsilon)t}.$$

Hence,

$$\|w\|_\infty \leq (1+T\|U\|_\infty)\,\varepsilon\, e^{(\|A\|_\infty+\varepsilon)T}. \tag{6.2.18}$$

Letting, $\varepsilon \to 0$, we obtain $\|w\|_\infty \to 0$.

From (6.2.18) and (6.2.17) we deduce

$$\|w'\|_\infty \leq \varepsilon\,\|u\|_\infty + (\|A\|_\infty+\varepsilon)(1+T\|u\|_\infty)\,\varepsilon\, e^{(\|A\|_\infty+\varepsilon)T}.$$

Thus, $\|w'\|_\infty \to 0$, for $\varepsilon \to 0$. Therefore, $\|w\|_{C^1([0,T])} \to 0$, for $\varepsilon \to 0$, i.e., the mapping (6.2.15) is continuous.

Furthermore, the mapping

$$\left(C^1\left([0,T],\mathbb{R}^J\right),\|\cdot\|_\infty\right) \ni U \mapsto U \in \left(C^1\left([0,T],\mathbb{R}^J\right),\|\cdot\|_2\right)$$

with

$$\|U\|_2^2 := \int_0^T \|U(t)\|_{\mathbb{R}^J}^2 + \|\frac{d}{dt}U(t)\|_{\mathbb{R}^J}^2 \leq T\|U\|_\infty^2$$

is continuous. Exploiting that the triangulation $\mathcal{T}_h(t)$ of $\Gamma_h(t)$ is quasi-uniform, uniformly in time, the continuity of the linear mapping

$$\left(C^1\left([0,T],\mathbb{R}^J\right),\|\cdot\|_2\right) \ni U \mapsto U_h \in \mathcal{C}^1_{S_h}$$

follows from the triangle inequality and the Cauchy-Schwarz inequality. We finally conclude that the function

$$\Omega \ni \omega \mapsto U_h(\omega) \in \mathcal{C}^1_{S_h}$$

is measurable as a composition of measurable and continuous mappings (see Lemma 2.1.12).

The next step is to prove the stability property (6.2.8). For each fixed $\omega \in \Omega$, path-wise stability results from [58, Lemma 4.3] imply

$$\|U_h(\omega)\|^2_{\mathcal{C}^1_{S_h}} \leq C\|U_{h,0}(\omega)\|^2_{H^1(\Gamma_h(0))} \tag{6.2.19}$$

where $C = C(\alpha_{\min},\alpha_{\max},V_h,T,\mathcal{G}_h^T)$ is independent of ω and $U_{h,0}(x) \in L^2(\Omega)$. Integrating (6.2.19) over Ω we get the bound

$$\|U_h\|_{W(V_h,H_h)} = \|U_h\|^2_{L^2(\Omega,\mathcal{C}^1_{S_h})} \leq C\|U_{h,0}\|^2_{V_h(0)}.$$

In particular, we have $U_h \in W_h(V_h, H_h)$.

It is left to show that U_h solves (6.2.9) and thus also Problem 6.2.1. Exploiting the tensor product structure of the test space $L^2(\Omega, S_h(t)) \cong L^2(\Omega) \otimes S_h(t)$ (see (6.2.6)), we find that

$$\{\varphi_h(x,t)\eta(\omega) \,|\, \varphi_h(t) \in S_h(t), \eta \in L^2(\Omega)\} \subset L^2(\Omega) \otimes S_h(t)$$

is a dense subset of $L^2(\Omega, S_h(t))$. Taking any test function $\varphi_h(x,t)\eta(\omega)$ from this dense subset, we first insert $\varphi_h(x,t) \in S_h(t)$ into the path-wise problem (6.2.10), then multiply with $\eta(\omega)$, and finally integrate over Ω to establish (6.2.9). This completes the proof. □

6.3. Lifted finite elements

We exploit (6.1.1) to define the lift $\eta_h^l(\cdot,t)\colon \Gamma(t) \to \mathbb{R}$ of functions $\eta_h(\cdot,t)\colon \Gamma_h(t) \to \mathbb{R}$ by

$$\eta_h^l(p,t) := \eta_h(x(p,t)), \quad p \in \Gamma(t).$$

Conversely, (2.4.4) is utilized to define the inverse lift $\eta^{-l}(\cdot,t)\colon \Gamma_h(t) \to \mathbb{R}$ of functions $\eta(\cdot,t)\colon \Gamma(t) \to \mathbb{R}$ by

$$\eta^{-l}(x,t) := \eta(p(x,t),t), \quad x \in \Gamma_h(t).$$

These operators are inverse to each other, i.e., $(\eta^{-l})^l = (\eta^l)^{-l} = \eta$, and, taking characteristic functions η_h, each element $E(t) \in \mathcal{T}_h(t)$ has its unique associated lifted element $e(t) \in \mathcal{T}_h^l(t)$. Recall that the inverse lift α^{-1} of the diffusion coefficient α was already introduced in (6.1.5).

The next lemma states equivalence relations between corresponding norms on $\Gamma(t)$ and $\Gamma_h(t)$ that follow directly from their deterministic counterparts (see [54]).

Lemma 6.3.1. Let $t \in [0,T]$, $\omega \in \Omega$, and let $\eta_h(\omega)\colon \Gamma_h(t) \to \mathbb{R}$ with the lift $\eta_h^l(\omega)\colon \Gamma \to \mathbb{R}$. Then for each plane simplex $E \subset \Gamma_h(t)$ and its curvilinear lift $e \subset \Gamma(t)$, there is a constant $c > 0$ independent of E, h, t, and ω such that

$$\frac{1}{c}\,\|\eta_h\|_{L^2(\Omega,L^2(E))} \le \|\eta_h^l\|_{L^2(\Omega,L^2(e))} \le c\,\|\eta_h\|_{L^2(\Omega,L^2(E))} \tag{6.3.1}$$

$$\frac{1}{c}\,\|\nabla_{\Gamma_h}\eta_h\|_{L^2(\Omega,L^2(E))} \le \|\nabla_\Gamma \eta_h^l\|_{L^2(\Omega,L^2(e))} \le c\,\|\nabla_{\Gamma_h}\eta_h\|_{L^2(\Omega,L^2(E))} \tag{6.3.2}$$

$$\frac{1}{c}\,\|\nabla^2_{\Gamma_h}\eta_h\|_{L^2(\Omega,L^2(E))} \le c\|\nabla^2_\Gamma \eta_h^l\|_{L^2(\Omega,L^2(e))} + ch\|\nabla_\Gamma \eta_h^l\|_{L^2(\Omega,L^2(e))}, \tag{6.3.3}$$

if the corresponding norms are finite and where $\nabla^2_{\Gamma_h}$ denotes the matrix of second order tangential derivatives.

The motion of the vertices of the triangles $E(t) \in \{\mathcal{T}_h(t)\}$ induces a discrete velocity v_h of the surface $\{\Gamma(t)\}$. More precisely, for a given trajectory $X(t)$ of a point on $\{\Gamma_h(t)\}$ with velocity $V_h(X(t),t)$ the associated discrete velocity $\mathbf{w}_h$ in $Y(t) = p(X(t),t)$ on $\Gamma(t)$ is defined by

$$\mathbf{w}_h(Y(t),t) = Y'(t) = \frac{\partial p}{\partial t}(X(t),t) + V_h(X(t),t) \cdot \nabla p(X(t),t). \tag{6.3.4}$$

The discrete velocity $\mathbf{w}_h$ gives rise to a discrete material derivative of functions $\varphi \in L_V^2$ in an element-wise sense, i.e., we set

$$\partial_h^\bullet \varphi|_{e(t)} := (\varphi_t + \mathbf{w}_h \cdot \nabla \varphi)|_{e(t)}$$

for all $e(t) \in \mathcal{T}_h^l(t)$, where $\nabla \varphi$ is defined via a smooth extension, analogous to the definition (3.4.1).

We introduce a lifted finite element space by

$$S_h^l(t) := \{\eta^l \in \mathcal{C}(\Gamma(t)) \mid \eta \in S_h(t)\}.$$

Note that there is a unique correspondence between each element $\eta \in S_h(t)$ and $\eta^l \in S_h^l(t)$. Furthermore, one can show that for every $\phi_h \in S_h(t)$ here holds

$$\partial_h^\bullet(\phi_h^l) = (\partial_h^\bullet \phi_h)^l. \tag{6.3.5}$$

Therefore, by (6.2.2) we get

$$\partial_h^\bullet \chi_j^l = 0.$$

We finally state an analogue to the transport Lemma 6.1.2 on simplicial surfaces.

Lemma 6.3.2. (Transport lemma for smooth triangulated surfaces.)

Let $\Gamma(t)$ be an evolving surface decomposed into curved elements $\{\mathcal{T}_h(t)\}$ whose edges move with velocity $\mathbf{w}_h$. Then the following relations hold for functions φ_h, u_h such that the following quantities exist

$$\frac{d}{dt} \int_\Omega \int_{\Gamma(t)} \varphi_h = \int_\Omega \int_{\Gamma(t)} \partial_h^\bullet \varphi_h + \varphi_h \nabla_\Gamma \cdot \mathbf{w}_h.$$

and

$$\frac{d}{dt} m(\varphi, u_h) = m(\partial_h^\bullet \varphi_h, u_h) + m(\varphi_h, \partial_h^\bullet u_h) + g(v_h; \varphi_h, u_h). \tag{6.3.6}$$

Remark 6.3.3. Let U_h be the solution of the semi-discrete Problem 6.2.1 with initial condition $U_h(0) = U_{h,0}$ and let $u_h = U_h^l$ with $u_h(0) = u_{h,0} = U_{h,0}^l$ be its lift. Then, as a consequence of Theorem 6.2.2, (6.3.5), and Lemma 6.3.1, the following estimate

$$\|u_h\|_{W(V,H)} \leq C_0 \|u_h(0)\|_{V(0)} \tag{6.3.7}$$

holds with C_0 depending on the constants C and c appearing in Theorem 6.2.2 and Lemma 6.3.1, respectively.

7. Error estimates

In this chapter we formulate the results concerning the approximation of the surface, which are in the deterministic setting proved in [55] and [58]. Our goal is to prove that they still hold in the random case. The main task is to keep track of constants that appear and show that they are independent of the realization. This conclusion mainly follows from the assumption (4.1.2) about the uniform distribution of the diffusion coefficient. Furthermore, we need to show that the extended definitions of the interpolation operator and the Ritz projection operator are integrable with respect to $\mathbb{P}$.

7.1. Interpolation and geometric error estimates

We start with an interpolation error estimate for functions $\eta \in L^2(\Omega, H^2(\Gamma(t)))$, where the interpolation $I_h\eta$ is defined as the lift of piecewise linear nodal interpolation $\widetilde{I}_h\eta \in L^2(\Omega, S_h(t))$. Note that $\widetilde{I}_h$ is well-defined, because the vertices $(X_j(t))_{j=1}^J$ of $\Gamma_h(t)$ lie on the smooth surface $\Gamma(t)$ and $n = 2, 3$.

Lemma 7.1.1. The interpolation error estimate

$$\begin{aligned} &\|\eta - I_h\eta\|_{H(t)} + h\|\nabla_\Gamma(\eta - I_h\eta)\|_{H(t)} \\ &\quad \leq ch^2 \left(\|\nabla_\Gamma^2\eta\|_{H(t)} + h\|\nabla_\Gamma\eta\|_{H(t)}\right) \end{aligned} \tag{7.1.1}$$

holds for all $\eta \in L^2(\Omega, H^2(\Gamma(t)))$ with a constant c depending only on the shape regularity of $\Gamma_h(t)$.

Proof. The proof of the lemma follows directly from the deterministic case and Lemma 6.3.1. □

We continue with estimating the geometric perturbation errors in the bilinear forms.

Lemma 7.1.2. Let $t \in [0, T]$ be fixed. For $W_h(\cdot, t)$ and $\phi_h(\cdot, t) \in L^2(\Omega, S_h(t))$ with corresponding lifts $w_h(\cdot, t)$ and $\varphi_h(\cdot, t) \in L^2(\Omega, S_h^l(t))$ we have the following estimates of the geometric error

$$|m(w_h, \varphi_h) - m_h(W_h, \phi_h)| \leq ch^2\|w_h\|_{H(t)}\|\varphi_h\|_{H(t)} \tag{7.1.2}$$

$$|a(w_h, \varphi_h) - a_h(W_h, \phi_h)| \leq ch^2\|\nabla_\Gamma w_h\|_{H(t)}\|\nabla_\Gamma\varphi_h\|_{H(t)} \tag{7.1.3}$$

$$|g(v_h; w_h, \varphi_h) - g_h(V_h; W_h, \phi_h)| \leq ch^2\|w_h\|_{V(t)}\|\varphi_h\|_{V(t)} \tag{7.1.4}$$

$$|m(\partial_h^\bullet w_h, \varphi_h) - m_h(\partial_h^\bullet W_h, \phi_h)| \leq ch^2\|\partial_h^\bullet w_h\|_{H(t)}\|\varphi\|_{H(t)}. \tag{7.1.5}$$

Proof. The assertion follows from uniform bounds of $\alpha(\omega, t)$ and $\partial_h^\bullet\alpha(\omega, t)$ with respect to $\omega \in \Omega$ together with corresponding deterministic results obtained in [58] and [95]. □

Since the velocity $\mathbf{w}$ of $\Gamma(t)$ is deterministic, we can use [58, Lemma 5.6] to control its deviation from the discrete velocity $\mathbf{w}_h$ on $\Gamma(t)$. Furthermore, [58, Corollary 5.7] provides the following error estimates for the continuous and discrete material derivative.

Lemma 7.1.3. For the continuous velocity $\mathbf{w}$ of $\Gamma(t)$ and the discrete velocity $\mathbf{w}_h$ defined in (6.3.4) the estimate

$$|\mathbf{w} - \mathbf{w}_h| + h\,|\nabla_\Gamma(\mathbf{w} - \mathbf{w}_h)| \leq ch^2 \tag{7.1.6}$$

holds pointwise on $\Gamma(t)$. Moreover, there holds

$$\|\partial^\bullet z - \partial_h^\bullet z\|_{H(t)} \leq ch^2 \|z\|_{V(t)}, \qquad z \in V(t), \tag{7.1.7}$$

$$\|\nabla_\Gamma(\partial^\bullet z - \partial_h^\bullet z)\|_{H(t)} \leq ch\|z\|_{L^2(\Omega,H^2(\Gamma))}, \qquad z \in L^2(\Omega, H^2(\Gamma(t))), \tag{7.1.8}$$

provided that the left-hand sides are well-defined.

Remark 7.1.4. Since $\mathbf{w}_h$ is a $\mathcal{C}^2$-velocity field by assumption, (7.1.6) implies a uniform upper bound for $\nabla_{\Gamma(t)} \cdot \mathbf{w}_h$ which in turn yields the estimate

$$|g(\mathbf{w}_h; w, \varphi)| \leq c\|w\|_{H(t)}\|\varphi\|_{H(t)}, \qquad \forall w, \varphi \in H(t) \tag{7.1.9}$$

with a constant c independent of h.

7.2. Ritz projection

The Ritz projection is a common tool in the error analysis of parabolic PDEs, in particular for surface PDEs. It is typically applied to split the overall error into a finite element error and a geometrical error according to

$$u - u_h = (u - u_p) + (u_p - u_h) = \rho + \theta.$$

In our case, the first term ρ is the error in an auxiliary elliptic problem that will be specified in this section and to which we will apply elliptic regularity and standard arguments in order to derive h^2 error estimates. The second term θ describes the geometric error introduced by approximating the surface - and will be analysed in the following section. The particular choice of projection means that we can again derive errors of order h^2.

Since we want to consider the $L^2(\Omega, S_h^l(t))$-error, we define the Ritz projection path-wise in a standard way and prove its L^2-regularity w.r.t. measure $\mathbb{P}$ and that a-orthogonality also holds in the whole space. At the end of the section we provide error estimates for Ritz projection .

For each fixed $t \in [0, T]$ and $\beta \in L^\infty(\Gamma(t))$ with $0 < \beta_{\min} \leq \beta(x) \leq \beta_{\max} < \infty$ a.e. on $\Gamma(t)$ the Ritz projection

$$H^1(\Gamma(t)) \ni v \mapsto \mathcal{R}^\beta v \in S_h^l(t)$$

is well-defined by the conditions $\int_{\Gamma(t)} \mathcal{R}^\beta v = 0$ and

$$\int_{\Gamma(t)} \beta \nabla_\Gamma \mathcal{R}^\beta v \cdot \nabla_\Gamma \varphi_h = \int_{\Gamma(t)} \beta \nabla_\Gamma v \cdot \nabla_\Gamma \varphi_h \qquad \forall \varphi_h \in S_h^l(t), \tag{7.2.1}$$

because $\{\eta \in S_h^l(t) \mid \int_{\Gamma(t)} \eta = 0\} \subset H^1(\Gamma(t))$ is finite dimensional and thus closed. Note that

$$\|\nabla_\Gamma R^\beta v\|_{L^2(\Gamma(t))} \leq \tfrac{\beta_{\max}}{\beta_{\min}} \|\nabla_\Gamma v\|_{L^2(\Gamma(t))}. \tag{7.2.2}$$

For fixed $t \in [0,T]$, the pathwise Ritz projection $u_p : \Omega \mapsto S_h^l(t)$ of $u \in L^2(\Omega, H^1(\Gamma(t)))$ is defined by

$$\Omega \ni \omega \to u_p(\omega) = R^{\alpha(\omega,t)} u(\omega) \in S_h^l(t). \tag{7.2.3}$$

In the following lemma, we state regularity and a-orthogonality.

Lemma 7.2.1. Let $t \in [0,T]$ be fixed. Then, the path-wise Ritz projection $u_p : \Omega \mapsto S_h^l(t)$ of $u \in L^2(\Omega, H^1(\Gamma(t)))$ satisfies $u_p \in L^2(\Omega, S_h^l(t))$ and the Galerkin orthogonality

$$a(u - u_p, \eta_h) = 0 \qquad \forall \eta_h \in L^2(\Omega, S_h^l(t)). \tag{7.2.4}$$

Proof. According to Assumption 4.1.2 the mapping

$$\Omega \ni \omega \mapsto \alpha(\omega, t) \in \mathcal{B} := \{\beta \in L^\infty(\Gamma(t)) \mid \alpha_{\min}/2 \leq \beta(x) \leq 2\alpha_{\max}\} \subset L^\infty(\Gamma(t))$$

is measurable. Hence by Lemma 2.1.12 it is sufficient to prove that the mapping

$$\mathcal{B} \ni \beta \mapsto R^\beta \in \mathcal{L}(H^1(\Gamma(t)), S_h^l(t))$$

is continuous with respect to the canonical norm in the space $\mathcal{L}(H^1(\Gamma(t)), S_h^l(t))$ of linear operators from $H^1(\Gamma(t))$ to $S_h^l(t)$. To this end, let β, $\beta' \in \mathcal{B}$ and $v \in H^1(\Gamma(t))$ be arbitrary and we will skip the dependence on t from now on. Then, inserting the test function $\varphi_h = (\mathcal{R}^\beta - \mathcal{R}^{\beta'})v \in S_h^l(t)$ into the definition (7.2.1), utilizing the stability (7.2.2), we obtain

$$\begin{aligned}
\alpha_{\min}/2 \quad & \|(\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v\|^2_{H^1(\Gamma)} \leq (1 + C_P^2) \int_\Gamma \beta |\nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v|^2 \\
= \quad & (1 + C_P^2)\Big(\int_\Gamma (\beta - \beta')\nabla_\Gamma \mathcal{R}^{\beta'} v \nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v \\
+ \quad & \int_\Gamma \beta' \nabla_\Gamma \mathcal{R}^{\beta'} v \nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v - \int_\Gamma \beta \nabla_\Gamma v \nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v\Big) \\
= \quad & (1 + C_P^2)\left(\int_\Gamma (\beta' - \beta)(\nabla_\Gamma v - \nabla_\Gamma \mathcal{R}^{\beta'} v)\nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v\right) \\
\leq \quad & (1 + C_P^2)\|\beta' - \beta\|_{L^\infty(\Gamma)} \|\nabla_\Gamma (v - \mathcal{R}^{\beta'} v)\|_{L^2(\Gamma)} \|\nabla_\Gamma (\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v\|_{L^2(\Gamma)} \\
\leq \quad & \left(1 + 4\frac{\alpha_{\max}}{\alpha_{\min}}\right)(1 + C_P^2)\|\beta' - \beta\|_{L^\infty(\Gamma)} \|v\|_{H^1(\Gamma)} \|(\mathcal{R}^{\beta'} - \mathcal{R}^\beta)v\|_{H^1(\Gamma)},
\end{aligned}$$

where C_P denotes the Poincaré constant in $\{\eta \in H^1(\Gamma) \mid \int_\Gamma \eta = 0\}$ (see, e.g., [59, Theorem 2.12]).

The norm of u_p in $L^2(\Omega, H^1(\Gamma(t)))$ is bounded, because Poincaré's inequality and (4.1.2) lead to

$$\begin{aligned}
\alpha_{\min} \int_\Omega \|u_p(\omega)\|^2_{H^1(\Gamma(t))} &\leq (1 + C_P^2) \int_\Omega \alpha(\omega, t) \|\nabla_\Gamma \mathcal{R}^{\alpha(\omega,t)}(u(\omega))\|^2_{L^2(\Gamma(t))} \\
\leq (1 + C_P^2)\alpha_{\max} \int_\Omega \|\nabla_\Gamma u(\omega)\|^2_{L^2(\Gamma(t))} &\leq (1 + C_P^2)\|\nabla_\Gamma u\|^2_{L^2(\Omega, H^1(\Gamma(t)))}.
\end{aligned}$$

This implies $u_p \in L^2(\Omega, S_h^l(t))$.

It is left to prove (7.2.4). For that purpose we select an arbitrary test function $\varphi_h(x)$ in (7.2.1), multiply with arbitrary $w \in L^2(\Omega)$, utilise $w(\omega)\nabla_\Gamma\varphi_h(x) = \nabla_\Gamma(w(\omega)\varphi_h(x))$, and integrate over Ω to obtain

$$\int_\Omega \int_{\Gamma(t)} \alpha(\omega, x)\nabla_\Gamma(u(\omega, x) - u_p(\omega, x))\nabla_\Gamma(\varphi_h(x)w(\omega)) = 0.$$

Since $\{v(x)w(\omega) \mid v \in S_h^l(t), w \in L^2(\Omega)\}$ is a dense subset of $V_h(t)$, the Galerkin orthogonality (7.2.4) follows. □

An error estimate for the path-wise Ritz projection u_p defined in (7.2.3) is established in the next theorem.

Theorem 7.2.2. For fixed $t \in [0, T]$, the path-wise Ritz projection $u_p \in L^2(\Omega, S_h^l(t))$ of $u \in L^2(\Omega, H^2(\Gamma(t)))$ satisfies the error estimate

$$\|u - u_p\|_{H(t)} + h\|\nabla_\Gamma(u - u_p)\|_{H(t)} \leq ch^2\|u\|_{L^2(\Omega,H^2(\Gamma(t)))} \tag{7.2.5}$$

with a constant c depending only on the properties of α as stated in Assumptions 4.1.2, 4.3.3 and 6.0.1 and the shape regularity of $\Gamma_h(t)$.

Proof. The Galerkin orthogonality (7.2.4) and (4.1.2) provide

$$\begin{aligned}\alpha_{\min}\|\nabla_\Gamma(u - u_p)\|_{H(t)} &\leq \alpha_{\max} \inf_{v\in L^2(\Omega,S_h^l(t))} \|\nabla_\Gamma(u - v)\|_{H(t)} \\ &\leq \alpha_{\max}\|\nabla_\Gamma(u - I_h v)\|_{H(t)}.\end{aligned}$$

Hence, the bound for the gradient follows directly from Lemma 7.1.1.

In order to get the second order bound, we will use a Aubin-Nitsche duality argument. For every fixed $\omega \in \Omega$, we consider the path-wise problem to find $w(\omega) \in H^1(\Gamma(t))$ with $\int_{\Gamma(t)} w = 0$ such that

$$\int_{\Gamma(t)} \alpha\nabla_\Gamma w(\omega) \cdot \nabla_\Gamma\varphi = \int_{\Gamma(t)} (u - u_p)\varphi \qquad \forall\varphi \in H^1(\Gamma(t)). \tag{7.2.6}$$

Since $\Gamma(t)$ is $\mathcal{C}^2$, it follows from [59, Theorem 3.3] that $w(\omega) \in H^2(\Gamma(t))$. Inserting the test function $\varphi = w(\omega)$ into (7.2.6) and utilizing Poincaré's inequality, we obtain

$$\|\nabla_\Gamma w(\omega)\|_{L^2(\Gamma(t))} \leq \frac{C_P}{\alpha_{\min}}\|u - u_p\|_{L^2(\Gamma(t))}.$$

Previous estimate together with the product rule for divergence imply

$$\|\Delta_\Gamma w(\omega)\|_{L^2(\Gamma(t))} \leq \frac{1}{\alpha_{\min}}\|u - u_p\|_{L^2(\Gamma(t))} + \frac{C_P}{\alpha_{\min}^2}\|\alpha(\omega)\|_{\mathcal{C}^1(\Gamma(t))}\|u - u_p\|_{L^2(\Gamma(t))}.$$

Hence. we have the following estimate

$$\|w(\omega)\|_{H^2(\Gamma(t))} \leq C\|u - u_p\|_{L^2(\Gamma(t))}, \tag{7.2.7}$$

with a constant C depending only on the properties of α as stated in Assumptions 4.1.2, 4.3.3 and 6.0.1. Furthermore, well-known results on random elliptic PDEs with uniformly bounded coefficients [27, 29] imply measurablility of $w(\omega)$, $\omega \in \Omega$. Integrating (7.2.7) over Ω, we therefore obtain

$$\|w\|_{L^2(\Omega,H^2(\Gamma(t)))} \leq C\|u - u_p\|_{H(t)}. \tag{7.2.8}$$

Once more using Lemma 7.1.1, Galerkin orthogonality (7.2.4), and (7.2.8), we get

$$\begin{aligned}
\|u - u_p\|^2_{H(t)} &= a(w, u - u_p) \ = a(w - I_h w, u - u_p) \\
&\leq \alpha_{\max}\|\nabla_\Gamma(w - I_h w)\|_{H(t)}\|\nabla_\Gamma(u - u_p)\|_{H(t)} \\
&\leq c' h^2 \|w\|_{L^2(\Omega,H^2(\Gamma(t)))}\|u\|_{L^2(\Omega,H^2(\Gamma(t)))} \\
&\leq c' c h^2 \|u - u_p\|_{H(t)}\|u\|_{L^2(\Omega,H^2(\Gamma(t)))}.
\end{aligned}$$

with a constant c' depending on the shape regularity of $\Gamma_h(t)$. This completes the proof. □

Remark 7.2.3. The first order error bound for $\|\nabla_\Gamma(u - u_p)\|_{H(t)}$ still holds, if spatial regularity of α as stated in Assumption 6.0.1 is not satisfied.

We conclude with an error estimate for the material derivative of u_p that can be proved as in the deterministic setting [58, Theorem 6.2].

Theorem 7.2.4. For each fixed $t \in [0, T]$, the discrete material derivative of the path-wise Ritz projection satisfies the error estimate

$$\begin{aligned}
&\|\partial_h^\bullet u - \partial_h^\bullet u_p\|_{H(t)} + h\|\nabla_\Gamma(\partial_h^\bullet u - \partial_h^\bullet u_p)\|_{H(t)} \\
&\quad \leq ch^2(\|u\|_{L^2(\Omega,H^2(\Gamma))} + \|\partial^\bullet u\|_{L^2(\Omega,H^2(\Gamma))})
\end{aligned} \tag{7.2.9}$$

with a constant C depending only on the properties of α as stated in Assumptions 4.1.2, 4.3.3 and 6.0.1.

7.3. Discretization error estimates for the evolving surface finite element

Now we are in the position to state an error estimate for the evolving surface finite element discretization of Problem 6.0.3 as formulated in Problem 6.2.1.

Theorem 7.3.1. Assume that the solution u of Problem 6.0.3 has the following regularity properties

$$\sup_{t\in(0,T)} \|u(t)\|_{L^2(\Omega,H^2(\Gamma(t)))} + \int_0^T \|\partial^\bullet u(t)\|^2_{L^2(\Omega,H^2(\Gamma(t)))} dt < \infty \tag{7.3.1}$$

and let $U_h \in W_h(V_h, H_h)$ be the solution of the approximating Problem 6.2.1 with an initial condition $U_h(0) = U_{h,0} \in V_h(0)$ such that

$$\|u(0) - U^l_{h,0}\|_{H(0)} \leq ch^2 \tag{7.3.2}$$

holds with a constant $c > 0$ independent of h. Then the lift $u_h := U_h^l$ satisfies the error estimate

$$\sup_{t\in(0,T)} \|u(t) - u_h(t)\|_{H(t)} \leq Ch^2 \tag{7.3.3}$$

with a constant C independent of h.

Proof. Utilizing the preparatory results from the preceding sections, the proof can be carried out in analogy to the deterministic version stated in [58, Theorem 4.4].

The first step is to decompose the error for fixed t into the path-wise Ritz projection error and the deviation of the path-wise Ritz projection u_p from the approximate solution u_h according to

$$\|u(t) - u_h(t)\|_{H(t)} \leq \|u(t) - u_p(t)\|_{H(t)} + \|u_p(t) - u_h(t)\|_{H(t)}, \quad t \in (0,T).$$

For ease of presentation the dependence on t is often skipped in the continuation.

As a consequence of Theorem 7.2.2 and the regularity assumption (7.3.1), we have

$$\sup_{t\in(0,T)} \|u - u_p\|_{H(t)} \leq ch^2 \sup_{t\in(0,T)} \|u\|_{L^2(\Omega,H^2(\Gamma(t)))} < \infty.$$

Hence, it is sufficient to show a corresponding estimate for

$$\theta := u_p - u_h \in L^2(\Omega, S_h^l).$$

Here and in the continuation we set $\varphi_h = \phi_h^l$ for $\phi_h \in L^2(\Omega, S_h)$.

Utilizing (6.2.7) and the transport formulae (6.1.6) in Lemma 6.1.2 and (6.3.6) in Lemma 6.3.2, respectively, we obtain

$$\frac{d}{dt} m(u_h, \varphi_h) + a(u_h, \varphi_h) - m(u_h, \partial_h^\bullet \varphi_h) = F_1(\varphi_h), \quad \forall \varphi_h \in L^2(\Omega, S_h^l) \tag{7.3.4}$$

denoting

$$\begin{aligned} F_1(\varphi_h) := {} & m(\partial_h^\bullet u_h, \varphi_h) - m_h(\partial_h^\bullet U_h, \phi_h) \\ & + a(u_h, \varphi_h) - a_h(U_h, \phi_h) + g(v_h; u_h, \varphi_h) - g_h(V_h; U_h, \phi_h). \end{aligned} \tag{7.3.5}$$

Exploiting that u solves Problem 6.0.3 and thus satisfies (6.0.3) together with the Galerkin orthogonality (7.2.4) and rearranging terms, we derive

$$\frac{d}{dt} m(u_p, \varphi_h) + a(u_p, \varphi_h) - m(u_p, \partial_h^\bullet \varphi_h) = F_2(\varphi_h), \quad \forall \varphi_h \in L^2(\Omega, S_h^l) \tag{7.3.6}$$

denoting

$$F_2(\varphi_h) := m(u, \partial^\bullet \varphi_h - \partial_h^\bullet \varphi_h) + m(u - u_p, \partial_h^\bullet \varphi_h) - \frac{d}{dt} m(u - u_p, \varphi_h). \tag{7.3.7}$$

We subtract (7.3.4) from (7.3.6) to get

$$\frac{d}{dt} m(\theta, \varphi_h) + a(\theta, \varphi_h) - m(\theta, \partial_h^\bullet \varphi_h) = F_2(\varphi_h) - F_1(\varphi_h) \quad \forall \varphi_h \in L^2(\Omega, S_h^l). \tag{7.3.8}$$

Inserting the test function $\varphi_h = \theta \in L^2(\Omega, S_h^l)$ into (7.3.8), utilizing the transport Lemma 6.3.2, and integrating in time, we obtain

$$\tfrac{1}{2}\|\theta(t)\|^2_{H(t)} - \tfrac{1}{2}\|\theta(0)\|^2_{H(0)} + \int_0^t a(\theta,\theta) + \int_0^t g(\mathbf{w}_h;\theta,\theta) = \int_0^t F_2(\theta) - F_1(\theta).$$

Hence, Assumption 4.1.2 together with (7.1.9) in Remark 7.1.4 provides the estimate

$$\begin{aligned}\tfrac{1}{2}\|\theta(t)\|^2 + \alpha_{\min}\int_0^t \|\nabla_\Gamma \theta\|^2_{H(t)} \leq \\ \tfrac{1}{2}\|\theta(0)\|^2 + \quad c\int_0^t \|\theta\|^2_{H(t)} + \int_0^t |F_1(\theta)| + |F_2(\theta)|.\end{aligned} \tag{7.3.9}$$

Lemma 7.1.2 allows to control the geometric error terms in $|F_1(\theta)|$ according to

$$|F_1(\theta)| \leq ch^2\|\partial_h^\bullet u_h\|_{H(t)}\|\theta_h\|_{H(t)} + ch^2\|u_h\|_{V(t)}\|\theta_h\|_{V(t)}.$$

The transport formula (6.3.6) provides the identity

$$F_2(\varphi_h) = m(u, \partial^\bullet\varphi_h - \partial_h^\bullet\varphi_h) - m(\partial_h^\bullet(u-u_p),\varphi_h) - g(v_h; u-u_p,\varphi_h)$$

from which Lemma 7.1.3, Theorem 7.2.4, and Theorem 7.2.2 imply

$$|F_2(\theta)| \leq ch^2\|u\|_{H(t)}\|\theta_h\|_{V(t)} + ch^2(\|u\|_{L^2(\Omega,H^2(\Gamma(t)))} + \|\partial^\bullet u\|_{L^2(\Omega,H^2(\Gamma(t)))})\|\theta_h\|_{H(t)}.$$

We insert these estimates into (7.3.9), rearrange terms, and apply Young's inequality to show that for each $\varepsilon > 0$ there is a positive constant $c(\varepsilon)$ such that

$$\begin{aligned}\frac{1}{2}\|\theta(t)\|^2_{H(t)} + (\alpha_{\min} - \varepsilon)\int_0^t \|\nabla_\Gamma\theta\|^2_{H(t)} \leq \frac{1}{2}\|\theta(0)\|^2_{H(0)} + c(\varepsilon)\int_0^t \|\theta\|^2_{H(t)} \\ + c(\varepsilon)h^4\int_0^t \left(\|u\|^2_{L^2(\Omega,H^2(\Gamma(t)))} + \|\partial^\bullet u\|^2_{L^2(\Omega,H^2(\Gamma(t)))} + \|\partial_h^\bullet u\|^2_{H(t)} + \|u_h\|^2_{V(t)}\right).\end{aligned}$$

For sufficiently small $\varepsilon > 0$, Gronwall's lemma implies

$$\sup_{t\in(0,T)} \|\theta(t)\|^2_{H(t)} + \int_0^T \|\nabla_\Gamma\theta\|^2_{H(t)} \leq c\|\theta(0)\|^2_{H(0)} + ch^4 C_h, \tag{7.3.10}$$

where

$$C_h = \int_0^T [\|u\|^2_{L^2(\Omega,H^2(\Gamma(t)))} + \|\partial^\bullet u\|^2_{L^2(\Omega,H^2(\Gamma(t)))} + \|\partial_h^\bullet u\|^2_{H(t)} + \|u_h\|^2_{V(t)}].$$

Now the consistency assumption (7.3.2) yields $\|\theta(0)\|^2_{H(0)} \leq ch^4$ while the stability result (6.3.7) in Remark 6.3.3 along with the regularity assumption leads to (7.3.1) $C_h \leq C < \infty$ with a constant C independent of h. This completes the proof. □

Remark 7.3.2. Observe that without Assumption 6.0.1 we still get the H^1-bound

$$\left(\int_0^T \|\nabla_\Gamma(u(t)-u_h(t))\|_{H(t)}^2\right)^{1/2} \leq Ch.$$

The following error estimate for the expectation

$$E[u] = \int_\Omega u$$

is an immediate consequence of Theorem 7.3.1 and the Cauchy-Schwarz inequality.

Theorem 7.3.3. Under the assumptions and with the notation of Theorem 7.3.1 we have the error estimate

$$\sup_{t\in(0,T)} \|E[u(t)] - E[u_h(t)]\|_{L^2(\Gamma(t))} \leq Ch^2. \tag{7.3.11}$$

We close this section with an error estimate for the Monte-Carlo approximation of the expectation $E[u_h]$. Note that $E[u_h](t) = E[u_h(t)]$, because the probability measure does not depend on time t. For each fixed $t \in (0,T)$ and some $M \in \mathbb{N}$, the Monte-Carlo approximation $E_M[u_h](t)$ of $E[u_h](t)$ is defined by

$$E_M[u_h(t)] := \frac{1}{M}\sum_{i=1}^{M} u_h^i(t) \in L^2(\Omega^M, L^2(\Gamma(t))), \tag{7.3.12}$$

where u_h^i are independent identically distributed copies of the random field u_h.

Proof of the following well-known result can be found, e.g. in [94, Theorem 9.22].

Lemma 7.3.4. For each fixed $t \in (0,T)$, $w \in L^2(\Omega, L^2(\Gamma(t)))$, and any $M \in \mathbb{N}$ we have the error estimate

$$\|E[w] - E_M[w]\|_{L^2(\Omega^M, L^2(\Gamma(t)))} = \tfrac{1}{\sqrt{M}}\mathrm{V}ar[w]^{\frac{1}{2}} \leq \tfrac{1}{\sqrt{M}}\|w\|_{L^2(\Omega, L^2(\Gamma(t)))} \tag{7.3.13}$$

with $\mathrm{V}ar[w]$ denoting the variance $Var[w] = E[\|E[w]-w\|^2_{L^2(\Omega,\Gamma(t))}]$ of w.

Theorem 7.3.5. Under the assumptions and with the notation of Theorem 7.3.1 we have the error estimate

$$\sup_{t\in(0,T)} \|E[u](t) - E_M[u_h](t)\|_{L^2(\Omega^M, L^2(\Gamma(t)))} \leq C\left(h^2 + \tfrac{1}{\sqrt{M}}\right)$$

with a constant C independent of h and M.

Proof. Let us first note that

$$\sup_{t\in(0,T)} \|u_h\|_{H(t)} \leq (1+C) \sup_{t\in(0,T)} \|u\|_{H(t)} < \infty \tag{7.3.14}$$

follows from the triangle inequality and Theorem 7.3.1. For arbitrary fixed $t \in (0,T)$ the triangle inequality yields

$$\begin{aligned}\|E[u](t) - E_M[u_h](t)\|_{L^2(\Omega^M, L^2(\Gamma(t)))} \leq \\ \|E[u](t) - E[u_h](t)\|_{L^2(\Gamma(t)))} \;+\; \|E[u_h(t)] - E_M[u_h(t)]\|_{L^2(\Omega^M, L^2(\Gamma(t)))}\end{aligned}$$

so that the assertion follows from Theorem 7.3.3, Lemma 7.3.4, and (7.3.14). □

8. Numerical Experiments

8.1. Computational aspects

In the following numerical computations we consider a fully discrete scheme as resulting from an implicit Euler discretization of the semi-discrete Problem 6.2.1. More precisely, we select a time step $\tau > 0$ with $K\tau = T$, set

$$\chi_j^k = \chi_j(t_k), \quad k = 0, \dots, K,$$

with $t_k = k\tau$, and approximate $U_h(\omega, t_k)$ by

$$U_h^k(\omega) = \sum_{j=1}^{J} U_j^k(\omega)\chi_j^k, \qquad k = 0, \dots, J,$$

with unknown coefficients $U_j^k(\omega)$ characterized by the initial condition

$$U_h^0 = \sum_{j=1}^{J} U_{h,0}(X_j(0))\chi_j^0$$

and the fully discrete scheme

$$\frac{1}{\tau}\big(m_h^k(U_h^k, \chi_j^k) - m_h^{k-1}(U_h^{k-1}, \chi_j^{k-1})\big) + a_h^k(U_h^k, \chi_j^k) = \int_\Omega \int_{\Gamma(t_k)} f(t_k)\chi_j^k \tag{8.1.1}$$

for $k = 1, \dots, J$. Here, for $t = t_k$ the time-dependent bilinear forms $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ are denoted by $m_h^k(\cdot,\cdot)$ and $a_h^k(\cdot,\cdot)$, respectively. The fully discrete scheme (8.1.1) is obtained from an extension of (6.2.7) to non-vanishing right-hand sides $f \in \mathcal{C}((0,T), H(t))$ by inserting $\varphi = \chi_j$, exploiting (6.2.2), and replacing the time derivative with the backward difference quotient. As α is defined on the whole ambient space in the subsequent numerical experiments, the inverse lift α^{-l} occurring in $a_h(\cdot,\cdot)$ is replaced by $\alpha|_{\Gamma_h(t)}$, and the integral is computed using a quadrature formula of degree 4.

The expectation $E[U_h^k]$ is approximated by the Monte-Carlo method

$$E_M[U_h^k] = \frac{1}{M}\sum_{i=1}^{M} U_h^k(\omega^i), \qquad k = 1, \dots, K,$$

with independent, uniformly distributed samples $\omega^i \in \Omega$. For each sample ω^i, the evaluation of $U_h^k(\omega^i)$ from the initial condition and (8.1.1) amounts to the solution of J linear systems which is performed iteratively by a preconditioned conjugate gradient method up to the accuracy 10^{-8}.

From our theoretical findings stated in Theorem 7.3.5 and the fully discrete deterministic results in [57, Theorem 2.4], we expect that the discretization error

$$\sup_{k=0,\dots,K} \|E[u](t_k) - E_M[U_h^k]\|_{L^2(\Omega^M, L^2(\Gamma_h(t_k)))} \tag{8.1.2}$$

behaves like $\mathcal{O}\left(h^2 + \frac{1}{\sqrt{M}} + \tau\right)$. This conjecture will be investigated in our numerical experiments. To this end, the integral over Ω^M in (8.1.2) is always approximated by the average of 20 independent, identically distributed sets of samples. We denote the error and a parameter at level l by E_l and P_l (for $P = h$, τ or M), respectively, to introduce the experimental order of convergence at level l according to

$$\operatorname{eoc}(P_l) = \frac{\log(E_l/E_{l-1})}{\log(P_l/P_{l-1})}.$$

The implementation was carried out in the framework of Dune (Distributed Unified Numerics Environment) [16, 17, 42], and the corresponding code for the first four examples and the last example is available at `https://github.com/tranner/dune-mcesfem`. Those examples are presented in [48].

8.2. Moving curve

We will consider five problems on a moving curve with different regularity of the random diffusion coefficients. For the first four problems we consider the ellipse

$$\Gamma(t) = \left\{ x = (x_1, x_2) \in \mathbb{R}^2 \;\middle|\; \frac{x_1^2}{a(t)} + \frac{x_2^2}{b(t)} = 1 \right\}, \qquad t \in [0, T],$$

with oscillating axes $a(t) = 1 + \frac{1}{4}\sin(t)$, $b(t) = 1 + \frac{1}{4}\cos(t)$, the velocity

$$\mathbf{w}(t) = \left(\frac{x_1 a(t)}{2a'(t)}, \frac{x_2 b(t)}{2b'(t)} \right)^T,$$

and $T = 1$. For the last problem, we consider the stationary circle, i.e. $a(t) = b(t) = 1$, for every $t \in [0, T]$.

The initial polygonal approximation $\Gamma_{h,0}$ of $\Gamma(0)$ is depicted in section 8.2 for the mesh sizes $h = h_j$, $j = 0, \dots, 4$, that are used in our computations. We select the corresponding time step sizes $\tau_j = \tau_{j-1}/4$ and the corresponding numbers of samples $M_1 = 1, M_j = 16M_{j-1}$ for $j = 2, 3, 4, 5$.

In the first four problems, the right-hand side f in eq. (8.1.1) is selected in such a way that for each $\omega \in \Omega$ the exact solution of the resulting path-wise problem is given by

$$u(x, t, \omega) = \sin(t)\big\{ \cos(3x_1) + \cos(3x_2) + Y_1(\omega)\cos(5x_1) + Y_2(\omega)\cos(5x_2) \big\},$$

which clearly has a path-wise strong material derivative for all $\omega \in \Omega$ and satisfies the regularity property (7.3.1). We set $u_0(x, \omega) = u(x, 0, \omega) = 0$ so that (7.3.2) obviously holds true. For

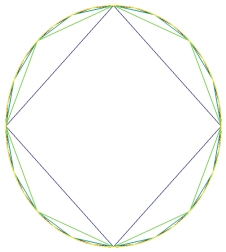


Figure 8.1.: Polygonal approximation $\Gamma_{h,0}$ of $\Gamma(0)$ for $h = h_0, \dots, h_4$.

each test problem, we choose a different random diffusion coefficient α occurring in $a_h(\cdot,\cdot)$. In the first four experiments, the coefficient and the solution depend on two random variables: Y_1 and Y_2, which are independent, uniformly distributed random variables on $\Omega = (-1, 1)$. In the last example of this section, we will consider a different solution and a random coefficient that depend on six independent, uniformly distributed random variables.

Spatially smooth coefficient

First, we consider a smooth problem. The random diffusion coefficient α is given by

$$\alpha(x,\omega) = 1 + \frac{Y_1(\omega)}{4}\sin(2x_1) + \frac{Y_2(\omega)}{4}\sin(2x_2)$$

and satisfies Assumptions 4.1.2, 4.3.3 and 6.0.1. The resulting approximate discretization errors eq. (8.1.2) are reported in table 8.1 and suggest the optimal behaviour $\mathcal{O}(h^2 + M^{-1/2} + \tau)$.

Table 8.1.: Discretization errors for a moving curve in $\mathbb{R}^2$ for a spatially smooth coefficient.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
1.500000	1	1	3.00350	—	—	—
0.843310	16	4^{-1}	$2.23278 \cdot 10^{-1}$	4.51325	−0.93743	1.87487
0.434572	256	4^{-2}	$1.86602 \cdot 10^{-1}$	0.27066	−0.06472	0.12944
0.218962	4 096	4^{-3}	$4.88096 \cdot 10^{-2}$	1.95642	−0.48368	0.96736
0.109692	65 536	4^{-4}	$1.29667 \cdot 10^{-2}$	1.91768	−0.47809	0.95618

Spatially less smooth coefficient

We consider the random diffusion coefficient α given by

$$\alpha(x,\omega) = 1 + \frac{Y_1(\omega)}{4}|x_1|x_1 + \frac{Y_2(\omega)}{4}|x_2|x_2.$$

Note that this coefficient is less smooth in x compared to the previous example. Namely, $\alpha(\cdot,\omega) \in C^1(\mathbb{R}^2)$ and its tangential gradient is uniformly bounded in ω so that Assumptions 4.1.2, 4.3.3 and 6.0.1 are satisfied, but $\alpha(\cdot,\omega) \notin C^2(\mathbb{R}^2)$. The resulting discretization errors eq. (8.1.2) reported in table 8.2 are suggesting the optimal behaviour $\mathcal{O}(h^2 + M^{-1/2} + \tau)$.

Table 8.2.: Discretization errors for a moving curve in $\mathbb{R}^2$ for a spatially less smooth coefficient.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
0.843082	16	$0.1 \cdot 4^1$	$2.28659 \cdot 10^{-1}$	—	—	—
0.434572	256	0.1	$2.14613 \cdot 10^{-1}$	0.09566	−0.02287	0.04573
0.218962	4 096	$0.1 \cdot 4^{-1}$	$5.14210 \cdot 10^{-2}$	2.08441	−0.51533	1.03065
0.109692	65 536	$0.1 \cdot 4^{-2}$	$1.37766 \cdot 10^{-2}$	1.90543	−0.47503	0.95007
0.054873	1 048 576	$0.1 \cdot 4^{-3}$	$3.86361 \cdot 10^{-3}$	1.83548	−0.45855	0.91710

Non-linear occurrence of randomness

The random coefficient α in the next experiment is spatially smooth, but now exhibits stronger stochastic fluctuations. It is given by

$$\alpha(x, \omega) = 1 + \frac{1}{4} \sin\left(4\pi Y_1(\omega) x_1 + 4\pi Y_2(\omega) x_2\right).$$

Again, Assumptions 4.1.2, 4.3.3 and 6.0.1 are fulfilled and the resulting discretization errors eq. (8.1.2) reported in table 8.3 suggest the optimal behaviour $\mathcal{O}(h^2 + M^{-1/2} + \tau)$.

Table 8.3.: Discretization errors for a moving curve in $\mathbb{R}^2$ for non-linear randomness.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
0.843082	16	$0.1 \cdot 4^1$	$2.70111 \cdot 10^{-1}$	—	—	—
0.434572	256	0.1	$2.22950 \cdot 10^{-1}$	0.28955	−0.06921	0.13842
0.218962	4 096	$0.1 \cdot 4^{-1}$	$5.82967 \cdot 10^{-2}$	1.95693	−0.48381	0.96762
0.109692	65 536	$0.1 \cdot 4^{-2}$	$1.48861 \cdot 10^{-2}$	1.97494	−0.49236	0.98473
0.054873	1 048 576	$0.1 \cdot 4^{-3}$	$3.74749 \cdot 10^{-3}$	1.99136	−0.49749	0.99498

Violating the assumptions

We finally test our algorithm with a problem that satisfies Assumptions 4.1.2 and 4.3.3, but not assumption 6.0.1. The random diffusion coefficient α is given by

$$\alpha(x, \omega) = 1 + \exp\left(\frac{-2x_1^2}{Y_1(\omega) + 1}\right) + \exp\left(\frac{-2x_2^2}{Y_2(\omega) + 1}\right).$$

The tangential gradient of α is not uniformly bounded in $\omega \in \Omega$. Hence, assumption 6.0.1 is violated and theorem 7.3.5 can not be applied. Only first order error bounds in h hold according to Remark 7.2.3. However, the resulting discretization errors eq. (8.1.2) reported in table 8.4 are still suggesting the optimal behaviour $\mathcal{O}(h^2 + M^{-1/2} + \tau)$.

Table 8.4.: Discretization errors for a moving curve in $\mathbb{R}^2$, when the assumptions are violated.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
0.844130	16	0.1	$4.14221 \cdot 10^{-1}$	—	—	—
0.434602	256	$0.1 \cdot 4^{-1}$	$2.72451 \cdot 10^{-1}$	0.63105	−0.15110	0.30220
0.218963	4 096	$0.1 \cdot 4^{-2}$	$7.50688 \cdot 10^{-2}$	1.88038	−0.46493	0.92985
0.109692	65 536	$0.1 \cdot 4^{-3}$	$1.88296 \cdot 10^{-2}$	2.00075	−0.49880	0.99760
0.054873	1 048 576	$0.1 \cdot 4^{-4}$	$4.95240 \cdot 10^{-3}$	1.92815	−0.48170	0.96340

More random variables influence the input data

In the next experiment we consider the uncertain coefficient defined on the unit sphere $\mathbb{S}^1$. The coefficient written in the polar coordinates has the form

$$\alpha(\omega, r, \varphi) = r_0 + \sum_{j=1}^{J} c_j Y_{2j-1} \cos(j\varphi) + s_j Y_{2j} \sin(j\varphi) \quad \varphi \in [0, 2\pi],\ J \in \mathbb{N},\ \omega \in \Omega. \tag{8.2.1}$$

This experiment was motivated by the example presented in [76]. We assume that $(Y_j)_1^{2J}$ are i.i.d. with $Y_j \sim \mathcal{U}[-1, 1]$, for every $1 \leq j \leq 2J$ and every $J \in \mathbb{N}$. Since the coefficient is defined on the circle, we will take $r_0 = 1$. As noted in [76, Ass. 2.3], to ensure $\mathbb{P}$-a.s. boundedness and positivity of α, we assume that coefficients $(c_j)_j$ and $(s_j)_j$ satisfy

$$\sum_{j \geq 1} (|c_j| + |s_j|) \leq \frac{1}{2},$$

which implies that $\alpha(\omega, \varphi) \in [\frac{1}{2}, \frac{3}{2}]$, for a.e. ω. In particular, we will choose $s_j = c_j = \frac{1}{100 j^2}$. To compute $\cos(j\varphi)$ and $\sin(j\varphi)$, we exploit formulae

$$\cos(j\varphi) = T_j(\cos(\varphi)) = T_j(x_1) \qquad \sin(j\varphi) = \sin\varphi U_{j-1}(\cos\varphi)) = x_2 U_{j-1}(x_1)$$

for $(x_1, x_2) \in \mathbb{S}^1$, where T_j and U_j are 1st and 2nd Chebyshev polynomials, respectively. For our computations we take $J = 3$. Hence, the random coefficient depends on 6 random variables. The formula for the random coefficient used for the numerical experiment is

$$\begin{aligned} \alpha_6(\omega, (x_1, x_2)) &= 1 + \sum_{j=1}^{3} \frac{1}{100 j^2} Y_{2j-1} T_j(x_1) + x_2 \sum_{j=1}^{3} \frac{1}{100 j^2} Y_{2j} U_{j-1}(x_1) \\ &= 1 + 0.01\, x_1\, Y_1 + 0.0025(-1 + 2x_1^2)\, Y_3 + 0.00111111(-3x_1 + 4x_1^3) Y_5 \\ &+ x_2(0.01 Y_2 + 0.005 x_1 Y_4 + 0.00111111(-1 + 4x_1^2) Y_6). \end{aligned} \tag{8.2.2}$$

Clearly, the coefficient satisfies Assumptions 4.1.2, 4.3.3 and 6.0.1. One realization of the coefficient α is represented in Figure 8.2, where artificial scaling of the coefficient is used in order to be able to better see the stochastic fluctuations.

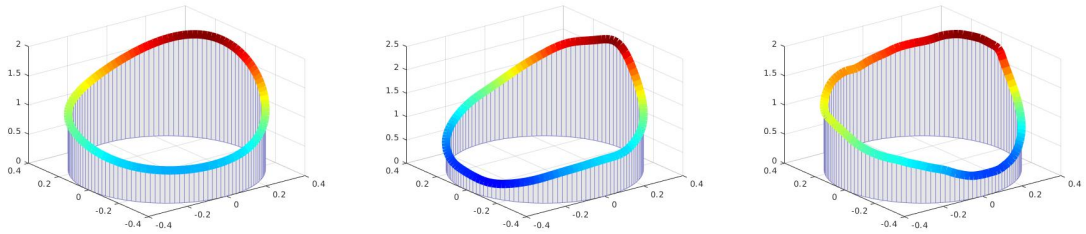


Figure 8.2.: Realizations of diffusion coefficient (8.2.1) for J=3, 12, 72.

The right-hand side f in eq. (8.1.1) is selected in such a way that for each $\omega \in \Omega$ the exact solution of the resulting path-wise problem is given by

$$\begin{aligned} u(x,t,\omega) &= \sin(t)\big\{\cos(3x_1)+\cos(3x_2)+Y_1(\omega)\cos(5x_1)+Y_2(\omega)\cos(5x_2)+ \\ &\qquad Y_3(\omega)\cos(7x_1)+Y_4(\omega)\cos(7x_2)+Y_5(\omega)\cos(9x_1)+Y_6(\omega)\cos(9x_2)\big\} \\ &= \sin(t)\big\{\cos(3x_1)+\cos(3x_2)+\sum_{j=1}^{6} Y_{2j-1}\cos(4+(2j-1))x_1+\sum_{j=1}^{6} Y_{2j}\cos(4+(2j-1))x_2\big\}. \end{aligned}$$

which clearly has a path-wise strong material derivative for all $\omega \in \Omega$ and satisfies the regularity property (7.3.1). Note that the exact solution also depends on six random variables. The resulting approximate discretization errors eq. (8.1.2) are reported in table 8.5 and suggest the optimal behaviour $\mathcal{O}(h^2 + M^{-1/2} + \tau)$.

Table 8.5.: Discretization errors for a stationary circle in $\mathbb{R}^2$ for the test case with more RVs.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
0.765367	16	0.1	$3.75394 \cdot 10^{-1}$	—	—	—
0.390181	256	$0.1 \cdot 4^{-1}$	$1.75117 \cdot 10^{-1}$	1.13125	−0.27528	0.55056
0.196034	4 096	$0.1 \cdot 4^{-2}$	$4.61383 \cdot 10^{-2}$	1.93774	−0.48106	0.96217
0.098135	65 536	$0.1 \cdot 4^{-3}$	$1.18367 \cdot 10^{-2}$	1.96640	−0.49074	0.98149

8.3. Moving surface

We consider the ellipsoid

$$\Gamma(t) = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \;\middle|\; \frac{x_1^2}{a(t)} + x_2^2 + x_3^2 = 1 \right\}, \qquad t \in [0, T],$$

with oscillating x_1-axis $a(t) = 1 + \frac{1}{4}\sin(t)$, the velocity

$$\mathbf{w}(t) = \left(\frac{x_1 a(t)}{2a'(t)}, 0, 0 \right)^T,$$

and $T = 1$. The random diffusion coefficient α occurring in $a_h(\cdot,\cdot)$ is given by

$$\alpha(x,\omega) = 1 + x_1^2 + Y_1(\omega)x_1^4 + Y_2(\omega)x_2^4,$$

where Y_1 and Y_2 denote independent, uniformly distributed random variables on $\Omega = (-1,1)$. Observe that Assumptions 4.1.2, 4.3.3 and 6.0.1 are satisfied for this choice. The right-hand side f in eq. (8.1.1) is chosen such that for each $\omega \in \Omega$ the exact solution of the resulting path-wise problem is given by

$$u(x,t,\omega) = \sin(t)x_1x_2 + Y_1(\omega)\sin(2t)x_1^2 + Y_2(\omega)\sin(2t)x_2,$$

which clearly has a path-wise strong material derivative for all $\omega \in \Omega$ and satisfies the regularity property (7.3.1). As before, we select the initial condition $u_0(x,\omega) = u(x,0,\omega) = 0$ so that (7.3.2) holds true.

The initial triangular approximation $\Gamma_{h,0}$ of $\Gamma(0)$ is depicted in section 8.3 for the mesh sizes $h = h_j$, $j = 0,\dots,3$. We select the corresponding time step sizes $\tau_0 = 1$, $\tau_j = \tau_{j-1}/4$ and

Figure 8.3.: Triangular approximation $\Gamma_{h,0}$ of $\Gamma(0)$ for $h = h_0,\dots,h_3$.

the corresponding numbers of samples $M_1 = 1$, $M_j = 16M_{j-1}$ for $j = 2,3,4$. The resulting discretization errors eq. (8.1.2) are shown in table 8.6. Again, we observe that the discretization error behaves like $\mathcal{O}(h^2 + M^{-1/2} + \tau)$. This is in accordance with our theoretical findings stated in theorem 7.3.5 and fully discrete deterministic results [57, Theorem 2.4].

Table 8.6.: Discretization errors for a moving surface in $\mathbb{R}^3$.

h	M	τ	Error	eoc(h)	eoc(M)	eoc(τ)
1.276870	1	1	$9.91189 \cdot 10^{-1}$	—	—	—
0.831246	16	4^{-1}	$1.70339 \cdot 10^{-1}$	4.10285	−0.63519	1.27037
0.440169	256	4^{-2}	$4.61829 \cdot 10^{-2}$	2.05293	−0.47075	0.94149
0.222895	4 096	4^{-3}	$1.18779 \cdot 10^{-2}$	1.99561	−0.48977	0.97954

9. Random moving domains

In the final chapter we study the situation where the uncertainty of the model comes from the geometrical aspect. For example, the domain is often given by scanning or some other digital imaging technique with limited resolution which leads to the variance between the shape of the real body and the model (for a mathematical model of this problem see [9]). As before, a well established and efficient way to deal with this problem is to adopt the probabilistic approach, construct models of geometrical uncertainty and describe the phenomena by PDEs on a random domain. More precisely, we consider the fixed initial deterministic domain $D_0 \subset \mathbb{R}^d$ and its evolution in a time interval $[0, \tau]$ by a random velocity V. In this way we obtain a non-cylindrical, i.e. time-dependent, random domain

$$Q(\omega) := \bigcup_{t\in(0,\tau)} D_t(\omega) \times \{t\},$$

also known as a tube domain. Random domains appear in many applications, such as biology, surface imaging, manufacturing of nano-devices etc. One particular application example is in wind engineering as presented in [24]. More precisely, the authors study how small uncertain geometric changes in the Sunshine Skyway Bridge deck affect its aerodynamic behaviour. The geometric uncertainty of the bridge is due to its specific construction and wind effect. This model results in a PDE on a random domain. The analysis and numerical analysis of random domains have been considered by many authors, see [25, 26, 73, 76, 124]. In general, there are two main approaches in this area: the perturbation method and the domain mapping method. The perturbation method (cf. [74]) is based on a perturbation field that is defined on the boundary of a reference domain

$$T(\omega) : \partial D_0 \mapsto \mathbb{R}^d$$

and uses a shape Taylor expansion with respect to T to present the solution of the considered equation. The main disadvantage of this method is that it is applicable only for small perturbations. The domain mapping method (cf. [73, 124]) does not have such a constraint, but instead, it requires one to also know the perturbation field T in the interior of the domain D_0:

$$T(\omega) : \overline{D_0} \mapsto \mathbb{R}^d.$$

The main idea of the domain mapping method is to reformulate the PDE on the random domain into to PDE with random coefficients on a fixed reference domain. This reformulation allows us to apply numerous available numerical methods for solving random PDEs and to avoid the construction of a new mesh for every realization of a random domain. The main difficulty in this approach is the construction of a random mapping T if we consider complex geometries.

Remark 9.0.1. An alternative approach is suggested in [101] and it is known as the eXtended stochastic FEM. It relies on two main steps: the implicit representation of complex geometries using random level-set functions and the application of a Galerkin approximation at both stochastic and deterministic levels.

Nevertheless, since we are not considering complex geometries and our problem is formulated differently, it is natural for us to apply the domain mapping method. Namely, we start with a random velocity and a fixed initial domain, and as a result we build a random tube. To a random velocity, we will associate its flow T that will map a domain D_0 into a random domain $D_t(\omega)$ at a time t.

Notice that in the previous work on random domains, mainly elliptic PDEs are considered. Very few papers consider parabolic PDEs on random domains, such as [24, 25]. In addition, to the best of our knowledge, there are no results on random domains that change in time. This is exactly the topic of this section. We will consider the well-posedness of the heat equation on a random time varying domain.

PDEs on so-called non-cylindrical domains, i.e. domains changing in time, are a well-established topic regarded analysis and numerical analysis, with numerous applications. Numerous physical examples concerning phenomena on time dependent spatial domains are represented in survey article [85]. Some of the examples are: fluid flows with a free or moving solid-fluid interface, the Friedmann model in astrophysics that describes the scaling of key densities in the Universe with its time-dependent expansion factor, and many examples of biological processes that involve time-dependent domains, such as the formation of patterns and shapes in biology. Concerning the mathematical analysis of non-cylindrical domains, there are many papers considering various types of equations, where regularity assumptions of the evolution and definition of proper function spaces are one of the main challenges (see [21, 23, 34, 38, 84, 90]). In particular [84, 90] focus on appropriate formulation of the heat equation on a random domain and on proving the existence and uniqueness of strong, weak and ultra weak solutions, as well as providing energy estimates. These papers use coordinate transformation to reformulate the PDE into a cylindrical domain and Lions' general theory for proving the well-posedness. Moreover, in [38, 52, 53] similar results were obtained but with a greater focus on the connection of the non-cylindrical domain and the velocity field. Since we are particularly interested in how the velocity field induces a non-cylindrical domain, we will in this chapter mainly follow calculations from these papers.

However, as our domain is random, if we were to merely apply the existing results to our setting, we would only get the path-wise existence of solutions. Since we are interested in the statistics of solutions, we will rather apply a more general theory of well-posedness of parabolic PDEs, represented for example in [123].

In Figure 9.1, we visualize the difference between the deterministic cylindrical domain, the random cylindrical domain and the random non-cylindrical domain. The first plot presents a standard cylindrical domain. The second one is a realization of a random tube given by

$$\mathbb{S}^1 \ni (x_0, y_0) \mapsto (x(\omega), y(\omega)) := (2x_0 Y_1(\omega), 3y_0 Y_2(\omega)) \in D(\omega)$$

where $Y_1, Y_2 \sim \mathcal{U}(0,1)$ are independent RVs. The last two plots are two realizations of a random

non-cylindrical tube defined by

$$\mathbb{S}^1 \ni (x_0, y_0) \mapsto (x(\omega, t), y(\omega, t)) :=$$
$$(Y_1(\omega)(\sin(Y_2(\omega))+1.5)x_0+0.3\cos(Y_3(\omega)t), Y_4(\omega)(\sin(Y_5(\omega))+1.5)y_0+0.3\sin(Y_6(\omega)t))\in D_t(\omega)$$

where $Y_1, \ldots, Y_6 \sim \mathcal{U}(0, 1)$ are independent RVs.

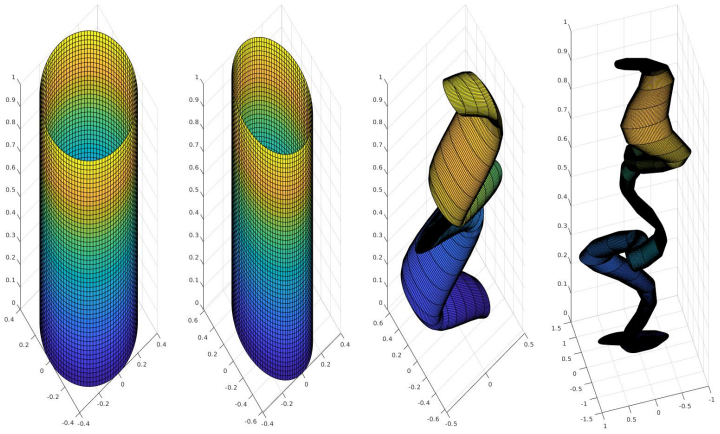


Figure 9.1.: Cylindrical domain, realization of a random cylindrical domain and realizations of random non-cylindrical domains, respectively.

9.1. Random tubes

Let $D_0 \subset \mathbb{R}^d$ be an open, bounded domain with a Lipschitz boundary. Furthermore, let $V : \Omega \times [0, \tau] \times \mathbb{R}^d \to \mathbb{R}^d$ be a random vector field. We would like to explain how $V(\omega)$ forms a random tube $Q_V(\omega)$, for any $\omega \in \Omega$. Furthermore, we will assume the existence of a hold-all domain i.e. we assume that there exists a bounded open set B such that $Q(\omega)$ remains in $(0, \tau) \times B$. Concerning the notation from the setting, (2.4.3), this means that we assume that there exists a uniform hold-all domain that contains all $\mathcal{N}_\tau(\omega)$. We will assume that the velocity field is defined on this domain B, and not on the whole space $\mathbb{R}^d$. How much set B varies from D_0, depends on how big the stochastic fluctuation of the initial domain is.

First, to a vector field $V(\omega)$, we can associate its flow $T_V(\omega)$. More precisely, for fixed $\omega \in \Omega$ and $X \in \overline{D}_0$ we consider the path-wise solution $x_{_V}(\omega, \cdot, X)$ of the ODE

$$\frac{dx_{_V}}{dt}(\omega, t, X) = V(\omega, t, x_{_V}(\omega, t, X)) \quad t \in [0, \tau] \tag{9.1.1}$$
$$x_{_V}(\omega, 0, X) = X. \tag{9.1.2}$$

Fro fixed t and X, by Fubini's theorem 2.2.3, $\omega \mapsto x_{_V}(\omega, t, X)$ is a measurable function.

Moreover, for any $\omega \in \Omega$ and $t \in [0, \tau]$, we consider the transformation

$$\begin{aligned} T_V(\omega, t) : \overline{B} &\to \overline{B} \\ X \mapsto T_V(\omega, t)(X) &:= x_{_V}(\omega, t, X). \end{aligned}$$

We denote the mapping $(\omega, t, X) \mapsto T_V(\omega, t)(X)$ by T_V. For brevity, and when there is no danger of confusion, we will not write the associate vector field V and we will write $T_t(\omega, X) \equiv T_V(\omega, t)(X)$. The measurability of x implies the measurability of $\omega \mapsto T_t(\omega, X)$, for fixed t and X.

Now, to a sufficiently smooth vector field $V(\omega)$ we can associate a tube $Q_{V,\tau}(\omega)$ defined by

$$Q_{V,\tau}(\omega) := \bigcup_{t\in(0,\tau)} D_t(\omega) \times \{t\} \qquad Q_0(\omega) := D_0,$$

where $D_t(\omega) := T_t(\omega)(D_0)$. Similarly as for the flow, we will use the notation Q and will not write the associate vector field V. Note that the notation in this chapter differs from that of previous chapters. Namely, before the flow was denote by ϕ and in this chapter we denote it by T, and it is a random function. According to this, in the Definition 3.3.1 of Bochner type spaces, one should consider this definition path-wise for $T_t(\omega)$ instead of ϕ_t.

Remark 9.1.1. Conversely, given a sample ω and a random tube $Q(\omega)$ with enough smoothness of a lateral boundary that ensures the existence of the outward normal, we can associate to $Q(\omega)$ a random smooth vector field $V(\omega)$ whose associated flow satisfies $T_V(\omega, t)(D_0) = D_t(\omega) \subset \mathbb{R}^d, \forall t \in [0, \tau]$.

The relation between the regularity of the velocity field $V(\omega)$ and the regularity of its associated flow $T_t(\omega)$ has been investigated using the general theory of shape calculus (for general results see for example [44, Ch 4, Th 5.1]). Here we will state weaker results that will be sufficient for our analysis. These results are also presented in [52, Proposition 2.1, Proposition 2.2] and [53].

First, let us state the assumptions about the velocity field.

Assumption 9.1.2. The velocity field satisfies the following regularity assumptions

$$V(\omega) \in C([0, \tau], W^{k,\infty}(B, \mathbb{R}^d)) \quad \text{for a.e. } \omega \text{ and } k \geq 1 \tag{9.1.3}$$

and

$$V(\omega, t) \cdot n_B = 0 \quad \text{on } \partial B, \quad \text{for a.e. } \omega \tag{9.1.4}$$

where $n_B \in \mathbb{R}^d$ is the unit outward normal field to B.

The assumption (9.1.4) ensures that the transformation T_t is one-to-one homeomorphism which maps $\overline{B}$ to $\overline{B}$ (cf. [50, pp. 87–88]). In particular, it maps the interior points onto interior points and the boundary points onto boundary points. Thus, for every $t \in [0, \tau]$ we can consider the transformation $T(V)_t^{-1} \equiv T_t^{-1} : \overline{B} \to \mathbb{R}^n$. Note that T_t^{-1} is the flow at $s = t$ of the velocity filed $\tilde{V}_t(s) := -V(t - s)$.

Remark 9.1.3. Instead of (9.1.4), one can make a more general assumption that $\pm V(\omega, t, x)$ belongs to a so-called s Bouligand's contingent cone. For more details see [44, Ch. 5]. Another option would be to assume that the velocity field V is defined on the whole $\mathbb{R}^d$. In this setting the assumption (9.1.4) is not needed and the analogue regularity results hold for the flow, see [44, Theorem 4.1].

For simplicity of notation, since the following result, stated in [52, Proposition 2.1, 2.2], is deterministic, and we will consider it path-wise, we will omit writing ω.

Lemma 9.1.4. Let Assumption 9.1.2 hold. Then there exists a unique associated flow $T(V)$ that is a solution of

$$\frac{d}{dt}T(t,\cdot) = V(t, T(t,\cdot)), \quad T(0) = Id. \tag{9.1.5}$$

such that

$$T_V \in C^1([0,\tau], W^{k-1,\infty}(B,\mathbb{R}^d)) \cap C([0,\tau], W^{k,\infty}(B,\mathbb{R}^d)).$$

Moreover,

$$T_V^{-1} \in C([0,\tau], W^{k,\infty}(B,\mathbb{R}^d)).$$

For our analysis we will need more regularity of the inverse transformation T_t^{-1}. Utilizing the implicit function theorem, better regularity result for T_t^{-1} can be obtained on some subinterval $[0,\tau']$. For the proof see [53, Proposition 2.2].

Lemma 9.1.5. There exists $\tau' \in (0,\tau]$ such that $T_V^{-1} \in C^1([0,\tau'], W^{k-1,\infty}(B,\mathbb{R}^d))$.

Observe that in our setting we consider Lemma 9.1.5 path-wise. Thus, for every ω there exists $\tau'(\omega) \in (0,\tau]$. For this reason we need to make an additional assumption to avoid that $\tau'(\omega)$ converges to zero. We assume the existence of a deterministic constant τ_0 such that

$$0 < \tau_0 \le \tau'(\omega) \le \tau \quad \forall\omega.$$

We then consider our problem on the time interval $[0,\tau_0]$. By abuse of notation, we continue to write τ for τ_0. Hence, we have that

$$T_V, T_V^{-1} \in C^1([0,\tau], W^{k-1,\infty}(B,\mathbb{R}^d)) \cap C([0,\tau], W^{k,\infty}(B,\mathbb{R}^d)). \tag{9.1.6}$$

Now we move to the probability setting and make Assumption 9.1.2 more concrete and suitable for our calculations. Assuming that $\overline{B}$ has enough regular shape, such as bounded, open, path-connected and locally Lipschitz subset of $\mathbb{R}^d$, from [44, Ch 2, Th 2.6], we infer

$$W^{k+1,\infty}(B,\mathbb{R}^d)) = C^{k,1}(\overline{B},\mathbb{R}^d) \quad \text{and} \quad C^{k,1}(\overline{B},\mathbb{R}^d) \hookrightarrow C^k(\overline{B},\mathbb{R}^d).$$

In particular, in our setting it is sufficient to assume that $k = 3$. Therefore the assumption on the regularity of the velocity field becomes the following:

Assumption 9.1.6. The velocity field satisfies the following regularity assumptions

$$V(\omega) \in C([0,\tau], C^2(\overline{B},\mathbb{R}^d)) \quad \text{for a.e. } \omega \tag{9.1.7}$$

and

$$V(\omega,t)\cdot n_B = 0 \quad \text{on } \partial B \quad \text{for a.e. } \omega. \tag{9.1.8}$$

Then, according to 9.1.6, we obtain the following regularity of the associated flow and its inverse

$$T(\omega), T^{-1}(\omega) \in C^1([0,\tau], C(\overline{B}, \mathbb{R}^d)) \cap C([0,\tau], C^2(\overline{B}, \mathbb{R}^d)). \tag{9.1.9}$$

Remark 9.1.7. In the literature, a standard assumption for non-cylindrical problems is a monotone or regular (Lipschitz) variation of the domain D_t. The weaker assumptions on time-regularity of the boundary are considered in [21] . Namely, the authors assume only the Hölder regularity for the variation of the domains. The motivating example for this kind of assumption is a stochastic evolution problem in the whole space $\mathbb{R}^d$.

In view of Assumption 9.1.6, spatial domains $D_t(\omega)$ in $\mathbb{R}^d$ are obtained from a base domain D_0 by a C^2-diffeomorphism, which is continuously differentiable in the time variable. The C^1 dependence in time indicates that we do not have an overly rough evolution in time, and C^2 regularity in space means that topological properties are preserved along time. In addition, to ensure the uniform bound and the coercivity of the bilinear form that will be considered, we suppose to have the uniform bound of the norm.

Assumption 9.1.8. We assume that there exists a constant $C_T > 0$ such that

$$\|T(\omega)\|_{C([0,\tau],C^2(\overline{B},\mathbb{R}^d))}, \|T^{-1}(\omega)\|_{C([0,\tau],C^2(\overline{B},\mathbb{R}^d))} \leq C_T \quad \text{for a.e. } \omega.$$

Let $DT_t(\omega)$ and $DT_t^{-1}(\omega)$ denote the Jacobian matrices of $T_t(\omega)$ and $T_t^{-1}(\omega)$, respectively. From (9.1.9) and (9.1.8), we infer

$$DT(\omega), DT^{-1}(\omega) \in C^1([0,\tau], C^1(\overline{B}, \mathbb{R}^d)).$$

$$\|DT(\omega)\|_{C([0,\tau],C^1(\overline{B},\mathbb{R}^d))}, \|DT^{-1}(\omega)\|_{C([0,\tau],C^1(\overline{B},\mathbb{R}^d))} \leq C_D \quad \text{for a.e. } \omega, \tag{9.1.10}$$

for a constant $C_D > 0$. Since for the operator norm $\|\cdot\|$ of any square matrix M, we have $\|MM^T\| = \|M^TM\| = \|M\|^2$, then by (9.1.10) it holds

$$\max_{t,x} \|DT_t(\omega,x)DT_t^\top(\omega,x)\| = \max_{x,t} \|DT_t^\top(\omega,x)DT_t(\omega,x)\| \leq C_D^2 \quad \text{for a.e. } \omega, \tag{9.1.11}$$

and the analogue holds for the inverse Jacobian matrix. Moreover, let $J_t(\omega) := \det(DT_t(\omega))$ and $J_t^{-1}(\omega) := \det(DT_t^{-1}(\omega))$. Since $J_t(\omega)$ does not vanish, $J_0(\omega) = 1$, and because it is continuous, it follows that $J_t(\omega) > 0$, a.e. and the same holds for its inverse. From (9.1.10) we conclude

$$J_{(\cdot)}(\omega), J_{(\cdot)}^{-1}(\omega) \in C^1([0,\tau], C^1(\overline{B}, \mathbb{R})) \quad \text{for a.e. } \omega. \tag{9.1.12}$$

In addition, we need to assume a uniform bound for the gradient of the inverse Jacobian. The regularity result (9.1.12) implies that the gradient of the inverse Jacobian is bounded, but not that this bound is uniform in ω.

Assumption 9.1.9. Assume that there exists a constant $C_J > 0$ independent of t and ω such that

$$\|\nabla_x J_t^{-1}(\omega,x)\|_{\mathbb{R}^d} \leq C_J.$$

Remark 9.1.10. Since $(M^\top)^{-1} = (M^{-1})^\top, M \in \mathbb{R}^{d\times d}$, inverse and transpose operations commute, we will just write $M^\top$ for transpose and $M^{-\top}$ for its inverse.

Furthermore, let $\sigma_i(\omega) = \sigma_i(DT_t(\omega, x)), i = 1, \dots, d$ denote the singular values of the Jacobian matrix, i.e. the square root of eigenvalues of the matrix $DT_t\, DT_t^\top$ or equivalently, the matrix $DT_t^\top\, DT_t$. If we consider a matrix which has continuous functions as entries, it follows that its eigenvalues are also continuous functions (see [126]). This argument is based on the fact that the eigenvalues are roots of the characteristic polynomial and roots of any polynomial are continuous functions of its coefficients. As the coefficients of the characteristic polynomial depend continuously on the entries of the matrix and singular values are the square roots of eigenvalues of $DT_t\, DT_t^\top$, it follows

$$\sigma_i(\omega) \in C([0,\tau], C(\overline{B}, \mathbb{R})).$$

Thus, for every i, $\sigma_i(\omega)$ achieves the minimum and maximum on $[0,\tau] \times \overline{B}$. The independence on ω of these minimal and maximal values follows from (9.1.10). To see this, recall that the Rayleigh quotient and the definition of the singular value imply

$$\sigma_i(\omega, x, t) \le \max_{\|y\|_{\mathbb{R}^d}=1} \|DT_t(\omega, x)y\|_{\mathbb{R}^d}.$$

Thus, for $\overline{\sigma} := C_D$, $\underline{\sigma} := C_D^{-1}$, every $i = 1, \dots, d$ and a.e. ω we have

$$0 < \underline{\sigma} \le \min_{x,t}\{\sigma_i(\omega, t, x)\} \le \max_{x,t}\{\sigma_i(\omega, t, x)\} \le \overline{\sigma} < \infty. \tag{9.1.13}$$

Since $J(\omega) = \prod_{i=1}^n \sigma_i(\omega)$, the bound (9.1.13) implies the uniform bound for the determinant of the Jacobian, i.e. for a.e. ω it holds

$$0 < \underline{\sigma}^n \le J_t(\omega, x) \le \overline{\sigma}^n < \infty \quad \text{for every } x \in \overline{B}, t \in [0,T]. \tag{9.1.14}$$

The analogue reciprocal bounds hold for the J_t^{-1}.

9.2. Heat equation on a random domain

We consider the following initial boundary value problem for the heat equation in the non-cylindrical domain $Q(\omega)$

$$\begin{aligned} u' - \Delta u &= f \quad \text{in } Q(\omega) \\ u &= 0 \quad \text{on } \cup_{t\in(0,\tau)} \partial D_t(\omega) \times \{t\} \\ u(\omega, x, 0) &= u_0(x, \omega) \quad x \in D_0. \end{aligned} \tag{9.2.1}$$

Note that we assume that the initial domain D_0 is deterministic and u' is a weak time derivative.

Remark 9.2.1. The general form point-wise conservation law on an evolving flat domain D_t, derived in [55], is given by

$$\partial^\bullet u + u\nabla \cdot V + \nabla \cdot q = 0$$

where V is the velocity of the evolution, q is the flux and $\partial^\bullet$ is the material derivative. Taking in particular $q = -\nabla u - Vu$, we obtain the form (9.2.1). Thus, although the material derivative does not explicitly appear in the formulation of the equation, as we have already commented, the material derivative is a natural notion for the derivative of a function defined on a moving domain. Thus, for the solution u, we will ask that its material derivative is in the appropriate space and we will use the solution space introduced in Section 3.5. Thus, $u' = \partial^\bullet u - \nabla u \cdot V$.

Assuming enough regularity for f and u_0, we specify the weak path-wise formulation of the boundary value problem (9.2.1).

Problem 9.2.2 (Weak path-wise form of the heat equation on $D_t(\omega)$)**.** For any ω, find $u(\omega) \in W(H_0^1(D_t(\omega)), L^2(D_t(\omega)))$ that point-wise a.e. satisfies the initial condition $u(0) = u_0 \in L^2(\Omega, H^1(D_0))$ and

$$\int_{D_t(\omega)} \big(u'(\omega,t)\varphi + \langle \nabla u(\omega,t), \nabla\varphi\rangle_{\mathbb{R}^n}\big) = \int_{D_t(\omega)} f(\omega,t)\varphi \tag{9.2.2}$$

for every $\varphi \in H_0^1(D_t(\omega))$ and a.e. $t \in [0,T]$.

Since (9.2.1) is posed on a random domain, we would like to show that the solution u is also a random variable and that it has finite moments. However, since the domain is random, we have $u(\omega,t) \in D_t(\omega)$. Thus finding an appropriate solution space for u and defining its expectation is not straightforward. The notion of a stochastic process with a random domain has already been analysed (see [51] and references therein). The authors begin by defining what is meant by a random open convex set in a probabilistic setting and and then go on to explain what a stochastic process with a random domain is. Moreover, in [35], the authors give a possible interpretation of the notions of noise and a random solution on time-varying domains. We believe that these ideas could be generalized to our setting, but they will not be analysed in this thesis.

Instead, as already announced, motivated by the domain mapping method, we consider the pull-back of the problem (9.2.1) on the fixed domain D_0 and study the solution $\hat{u}$ of the reformulated problem. We will first derive the path-wise formulation for the function $\hat{u}$ that is equivalent to Problem 4.3.1. Now for the function $\hat{u}$ it makes sense to ask $\hat{u} \in \mathcal{W}(H_0^1(D_0), L^2(D_0))$ and it is clear what its expectation is. Thus, using the domain mapping method, we translate the PDE on the random domain into a PDE with random coefficients on the fixed domain D_0.

Let $\hat{u}(\omega) : [0,\tau] \times D_0 \to \mathbb{R}$ be defined by

$$\hat{u}(\omega,t,y) := u(\omega,t,T_t(\omega,y)) \quad \text{for every } y \in D_0, t \in [0,\tau]. \tag{9.2.3}$$

Thus, $\hat{u} : \Omega \times \hat{Q} \to \mathbb{R}$, where $\hat{Q} := [0,\tau] \times D_0$.

Lemma 9.2.3 (Formulae for transformed ∇ and ∂_t)**.** Let $f(\omega) \in L^2_{H^1(D_t(\omega))}$ and $\hat{f}(\omega,t,X) := f(\omega,t,T_t(\omega,X))$, $\hat{f}(\omega) : \hat{Q} \to \mathbb{R}$, for every $\omega \in \Omega$. Then

$$\nabla_x f(\omega,t,T_t(\omega,y)) = DT_t^{-\top}(\omega,y)\,\nabla_y \hat{f}(\omega,t,y) \quad y \in D_0 \tag{9.2.4}$$

$$f'(\omega,t,T_t(\omega,y)) = \hat{f}'(\omega,t,y) - V(\omega,t,T_t(y)) \cdot (DT_t^{-\top}(\omega,t,y)\nabla_y \hat{f}(\omega,t,y)) \quad y \in D_0. \tag{9.2.5}$$

Proof. Since ω dependence doesn't play a role in the previous formulae, we will not write it in order to simplify the notation. The identity (9.2.4) follows directly from the chain rule (see [22, Proposition IX.6]) and definition (9.2.3):

$$\nabla_y \hat{f}(t,y) = DT_t^\top(y) \nabla_x f(t, T_t(y)).$$

Utilizing once more the chain rule for the derivative w.r.t. time, the relation (9.2.4), and (9.1.5), we get

$$\begin{aligned} \hat{f}'(t,y) &= f'(t,T_t(y)) + DT_t^{-\top}(t,y)\nabla_y \hat{f}(\omega,t,y)) \cdot \frac{\partial T}{\partial t}(t,y) \\ &= f'(t,T_t(y)) + (DT_t^{-\top}(\omega,t,y)\nabla_y \hat{f}(\omega,t,y)) \cdot V(t,T_t(y)) \end{aligned}$$

which implies the relation (9.2.5). □

Now we can formulate the weak path-wise problem on the reference domain. For simplicity of notation, we don't write the dependence on ω explicitly here.

Problem 9.2.4 (Weak path-wise form of the heat equation on D_0)**.** For every ω, find $\hat{u}(\omega) \in W(H_0^1(D_0), L^2(D_0))$ that point-wise a.e. satisfies the initial condition $u(0) = u_0 \in L^2(\Omega, H^1(D_0))$ and

$$\begin{aligned} \int_{D_0} \Big(\hat{u}'(t,y) - \Big\langle DT_t^{-\top}(t,y)\nabla\hat{u}(t,y), V(t,T_t(y))\Big\rangle_{\mathbb{R}^d}\Big) J_t(y)\hat{\varphi}(y) \\ + \langle A(t,y)\nabla\hat{u}(t,y), \nabla\hat{\varphi}(y)\rangle_{\mathbb{R}^d}\, dy = \int_{D_0} \hat{f}(t,y)\hat{\varphi}(y)J_t(y)dy \end{aligned} \quad (9.2.6)$$

for every $\hat{\varphi} \in H_0^1(D_0)$ and a.e. $t \in [0,T]$, where

$$A(\omega,t,y) = J_t(\omega,y)DT_t^{-1}(\omega,y)\, DT_t^\top(\omega,y)^{-1} \quad y \in D_0. \quad (9.2.7)$$

Lemma 9.2.5 (Path-wise formulations on $Q_T(\omega)$ and $\hat{Q}_T$)**.** Letting $f \in L^2_{L^2(\Omega,L^2(D_t(\omega)))}$, the following are equivalent.

i) $u(\omega)$ is a path-wise weak solution to Problem 9.2.2

ii) $\hat{u}(\omega)$ is a path-wise weak solution to Problem 9.2.4.

Proof. Let us assume that i) holds. From the substitution rule $x = T_t(y)$ and Lemma 9.2.3, we obtain

$$\begin{aligned} \int_{D_0} u'(t,T_t(y))\varphi(t,T_t(y))J_t(y)dy + \int_{D_0} \nabla u(t,T_t(y)) \cdot \nabla\varphi(t,T_t(y))J_t(y)dy = \\ \int_{D_0} \Big(\hat{u}'(t,y) - DT_t^{-\top}(t,y)\nabla\hat{u}(t,y) \cdot V(t,T_t(y))\Big)\hat{\varphi}(t,y)J_t(y)dy + \\ \int_{D_0} \Big\langle DT_t^{-\top}(y)\nabla\hat{u}(t,y), DT_t^{-\top}(y)\nabla\hat{\varphi}(t,y)\Big\rangle_{\mathbb{R}^d} J_t(y)dy = \\ \int_{D_0} \hat{f}(t,y)\hat{\varphi}(t,y)J_t(y)dy. \end{aligned}$$

Since

$$\int_{D_0} \Big\langle DT_t^{-\top}(y)\nabla\hat{u}(t,y), DT_t^{-\top}(y)\nabla\hat{\varphi}(t,y)\Big\rangle_{\mathbb{R}^d} J_t(y)dy = \int_{D_0} \langle A(t,y)\nabla\hat{u}(t,y), \nabla\hat{\varphi}(y)\rangle_{\mathbb{R}^d}\, dy,$$

where the matrix A is defined by (9.2.7), it follows that that $\hat{u}$ is a path-wise weak solution of Problem 9.2.4. The proof of implication ii) $\Rightarrow$ i) is similar. □

Note that according to Lemma 3.3.3, it holds

$$u(\omega) \in L^2_{H_0^1(D_t(\omega))} \Leftrightarrow \hat{u}(\omega) \in L^2(0,\tau; H_0^1(D_0)) \quad \text{for a.e. } \omega.$$

Remark 9.2.6. The spaces $H_0^1(D_0)$ and $H_0^1(D_t(\omega))$ are isomorphic due to the isomorphism $\eta \mapsto \eta \circ T_t(\omega)^{-1}$. This implies that the space of test functions is independent of ω. For more details see [73, Lemma 2.2].

9.3. Well-posedness of the transformed equation

In order to get (9.2.6) in a standard form, which is more convenient to apply the general theory of well-posedness for parabolic PDEs presented in [123], we need to remove the weight J_t^{-1} in front of the time derivative $\hat{u}'$. This form we can achieve by testing the equation (9.2.6) with functions $\hat{\varphi}(t,y) = J_t^{-1}(y)\tilde{\varphi}(t,y)$. The spatial regularity of J_t stated in (9.1.12), implies

$$\forall\hat{\varphi} \in H_0^1(D_0) \Leftrightarrow \forall\tilde{\varphi} \in H_0^1(D_0).$$

In this way we obtain the equivalent form of (9.2.6) given by

$$\int_{D_0} \Big(\hat{u}'(t,y) - \Big\langle DT_t^{-\top}(t,y)\nabla\hat{u}(t,y), V(t,T_t(y))\Big\rangle_{\mathbb{R}^d}\Big)\tilde{\varphi}(y) + \big\langle A(t,y)\nabla\hat{u}(t,y), \nabla(J_t^{-1}(y)\tilde{\varphi}(y))\big\rangle_{\mathbb{R}^d}\, dy = \int_{D_0} \hat{f}(t,y)\hat{\varphi}(y)dy, \quad (9.3.1)$$

for all $\tilde{\varphi} \in H_0^1(D_0)$. Utilizing the product rule for the gradient and symmetry of the matrix A, we arrive at the equivalent weak path-wise form of the heat equation:

Problem 9.3.1 (Weak path-wise form of the heat equation on $D_t(\omega)$)**.** For every ω, find $\hat{u}(\omega) \in W(H_0^1(D_0), L^2(D_0))$ that point-wise a.e. satisfies the initial condition $u(0) = u_0 \in L^2(\Omega, H^1(D_0))$ and

$$\int_{D_0} \big(\hat{u}'(t,y) + \big\langle A(t,y)\nabla J_t^{-1}(y) - DT_t^{-1}(y)V(t,T_t(y)), \nabla\hat{u}(t,y)\big\rangle_{\mathbb{R}^d}\big)\tilde{\varphi}(y) + \Big\langle DT_t^{-1}(y)DT_t^{-\top}(t,y)\nabla\tilde{\varphi}(t,y), \nabla\hat{u}(t,y)\Big\rangle_{\mathbb{R}^d} dy = \int_{D_0} \hat{f}(t,y)\tilde{\varphi}(y)dy \quad (9.3.2)$$

for every $\tilde{\varphi} \in H_0^1(D_0)$ and a.e. $t \in [0,T]$.

Observe that the partial integration and the fact that a test function vanishes on the boundary ∂D_0 imply

$$\int_{D_0} \Big\langle DT_t^{-1}(y)DT_t^{-\top}(y)\nabla\tilde{u}(t,y), \nabla\hat{\varphi}(t,y)\Big\rangle_{\mathbb{R}^d} dy = \\ -\int_{D_0} \operatorname{div}(DT_t^{-1}(y)DT_t^{-\top}(y)\nabla\hat{u}(t,y))\tilde{\varphi}(t,y)dy.$$

Let us comment on the boundary condition and initial condition. Since T_0 is the identity and D_0 is the deterministic initial domain, the initial condition stays the same:

$$u(\omega,x,0) = u_0(\omega,x) \Leftrightarrow \hat{u}(\omega,x,0) = u_0(\omega,x), \quad \forall x \in D_0,$$

for a.e. $\omega \in \Omega$. Moreover, as the boundary of $\partial D_t(\omega)$ is mapped to ∂D_0, the reformulated boundary condition stays the same:

$$\begin{aligned} u(\omega,t,x) &= 0 \quad \forall (x,t) \in \cup_{t\in(0,\tau)}\partial D_t(\omega)\times\{t\} \Leftrightarrow \\ \hat{u}(\omega,t,y) &= 0 \quad \forall (y,t) \in \partial D_0 \times (0,\tau) \end{aligned}$$

for a.e. $\omega \in \Omega$. Hence, in the distribution sense, we are led to consider for a.e. ω

$$\begin{aligned} \hat{u}' - \operatorname{div}(J_t^{-1}A\nabla\hat{u}) + \big\langle \nabla\hat{u}, A\nabla J_t^{-1} - DT_t^{-1}V\circ T_t\big\rangle_{\mathbb{R}^d} &= \hat{f} \quad \text{in } (0,\tau)\times D_0 \\ \hat{u}(\omega,x,t) &= 0 \quad \text{on } \partial D_0 \times (0,\tau) \\ \hat{u}(\omega,x,0) &= u_0(\omega,x) \quad \text{on } D_0. \end{aligned}$$

Our goal is to show that $\hat{u}$ is a random variable and that it has finite moments, under suitable assumptions on the initial data. Thus, we will formulate a mean-weak formulation for $\hat{u}$. Furthermore, we will prove a more general result, when we have less regularity in the initial data. The regularity results can be obtained from the general theory on parabolic PDEs. In particular, assuming more regularity on $\hat{f}$ and u_0, we obtain better regularity of the time derivative of $\hat{u}$.

Observe that since $L^2(\Omega)$ is separable, utilizing tensor product isomorphisms stated in Theorem 2.5.5, we conclude

$$\begin{aligned} L^2(\Omega)\otimes L^2(0,\tau;H) &\cong L^2(\Omega, L^2(0,\tau;H)) \cong L^2(\Omega\times(0,\tau);H) \\ &\cong L^2(0,\tau;L^2(\Omega;H)) \cong L^2(0,\tau)\otimes L^2(\Omega,H) \end{aligned}$$

for any Hilbert space H. Thus, it holds

$$L^2(\Omega)\otimes\mathcal{W}(H_0^1(D_0),H^{-1}(D_0)) \cong \mathcal{W}(L^2(\Omega,H_0^1(D_0)),L^2(\Omega,H^{-1}(D_0))),$$

where $\mathcal{W}(L^2(\Omega,H_0^1(D_0)),L^2(\Omega,H^{-1}(D_0)))$ is a standard Bochner space defined by (2.2.2).

Problem 9.3.2 (Mean-weak formulation on D_0)**.** Find $\hat{u}\in\mathcal{W}_0(L^2(\Omega,H_0^1(D_0)),L^2(\Omega,H^{-1}(D_0)))$ such that a.e. in $[0,T]$ it holds

$$\int_\Omega\int_{D_0}\langle\hat{u}',\varphi\rangle_{H^{-1}(D_0),H^1(D_0)}\,dyd\mathbb{P} + \int_\Omega\int_{D_0}\Big\langle DT_t^{-1}(\omega,y)\,DT_t^{-\top}(\omega,y)\nabla\hat{u},\nabla\varphi\Big\rangle_{\mathbb{R}^d}dyd\mathbb{P} + \\ \int_\Omega\int_{D_0}\big\langle A(\omega,t,y)\nabla J_t^{-1}(\omega,y) - DT_t^{-1}(\omega,y)V(t,T_t(y)),\nabla\hat{u}\big\rangle_{\mathbb{R}^d}\varphi dyd\mathbb{P} = \int_\Omega\int_{D_0}\hat{f}\varphi dyd\mathbb{P}$$

for every $\varphi\in L^2(\Omega,H^1(D_0))$.

Theorem 9.3.3. Let Assumptions 9.1.6, 9.1.8 and 9.1.9 hold and $f \in L^2_{L^2(\Omega,H^{-1}(D_0))}$. Then, there is a unique solution $\hat{u} \in \mathcal{W}(L^2(\Omega, H^1_0(D_0)), L^2(\Omega, H^{-1}(D_0)))$ of Problem 9.3.2 and we have the a priori bound

$$\|\hat{u}\|_{\mathcal{W}(L^2(\Omega,H^1_0(D_0)),L^2(\Omega,H^{-1}(D_0)))} \leq C\|f\|_{L^2_{L^2(\Omega,H^{-1}(D_0))}} \tag{9.3.3}$$

with some $C \in \mathbb{R}$.

Proof. Let $V := L^2(\Omega, H^1_0(D_0))$ and $H := L^2(\Omega, L^2(D_0))$. Then $V \subset H \subset V^*$ is a Gelfand triple. Furthermore, for every $t \in [0, \tau]$ we introduce the bilinear form $a(t; \cdot, \cdot) : V \times V \to \mathbb{R}$ by

$$a(t;\varphi,\psi) := \int_\Omega \int_{D_0} \Big(\Big\langle DT_t^{-1}\, DT_t^{-\top} \nabla\varphi, \nabla\psi \Big\rangle_{\mathbb{R}^n} + \big\langle A\nabla J_t^{-1} - DT_t^{-1} V \circ T_t, \nabla\varphi \big\rangle_{\mathbb{R}^n} \psi \Big)\, dX d\mathbb{P}. \tag{9.3.4}$$

We will prove that $a(t; \varphi, \psi)$ satisfies the following assumptions, which are necessary conditions for the well-posedness of the parabolic PDE stated in [123, Theorem 26.1].

i) $a(t; \varphi, \psi)$ is measurable on $[0, \tau]$, for fixed $\varpi, \psi \in V$.

ii) There exists some $c > 0$, independent of t, such that

$$|a(t;\varphi,\psi)| \leq c\|\varphi\|_V \|\psi\|_V \quad \text{for all } t \in [0,\tau], \varphi, \psi \in V. \tag{9.3.5}$$

iii) There exist real $k_0, \alpha \geq 0$ independent of t and φ, with

$$a(t;\varphi,\varphi) + k_0\|\varphi\|^2_H \geq \alpha\|\varphi\|^2_V \quad \text{for all } t \in [0,\tau], \varphi \in V. \tag{9.3.6}$$

i) Due to assumption (9.1.6) and regularity results (9.1.10) and (9.1.12), the integrand in the definition (9.3.4) is $\mathcal{B}([0, \tau])$-measurable. Consequently, according to Fubini's theorem 2.2.3, we obtain the Borel measurability on $[0, \tau]$ of the mapping

$$t \mapsto a(t; \varphi, \psi) \tag{9.3.7}$$

for fixed $\varphi, \psi \in V$. Thus $i)$ is satisfied.

ii) Our next goal is to prove $ii)$. Applying the Cauchy-Schwartz inequality for $\mathbb{R}^d$, we infer

$$\begin{aligned} &\int_\Omega \int_{D_0} |\, \Big\langle DT_t^{-1}\, DT_t^{-\top} \nabla\varphi, \nabla\psi \Big\rangle_{\mathbb{R}^n} |\ \leq \\ &\int_\Omega \int_{D_0} \|DT_t^{-1}\, DT_t^{-\top} \nabla\varphi\|_{\mathbb{R}^d} \|\nabla\psi\|_{\mathbb{R}^d} \ \leq\ C_1 \|\nabla\varphi\|_H \|\nabla\psi\|_H, \end{aligned} \tag{9.3.8}$$

where the last inequality follows from (9.1.11), for $C_1 = \underline{\sigma}^2$.

Using the Cauchy-Schwartz inequality for $\mathbb{R}^d$ one more time, we get

$$\begin{aligned} &\int_\Omega \int_{D_0} |\, \big\langle \nabla\varphi(\omega,t,y), A(\omega,t,y)\nabla J_t^{-1}(\omega,y) - DT_t^{-1}(\omega,y) V(t, T_t(y)) \big\rangle_{\mathbb{R}^d} |\, |\psi(\omega,t,y)| \\ &\leq \int_\Omega \int_{D_0} \|A(\omega,t,y)\nabla J_t^{-1}(\omega,y) - DT_t^{-1}(\omega,y) V(t,T_t(y))\|_{\mathbb{R}^d} \|\nabla\varphi(\omega,t,y)\|_{\mathbb{R}^d} |\psi(\omega,t,y)| \\ &\leq C_2 \||\nabla\varphi|\|_H \|\psi\|_H. \end{aligned} \tag{9.3.9}$$

Let us explain the following bound that we used

$$\|A(\omega,t,\cdot)\nabla J_t^{-1}(\omega,\cdot) - DT_t^{-1}(\omega,\cdot)V(t,T_t(\cdot))\|_\infty \\ := \max_{y\in D_0} \|A(\omega,t,y)\nabla J_t^{-1}(\omega,y) - DT_t^{-1}(\omega,y)V(t,T_t(y))\|_{\mathbb{R}^d} \le C_2, \quad (9.3.10)$$

for some $C_2 > 0$ independent of t. Namely, according to triangular inequality we have

$$\|A(\omega,t,\cdot)\nabla J_t^{-1}(\omega,\cdot) - DT_t^{-1}(\omega,\cdot)V(t,T_t(\cdot))\|_\infty \le \\ \|A(\omega,t,\cdot)\nabla J_t^{-1}(\omega,\cdot)\|_\infty + \|DT_t^{-1}(\omega,\cdot)V(t,T_t(\cdot))\|_\infty.$$

The uniform bound of the second term follows from (9.1.8) and (9.1.10). Concerning the first term, utilizing Assumption 9.1.9 we get

$$\|A(\omega,t,\cdot)\nabla J_t^{-1}(\omega,\cdot)\|_\infty \le C_J\|A\|_\infty.$$

Since, from (9.1.14) and (9.1.13) we conclude

$$\|A\|_\infty \le \lambda_{\max}A \le \overline{\sigma}^d\lambda_{\max}(DT_t^{-1}\,DT_t^{-\top}) \le \overline{\sigma}^d\underline{\sigma}^2,$$

and the bound (9.3.10) follows. Finally, inequalities (9.3.8) and (9.3.9), ensure condition ii).

iii) To prove iii) we use the bound (9.1.13) that implies the bound for the eigenvalue $\lambda_{\min}(DT_t^{-1}\,DT_t^{-\top}) \ge \frac{1}{\overline{\sigma}^2} =: C_3$. Thus, exploiting this bound and the Rayleigh quotient of the minimal eigenvalue of the symmetric matrix $DT_t^{-1}\,DT_t^{-\top}$, we obtain

$$\begin{aligned}
C_3\|\nabla\varphi\|_H^2 &\le \int_\Omega\int_{D_0} \lambda_{\min}(DT_t^{-1}\,DT_t^{-\top})\|\nabla\varphi\|_{\mathbb{R}^d}^2 \\
&\le \int_\Omega\int_{D_0} \left\langle DT_t^{-1}\,DT_t^{-\top}\nabla\varphi,\nabla\varphi\right\rangle_{\mathbb{R}^d} \\
&\le a(t;\varphi,\varphi) + \int_\Omega\int_{D_0} \|\nabla\varphi\|_{\mathbb{R}^d}\|DT_t^{-1}V\circ T_t - A\nabla J_t^{-1}\|_{\mathbb{R}^d}|\varphi| \\
&\le a(t;\varphi,\varphi) + C_2\int_\Omega\int_{D_0} \|\nabla\varphi\|_{\mathbb{R}^d}|\varphi| \\
&\le a(t;\varphi,\varphi) + C_2\|\nabla\varphi\|_H\|\varphi\|_H \\
&\le a(t;\varphi,\varphi) + C_2\left(2\varepsilon\|\nabla\varphi\|_H^2 + \frac{1}{2\varepsilon}\|\varphi\|_H^2\right),
\end{aligned}$$

where we used Young's inequality in the last step. For small enough $\varepsilon > 0$, we get

$$(C_3 - 2\varepsilon)\|\nabla\varphi\|_H^2 \le a(t;\varphi,\varphi) + k_0\|\varphi\|_H^2,$$

for $k_0 := C_2\frac{1}{2\varepsilon}$. Applying Poincare's inequality with the constant C_P from

$$\frac{C_3 - 2\varepsilon}{1+C_P^2}\|\varphi\|_V^2 \le (C_3-2\varepsilon)\|\nabla\varphi\|_H^2 \le a(t;\varphi,\varphi) + k_0\|\varphi\|_H^2,$$

we conclude that iii) holds with $\alpha = \frac{C_3-2\varepsilon}{1+C_P^2}$.

After proving i), ii) and iii), the classical results can be applied. Hence, [123, Theorem 26.1] yields the existence and uniqueness of the solution $\hat{u}$ that satisfies an a priori bound (9.3.3). □

Regularity results for the considered problem can be obtained using the general theory from [123] for parabolic PDEs. Moreover, numerical analysis and numerical experiments can be considered for the analyzed problem. In particular, the representation of the random velocity V and what are sufficient regularity assumptions on V that would ensure the well-posedness of the problem, are two interesting questions. Furthermore, it is natural to ask: what happens if the random domain on which the equation is posed is curved? These questions will not be answered in this thesis, but will be topics of further research.

A. Appendix

A.1. Dual operator

Let H be a Hilbert space. The following important theorem that characterizes the dual space space H^* is due to Riesz and Fréchet. For the proof we refer the reader to [109, Theorem II.4].

Theorem A.1.1. For each $f \in H^*$, there is a unique $y_f \in H$ such that $f(x) = (y_f, x)_H$ for all $x \in H$. In addition $\|y_f\|_H = \|f\|_{H^*}$.

Hence, every $x \in H$ can be identified with the functional $y \mapsto (y, x)_H$. To avoid writing x on the right side, it is more convenient to identify H with its antidual space $(H^*)'$, where x now corresponds to the anti-linear functional $y \mapsto (x, y)_H$. The concept of antidual space can be extended to any Banach space X. Then the antidual $(X^*)'$ is the space of all anti-linear continuous functionals $f : X \to \mathbb{C}$ and we write $f \in (X^*)'$, $f(x) = \langle f, x\rangle$. In the situations that we will consider, namely real Hilbert space, there is no difference between antidual and dual space. Therefore, from now on we will drop the extra prime in the notation and just write X^*. However, we need to note that in the case of general Banach space one should be careful since there is a slight difference between dual and antidual space.

Let X and Y be normed spaces and $A \in \mathcal{B}(X, Y)$. To every functional $f' \in Y^*$, we can associate a new functional $g' \in X^*$, defined by $g'(x) := f'(Ax)$. By means of this, we define a mapping $f' \mapsto g' = f'A$ that is called the (Hermitian) conjugate operator and denoted by A':

$$A'f'(x) \equiv \left\langle A'f', x\right\rangle_X = \left\langle f', Ax\right\rangle_Y .$$

One could see this definition as a trade of the dual pairing for the inner product in the definition of an adjoint operator

$$\langle Ah_1, h_2\rangle_{H_1} = \langle h_1, A^*h_2\rangle_{H_2} ,$$

where H_1, H_2 are Hilbert spaces and $A \in \mathcal{B}(H_1, H_2)$. It is obvious that $A' \in \mathcal{B}(Y^*, X^*)$ and $\|A\| = \|A'\|$. In a special case, when $A \in \mathcal{B}(H)$, where H is a Hilbert space, the relation between conjugate operator A' and adjoint operator A^* is given by

$$A' \langle \cdot, g\rangle = \langle \cdot, A^*g\rangle , \quad \forall g \in H.$$

Let X also be a reflexive Banach space ($X = X^*{*}$), then from [123, Theorem 17.1] we get the following result

$$A \text{ is injective} \Leftrightarrow A \text{ is dense in } X'. \tag{A.1.1}$$

Recall that every Hilbert space is reflexive, because of Riesz' theorem.

A.2. Duality pairing and the inner product

Following [123], we will present the justification for the formula (3.1.2), i.e. that duality pairing is compatible with the inner product on the pivot space. This means that duality pairing $\langle \cdot, \cdot \rangle_{V^*,V}$ can be seen as a continuous extension of the inner product on H. More precisely, that every element $h \in H$ can be seen as a linear continuous functional on V^*. Since we want to interpret elements from H as elements from V^*, we consider embedding i' of H into V^*, i.e. we identify elements in H with elements in V^* by i' and we consider $(i'\cdot, \cdot)_H$. We have that Im $i' = i'H$ is dense in V^* and we will show that $(\cdot, \cdot)_H$ i.e. $(i'h, v)_H$ for $h \in H$, $v \in V$ is a linear continuous functional on Im $i' \times V$, which would imply the unique continuous extension on the $V^* \times V$. Linearity is clear and to prove the continuity, we use the continuity of i which yields

$$\|ix\|_H \leq C\|x\|_V, \quad x \in V.$$

Using the equivalent norm, we can re-norm V such that we obtain

$$\|ix\|_H \leq \|x\|_V, \quad \forall x \in V, \tag{A.2.1}$$

which implies $\|i\| \leq 1$. Since the (anti)dual operator has the same norm i.e. $\|i'\| = \|i\| \leq 1$, we get

$$\|i'h\|_{V^*} \leq \|h\|_H, \quad \forall h \in H. \tag{A.2.2}$$

Combining (A.2.1) and (A.2.2), we obtain

$$\|i'ix\|_{V^*} \leq \|ix\|_H \leq \|x\|_V, \quad \forall x \in V$$

which by omitting i and i' yields

$$\|x\|_{V^*} \leq \|x\|_H \leq \|x\|_V, \quad \forall x \in V. \tag{A.2.3}$$

Utilizing the definition of i', we have $\langle i'h, x \rangle_V = (h, ix)_H$, which together with (A.2.3) gives us

$$|(h, ix)_H| = |\left\langle i'h, x \right\rangle| \leq \|i'h\|_{V^*}\|x\|_V \leq \|h\|_H\|x\|_V.$$

Hence every functional $\langle x', \cdot \rangle_V$ on the unit ball in V can be uniformly approximated by the scalar product $(i'h, \cdot)_H = (h, i\cdot)_H$ i.e.

$$\left\langle x', x \right\rangle_V = \lim_{i'h \to \infty} (h, ix)_H, \quad \forall x \in V.$$

The last formula tells us that we can consider functionals from V^* utilizing the continuous extension of $(\cdot, \cdot)_H$ on $V^* \times V$.

A.3. Doob–Dynkin lemma

The Doob-Dynkin lemma is often exploited in the UQ community to explain that if a random coefficient depends on a finite number of RVs, then so does the solution. The following results can be found in [69, Apendix A] and [94, Ch. 4].

Lemma A.3.1. (Doob–Dynkin). Let $(\Omega, \mathcal{F})$ and $(\Theta, \mathcal{A})$ denote measure spaces and let $X : \Omega \to \Theta$ be measurable. Furthermore, let $\sigma(X) := \{X^{-1}(A) : A \in \mathcal{A}\}$ be the σ-algebra generated by X. Then, a function $Y : \Omega \to \mathbb{R}$ is $\sigma(X)$-measurable if and only if there exists a function $g : \Theta \to \mathbb{R}$ such that $Y = g(X)$.

Corollary A.3.2. Let $(\Omega, \mathcal{F})$ be a measure space. If $X, Y : \Omega \to \mathbb{R}$ are two given measurable functions, then Y is $\sigma(X)$-measurable if and only if there exists a Borel-measurable function $g : \mathbb{R} \to \mathbb{R}$ such that $Y = g(X)$.

Proof. The proof follows directly from the previous lemma, by setting $\Theta := \mathbb{R}$ and $\mathcal{A} := \mathcal{B}(\mathbb{R})$. □

To understand how this is applied for RPDEs we recall the notion of a conditional expectation given a σ-algebra.

Definition A.3.3. Let $X \in L^2(\Omega, \mathcal{F}, H)$, where H is a separable Hilbert space. If $\mathcal{A}$ is a sub σ-algebra of $\mathcal{F}$, *the conditional expectation of X given $\mathcal{A}$*, denoted $\mathbb{E}[X|\mathcal{A}]$, is defined as $\mathbb{E}[X|\mathcal{A}] := PX$, where P is the orthogonal projection from $L^2(\Omega, \mathcal{F}, H)$ to $L^2(\Omega, \mathcal{A}, H)$.

Note that according to the definition, the conditional expectation $\mathbb{E}[X|\mathcal{A}]$ is an H-valued $\mathcal{A}$-measurable random variable. In particular, we are interested in the case when $\mathcal{A} := \sigma(Y)$, where Y is a second order RV. In most cases Y will be a random coefficient represented by the truncated KL expansion, i.e., it will depend on a finite number of random variables. Since $\mathbb{E}[X|\sigma(Y)]$ is $\sigma(Y)$-measurable, by the Doob-Dynkin Lemma A.3.1 it follows that $\mathbb{E}[X|\sigma(Y)]$ is a function of Y, i.e. a solution of a PDE also depends on a finite number of RVs.

A.4. Kolmogorov test

The Kolmogorov test, also known as the Kolmogorov–Chentsov Theorem, is an important result on the existence of regular modifications of a stochastic process. We first state the standard result that is proved for example in [37, Theorem 3.3].

Theorem A.4.1. Let $X(t)$, $t \in [0, T]$ be a stochastic process with values in a separable Banach space E, such that, for some positive constants $C > 0$, $\varepsilon > 0$, $\delta > 0$ and all $t, s \in [0, T]$,

$$\mathbb{E}\|X(t) - X(s)\|^\delta \leq C|t - s|^{1+\varepsilon}.$$

Then there exists a version of X with $\mathbb{P}$-almost all trajectories being Hölder continuous functions with an arbitrary exponent smaller than ε/δ. In particular, X has a continuous version.

There exists a generalization of the Kolmogorov test for processes with values in some function spaces [37, Theorem 3.4]. This result can be extended in many ways, for example to random fields on cubes, random fields on spheres, random fields on metric spaces etc. For a review on the literature we refer the reader to [6, 104]. In [6, Theorem 3.5] the authors extend these results on random fields on manifolds and they also prove the sample differentiability under suitable further assumptions on the random fields. These results can be exploited to improve the order of convergence of numerical methods.

The Kolmogorov test can be specially adapted to Gaussian measures, as presented in [72, Theorem 3.17]. Combining this result with Fernique's theorem, we deduce that we can apply the Kolmogorov test to any Gaussian Banach-space valued process, cf. [72, Proposition 3.18]. As a useful consequence of this result we obtain a criterion for the Hölder regularity of $f := \sum_k \eta_k f_k$, where $\eta_k \sim \mathcal{N}(0,1)$ are i.i.d. and $\{f_k\}_k$ are Lipschitz functions.

Lemma A.4.2. [72, Corollary 3.22] Let $\{\eta_k\}_{k\geq 0}$ be countably many i.i.d. standard Gaussian RVs. Moreover, let $\{f_k\}_{k\geq 0} \subset \mathrm{Lip}(G)$ where the domain $G\mathbb{R}^d$ is sufficiently regular for Kolmogorov's continuity theorem to hold. Suppose there is some $\delta \in (0,2)$ such that

$$S_1^2 = \sum_{k\in I} \|f_k\|_{L^\infty}^2 < \infty \quad \text{and} \quad S_2^2 = \sum_{k\in I} \|f_k\|_{L^\infty}^{2-\delta} \mathrm{Lip}(f_k)^\delta < \infty,$$

and define $f := \sum_k \eta_k f_k$.Then f is almost surely bounded and Hölder continuous for every Hölder exponent smaller than $\delta/2$.

Summary

The aim of this work is to merge the fields of uncertainty quantification and surface partial differential equations. We present the analysis and numerical analysis of advection-diffusion equations with random coefficients on moving hypersurfaces. First, we develop an appropriate setting and formulation of the random equation on evolving hypersurfaces. This consists of defining a weak and a strong material derivative, which account for the spatial movement. Then we define the solution space for these kind of equations, which is the Bochner-type space of random functions defined on a moving domain. These results are based on the general framework results presented in [4]. We consider two cases: uniformly bounded and log-normal distributions of the diffusion coefficient. For both cases we prove the well-posedness of the considered problem. In the case when the coefficient is uniformly bounded from above and below, the proof relies on the Banach-Nečas-Babuška theorem. If the coefficient has log-normal distribution, this approach is not possible since the integration over the probability space would lead to an ill-posed problem. Instead, we consider the path-wise approach and in addition we prove the measurability of the solution and boundedness of its L^p-norm, $1 \leq p \leq \infty$ w.r.t. the probability measure. In these proofs we particularly utilize results concerning tensor spaces and the Karhunen-Loève expansion. For this reason we specifically clarify the notions of these concepts.

Next, we introduce and analyse a surface finite element discretization of advection-diffusion equations with uniformly bounded random coefficients on evolving hypersurfaces. After proving unique solvability of the resulting semi-discrete problem, we prove optimal error bounds for the semi-discrete solution and Monte-Carlo samplings of its expectation in appropriate Bochner spaces. Our theoretical findings are illustrated by numerical experiments in two and three space dimensions.

We conclude the thesis by providing an outlook for further development. Namely, we consider what happens when the velocity of the evolution is random. We show that this leads to a PDE on a random non-cylindrical domain. Under precisely stated assumptions concerning the velocity field and its associated flow, we prove the well-posedness of the heat equation on a random flat domain that changes in time.

Zusammenfassung

Diese Arbeit behandelt die Schnittstelle von zwei mathematischen Gebieten: Uncertainty Quantification und Surface Partial Differential Equations. Dazu betrachten wir Advektions-Diffusions-Gleichungen mit zufälligen Koeffizienten auf zeitabhängigen, sich bewegenden Hyperflächen und präsentieren sowohl theoretische als auch numerische Resultate. Zunächst schaffen wir die nötigen Grundlagen und entwickeln eine Formulierung von zufälligen Gleichungen auf sich bewegenden Hyperflächen. Dafür definieren wir eine schwache und starke Material Derivative, die räumliche Zeitabhängigkeit berücksichtigt, und einen Lösungsraum für die betrachteten Gleichungen, eine Art Bochner-Raum bestehend aus zufälligen Funktionen auf sich bewegenden Gebieten. Diese Herleitung basiert auf allgemeinen Resultaten aus [4]. Wir unterscheiden anschließend zwei Fälle: gleichmäßig beschränkte und log-normalverteilte Diffusionskoeffizienten. Für beide Fälle zeigen wir, dass das Problem korrekt gestellt ist. Für den Fall, dass der Diffusionskoeffizient gleichmäßig von oben und unten beschränkt ist, verwenden wir das Banach-Babuška-Theorem. Im Falle einer Log-Normalverteilung wählen wir einen anderen Ansatz, da die Integration über den Wahrscheinlichkeitsraum zu einem schlecht gestellten Problem führen würde. Wir betrachten das Problem daher pfadweise und zeigen die Messbarkeit der Lösung und die Beschränktheit der L^p-Norm für $1 \leq p \leq \infty$ bezüglich des Wahrscheinlichkeitsmaßes. Da die Beweise insbesondere Resultate zu Tensor-Räumen und zur Karhunen-Loève-Expansion verwenden, erläutern wir die entsprechenden Konzepte und Notationen.

Anschließend führen wir die Finite-Elemente-Diskretisierung für Oberflächen ein und analysieren sie für Advektions-Diffusions-Gleichungen mit gleichmäßig beschränkten zufälligen Koeffizienten auf sich bewegenden Hyperflächen. Nachdem wir die eindeutige Lösbarkeit des resultierenden semi-diskreten Problems bewiesen haben, leiten wir optimale Fehlerschranken für die semi-diskrete Lösung und für Monte-Carlo Schätzungen des Erwartungswertes in geeigneten Bochner-Räumen her. Die theoretischen Ergebnisse werden durch zwei- und dreidimensionale numerische Experimente illustriert.

Zum Abschluss dieser Arbeit geben wir einen Ausblick für zukünftige Weiterentwicklung. Dazu betrachten wir, welche Auswirkungen es hat, wenn die Geschwindigkeit der räumlichen Bewegung der Hyperflächen zufällig ist. Wir zeigen, dass dies zu partiellen Differentialgleichungen auf zufälligen nicht-zylindrischen Gebieten führt. Unter Annahmen an das Geschwindigkeitsfeld und den resultierenden Fluss können wir zudem beweisen, dass die Wärmeleitungsgleichung auf einem sich bewegenden, zufälligen, ebenen Gebiet korrekt gestellt ist.

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