Stability of Arakelov bundles over arithmetic curves and Bridgeland stability conditions on holomorphic triples

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Introduction

In algebraic geometry, moduli spaces naturally arise as spaces where each closed point corresponds to an equivalence class of certain geometric objects. In general, the moduli space of all such objects is too big. In order to get a finite-type scheme parametrizing these objects, we restrict ourselves to *stable* objects.

In this thesis we are particularly interested in stability in two different contexts:

- Part I - Arakelov bundles over arithmetic curves.
- Part II - Bridgeland stability conditions on holomorphic triples over curves.

Arakelov bundles over arithmetic curves

Arakelov geometry is a theory to study varieties over rings of integers of number fields by putting smooth hermitian metrics at infinity. Roughly, one can interpret Arakelov theory as a dictionary to translate (projective) algebraic geometry concepts into number theory and complex analysis. This identification is highly nontrivial and sometimes it might not be even possible. Part I of this thesis is inspired by this philosophy.

Although S. Arakelov developed the theory for arithmetic surfaces (see [Ara74, Ara75]), our setting is on arithmetic curves. Our main goal is to provide a notion of stability for Arakelov principal bundles over arithmetic curves which extends semistability for Arakelov vector bundles and agrees with the analogue for Arakelov group schemes. We provide a gentle introduction to the formalism below.

Let K be a number field and denote by $\mathcal{O}_K$ its ring of integers. As $\mathcal{O}_K$ is a Dedekind domain, $\mathrm{Spec}(\mathcal{O}_K)$ is a smooth affine algebraic curve. Let $\eta = (0)$ denote the generic point.

Each element $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K) \setminus \{\eta\}$ defines a valuation $\nu_{\mathfrak{p}} \colon K \to \mathbb{R}$ and its equivalence class gives a so-called finite place of K. The finite places correspond to nonarchimedean ($\mathfrak{p}$-adic) valuations. On the other hand, the *infinite places* of K are the archimedean valuations of K. These archimedean valuations correspond to complex embeddings $\iota \colon K \longrightarrow \mathbb{C}$ up to complex conjugation, we denote this (finite) set by X_∞. The *arithmetic curve* associated to K is defined as the disjoint union

$$X \coloneqq \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty.$$

It is referred to as a curve due to the classical analogy between number fields and function fields. Moreover, the well-known product formula

$$\prod_{\nu \in X} |a|_\nu = 1 \text{ for all } a \in K \setminus \{0\}$$

with ν running over all suitably normalized valuations of K, leads to think of X as a projective algebraic curve, i.e. a compactification of the affine curve $\mathrm{Spec}(\mathcal{O}_K)$ by adding the finite set of points X_∞ at infinity.

Now, an *Arakelov vector bundle*

$$\bar{E} = (E, \{\langle \cdot, \cdot \rangle_{E,\nu}\}_{\nu \in X_\infty})$$

on X consists of the following data:

1. a locally free $\mathcal{O}_K$-module E of finite rank,
2. a family of scalar products $\{\langle \cdot, \cdot \rangle_{E,\nu}\}_{\nu \in X_\infty}$ defined on the K_ν-vector space

   $$E_\nu := E \otimes K_\nu,$$

   where $K_\nu \cong \mathbb{R}$ or $\mathbb{C}$ denotes the completion of K with respect to ν.

There exists a notion of *degree* of Arakelov vector bundles which leads to a definition of slope semistability. Moreover, as in classical algebraic geometry, every unstable Arakelov vector bundle has a unique Grayson-Stuhler filtration [38] analogous to the Harder-Narasimhan filtration.

In [16] and [14] J.B. Bost raised the question whether the tensor product of two semistable Arakelov bundles is again semistable. There are numerous proofs of the analogous fact in algebraic geometry. In [49], Narasimhan and Seshadri related the theory of polystable vector bundles on a compact Riemann surface to the theory of unitary representations of the fundamental group of that Riemann surface and they obtained that the tensor product of two semistable vector bundles is semistable as a corollary. In [40] (and [45]) Hartshorne related semistability of a vector bundle to its nefness.

In Arakelov geometry, Y. André [2] provided examples of nef Arakelov bundles whose tensor product is not nef, meaning that Hartshorne's approach does not work in the Arakelov world. However, an affirmative answer was given for low ranks by de Shalit and Parzanchevski in [29] and by Chen in [23]. Bost also observed that different choices of the metric on the tensor product might lead to better results [15].

On the other hand, Ramanan and Ramanathan [52] put the tensor product theorem for vector bundles in terms of the behavior of semistability of principal bundles under extension of the structure group.

In view of the numerous occurrences of tensor product theorems in various areas of algebraic and arithmetic geometry, this thesis provides the building stone to tackle the problem in terms of principal bundles.

The concept of an Arakelov principal bundle over an arithmetic scheme appeared explicitly in the work [58] of A. Chambert-Loir and Y. Tschinkel. It is closely related to the notion of decorated principal bundle by A. Schmitt [54].

Let $G \subset \mathrm{GL}(n, \mathcal{O}_K)$ be a reductive connected affine algebraic group. An *Arakelov principal G-bundle*

$$\bar{\mathcal{X}} := (\mathcal{X}, \{\sigma_\nu\}_{\nu \in X_\infty})$$

on X consists of the following data:

1. a principal G-bundle $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$,
2. reductions σ_ν of the structure group of $\mathcal{X}_\nu$ to a maximal compact subgroup H_ν of $G(K_\nu) := G \otimes_K K_\nu$, i.e. $\mathrm{Spec}(K_\nu) \xrightarrow{\sigma_\nu} \mathcal{X}_\nu / H_\nu$.

We say that an Arakelov principal G-bundle $\bar{\mathcal{X}}$ is *semistable* if for all reductions

$$\bar{\mathcal{X}}_P := (\mathcal{X}_P, \{\sigma_{P,\nu}\}_{\nu \in X_\infty})$$

to parabolic subgroups $P \subset G$ the following inequality holds

$$\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \leq 0.$$

The non-negative real number

$$\mathrm{ideg}(\bar{\mathcal{X}}) := \max\{\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \mid \bar{\mathcal{X}}_P \text{ reduction to parabolic } P\}$$

is called the *Arakelov degree of instability* of $\bar{\mathcal{X}}$. A *canonical Arakelov reduction* is a reduction $\bar{\mathcal{X}}_P$ to a parabolic subgroup P such that $\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) = \mathrm{ideg}(\bar{\mathcal{X}})$.

Let G_0 be a split reductive group scheme over $\mathcal{O}_K$ and let $\mathcal{X}$ be a principal G_0-bundle on $\mathrm{Spec}(\mathcal{O}_K)$. We define a group scheme

$$\mathrm{Aut}_{G_0}(\mathcal{X}) := \mathcal{X} \times^{G_0, \mathrm{conj}} G_0$$

where G_0 acts by conjugation on G_0. It is well known that parabolic subgroups of $\mathrm{Aut}_{G_0}(\mathcal{X})$ are the same as reductions $\mathcal{X}_{P_0}$ of $\mathcal{X}$ to P_0 [30, Exposé XXVI, Lemme 3.20].

Moreover, given $\nu \in X_\infty$, consider a maximal compact subgroup $K_{0,\nu} \subset G_{0,\nu}$. We show that a section of $\mathcal{X}_\nu / H_{0,\nu}$ is equivalent to giving a maximal compact subgroup $H_\nu \subset \mathcal{G}_\nu$. This shows that an Arakelov principal G_0-bundle $\bar{\mathcal{X}}$ is equivalent to giving the Arakelov group scheme

$$\bar{\mathcal{G}} := (\mathrm{Aut}_{G_0}(\mathcal{X}), \{H_\nu\}_{\nu \in X_\infty}).$$

Then, the key fact in Part I is the following result.

Proposition 0.0.1. *Let G_0 be a split reductive group scheme over $\mathcal{O}_K$ and let $\bar{\mathcal{X}}$ be an Arakelov principal G_0-bundle. Then the canonical parabolic subgroup of the Arakelov group scheme $\bar{\mathcal{G}}$ is equivalent to giving a canonical reduction for $\bar{\mathcal{X}}$.*

Proposition 0.0.1 allows then to adapt the constructions of Harder and Stuhler [39] to our context and prove the main theorem of Part I.

Theorem 0.0.2 (Main theorem)**.** *Every Arakelov principal G-bundle $\bar{\mathcal{X}}$ has a unique Arakelov canonical reduction $\bar{\mathcal{X}}_P$.*

Furthermore, when $G = \mathrm{GL}(n)$ the Arakelov canonical reduction $\bar{\mathcal{X}}_P$ corresponds to the Grayson-Stuhler filtration of the Arakelov vector bundle associated to the Arakelov principal G-bundle $\bar{\mathcal{X}}$.

A natural next step might be to study the question how semistability behaves under extension of the structure group. It could be interesting to investigate how to adapt the techniques of Balaji and Parameswaran [5] to study the behavior of semistability of decorated principal bundles [54] under extension of the structure group in a general context. Furthermore, it would also be interesting to see applications of Harder-Stuhler's techniques to automorphism groups of indefinite lattices.

Bridgeland stability conditions on holomorphic triples over curves

This is joint work with A. Rincón Hidalgo (Freie Universität Berlin) and A. Rüffer (University of Limerick) [47].

Stability conditions on triangulated categories were introduced by Bridgeland in [19] as a mathematical formalization of Douglas' work on Π-stability of D-branes for super conformal field theories (SCFT) in [31], [32].

Given a triangulated category $\mathcal{D}$, a Bridgeland stability condition on $\mathcal{D}$ consists of a bounded t-structure on $\mathcal{D}$ and a stability function on its heart with the Harder-Narasimhan property. Such stability condition can be viewed as an abstraction of classical slope-stability for vector bundles on a smooth projective curve. In [19], Bridgeland proves that the set of stability conditions has a natural topology and is a complex manifold, possibly infinite dimensional. We are particularly interested in the finite dimensional submanifold of numerical stability conditions, denoted by $\mathrm{Stab}(\mathcal{D})$. A key fact is that the stability manifold $\mathrm{Stab}(\mathcal{D})$ carries naturally a right action of $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$, the universal covering of $\mathrm{GL}^+(2,\mathbb{R})$, and a commuting left action by isometries of the group of exact autoequivalences of $\mathcal{D}$. In addition, we will require our stability conditions to satisfy the support property, which ensures convenient deformation properties.

The stability manifolds of smooth projective curves were determined in [19], [46], [51]. In the case $\mathcal{D} = D^b(C)$, i.e. the bounded derived category of coherent sheaves on a curve C of genus $g \geq 1$, the action of $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ is free and transitive, which means that the stability manifold $\mathrm{Stab}(C)$ can be thought as $\tilde{\mathrm{GL}}^+(2,\mathbb{R})$ itself. Some stability conditions have been constructed on projective surfaces as well as a connected component of the stability manifold for K3 surfaces [3], [20]. Macrì gives a method for constructing stability conditions from Ext-exceptional collections in

[46]. Our example will probably be the first completely described stability manifolds for a triangulated category with (finite) homological dimension greater than 1.

We study $\mathrm{Stab}(\mathcal{T}_C)$ the stability manifold for the bounded derived category of $\mathrm{TCoh}(C)$, the abelian category of holomorphic triples on curves of genus $g \geq 1$, i.e. triples $T = (E_1, E_2, \Phi)$ where $E_1, E_2 \in \mathrm{Coh}(C)$ and $\Phi\colon E_1 \to E_2$ is a morphism between them. Holomorphic triples were first introduced by García-Prada et al. in [35] and [18] for vector bundles over a smooth projective curve of genus g. It was shown in [35] and [18] that projective moduli spaces for holomorphic triples exist. Later, a precise construction via GIT of the moduli spaces was given by A. Schmitt in [55]. A variation of moduli with respect to the parameter α is found in [17]. Moreover, after the work of C. Daly in [28], we know that the submanifold of stability conditions corresponding to the heart $\mathrm{TCoh}(C)$ is isomorphic to $\mathrm{Stab}(C)^\circ \times \mathrm{Stab}^\circ(C)$, with $\mathrm{Stab}^\circ(C)$ the connected component of stability conditions corresponding to the heart $\mathrm{Coh}(C)$. In her work, Daly was implicitly using the description $\mathcal{T}_C$ as semiorthogonal decomposition of $D^b(C)$.

Recently, Bayer et al. in [6] introduced a very general method to induce Bridgeland stability conditions on semiorthogonal decompositions. In particular, they proved the existence of Bridgeland stability conditions on the Kuznetsov component of the derived category of many Fano 3-folds and extended it to a Bridgeland stability condition on the whole cubic fourfold X using J. Collins and A. Polishchuk's results in [25].

In our case, we use the complete description of $\mathrm{Stab}(C)$ to construct stability conditions on $\mathcal{T}_C$. First of all, we follow A. Bondal and Kapranov's results in [12] to show the precise structure of $\mathcal{T}_C$ as semiorthogonal decomposition of $D^b(C)$:

i) $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$,

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"] & E_1 \arrow[r] \arrow[d] & 0 \arrow[d] \\ E_2 \arrow[r] & E_2 \arrow[r] & 0 \arrow[r] & E_2[1]. \end{tikzcd}$$

ii) $\mathcal{T}_C = \langle \mathcal{C}_3, \mathcal{C}_1 \rangle$,

$$\begin{tikzcd} C(\varphi)[-1] \arrow[r] \arrow[d] & E_1 \arrow[r] \arrow[d, "\varphi"] & E_2 \arrow[r] \arrow[d, "\mathrm{id}"] & C(\varphi) \arrow[d] \\ 0 \arrow[r] & E_2 \arrow[r] & E_2 \arrow[r] & 0. \end{tikzcd}$$

iii) $\mathcal{T}_C = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$,

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\mathrm{id}"] & E_1 \arrow[r] \arrow[d, "\varphi"] & 0 \arrow[r] \arrow[d] & E_1[1] \arrow[d, "\mathrm{id}[1]"] \\ E_1 \arrow[r] & E_2 \arrow[r] & C(\varphi) \arrow[r] & E_1[1]. \end{tikzcd}$$

Moreover, following the BK-constructions we obtain a Serre functor $S_{\mathcal{T}_C}$ on $\mathcal{T}_C$ which at the level of objects is given by

$$S_{\mathcal{T}_C}(E_1^\bullet \xrightarrow{\varphi} E_2^\bullet) = E_2^\bullet \otimes \omega_C[1] \to C(\varphi) \otimes \omega_C[1]. \tag{0.0.1}$$

Moreover, $S_{\mathcal{T}_C}$ alternates between the above semiorthogonal decompositions:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, bend left] \arrow[rr, bend left=40, "S_{\mathcal{T}_C}"] & \mathcal{C}_2 \arrow[l, bend left] \arrow[r, bend left] & \mathcal{C}_3 \arrow[l, bend left] \arrow[ll, bend left=40, "S_{\mathcal{T}_C}^{-1}"] \end{tikzcd}$$

In particular, if $C = E$ is an elliptic curve, then $S^3_{\mathcal{T}_E} = [4]$. This implies that $\mathcal{T}_E$ is a fractional Calabi-Yau category of fractional dimension $4/3$.

Next, we glue hearts from the smaller subcategories into hearts of $\mathcal{T}_C$ but before that we explore the relation between the classical construction by recollement of A. Beilinson et al. in [10] and [25].

Given a semiorthogonal decomposition $(\mathcal{D}_1, \mathcal{D}_2)$ of a triangulated category $\mathcal{D}$, a stability condition $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$ is glued from $\sigma_1 = (Z_1, \mathcal{A}_1)$ on $\mathcal{D}_1$ and $\sigma_2 = (Z_2, \mathcal{A}_2)$ on $\mathcal{D}_2$ if and only if $Z_i = Z|_{\mathcal{D}_i}$ for $i = 1, 2$,

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}_1, \mathcal{A}_2) = 0 \tag{0.0.2}$$

and $\mathcal{A}_i \subset \mathcal{A}$ for $i = 1, 2$.

We provide an explicit proof that the hearts obtained by CP-gluing and by recollement are the same when the CP-gluing condition (0.0.2) is fulfilled. After this, we explore CP-constructions of stability conditions and we prove the following result.

Lemma 0.0.3 (Jealousy Lemma)**.** *Let $\mathcal{A} \subset \mathcal{T}_C$ be a heart constructed by recollement of hearts $\mathcal{A}_i, \mathcal{A}_j \subset \mathcal{C}$, which do not satisfy CP-gluing conditions. Then, $\mathcal{A}$ does not accept a stability function defined on $K(\mathcal{A})$, i.e. $Z(\mathcal{A}) \not\subset \overline{\mathbb{H}}$ for every $Z\colon K(\mathcal{A}) \to \mathbb{C}$.*

This result highlights the need of satisfying CP-gluing condition to ensure the existence of a stability function, which leads to a stability condition provided that it satisfies the Harder-Narasimhan and support properties.

The last step is to show these remaining properties. We show in general that:

Proposition 0.0.4. *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose that σ is obtained from CP-gluing of pre-stability conditions σ_i on $\mathcal{D}_i$ for $i = 1, 2$. If σ is a pre-stability condition on $\mathcal{D}$, then there exists a quadratic form Q such that*

a) for every σ-semistable object $E \in \mathcal{P}(\phi)$, we have $Q(v(E)) \geq 0$.

b) Q is negative semi-definite with respect to the kernel of Z.

Moreover, there exists a quadratic form Q' such that Q' is negative definite with respect to the kernel of Z.

Although we can not guarantee that σ will satisfy the support property with respect to Q' for an arbitrary triangulated category $\mathcal{D}$, Proposition 0.0.4 shows that the CP-gluing procedure will be quite close to give stability conditions satisfying support property independently on whether the former (pre-)stability conditions satisfy it or not.

Given a candidate pre-stability condition σ on $\mathcal{T}_C$, obtained from CP-gluing of pre-stability conditions σ_i on $\mathcal{C}_i$ for $i = 1, 2$, we can directly ensure the Harder-Narasimhan property only under certain conditions:

1. When there exists a real number $a \in (0, 1)$ such that $(\sigma_1, \sigma_2) \in S(a)$, where
$$S(a) \cong \left\{(T_1, f_1), (T_2, f_2) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})\colon f_1(0) \geq f_2(0) \text{ and } f_1(a) \geq f_2(a)\right\}.$$
2. For σ_1 and σ_2 being a discrete stability condition on $\mathcal{C}_i$ for $i = 1, 2$ respectively.

We note that condition 1. is non-trivial and it consists of CP-glued pre-stability conditions that behave well under the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action. Roughly, it states that there exists some $a \in (0, 1)$ such that the rotation by $a\pi$ of σ is again a CP-glued pre-stability condition.

On one hand, we show that the CP-gluing of the standard slope stability condition on $\mathcal{C}_1$ and on $\mathcal{C}_2$ (denoted by σ_0) belongs to $S(a)$ for every a and on the other hand we provide explicit examples of CP-glued conditions which are not CP-glued after acting by some of these rotations.

At this point, we show that we obtain the following result.

Theorem 0.0.5. *σ_0 satisfies the support property with respect to Q'. Moreover, since $\sigma_0 \in S(a)$ for every $a \in (0, 1)$ all elements in the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit of σ_0 are stability conditions on $\mathcal{T}_C$.*

Finally, all the evidence shown leads us to formulate the following conjecture.

Conjecture 0.0.6. *All gluing of stability conditions $\sigma_i = (Z_i, \mathcal{P}_i)$ on $\mathcal{C}_i$ for $i = 1, 2, 3$ give stability conditions on $\mathcal{T}_C$.*

However, we have proven in [47] the following result which implies that we can describe the whole stability manifold.

Theorem 0.0.7. *Let i_* (resp. j_*, resp. l_*) denote the 3 possible inclusions of $\mathcal{C}$ in $\mathcal{T}_C$. We define the following subsets of* $\mathrm{Stab}(\mathcal{T}_C)$*:*

$$\Theta_{12} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)),\, i_*(\mathcal{O}_C),\, j_*(\mathbb{C}(x)) \text{ and } j_*(\mathcal{O}_C) \text{ are } \sigma\text{-stable}\}$$
$$\Theta_{31} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)),\, i_*(\mathcal{O}_C),\, l_*(\mathbb{C}(x)) \text{ and } l_*(\mathcal{O}_C) \text{ are } \sigma\text{-stable}\}$$
$$\Theta_{23} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid j_*(\mathbb{C}(x)),\, j_*(\mathcal{O}_C),\, l_*(\mathbb{C}(x)) \text{ and } l_*(\mathcal{O}_C) \text{ are } \sigma\text{-stable}\}$$

then,
$$\mathrm{Stab}(\mathcal{T}_C) = \Theta_{12} \cup \Theta_{23} \cup \Theta_{13}.$$

Moreover, we have given a precise description of all the stability conditions in $\mathrm{Stab}(\mathcal{T}_C)$ as either constructed by CP-gluing or tilting $\mathrm{TCoh}(C)$ by a stability function which fails to be a Bridgeland stability condition (as in [20]). At this point, we follow Bridgeland's construction for K3-surfaces as well as Bayer-Macrì's [7] for the local projective plane to extend the Harder-Narasimhan condition to the whole manifold via the support property.

Organization of the thesis

Finally, we outline the contents of the thesis.

Part I - *Arakelov bundles over arithmetic curves*. In Chapter I we compile the basics about Arakelov geometry that we briefly described above. We define Arakelov vector bundles on arithmetic curves and we explore the relationship of nefness and the tensor product problem as evidence of the pathologies of the Arakelov setting. Chapter II reproduces Behrend's construction of complementary polyhedra for stability of group schemes and the later adaptation to Arakelov geometry by Harder and Stuhler. The main results of this part are contained in chapter III, where we define Arakelov principal bundles. We provide a notion of stability and prove Theorem 0.0.2 i.e. we show that our definition agrees with all the previous constructions.

Part II - *Bridgeland stability conditions on holomorphic triples over curves*. Chapter IV gathers basic facts about triangulated and derived categories. In Chapter V we introduce the general definition of Bridgeland stability conditions and explore few examples of constructions of stability conditions that are interesting for our constructions. Finally, chapter VI contains all our constructions of Bridgeland stability conditions on holomorphic triples over curves. First we describe the bounded derived category of holomorphic triples on curves $\mathcal{T}_C$ as semiorthogonal decomposition of the bounded derived category of coherent sheaves on the curve and we show the existence of the Serre functor $S_{\mathcal{T}_C}$ (0.0.1). Next, we compare recollement and CP-gluing to construct hearts via semiorthogonal decompositions, by gluing hearts in the smaller categories and we compute the necessary numerical conditions for triples. Finally, we construct stability conditions on $\mathcal{T}_C$ by gluing stability conditions from $\mathrm{Stab}(C)$. We study the Harder-Narasimhan and the support properties of glued stability conditions in general and for triples. The very last section shows the sketch of how we finally come up with the full description of the stability manifold $\mathrm{Stab}(\mathcal{T}_C)$.

The results contained in chapter VI will appear soon in the co-authored paper *Bridgeland stability conditions on holomorphic triples over curves* as a preprint on the Mathematics ArXiv, [47]. The sections reproduced here are those that existed in similar form in my research before the paper was finished. The proofs in the final section of that chapter have not been included as they will be presented in the co-author's PhD theses (mainly from Theorem 0.0.7 onwards in the exposition above).

Part I

Arakelov bundles over arithmetic curves

Arakelov vector bundles on arithmetic curves

This chapter provides basic facts about arithmetic curves and Arakelov vector bundles defined on these arithmetic curves that we will use afterwards.

1.1 Arithmetic curves

Definition 1.1.1. An *absolute value* of a field K is a function

$$|\cdot|\colon K \to \mathbb{R}$$

satisfying the following properties

i) $|x| \geq 0$ for all $x \in K$ and $|x| = 0$ if and only if $x = 0$

ii) $|xy| = |x||y|$, for all $x, y \in K$

iii) $|x + y| \leq |x| + |y|$, for all $x, y \in K$.

Definition 1.1.2. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are equivalent if and only if there exists a real number $s > 0$ such that

$$|x|_1 = |x|_2^s$$

for all $x \in K$.

Remark 1.1.3. Given an absolute value $|\cdot|$ on a field K, we define the distance between two points $x, y \in K$ as

$$d(x, y) = |x - y|.$$

This distance function turns K into a topological space, where two absolute values $|\cdot|_1$ and $|\cdot|_2$ on K are equivalent if and only if they define the same topology on K. See [50, Proposition II.3.3] for the details.

Definition 1.1.4. An absolute value $|\cdot|$ on a field K, is called *nonarchimedean* if it satisfies the *strong triangle inequality*

$$|x + y| \leq \max\{|x|, |y|\}.$$

Otherwise, $|\cdot|$ is called *archimedean*.

Example 1.1.5. When $K = \mathbb{Q}$ is the field of the rational numbers, we have the usual absolute value

$$|\cdot|_\infty = |\cdot|,$$

which is archimedean, and for each prime number $p \in \mathbb{Z}$ the p-adic absolute value

$$|x|_p = 1/p^m,$$

where if $x = a/b \in \mathbb{Q}$, with $a, b \in \mathbb{Z}$, then m is the highest power extracted from a and b, i.e.

$$x = p^m a'/b'$$

with $\gcd(a'b', p) = 1$. The p-adic absolute value $|x|_p$ is nonarchimedean.

Proposition 1.1.6 (Ostrowski's theorem [50, Proposition II.3.7])**.** *Every absolute value of $\mathbb{Q}$ is equivalent to one of the absolute values $|\cdot|_\infty$ or $|\cdot|_p$ for $p \in \mathbb{Z}$ prime.*

Given a prime number $p \in \mathbb{Z}$ and a rational number $x \in \mathbb{Q}$, we denote the exponent m in the definition of $|x|_p$ as $\nu_p(x)$, so that we have

$$|x|_p = p^{-\nu_p(x)}.$$

In general, given a nonarchimedean absolute value $|\cdot|$ on a field K, we put

$$\nu(x) := -\log|x|$$

for $x \in K \backslash \{0\}$ and $\nu(0) := \infty$. In this way we obtain a function

$$\nu\colon K \to \mathbb{R} \cup \{\infty\}$$

satisfying the following properties

i) $\nu(x) = \infty$ if and only if $x = 0$

ii) $\nu(xy) = \nu(x) + \nu(y)$

iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

Definition 1.1.7. A function ν satisfying these properties is called a *valuation* of K. Two valuations ν_1 and ν_2 of K are *equivalent* if $\nu_1 = s\nu_2$ for some real number $s > 0$.

Conversely, given a valuation ν of a field K we obtain an absolute value by putting

$$|x|_\nu = q^{-\nu(x)}$$

for some fixed real number $q > 1$.

Remark 1.1.8. Note that replacing ν by an equivalent valuation $s\nu$ changes $|\cdot|$ into the equivalent absolute value $|\cdot|^s$.

Definition 1.1.9. Let $K \supset \mathbb{Q}$ be a number field. A *place* of K is a class of equivalent valuations of K. By abuse of notation, we will denote by ν a place of K, even though it refers to a representative of the equivalence class.

Let $K \supset \mathbb{Q}$ be a number field. Define $\mathcal{O}_K \subset K$ to be its ring of integers, i.e. the integral closure of $\mathbb{Z} \subset K$,

$$\mathcal{O}_K \coloneqq \{t \in K \mid p(t) = 0 \text{ for some monic } p(x) \in \mathbb{Z}[x]\}.$$

It is easy to see that $\mathcal{O}_K$ is a Dedekind domain, that is a noetherian, integrally closed domain of dimension 1. Thus, we have a smooth affine algebraic curve

$$\operatorname{Spec}(\mathcal{O}_K) \coloneqq \{\mathfrak{p} \subset \mathcal{O}_K \text{ prime ideal}\}$$

and we denote by $\eta = (0)$ the generic point.

Each element $\mathfrak{p} \neq \eta$ defines a valuation $\nu_{\mathfrak{p}} \colon K \to \mathbb{R}$ and (its equivalence class) gives a so-called *finite place* of K. They correspond to nonarchimedean ($\mathfrak{p}$-adic) valuations. The *infinite places* of K are the archimedean valuations of K. These archimedean valuations correspond to complex embeddings $\iota\colon K \longrightarrow \mathbb{C}$ of K up to complex conjugation and we denote this (finite) set by X_∞.

Remark 1.1.10. By [50, Theorem II.8.1] there are 2 sorts of infinite places:

- *Real places* are given by embeddings

$$\iota\colon K \longrightarrow \mathbb{R}\,.$$

- *Complex places* are given by pairs of complex-conjugate embeddings

$$\iota\colon K \longrightarrow \mathbb{C}\,.$$

An infinite place ν is real or complex depending whether the completion K_ν is isomorphic to $\mathbb{R}$ or to $\mathbb{C}$.

Definition 1.1.11. Given a number field K, the *arithmetic curve* associated to K is defined as the disjoint union

$$X \coloneqq \operatorname{Spec}(\mathcal{O}_K) \cup X_\infty.$$

Note that the set $X \setminus \eta$ of all places of K is in canonical bijection to the set of all valuations of K, up to equivalence of valuations. Moreover, one can think X as a finitely many points compactification of $\operatorname{Spec}(\mathcal{O}_K)$, since there is the well-known *product formula*

$$\prod_{\nu \in X} |a|_\nu = 1 \text{ for all } a \in K \setminus \{0\} \tag{1.1.1}$$

with ν running over all suitably normalized valuations of K (see [50, Proposition III.1.3]).

Remark 1.1.12. By *suitably normalized valuations of K* we mean the following:

To each prime $\mathfrak{p}$ of K we associate a canonical homomorphism $\nu_{\mathfrak{p}}\colon K^* \longrightarrow \mathbb{R}$ from the multiplicative group K^* of K.

- If $\mathfrak{p}$ is finite, $\nu_{\mathfrak{p}}$ is the $\mathfrak{p}$-adic valuation which is normalized by the condition $\nu_{\mathfrak{p}}(K^*) = \mathbb{Z}$.
- If $\mathfrak{p}$ is infinite, $\nu_{\mathfrak{p}}(x) = -\log|\iota x|$, where $\iota\colon K \longrightarrow \mathbb{C}$ is a complex embedding defining $\mathfrak{p}$.

Now, for a finite prime $\mathfrak{p}$, denote by p the prime number corresponding to the characteristic of its residue field $\kappa(\mathfrak{p})$ and put $f_{\mathfrak{p}} := [\kappa(\mathfrak{p})\colon \kappa(p)]$ and $\mathcal{N}(\mathfrak{p}) := p^{f_{\mathfrak{p}}}$. For an infinite prime $\mathfrak{p}$ put $f_{\mathfrak{p}} := [K_{\mathfrak{p}}\colon \mathbb{R}]$ and $\mathcal{N}(\mathfrak{p}) := e^{f_{\mathfrak{p}}}$. Define then the absolute value $v_{\mathfrak{p}}\colon K \longrightarrow \mathbb{R}$ by $v_{\mathfrak{p}}(x) := \mathcal{N}(\mathfrak{p})^{-\nu_{\mathfrak{p}}(x)}$ for $x \neq 0$ and $v_{\mathfrak{p}}(0) := 0$.

With these notations, the product formula says that for any $x \in K^*$, one has $v_{\mathfrak{p}}(x) = 1$ for almost all $\mathfrak{p}$, and

$$\prod_{\mathfrak{p}} v_{\mathfrak{p}}(x) = 1.$$

1.2 Arakelov vector bundles

Now we define vector bundles on arithmetic curves and introduce their notion of stability.

Definition 1.2.1. Let $X = \operatorname{Spec}(\mathcal{O}_K) \cup X_\infty$ be an arithmetic curve. An *Arakelov vector bundle*

$$\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu \in X_\infty})$$

on X consists of the data of a locally free $\mathcal{O}_K$-module E of finite rank and of a family of scalar products $\{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu \in X_\infty}$ defined on the K_ν-vector space $E_\nu := E \otimes K_\nu$, where $K_\nu \cong \mathbb{R}$ or $\mathbb{C}$ denotes the completion of K with respect to ν.

Remark 1.2.2. Note that different choices of the scalar products $\langle\cdot,\cdot\rangle_{E,\nu}$ defined on E_ν for $\nu \in X_\infty$ give rise to different Arakelov vector bundles.

Remark 1.2.3. In the definition of Arakelov vector bundles in [39] they consider certain families of norms $\|.\|_\nu$ for $\nu \in X_\infty$, which correspond to $\|s\|_\nu^2 := \langle s, s\rangle_{E,\nu}$, for every $s \in E_\nu$. We will use both notations indistinctly.

Definition 1.2.4. Let $\bar{E}$ be an Arakelov vector bundle over X. The elements of the finite pointed set

$$\Gamma(X, \bar{E}) := \{s \in E \mid \langle s, s\rangle_{E,\nu} \leq 1 \text{ for all } \nu \in X_\infty\}$$

are called *global sections* of $\bar{E}$.

Basic linear algebra operations apply to Arakelov vector bundles by defining the scalar products in the infinite places.

Subbundles. Let $\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu\in X_\infty})$ be an Arakelov vector bundle on X, a *subbundle* $\bar{F} \subsetneq \bar{E}$ is given by a subbundle $F \subsetneq E$ equipped with the restriction of $\langle\cdot,\cdot\rangle_{E,\nu}$ to F_ν, for $\nu \in X_\infty$.

Quotients. Let $\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu\in X_\infty})$ be an Arakelov vector bundle on X, and $\bar{F} \subsetneq \bar{E}$ denote a subbundle of $\bar{E}$. For $\nu \in X_\infty$, the orthogonal projections $p_\nu \colon E_\nu \to F_\nu^\perp$ provide isomorphisms $(E/F) \otimes K_\nu \to F_\nu^\perp$ which can be used to make E/F into an Arakelov vector bundle on X denoted $\bar{E}/\bar{F}$.

Direct sums. Let $\bar{E}_i = (E_i, \{\langle\cdot,\cdot\rangle_{E_i,\nu}\}_{\nu\in X_\infty})$ for $i = 1, 2$ be 2 Arakelov vector bundles on X. One define their direct sum

$$\bar{E}_1 \oplus \bar{E}_2$$

by considering the direct sum $E_1 \oplus E_2$ of the corresponding locally free $\mathcal{O}_K$-modules equipped with the scalar products

$$\langle x_1 \oplus x_2, y_1 \oplus y_2\rangle_{E_1\oplus E_2,\nu} = \langle x_1, y_1\rangle_{E_1,\nu} + \langle x_2, y_2\rangle_{E_2,\nu}$$

for $\nu \in X_\infty$, defined on $(E_1 \oplus E_2)_\nu = E_{1\nu} \oplus E_{2\nu}$.

Tensor products. Let $\bar{E}_i = (E_i, \{\langle\cdot,\cdot\rangle_{E_i,\nu}\}_{\nu\in X_\infty})$ for $i = 1, 2$ be 2 Arakelov vector bundles on X. One define their tensor product

$$\bar{E}_1 \otimes \bar{E}_2$$

by considering the tensor product $E_1 \otimes E_2$ of the corresponding locally free $\mathcal{O}_K$-modules equipped with the scalar products

$$\langle x_1 \otimes x_2, y_1 \otimes y_2\rangle_{E_1\otimes E_2,\nu} = \langle x_1, y_1\rangle_{E_1,\nu} \cdot \langle x_2, y_2\rangle_{E_2,\nu}$$

for $\nu \in X_\infty$, defined on $(E_1 \otimes E_2)_\nu = E_{1\nu} \otimes E_{2\nu}$.

Dual. Let $\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu\in X_\infty})$ be an Arakelov vector bundle on X. One define its dual $\bar{E}^*$ by considering the dual $E^* = \mathrm{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$ of the corresponding locally free $\mathcal{O}_K$-module equipped with the scalar product

$$\langle x^*, y^*\rangle_{E^*,\nu} = \overline{\langle x, y\rangle}_{E,\nu}$$

for $\nu \in X_\infty$, defined on $E_\nu^* = \mathrm{Hom}(E_\nu, K_\nu)$ and where x^* denotes the homomorphism $\langle\ , x\rangle_{E,\nu} \in \mathrm{Hom}(E_\nu, K_\nu)$.

Exterior powers. Let $\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu\in X_\infty})$ be an Arakelov vector bundle on X. One define its n-th exterior power $\bigwedge^n \bar{E}$ by considering the n-th exterior power $\bigwedge^n E$ of the corresponding locally free $\mathcal{O}_K$-module equipped with the scalar product

$$\langle x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_n\rangle_{\bigwedge^n E,\nu} = \det\left(\langle x_i, y_i\rangle_{E,\nu}\right)$$

for $\nu \in X_\infty$, defined on

$$(\bigwedge^n E)_\nu = \bigwedge^n E_\nu.$$

For the highest exterior power, $n = \mathrm{rk}(E)$, the Arakelov line bundle obtained is called the *determinant* of $\bar{E}$,

$$\det(\bar{E}) := \wedge^{\mathrm{rk}\, E}\bar{E}.$$

Base change. Let K be a number field and let X_K be the arithmetic curve associated to K. Furthermore, let $\bar{E} = (E, \{\langle\cdot,\cdot\rangle_{E,\nu}\}_{\nu\in X_{K,\infty}})$ be an Arakelov vector bundle over X_K. Now, given a finite extension L of K, let X_L be the arithmetic curve associated to L. We define

$$\bar{E}_{X_L} = (E_{\mathcal{O}_L}, \{\langle\cdot,\cdot\rangle_{E_{\mathcal{O}_L},\nu'}\}_{\nu'\in X_{L,\infty}})$$

the base change of $\bar{E}$ to X_L, as follows. Consider the base change $E_{\mathcal{O}_L} := E \otimes_{\mathcal{O}_K} \mathcal{O}_L$ of the corresponding locally free $\mathcal{O}_K$-module. To define the scalar product, note that we have a surjection

$$f\colon X_{L,\infty} \twoheadrightarrow X_{K,\infty}$$

where, given $\nu \in X_{K,\infty}$ (resp. $\nu' \in X_{L,\infty}$) with $\nu = f(\nu')$, then $d_{\nu'} := [L_{\nu'} : K_\nu]$ is either $d_{\nu'} = 1$ or $d_{\nu'} = 2$. Hence, we set

$$\langle,\rangle_{E_{\mathcal{O}_L},\nu'} := d_{\nu'}\langle,\rangle_{E,f(\nu')}$$

for $\nu' \in X_{L,\infty}$, defined on $E_{\mathcal{O}_L,\nu'}$.

Restriction of scalars. Let $\pi\colon \mathrm{Spec}(\mathcal{O}_K) \to \mathrm{Spec}(\mathbb{Z})$ be the natural morphism. The (locally free) module $E_{\mathbb{Z}} := \pi_* E$ is simply the locally free $\mathcal{O}_K$-module E, viewed as a $\mathbb{Z}$-module. Let $\nu_{\mathbb{Q}}$ denote the only infinite place of $\mathbb{Q}$ and denote by $E_{\mathbb{R}}$ the completion of $E_{\mathbb{Z}}$ with respect to $\nu_{\mathbb{Q}}$. Note that there is a natural isomorphism

$$\begin{aligned} E_{\mathbb{R}} &\longrightarrow \bigoplus_{\nu\in X_\infty} E_\nu \ , \\ x &\longmapsto (x_\nu)_{\nu\in X_\infty} \end{aligned}$$

see [38, section 1] and [22, Lemma 1.3.1] for more details. Then we take the following scalar product

$$\langle x, y\rangle_{E_{\mathbb{Z}},\nu_{\mathbb{Q}}} = \sum_{\nu\in X_\infty} \langle x_\nu, y_\nu\rangle_{E,\nu}$$

for $x, y \in E_{\mathbb{R}}$. All together, we obtain an Arakelov vector bundle which we denote by $\pi_*\bar{E}$.

Definition 1.2.5. A *morphism of Arakelov vector bundles*

$$\Phi\colon \bar{E} \to \bar{F}$$

consists of an $\mathcal{O}_K$-linear map $\Phi_{\mathcal{O}_K}\colon E \to F$ such that for each place $\nu \in X_\infty$ the induced map

$$\Phi_{K_\nu} := \Phi_{\mathcal{O}_K} \otimes 1\colon E_\nu \to F_\nu$$

satisfies

$$\langle \Phi_{K_\nu}(s), \Phi_{K_\nu}(s)\rangle_{F,\nu} \leq \langle s, s\rangle_{E,\nu}$$

for all $s \in E_\nu$.

Moreover, an *isomorphism of Arakelov vector bundles* $\Phi\colon \bar{E} \to \bar{F}$ is an $\mathcal{O}_K$-linear isomorphism $\Phi_{\mathcal{O}_K}\colon E \to F$ such that for each place $\nu \in X_\infty$ the induced map $\Phi_{K_\nu}\colon E_\nu \to F_\nu$ is an isometry.

Remark 1.2.6. The resulting category of Arakelov vector bundles on X is not additive, since the sum of morphisms is not always a morphism. For example, $\mathrm{id}_{\bar{E}} + \mathrm{id}_{\bar{E}}$ is not a morphism of Arakelov vector bundles.

Definition 1.2.7. Given an Arakelov line bundle $\bar{L}$ on X, the *degree* of $\bar{L}$ is

$$\deg(\bar{L}) \coloneqq \log(\#L/s\mathcal{O}_K) - \sum_{\nu\in X_\infty} \epsilon_\nu \log(\langle s, s\rangle_{L,\nu}^{1/2}) \in \mathbb{R} \tag{1.2.1}$$

where $s \in L \setminus \{0\}$ is arbitrary and $\epsilon_\nu = [K_\nu\colon \mathbb{R}]$, i.e. $\epsilon_\nu = 1$ or 2 if $K_\nu \cong \mathbb{R}$ or $\mathbb{C}$ respectively. For an Arakelov vector bundle $\bar{E}$ of higher rank it is defined as the degree of its determinant, i.e. $\deg(\bar{E}) \coloneqq \deg(\det(\bar{E}))$.

Now we see that the degree is well defined.

Lemma 1.2.8. *Let $\bar{L}$ be an Arakelov line bundle on X. Its degree* $\deg(\bar{L})$ (1.2.1) *is well defined, i.e. it is independent of the choice of $s \in L \setminus \{0\}$.*

Proof. We first claim that

$$\log(\#L/s\mathcal{O}_K) = -\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \log(\|s\|_{L,\mathfrak{p}}). \tag{1.2.2}$$

Then, note that for any $t \in L \setminus \{0\}$, there exists $a \in K^*$ such that $t = as$. Now, by the product formula (1.1.1)

$$\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \log(\|a\|_{L,\mathfrak{p}}) + \sum_{\nu\in X_\infty} \log(\|a\|_{L,\nu}^{\epsilon_\nu}) = 0.$$

On the other hand, we observe that

$$\sum_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \log(\|t\|_{L,\mathfrak{p}}^{-1}) + \sum_{\nu\in X_\infty} \log(\|t\|_{L,\nu}^{-\epsilon_\nu}) = \sum_{\mathfrak{p}\in\mathrm{Spec}(\mathcal{O}_K)} \log(\|as\|_{L,\mathfrak{p}}^{-1}) + \sum_{\nu\in X_\infty} \log(\|as\|_{L,\nu}^{-\epsilon_\nu})$$

which shows the desired equality

$$\log(\#L/t\mathcal{O}_K) - \sum_{\nu\in X_\infty} \epsilon_\nu \log(\|t\|_{L,\nu}) = \log(\#L/s\mathcal{O}_K) - \sum_{\nu\in X_\infty} \epsilon_\nu \log(\|s\|_{L,\nu}).$$

We finish by proving the claim (1.2.2). First, note that

$$L/s\mathcal{O}_K \cong \bigoplus_{\nu} L_{\mathfrak{p}}/s(\mathcal{O}_K)_{\mathfrak{p}}.$$

Furthermore, let $f_\mathfrak{p}$ denote the isomorphism $f_\mathfrak{p}\colon L_\mathfrak{p} \to (\mathcal{O}_K)_\mathfrak{p}$ and apply it to each direct summand to obtain

$$L_\mathfrak{p}/s(\mathcal{O}_K)_\mathfrak{p} \cong (\mathcal{O}_K/f_\mathfrak{p}(s))^{v_\mathfrak{p}(s)}.$$

This implies that

$$\#L/s\mathcal{O}_K = \prod_\mathfrak{p} \|s\|_\mathfrak{p}^{-1}.$$

□

Next, we show the computation of the degree of an Arakelov vector bundle as a $\mathbb{Z}$-lattice.

Let $K \supset \mathbb{Q}$ be a number field with ring of integers $\mathcal{O}_K$. Then, if we interpret an Arakelov vector bundle $\bar{E}$ as a hermitian lattice, one has that

$$\deg(\bar{E}) = -\log(\mathrm{vol}(E)),$$

where we define volume of E to be the covolume of the $\mathbb{Z}$-module $E_\mathbb{Z}$ inside its inner product space $E_\mathbb{R}$ (where the scalar product is the one defined by restriction of scalars).

This is a consequence of a classical result in algebraic number theory, that states the following.

Lemma 1.2.9. *Let $\mathfrak{a}$ be a nonzero ideal of $\mathcal{O}_K$, denote by F its fundamental domain as lattice in $\mathbb{R}^N$ for $N = [K\colon \mathbb{Q}] \in \mathbb{Z}_{>0}$. Let r_2 denote the number of complex embeddings of K. Then,*

$$\mathrm{vol}(F) = 2^{-r_2}\sqrt{|\,\mathrm{D}_{K/\mathbb{Q}}(\mathfrak{a})|}$$

where $\mathrm{D}_{K/\mathbb{Q}}(\mathfrak{a}) \coloneqq (\mathcal{O}_K\colon \mathfrak{a})^2 D_K$, with $D_K = \mathrm{D}_{K/\mathbb{Q}}(\mathcal{O}_K)$ denoting the discriminant of $\mathcal{O}_K$.

Proof. Consider an integral basis $\alpha_1, \cdots, \alpha_N$ of $\mathfrak{a}$ over $\mathbb{Z}$. Denote the real (resp. complex) embeddings of K by $\sigma_1, \cdots, \sigma_{r_1}$ (resp. $\tau_1, \cdots, \tau_{r_2}$ and their conjugates), where $r_1 + 2r_2 = N$. In this way each element $\alpha \in K$ maps to a vector

$$(\sigma_1\alpha, \cdots, \sigma_{r_1}\alpha, \tau_1\alpha, \cdots, \tau_{r_2}\alpha, \bar{\tau}_1\alpha, \cdots, \bar{\tau}_{r_2}\alpha) \in \mathbb{C}^N.$$

By definition, $\mathrm{D}_{K/\mathbb{Q}}(\mathfrak{a})$ is the square of the determinant of the $N \times N$-matrix

$$\begin{pmatrix} \sigma_1\alpha_1 & \cdots & \sigma_1\alpha_N \\ \vdots & & \vdots \\ \sigma_{r_1}\alpha_1 & \cdots & \sigma_{r_1}\alpha_N \\ \tau_1\alpha_1 & \cdots & \tau_1\alpha_N \\ \vdots & \circledast_1 & \vdots \\ \tau_{r_1}\alpha_1 & \cdots & \tau_{r_1}\alpha_N \\ \bar{\tau}_1\alpha_1 & \cdots & \bar{\tau}_1\alpha_N \\ \vdots & \circledast_2 & \vdots \\ \bar{\tau}_{r_2}\alpha_1 & \cdots & \bar{\tau}_{r_2}\alpha_N \end{pmatrix}. \tag{1.2.3}$$

Now, let $x_j := \Re(\tau_j\alpha)$ be the real part and $y_j := \Im(\tau_j\alpha)$ the imaginary part of $\tau_j\alpha$ for $j = 1, \cdots, r_2$. Moreover, put $\tau_j\alpha_\iota = x_{j,\iota} + iy_{j,\iota}$, for $\iota = 1, \cdots, N$, so that (1.2.3) reads as follows

$$\begin{pmatrix} \sigma_1\alpha_1 & \cdots & \sigma_1\alpha_N \\ \vdots & & \vdots \\ \sigma_{r_1}\alpha_1 & \cdots & \sigma_{r_1}\alpha_N \\ x_{1,1}+iy_{1,1} & \cdots & x_{1,N}+iy_{1,N} \\ \vdots & \circledast_1 & \vdots \\ x_{r_2,1}+iy_{r_2,1} & \cdots & x_{r_2,N}+iy_{r_2,N} \\ x_{1,1}-iy_{1,1} & \cdots & x_{1,N}-iy_{1,N} \\ \vdots & \circledast_2 & \vdots \\ x_{r_2,1}-iy_{r_2,1} & \cdots & x_{r_2,N}-iy_{r_2,N} \end{pmatrix}. \tag{1.2.4}$$

Replacing the set of rows $\circledast_1$ by $\circledast_1 + \circledast_2$ we obtain

$$\begin{pmatrix} \sigma_1\alpha_1 & \cdots & \sigma_1\alpha_N \\ \vdots & & \vdots \\ \sigma_{r_1}\alpha_1 & \cdots & \sigma_{r_1}\alpha_N \\ 2x_{1,1} & \cdots & 2x_{1,N} \\ \vdots & \circledast_1' & \vdots \\ 2x_{r_2,1} & \cdots & 2x_{r_2,N} \\ x_{1,1}-iy_{1,1} & \cdots & x_{1,N}-iy_{1,N} \\ \vdots & \circledast_2 & \vdots \\ x_{r_2,1}-iy_{r_2,1} & \cdots & x_{r_2,N}-iy_{r_2,N} \end{pmatrix}. \tag{1.2.5}$$

and replacing $\circledast_2$ by $\frac{\circledast_1'}{2} - \circledast_2$ in (1.2.5), we get that the absolute value of the determinant of (1.2.4) equals

$$2^{r_2} \det \begin{pmatrix} \sigma_1\alpha_1 & \cdots & \sigma_1\alpha_N \\ \vdots & & \vdots \\ \sigma_{r_1}\alpha_1 & \cdots & \sigma_{r_1}\alpha_N \\ x_{1,1} & \cdots & x_{1,N} \\ \vdots & & \vdots \\ x_{r_2,1} & \cdots & x_{r_2,N} \\ y_{1,1} & \cdots & y_{1,N} \\ \vdots & & \vdots \\ y_{r_2,1} & \cdots & y_{r_2,N} \end{pmatrix}$$

i.e. the right hand side is the determinant of a set of basis vectors for $\mathfrak{a}$ as a lattice in $\mathbb{R}^N$ having all their components in the direction of the canonical unit vectors of $\mathbb{R}^N$.

Therefore, the claim follows

$$\sqrt{|\operatorname{D}_{K/\mathbb{Q}}(\mathfrak{a})|} = 2^{r_2}\operatorname{vol}(F).$$

□

Lemma 1.2.10. *Given $n \in \mathbb{Z}_{>0}$,*

$$\operatorname{vol}(\mathcal{O}_K^n) = \prod_{\nu\in X_\infty} (\det z_\nu)^{\epsilon_\nu/2}(\operatorname{vol}(\mathcal{O}_K))^n$$

where $\epsilon_\nu = [K_\nu : \mathbb{R}]$ and z_ν denotes the matrix of the scalar product $\langle,\rangle_\nu$, for $\nu \in X_\infty$, evaluated on the standard basis.

Proof. If all $z_\nu = \operatorname{Id}$, then the direct sum $\mathcal{O}_K^n$ is orthogonal and then it has volume

$$\operatorname{vol}(\mathcal{O}_K^n) = (\operatorname{vol}(\mathcal{O}_K))^n.$$

Otherwise, we choose an orthonormal basis $\{e_{j,\nu}\}$ of K_ν^n. Let y_ν be the K_ν-automorphism of K_ν^n such that the standard basis $\{b_j\}$ is given by $b_j = y_\nu e_{j,\nu}$. Denote by y the direct product of the y_ν's for $\nu \in X_\infty$, it is a $\mathbb{R}$-automorphism of $\prod_{\nu\in X_\infty} K_\nu^n$. If we let $E' = y^{-1}\mathcal{O}_K^n$, then its volume

$$\operatorname{vol}(E') = (\operatorname{vol}(\mathcal{O}_K))^n.$$

Thus,

$$\operatorname{vol}(\mathcal{O}_K^n) = |\det y|(\operatorname{vol}(\mathcal{O}_K))^n.$$

Now, since endomorphisms of complex vector spaces have extra multiplicity, i.e. any endomorphism h of a $\mathbb{C}$-vector space has that $\det_{\mathbb{R}} h = (\det h)^2$, and

$$|\det y| = \prod_{\nu\in X_\infty} |\det y_\nu|^{\epsilon_\nu}.$$

Moreover, if we denote Y_ν the matrix of the y_ν with respect the orthonormal basis $\{e_{j,\nu}\}$, then $z_\nu =^t Y_\nu Y_\nu$ and we find

$$\operatorname{vol}(\mathcal{O}_K^n) = \prod_{\nu\in X_\infty} (\det z_\nu)^{\epsilon_\nu/2}(\operatorname{vol}(\mathcal{O}_K))^n.$$

□

Finally, we see the relation between the Arakelov degree and the volume as hermitian lattice.

Proposition 1.2.11. *Let $\bar{E}$ be an Arakelov vector bundle over X. Then,*

$$\deg(\bar{E}) = -\log\operatorname{vol}(E).$$

Proof. For every Arakelov vector bundle $\bar{E}$ there exists a subbundle $\bar{E}' \subset \bar{E}$ with same rank which is free, i.e. $E' = s\mathcal{O}_k^n$ for $s \in E \setminus \{0\}$. Therefore,

$$\mathrm{vol}(E) = \prod_{\nu \in X_\infty} \frac{(\det z_\nu)^{\epsilon_\nu/2}(\mathrm{vol}(\mathcal{O}_K))^n}{\#(E/s\mathcal{O}_K)}$$

where z_ν denotes again the matrix of the scalar product $\langle , \rangle_\nu$, for $\nu \in X_\infty$, evaluated on the standard basis. A direct computation applying logarithm to both sides shows

$$-\log \mathrm{vol}(E) = \deg(\bar{E}).$$

□

Remark 1.2.12. The claims in Lemma 1.2.10 and Proposition 1.2.11 use the scalar product matrices z_ν for $\nu \in X_\infty$. This is independent of a choice of a basis, since the determinant of a base change is a unit of $\mathcal{O}_K$ and the product of all archimedean norms of a unit of $\mathcal{O}_K$ is one.

Next, we show some properties of the Arakelov degree.

Proposition 1.2.13. *Let K be a number field and denote by X the arithmetic curve associated to K. Let $\bar{E}$ and $\bar{F}$ be an Arakelov vector bundles over X.*

1. *Given a finite extension L of K, let $\bar{E}_{X_L}$ denote the base change to X_L, the arithmetic curve associated to L. Then,*

$$\deg(\bar{E}_{X_L}) = [L : K] \deg(\bar{E})$$

2. $\deg(\bar{E} \otimes \bar{F}) = \mathrm{rk}(\bar{F}) \deg(\bar{E}) + \mathrm{rk}(\bar{E}) \deg(\bar{F})$
3. $\deg(\bar{E}^*) = -\deg(\bar{E})$
4. $\deg(\bar{E} \oplus \bar{F}) = \deg(\bar{E}) + \deg(\bar{F})$.

Proof. 1. Recall that we have a surjection

$$f \colon X_{L,\infty} \twoheadrightarrow X_{K,\infty}$$

where, given $\nu \in X_{K,\infty}$ (resp. $\nu' \in X_{L,\infty}$) with $\nu = f(\nu')$, then $d_{\nu'} := [L_{\nu'} : K_\nu]$ is either $d_{\nu'} = 1$ or $d_{\nu'} = 2$. Moreover, note that

$$[L : K] = \sum_{\nu'} d_{\nu'}.$$

Hence, given $s' \in \det(E_{\mathcal{O}_L}) \setminus \{0\}$, we have

$$\deg(\bar{E}_{X_L}) = -\sum_{\mathfrak{p}' \in \mathrm{Spec}(\mathcal{O}_L)} \log(\|s'\|_{\det(E_{\mathcal{O}_L}),\mathfrak{p}'}) - \sum_{\nu' \in X_{L,\infty}} \epsilon_{\nu'} \log(\|s'\|_{\det(E_{\mathcal{O}_L}),\nu'}).$$

Finally, the claim follows from the definition of the scalar product $\langle , \rangle_{E_{\mathcal{O}_L,\nu'}}$, defined on $E_{\mathcal{O}_L,\nu'}$.

2. This is a consequence of

$$\det(\bar{E} \otimes \bar{F}) \cong \det(\bar{E})^{\otimes \operatorname{rk}(\bar{F})} \otimes \det(\bar{F})^{\otimes \operatorname{rk}(\bar{E})}.$$

3. This is a consequence of

$$\bar{E} \otimes \bar{E}^* \cong \bar{\mathcal{O}}_K.$$

4. This is a consequence of

$$\det(\bar{E} \oplus \bar{F}) \cong \det(\bar{E}) \otimes \det(\bar{F}).$$

□

Remark 1.2.14. Let $\bar{E}$ be an Arakelov vector bundle over the arithmetic curve associated to a number field K. After Proposition 1.2.13 1. we will consider the degree of $\bar{E}$ to be *normalized* i.e. in what follows, we write $\deg(\bar{E})$ to denote $\deg(\bar{E})/[K : \mathbb{Q}]$. This will not affect semistability and will make the value invariant under finite extension of the number field K.

If Y is a smooth projective curve of genus g over an algebraically closed field of characteristic 0 and E is a vector bundle over Y, the Riemann-Roch formula states

$$h^0(Y, E) - h^1(Y, E) = \deg(E) + (1 - g)\operatorname{rk}(E). \tag{1.2.6}$$

Moreover, by duality we have $h^1(Y, E) = h^0(Y, \omega_Y^{-1} \otimes E^*)$, where ω_Y denotes the canonical bundle of Y.

Let X be an arithmetic curve associated to a number field K. There exists a dualizing Arakelov bundle $\mathcal{D}_K^{-1}$ that yields a duality theorem as shown in [22, Theorem 1.3.2]. Unfortunately, with the definition of global sections given for Arakelov vector bundles in Definition 1.2.4 there is no such Riemann-Roch equality as in (1.2.6). However, Gillet and Soulé [36] established an inequality as an approximate analogue of the Riemann-Roch formula.

Proposition 1.2.15. *Let $\bar{E}$ be an Arakelov vector bundle over the arithmetic curve associated to a number field K, of rank $n \in \mathbb{Z}_{>0}$. Then, one has*

$$\left| h^0(\bar{E}) - h^0(\mathcal{D}_K^{-1} \otimes \bar{E}^*) - \deg(\bar{E}) - \frac{1}{2} n \log |D_K| \right| \leq c(r_1, r_2, n)$$

where $h^0(\bar{E}) := \log(\#\Gamma(X, \bar{E}))$, D_K denotes the discriminant of K, r_1 (resp. r_2) denotes the number of real (resp. complex) embeddings of K in $\mathbb{C}$ and $c(r_1, r_2, n)$ is a constant depending only on r_1, r_2, and n.

In [22], Chambert-Loir recalls an alternative definition of h^0 which allows for an exact Riemann-Roch equality as in (1.2.6). Moreover, he shows the following inequality that we will use later.

Proposition 1.2.16 ([22, Proposition 1.4.12])**.** *Let $\bar{E}$ be an Arakelov vector bundle over the arithmetic curve associated to a number field K, of rank $n \in \mathbb{Z}_{>0}$. Then, one has*

$$h^0(\bar{E}) \geq \deg(\bar{E}) - \frac{1}{2} n \log |D_K| - \frac{2+dn}{2} \log \left(\frac{2+dn}{2\pi} \right) - \frac{1}{2} \log \pi$$

where $d := [K : \mathbb{Q}]$.

Next, we define semistability of an Arakelov vector bundle $\bar{E}$ analogously to semistability of algebraic vector bundles on curves.

Definition 1.2.17. Let $\bar{E}$ be an Arakelov vector bundle of rank $r \in \mathbb{Z}_{>0}$, then the *slope* of $\bar{E}$ is

$$\mu(\bar{E}) := \frac{\deg(\bar{E})}{r}.$$

Then, $\bar{E}$ is called *semistable*, if for all non-trivial subbundles $0 \subsetneq \bar{F} \subsetneq \bar{E}$ it holds that $\mu(\bar{F}) \leq \mu(\bar{E})$.

Example 1.2.18. Consider $K = \mathbb{Q}$. Let $\bar{L} = (L, \langle ., . \rangle)$ be a rank r Arakelov vector bundle on $\mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$, i.e. L is a locally free $\mathbb{Z}$-module of rank r and we let $\langle ., . \rangle$ denote an euclidean metric in $\mathbb{R}^r$. By Proposition 1.2.11, its degree is

$$\deg(\bar{L}) = -\log(\mathrm{vol}(L)).$$

Now, let $\mathcal{H}$ denote the upper half plane of the euclidean plane and

$$\mathcal{D} = \{z \in \mathcal{H} \mid |z| \geq 1, |\Re(z)| \leq 1/2\}$$

the fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{H}$. See the gray region in Figure 1.1.

Given $\tau \in \mathcal{D}$, consider the lattice

$$L_\tau := \mathbb{Z} + \mathbb{Z}\tau$$

and together the standard euclidean metric, denote $\bar{L}_\tau$ the corresponding rank 2 Arakelov vector bundle.

Hence, $\bar{L}_\tau$ is semistable if and only if for every proper Arakelov line subbundle of $\bar{L}_\tau$, i.e. $0 \neq \bar{L}' \subsetneq \bar{L}_\tau$,

$$\begin{aligned} -\log(\mathrm{vol}(L')) &\leq \frac{-\log(\mathrm{vol}(L_\tau))}{2} \\ &= \frac{-\log(\Im(\tau))}{2}. \end{aligned}$$

Note that 1 is a vector of minimal length in L_τ. Therefore, $\bar{L}_\tau$ is semistable if and only if $1 \geq \Im(\tau)$. See the darkest region in Figure 1.1.

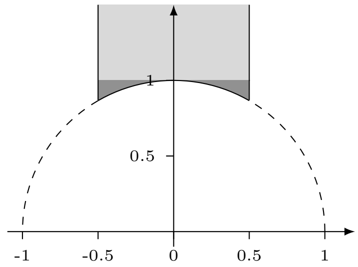

Figure 1.1: Stability region of $\bar{L}_\tau$.

Example 1.2.19. Put $K = \mathbb{Q}$. Let

$$\mathbb{A}_n = \mathbb{Z}^{n+1} \cap \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1 + \cdots + x_{n+1} = 0\}$$

denote the root lattice with Gram matrix

$$B_{\mathbb{A}_n} := \begin{pmatrix} 2 & 1 & \cdots & 0 & 0 \\ 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & \cdots & 1 & 2 \end{pmatrix}$$

in the standard basis $e_1, \cdots, e_n$ and standard euclidean norm. We note that $\mathbb{A}_n$ is semistable since the natural representation by permutation of coordinates of the symmetric group on $\mathbb{A}_n \times_{\mathbb{Z}} \mathbb{R}$ is irreducible and preserves $\mathbb{A}_n$. Furthermore,

$$\begin{aligned} \mu(\mathbb{A}_n) &= -\frac{1}{n} \log(\sqrt{\det B_{\mathbb{A}_n}}) \\ &= -\frac{1}{2n} \log(n+1) \end{aligned}$$

where the last inequality can be shown by induction. Indeed, we have $\det B_{\mathbb{A}_1} = 2$ (resp. $\det B_{\mathbb{A}_2} = 3$) for $n = 1$ (resp. $n = 2$) and in general for $n \geq 3$ we have

$$\begin{aligned} \det B_{\mathbb{A}_n} &= 2 \det B_{\mathbb{A}_{n-1}} - \det B_{\mathbb{A}_{n-2}} \\ &= 2n - (n-1) \\ &= n + 1. \end{aligned}$$

Now, given a number number field K and a rank 1 subbundle $L \subset \mathbb{A}_n \otimes_{\mathbb{Z}} \mathcal{O}_K$, we consider the morphisms

$$X_{i,L} \colon L \to \mathcal{O}_K$$

that given $(x_1, \cdots, x_{n+1}) \in L$,

$$X_{i,L}(x_1, \cdots, x_{n+1}) := x_i$$

for each $i \in \{1, \cdots, n+1\}$. Let $\alpha(L)$ denote the cardinality of the set

$$\{i \in \{1, \cdots, n+1\} \mid X_{i,L} \neq 0\}.$$

Then, we claim

$$\deg(\bar{L}) \leq -\frac{1}{2}\log(\alpha(L)). \tag{1.2.7}$$

Note that $\alpha(L) \geq 2$. Indeed, for $I := \{i_1, \cdots, i_{n-1}\} \subset \{1, \cdots, n+1\}$ denote by $L_I \subset \mathbb{A}_n$ the $\mathbb{Z}$-subbundle defined by $X_{i_1} = \cdots = X_{i_{n-1}} = 0$, it satisfies

$$\deg(\bar{L}_I) = -\frac{1}{2}\log 2$$

and in this case we verify that $\alpha(L_I) = 2$.

Now we take a rank 1 subbundle $L \subset \mathbb{A}_n \otimes_{\mathbb{Z}} \mathcal{O}_K$, with $L \neq L_I \otimes_{\mathbb{Z}} \mathcal{O}_K$ for any I as above. Note that this assumption implies that $X_{i,L} \neq 0$ for at least 3 different $i \in \{1, \cdots, n+1\}$. Hence, for each $i \in \{1, \cdots, n+1\}$ such that $X_{i,L} \neq 0$, by semistability of line bundles we know that

$$\deg \bar{L} \leq -\frac{1}{[K:\mathbb{Q}]}\sum_{\nu \in X_\infty} \log \|X_{i,L}\|_\nu.$$

Then,

$$\alpha(L)\deg \bar{L} \leq -\frac{1}{[K:\mathbb{Q}]}\sum_{\nu \in X_\infty} \log\left(\prod_{i=1}^{\alpha(L)} \|X_{i,L}\|_\nu\right). \tag{1.2.8}$$

Finally, we recall the inequality of arithmetic and geometric means, i.e. given positive real numbers $x_1, \cdots, x_n$,

$$\sqrt[n]{x_1 \cdots x_n} \leq \frac{x_1 + \cdots + x_n}{n}.$$

This implies that

$$\log\left(\prod_{i=1}^{\alpha(L)} \|X_{i,L}\|_\nu\right) \leq -\frac{\alpha(L)}{2}\log \alpha(L) + \frac{\alpha(L)}{2}\log\left(\sum_{i=1}^{\alpha(L)} \|X_{i,L}\|_\nu\right)$$

for every $\nu \in X_\infty$. Now, by definition

$$\log\left(\sum_{i=1}^{\alpha(L)} \|X_{i,L}\|_\nu\right) \leq 0$$

and the claim (1.2.7) follows by plugging in these inequalities in (1.2.8).

The following lemma provides the main tool in the construction of the analogue to the Harder Narasimhan filtration.

Lemma 1.2.20. *Let $\bar{E}$ be an Arakelov vector bundle on X. For every $c \in \mathbb{R}$, there exist finitely many subbundles $\bar{F} \subset \bar{E}$ such that* $\deg \bar{F} \geq c$.

Proof. First of all, we restrict to the case where $K = \mathbb{Q}$ by restriction of scalars. Let $\pi\colon \operatorname{Spec}(\mathcal{O}_K) \to \operatorname{Spec}(\mathbb{Z})$ be the natural morphism. Moreover, the functor π_* induces an injection from the set of Arakelov subbundles $\bar{F}$ of $\bar{E}$ to those of $\pi_*\bar{E}$ (see for example [38, Lemma 1.2]).

Now, recall that if we consider $\bar{E}$ as an hermitian lattice, by Proposition 1.2.11, we have $\deg \bar{E} = -\log \operatorname{vol}(E)$. Hence it is equivalent to see that for every $c \in \mathbb{R}$, there are finitely many submodules $F \subset E$ with $\operatorname{vol}(F) \leq -c$.

Now, after all this assumptions, we have $\mathcal{O}_K = \mathbb{Z}$ and let $r \in \mathbb{Z}_{>0}$ denote the rank of F. If $r = 1$ it is clear since the F are discrete in $F_{\mathbb{R}}$ and the ball of radius c is compact. If $r > 1$, Let $\bar{F}_1$ be the Arakelov subbundle defined by $F_1 \coloneqq F_K \cap E$. Both $\bar{F}$ and $\bar{F}_1$ have the same rank r and satisfy $F \subset F_1$, hence

$$\deg(\bar{F}_1) = \deg(\bar{F}) + \log(\#(F_1/F)).$$

In particular, $\deg(\bar{F}_1) \geq c$. On the other hand, the rank r subspace $F_K \subset E_K$ is determined by the line $\wedge^r F_K$ in $\wedge^r E_K$. By the rank 1 case, it follows that $\bar{F}_1$ belongs to a finite set of Arakelov subbundles and $\log(\#(F_1/F)) \leq \deg(\bar{F}_1) - c$.

Our claim follows since, given any positive integer n, the set of submodules F' of F_1 such that $\#(F_1/F')$ is bounded is finite. □

In [38, Discussion 1.16] it is shown that as a consequence, together with a discussion about the degrees of the subbundles, that every Arakelov vector bundle $\bar{E}$ on X has a unique filtration analogue to the Harder-Narasimhan filtration.

Proposition 1.2.21 ([38])**.** *For an Arakelov vector bundle $\bar{E}$ on X there exists a unique filtration*

$$0 = \bar{E}_0 \subsetneq \bar{E}_1 \subsetneq \cdots \subsetneq \bar{E}_r = \bar{E}$$

satisfying the following properties:

i) All quotients $\bar{E}_j/\bar{E}_{j-1}$, with $j = 1, \cdots, r$, are semistable of slope $\mu_j(\bar{E})$.

ii) These slopes satisfy

$$\mu_1(\bar{E}) > \mu_2(\bar{E}) > \cdots > \mu_r(\bar{E}).$$

Moreover, this filtration also satisfies

iii) If we write

$$\mu_{\max}(\bar{E}/\bar{E}_{j-1}) \coloneqq \max_{0 \subsetneq \bar{F} \subsetneq \bar{E}/\bar{E}_{j-1}} \mu(\bar{F}), \tag{1.2.9}$$

then $\bar{E}_j/\bar{E}_{j-1}$ is the largest subbundle of $\bar{E}/\bar{E}_{j-1}$ such that

$$\mu_j(\bar{E}) = \mu_{\max}(\bar{E}/\bar{E}_{j-1}).$$

iv) If we write

$$\mu_{\min}(\bar{E}_j) := \min_{0 \subsetneq \bar{F} \subsetneq \bar{E}_j} \mu(\bar{E}_j/\bar{F}),$$

then $\bar{E}_j/\bar{E}_{j-1}$ is the smallest quotient bundle of $\bar{E}_j$ such that

$$\mu_j(\bar{E}) = \mu_{\min}(\bar{E}_j).$$

Definition 1.2.22. For an Arakelov vector bundle $\bar{E}$ on X the canonical filtration

$$0 = \bar{E}_0 \subsetneq \bar{E}_1 \subsetneq \cdots \subsetneq \bar{E}_r = \bar{E}$$

given by Proposition 1.2.21 is also called the *Grayson-Stuhler* (GS-)filtration of $\bar{E}$.

Remark 1.2.23. Note that an Arakelov vector bundle $\bar{E}$ on X is semistable if and only if its corresponding GS-filtration contains only 0 and $\bar{E}$, i.e. $r = 1$.

Let E be a locally free $\mathcal{O}_K$-module of rank n, let $\Gamma = \mathrm{GL}(E)$. Denote $\widetilde{X} = \widetilde{X}(E)$ the space of Arakelov vector bundles on X whose underlying locally free $\mathcal{O}_K$-module is E. If we let $\widetilde{X}_\nu$ be the space of scalar products on E_ν, for $\nu \in X_\infty$, we can see $\widetilde{X}_\nu$ as an open subspace of a real or complex vector space (up to fixing a basis for E_ν). We have $\widetilde{X} = \prod_{\nu \in X_\infty} \widetilde{X}_\nu$ which defines a natural topology on $\widetilde{X}$.

Given $\bar{E} \in \widetilde{X}$, let $\langle v, w \rangle_{E,\nu}$ denote the value of the scalar product on $v, w \in E_\nu$. For $g \in \Gamma$, we define a new Arakelov vector bundle $g\bar{E}$ by the formula

$$\langle v, w \rangle_{gE,\nu} = \langle g^{-1}v, g^{-1}w \rangle_{E,\nu}$$

and this defines a left-action of Γ on $\widetilde{X}$.

This left-action provides an isomorphism $g\colon\ \bar{E} \xrightarrow{\sim} g\bar{E}$ also denoted by g. Conversely, given an isomorphism $g\colon\ \bar{E}_1 \xrightarrow{\sim} \bar{E}_2$ of Arakelov vector bundles in $\widetilde{X}$, since both of them have the same underlying locally free $\mathcal{O}_K$-module E, it gives rise to an element $g \in \Gamma$. Then, it is clear that $\bar{E}_1 = g\bar{E}_2$.

Thus, the orbit set

$$\Gamma \setminus \widetilde{X}$$

can be regarded as the set of isomorphism classes of Arakelov vector bundles on X with underlying locally free $\mathcal{O}_K$-module E. In [38] is shown that when restricting to the semistable points the quotient $\Gamma \setminus \widetilde{X}$ is compact so that it is the analogue of the moduli space for vector bundles on an algebraic curve.

1.3 Nefness and the tensor product problem in Arakelov geometry

This section compiles results from [2]. It provides interesting evidence of the pathologies of the Arakelov setting compared to the classical setting.

Given an Arakelov vector bundle $\bar{E}$ on X, recall the notation above (1.2.9). Hence, $\bar{E}$ is semistable if and only if $\mu_{\max}(\bar{E}) = \mu(\bar{E})$.

Now, given $\bar{E}_1$ and $\bar{E}_2$ two Arakelov vector bundles on X, note that

$$\mu(\bar{E}_1 \otimes \bar{E}_2) = \mu(\bar{E}_1) + \mu(\bar{E}_2).$$

This implies that

$$\mu_{\max}(\bar{E}_1 \otimes \bar{E}_2) \geq \mu_{\max}(\bar{E}_1) + \mu_{\max}(\bar{E}_2)$$

and one can ask whether it is an equality.

Conjecture 1.3.1 ([16, Problem 1.3])**.** *Let $\bar{E}_1$ and $\bar{E}_2$ be two Arakelov vector bundles on X,*

$$\mu_{\max}(\bar{E}_1 \otimes \bar{E}_2) = \mu_{\max}(\bar{E}_1) + \mu_{\max}(\bar{E}_2).$$

Remark 1.3.2. Equivalently, if $\bar{E}_1$ and $\bar{E}_2$ are semistable, then $\bar{E}_1 \otimes \bar{E}_2$ is also semistable.

There are numerous proofs of this fact in the classical setting. In the following lines we explore the pathologies found in the Arakelov setting when we relate the notions of semistability and nefness.

Definition 1.3.3. An Arakelov vector bundle $\bar{E}$ on X is *numerically effective* (we will write *nef*) if for every finite extension $K' \supset K$, any quotient line bundle on the pull-back $\bar{E}' := \bar{E}_{\mathrm{Spec}(\mathcal{O}_{K'})}$ has non-negative degree.

Example 1.3.4 (Direct sum of nef is nef)**.** Let $\bar{E}_1$ and $\bar{E}_2$ be two nef Arakelov vector bundles on X, we want to see that $\bar{E}_1 \oplus \bar{E}_2$ is also nef.

Indeed, fix a finite extension $K' \supset K$. Consider a rank one quotient of $\bar{E}_1' \oplus \bar{E}_2'$, $\bar{L}$. Assume (without loss of generality) that the restriction of the quotient morphism to $\bar{E}_1'$ is nonzero and denote by $\bar{L}'$ the image in $\bar{L}$ under this restriction. Since $\bar{E}_1$ is nef, $\deg \bar{L}' \geq 0$. Therefore,

$$\deg \bar{L} \geq \deg \bar{L}' \geq 0.$$

In the classical setting, Kleiman's Theorem states that the nefness of a vector bundle implies the non-negativity of its degree (see for example [45, Lemma 6.4.10]). As a consequence, it follows that a vector bundles of degree 0 is nef if and only if it is semistable. However, in Arakelov geometry we have a bound for the degree of a nef Arakelov vector bundle which allows negative degrees, as the following example illustrates.

Example 1.3.5 (Example of nef Arakelov vector bundle with negative degree)**.** On $K = \mathbb{Q}$, let

$$\mathbb{A}_2 = \mathbb{Z}^3 \cap \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$$

denote the root lattice with Gram matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

in some basis e_1, e_2, with angle $\widehat{e_1 e_2} = \pi/3$.

Fix $\lambda \in [\frac{\log(3/2)}{2}, \frac{\log 3}{4})$, we denote by $\mathbb{A}_2(\lambda)$ the resulting lattice by multiplying all norms by $e^{-\lambda}$. Hence,

$$\begin{aligned}\deg(\mathbb{A}_2(\lambda)) &= \deg(\mathbb{A}_2) + \lambda \operatorname{rk}(\mathbb{A}_2)\\ &= -\frac{\log 3}{2} + 2\lambda \in [\frac{\log 3}{2} - \log 2, 0).\end{aligned}$$

Let us see that $\mathbb{A}_2(\lambda)$ is nef. Note that the shortest length of a vector in $\mathbb{A}_2\langle\lambda\rangle$ is $\sqrt{2}e^{-\lambda}$, so its degree as rank one sublattice is

$$-\log(\sqrt{2}e^{-\lambda}) = \lambda - \frac{\log 2}{2}.$$

Thus, any rank one sublattice of $\mathbb{A}_2(\lambda)$ has degree $\leq \lambda - \frac{\log 2}{2}$.

Next, let $\bar{L} \coloneqq \mathbb{A}_2(\lambda)/\bar{L}'$ be the quotient of $\mathbb{A}_2(\lambda)$ by a line subbundle $\bar{L}'$. The previous observation together with the additivity of the degree imply

$$\begin{aligned}\deg(\bar{L}) &= \deg(\mathbb{A}_2(\lambda)) - \deg(\bar{L}')\\ &\geq \deg(\mathbb{A}_2(\lambda)) - (\lambda - \frac{\log 2}{2})\end{aligned}$$

and the fact that $\lambda \geq 1/2\log(3/2)$ implies that any such quotient of rank 1 of $\mathbb{A}_2(\lambda)$ has non-negative degree, i.e.

$$\begin{aligned}\deg(\bar{L}) &\geq -\frac{\log 3}{2} + 2\lambda - (\lambda - \frac{\log 2}{2})\\ &= \lambda - 1/2\log(3/2) \geq 0.\end{aligned}$$

Now, let $K \supset \mathbb{Q}$ be an arbitrary number field. Take $a, b \in \mathcal{O}_K$, with $ab \neq 0$ such that $l = ae_1 + be_2$ is a non-zero vector in $\mathbb{A}_2(\lambda)_{\mathcal{O}_K}$. If we see that the degree of the $\mathcal{O}_K$-lattice that it spans is $\leq [K\colon \mathbb{Q}](\lambda - \frac{\log 2}{2})$, by the same arguments as before, it implies that any rank 1 quotient of $\mathbb{A}_2(\lambda)$ has positive degree. Equivalently, we are going to see that

$$\prod_{\sigma K \hookrightarrow \mathbb{C}} \|\sigma(a)e_1 + \sigma(b)e_2\|^2 \geq (2e^{-2\lambda})^{[K\colon \mathbb{Q}]}.$$

Indeed, given $\sigma\colon K \hookrightarrow \mathbb{C}$,

$$\begin{aligned}\frac{e^{2\lambda}}{2}\|\sigma(a)e_1 + \sigma(b)e_2\|^2 &= |\bar{\sigma}(a)|^2 + |\sigma(b)|^2 + \Re(\bar{\sigma}(a)\sigma(b))\\ &= (|\bar{\sigma}(a)| - |\sigma(b)|)^2 + 2|\bar{\sigma}(a)\sigma(b)| + \Re(\bar{\sigma}(a)\sigma(b))\\ &\geq |\bar{\sigma}(a)\sigma(b)|\\ &= |\sigma(ab)|\end{aligned}$$

and since $a, b \in \mathcal{O}_K$ with $ab \neq 0$,

$$\prod_{\sigma K \hookrightarrow \mathbb{C}} |\sigma(ab)| \geq 1.$$

Remark 1.3.6. Note that in particular we have seen that $\mathbb{A}_2(\lambda)$ is semistable of negative degree. Indeed, we have found that any rank one sublattice $\bar{L}$ of $\mathbb{A}_2(\lambda)$ has

$$\mu(\bar{L}) \leq \lambda - \frac{\log 2}{2}.$$

Now observe that we have

$$\mu(\mathbb{A}_2(\lambda)) = \lambda - \frac{\log 3}{4}$$

and

$$\lambda - \frac{\log 3}{4} \geq \lambda - \frac{\log 2}{2},$$

implying $\mu(\bar{L}) \leq \mu(\mathbb{A}_2(\lambda))$.

Remark 1.3.7. Moreover, the lattice $\mathbb{Z}\langle\lambda'\rangle$ is nef for $\lambda' \geq 0$. Then, Example 1.3.4 implies that $\mathbb{A}_2(\lambda) \oplus \mathbb{Z}(\lambda')$ is also nef. In particular, if one takes

$$\lambda' := \frac{\log 3}{2} - 2\lambda$$

then $\mathbb{A}_2(\lambda) \oplus \mathbb{Z}(\lambda')$ is nef, of degree 0 but not semistable, contradicting the classical result (every nef vector bundle over a smooth projective curve defined over a field of characteristic 0, of degree 0 is semistable).

Proposition 1.3.8 (Arithmetic Kleiman theorem [2, Lemma 3.2])**.** *Let $\bar{E}$ be a nef rank r Arakelov vector bundle on X. Then,*

$$\deg \bar{E} \geq -[K\colon \mathbb{Q}] \log r.$$

Theorem 1.3.9 ([2, Theorem 0.4])**.** *Let $\bar{E}_1$ and $\bar{E}_2$ be two Arakelov vector bundles on X,*

$$\mu_{\max}(\bar{E}_1 \otimes \bar{E}_2) \leq \mu_{\max}(\bar{E}_1) + \mu_{\max}(\bar{E}_2) + \frac{[K\colon \mathbb{Q}]}{2} \log(\operatorname{rk}(\bar{E}_1 \otimes \bar{E}_2)). \tag{1.3.1}$$

Proof. First of all, we can assume without loss of generality that $\mu_{\max}(\bar{E}_i) = 0$ for $i = 1, 2$ by replacing $\bar{E}_i$ by $\bar{E}_i(\lambda)$ if needed. Hence, $\bar{E}_i^*$ are nef for $i = 1, 2$.

Indeed, put $\bar{E} := \bar{E}_i$ for $i = 1, 2$. The fact that $\mu_{\max}(\bar{E}) = 0$ implies that $\deg \bar{F}^* \geq 0$ for all non-trivial subbundles $\bar{F}^*$ of $\bar{E}^*$. In particular, any rank one quotient subbundle of $\bar{E}^*$ will have positive degree. Now, if we consider a finite extension $K' \supset K$, the pull-back $\bar{E}' = \bar{E}_{\operatorname{Spec}(\mathcal{O}_{K'})}$ satisfies

$$\mu_{\max}(\bar{E}') = [K'\colon K]\mu_{\max}(\bar{E}) = 0 \tag{1.3.2}$$

and the claim follows by the previous argument.

We will show that for every rank r subbundle, $0 \neq \bar{E} \subset \bar{E}_1 \otimes \bar{E}_2$,

$$\deg \bar{E} \leq [K\colon \mathbb{Q}] \log r.$$

Indeed, given a finite extension $K' \supset K$, a line subbundle $\bar{L}$ of $\bar{E}'_1 \otimes \bar{E}'_2$ gives rise to a nonzero morphism

$$f' \colon (\bar{E}'_2)^* \to \bar{L}^* \otimes \bar{E}'_1.$$

Hence, the normalization of the maximal slopes, together with (1.3.2) and duality implies that any quotient of $(\bar{E}'_2)^*$ has positive degree and any subbundle

$$0 \neq \bar{M} \subset \bar{L}^* \otimes \bar{E}'_1$$

has $\deg \bar{M} \leq \deg \bar{L}^*$. Finally, factor f' through the quotient by its kernel and we get $\deg \bar{L}^* \geq 0$. Putting all together, this implies that $(\bar{E}_1 \otimes \bar{E}_2)^*$ is nef, so its quotient $\bar{E}^*$ is nef and therefore, by Proposition 1.3.8,

$$\deg \bar{E} \leq [K \colon \mathbb{Q}] \log r.$$

□

Remark 1.3.10. If we consider the classical setting, say let k be an algebraically closed field of characteristic 0 and let Y be a smooth projective curve on k. Then we recall that Kleiman's theorem (i.e. the classical result on which Theorem 1.3.8 is based) states that if a vector bundle E is nef, then $\deg E \geq 0$.

This different lower bound compared to the given one in Theorem 1.3.8, is responsible of the extra summand in (1.3.1). Given $\bar{E}_1$ and $\bar{E}_2$ two Arakelov vector bundles on the arithmetic X associated to a number field K, the mentioned extra summand in (1.3.1) does not allow to conclude that

$$\mu_{\max}(\bar{E}_1 \otimes \bar{E}_2) \leq \mu_{\max}(\bar{E}_1) + \mu_{\max}(\bar{E}_2)$$

unless both $\bar{E}_1$ and $\bar{E}_2$ are Arakelov line bundles.

Arakelov group schemes

This chapter provides an introduction to Arakelov group schemes and the construction of Behrend's complementary polyhedra in the lines of [39] (after [9]) to study their semistability.

1.4 Behrend's complementary polyhedra

1.4.1 Complementary polyhedra

Let $(V, \langle, \rangle)$ be an Euclidean $\mathbb{R}$-vector space of dimension n and denote $V^\vee$ its dual.

Definition 1.4.1. A *reduced root system* of V consists of a finite set Φ of elements of V such that

i) $0 \notin \Phi$ and Φ generates V.

ii) For every $\alpha \in \Phi$, there exists a unique $\alpha^\vee \in V^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$ and if for $x \in V$
$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha$$
denotes the reflection at the line spanned by α, then $s_\alpha(\Phi) = \Phi$.

iii) $\alpha^\vee(\Phi) \subset \mathbb{Z}$ for every $\alpha \in \Phi$.

iv) $2\alpha \notin \Phi$ for every $\alpha \in \Phi$.

The elements of the root system are called *roots*.

Definition 1.4.2. A subset $\Delta \subset \Phi$ is called *basis* of Φ if it satisfies

i) Δ generates V.

ii) Every root is an integral linear combination of elements of Δ and the coefficients are either all non-negative or non-positive.

The elements of a basis are called *simple* roots.

Remark 1.4.3. The choice of a basis determines a partition
$$\Phi = \Phi^+ \cup \Phi^-$$
into positive and negative roots.

Definition 1.4.4. Let $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ be a basis of Φ. The $n \times n$-matrix

$$A := \left(\langle \alpha_i, \alpha_j^\vee \rangle\right)_{ij}$$

is called the *Cartan matrix.*

Remark 1.4.5. Note that all the entries of A are integers because of property iii) in Definition 1.4.1.

Now, for each $\alpha \in \Phi$, consider

$$H(\alpha) := \{x \in V \mid \langle x, \alpha \rangle = 0\}$$

the hyperplane orthogonal to α.

The collection of such $H(\alpha)$ gives a decomposition of V into *facets.*

Definition 1.4.6. Two points $v, w \in V$ are in the same *facet* if and only if for every $\alpha \in \Phi$ either $v, w \in H(\alpha)$ or they lie in the same side of $H(\alpha)$, i.e. $\langle v, \alpha \rangle \langle w, \alpha \rangle > 0$. Facets of maximal dimension are called *(Weyl) chambers.* We denote by $\mathcal{C}(V, \Phi)$ the set of Weyl chambers.

Remark 1.4.7. The definitions of facet and chamber are as in [9]. However, the reader should not confuse them with facets in classical convex geometry, where our facets are simply called k-faces, where $k \in \mathbb{Z}$ denotes its dimension and our chambers are called facets. See for example [34, Definition 4.1].

Definition 1.4.8. A subset $R \subset \Phi$ is *parabolic* if it satisfies the following properties

1. For all $\alpha \in \Phi$, $\alpha \in R$ or $-\alpha \in R$.

2. If $\alpha, \beta \in R$ with $\alpha + \beta \in \Phi$, then $\alpha + \beta \in R$.

Lemma 1.4.9 ([9, Corollary 1.8])**.** *For every facet F,*

$$R(F) := \{\alpha \in \Phi \mid \langle \alpha, \beta \rangle \geq 0 \ \forall \beta \in F\}$$

defines a bijective correspondence between facets of Φ and parabolic subsets of Φ. Moreover, this correspondence inverts inclusions, meaning that $R(F_1) \subset R(F_2)$ whenever $F_2 \subset F_1$.

Definition 1.4.10. For every facet F, we call the *reduction of Φ to F* to be the subspace

$$(V_F := (\operatorname{span}(F))^\perp \subset V, \Phi_F := \Phi \cap V_F).$$

Remark 1.4.11. If we denote

$$U(F) := \{\alpha \in \Phi \mid \exists \lambda \in F \colon \langle \alpha, \lambda \rangle > 0\},$$

then we have

$$R(F) = U(F) \cup \Phi_F.$$

Definition 1.4.12. Let Λ denote the set of *weights*, i.e. the set of $\lambda \in V$ such that $\alpha^\vee(\lambda) \in \mathbb{Z}$ for all $\alpha \in \Phi$. For a basis $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ of Φ, a weight $\lambda \in \Lambda$ is *dominant* if $\alpha_i^\vee(\lambda) \geq 0$ for all $i = 1, \cdots, n$. Denote by

$$\Lambda_{\mathrm{fd}} := \{\lambda_1, \cdots, \lambda_n\} \subset \Lambda$$

the system of fundamental dominant weights, i.e. such that $\alpha_j^\vee(\lambda_i) = \delta_{ij}$.

For any facet F of V, define the set of vertices of F as

$$\mathrm{vert}(F) := \{\lambda \in \Lambda_{\mathrm{fd}} \mid \lambda \in \overline{F}\}.$$

Hence,

$$F = \{\sum a_i\lambda_i,\ a_i > 0,\ \lambda_i \in \mathrm{vert}(F)\}.$$

Furthermore, if $C \in \mathcal{C}(V, \Phi)$ corresponds to a basis $\{\alpha_1, \cdots, \alpha_n\}$, then

$$\mathrm{vert}(C) = \{\lambda_1, \cdots, \lambda_n\}$$

is the set of fundamental dominant weights with respect to this basis.

Definition 1.4.13. Given two chambers $C, D \in \mathcal{C}(V, \Phi)$ are (α-)conjugated if they have $n - 1$ vertices in common and there exists a unique root $\alpha \in \Phi$ such that $\alpha^\vee |_C > 0$ and $\alpha^\vee |_D < 0$.

Definition 1.4.14. A *complementary polyhedron* for (V, Φ) consists of a map

$$\begin{aligned} d\colon\ \mathcal{C}(V, \Phi) &\longrightarrow V^\vee \\ C &\longmapsto d(C) \end{aligned}$$

such that, for every pair of chambers $C, D \in \mathcal{C}(V, \Phi)$,

C1. For every common vertex $\lambda \in \mathrm{vert}(C) \cap \mathrm{vert}(D)$, we have

$$d(C)(\lambda) = d(D)(\lambda).$$

C2. If they are α-conjugated, then

$$d(C)(\alpha) \leq d(D)(\alpha).$$

Thanks to property C1 in the definition of d, given a facet F, we can choose $C \in \mathcal{C}(V, \Phi)$ such that $\overline{C} \supset F$ and define the *dual polyhedron* of F as

$$d(F) := \mathrm{ConvexHull}\{d(C) \mid \overline{C} \supset F\}.$$

Note that

$$d(\{0\}) = \mathrm{ConvexHull}\{d(C) \mid C \in \mathcal{C}(V, \Phi)\}.$$

Moreover,

$$\begin{aligned} \deg F &:= \sum_{\alpha \in U(F)} d(C)(\alpha) \\ &= \sum_{\alpha \in R(F)} d(C)(\alpha). \end{aligned}$$

Definition 1.4.15. We say that d is *semistable* if any of these equivalent conditions hold

i) $\deg F \leq 0$ for every facet F.

ii) $\deg F \leq 0$ for every one-dimensional facet F.

iii) $0 \in d(\{0\})$

Let F be a facet and $\lambda \in \mathrm{vert}(F)$. Define

$$\psi(F, \lambda) := \{\alpha \in \Phi \mid \lambda^\vee(\alpha) = 1,\ \mu^\vee(\alpha) = 0,\ \forall \mu \in \mathrm{vert}(F) \setminus \{\lambda\}\}.$$

The *numerical invariant* of F with respect to λ and d is

$$n(F, \lambda) := \sum_{\alpha \in \psi(F,\lambda)} d(C)(\alpha).$$

Definition 1.4.16. We say that a facet F is *special* with respect to d if

i) $n(F, \lambda) > 0$ for every $\lambda \in \mathrm{vert}(F)$.

ii) The reduction (V_F, Φ_F) with

$$d_F \colon \mathcal{C}(V_F, \Phi_F) \to V_F^\vee$$

is semistable.

The following proposition gives a characterization of special facets in terms of the dual polyhedron.

Proposition 1.4.17 ([9, Proposition 3.13])**.** *Let*

$$d \colon \mathcal{C}(V, \Phi) \to V^\vee$$

be a complementary polyhedron. Let $y(d)$ be the unique point of $V^\vee$ in $d(\{0\})$ closest to 0*. Then, F is special if $y(d) \in d(F)$. In fact,*

$$F^\vee \cap d(F) = \{y(d)\}.$$

Theorem 1.4.18 ([9, Corollary 3.14])**.** *Every root system Φ with complementary polyhedron d has a unique special facet.*

Example 1.4.19. Type A_1
Let $V = \mathbb{R}$ endowed with the standard inner product and denote the set of roots $\Phi = \{\pm 1\}$, with Cartan Matrix $A = (2)$ and $\Lambda_{fd} = \{\pm 1/2\}$ is the set of fundamental weights.

The vector space V decomposes into 3 facets: $\{0\}$ and $P^\pm := \mathbb{R}_{>0}(\pm 1)$.

To give a complementary polyhedron

$$d \colon \mathcal{C}(V, \Phi) \to V^\vee$$

on Φ, we fix $d^\pm := d(P^\pm) \in V^\vee$. Suppose $d^\pm = x^\pm$ (constant). Now, we check the properties of Definition 1.4.14:

C1. Doesn't apply in this case since the facets are all pair-wise disjoint.

C2. Since $P^{\pm}$ are 1-conjugate, i.e. they are disjoint, $1\,|_{P^+} > 0$ and $1\,|_{P^-} < 0$, we have $d^+(1) \leq d^-(1)$, i.e. $2x^+ \leq 2x^-$.

Hence, it corresponds to the interval

$$d(\{0\}) = [x^+, x^-].$$

Then, (Φ, d) is semistable if and only if $0 \in d(\{0\})$, i.e. if and only if

$$x^+ \leq 0 \leq x^-.$$

If $x^+ > 0$ (resp. $x^- < 0$), then P^+ (resp. P^-) is the special facet.

1.4.2 Root data and linear algebraic groups

Let G be a connected reductive linear algebraic group over an algebraically closed field K of characteristic 0, we denote by $\mathfrak{g}$ its Lie algebra. Given a maximal torus $T \subset G$, with Lie algebra $\mathfrak{t}$, let

$$X^*(T) = \mathrm{Hom}(T, K^*)$$

be its character group. The adjoint representation

$$\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$$

gives a decomposition of $\mathfrak{g}$ into root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in X^*(T)} \mathfrak{g}_\alpha$$

where

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}(t)X = \alpha(t)X,\ \forall t \in T\}.$$

Then, we write

$$\Phi(G, T) \coloneqq \{\alpha \in X^*(T) \mid \mathfrak{g}_\alpha \neq 0\}$$

to denote the roots of G with respect to T. Note that this set is finite since the dimension of $\mathfrak{g}$ is finite. Define

$$V \coloneqq \mathrm{span}(\Phi(G, T)) \otimes_{\mathbb{Z}} \mathbb{R}$$

equipped with euclidean scalar product. Then, $\Phi \coloneqq \Phi(G, T)$ is a reduced root system in V of rank $= \dim(V)$, equal to the semisimple rank of G.

Given a root $\alpha \in \Phi$, we will denote by U_α the *root subgroup* of G i.e. the unique one-dimensional connected unipotent subgroup of G normalized by T with Lie algebra $\mathrm{Lie}(U_\alpha) = \mathfrak{g}_\alpha$.

Now, let $T \subset B \subset G$ be a Borel subgroup containing the fixed torus. It corresponds to give a basis $\Delta \subset \Phi(G,T)$. For a root $\alpha \in \Phi$, denote by s_α its corresponding reflection. The s_α are in bijection with elements $S_\alpha \in N_G(T)/T$, where $N_G(T)$ denotes the normalizer of T in G. Given a subset $I \subset \Delta$, define W_I to be the subgroup generated by all S_α for $\alpha \in I$. Then,

$$P_I := BW_IB$$

is a parabolic subgroup of G. In particular, $P_\emptyset = B$ and $P_\Delta = G$. Furthermore, the roots of P_I with respect to T are

$$(\Phi^+ \cup \Phi^-) \cap \Phi_I$$

where Φ_I is the set of roots that integral linear combinations of elements of I. Note that all parabolic subgroups of G containing B are of this form.

Remark 1.4.20. Here we are basically using the bijective correspondence stated in [9, Lemma 5.2], between parabolic subgroups of G containing T and facets of $\Phi(G,T)$.

More details also found in [57].

Example 1.4.21. Type A_{n-1}, for $n > 1$.

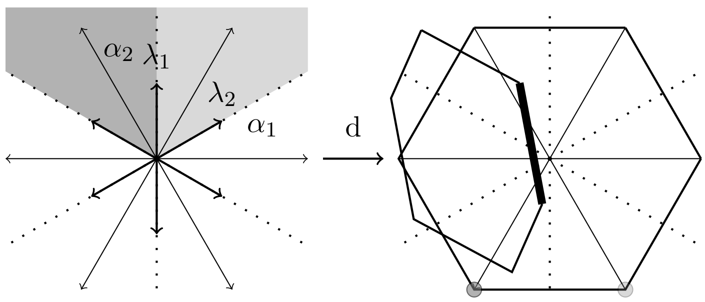

Figure 1.2: Semistable and **unstable** complementary polyhedra for A_2.

Let $G = \mathrm{GL}(n)$ with Lie algebra $\mathfrak{g} = \mathrm{Mat}(n \times n)$. Denote by $T \subset G$ the maximal torus of diagonal matrices. Let $E_{ij} \in \mathfrak{g}$ be the matrix with 1 at (i,j) and 0 elsewhere. The set of roots of G corresponding to T is

$$\Phi(G,T) = \{\alpha_{i,j} \mid 1 \leq i \neq j \leq n\}$$

where

$$\begin{aligned} \alpha_{i,j}\colon \quad & T \longrightarrow \mathbb{G}_m \\ & \mathrm{diag}(t_1, \cdots, t_n) \longmapsto t_i t_j^{-1} \end{aligned}$$

for $1 \leq i \neq j \leq n$. The Weyl group W is isomorphic to the symmetric group $\mathcal{S}_n$. For $\{e_i\}_{i=1,\cdots,n}$ the standard orthonormal basis of $\mathbb{R}^n$ and $\sigma \in \mathcal{S}_n$, denote the corresponding permutation matrix

$$P_\sigma := (e_{\sigma(1)}, \cdots, e_{\sigma(n)}) \in W.$$

Let $B \subset G$ denote the Borel subgroup of upper triangular matrices and consider a basis of $\Phi(G,T)$ corresponding to B by

$$\Delta_B := \{\alpha_{i,i+1}\}_{1\leq i\leq n-1}.$$

If we consider

$$I := \{\alpha_{i_0,i_0+1}\} \subset \Delta_B$$

for some $1 \leq i_0 \leq n-1$, then the corresponding parabolic subgroup is of the form

$$P_I = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i_0} & a_{1i_0+1} & a_{1i_0+2} & \cdots & a_{1n-1} & a_{1n} \\ 0 & a_{22} & \cdots & a_{2i_0} & a_{2i_0+1} & a_{2i_0+2} & \cdots & a_{2n-1} & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{i_0i_0} & a_{i_0i_0+1} & a_{i_0i_0+2} & \cdots & a_{i_0n-1} & a_{i_0n} \\ 0 & 0 & \cdots & a_{i_0+1i_0} & a_{i_0+1i_0+1} & a_{i_0+1i_0+2} & \cdots & a_{i_0+1n-1} & a_{i_0+1n} \\ 0 & 0 & \cdots & 0 & 0 & a_{i_0+2i_0+2} & \cdots & a_{i_0+2n-1} & a_{i_0+2n} \\ \vdots & \vdots & \ddots & & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & a_{n-1n-1} & a_{n-1n} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & a_{nn} \end{pmatrix}.$$

By considering the permutation matrix P_σ and

$$B_\sigma := P_\sigma B P_\sigma^{-1}$$

we obtain exactly $n!$ Borel subgroups containing T (Note that $B_{\mathrm{id}} = B$). Hence, to give a complementary polyhedron on $\Phi(G,T)$ consists to define $n!$ points in

$$V^\vee = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_1 + \cdots + x_n = 0\}$$

satisfying the 2 conditions in Definition 1.4.14.

For example, see Figure 1.2 for the case $n = 1$. There, at left hand side one sees a reduced root system of $\mathbb{R}^2$ with two α_2-conjugated chambers shaded. At right hand side, one finds an example of semistable complementary polyhedron (with the shape of a regular hexagon, whose convex hull contains the origin) with the fat points corresponding to the previous α_2-conjugated chambers, as well as an example of unstable complementary polyhedron (with the shape of an irregular hexagon, whose convex hull does not contain the origin and where the bold edge corresponds to the image of the special facet under the complementary polyhedron).

1.4.3 Semi-simple and reductive group schemes

We recall some facts from the theory of reductive group schemes.

Definition 1.4.22. Let K be a number field with ring of integers $\mathcal{O}_K$. A *split torus* over K is a group scheme $T/\operatorname{Spec}(K)$ that is isomorphic to $\mathbb{G}_m^d/\operatorname{Spec}(K)$. Such a split torus has a unique extension to a split torus $\mathcal{T}$ over $\mathcal{O}_K$. Then, we have the character module

$$X^*(T) = X^*(\mathcal{T}) = \operatorname{Hom}(\mathcal{T}, \mathbb{G}_m) \cong \mathbb{Z}^d.$$

A *torus* over K, denoted $T/\operatorname{Spec}(K)$, is a group scheme, which splits over a finite extension L/K, i.e. $T_L := T \times_{\operatorname{Spec}(K)} \operatorname{Spec}(L)$ is a split torus over L. Let $\overline{K}$ denote an algebraic closure of K, there is a smallest extension (which is normal and separable) $K \subset K' \subset \overline{K}$ such that $T_{K'}$ splits. Such a torus extends to a torus over $\operatorname{Spec}(\mathcal{O}_K)$ if and only if the extension K'/K is unramified.

A smooth connected group scheme G/F over an arbitrary field F is called *reductive* if it has no nontrivial unipotent normal subgroups. It is called *semisimple* if its connected center is trivial.

Definition 1.4.23. A group scheme $\mathcal{G}/\operatorname{Spec}(\mathcal{O}_K)$ is *semisimple* (resp. *reductive*) if it is a smooth, affine group scheme of finite type over $\operatorname{Spec}(\mathcal{O}_K)$ whose fibers $\mathcal{G} \times_{\operatorname{Spec}(\mathcal{O}_K)} \operatorname{Spec}(K(\mathfrak{p}))$ are semisimple (resp. reductive) over $\operatorname{Spec}(K(\mathfrak{p}))$ for all points $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K)$. The group scheme $\mathcal{G}$ is called *split* if it has a split maximal torus $\mathcal{T} \subset \mathcal{G}$ i.e. such that every fiber

$$\mathcal{T} \times_{\operatorname{Spec}(\mathcal{O}_K)} \operatorname{Spec}(K(\mathfrak{p})) \subset \mathcal{G} \times_{\operatorname{Spec}(\mathcal{O}_K)} \operatorname{Spec}(K(\mathfrak{p}))$$

is a split maximal torus for all $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_K)$.

Remark 1.4.24. Any reductive group scheme is locally split for the étale topology.

In general, the type of a reductive group scheme is the set of simple roots Δ, together with its structure as Dynkin diagram and the action of the Galois group on it. In the following lines, we describe this latter action.

Let $S := \operatorname{Spec}(F)$ where F is a field. Then, a reductive group scheme $\mathcal{G}/S$ has a maximal torus $\mathcal{T}/S \subset \mathcal{G}/S$ that splits over a finite unramified extension F_1/F. We denote

$$\mathcal{T}_1 := \mathcal{T} \times_S \operatorname{Spec}(F_1).$$

Then, the Galois group $\operatorname{Gal}(F_1/F)$ acts on the characters $X^*(\mathcal{T}_1)$ and on the set of roots $\Phi(\mathcal{G}, \mathcal{T})$ by permutations.

Now, if we have two tori $\mathcal{T}$, $\mathcal{T}'$, then there exists an unramified extension F'/F which splits both tori. Given two Borel subgroups $\mathcal{B} \supset \mathcal{T}$ and $\mathcal{B}' \supset \mathcal{T}'$, denote the corresponding basis of (simple) positive roots as $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{B}'}$ respectively. Then, there is a unique element $g \in \mathcal{G}(F')$ (unique up to an element in $\mathcal{T}(F')$) which conjugates the tori and the Borel subgroups and hence it provides a bijection between $\Delta_{\mathcal{B}}$ and $\Delta_{\mathcal{B}'}$. Hence, we identify them and omit the reference to the Borel subgroup by writing just Δ.

Take now $\sigma \in \mathrm{Gal}(F_1/F)$, we can consider $\mathcal{T}^\sigma = \mathcal{T}'$ and $\mathcal{B}^\sigma = \mathcal{B}'$ and σ defines a bijection

$$\hat{\sigma} \colon \Delta = \Delta_{\mathcal{B}} \to \Delta_{\mathcal{B}'} = \Delta$$

and hence we get an action of the Galois group on Δ. If this action is trivial, then $\mathcal{G}$ is of *inner type*, otherwise, it is of *outer type*.

1.5 The data at infinity

In this section we will focus on the local archimedian case, i.e. we look at the additional datum at infinity attached to Arakelov group schemes (i.e. group schemes on arithmetic curves that will be properly defined in the next section).

Let K_ν denote an archimedian local field, so it is either isomorphic to $\mathbb{R}$ or $\mathbb{C}$. Let G be a split reductive algebraic group over K_ν (with split maximal torus T) and denote by $\mathfrak{g}$ (resp. $\mathfrak{t}$) its associated Lie algebra.

Moreover, we fix an underlying Lie algebra structure $\mathfrak{g}_0$ over the real subfield $F_0 \subset K_\nu$ (also assumed to be fixed). We consider a reduced root system Φ of G_0 associated to T_0.

Definition 1.5.1. Le V be a finite-dimensional complex vector space. A subset R of V is called a *complex root system* if it satisfies the conditions of Definition 1.4.1 where the only difference is that now R spans V as complex vector space.

Remark 1.5.2. In general, let Φ be a root system of a real vector space V_0 and let V be the complexification $V_0 \otimes \mathbb{C}$ of V_0. The space V_0 is embedded in V and Φ is a (complex) root system in V. This can be seen by extending the reflections s_α^0 of V_0 by linearity to V. Actually, [56, Theorem V.5] states that every complex root system can be obtained in this way, reducing the theory of complex root systems to that of real root systems.

Definition 1.5.3. A *Chevalley basis* of $\mathfrak{g}_0$ consists of a set

$$\{X_\alpha \mid \alpha \in \Phi\} \cup \{H_\alpha \mid \alpha \in \Phi\}$$

satisfying the commutation rules:

i) $[X_\alpha, X_{-\alpha}] = H_\alpha$.

ii) $[t, X_\alpha] = \alpha(t)X_\alpha$, for $t \in \mathfrak{t}_0$. In particular, $[H_\alpha, X_\alpha] = 2X_\alpha$.

iii) $[X_\alpha, X_\beta] = c_{\alpha,\beta}X_{\alpha+\beta}$, for $\alpha, \beta \in \Phi$ such that $\alpha+\beta \in \Phi$, with structure constants $c_{\alpha,\beta} \in \mathbb{Z}$.

Remark 1.5.4. A Chevalley basis for a (complex) Lie algebra is a basis constructed by C. Chevalley with the property that all structure constants are integers. This is the starting point for the construction of the so-called *Chevalley groups*, which are analogous to Lie groups over finite fields.

Furthermore, we note that a Chevalley basis is a *Weyl basis*, but with a different normalization. See the definition of Weyl basis in [56, Theorem VI.6] and Chevalley's normalization in [56, Theorem VI.11].

Example 1.5.5. Put $K_\nu = \mathbb{R}$, for ν the only infinite place of $\mathbb{Q}$. For the associated Lie algebra

$$\mathfrak{g} = \mathfrak{sl}(n) \otimes \mathbb{R} = \{X \in \mathrm{Mat}(n, \mathbb{R}) \mid \mathrm{tr}(X) = 0\},$$

we denote by Φ the set of roots associated to the maximal torus T of determinant 1 diagonal matrices. Its character group is

$$\begin{aligned} X^*(T) &= \mathrm{span}\{\epsilon_i - \epsilon_j \mid 1 \leq i < j \leq n\} \\ &\cong \{(a_1, \cdots, a_n) \in \mathbb{Z}^n \mid a_1 + \cdots + a_n = 0\} \cong \mathbb{Z}^{n-1} \end{aligned}$$

where ϵ_j denotes the linear form on the diagonal matrices which assigns to a diagonal matrix its j-th entry. The corresponding root system in

$$V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{n-1}$$

takes the form

$$\begin{aligned} \Phi &= \{\pm(\epsilon_i - \epsilon_j) \mid 1 \leq i < j \leq n\} \\ &\cong \{\pm(e_i - e_j) \mid 1 \leq i < j \leq n\} \subset \mathbb{Z}^n \end{aligned}$$

where e_j denotes the j-th vector in the natural orthonormal basis of $\mathbb{R}^n$.

We fix a Chevalley basis

$$\{X_{\alpha_{ij}} \mid \alpha_{ij} \in \Phi\} \cup \{H_{\alpha_{ij}} \mid \alpha_{ij} \in \Phi\}$$

with

- $X_{\alpha_{ij}} \coloneqq E_{ij}$
- $H_{\alpha_{ij}} \coloneqq E_{ii} - E_{jj}$

for $i < j$ or $j < i$, $1 \leq i, j \leq n$, where E_{ij} denotes the matrix with 1 at entry (i, j) and 0 elsewhere.

In fact, if we let $X_\alpha = E_{i_1 j_1}$ and $X_\beta = E_{i_2 j_2}$, then $\alpha + \beta \in \Phi$ if and only if one of the following two cases occur:

- $i_1 = j_2$: in this case $[X_\alpha, X_\beta] = -E_{i_2 j_1}$ and hence $c_{\alpha,\beta} = -1$
- $i_2 = j_1$: in this case $[X_\alpha, X_\beta] = E_{i_1 j_2}$ and hence $c_{\alpha,\beta} = 1$.

Definition 1.5.6. Now we define the standard involution

$$\Theta_0 \colon \mathfrak{g}_0 \to \mathfrak{g}_0$$

as follows:

1. $\Theta_0(t) = -t$, for $t \in \mathfrak{t}_0$
2. $\Theta_0(X_\alpha) = -X_{-\alpha}$, for $\alpha \in \Phi$.

Remark 1.5.7. This extends to an F_0-linear involutive Lie algebra automorphism of $\mathfrak{g}_0$. If $K_\nu \cong \mathbb{C}$, it extends to an antilinear automorphism of $\mathfrak{g}$.

Now, we note that the corresponding Cartan decomposition of $\mathfrak{g}$ associated with Θ_0 is of the form $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{p}_0$, where

$$\mathfrak{h}_0 = \{X \in \mathfrak{g} \mid \Theta_0(X) = X\}$$

and

$$\mathfrak{p}_0 = \{X \in \mathfrak{g} \mid \Theta_0(X) = -X\}.$$

Denote by H_0, the maximal compact subgroup of G whose Lie algebra is $\mathfrak{h}_0$.

The space G/H_0 parametrizes the maximal compact subgroups of G by associating $H \coloneqq gH_0g^{-1}$ to every class $x = gH_0$. In the same way, one associates to Θ_0 the involution Θ_x,

$$\Theta_x(X) \coloneqq g\Theta_0(X)g^{-1}.$$

which is compatible with the $\mathrm{SL}(n,\mathbb{R})$-action, i.e. if $x' = g'x$, then

$$\Theta_{x'}(X) \coloneqq g'\Theta_x(X)g'^{-1}.$$

On $\mathfrak{g}$ we have the Killing form

$$(X,Y) \coloneqq \mathrm{tr}(\mathrm{ad}(X)\,\mathrm{ad}(Y))$$

which is invariant on $\mathfrak{g}$ in the sense that

$$([Z,X],Y) + (X,[Z,Y]) = 0.$$

Combining it with Θ_x, we obtain a symmetric bilinear (if $K_\nu \cong \mathbb{R}$) resp. a hermitean form (if $K_\nu \cong \mathbb{C}$),

$$(X,Y)_x \coloneqq -(X,\Theta_x(Y)), \tag{1.5.1}$$

for $X,Y \in \mathfrak{g}$.

Proposition 1.5.8 ([56, section V.1])**.** *With the previous notations.*

i) The radical of $(,)_x$ is the center $Z(\mathfrak{g})$ of $\mathfrak{g}$.

ii) $(,)_x$ is positive definite and nondegenerate on the semisimple part of $\mathfrak{g}$.

iii) One has an orthogonal decomposition with respect to the Killing form $(,)$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi^+} (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha})$$

and with respect to the form $(,)_0$

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} (\mathfrak{g}_\alpha)$$

where $\mathfrak{g}_\alpha$ denotes the root subspace $\mathbb{R}X_\alpha$ of $\mathfrak{g}$.

iv) For every $\alpha \in \Phi$, one has

$$\begin{aligned}(H_\alpha, H_\alpha)_0 &= 2(X_\alpha, X_\alpha)_0 \\ &= \sum_{\beta\in\Phi} \beta(H_\alpha)^2.\end{aligned}$$

Proof. Parts *i)* and *ii)* follow directly from the definition, i.e. $(,)_x$ inherits these properties from the Killing form.

Part *iii)* is a standard result (see [56, Theorems VI.1 - V.3]).

To see part *iv)*, note that for the Killing form we have

$$\begin{aligned}(H_\alpha, H_\alpha) &= (H_\alpha, [X_\alpha, X_{-\alpha}]) \\ &= -([X_\alpha, H_\alpha], X_{-\alpha}) \\ &= (2X_\alpha, X_{-\alpha}) \\ &= 2(X_\alpha, X_{-\alpha}).\end{aligned}$$

Hence,

$$\begin{aligned}(H_\alpha, H_\alpha)_0 &= -(H_\alpha, \Theta_0(H_\alpha)) \\ &= -(H_\alpha, -H_\alpha) \\ &= (H_\alpha, H_\alpha) \\ &= 2(X_\alpha, X_{-\alpha}) \\ &= -2(X_\alpha, -X_{-\alpha}) \\ &= -2(X_\alpha, \Theta_0(X_\alpha)) \\ &= 2(X_\alpha, X_\alpha)_0.\end{aligned}$$

Furthermore,

$$\begin{aligned}(H_\alpha, H_\alpha)_0 &= (H_\alpha, H_\alpha) \\ &= \operatorname{tr}(\operatorname{ad}(H_\alpha)\operatorname{ad}(H_\alpha)) \\ &= \sum_{\beta\in\Phi} \beta(H_\alpha)^2.\end{aligned}$$

where the last equality holds since we have

$$\operatorname{ad}(H_\alpha)\operatorname{ad}(H_\alpha)(X_\beta) = \beta(H_\alpha)^2$$

for all $\beta \in \Phi$, and

$$\operatorname{ad}(H_\alpha)\operatorname{ad}(H_\alpha)(H_\beta) = 0.$$

$\square$

Example 1.5.9. Assume notations of Example 1.5.5. Define the standard involution

$$\Theta_0 \colon \mathfrak{g} \to \mathfrak{g}$$

as follows:

1. $\Theta_0(t) = -t$, for $t \in \mathfrak{t}$
2. $\Theta_0(X_\alpha) = \Theta_0(E_{ij}) = -E_{ji} = -X_{-\alpha}$, for $\alpha \in \Phi$.

Now, associated with Θ_0 the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{h}_0 \oplus \mathfrak{p}_0$, where

$$\begin{aligned}\mathfrak{h}_0 &= \{X \in \mathfrak{g} \mid \Theta_0(X) = X\}\\ &= \{X \in \mathfrak{g} \mid X^t = -X\}\\ &= \mathrm{Lie}(\mathrm{SO}(n, \mathbb{R})).\end{aligned}$$

and

$$\begin{aligned}\mathfrak{p}_0 &= \{X \in \mathfrak{g} \mid \Theta_0(X) = -X\}\\ &= \{X \in \mathfrak{g} \mid X^t = X\}\\ &= \text{symmetric matrices.}\end{aligned}$$

Denote by $H_0 \coloneqq \mathrm{SO}(n, \mathbb{R})$, the maximal compact subgroup of $\mathrm{SL}(n, \mathbb{R})$ whose Lie algebra $\mathrm{Lie}(H_0) = \mathfrak{h}_0$.

In the case $n = 2$, we have

$$\Phi = \{\alpha_1 = (1, -1), \alpha_2 = -\alpha_1\}.$$

A Chevalley basis is given by

$$e_1 \coloneqq X_{\alpha_1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ e_2 \coloneqq X_{\alpha_2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ e_3 \coloneqq H_{\alpha_1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Furthermore,

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},\ \mathrm{ad}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix},\ \mathrm{ad}(e_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, the Killing form has matrix form

$$\begin{pmatrix} 0 & 4 & 0 \\ 4 & 0 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Therefore, combining it with the standard Cartan involution we get a symmetric bilinear form

$$(X, Y)_0 = -(X, \Theta_0(Y))$$

which in this particular case has diagonal matrix form

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

For any pair (B, Θ) where B is a Borel subgroup of G over K_ν and $\Theta = \Theta_x$ is a Cartan involution corresponding to a maximal compact subgroup H (corresponding to a point $x \in G/F_0$) one can associate in particular the restriction of the form $(,)_x$ to the unipotent radical $\mathrm{rad}_u(B)$ of $\mathrm{Lie}(B)$ over K_ν as well as the induced forms on the various subquotients u_α of $\mathrm{rad}_u(B)$ belonging to the roots. One should remark that these subquotients depend only upon the Borel group B resp. its Lie algebra $\mathrm{Lie}(B)$ and do not require to fix a maximal split torus in B.

Remark 1.5.10 (Normalization)**.** The process of fixing the above metrics requires some normalization because later on we want the Chevalley schemes with standard base and standard involution as above to be semistable Arakelov group schemes, which would not be the case, if we worked just with the forms or metrics $(,)_x$. In particular, with respect to the standard Borel subgroup B_0 given by the Chevalley structure $\{X_\alpha, H_\alpha \mid \alpha \in \Phi\}$ above and $\Theta = \Theta_0$ we obtain

$$(X_\alpha, X_\alpha)_0 = \frac{1}{2}(H_\alpha, H_\alpha)_0.$$

Hence, the metric on a subquotient $\underline{u}_\alpha$ of the unipotent radical $\mathrm{rad}_u(B)$ induced by $(,)_x$ will be multiplied by $2(H_\alpha, H_\alpha)_0^{-1}$, i.e.

$$h_x \mid_{\underline{u}_\alpha} = 2(H_\alpha, H_\alpha)_0^{-1}(,)_x \mid_{\underline{u}_\alpha} .$$

In particular, with respect to the standard Borel subgroup B_0 given by the Chevalley structure above and $\Theta = \Theta_0$, we obtain $h_0 = 1$, so that restricting h_0 to $\mathrm{rad}_u(B_0)$ it is the standard (symmetric bilinear, resp. hermitean) form $\bigoplus_{\alpha \in \Phi^+} \langle 1 \rangle$.

Example 1.5.11. With the notations of Example 1.5.5 and Example 1.5.9, let us consider the case $n = 2$. Let B be standard Borel subgroup of $\mathrm{SL}(2, \mathbb{R})$, consisting of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{11}^{-1} \end{pmatrix}$$

with $a_{11}, a_{12} \in \mathbb{R}$ and $a_{11} \neq 0$. Its unipotent radical $\mathrm{Rad}_u(B)$ consists of matrices of the form

$$\begin{pmatrix} 1 & a_{12} \\ 0 & 1 \end{pmatrix}$$

with $a_{12} \in \mathbb{R}$. In this case, there is only one subquotient of $\mathrm{rad}_u(B)$, which corresponds to α_1, denoted u_{α_1}. Hence,

$$\begin{aligned} h_0 \mid_{\underline{u}_{\alpha_1}} &= 2(H_{\alpha_1}, H_{\alpha_1})_0^{-1}(X_{\alpha_1}, X_{\alpha_1})_0 \\ &= (2/8)4 = 1. \end{aligned}$$

Example 1.5.12. With the notations of Example 1.5.5 and Example 1.5.9, let us consider the case $n = 3$. Let B be standard Borel subgroup of $\mathrm{SL}(3, \mathbb{R})$, consisting of matrices of the form

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11}^{-1}a_{22}^{-1} \end{pmatrix}$$

with $a_{ij} \in \mathbb{R}$ and $a_{11}a_{22} \neq 0$. Its unipotent radical $\mathrm{Rad}_u(B)$ consists of matrices of the form

$$\begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix}$$

with $a_{ij} \in \mathbb{R}$. In this case, there are two subquotients of $\mathrm{rad}_u(B)$ corresponding to the filtration:

$$\mathrm{Rad}_u(B) = U_0 \supset U_1 = \left\{ \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid a_{13} \in \mathbb{R} \right\} \supset U_2 = (1)$$

α_1, denoted u_{α_1}. Hence,

$$\begin{aligned} h_0 \mid_{\underline{u}_{\alpha_1}} &= 2(H_{\alpha_1}, H_{\alpha_1})_0^{-1}(X_{\alpha_1}, X_{\alpha_1})_0 \\ &= (2/12)6 = 1. \end{aligned}$$

1.6 Arakelov group schemes

Given a number field K, let $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ be the arithmetic curve associated to K. Let $G \subset \mathrm{GL}(n, K)$ be a reductive connected affine algebraic group. We first recall the definition of Arakelov group scheme.

Definition 1.6.1 ([39])**.** An *Arakelov group scheme* (of type G)

$$\bar{\mathcal{G}} := (\mathcal{G}, \{H_\nu\}_{\nu \in X_\infty})$$

on X is given by a group scheme $\mathcal{G}$ on $\mathrm{Spec}(\mathcal{O}_K)$ of type G and a maximal compact subgroup $H_\nu \subset G(K_\nu)$ for every $\nu \in X_\infty$.

Remark 1.6.2. The maximal compact subgroups $H_\nu \subset G(K_\nu)$ are unique up to conjugation [13]. This fact makes them *essentially unique* so that for example one may just take $U(n)$ (resp. $O(n)$) as maximal compact subgroup of $\mathrm{GL}(n, \mathbb{C})$ (resp. $\mathrm{GL}(n, \mathbb{R})$).

Proposition 1.6.3. *An Arakelov group scheme*

$$\bar{\mathcal{G}} = (\mathcal{G}, \{H_\nu\}_{\nu \in X_\infty})$$

determines the structure of an Arakelov vector bundle on $\mathrm{Lie}(G|K)$*, given as*

$$(\mathrm{Lie}(\mathcal{G}), \{||.||_\nu\}_{\nu \in X_\infty}).$$

Proof. The inclusion

$$\mathrm{Lie}(\mathcal{G}) \hookrightarrow \mathrm{Lie}(G|K)$$

gives an $\mathcal{O}_K$-lattice in $\mathrm{Lie}(G|K)$. The norms $||.||_\nu$ are defined on $\mathrm{Lie}(G(K_\nu))$ as

$$||X||_\nu^2 = (X, X)_{H_\nu}$$

where $(,)_{H_\nu}$ denotes an euclidean (resp. hermitian) scalar product defined from a Cartan involution associated to H_ν as described in the previous section. $\square$

1.6.1 The numerical invariants of a parabolic subgroup

Let $P \subset G$ be a parabolic subgroup defined over K. Given an Arakelov group scheme $\bar{\mathcal{G}}$ of type G, P extends in a unique way to a parabolic subgroup scheme $\mathcal{P} \hookrightarrow \mathcal{G}$ on $\mathrm{Spec}(\mathcal{O}_K)$. In particular, this has as unipotent radical the subgroup scheme $\mathrm{Rad}_u(\mathcal{P}) \subset \mathcal{P}$ and denote by $\mathrm{rad}_u(\mathcal{P})$ its Lie algebra. Further, the norms $||.||_\nu$ are defined on $\mathrm{Lie}(G(K_\nu))$ and can be restricted to $\mathrm{rad}_u(\mathcal{P}) \otimes_K K_\nu$, so that we obtain an Arakelov subbundle

$$\overline{\mathrm{rad}}_u(P) := (\mathrm{rad}_u(\mathcal{P}), \{||.||_\nu\}_{\nu \in X_\infty})$$

of

$$(\mathrm{Lie}(\mathcal{G}), \{||.||_\nu\}_{\nu \in X_\infty}).$$

Next, we describe the filtration

$$\mathrm{Rad}_u(\mathcal{P}) = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \cdots \supset \mathcal{U}_r \supset (1) \tag{1.6.1}$$

by normal unipotent subgroup schemes extending the corresponding filtration by unipotent subgroups $\{U_i\}_{i=0,\cdots,r}$ of $\mathrm{Rad}_u(P)$ over K.

Take a split maximal torus $T \subset P$ and let B be a Borel subgroup such that

$$T \subset B \subset P$$

are all defined over K. Given a root system $\Phi(G,T)$ associated with T, denote by

$$\Delta := \{\alpha_1, \cdots, \alpha_n\}$$

a basis associated with B. Note that $P/\mathrm{Rad}_u(P)$ is a split reductive group over K with split maximal torus $\overline{T}$ isomorphic to T under the projection morphism

$$P \to P/\mathrm{Rad}_u(P).$$

We say the type of the parabolic subgroup P is

$$t(P) = \{\alpha_1, \cdots, \alpha_r\}.$$

Denote by $\{\lambda_1^\vee, \cdots, \lambda_r^\vee\}$ the set of coweights corresponding to the set of roots $t(P)$, consider the free abelian group $\bigoplus\limits_{i=1}^{r} \mathbb{Z}\lambda_i^\vee$.

Definition 1.6.4. A vector

$$v = \sum_{i=1}^{r} a_i \lambda_i^\vee \in \bigoplus_{i=1}^{r} \mathbb{Z}\lambda_i^\vee$$

is called positive if $a_i \geq 0$ for all i.

For two vectors $v = \sum\limits_{i=1}^{r} a_i \lambda_i^\vee$ and $v' = \sum\limits_{i=1}^{r} a_i' \lambda_i^\vee$, we write $v' > v$ if $v' - v$ is positive and $v' \neq v$.

Definition 1.6.5. We denote the set of positive roots by

$$\Phi^+ = \{\alpha \in \Phi(G,T) \mid \alpha = \sum_{i=1}^{n} r_i\alpha_i;\ r_i \geq 0\ \forall i\}.$$

1. Given a positive vector $v = \sum_{i=1}^{r} a_i\lambda_i^\vee \in \bigoplus_{i=1}^{r} \mathbb{Z}\lambda_i^\vee$ we set

$$\Omega(v) \coloneqq \{\alpha \in \Phi^+ \mid \langle \alpha, \lambda_i^\vee \rangle \geq a_i\ \forall i\}.$$

2. Let $U(v)$ denote the unipotent subgroup generated by the root subgroups U_α, for $\alpha \in \Omega(v)$ and

$$W(P,v) \coloneqq U(v)/\langle U(v') \mid v' > v\rangle.$$

3. Given a root $\alpha \in \Phi(G,T)$, its *height* is

$$\mathrm{ht}(\alpha) \coloneqq \sum_{i=1}^{r} \langle \alpha, \lambda_i^\vee \rangle.$$

Lemma 1.6.6 ([27, Lemma 5.4.4])**.** *Let U_j be normal unipotent subgroups with $j = 0, \cdots, r$ appearing in the filtration of* $\mathrm{Rad}_u P$ (1.6.1)*. Given $j = 0, \cdots, r$, U_j coincides with the algebraic subgroup of* $\mathrm{Rad}_u(P)$ *generated by all root subgroups $U_\alpha \subset \mathrm{Rad}_u(P)$ such that* $\mathrm{ht}(\alpha) \geq j+1$*.*

Example 1.6.7. With the notations of Example 1.4.21 for $G = \mathrm{GL}(n+1)$, consider the case $n = 3$. Denote by $T \subset G$ the maximal torus of diagonal matrices. Let $E_{ij} \in \mathfrak{g}$ be the matrix with 1 at (i,j) and 0 elsewhere. The set of roots of G corresponding to T is

$$\Phi(G,T) = \{\alpha_{i,j} \mid 1 \leq i \neq j \leq 4\}$$

where

$$\begin{aligned} \alpha_{i,j}\colon \quad & T \longrightarrow \mathbb{G}_m \\ \mathrm{diag}(t_1, \cdots, t_4) & \longmapsto t_i t_j^{-1} \end{aligned}$$

for $1 \leq i \neq j \leq 4$.

Then, its lie algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha_{ij} \in \Phi} \mathfrak{g}_{\alpha_{ij}}$$

where in this case $\mathfrak{g}_{\alpha_{ij}} = E_{ij}$ and the corresponding root subgroups are of the form

$$U_{\alpha_{ij}} = \{\mathrm{Id} + cE_{ij} \mid c \neq 0\}$$

for $1 \leq i \neq j \leq 4$.

Let $B \subset G$ denote the Borel subgroup of upper triangular matrices and consider a basis of $\Phi(G,T)$ corresponding to B by

$$\Delta_B \coloneqq \{\alpha_{i,i+1}\}_{1 \leq i \leq 3}.$$

If we consider

$$I \coloneqq \{\alpha_{1,2}\} \subset \Delta_B$$

then, the corresponding parabolic subgroup is of the form

$$P \coloneqq P_I = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

and $t(P) = \{\alpha_{2,3}, \alpha_{3,4}\}$. Its unipotent radical $\operatorname{Rad}_u P$ consists of matrices of the form

$$\begin{pmatrix} 1 & 0 & a_{13} & a_{14} \\ 0 & 1 & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The filtration of $\operatorname{Rad}_u P$ given by (1.6.1) has the following shape

$$\operatorname{Rad}_u P = U_0 \supset U_1 \supset U_2 \supset (1)$$

and consists of the following normal unipotent subgroups

i) $U_0 = \operatorname{Rad}_u P$

ii) $U_1 = \{A \in U_0 \mid a_{23} = 0\}$

iii) $U_2 = \{A \in U_1 \mid a_{13} = a_{24} = 0\}$.

Let us consider U'_j the algebraic subgroup of $\operatorname{Rad}_u(P)$ generated by all root subgroups U_α with $\operatorname{ht}(\alpha) \geq j+1$. We see that $U'_j = U_j$ for $j = 1, 2$.

For the case $j = 1$, the set of roots with α with $\operatorname{ht}(\alpha) \geq 2$ is $\{\alpha_{13}, \alpha_{14}, \alpha_{24}\}$. Then, it follows that $U_1 = U'_1$.

For the case $j = 2$, the set of roots with α with $\operatorname{ht}(\alpha) \geq 3$ is $\{\alpha_{14}\}$. Then, it follows that $U_2 = U'_2$.

Remark 1.6.8. 1. There is a decomposition

$$U_j/U_{j+1} = \bigoplus_{v \in \mathcal{T}_{j+1}} W(P, v)$$

where we put

$$\mathcal{T}_{j+1} \coloneqq \{v = \sum_{i=1}^{r} a_i \lambda_i^\vee \mid \sum_{i=1}^{r} a_i = j+1\}.$$

2. The definition of all these unipotent groups depends only on the choice of a pair $B \subset P$, i.e. upon fixing a basis of roots, but it is independent of the choice of the split torus T.

3. All these groups can be extended to $\mathrm{Spec}(\mathcal{O}_K)$ since we assume $G \subset \mathrm{GL}(n, K)$ be a split reductive connected affine algebraic group [26, Theorem 1.2] and [30, XXV]. The extension is uniquely determined up to group isomorphism.

Finally, recall the Arakelov vector bundle $\overline{\mathrm{rad}}_u(P)$ on $\mathrm{rad}_u(P)$ induces Arakelov vector bundles $\underline{\underline{u_j}}$ on

$$\underline{u_j} \coloneqq Lie(U_j),$$

$\underline{\underline{u_j}}/\underline{\underline{u_{j+1}}}$ on

$$\underline{u_j}/\underline{u_{j+1}} \coloneqq Lie(U_j/U_{j+1})$$

and $\underline{\underline{w}}(P, v)$ on

$$\underline{w}(P, v) \coloneqq Lie(W(P, v)).$$

Definition 1.6.9. For a positive vector $v \in \bigoplus_{i=1}^{r} \mathbb{Z}\lambda_i^\vee$, we define the *numerical invariant* of P with respect to v as

$$n(P, v) \coloneqq \deg(\underline{\underline{w}}(P, v)).$$

1.6.2 The numerical invariants of a Borel subgroup

Next, we show an alternative construction of the numerical invariants for the case that $P = B$ is a Borel subgroup.

Let C be a (Weyl) chamber in

$$\mathcal{C}(X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}, \Phi(G, T))$$

with corresponding Borel subgroup $B \subset G$ and $\{\alpha_1, \cdots, \alpha_n\}$ its associated basis of $\Phi(G, T)$. Note that C defines an ordering of Φ by requiring that $\alpha <_C \beta$ if and only if $\beta - \alpha$ is positive with respect to C. Note that we will omit the reference to the chamber C when it is clear from the context.

Definition 1.6.10. Let α_0 be a root. Then,

$$v \coloneqq \sum_{i=1}^{n} \langle \alpha_0, \lambda_i^\vee \rangle \lambda_i^\vee$$

is a vector in $\bigoplus_{i=1}^{n} \mathbb{Z}\lambda_i^\vee$ which is either positive or negative, according to whether α_0 is positive or negative (with respect to the chamber C). We distinguish between these two cases:

i) The root α_0 is positive. We define

$$V_C(\alpha_0) \coloneqq \bigoplus_{\beta \geq_C \alpha_0} \underline{u}_\beta$$

and

$$V'_C(\alpha_0) \coloneqq \bigoplus_{\beta >_C \alpha_0} \underline{u}_\beta.$$

ii) The root α_0 is negative. We define

$$V_C(\alpha_0) \coloneqq \mathrm{Lie}(T) \oplus \bigoplus_{\beta \not<_C \alpha_0} \underline{u}_\beta$$

and

$$V'_C(\alpha_0) \coloneqq \mathrm{Lie}(T) \oplus \bigoplus_{\beta \not\leq_C \alpha_0} \underline{u}_\beta.$$

The linear subspaces of $\mathrm{Lie}(\mathcal{G})$ defined above extend uniquely to Arakelov subbundles of $\mathrm{Lie}(\mathcal{G})$. Moreover, the quotient bundle $V_C(\alpha_0)/V'_C(\alpha_0)$ is an Arakelov line bundle over X.

Proposition/Definition 1.6.11. *[39, Proposition/Definition 5.9] Let α be a root. Again, we distinguish between these two cases:*

i) The root α is positive. Then, there is a canonical isomorphism

$$\underline{\underline{u}}_\alpha \cong V_C(\alpha)/V'_C(\alpha).$$

ii) The root α_0 is negative. We then define

$$\underline{\underline{u}}_\alpha \coloneqq V_C(\alpha)/V'_C(\alpha).$$

Proposition 1.6.12. *[39, Proposition 5.10] With the above notations, we have the following isomorphisms of Arakelov line bundles:*

1.

$$\underline{\underline{u}}_\alpha \otimes \underline{\underline{u}}_\beta \cong \underline{\underline{u}}_{\alpha+\beta}$$

2.

$$\underline{\underline{u}}_\alpha \otimes \underline{\underline{u}}_{-\alpha} \cong \overline{\mathcal{O}}$$

Remark 1.6.13. We have the orthogonal decomposition

$$\overline{\mathrm{rad}}_u(B) = \bigoplus_{i=1}^{n} \underline{\underline{u}}_{\alpha_i}$$

where $\underline{\underline{u}}_{\alpha_i} = \underline{\underline{w}}(B, \lambda_i^\vee)$. This implies that

$$n(B, \lambda_i^\vee) = \deg(\underline{\underline{u}}_{\alpha_i}).$$

1.6.3 Construction of the complementary polyhedron

Definition 1.6.14. With the above notations, we define a map $d\colon \mathcal{C}(V,\Phi) \to V^*$ as follows

$$d(C) \coloneqq \sum_{i=1}^{n} n(B, \lambda_i^\vee)\lambda_i^\vee \tag{1.6.2}$$

$$= \sum_{i=1}^{n} \deg(\underline{\underline{u}}_{\alpha_i})\lambda_i^\vee \in X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}. \tag{1.6.3}$$

Theorem 1.6.15 ([9, Proposition 6.6] and [39, Theorem 5.3])**.** *The map defined as* (1.6.2) *with* $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$, *is a complementary polyhedron for the root system* $\Phi(G,T)$ *in the sense of Definition 1.4.14.*

Proof. We have to check that the map defined as (1.6.2) satisfies the conditions in Definition 1.4.14 i.e. for every pair of chambers $C, D \in \mathcal{C}(V,\Phi)$,

C1. If there is a common vertex

$$\lambda \in \mathrm{vert}(C) \cap \mathrm{vert}(D),$$

then

$$d(C)(\lambda) = d(D)(\lambda). \tag{1.6.4}$$

C2. If they are α-conjugated, then

$$d(C)(\alpha) \leq d(D)(\alpha). \tag{1.6.5}$$

1. Suppose λ is a fundamental weight for $\Phi = \Phi(G,T)$. Let P be the maximal parabolic subgroup associated to λ such that $T \subset P$ and let B (resp. B') be a Borel subgroup with $T \subset B \subset P$ (resp. $T \subset B' \subset P$).

Let $C = C(B)$ (resp. $D = C(B')$) be the chamber in $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to the Borel subgroup B (resp. B'). We want to verify (1.6.4), i.e. that the calculation of $\deg(\overline{\mathrm{rad}}_u(P))$ is independent of the choice of Borel subgroup and it only depends on the choice of the fundamental weight λ.

For that, suppose $\{\alpha_1, \cdots, \alpha_n\}$ is the basis of Φ corresponding to B such that $t(P) = \{\alpha_1\}$ and $\{\lambda_1^\vee, \cdots, \lambda_n^\vee\}$ denotes the set of coweights. We compute

$$\begin{aligned}
\deg(\overline{\mathrm{rad}}_u(P)) &= \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P)} \deg(\underline{\underline{u}}_\alpha) \\
&= \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P)} \sum_{i=1}^{n} \langle \alpha, \lambda_i^\vee \rangle \deg(\underline{\underline{u}}_{\alpha_i}) \\
&= \langle \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P)} \alpha, \sum_{i=1}^{n} \deg(\underline{\underline{u}}_{\alpha_i})\lambda_i^\vee \rangle \\
&= \langle \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P)} \alpha, d(C) \rangle.
\end{aligned}$$

by [39, Lemma 3.5],

$$\sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P)} \alpha = c\lambda$$

for some $c \in \mathbb{R}_{\geq 0}$. Note that c is independent on the choice of B, it only depends on the choice of the fundamental weight λ. Therefore, this implies

$$\deg(\overline{\mathrm{rad}}_u(P)) = c\langle \lambda, d(C)\rangle$$

and this shows that $\langle \lambda, d(C)\rangle$ is independent on the choice of B.

2) We denote by $\{\alpha_1, \cdots, \alpha_n\}$ (resp. $\{\alpha'_1, \cdots, \alpha'_n\}$) the simple roots defined by the chamber C (resp. D) and the corresponding sets of vertices

$$\mathrm{vert}(C) = \{\lambda_1, \cdots, \lambda_n\}$$

and

$$\mathrm{vert}(D) = \{\lambda'_1, \cdots, \lambda'_n\}.$$

Let B (resp. B') be the Borel subgroup corresponding to the chamber C (resp. D). Moreover, we assume without loss of generality that C and D are α_1-conjugated i.e. $D = s_1(C)$, where

$$s_1(x) = x - \langle x, \alpha_1^\vee\rangle \alpha_1$$

denotes the reflection corresponding to α_1. Therefore, the weights

$$\mathrm{vert}(D) = \{s_1(\lambda_1), \cdots, s_1(\lambda_n)\}$$

are given as

$$\{\lambda_1 - \alpha_1, \lambda_2, \cdots, \lambda_n\}.$$

We write again s_1 for the reflection induced by s_1 in V^*, i.e.

$$s_1(\lambda_1^\vee) = \lambda_1^\vee - \alpha_1^\vee$$

and $s_1(\lambda_j^\vee) = \lambda_j^\vee$ for $j = 2, \cdots, n$.

Now, we note that by definition of the map (1.6.2) we have

$$\begin{aligned} d(D) &= \sum_{i=1}^n n(D, \lambda'_i)\lambda_i'^\vee \\ &= n(D, \lambda_1 - \alpha_1)(\lambda_1^\vee - \alpha_1^\vee) + \sum_{i=2}^n n(D, \lambda_i)\lambda_i^\vee. \end{aligned}$$

Therefore,

$$\begin{aligned} \langle \alpha_1, d(D)\rangle &= n(D, \lambda_1 - \alpha_1)\langle \alpha_1, \lambda_1 - \alpha_1\rangle + \sum_{i=2}^n n(D, \lambda_i)\langle \alpha_1, \lambda_i^\vee\rangle \\ &= -n(D, \lambda_1 - \alpha_1) \end{aligned}$$

since $\langle \alpha_1, \lambda_j^\vee \rangle = \delta_{1,j}$. Further, again by definition we have

$$\begin{aligned}
\langle \alpha_1, d(D) \rangle &= -n(D, \lambda_1 - \alpha_1) \\
&= -\deg(\underline{\underline{u}}_{-\alpha_1}) \\
&= \deg(\underline{\underline{u}}_{\alpha_1}) \\
&= \deg\left(\mathcal{V}_D(\alpha_1)/\mathcal{V}'_D(\alpha_1)\right).
\end{aligned}$$

Next we claim that

$$\{\alpha \in \Phi \mid \alpha_1 \leq_C \alpha\} \subset \{\alpha \in \Phi \mid \alpha \not<_D \alpha_1\}. \tag{1.6.6}$$

If the claim is true, it implies the following inclusion

$$\{\alpha \in \Phi \mid \alpha_1 <_C \alpha\} \subset \{\alpha \in \Phi \mid \alpha \not\leq_D \alpha_1\}. \tag{1.6.7}$$

Together, inclusions (1.6.6) and (1.6.7) imply that we have inclusions

$$\begin{aligned}
V_C(\alpha_1) &\hookrightarrow V_D(\alpha_1), \\
V'_C(\alpha_1) &\hookrightarrow V'_D(\alpha_1),
\end{aligned}$$

inducing a homomorphism of Arakelov line bundles

$$V_C(\alpha_1)/V'_C(\alpha_1) \to V_D(\alpha_1)/V'_D(\alpha_1). \tag{1.6.8}$$

Note that the homomorphism (1.6.8) is equal to the identity on the generic fiber, implying that it is not zero. Then, we obtain the following inequality of degrees

$$\deg(V_C(\alpha_1)/V'_C(\alpha_1)) \leq \deg(V_D(\alpha_1)/V'_D(\alpha_1))$$

which it is exactly the inequality (1.6.5).

Finally, the claim (1.6.6) follows by contradiction, i.e. assume $\alpha_1 \leq_C \alpha$ and $\alpha <_D \alpha_1$. Then, this implies that $\langle \alpha_1, \lambda_i^\vee \rangle = 0$ for $i = 2, \cdots, n$. Therefore, α is a scalar multiple of α_1, i.e. either $\alpha = \alpha_1$ or $\alpha = -\alpha_1$. But $\alpha = \alpha_1$ (resp. $\alpha = -\alpha_1$) contradicts $\alpha <_D \alpha_1$ (resp. $\alpha_1 \leq_C \alpha$). □

1.6.4 Semistability and canonical parabolic subgroup

Given a number field K, let $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ be the arithmetic curve associated to K. Let $G \subset \mathrm{GL}(n, K)$ be a reductive connected affine algebraic group.

Definition 1.6.16. An Arakelov group scheme

$$\bar{\mathcal{G}} = (\mathcal{G}, \{H_\nu\}_{\nu \in X_\infty})$$

(of type G) is *semistable* if for every parabolic subgroup $P \subset G$, we have

$$\deg(\mathcal{G}, \mathcal{P}) := \sum_{u_\alpha \subset \mathrm{rad}_u(P)} \deg(\underline{\underline{u}}_\alpha) \leq 0.$$

Lemma 1.6.17 ([39, Lemma 6.1])**.** *The set of real numbers*

$$\{\deg(\mathcal{G}, \mathcal{P}) \mid P \subset G \text{ parabolic}\}$$

is bounded from above and attains its maximum.

Proof. Follows from Lemma 1.2.20 applied to the associated Arakelov vector bundle on $\mathrm{Lie}(G|K)$, given as

$$(\mathrm{Lie}(\mathcal{G}), \{||.||_\nu\}_{\nu \in X_\infty}).$$

□

Definition 1.6.18. We call *degree of instability* of $\bar{\mathcal{G}}$ the largest degree of its parabolic subgroups, i.e.

$$\deg_i(\bar{\mathcal{G}}) := \max_{P \subset G \text{ parabolic}} \deg(\mathcal{G}, \mathcal{P})$$

which by Lemma 1.6.17 is finite. Hence, $\bar{\mathcal{G}}$ is semistable if $\deg_i(\bar{\mathcal{G}}) \leq 0$.

Theorem 1.6.19 ([39, Proposition 6.2-3])**.** *Every Arakelov group scheme contains a unique canonical parabolic subgroup.*

Proof. **Existence.** If an Arakelov group scheme $\bar{\mathcal{G}}$ is not semistable, then let P be a parabolic subgroup such that it is the largest element in the set of parabolic subgroups of maximal degree in $\mathcal{G}$ with respect to $\bar{\mathcal{G}}$. Then, it is canonical with respect to the complementary polyhedron constructed in Theorem 1.6.15, i.e. $P/\operatorname{Rad}_u(P)$ is semistable and the numerical invariants of P are positive.

If $\tilde{P} := P/\operatorname{Rad}_u(P)$ is not semistable, there exists a parabolic $Q \subset P$ such that $\tilde{Q} := Q/\operatorname{Rad}_u(P)$ has positive degree i.e. $\deg(\overline{\mathrm{rad}}_u(\tilde{Q})) > 0$. However, since

$$\deg(\overline{\mathrm{rad}}_u(Q)) = \deg(\overline{\mathrm{rad}}_u(\tilde{Q})) + \deg(\overline{\mathrm{rad}}_u(P))$$

this would mean that $\deg(\overline{\mathrm{rad}}_u(Q)) > \deg(\overline{\mathrm{rad}}_u(P))$ and this contradicts the assumption of the maximality of degree of P.

Fix a split maximal torus T and a Borel subgroup B (both over K) such that $T \subset B \subset P$. We denote by $\{\alpha_1, \cdots, \alpha_n\}$ a basis for $\Phi(G, T)$ and by $\{\lambda_1^\vee, \cdots, \lambda_n^\vee\}$ the set of coweights such that

$$\mathrm{vert}(P) := \mathrm{vert}(F(P)) = \{\lambda_1, \cdots, \lambda_r\}$$

where $F(P)$ is the facet in $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ associated to P.

Next, consider the unique parabolic subgroup $Q \supset P$ such that the type

$$t(Q) = T(P) \backslash \{\alpha_1\}$$

and therefore

$$\mathrm{vert}(Q) := \mathrm{vert}(F(Q)) = \{\lambda_2, \cdots, \lambda_r\}.$$

By assumption, we have that $\deg(\overline{\mathrm{rad}}_u(P)) > \deg(\overline{\mathrm{rad}}_u(Q))$. Now, let us consider the reduction of the complementary polyhedron from Theorem 1.6.15 to $F(Q)$. We put

$$(V_{F(Q)} \coloneqq ((F(Q))^\vee)^\perp \subset V, \Phi_{F(Q)} \coloneqq \Phi \cap V_{F(Q)})$$

and consider $\mathrm{pr}\colon V \to V_{F(Q)}$ the projection to $V_{F(Q)}$. Then, we compute

$$\begin{aligned}\deg(\overline{\mathrm{rad}}_u(P')) &= \langle \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P')} \alpha, \mathrm{pr}(d(C))\rangle \\ &= \sum_{\lambda' \in \mathrm{vert}(\mathrm{pr}(F(P)))} \frac{n(P',\lambda)}{\#\psi(P',\lambda)} \langle \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P')} \alpha, \lambda^\vee\rangle.\end{aligned}$$

Finally, since $\mathrm{vert}(\mathrm{pr}(F(P))) = \{\lambda_1\}$ and

$$\langle \sum_{\underline{u}_\alpha \subset \mathrm{rad}_u(P')} \alpha, \lambda_1^\vee\rangle > 0$$

it follows that $n(P', \lambda_1) = n(P, \lambda_1) > 0$.

Unicity. Suppose that P and Q are two such canonical subgroups. We find a split maximal torus T such that $T \subset P$ and $T \subset Q$. Since both of them are canonical, it implies that their corresponding facets $F(P)$ and $F(Q)$ in $V = X^*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ are both special. Then, by Theorem 1.4.18 $F(P) = F(Q)$, which implies that $P = Q$. □

Definition 1.6.20. If $\bar{\mathcal{G}}$ is not semistable, then a parabolic subgroup P is called *canonical* when it is the largest element in the set of parabolic subgroups of maximal degree in $\mathcal{G}$ with respect to $\bar{\mathcal{G}}$.

In the next chapter we will see examples of canonical parabolic subgroups (and of their induced filtrations) and compare them to Grayson-Stuhler filtrations for Arakelov vector bundles over arithmetic curves.

Stability of Arakelov bundles via complementary polyhedra

1.7 Semistability of Arakelov vector bundles

Next we use the previous constructions to investigate the notion of semistability for Arakelov vector bundles. Given a number field K, let

$$X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$$

be the arithmetic curve associated to K. Denote by η the generic point.

Theorem 1.7.1. *For an Arakelov vector bundle*

$$\bar{E} = (E, \{\langle \cdot, \cdot \rangle_{E,\nu}\}_{\nu \in X_\infty})$$

of rank n on X, consider the following settings:

1. *By Proposition 1.2.21, $\bar{E}$ has a canonical filtration*

$$0 = \bar{E}_0 \subsetneq \bar{E}_1 \subsetneq \cdots \subsetneq \bar{E}_r = \bar{E},$$

   the so-called GS-filtration of $\bar{E}$.

2. *We consider the group of automorphisms of $\bar{E}$*

$$\mathrm{Aut}(\bar{E}) \coloneqq (\mathrm{Aut}(E), \{H_\nu\}_{\nu \in X_\infty}),$$

   which is an Arakelov group scheme (of type $\mathrm{GL}(n)$*) in the sense of [39]. By Theorem 1.6.19, it has a canonical parabolic subgroup.*

Then, these two settings are equivalent.

Indeed, note that groups of automorphisms of Arakelov vector bundles correspond to Arakelov group schemes of type A_{n-1}. An Arakelov group scheme of type A_{n-1} is semistable if for all maximal parabolic subgroups P (i.e. such that they stabilize flags of length 2) we have

$$\deg(\mathcal{G}, \mathcal{P}) \leq 0.$$

Then, we will see that the GS-filtration has a translation in this context in terms of the canonical parabolic subgroup.

Let us consider the Lie algebra of $\bar{\mathcal{G}} := \mathrm{Aut}(\bar{E})$, as $\bar{\mathfrak{g}} = (\bar{E})^\vee \otimes \bar{E}$. It is indeed an Arakelov vector bundle with the Cartan-Killing metrics. Furthermore, the choice of a maximal torus of $T \subset \mathrm{GL}(\eta)$ corresponds to choosing a generic splitting

$$\bar{E}_\eta \cong (\bar{L}_1 \oplus \cdots \oplus \bar{L}_n)_\eta.$$

Then, we have a decomposition

$$\bar{\mathfrak{g}}_\eta = \left(\mathcal{O}_K \oplus \bigoplus_{\alpha \in \Phi} L_\alpha \right)_\eta$$

where Φ denotes the set of roots with respect to $T \subset \mathrm{GL}(\eta)$ and the root bundles can be seen as Arakelov line bundles on X, with the corresponding restricted metrics at infinity.

Given a basis $\Delta = \{\alpha_1, \cdots, \alpha_r\}$ of Φ, with vertices $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$, we consider the corresponding Borel subgroup $T \subset B \subset \mathrm{GL}(\eta)$.

The complementary polyhedron of $\bar{E}$ with respect to T is then given by

$$d(B) := \sum_{i=1}^{r} n(B, \lambda_i) \lambda_i^\vee$$

where $n(B, \lambda_i) = \deg(L_{\alpha_i})$.

Example 1.7.2. Let K be a number field with ring of integers $\mathcal{O}_K$. Let

$$\bar{E} = \mathcal{O}_K(m) \oplus \mathcal{O}_K(-m)$$

for $m \in \mathbb{R}_{\geq 0}$, with the corresponding re-scaled standard euclidean/hermitian metrics at infinity. The generic fiber of its automorphism group is $\mathrm{Aut}(\bar{E})_\eta \cong \mathrm{GL}(2)$. Its root system is

$$\Phi(\mathrm{GL}(2), T)) = \{\alpha_1, \alpha_2\},$$

with $\alpha_1 = (1, -1)$ and $\alpha_2 = (-1, 1)$, which have weights $\Lambda = \{\lambda_1, \lambda_2\}$ with $2\lambda_i = \alpha_i$. The associated group scheme $\mathrm{Aut}(E)$ has Lie algebra

$$\begin{aligned} \mathrm{Lie}(\mathrm{Aut}(\bar{E})) &= \mathrm{Hom}(\bar{E}, \bar{E}) \\ &= \mathcal{O}_K^2 \oplus \mathcal{O}_K(2m) \oplus \mathcal{O}_K(-2m). \end{aligned}$$

The corresponding Borel subgroups are:

- B^+ upper triangular, corresponding to the basis $\Delta_{B^+} = \alpha_1$.
- B^- lower triangular, corresponding to the basis $\Delta_{B^-} = \alpha_2$.

Next, we compute the numerical invariants:

$$\begin{aligned} n(B^+, \lambda_1) &= \deg \mathcal{O}_K(2m) \\ &= 2m \end{aligned}$$

and

$$\begin{aligned} n(B^-, \lambda_2) &= \deg \mathcal{O}_K(-2m) \\ &= -2m. \end{aligned}$$

Finally, the complementary polyhedron is given by

$$\begin{aligned} d(B^+) &= n(B^+, \lambda_1)\lambda_1^\vee \\ &= m\alpha_1^t \\ &= (m, -m)^t \end{aligned}$$

and

$$\begin{aligned} d(B^-) &= n(B^-, \lambda_2)\lambda_2^\vee \\ &= -m\alpha_2^t \\ &= (m, -m).^t \end{aligned}$$

Hence, it reduces to one point

$$F = \{(m, -m)^t\}.$$

Hence, $\bar{E}$ semistable if and only if $m = 0$. Note that if $m > 0$, then B^+ is canonical. Furthermore, B^+ corresponds to the filtration

$$0 \subsetneq \mathcal{O}_K(m) \subsetneq \bar{E}$$

which is indeed the GS-filtration.

Remark 1.7.3. Note that the normalization of the Cartan-Killing metrics given in 1.5.10 ensures that for $\bar{E} = \bar{\mathcal{O}_K}^n$ with trivial metrics at infinity, all line bundles L_α are trivial and therefore, the numerical invariants are all 0 (which indeed implies that $\bar{E}$ is semistable).

1.8 Semistability of Arakelov principal bundles

1.8.1 Stability of principal bundles over smooth projective curves

Let K be an algebraically closed field of characteristic 0. Let $\mathcal{X}$ be a principal G-bundle on a smooth projective algebraic curve Y, defined over K. The stability of $\mathcal{X}$ was introduced in [53] and depends on reductions to parabolic subgroups. Since a principal $\mathrm{GL}(n)$-bundle corresponds to a rank n vector bundle $E_{\mathcal{X}}$, we want to determine a canonical reduction of $\mathcal{X}$ generalizing the notion of Harder-Narasimhan filtration of $E_{\mathcal{X}}$.

In [9], Behrend uses the previous construction to identify the canonical reduction with a special facet.

Definition 1.8.1. Let $\mathcal{X}$ be a principal G-bundle. The non-negative integer

$$\mathrm{ideg}(\mathcal{X}) := \max\{\deg(\sigma^*\mathcal{X} \times^{\mathrm{Ad}} \mathfrak{p}) \mid P \subset G \text{ parabolic and } \sigma\colon Y \to \mathcal{X}/P \text{ reduction}\}$$

(which exists by [9, Lemma 4.3]) is called the *degree of instability* of $\mathcal{X}$. A pair (β, Q) consisting of a parabolic subgroup $Q \subset G$ and a reduction $\sigma\colon Y \to \mathcal{X}/Q$ is called a *canonical reduction* if

$$\deg(\beta^*\mathcal{X} \times^{\mathrm{Ad}} \mathfrak{q}) = \mathrm{ideg}(\mathcal{X}).$$

Proposition 1.8.2 ([9, Proposition 7.2], [41, Lemma 4])**.** *A canonical reduction (β, Q) of a principal G-bundle $\mathcal{X}$ satisfies the following properties*

1. *For any dominant character $\chi\colon Q \to K^*$, let*

$$L(\beta, \chi) := \beta^*\mathcal{X} \times^{\chi} K$$

be the associated line bundle. Then, $\deg(L(\beta, \chi)) > 0$.

2. *The extension of the Q-bundle $\beta^*\mathcal{X}$ to the Levi quotient $L = Q/R_u(Q)$ is a semistable principal L-bundle.*

Theorem 1.8.3 ([9, Theorem 8.2])**.** *Any principal G-bundle has a unique canonical reduction (β, Q) to a parabolic subgroup $Q \subset G$.*

In this chapter we introduce the concept of Arakelov principal bundles on arithmetic curves. We later use Behrend's complementary polyhedra to study their semistability in the same way as in this subsection by considering the numerical invariants for Borel subgroups of $G(\eta)$.

1.8.2 Semistability of Arakelov principal bundles

Given a number field K, let $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$ be the arithmetic curve associated to K. Let $G \subset \mathrm{GL}(n, \mathcal{O}_K)$ be a reductive connected affine algebraic group. We define Arakelov principal G-bundles on X as a generalization of Arakelov vector bundles, along the same lines as [58].

Definition 1.8.4. An *Arakelov principal G-bundle*

$$\bar{\mathcal{X}} := (\mathcal{X}, \{\sigma_\nu\}_{\nu \in X_\infty})$$

on X consists of the data of some principal G-bundle $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ and σ_ν reductions of structure group of $\mathcal{X}_\nu$ to $H_\nu \subset G(K_\nu)$ a maximal compact subgroup of $G(K_\nu) := G \otimes_K K_\nu$, i.e. $\mathrm{Spec}(K_\nu) \xrightarrow{\sigma_\nu} \mathcal{X}_\nu/H_\nu$.

Remark 1.8.5. We assume that for conjugated complex embeddings the corresponding sections σ_ν are complex conjugated, meaning that the corresponding maximal compact subgroups are complex conjugated. Hence, the reductions σ_ν are well-defined for every $\nu \in X_\infty$.

We denote by $\hat{H}^1(X, (G, H_\infty))$ the set of isomorphism classes of Arakelov principal G-bundles on X. Moreover, we denote by $\hat{H}^0(X, (G, H_\infty))$ the set of sections $s \in H^0(X, G)$ such that for every $\nu \in X_\infty$, s defines a section

$$\mathrm{Spec}(K_\nu) \to H_\nu.$$

Remark 1.8.6. In terms of cocycles, given an Arakelov principal G-bundle $\bar{\mathcal{X}}$, consider a trivializing (Zariski) covering $\{U_i\}_{i\in I}$. Then, its defining cocycle corresponds to giving $\varphi_{ij} \in \Gamma(U_i \cap U_j, G)$ taking values in H_ν for every $\nu \in X_\infty$. Recall that one may assume Zariski local triviality after [33, Theorem 2].

Example 1.8.7. 1) For $G = \mathrm{GL}(n)$, a principal G-bundle $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$ corresponds to a rank n vector bundle $E \to \mathrm{Spec}(\mathcal{O}_K)$ by

$$\mathcal{X} = \mathcal{I}som(\mathcal{O}^{\oplus n}_{\mathrm{Spec}(\mathcal{O}_K)}, E).$$

Furthermore, for $\nu \in X_\infty$ a complex place, we fix $H_\nu = \mathrm{U}(n) \subset G(\mathbb{C})$ and to give a section of

$$\mathcal{X}_\nu\,(\mathrm{GL}(n,\mathbb{C})/\,\mathrm{U}(n))$$

corresponds to give an hermitian metric over E_ν. On the other hand, for $\nu \in X_\infty$ a real place, we fix $H_\nu = \mathrm{O}(n) \subset G(\mathbb{R})$ and to give a section of

$$\mathcal{X}_\nu\,(\mathrm{GL}(n,\mathbb{R})/\,\mathrm{O}(n))$$

corresponds to give an euclidean metric over E_ν. Therefore, Arakelov principal $\mathrm{GL}(n)$-bundles are in natural bijection with rank n Arakelov vector bundles.

2) In particular, for $G = \mathbb{G}_m$, the group of isomorphism classes of Arakelov principal G-bundles corresponds to the group of isomorphism classes of Arakelov line bundles.

Remark 1.8.8. The choice of the maximal compact subgroup in the previous example is done without loss of generality by Remark 1.6.2.

Given two Arakelov principal G-bundles $\bar{\mathcal{X}}_1$ and $\bar{\mathcal{X}}_2$ on $X = \mathrm{Spec}(\mathcal{O}_K) \cup X_\infty$, a *homomorphism* between them consists of an homomorphism $f\colon \mathcal{X}_1 \to \mathcal{X}_2$ that preserves the reductions at the infinite places, meaning that the following diagram

$$\begin{tikzcd}
\mathcal{X}_{1,\nu}/H_\nu \arrow[rr, "\bar{f}_{K_\nu}"] & & \mathcal{X}_{2,\nu}/H_\nu \\
& \mathrm{Spec}(K_\nu) \arrow[ul, "\sigma_{1,\nu}"] \arrow[ur, "\sigma_{2,\nu}"'] &
\end{tikzcd}$$

is commutative.

Note that to define now extension and reduction of structure group of an Arakelov principal G-bundle $\bar{\mathcal{X}}$ it is not enough to have group homomorphisms $\rho\colon G \to G'$, we need them to agree with the extra-structure for every $\nu \in X_\infty$. Indeed, let ρ be a group homomorphism as before such that, for every $\nu \in X_\infty$, we have $\rho_\nu(H_\nu) \subset H'_\nu$ where H_ν (resp. H'_ν) denote the given maximal compact subgroups of G_ν (resp. G'_ν). We denote by

$$\rho_*\bar{\mathcal{X}} = (\rho_*\mathcal{X}, \{\rho_*\sigma_\nu\}_{\nu\in X_\infty})$$

the Arakelov principal G'-bundle obtained from $\bar{\mathcal{X}}$ by *extension of structure group*, consisting of the extension of structure group $\rho_*\mathcal{X}$ of $\mathcal{X}$ and $\rho_*\sigma_\nu$ a reduction of structure group of $\rho_*\mathcal{X}$ to H'_ν.

Conversely, given a closed subgroup $G_1 \subset G$, a reduction of structure group of $\bar{\mathcal{X}}$ to G_1 consists of

$$\bar{\mathcal{X}}_1 \coloneqq (\mathcal{X}_1, \{\sigma_{1,\nu}\}_{\nu\in X_\infty})$$

with $\mathcal{X}_1$ a reduction of structure group of $\mathcal{X}$ to G_1 and, for every $\nu \in X_\infty$, reductions

$$\sigma_{1,\nu}\colon \operatorname{Spec}(K_\nu) \longrightarrow \mathcal{X}_\nu(G_{1,\nu}/H_{1,\nu})$$

where $H_{1,\nu}$ denotes a maximal compact subgroup of $G_{1,\nu}$.

Definition 1.8.9. An Arakelov principal G-bundle $\bar{\mathcal{X}}$ is *semistable* if for all reductions

$$\bar{\mathcal{X}}_P \coloneqq (\mathcal{X}_P, \{\sigma_{P,\nu}\}_{\nu\in X_\infty})$$

to parabolic subgroups $P \subset G$ the following inequality holds

$$\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \le 0.$$

Lemma 1.8.10. *Let $\bar{\mathcal{X}}$ be an Arakelov principal G-bundle on X. Then, there exists a constant C such that the degree of the vector bundle*

$$\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \le C$$

for every parabolic subgroup $P \subset G$ and reduction $\bar{\mathcal{X}}_P$ of structure group of $\bar{\mathcal{X}}$ to P.

Proof. This is a direct consequence of the arithmetic Riemann-Roch inequality in Proposition 1.2.16. Indeed, recall it reads

$$\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \le h^0(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) + \frac{1}{2}n\log|D_K| + \frac{2+dn}{2}\log\left(\frac{2+dn}{2\pi}\right) + \frac{1}{2}\log\pi$$

where $n = \operatorname{rk}(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p})$ and $d = [K:\mathbb{Q}]$. On one hand,

$$h^0(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \le h^0(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{g}).$$

On the other hand, recall that by Minkowski's theorem $|D_K| \ge 1$ [22, Theorem 1.4.14] and this implies that $\log|D_K| \ge 0$. Therefore,

$$\frac{1}{2}n\log|D_K| + \frac{2+dn}{2}\log\left(\frac{2+dn}{2\pi}\right) \le \frac{1}{2}n'\log|D_K| + \frac{2+dn'}{2}\log\left(\frac{2+dn'}{2\pi}\right)$$

where $n' = \mathrm{rk}(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{g})$. The claim follows since we have seen that

$$\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \leq h^0(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{g}) + \frac{1}{2}n' \log|D_K| + \frac{2+dn'}{2}\log\left(\frac{2+dn'}{2\pi}\right) + \frac{1}{2}\log\pi$$

which is now independent of the choice of parabolic subgroup $P \subset G$. □

Definition 1.8.11. The non-negative real number

$$\mathrm{ideg}(\bar{\mathcal{X}}) := \max\{\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) \mid \bar{\mathcal{X}}_P \text{ reduction to parabolic } P\}$$

is called the *Arakelov degree of instability* of $\bar{\mathcal{X}}$. A *canonical Arakelov reduction* is a reduction $\bar{\mathcal{X}}_P$ to a parabolic subgroup P such that $\deg(\bar{\mathcal{X}}_P \times^{\mathrm{Ad}} \mathfrak{p}) = \mathrm{ideg}(\bar{\mathcal{X}})$.

Now, we want to adapt the constructions of Harder and Stuhler to our context, i.e. we will prove that the canonical parabolic subgroup of Arakelov group schemes $\bar{\mathcal{G}}$ which are inner forms of G_0, a split reductive group scheme over $\mathcal{O}_K$, is equivalent to giving a canonical reduction for Arakelov principal G_0-bundles.

Lemma 1.8.12. *Let G_0 be a split reductive group scheme over $\mathcal{O}_K$ and $P_0 \subset G_0$ a parabolic subgroup. To give a reduction of an Arakelov principal G_0-bundle $\bar{\mathcal{X}}$ to P_0 is equivalent to giving a parabolic subgroup of the Arakelov group scheme*

$$(\mathrm{Aut}_{G_0}(\mathcal{X}) = \mathcal{X} \times^{G_0,\mathrm{conj}} G_0, \{H_\nu\}_{\nu \in X_\infty})$$

Proof. Given a principal G_0-bundle $\mathcal{X}$ on $\mathrm{Spec}(\mathcal{O}_K)$, one defines a group scheme

$$\mathrm{Aut}_{G_0}(\mathcal{X}) = \mathcal{X} \times^{G_0,\mathrm{conj}} G_0$$

. where G_0 acts by conjugation on G_0.

In particular, given a type $t(P)$ of parabolic subgroups of $\mathrm{Aut}_{G_0}(\mathcal{X})$, let $P_0 \subset G_0$ be a parabolic subgroup of the same type. Then, parabolic subgroups of $\mathrm{Aut}_{G_0}(\mathcal{X})$ of type $t(P)$ are the same as reductions $\mathcal{X}_{P_0}$ of $\mathcal{X}$ to P_0 (see [30, Exposé XXVI, Lemme 3.20]), since both are parametrized by sections of

$$\begin{aligned}\mathrm{Aut}_{G_0}(\mathcal{X})/P &= \mathcal{X} \times^{G_0} G_0/P_0\\ &= \mathcal{X}/P_0.\end{aligned}$$

Moreover, under this equivalence,

$$\mathrm{Lie}(P) = \mathcal{X}_{P_0} \times^{P_0} \mathrm{Lie}(P_0)$$

and

$$\mathrm{Lie}(\mathrm{Aut}_{G_0}(\mathcal{X})) = \mathcal{X} \times^{\mathrm{Ad}} \mathrm{Lie}(G_0). \tag{1.8.1}$$

On the other hand, given $\nu \in X_\infty$, consider a maximal compact subgroup $K_{0,\nu} \subset G_{0,\nu}$. Then, using the equality (1.8.1), a section of $\mathcal{X}_\nu/H_{0,\nu}$ is equivalent to giving a maximal compact subgroup $H_\nu \subset \mathrm{Aut}_{G_0}(\mathcal{X})_\nu$. This shows the claim. □

Finally, we check that the notions of canonical reduction and canonical parabolic subgroup coincide under the above equivalence.

Proposition 1.8.13. *Under the equivalence given in Lemma 1.8.12, a reduction $\bar{\mathcal{X}}_{P_0}$ to a parabolic subgroup $P_0 \subset G_0$ is canonical if and only if the corresponding parabolic subgroup P of $\mathcal{G}$, together with the collection of maximal compact subgroups $H_\nu \subset \mathcal{G}_\nu$ with $\nu \in X_\infty$ as described above is canonical.*

Proof. We assume the same notations as in in Lemma 1.8.12.

The connected components $v \subset t(P)$ parametrize the parabolic subgroups Q_0 containing P which are minimal with respect to this property. The numerical invariant

$$n(P, v) = \deg(\bar{\mathcal{X}}(\chi))$$

is given by a character of the form

$$\chi = n\left(\sum_{\alpha_i \in v} \alpha_i\right) + \sum_{j>s} n_j \alpha_j$$

i.e. the character χ is a multiple of the orthogonal projection of

$$\left(\sum_{\alpha_i \in v} \alpha_i\right) \in X^*(T)$$

onto $X^*(P)$. Thus,

$$n(P, v) = -c \deg\left(\bar{\mathcal{X}} \times^{P_0} \operatorname{Lie}(Q_0)/\operatorname{Lie}(P_0)\right)$$

for some $c > 0$. This shows that the two notions coincide. □

Lemma 1.8.14. *A canonical Arakelov reduction $\bar{\mathcal{X}}_P$ satisfies the following properties:*

1. *For any character $\chi\colon P \to \mathbb{G}_m$ whose restriction to the chosen maximal torus $T \subset P$ is a non-negative linear combination $\sum n_i \alpha_i$ of simple roots $\alpha_i \in \Delta$ (where $n_i \geq 0$, and at least one $n_i \neq 0$), if we let $\mathcal{L}(P, \chi)$ be the associated line bundle to $\bar{\mathcal{X}}_P$, then $\deg \mathcal{L}(P, \chi) > 0$.*
2. *The extension of $\bar{\mathcal{X}}$ to the Levi quotient $L = P/\operatorname{Rad}_u(P)$ is semistable.*

Proof. By Proposition 1.8.13 it is equivalent to [39, 6.2] applied to the group scheme

$$\mathcal{X} \times^{G,\mathrm{conj}} G.$$

□

All this together concludes our final result.

Theorem 1.8.15. *Every Arakelov principal G-bundle $\bar{\mathcal{X}}$ has a unique Arakelov canonical reduction $\bar{\mathcal{X}}_P$.*

Moreover, when $G = \mathrm{GL}(n)$ the canonical parabolic subgroup P corresponds to the Grayson-Stuhler filtration of the Arakelov vector bundle associated to the Arakelov principal G-bundle $\bar{\mathcal{X}}$.

Part II

Bridgeland stability conditions on holomorphic triples over curves

Triangulated categories

This chapter provides basic facts about triangulated and derived categories that we will use afterwards. We recommend [42] for further details.

2.1 Triangulated and derived categories

Definition 2.1.1. A category $\mathcal{A}$ is called *additive* if for every two objects $A, B \in \mathcal{A}$ the set $\mathrm{Hom}_{\mathcal{A}}(A, B)$ is endowed with the structure of an abelian group such that the following conditions are satisfied

i) The compositions

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}}(A_1, A_2) \times \mathrm{Hom}_{\mathcal{A}}(A_2, A_3) &\longrightarrow \mathrm{Hom}_{\mathcal{A}}(A_1, A_3) \\ (f, g) &\longmapsto g \circ f \end{aligned}$$

are bilinear.

ii) There exists a zero object $0 \in \mathcal{A}$, i.e. an object 0 such that $\mathrm{Hom}_{\mathcal{A}}(0, 0)$ is the trivial group with one element.

iii) For any two objects $A_1, A_2 \in \mathcal{A}$ there exists an object $B \in \mathcal{A}$ with morphisms $j_i \colon A_i \to B$ and $p_i \colon B \to A_i$, $i = 1, 2$, which make B the direct sum and the direct product of A_1 and A_2.

An additive category $\mathcal{A}$ is called *abelian* if it satisfies the following additional condition

iv) Every morphism $f \in \mathrm{Hom}(A, B)$ admits a kernel and a cokernel and the natural map $\mathrm{Coim}(f) \to \mathrm{Im}(f)$ is an isomorphism.

In general, the (bounded) derived category of an abelian category will fail to be again abelian, but it will still be additive. Then we give an extra structure (so called triangulated structure) in order to have an analogous notion of short exact sequences in this context.

Definition 2.1.2. Let $\mathcal{D}$ be an additive category. The structure of a *triangulated category* on $\mathcal{D}$ is given by an additive equivalence $[1] \colon \mathcal{D} \to \mathcal{D}$, the *shift functor*, and a set of *distinguished triangles*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

subject to the following axioms

A.1 i) Any triangle of the form

$$A \xrightarrow{\mathrm{id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

ii) Any triangle isomorphic to a distinguished triangle is distinguished.

iii) Any morphism $f\colon A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1].$$

A.2 A triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is distinguished if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is distinguished. Hence, we visualize distinguished triangles as

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul, dashed, "{[1]}"] & \end{tikzcd}.$$

A.3 Suppose there exists a commutative diagram of distinguished triangles with vertical arrows f and g:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, dotted, "h"] & A[1] \arrow[d, "{f[1]}"] \\ A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]. \end{tikzcd}$$

Then the diagram can be completed to a commutative diagram, i.e. to a morphism of triangles, by a (not necessarily unique) morphism h.

A.4 Octahedral axiom. Given distinguished triangles

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} A[1]$$
$$B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} B[1]$$
$$A \xrightarrow{vu} C \xrightarrow{m} B' \xrightarrow{n} A[1].$$

There exists a distinguished triangle

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} C'[1]$$

such that $l = gm$, $k = nf$, $h = j[1]i$, $ig = u[1]n$ and $fj = mv$.

Proposition 2.1.3 ([42, Proposition 1.34]). *Let $\mathcal{D}$ be a triangulated category and let*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

be a distinguished triangle. Then, for any object $X \in \mathcal{D}$ the following sequences are exact:

$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X,A) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X,B) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X,C) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(X,A[1]) \cdots$$
$$\cdots \longrightarrow \mathrm{Hom}_{\mathcal{D}}(C,X) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(B,X) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(A,X) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(C[-1],X) \cdots$$

Definition 2.1.4. Let $\mathcal{D}$ and $\mathcal{D}'$ be triangulated categories with shift functors $[1]_{\mathcal{D}}$ and $[1]_{\mathcal{D}'}$ respectively. An additive functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called *exact* if it satisfies the following conditions

i) There exists a functor isomorphism

$$F \circ [1]_{\mathcal{D}} \to [1]_{\mathcal{D}'} \circ F.$$

ii) Any distinguished triangle

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]_{\mathcal{D}}$$

in $\mathcal{D}$ is mapped to a distinguished triangle

$$F(A) \longrightarrow F(B) \longrightarrow F(C) \longrightarrow F(A)[1]_{\mathcal{D}'}$$

in $\mathcal{D}'$. Here we identify $F(A[1]_{\mathcal{D}})$ with $F(A)[1]_{\mathcal{D}'}$ via the functor isomorphism in i).

Definition 2.1.5. Let $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ be a functor between arbitrary categories.

- A functor $H\colon \mathcal{C}_2 \to \mathcal{C}_1$ is *right adjoint* to F (one writes $F \dashv H$ or (F,H)) if there exist isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_2}(F(A_1),A_2) \cong \mathrm{Hom}_{\mathcal{C}_1}(A_1,H(A_2))$$

for any two objects $A_i \in \mathcal{C}_i$ for $i = 1,2$ which are functorial in A_1 and A_2.

- A functor $G\colon \mathcal{C}_2 \to \mathcal{C}_1$ is *left adjoint* to F (one writes $G \dashv F$ or (G,F)) if there exist isomorphisms

$$\mathrm{Hom}_{\mathcal{C}_2}(A_2,F(A_1)) \cong \mathrm{Hom}_{\mathcal{C}_1}(G(A_2),A_1)$$

for any two objects $A_i \in \mathcal{C}_i$ for $i = 1,2$ which are functorial in A_1 and A_2.

Clearly, H is right adjoint to F if and only if F is left adjoint to H.

Remark 2.1.6. i) Suppose $F \dashv H$. Then, $\mathrm{id}_{F(A)} \in \mathrm{Hom}_{\mathcal{C}_2}(F(A), F(A))$ induces a morphism $A \to H(F(A))$. The naturality of isomorphisms in the definition of the adjoint functor ensures that these morphisms define a functor morphism

$$h\colon \mathrm{id}_{\mathcal{C}_1} \longrightarrow H \circ F$$

and one can easily see that F is fully faithful if and only if h is an isomorphism. Similarly, there is a functor morphism

$$g\colon F \circ H \longrightarrow \mathrm{id}_{\mathcal{C}_2}$$

and one can easily see that H is fully faithful if and only if g is an isomorphism. See [42, Remark 1.24] for details.

ii) Using the Yoneda lemma, one verifies that a left (or right) adjoint functor is unique up to isomorphism whenever it exists. See [42, Remark 1.16] for details.

iii) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are triangulated categories, then F is exact if and only if H is exact. But this is not true for example if $\mathcal{C}_1$ and $\mathcal{C}_2$ are abelian categories. When F is left and right exact, in general its right adjoint is only left exact. See [42, Remark 1.16 and Proposition 1.40] for details.

The triangulated categories we will work with will be (bounded) derived categories of certain abelian categories.

Let $\mathcal{A}$ be an abelian category, and let $\mathrm{Kom}(\mathcal{A})$ denote the category of (bounded) complexes of objects in $\mathcal{A}$. A morphism of complexes $f\colon A^\bullet \to B^\bullet$ is called a *quasi-isomorphism* if the induced morphisms $H^i(f)\colon H^i(A^\bullet) \to H^i(B^\bullet)$ are isomorphisms for all $i \in \mathbb{Z}$.

Definition 2.1.7. With the previous notations, the (bounded) derived category of $\mathcal{A}$, denoted $D(\mathcal{A})$ (resp. $D^b(\mathcal{A})$) consists of (bounded) complexes of elements in $\mathcal{A}$ up to quasi-isomorphism, i.e. it is obtained by formally inverting all quasi-isomorphisms. Further details about the construction of $D(\mathcal{A})$ are found in section 2.1 in [42].

Remark 2.1.8. Viewing any object in $\mathcal{A}$ as a complex concentrated in degree zero yields an equivalence between $\mathcal{A}$ and the full subcategory of $D(\mathcal{A})$ that consists of all complexes $A^\bullet$ with $H^i(A^\bullet) = 0$, $\forall i \neq 0$.

The triangulated structure of $D(\mathcal{A})$ can be seen as follows. The shift functor [1] simply shifts the complex. For example, under the above equivalence, $\mathcal{A}[1]$ seen as a full subcategory of $D(\mathcal{A})$, consists of all complexes $A^\bullet$ with $H^i(A^\bullet) = 0$, $\forall i \neq -1$.

The derived analogue of kernels and cokernels are *cones*. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes, then the cone of f is defined as

$$C(f)^i := B^i \oplus A^{i+1}$$

with differential

$$d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

Distinguished triangles in $D(\mathcal{A})$ are triangles isomorphic to

$$A \xrightarrow{f} B \longrightarrow C(f) \longrightarrow A[1].$$

Remark 2.1.9. Given a short exact sequence

$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow C \longrightarrow 0$$

in an abelian category $\mathcal{A}$. Under the full embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$ it becomes a distinguished triangle

$$A \longrightarrow B \longrightarrow C \xrightarrow{\delta} A[1]$$

in $D(\mathcal{A})$ with δ given as the composition of the inverse of the quasi-isomorphism $C(f) \to C$ and the natural morphism $C(f) \to A[1]$.

Conversely, if

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

is a distinguished triangle with objects $A, B, C \in \mathcal{A}$, then

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is a short exact sequence in $\mathcal{A}$.

In what follows, we will work with abelian (resp. triangulated) categories that are essentially small, that is, that it is equivalent to an abelian (resp. triangulated) category such that the class of objects is a set.

2.2 Torsion pairs and t-structures

Let $\mathcal{D}$ be a triangulated category. T-structures are a tool which allows us to see the different abelian categories embedded in $\mathcal{D}$.

Definition 2.2.1. A *t-structure* on $\mathcal{D}$ consists of a pair of full additive subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$, with $\mathcal{D}^{\leq i} := \mathcal{D}^{\leq 0}[-i]$ and $\mathcal{D}^{\geq i} := \mathcal{D}^{\geq 0}[-i]$ for $i \in \mathbb{Z}$, such that:

i) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1}) = 0$.

ii) For all $E \in \mathcal{D}$, there is a distinguished triangle

$$T \longrightarrow E \longrightarrow F \longrightarrow T[1]$$

with $T \in \mathcal{D}^{\leq 0}$ and $F \in \mathcal{D}^{\geq 1}$.

iii) $\mathcal{D}^{\leq 0} \subset \mathcal{D}^{\leq 1}$ and $\mathcal{D}^{\geq 0} \supset \mathcal{D}^{\geq 1}$.

For each $n \in \mathbb{Z}$ there exist truncation functors $\tau_{\leq n} \colon \mathcal{D} \to \mathcal{D}^{\leq n}$ and $\tau_{\geq n} \colon \mathcal{D} \to \mathcal{D}^{\geq n}$ satisfying that for every non-zero object $E \in \mathcal{D}$ there exists a distinguished triangle

$$\tau_{\leq n}E \longrightarrow E \longrightarrow \tau_{\geq n}E \longrightarrow \tau_{\leq n}E[1]\ .$$

Definition 2.2.2. A t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$ is said to be *bounded* if

$$\mathcal{D} = \bigcup_{i,j\in\mathbb{Z}} \mathcal{D}^{\leq 0}[i] \cap \mathcal{D}^{\geq 0}[j].$$

Definition 2.2.3. The *heart* $\mathcal{A}$ of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 1})$ on $\mathcal{D}$ is defined as $\mathcal{A} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

The heart of a bounded t-structure is an abelian category $\mathcal{A}$ and short exact sequences in $\mathcal{A}$ are precisely distinguished triangles in $\mathcal{D}$ with objects in $\mathcal{A}$.

Example 2.2.4. Let $\mathcal{D} = D^b(\mathcal{A})$ be the bounded derived category of an abelian category $\mathcal{A}$. The standard t-structure is given by

$$\begin{aligned} \mathcal{D}^{\leq 0} &= \{E \in \mathcal{D} \colon H^i(E) = 0, \text{ for all } i > 0\} \\ \mathcal{D}^{\geq 0} &= \{E \in \mathcal{D} \colon H^i(E) = 0, \text{ for all } i < 0\} \end{aligned}$$

and its heart is the original abelian category $\mathcal{A} \subset \mathcal{D}$ in degree zero.

Lemma 2.2.5 ([19, Lemma. 3.2],[43, Remark 1.16])**.** *Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory of a triangulated category $\mathcal{D}$. Then $\mathcal{A}$ is the heart of a bounded t-structure if and only if*

i) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}[k_1], \mathcal{A}[k_2]) = 0$ *for* $k_1 > k_2$.

ii) For every nonzero $E \in \mathcal{D}$ there exists a finite sequence of integers

$$k_1 > k_2 > \cdots > k_m$$

and a collection of distinguished triangles

$$\begin{tikzcd}[column sep=small] 0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{m-1} \arrow[rr] & & E_m = E \arrow[dl] \\ & A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_m \arrow[ul, dashed] & \end{tikzcd}$$

with $A_j \in \mathcal{A}[k_j]$ for all j.

Remark 2.2.6. In other words, the lemma above shows that a bounded t-structure on a triangulated category $\mathcal{D}$ is determined by its heart $\mathcal{A}$. In fact, $\mathcal{D}^{\leq 0}$ is the extension-closed subcategory generated by the subcategories $\mathcal{A}[k]$ for integers $k \geq 0$.

A way to construct many non-trivial t-structures is by tilting $\mathcal{A}$ at a torsion pair.

Definition 2.2.7. Let $\mathcal{A}$ be an abelian category. A *torsion pair* for $\mathcal{A}$ consists of a pair $(\mathcal{T}, \mathcal{F})$ of full subcategories such that

i) $\mathrm{Hom}_{\mathcal{A}}(\mathcal{T}, \mathcal{F}) = 0$.

ii) For all $E \in \mathcal{A}$, there is a short exact sequence

$$0 \to T \to E \to F \to 0$$

with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

Definition 2.2.8. Let $\mathcal{A}$ be an abelian category with a torsion pair $(\mathcal{T}, \mathcal{F})$. Then the *tilt* of $\mathcal{A}$ with respect to $(\mathcal{T}, \mathcal{F})$ is defined as the full additive subcategory $\mathcal{A}^\sharp \subset D^b(\mathcal{A})$ of all objects $E \in D^b(\mathcal{A})$ with

$$H^0(E) \in \mathcal{T},\ H^{-1}(E) \in \mathcal{F}, \text{ and } H^i(E) = 0 \text{ for } i \neq 0, -1.$$

Lemma 2.2.9 ([43, Proposition 1.17])**.** *The category $\mathcal{A}^\sharp$ is the heart of a bound- ed t-structure on $D^b(\mathcal{A})$.*

Remark 2.2.10. A torsion pair $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$ gives rise to a torsion pair $(\mathcal{F}[1], \mathcal{T})$ for the tilt $\mathcal{A}^\sharp$. If $(\mathcal{T}, \mathcal{F})$ is non-trivial, then $\mathcal{A} \neq \mathcal{A}^\sharp$. In other words, objects in $\mathcal{A}$ can be thought as an extension of F by T, with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, determined by an element $\mathrm{Ext}^1(F, T)$. On the other hand, objects in $\mathcal{A}^\sharp$ are an extension of T by $F[1]$, i.e. determined by an element $\mathrm{Ext}^1(T, F[1]) = \mathrm{Ext}^2(T, F)$. More concretely, every object in $\mathcal{A}^\sharp$ can be represented by a 2-term complex $E^{-1} \xrightarrow{d} E^0$ with $\ker(d) \in \mathcal{F}$ and $\mathrm{coker}(d) \in \mathcal{T}$.

The following proposition shows the relation between tilts of the heart of a t-structure on $\mathcal{D}$ and new t-structures on $\mathcal{D}$.

Proposition 2.2.11 ([43, Proposition 1.20])**.** *If $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$ is the heart of a bounded t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ on a triangulated category $\mathcal{D}$, then there exists a natural bijection between:*

i) Torsion pairs $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$.

ii) t-structures on $\mathcal{D}$ given by $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$ satisfying $\mathcal{D}^{\leq 0}[1] \subset \mathcal{D}'^{\leq 0} \subset \mathcal{D}^{\leq 0}$.

Proof. A torsion pair $(\mathcal{T}, \mathcal{F})$ for $\mathcal{A}$ yields a t-structure

$$\begin{aligned}\mathcal{D}'^{\leq 0} &:= \{E \in \mathcal{D} \colon H^0(E) \in \mathcal{T}, \text{ and } H^i(E) = 0 \text{ for } i > 0\}\\ \mathcal{D}'^{\geq 0} &:= \{E \in \mathcal{D} \colon H^{-1}(E) \in \mathcal{F}, \text{ and } H^i(E) = 0 \text{ for } i < -1\}.\end{aligned}$$

Conversely, given such t-structure $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$, define a torsion pair by $\mathcal{T} := \mathcal{A} \cap \mathcal{D}'^{\leq 0}$ and $\mathcal{F} := \mathcal{A} \cap \mathcal{D}'^{\geq 1}$. □

Finally, note that the group $\mathrm{Aut}(\mathcal{D})$ of exact autoequivalences of $\mathcal{D}$ acts on the set of bounded t-structures: if $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure and $\Phi \in \mathrm{Aut}(\mathcal{D})$, then $\Phi(\mathcal{A}) \subset \mathcal{D}$ is also the heart of a bounded t-structure (in general different from the original one).

2.3 The Grothendieck and numerical groups

Now we recall a few definitions from the introduction of [19].

Definition 2.3.1. Let $\mathcal{A}$ be an abelian category. The *Grothendieck group* $K(\mathcal{A})$ is defined as an abelian group generated by isomorphism classes $[A]$ for $A \in \mathcal{A}$ such that for each short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

in $\mathcal{A}$, $[A] = [A'] + [A'']$.

Analogously, given a triangulated category $\mathcal{D}$, the *Grothendieck group* $K(\mathcal{D})$ is defined as an abelian group generated by isomorphism classes $[A]$ for $A \in \mathcal{D}$ such that for each distinguished triangle

$$A' \longrightarrow A \longrightarrow A'' \longrightarrow A'[1]$$

in $\mathcal{D}$, $[A] = [A'] + [A'']$.

Proposition 2.3.2 ([48, Theorem 3.5.2])**.** *If $\mathcal{A}$ is the heart of a bounded t-structure on a triangulated category $\mathcal{D}$, the natural morphism of abelian groups*

$$K(\mathcal{A}) \to K(\mathcal{D})$$

is an isomorphism.

Suppose now that $\mathcal{D}$ is linear over a field k. This means that the morphisms of $\mathcal{D}$ have the structure of a k-vector space with respect to which the composition law is bilinear. Suppose further that $\mathcal{D}$ is of finite type, that is that for every pair of objects E and F of $\mathcal{D}$ the vector space

$$\bigoplus_i \operatorname{Hom}_{\mathcal{D}}(E, F[i])$$

is finite-dimensional. In this situation one can define the Euler form on $K(\mathcal{D})$ via

$$\chi(E, F) = \sum_i (-1)^i \dim_k \operatorname{Hom}_{\mathcal{D}}(E, F[i])$$

and a free abelian group $\mathcal{N}(\mathcal{D}) := K(\mathcal{D})/K(\mathcal{D})^{\perp}$ called the *numerical Grothendieck group* of $\mathcal{D}$, where $K(\mathcal{D})^{\perp}$ denotes the right orthogonal with respect to the Euler form. If this group $\mathcal{N}(\mathcal{D})$ has finite rank the category $\mathcal{D}$ is said to be *numerically finite.*

Example 2.3.3. As pointed out by Bridgeland in [19], there are two large classes of examples of numerically finite triangulated categories.

1. If A is a finite-dimensional algebra over a field, then the bounded derived category $D^b(A)$ of finite-dimensional left A-modules is numerically finite.

2. If X is a smooth projective variety over $\mathbb{C}$, then the bounded derived category $D^b(X)$ of coherent sheaves on X is numerically finite.

From now on, all triangulated categories will be presumed to be numerically finite.

Bridgeland stability conditions on triangulated categories

2.4 The space of stability conditions

Let $\mathcal{D}$ be a triangulated category, equipped with a surjective group homomorphism

$$v\colon K(\mathcal{D}) \twoheadrightarrow \Lambda$$

from its Grothendieck group to a finite rank lattice Λ. Let us recall the definitions from [19].

Definition 2.4.1. A *slicing* $\mathcal{P}$ on $\mathcal{D}$ is a collection of full subcategories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$ satisfying:

a) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$, for all $\phi \in \mathbb{R}$.

b) If $\phi_1 > \phi_2$ and $E_i \in \mathcal{P}(\phi_i)$, $i = 1, 2$, then $\mathrm{Hom}(E_1, E_2) = 0$.

c) For every nonzero object $E \in \mathcal{D}$ there exists a finite sequence of maps

$$0 = E_0 \xrightarrow{f_1} E_1 \longrightarrow \cdots \longrightarrow E_{m-1} \xrightarrow{f_m} E_m = E$$

and of real numbers $\phi_1 > \cdots > \phi_m$ such that for every $j = 1, \cdots, m$, we have a distinguished triangle

$$E_{j-1} \xrightarrow{f_j} E_j \longrightarrow A_j \longrightarrow E_{j-1}[1]$$

with $A_j \in \mathcal{P}(\phi_j)$.

The objects of $\mathcal{P}(\phi)$ are called *semistable* of phase ϕ; its simple objects are called *stable*. The sequence of maps in c) is called the *HN-filtration* of E. We write $\phi^{\pm}(E)$ for the largest and the smallest phase in the associated HN-filtration of E.

Definition 2.4.2. A *pre-stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where $\mathcal{P}$ is a slicing and $Z\colon \Lambda \to \mathbb{C}$ is a group homomorphism such that $Z(v(E)) \in \mathbb{R}_{>0} \cdot e^{i\pi\phi}$ for all nonzero $E \in \mathcal{P}(\phi)$, for every $\phi \in \mathbb{R}$.

Remark 2.4.3. Let $\mathcal{A} \subset \mathcal{D}$ be a full additive subcategory and set $\mathcal{P}(\phi) \coloneqq \mathcal{A}[\phi]$ for $\phi \in \mathbb{Z}$ and $\emptyset$ otherwise. Then, $\mathcal{A}$ is the *heart* of a bounded t-structure if $\mathcal{P}$ is a slicing on $\mathcal{D}$.

Definition 2.4.4. A *stability function* on an abelian category $\mathcal{A}$ is a group homomorphism $Z\colon K(\mathcal{A}) \to \mathbb{C}$ such that for all $0 \neq E \in \mathcal{A}$, $Z(E)$ lies in the semi-closed upper half plane

$$\overline{\mathbb{H}} \coloneqq \{z \in \mathbb{C}\colon z = r \cdot e^{i\pi\phi} \text{ with } r \in \mathbb{R}_{>0} \text{ and } \phi \in (0, 1]\}.$$

For $0 \neq E \in \mathcal{A}$ we define its phase by $\phi(E) \coloneqq \frac{1}{\pi} \arg(Z(E)) \in (0, 1]$. Furthermore, E is called *Z-semistable* if for all nonzero subobjects $F \hookrightarrow E$, we have $\phi(F) \leq \phi(E)$.

Definition 2.4.5. We say that a stability function Z on an abelian category $\mathcal{A}$ satisfies the *HN-property* if every $0 \neq E \in \mathcal{A}$ admits a sequence

$$0 = E_0 \hookrightarrow E_1 \hookrightarrow \cdots \hookrightarrow E_m = E$$

such that E_i/E_{i-1} is Z-semistable for $i = 1, \cdots, m$ and

$$\phi(E_1/E_0) > \cdots > \phi(E_m/E_{m-1}).$$

Proposition 2.4.6 ([19, Proposition 5.3])**.** *To give a pre-stability condition on $\mathcal{D}$ is equivalent to giving a heart $\mathcal{A}$ of a bounded t-structure and a stability function Z on $\mathcal{A}$ with the HN-property.*

Remark 2.4.7. Here we use Proposition 2.3.2, i.e. that if $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ then $K(\mathcal{A})$ can be identified with $K(\mathcal{D})$.

Definition 2.4.8. A pre-stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ is called *locally finite* if there exists some $\epsilon > 0$ such that, for all $\phi \in \mathbb{R}$, each subcategory $\mathcal{P}((\phi - \epsilon, \phi + \epsilon))$ is of finite length, i.e. any sequence of epimorphisms (resp. monomorphisms) stabilizes.

Remark 2.4.9. In this way $\mathcal{P}(\phi)$ has finite length so that every object in $\mathcal{P}(\phi)$ has a finite Jordan-Hölder filtration into stable factors of the same phase.

We denote by $\mathrm{Stab}(\mathcal{D})$ the space of locally finite pre-stability conditions that are numerical, that is, those for which the stability function Z factors through the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$.

Remark 2.4.10. More generally, as pointed out in [8], one fixes a finite-dimensional lattice Λ with a map $v\colon K(\mathcal{D}) \twoheadrightarrow \Lambda$ and focus on stability conditions for which Z factors via Λ. Obviously, this is no restriction in case $K(\mathcal{D})$ of finite dimension; in this case a typical choice for Λ might be the numerical Grothendieck group $\mathcal{N}(\mathcal{D})$.

The stability conditions we consider also satisfy the additional conditions in the definition given in [44, Section 2] (in particular the support property, introduced below). The local-finiteness condition will then be automatic. We will follow [8, Appendix A], that contains a quite transparent and extended description of the support property.

Definition 2.4.11. Let $Q\colon \Lambda_{\mathbb{R}} = \Lambda \otimes \mathbb{R} \to \mathbb{R}$ be a quadratic form. A pre-stability condition $\sigma = (Z, \mathcal{P})$ satisfies the *support property* with respect to Q if

a) Q is negative definite with respect to the kernel of Z.

b) for every σ-semistable object $E \in \mathcal{P}(\phi)$, we have $Q(v(E)) \geq 0$.

A *stability condition* on $\mathcal{D}$ is a pre-stability condition that satisfies the support property with respect to some quadratic form Q.

Remark 2.4.12. This property ensures that stability conditions deform freely, and exhibit well-behaved wall-crossing.

The following lemma provides an equivalent definition of support property.

Lemma 2.4.13 ([8, Lemma A.4])**.** *A pre-stability condition $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to some quadratic form Q on $\Lambda_{\mathbb{R}}$ if and only if there exists a constant $C \in \mathbb{R}_{>0}$ such that for every σ-semistable object $E \in \mathcal{P}(\phi)$, $\|v(E)\| \leq C|Z(v(E))|$ for some norm $\|\cdot\|$ on $\Lambda_{\mathbb{R}}$.*

There is a generalized metric (and thus a topology) on the set $\mathrm{Slice}(\mathcal{D})$ of slicings on $\mathcal{D}$ given as follows. Given two slicings $\mathcal{P}$ and $\mathcal{Q}$, we write $\phi^{\pm}_{\mathcal{P}}(E)$ and $\phi^{\pm}_{\mathcal{Q}}(E)$ for the largest and smallest phase in the associated HN-filtration of E for $\mathcal{P}$ and $\mathcal{Q}$ respectively. Then, we define the distance of $\mathcal{P}$ and $\mathcal{Q}$ as

$$d_S(\mathcal{P}, \mathcal{Q}) \coloneqq \sup_{E \in \mathcal{D}} \{|\phi^{+}_{\mathcal{P}}(E) - \phi^{+}_{\mathcal{Q}}(E)|, |\phi^{-}_{\mathcal{P}}(E) - \phi^{-}_{\mathcal{Q}}(E)|\} \in [0, +\infty]. \tag{2.4.1}$$

Remark 2.4.14.

1. The term *generalised metric* is used to mean a distance function on a set X satisfying all the usual metric space axioms except that it need not be finite. Any such function defines a topology on X in the usual way and induces a metric space structure on each connected component of X.
2. The (generalized) distance (2.4.1) can be computed by considering $\mathcal{P}$-semistable objects alone, i.e. $E \in \mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$.

On the other hand, the function

$$\|W\|_{\sigma} \coloneqq \sup \left\{ \frac{|W(E)|}{|Z(E)|} \colon E \text{ is } \sigma\text{-semistable} \right\}$$

has all the properties of a norm on $\mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$, except that it may not be finite.

For each real number $\epsilon \in (0, 1)$, define a subset

$$B_{\epsilon}(\sigma) \coloneqq \{\tau = (W, \mathcal{Q}) \colon \|W - Z\|_{\sigma} < \sin(\pi\epsilon) \text{ and } d_S(\mathcal{P}, \mathcal{Q}) < \epsilon\} \subset \mathrm{Stab}(\mathcal{D}).$$

Remark 2.4.15. Note that the condition $\|W - Z\|_{\sigma} < \sin(\pi\epsilon)$ implies that for all σ-semistable objects E, the phase of $W(E)$ differs from the phase of $Z(E)$ by less than ϵ.

In [19] it is shown that as σ varies in $\mathrm{Stab}(\mathcal{D})$ the subsets $B_\epsilon(\sigma)$ form a basis for a topology on $\mathrm{Stab}(\mathcal{D})$.

The following theorem is the main theorem in [19].

Theorem 2.4.16 ([19, Theorem 1.2]). *For each connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ there is a linear subspace $V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ with a well-defined linear topology and a local homeomorphism $\mathcal{Z} \colon \Sigma \longrightarrow V(\Sigma)$, $(Z, \mathcal{P}) \mapsto Z$.*

Remark 2.4.17. Given a connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$, its corresponding linear subspace $V(\Sigma) \subset \mathrm{Hom}(K(\mathcal{D}), \mathcal{C})$ is defined as the set of $U \in \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$ such that $\|U\|_\sigma < \infty$, for some $\sigma \in \Sigma$.

Definition 2.4.18. A connected component $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ is called *full* if it has maximal dimension, i.e. $V(\Sigma) = \mathrm{Hom}(K(\mathcal{D}), \mathbb{C})$. A pre-stability condition σ is called *full* if it belongs to a full connected component.

We also want to clarify the relation between full pre-stability conditions and the support property introduced above, in the situation of finite-rank Grothendieck group. More precisely, we choose a metric $|\cdot|$ on $K(\mathcal{D})_\mathbb{R}$.

Proposition 2.4.19 ([7, Proposition B.4]). *Assume that $K(\mathcal{D})$ has finite rank. Then a pre-stability condition $\sigma = (Z, \mathcal{P})$ is full if and only if it has the support property.*

Moreover, the next theorem is a very nice result stating that the support property of an element in $\mathrm{Stab}(\mathcal{D})$ is extended to the whole connected component containing it.

Proposition 2.4.20 ([8, Proposition A.5]). *Given $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, assume that σ satisfies the support property with respect to a quadratic form Q on $\Lambda_\mathbb{R}$. Consider the open subset of $\mathrm{Hom}(\Lambda, \mathbb{C})$ consisting of central charges whose kernel is negative definite with respect to Q and let U be the connected component containing Z. Let $\mathcal{U} \subset \mathrm{Stab}(\mathcal{D})$ be the connected component of the pre-image $\mathcal{Z}^{-1}(U)$ containing σ. Then the following statements are true.*

1. *The restriction $\mathcal{Z}|_\mathcal{U} \colon \mathcal{U} \to U$ is a covering map.*

2. *Any stability condition $\sigma' \in \mathcal{U}$ satisfies the support property with respect to the same quadratic form Q.*

The following lemma compares stability conditions with the same stability function.

Lemma 2.4.21 ([19, Lemma 6.4]). *Let $\sigma = (Z, \mathcal{P})$ and $\tau = (Z, \mathcal{Q})$ be two stability conditions on $\mathcal{D}$ with the same stability function Z. If $d_S(\mathcal{P}, \mathcal{Q}) < 1$, then $\sigma = \tau$.*

Next, we state Bridgeland's deformation result for full stability conditions, since we will require this particular formulation in our case of study.

Theorem 2.4.22 ([19, Theorem 7.1], [20, Theorem 2.4])**.** *Let $\Sigma \subset \mathrm{Stab}(\mathcal{D})$ be a full connected component. Take $\sigma = (Z, \mathcal{P}) \in \Sigma$ and $0 < \epsilon < 1/2$. Then, for any group homomorphism $W\colon K(\mathcal{D}) \to \mathbb{C}$ with*

$$\|W - Z\|_\sigma < \sin(\pi\epsilon),$$

the exists a unique stability condition $\tau = (W, \mathcal{Q}) \in \Sigma$ such that $d_S(\mathcal{P}, \mathcal{Q}) < \epsilon$.

We finish this section by showing that there are two commuting actions on the set of stability conditions on $\mathcal{D}$.

- A left-action of the group of autoequivalences of $\mathcal{D}$, $\mathrm{Aut}(\mathcal{D})$: Given $\Phi \in \mathrm{Aut}(\mathcal{D})$ and $\sigma = (Z, \mathcal{P})$,
$$\Phi.\sigma = (Z', \mathcal{P}')$$
with $Z'(E) = Z(\Phi^{-1}(E))$ and $\mathcal{P}'(\phi) = \Phi(\mathcal{P}(\phi))$.

- A right-action of the universal cover of the group of 2×2 real matrices with positive determinant, $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$: An element $(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, consists of $T \in \mathrm{GL}^+(2, \mathbb{R})$ orientation preserving and $f\colon \mathbb{R} \to \mathbb{R}$ an increasing map with $f(x + 1) = f(x) + 1$ such that the induced maps on S^1 agree. Then,
$$\sigma.(T, f) = (Z', \mathcal{P}')$$
with $Z' = T^{-1}Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

Remark 2.4.23. Note that the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ preserves the semistable objects, but relabels their phases (so the heart can change).

Moreover, for any $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$, the stabilizer group $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})_\sigma \neq \mathrm{Id}$ if and only the image of Z is contained in a real line through 0 in $\mathbb{R}^2 \cong \mathbb{C}$.

Remark 2.4.24. Relation of the actions and support property:

- Given $\Phi \in \mathrm{Aut}(\mathcal{D})$, if $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to a quadratic form Q, then it follows from the definition of the action $\sigma' := \Phi.\sigma$ satisfies the support property with respect to $Q' := Q \circ \Phi^{-1}$.

- Given $(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, if $\sigma = (Z, \mathcal{P})$ satisfies the support property with respect to a quadratic form Q, then it follows from the definition of the action that $\sigma' := \sigma.(T, f)$ satisfies the support property with respect to the same quadratic form Q.

2.5 Examples of spaces of stability conditions

2.5.1 Curves

Let C be a curve of genus $g > 0$ over $\mathbb{C}$, consider $\mathcal{C} = D^b(C)$ the bounded derived category of coherent sheaves on C. Note that $Z : \mathbb{Z}^2 = \mathcal{N}(C) \to \mathbb{C} \cong \mathbb{R}^2$ are of the form

$$Z(d, r) = Ad + Br + i(Cr + Dd)$$

for certain A, B, C, $D \in \mathbb{R}$, where d and r stand for degree and rank respectively.

The next lemma is a strong consequence of $\mathcal{C}$ being hereditary, i.e. it has cohomological dimension 1.

Lemma 2.5.1 ([37]). *Given a distinguished triangle in $\mathcal{C}$,*

$$A \longrightarrow E \longrightarrow B \longrightarrow A[1]$$

with $E \in \mathrm{Coh}(C)$ and $\mathrm{Hom}_{\mathcal{C}}^{\leq 0}(A, B) = 0$, then $A, B \in \mathrm{Coh}(C)$.

Proposition 2.5.2 ([43, Lemma 2.16]). *For any $\sigma \in \mathrm{Stab}(\mathcal{C})$, every line bundle $\mathcal{L}$ and skyscraper sheaf $\mathbb{C}(x)$ of points $x \in C$ are σ-stable.*

Sketch of proof. Let X be either $\mathcal{L}$ or $\mathbb{C}(x)$. Given a stability condition $\sigma \in \mathrm{Stab}(\mathcal{C})$, if X is not σ-semistable, consider the final triangle of its HN-filtration.

$$E \longrightarrow X \longrightarrow A \longrightarrow E[1] \tag{2.5.1}$$

were A is σ-semistable and $\phi_\sigma(E) > \phi_\sigma(A)$. Then, $\mathrm{Hom}_{\mathcal{C}}^{\leq 0}(E, A) = 0$ and by Lemma 2.5.1, $E, A \in \mathrm{Coh}(C)$. Hence, the triangle (2.5.1) is in fact a short exact sequence in $\mathrm{Coh}(C)$. If $X = \mathbb{C}(x)$, there is not such exact sequence in $\mathrm{Coh}(C)$. If $X = \mathcal{L}$, then E is a line bundle and A is torsion, but then $\mathrm{Hom}_{\mathcal{C}}(E, A) \neq 0$ and yields a contradiction. Therefore, X is σ-semistable.

Now let A_0 be a stable factor of X with $\mathrm{Hom}(A_0, X) \neq 0$. Then, there exists a distinguished triangle

$$A \longrightarrow X \longrightarrow B \longrightarrow A[1]$$

where A, B are semistable and such that all stable factors of A are isomorphic to A_0 and $\mathrm{Hom}(A, B) = 0$. Moreover, by semistability, $\mathrm{Hom}^{<0}(A, B) = 0$ which by Lemma 2.5.1 implies that $A, B \in \mathrm{Coh}(C)$. As before, this implies that $B = 0$ and that all stable factors of E are isomorphic to A_0. Hence, $[E] = n[A_0]$, where n is the number of stable factors. Since $[k(x)] = (0, 1)$ and $[\mathcal{L}] = (1, \deg(\mathcal{L}))$, it implies that $n = 1$, i.e. X must be stable. $\square$

Moreover, there is a distinguished stability condition given by the standard slope stability:

$$\sigma_\mu := (Z_\mu, \mathrm{Coh}(C))$$

where $Z_\mu = -d + ir$. The next theorem states that in fact this is the only one up to the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action.

Theorem 2.5.3 ([19, Theorem 9.1], [46, Theorem 2.7]). *The action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on $\mathrm{Stab}(\mathcal{C})$ is free and transitive. In particular,*

$$\mathrm{Stab}(\mathcal{C}) \cong \widetilde{\mathrm{GL}}^+(2, \mathbb{R}). \tag{2.5.2}$$

Sketch of proof. By Proposition 2.5.2, all skyscraper sheaves $\mathbb{C}(x)$ and line bundles $\mathcal{L}$ are σ-stable for every $\sigma \in \mathrm{Stab}(\mathcal{C})$, but not isomorphic. Therefore, the existence of a non-zero morphism $\mathcal{L} \to \mathbb{C}(x)$ (and by Serre duality $\mathbb{C}(x) \to \mathcal{L}[1]$) give inequalities:

$$\phi_\sigma(\mathcal{L}) < \phi_\sigma(\mathbb{C}(x)) < \phi_\sigma(\mathcal{L}) + 1. \tag{2.5.3}$$

Hence, the image of Z is not contained in a real line, which implies that the $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-action is free.

Now, to show that it is transitive, we show that for every $\sigma \in \mathrm{Stab}(\mathcal{C})$ there exists $(T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that $\sigma.(T,f) = \sigma_\mu$. For σ_μ we find the following inequalities

$$0 < \phi_{\sigma_\mu}(\mathcal{L}) < \phi_{\sigma_\mu}(\mathbb{C}(x)) < \phi_{\sigma_\mu}(\mathcal{L}) + 1$$

where now $\phi_{\sigma_\mu}(\mathbb{C}(x)) = 1$. Comparing these inequalities with (2.5.3) we can find an element $(T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$ such that

$$Z'(\mathbb{C}(x)) = -1, \quad Z'(\mathcal{O}_C) = i, \quad \phi_{\sigma'}(\mathbb{C}(x)) = 1$$

where $\sigma' = \sigma.(T,f)$. Since both Z and Z' factor via $\mathcal{N}(C) = \mathbb{Z}^2$, with generators $\mathbb{C}(x)$ and $\mathcal{O}_C$, we see $Z' = Z_\mu$. Since $\phi_{\sigma'}(\mathbb{C}(x)) = 1$, all torsion sheaves have phase 1. Moreover, inequalities (2.5.3) imply that line bundles have phase in $(0,1)$ and therefore every coherent sheaf has HN-filtration with factor of slope in $(0,1]$ i.e. $\mathrm{Coh}(C) \subset \mathcal{P}'((0,1])$. Furthermore, since both $\mathrm{Coh}(C)$ and $\mathcal{P}'((0,1])$ are hearts of bounded t-structures on $\mathcal{C}$, $\mathrm{Coh}(C) = \mathcal{P}'((0,1])$, showing that $\sigma' = \sigma_\mu$. □

Remark 2.5.4. Since for every stability condition $\sigma \in \mathrm{Stab}(\mathcal{C})$ there exists a unique pair $\bar{g} = (T,f)$ such that $\sigma = \sigma_\mu \bar{g}$, in (2.5.2) we identify σ with $\bar{g}$.

For our purpose it is important to understand the isomorphism 2.5.2. First of all, note that the Iwasawa decomposition of a matrix $T \in \mathrm{GL}^+(2,\mathbb{R})$, is of the form $T = kKNA$ with $k \in \mathbb{R}_{>0}$, where K is a rotation matrix of certain degree $\phi \in [0, 2\pi)$, i.e.

$$K = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix},$$

the matrix A is of the form

$$A = \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix},$$

with $a \in \mathbb{R}_{>0}$ and N is a horizontal shear transformation which fixes the real-axis and acts as a stretching along each horizontal line

$$N = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

with $x \in \mathbb{R}$. Therefore, $\mathrm{GL}^+(2,\mathbb{R}) \cong \mathbb{C}^* \times \mathbb{H}$.

Next, we use the Iwasawa decomposition to study stability conditions $(Z, \mathcal{P})$ on $\mathcal{C}$ as pairs $(T,f) \in \widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

- The identity element corresponds to slope-stability $(\mathrm{Id}, \mathrm{id}) = \sigma_\mu$.
- When $T = K$, $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto x + \theta$ with $\theta \coloneqq \phi/\pi$. We find $(Z_\theta, \mathcal{P}_\theta)$ where

$$\begin{aligned} Z_\theta &= K^{-1} Z_\mu \\ &= -d\cos\phi + r\sin\phi + i(r\cos\phi + d\sin\phi). \end{aligned}$$

When $\theta \in [0,1)$,

$$\begin{aligned} \mathcal{P}_\theta((0,1]) &= \mathcal{P}(f((0,1])) \\ &= \mathcal{P}((\theta, \theta+1]) \\ &= \mathrm{Coh}^\theta(C) \end{aligned}$$

where $\mathrm{Coh}^\theta(C)$ stands for the tilting of $\mathrm{Coh}(C)$ with respect to the torsion pair

$$\begin{aligned} F_\theta &= \mathcal{P}((0,\theta]) \\ T_\theta &= \mathcal{P}((\theta,1]). \end{aligned}$$

When $\theta \in [1,2)$, put $\theta' \coloneqq \theta - 1 \in [0,1)$. Now,

$$\begin{aligned} \mathcal{P}_\theta((0,1]) &= \mathcal{P}(f((0,1])) \\ &= \mathcal{P}((\theta, \theta+1]) \\ &= \mathcal{P}((\theta'+1, \theta'+2]) \\ &= \mathrm{Coh}^{\theta'}(C)[1]. \end{aligned}$$

- When $T = A$, $f_a\colon \mathbb{R} \to \mathbb{R}$, we find $(Z_a, \mathcal{P}_a)$ for $a \in \mathbb{R}_{>0}$ where

$$\begin{aligned} Z_a &= A^{-1} Z_\mu \\ &= -d/a + iar \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_a((0,1]) &= \mathcal{P}(f_a((0,1])) \\ &= \mathcal{P}((0,1]) \\ &= \mathrm{Coh}(C) \end{aligned}$$

so it does not affect the heart.

- When $T = N$, $f_s\colon \mathbb{R} \to \mathbb{R}$, we find $(Z_r, \mathcal{P}_x)$ for $x \in \mathbb{R}$ where

$$\begin{aligned} Z_x &= N^{-1} Z_\mu \\ &= -d - xr + ir \end{aligned}$$

and

$$\begin{aligned} \mathcal{P}_x((0,1]) &= \mathcal{P}(f_x((0,1])) \\ &= \mathcal{P}((0,1]) \\ &= \mathrm{Coh}(C) \end{aligned}$$

so it does not affect the heart.

- For arbitrary (T, f) with decomposition $T = KAN$, we find

  $$\begin{aligned} Z &= T^{-1}Z_\mu \\ &= N^{-1}A^{-1}K^{-1}Z_\mu, \end{aligned}$$

  which by the previous observations, it means that the action first determines the heart and relabels the slicing afterwards.

Remark 2.5.5. Last but not least, we analyze the action of the Serre functor $S_{\mathcal{C}} \in \mathrm{Aut}(\mathcal{C})$. Note that

$$S_{\mathcal{C}}(E) = E \otimes \omega_C[1]$$

for all $E \in \mathcal{C}$, where ω_C denotes the canonical line bundle of C. Recall that the line bundle ω_C is trivial when $C = E$ is an elliptic curve and in general it has degree $2g-2$, where g denotes the genus of the curve.

Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{C}$. Recall also that $S_{\mathcal{C}}.\sigma = (Z', \mathcal{P}')$ with $Z'(E) = Z(S_{\mathcal{C}}^{-1}(E))$ and $\mathcal{P}'(\phi) = S_{\mathcal{C}}(\mathcal{P}(\phi))$. We note that if we have

$$Z(d, r) = Ad + Br + i(rC + dD),$$

then $Z'(d, r) = -Z(d, r) + r(2g-2)(A + iD)$. This shows that the action of the Serre functor maps a heart Coh^θ to $\mathrm{Coh}^{\theta'}[1]$ with $\theta, \theta' \in [0, 1)$ and they are not necessarily equal, i.e. in general $\theta \neq \theta'$. The only exceptions are when the curve is elliptic ($g = 1$) or when $g > 1$ and $\theta = 0$.

Remark 2.5.6. Burban and Kreussler described in [21] the stability manifold for the bounded derived category of coherent sheaves on a singular irreducible projective curve of arithmetic genus 1.

The case of $\mathbb{P}^1$ was treated independently by Okada [51] and Macri [46]. We say a bit about it in the next sections since it is an example of stability conditions constructed by means of exceptional collections.

2.5.2 Stability conditions and exceptional objects

Here we want to recall Macrì's construction [46] of stability conditions on triangulated categories generated by finitely many exceptional objects.

Definition 2.5.7. Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category.

1. An object $E \in \mathcal{D}$ is *exceptional* if $\mathrm{Hom}^0_{\mathcal{D}}(E, E) = \mathbb{C}$ and $\mathrm{Hom}^k_{\mathcal{D}}(E, E) = 0$ for all $k \neq 0$.

2. A (finite) sequence $\{E_i\}_{i=1}^n$ of exceptional objects in $\mathcal{D}$ is an *exceptional collection* if additionally $\mathrm{Hom}^k_{\mathcal{D}}(E_i, E_j) = 0$ for all k and $i > j$. It is usually denoted by $(E_1, \cdots, E_n)$.

3. An exceptional collection $(E_1, \cdots, E_n)$ is *strong* if $\mathrm{Hom}^k_{\mathcal{D}}(E_i, E_j) = 0$ for all i, j with $k \neq 0$.

4. An exceptional collection $(E_1, \cdots, E_n)$ is *complete* if it generates $\mathcal{D}$ by shifts and extensions.

5. An exceptional collection $(E_1, \cdots, E_n)$ is *Ext-exceptional* if $\mathrm{Hom}^k_{\mathcal{D}}(E_i, E_j) = 0$ for all $k \leq 0$ and $i \neq j$.

Definition 2.5.8. 1. Let E and F be exceptional objects. We define *left mutation* of F by E, $\mathcal{L}_E F$ and *right mutation* of E by F, $\mathcal{R}_F E$ by the following distinguished triangles

$$\mathcal{L}_E F \longrightarrow \mathrm{Hom}^\bullet_{\mathcal{D}}(E, F) \otimes E \longrightarrow F,$$
$$E \longrightarrow \mathrm{Hom}^\bullet_{\mathcal{D}}(E, F) \otimes F \longrightarrow \mathcal{R}_F E.$$

2. Let $\mathcal{E} = (E_1, \cdots, E_n)$ be an exceptional collection. We define a *left (resp. right) mutation* of $\mathcal{E}$ is defined as a mutation of a pair of adjacent objects in this collection:

$$\mathcal{L}_i\mathcal{E} = (E_1, \cdots E_{i-1}, \mathcal{L}_{E_i}E_{i+1}, E_i, E_{i+2}, \cdots, E_n)$$
$$\mathcal{R}_i\mathcal{E} = (E_1, \cdots E_{i-1}, E_{i+1}, \mathcal{R}_{E_{i+1}}E_i, E_{i+2}, \cdots, E_n)$$

For $i = 1, \cdots, n-1$. We can do mutations again in the mutated collection. We call any composition of mutations an *iterated mutation*.

Proposition 2.5.9 ([11, Proposition 4.9]). *i) A mutation of a (complete) exceptional collection is also a (complete) exceptional collection.*

ii) The following relations hold:

$$\mathcal{R}_i\mathcal{L}_i = \mathcal{L}_i\mathcal{R}_i = 1, \quad \mathcal{R}_i\mathcal{R}_{i+1}\mathcal{R}_i = \mathcal{R}_{i+1}\mathcal{R}_i\mathcal{R}_{i+1}, \quad \mathcal{L}_i\mathcal{L}_{i+1}\mathcal{L}_i = \mathcal{L}_{i+1}\mathcal{L}_i\mathcal{L}_{i+1}.$$

For an exceptional collection $(E_1, \cdots, E_n)$ on $\mathcal{D}$, we denote by $\langle E_1, \cdots, E_n\rangle$ the smallest extension-closed full subcategory of $\mathcal{D}$ containing the E_i's.

Lemma 2.5.10 ([46, Lemma 3.14]). *Let $(E_1, \cdots, E_n)$ be a complete Ext-excep-tional collection on $\mathcal{D}$. Then, $\langle E_1, \cdots, E_n\rangle$ is the heart of a bounded t-structure on $\mathcal{D}$.*

Lemma 2.5.11 ([46, Lemma 3.16]). *Let $(E_1, \cdots, E_n)$ be a complete Ext-exceptional collection on $\mathcal{D}$ and let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$. Assume that $E_i \in \mathcal{P}((0,1])$ for $i = 1, \cdots, n$. Then, $\langle E_1, \cdots, E_n\rangle = \mathcal{P}((0,1])$ and E_i is σ-stable for all $i = 1, \cdots, n$.*

Remark 2.5.12. Given a complete exceptional collection $(E_1, \cdots, E_n)$ on $\mathcal{D}$, the Grothendieck group is a free abelian group of finite rank $K(\mathcal{D}) \cong \mathbb{Z}^n$ generated by the isomorphism classes $[E_i]$ for $i = 1, \cdots, n$.

Let $\mathcal{E} = (E_1, \cdots, E_n)$ be a complete exceptional collection. We summarize Macri's construction of stability conditions from $\mathcal{E}$. For $i = 1, \cdots, n$, choose integers p_i such that $(E_1[p_1], \cdots, E_n[p_n])$ is Ext-exceptional. Denote by

$$Q_p := \langle E_1[p_1], \cdots, E_n[p_n] \rangle$$

the heart of a bounded t-structure. Pick n points $z_1, \cdots, z_n \in \overline{\mathbb{H}}$ and define a homomorphism

$$\begin{aligned} Z_p \colon\ K(Q_p) &\longrightarrow \mathbb{C} \\ [E_i[p_i]] &\longmapsto z_i. \end{aligned}$$

The pair (Z_p, Q_p) is a (locally finite) stability condition on $\mathcal{D}$ (see [46, Remark 2.2]).

Let $\Theta_{\mathcal{E}}$ be the subset of $\mathrm{Stab}(\mathcal{D})$ consisting of all stability conditions constructed with the previous procedure, up to the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$. The following lemma is an immediate consequence of Lemma 2.5.11.

Lemma 2.5.13 ([46])**.** *Let $\mathcal{E} = (E_1, \cdots, E_n)$ be a complete exceptional collection. Then, the E_i's are stable in each stability condition of $\Theta_{\mathcal{E}}$.*

Remark 2.5.14. The converse is not true in general, i.e. $\Theta_{\mathcal{E}}$ is not the subspace consisting of stability conditions in which the E_i's are stable (see [46, Remark 4.8]).

Lemma 2.5.15 ([46, Lemma 3.18])**.** *The subspace $\Theta_{\mathcal{E}} \subset \mathrm{Stab}(\mathcal{D})$ is an open, connected simply connected n-dimensional submanifold.*

Sketch of proof. Given an (arbitrary) exceptional collection $\mathcal{F}_s := (F_1, \cdots, F_s)$, with $s > 1$, we define, for $i < j$,

$$k_{i,j}^{\mathcal{F}_s} := \begin{cases} +\infty & \text{if } \mathrm{Hom}^k(F_i, F_j) = 0\ \forall k, \\ \min\{k \mid \mathrm{Hom}^k(F_i, F_j) \neq 0\} & \text{otherwise.} \end{cases}$$

Then, we define $\alpha_s^{\mathcal{F}_s} := 0$ and inductively for $i < s$,

$$\alpha_i^{\mathcal{F}_s} := \min_{j>i}\{k_{i,j}^{\mathcal{F}_s} + \alpha_j^{\mathcal{F}_s}\} - (s - i - 1).$$

Consider $\mathbb{R}^n$ with coordinates $\phi_1, \cdots, \phi_n$. Let $\mathcal{F}_s \subset (E_1, \cdots, E_n)$ be of the form $\mathcal{F}_s = (E_{l_1}, \cdots, E_{l_s})$ for $s > 1$. Define $R^{\mathcal{F}_s}$ on $\mathbb{R}^n$ as the relation $\phi_{l_1} < \phi_{l_s} + \alpha_1^{\mathcal{F}_s}$. Finally, define

$$\mathcal{C}_{\mathcal{E}} := \left\{ (m_1, \cdots, m_n, \phi_1, \cdots, \phi_n) \in \mathbb{R}^{2n} \mid \begin{array}{l} m_i > 0\ \forall i \text{ and} \\ R^{\mathcal{F}_s}\ \forall \mathcal{F}_s \subset \mathcal{E},\ s > 1 \end{array} \right\}.$$

Then, there is a homeomorphism from $\Theta_{\mathcal{E}}$ to $\mathcal{C}_{\mathcal{E}}$ given by

$$\begin{aligned} \rho \colon\ \Theta_{\mathcal{E}} &\longrightarrow \mathcal{C}_{\mathcal{E}} \\ \sigma = (Z, \mathcal{P}) &\longmapsto (|Z(E_1)|, \cdots, |Z(E_n)|, \phi_\sigma(E_1), \cdots, \phi_\sigma(E_n)). \end{aligned}$$

See [46] for more details. □

2.5.3 Quivers

Now let us consider a finite quiver $Q = (Q_0, Q_1, s, t)$, i.e. a directed graph. It consists of a finite set of vertices $Q_0 = \{0, 1, \cdots, n\}$ and a finite set $\mathbb{Q}_1$ of arrows between them, together with maps $s, t \colon Q_1 \to Q_0$ called source and target respectively. A finite dimensional representation V of a quiver Q consists of a family $V = \{V_a, f_\alpha\}$ where V_a is a finite dimensional vector space for each vertex $a \in Q_0$ and a linear map $f_\alpha \colon V_{s(\alpha)} \to V_{t(\alpha)}$ for each arrow $\alpha \in Q_1$. Let $\mathrm{Rep}(Q)$ be the abelian category of finite dimensional representations of Q and denote by $\mathcal{Q} := D^b(\mathrm{Rep}(Q))$ its bounded derived category.

Example 2.5.16. The P_n-quiver is given by two vertices $Q_0 = \{0, 1\}$ and n arrows $Q_1 = \{\alpha_1, \cdots, \alpha_n\}$ from 0 to 1:

$$\begin{tikzcd} {}^{0}\bullet \arrow[r, bend left, "\alpha_1"] \arrow[r, phantom, "\vdots"] \arrow[r, bend right, "\alpha_n"'] & \bullet^{1}. \end{tikzcd}$$

A representation of P_n is a pair of vector spaces V_0, V_1 together with n linear morphisms $f_{\alpha_1}, \cdots, f_{\alpha_n} \colon V_0 \to V_1$.

If we assume a quiver Q to be finite, with no loops and no oriented cycles, the following results are well-known [46, Section 3.1], [4]:

- The collection of objects $\{S_a\}_{a \in Q_0}$ where S_a is given by assigning $\mathbb{C}$ to the vertex a and 0 to the rest is a complete set of simple objects in $\mathrm{Rep}(Q)$ [4, Chapter III, Lemma 2.1].
- We have $\mathrm{Hom}(S_a, S_b) = \mathbb{C}$ for all $a = b \in Q_0$ and $\mathrm{Hom}(S_a, S_b) = 0$ otherwise. Moreover, $\mathrm{Ext}^1(S_a, S_b) = \mathbb{C}^n$ where n denotes the number of arrows from a to b.
- If we embed the S_a's in $\mathcal{Q}$ in degree 0, they are exceptional objects for all $a \in Q_0$ and by a suitable ordering $(S_a)_{a \in Q_0}$ becomes a complete exceptional collection on $\mathcal{Q}$.
- The Grothendieck group $K(\mathcal{Q})$ is generated by $(S_a)_{a \in Q_0}$.
- The category $\mathrm{Rep}(Q)$ is hereditary [4, Chapter VII, Theorem 1.7].

In [46], Macrì studied $\mathrm{Stab}(\mathcal{Q}_n)$, with $\mathcal{Q}_n := D^b(\mathrm{Rep}(P_n))$, i.e. the stability manifold for the bounded derived category of finite dimensional representations of the P_n-quiver. He sets $\{S_i\}_{i \in \mathbb{Z}}$ the family of exceptional objects on $\mathcal{Q}_n$, where $S_0[1]$ and S_1 are the minimal objects in $\mathrm{Rep}(P_n)$ and the other exceptional objects are defined inductively by

$$\begin{aligned} S_i &:= \mathcal{L}_{S_{i+1}} S_{i+2}, \quad i < 0, \\ S_i &:= \mathcal{R}_{S_{i-1}} S_{i-2}, \quad i \geq 2. \end{aligned}$$

According to [46] these are (up to shifts) the only exceptional objects in $\mathcal{Q}_n$. Each adjacent pair (S_i, S_{i+1}) is an exceptional pair. Moreover, each (S_i, S_{i+1}) is the right mutation of (S_{i-1}, S_i).

Example 2.5.17. If $n = 1$, there are only 3 exceptional objects up to shifts:

$$S_0[1] = (\mathbb{C} \to 0), \quad S_1 = (0 \to \mathbb{C}), \quad S_2 = (\mathbb{C} \to \mathbb{C}).$$

Furthermore, the right mutation satisfies that

$$\begin{aligned} \mathcal{R}_1\mathcal{R}_1\mathcal{R}_1(S_0, S_1) &= \mathcal{R}_1\mathcal{R}_1(S_1, S_2) \\ &= \mathcal{R}_1(S_2, S_0[1]) \\ &= (S_0[1], S_1[1]). \end{aligned}$$

Lemma 2.5.18 ([46, Lemma 4.1])**.** *Assume $n > 1$. If $i < j$, then*

- $\mathrm{Hom}^k_{\mathcal{Q}_n}(S_i, S_j) \neq 0$ *only if* $k = 0$*;*
- $\mathrm{Hom}^k_{\mathcal{Q}_n}(S_j, S_i) \neq 0$ *only if* $k = 1$*.*

In particular, the pair (S_i, S_{i+1}) is a complete strong exceptional pair.

Lemma 2.5.19 ([46, Lemma 4.2])**.** *In every stability condition on $\mathcal{Q}_n$ there exists a stable exceptional pair (E, F).*

Remark 2.5.20. This result is a consequence of the category $\mathrm{Rep}(Q)$ being hereditary.

Let Θ_i, for $i \in \mathbb{Z}$, be the subset of $\mathrm{Stab}(\mathcal{Q}_n)$ consisting of all stability conditions constructed from the complete exceptional pair (S_i, S_{i+1}) as in the discussion in the previous section.

Lemma 2.5.21 ([46, Section 4])**.** *For every $i \in \mathbb{Z}$, the set Θ_i coincides with the subset of $\mathrm{Stab}(\mathcal{Q}_n)$ consisting of all stability conditions in which S_i and S_{i+1} are stable.*

All these lemmas together imply the following description of stability manifolds.

Proposition 2.5.22.

- *If $n > 1$, $\mathrm{Stab}(\mathcal{Q}_n) = \bigcup_{i \in \mathbb{Z}} \Theta_i$.*
- *For $n = 1$, $\mathrm{Stab}(\mathcal{Q}_1) = \Theta_0 \cup \Theta_1 \cup \Theta_2$.*

The next proposition describes the intersection of these sets.

Proposition 2.5.23 ([46, Proposition 4.4])**.** *For all integers $k \neq h$ we have*

$$\Theta_k \cap \Theta_h = O_{-1}$$

Where O_{-1} is the $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-orbit of the stability condition $\sigma_{-1} = (Z_{-1}, \mathcal{P}_{-1})$ given by $Z_{-1}(S_0[1]) = -1$ and $Z_{-1}(S_1) = 1 + i$.

Remark 2.5.24. The $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$-orbit O_{-1} is in fact an open subset of $\mathrm{Stab}(\mathcal{Q}_n)$ homeomorphic to $\widetilde{\mathrm{GL}}^+(2,\mathbb{R})$.

Now, applying the proof of Lemma 2.5.15, we find an isomorphism

$$\Theta_k \cong \mathcal{C}_k \coloneqq \{(m_1, m_2, \phi_1, \phi_2) \in \mathbb{R}^4 \mid m_i > 0 \textit{ and } \phi_1 < \phi_2\}$$

for every $k \in \mathbb{Z}$. Since the $\mathcal{C}_k$'s are connected and simply connected and they glue on O_{-1}, which is contractible, $\mathrm{Stab}(\mathcal{Q}_n)$ is also connected and simply connected by Seifert-van Kampen's theorem.

Theorem 2.5.25 ([46, Theorem 4.5])**.** *The stability manifold* $\mathrm{Stab}(\mathcal{Q}_n)$ *is a connected and simply connected 2-dimensional complex manifold.*

Remark 2.5.26. We have been pointing out the case $n = 1$ since it gives the description we will use later on, but the other cases $n \geq 2$ are also quite interesting since it is well-known that $\mathcal{Q}_n$ is equivalent to $D^b(\mathbb{P}^{n-1})$ the bounded derived category of coherent sheaves on the $(n-1)$-projective space.

For $n = 2$, Okada [51] proved a stronger result, in fact he proved that

$$\mathrm{Stab}(\mathbb{P}^1) \cong \mathbb{C}^2.$$

Bridgeland stability conditions on holomorphic triples

2.6 Holomorphic triples over curves

Let C denote a smooth projective curve of genus $g > 0$ over $\mathbb{C}$.

Definition 2.6.1. A *holomorphic triple* $T = (E_1, E_2, \varphi)$ on C consists of two coherent sheaves $E_1, E_2 \in \mathrm{Coh}(C)$ and a sheaf morphism between them $\varphi\colon E_1 \to E_2$.

Definition 2.6.2. Let $T = (E_1, E_2, \varphi)$ and $T' = (E_1', E_2', \varphi')$ be two holomorphic triples on C. A *morphism* between them $f = (f_1, f_2)\colon T \to T'$ consists of a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "\varphi"'] & E_1' \arrow[d, "\varphi'"] \\ E_2 \arrow[r, "f_2"] & E_2'. \end{tikzcd}$$

Denote the category of holomorphic triples on C by $\mathrm{TCoh}(C)$.

Definition 2.6.3. Let $T = (E_1, E_2, \varphi)$ and $T' = (E_1', E_2', \varphi')$ be two holomorphic triples on C. We say that T' is a *subtriple* of T, $T' \subset T$, if $E_i' \subset E_i$ is a subsheaf for $i = 1, 2$ and the following diagram commutes

$$\begin{tikzcd} E_1 \arrow[r, hook] \arrow[d, "\varphi"'] & E_1' \arrow[d, "\varphi'"] \\ E_2 \arrow[r, hook] & E_2'. \end{tikzcd}$$

There is also a notion of semistability for holomorphic triples. For that, we need to introduce the definition of slope.

Definition 2.6.4. Let $\alpha \in \mathbb{R}$ be arbitrary. For a holomorphic triple $T = (E_1, E_2, \varphi)$, we define its *α-degree* as

$$\begin{aligned} \deg_\alpha(T) &\coloneqq \deg(E_1 \oplus E_2) + \alpha r_1 \\ &= d_1 + d_2 + \alpha r_1 \end{aligned}$$

where $d_i \coloneqq \deg(E_i)$, $r_i \coloneqq \mathrm{rk}(E_i)$ for $i = 1, 2$.

The *rank* of T is $\mathrm{rk}(T) = r_1 + r_2$ and the α-*slope* is

$$\mu_\alpha(T) = \frac{\deg_\alpha(T)}{\mathrm{rk}(T)} \in \mathbb{R} \cup \{\infty\}.$$

A holomorphic triple T is α-*(semi)stable* if for all non-trivial subtriples $T' \subsetneq T$

$$\mu_\alpha(T') < \mu_\alpha(T) \text{ (resp. } \mu_\alpha(T') \leq \mu_\alpha(T)).$$

Remark 2.6.5. Holomorphic triples were first introduced by García-Prada et al. in [35] and [18] for vector bundles over a smooth projective curve of genus g. In [1, Definition 2.2], Álvarez-Cónsul and García-Prada gave general notions of degree, rank, slope and semistability for quiver-bundles, depending on multiple parameters. For holomorphic triples, the resulting notions of semistability are more general than α-semistability (compare Proposition 2.6.11).

According to the definition of α-stability from [35] and [18], the parameter α can be any real number. However, it turns out that α-stable triples exists only under certain constraints, as shown in the following result.

Proposition 2.6.6 ([18, Proposition 3.13 and 3.14])**.** *Let $T = (E_1, E_2, \varphi)$ be an α-stable triple, with E_1 and E_2 vector bundles over C. Then,*

$$0 < \mu(E_1) - \mu(E_2) < \alpha,$$

where $\mu(E_i) = d_i/r_i$ denotes the slope of E_i for $i = 1, 2$. Moreover, if $r_1 \neq r_2$, then

$$\alpha < \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\mu(E_1) - \mu(E_2)).$$

Remark 2.6.7. Note that in the previous proposition we can replace stable by semistable with the inequalities allowing equality.

Remark 2.6.8. In the following we will write

$$\begin{aligned} \alpha_m &:= \mu(E_1) - \mu(E_2), \\ \alpha_M &:= \left(1 + \frac{r_1 + r_2}{|r_1 - r_2|}\right)(\mu(E_1) - \mu(E_2)). \end{aligned}$$

Note that if $\mu(E_1) = \mu(E_2)$, then $\alpha_m = \alpha_M = 0$, hence α-stable triples of vector bundles cannot exist and α-semistable triples exist only for $\alpha = 0$.

It was shown in [35] and [18] that projective moduli spaces for holomorphic triples of vector bundles exist. Later, a precise construction via GIT of the moduli spaces was given by Schmitt in [55]. A variation of moduli with respect to the parameter α is found in [17], where the main theorem reads as follows.

Theorem 2.6.9 ([17, Theorem])**.** *For every $\alpha \in (\alpha_m, \alpha_M)$ (resp. any value $\alpha > \alpha_m$ in the case $r_1 = r_2$) and $\alpha \geq 2g - 2$, the moduli space of α-semistable holomorphic triples is non-empty, smooth and irreducible,* Note that the category $\mathrm{TCoh}(C)$ is abelian since it is the quiver category of an abelian category. Denote by $K(T)$ (resp. $K(C)$) the Grothendieck group of $\mathrm{TCoh}(C)$ (resp. $\mathrm{Coh}(C)$).

Proposition 2.6.10 ([28, Lemma 5.3.4])**.** $K(T) \cong K(C) \oplus K(C)$.

Proof. Each $T \in \mathrm{TCoh}(C)$ defines a class $[T] \in K(T)$. We can place $T = (E_1, E_2, \varphi)$ in a short exact sequence as follows:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, "\phi"] & E_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_2 \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd}$$

Hence, $[T] = [0 \to E_2] + [E_1 \to 0]$ in $K(T)$ and we can define the isomorphism

$$\begin{aligned} K(T) &\longrightarrow K(C) \oplus K(C) \\ [T] &\longmapsto ([E_1], [E_2]). \end{aligned}$$

□

We denote by $\mathcal{N}(C) = \dfrac{K(C)}{K(C)^{\perp}} \cong \mathbb{Z}^2$ the numerical Grothendieck group of $\mathrm{Coh}(C)$ and by $\mathcal{T}_C$ the bounded derived category of holomorphic triples on C.

Note that the Euler form on $\mathrm{TCoh}(C)$

$$\chi(T, T') \coloneqq \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Hom}_{\mathcal{T}_C}(T, T'[i])$$

vanishes if and only if $r_1 = r_2 = 0$ and $d_1 = d_2 = 0$, and hence the numerical Grothendieck group of $\mathcal{T}_C$ is isomorphic to $\mathbb{Z}^4$ by

$$\begin{aligned} \mathcal{N}(\mathcal{T}_C) &\xrightarrow{\ \sim\ } \mathbb{Z}^4 \\ [E_1 \to E_2] &\longmapsto (r_1, d_1, r_2, d_2). \end{aligned}$$

2.6.1 Stability conditions with standard heart

For $\nu = 1, 2$, the inclusion functors $I_\nu \colon \mathrm{Coh}(C) \to \mathrm{TCoh}(C)$, induce group homomorphisms $i_\nu \colon K(C) \to K(T)$ where $i_1([E]) = [E \to 0]$ and $i_2([E]) = [0 \to E]$.

If $Z \colon K(T) \to \mathbb{C}$ is a stability function, denote the composition

$$Z_\nu \coloneqq Z \circ i_\nu \colon K(C) \to \mathbb{C}$$

for $\nu = 1, 2$. The Z_ν are again stability functions and if Z has the Harder-Narasimhan property, then the Z_ν have the Harder-Narasimhan property too.

Proposition 2.6.11 ([28, Proposition 5.3.10])**.** *For $i = 1, 2$, let $A_i, B_i, C_i \in \mathbb{R}$ be such that $A_i, C_i > 0$. Then,*

$$Z(r_1, d_1, r_2, d_2) \coloneqq -A_1 d_1 - A_2 d_2 + B_1 r_1 + B_2 r_2 + i(C_1 r_1 + C_2 r_2)$$

is a stability function on $\mathrm{TCoh}(C)$ *which has the Harder-Narasimhan property and the corresponding slicing is locally finite.*

Denote by $\mathrm{Stab}^\circ(\mathcal{T}_C)$ (resp. $\mathrm{Stab}^\circ(\mathcal{C})$) the set of numerical locally finite pre-stability conditions on $\mathcal{T}_C$ (resp. $\mathcal{C}$) with heart $\mathrm{TCoh}(C)$ (resp. $\mathrm{Coh}(C)$).

Theorem 2.6.12 ([28, Theorem 5.3.11])**.** *With the previous notations,*

$$\begin{aligned} \mathrm{Stab}^\circ(\mathcal{T}_C) &\xrightarrow{\sim} \mathrm{Stab}^\circ(\mathcal{C}) \times \mathrm{Stab}^\circ(\mathcal{C}) \\ (Z, \mathrm{TCoh}) &\longmapsto ((Z_1, \mathrm{Coh}), (Z_2, \mathrm{Coh})). \end{aligned}$$

A natural question arises from Theorem 2.6.12 is the following.

Question 2.6.13. *Can all stability conditions on $\mathcal{T}_C$ be constructed from the ones from $\mathcal{C}$?*

We will answer it in the following sections.

2.7 Semiorthogonal decompositions

From here, it is joint work with A. Rincón Hidalgo (Freie Universität Berlin) and A. Rüffer (University of Limerick) [47].

We note that we have three different ways to see $\mathrm{Coh}(C)$ embedded in $\mathrm{TCoh}(C)$:

$$\begin{aligned} i_*(\mathrm{Coh}) = \mathrm{Coh}_1 &\coloneqq \{E \to 0 \colon E \in \mathrm{Coh}(C)\} \subset \mathrm{TCoh} \\ j_*(\mathrm{Coh}) = \mathrm{Coh}_2 &\coloneqq \{0 \to E \colon E \in \mathrm{Coh}(C)\} \subset \mathrm{TCoh} \\ l_*(\mathrm{Coh}) = \mathrm{Coh}_3 &\coloneqq \{E \xrightarrow{\mathrm{id}} E \colon E \in \mathrm{Coh}(C)\} \subset \mathrm{TCoh} \end{aligned}$$

as well as three different ways to see $\mathcal{C}$ as strictly full subcategories of $\mathcal{T}_C$, where we will adopt the same notation i_*, j_*, l_*. As before, we denote by $\mathcal{C}_i$ for $i = 1, 2, 3$, to refer to the strictly full subcategories of $\mathcal{T}_C$ obtained as the image of $\mathcal{C}$ in $\mathcal{T}_C$ under each embedding:

$$\begin{aligned} \mathcal{C}_1 &\coloneqq i_*\mathcal{C} \subset \mathcal{T}_C \\ \mathcal{C}_2 &\coloneqq j_*\mathcal{C} \subset \mathcal{T}_C \\ \mathcal{C}_3 &\coloneqq l_*\mathcal{C} \subset \mathcal{T}_C. \end{aligned}$$

2.7.1 Admissible subcategories and semiorthogonal decompositions

We first introduce the concepts of semiorthogonal decomposition and admissible subcategories of an arbitrary triangulated category.

Definition 2.7.1. Let $\mathcal{D}$ be a triangulated category. A *semiorthogonal decomposition* of $\mathcal{D}$ consists of a collection $\mathcal{A}_1, \cdots, \mathcal{A}_n$ of full triangulated subcategories such that

1. $\mathrm{Hom}_{\mathcal{D}}(\mathcal{A}_i, \mathcal{A}_j) = 0$ for every $1 \leq j < i \leq n$.
2. $\mathcal{D}$ is generated by the $\mathcal{A}_i$.

We write $\mathcal{D} = \langle \mathcal{A}_1, \cdots, \mathcal{A}_n \rangle$.

Lemma 2.7.2 ([12, Proposition 1.5])**.** *Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be strictly full triangulated subcategories of $\mathcal{D}$. Assume that* $\mathrm{Hom}_{\mathcal{D}}(\mathcal{B}, \mathcal{A}) = 0$. *Then, the following are equivalent:*

1. *The category $\mathcal{D}$ is generated by $\mathcal{A}$ and $\mathcal{B}$ i.e. for each $X \in \mathcal{D}$, there exists a distinguished triangle*
$$B \longrightarrow X \longrightarrow A \longrightarrow B[1]$$
with $A \in \mathcal{A}$ and $B \in \mathcal{B}$.
2. $\mathcal{B} = {}^{\perp}\mathcal{A} := \{D \in \mathcal{D} \mid \mathrm{Hom}_D(D, \mathcal{A}) = 0\}$ *and there exists a functor $i^*\colon \mathcal{D} \to \mathcal{A}$ which is left adjoint to the inclusion $i\colon \mathcal{A} \hookrightarrow \mathcal{D}$.*
3. $\mathcal{A} = \mathcal{B}^{\perp} := \{D \in \mathcal{D} \mid \mathrm{Hom}_D(\mathcal{A}, D) = 0\}$ *and there exists a functor $j^!\colon \mathcal{D} \to \mathcal{B}$ which is right adjoint to the inclusion $j\colon \mathcal{B} \hookrightarrow \mathcal{D}$.*

Remark 2.7.3. When the previous conditions are satisfied, we have a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{A}, \mathcal{B} \rangle$. In this case, given $X \in \mathcal{D}$ the components $A \in \mathcal{A}$ and $B \in \mathcal{B}$ in (2.) are unique up to isomorphism.

Definition 2.7.4. Let $\mathcal{D}$ be a triangulated category. Let $\mathcal{A}$ and $\mathcal{B}$ be full triangulated subcategories of $\mathcal{D}$. If the conditions of Lemma 2.7.2 are satisfied we say that $\mathcal{A}$ is *left admissible*, $\mathcal{B}$ is *right admissible*. We say that a full subcategory of $\mathcal{D}$ is *admissible* if it is both left and right admissible.

Lemma 2.7.5 ([12])**.** *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{D}$. If $\mathcal{A}$ is admissible, we find an equivalence of triangulated categories ${}^{\perp}\mathcal{A} \cong \mathcal{A}^{\perp}$.*

Proof. We first define the following functors $F\colon {}^{\perp}\mathcal{A} \to \mathcal{A}^{\perp}$, $G\colon \mathcal{A}^{\perp} \to {}^{\perp}\mathcal{A}$. Given an object $E \in {}^{\perp}\mathcal{A}$, considered as element of $\mathcal{D}$, it has a unique triangle of the form

$$A \longrightarrow E \longrightarrow A' \longrightarrow A[1] \tag{2.7.1}$$

with $A \in \mathcal{A}$ and $A' \in \mathcal{A}^{\perp}$. We define $F(E) := A'$. Conversely, given an object $E' \in \mathcal{A}^{\perp}$, as element of $\mathcal{D}$, it has a unique triangle of the form

$$B' \longrightarrow E' \longrightarrow B \longrightarrow B'[1] \tag{2.7.2}$$

with $B' \in {}^\perp\mathcal{A}$ and $B \in \mathcal{A}$. We define $G(E') := B'$. We now show that FG is isomorphic to the identity (the opposite is analogous). Given $E' \in \mathcal{A}^\perp$, $G(E') \in {}^\perp\mathcal{A}$ is obtained by means of the distinguished triangle (2.7.2). Now apply (2.7.1) to $G(E')$

$$C \longrightarrow G(E') \longrightarrow C' \longrightarrow C[1]$$

and get $F(G(E')) = C' \in \mathcal{A}^\perp$. On the other hand, from (2.7.2) we also have

$$B[-1] \longrightarrow G(E') \longrightarrow E' \longrightarrow B$$

from where we obtain $C \cong B[-1]$ and $F(G(E')) \cong E'$. □

Now we see that in our context, $\mathcal{T}_C = D^b(\mathrm{TCoh}(C))$ admits the following semi-orthogonal decompositions.

Proposition 2.7.6. *The triangulated category $\mathcal{T}_C$ admits three semiorthogonal decompositions:*

$$\begin{aligned} \mathcal{T}_C &= \langle \mathcal{C}_3, \mathcal{C}_1 \rangle \\ &= \langle \mathcal{C}_2, \mathcal{C}_3 \rangle \\ &= \langle \mathcal{C}_1, \mathcal{C}_2 \rangle. \end{aligned}$$

Proof. We start by proving that the triangulated category $\mathcal{C}_3$ is admissible and its orthogonal categories are:

1. $\mathcal{C}_3^\perp = \mathcal{C}_2$,
2. ${}^\perp\mathcal{C}_3 = \mathcal{C}_1$.

Thus, by means of Lemma 2.7.2 we will have $\mathcal{T}_C = \langle \mathcal{C}_3, \mathcal{C}_1 \rangle = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$.

In the abelian categories TCoh and Coh, we define the following functors

$$\begin{aligned} l_*\colon \mathrm{Coh} \to \mathrm{TCoh} &\qquad E \mapsto E \xrightarrow{\mathrm{id}} E \\ l^*\colon \mathrm{TCoh} \to \mathrm{Coh} &\qquad E_1 \xrightarrow{\varphi} E_2 \mapsto E_2 \\ l^!\colon \mathrm{TCoh} \to \mathrm{Coh} &\qquad E_1 \xrightarrow{\varphi} E_2 \mapsto E_1. \end{aligned}$$

The definition on morphisms is addressed in Remark 2.7.9.

First, we see left adjointness, (l^*, l_*), i.e.

$$\mathrm{Hom}_{\mathrm{Coh}}(l^*(E_1 \xrightarrow{\varphi} E_2), F_1) = \mathrm{Hom}_{\mathrm{TCoh}}(E_1 \xrightarrow{\varphi} E_2, l_*(F_1)).$$

Indeed, a morphism $f \in \mathrm{Hom}_{\mathrm{TCoh}}(E_1 \xrightarrow{\varphi} E_2, l_*(F_1))$ consists of a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "\varphi"] & F_1 \arrow[d, "\mathrm{id}"] \\ E_2 \arrow[r, "f_2"] & F_1, \end{tikzcd}$$

which shows that f is uniquely determined by $f_2 \in \mathrm{Hom}_{\mathrm{Coh}}(l^*(E_1 \xrightarrow{\varphi} E_2), F_1)$.

Now we see right adjointness, $(l_*, l^!)$, i.e.

$$\mathrm{Hom}_{\mathrm{TCoh}}(l_*(E_1), F_1 \xrightarrow{\varphi} F_2) = \mathrm{Hom}_{\mathrm{Coh}}(E_1, l^!(F_1 \xrightarrow{\varphi} F_2)).$$

Indeed, a morphism $f \in \mathrm{Hom}_{\mathrm{TCoh}}(l_*(E_1), F_1 \xrightarrow{\varphi} F_2)$ consists of a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "\mathrm{id}"] & F_1 \arrow[d, "\varphi"] \\ E_1 \arrow[r, "f_2"] & F_2, \end{tikzcd}$$

which shows that f is uniquely determined by $f_1 \in \mathrm{Hom}_{\mathrm{Coh}}(E_1, l^!(F_1 \xrightarrow{\varphi} F_2))$.

Finally, since all these functors are exact, we extend them to the derived categories and we will keep the same notations.

To prove (1.) we have to see that

$$\mathcal{C}_3^{\perp} = \{F \in \mathcal{T}_C \mid \mathrm{Hom}_{\mathcal{T}_C}(l_*\mathcal{C}, F) = 0\} = \mathcal{C}_2.$$

Indeed, for any $F_1 \xrightarrow{\varphi} F_2 \in \mathcal{T}_C$ and any $E \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{T}_C}(l_*(E), F_1 \xrightarrow{\varphi} F_2) = \mathrm{Hom}_{\mathcal{C}}(E, l^!(F_1 \xrightarrow{\varphi} F_2))$$

by adjointness. This means that $\mathrm{Hom}_{\mathcal{C}}(E, F_1) = 0$ for all $E \in \mathcal{C}$. This happens if and only if $F_1 = 0$.

Now, to prove (2.) we have to see that

$${}^{\perp}\mathcal{C}_3 = \{E \in \mathcal{T}_C \mid \mathrm{Hom}_{\mathcal{T}_C}(E, l_*\mathcal{C}) = 0\} = \mathcal{C}_1.$$

Indeed, for any $E_1 \xrightarrow{\varphi} E_2 \in \mathcal{T}_C$ and any $F \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{T}_C}(E_1 \xrightarrow{\varphi} E_2, l_*(F)) = \mathrm{Hom}_{\mathcal{C}}(l^*(E_1 \xrightarrow{\varphi} E_2), F),$$

by adjointness. This means that $\mathrm{Hom}_{\mathcal{C}}(E_2, F) = 0$ for all $F \in \mathcal{C}$ and for all $t \in \mathbb{Z}$. This happens if and only if $E_2 = 0$.

We finish by proving that $\mathcal{C}_1$ is left-admissible and ${}^{\perp}\mathcal{C}_1 = \mathcal{C}_2$ which again by Lemma 2.7.2 will give the remaining equality, $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$.

In the abelian categories TCoh and Coh, we define the following functors

$$\begin{aligned} i_*\colon \mathrm{Coh} &\to \mathrm{TCoh} \qquad & E &\mapsto E \to 0 \\ i^*\colon \mathrm{TCoh} &\to \mathrm{Coh} \qquad & E_1 \xrightarrow{\varphi} E_2 &\mapsto E_1 \end{aligned}$$

The definition on morphisms is addressed in Remark 2.7.9.

First, we see left adjointness, (i^*, i_*), i.e.

$$\mathrm{Hom}_{\mathrm{Coh}}(i^*(E_1 \xrightarrow{\varphi} E_2), F_1) = \mathrm{Hom}_{\mathrm{TCoh}}(E_1 \xrightarrow{\varphi} E_2, i_*(F_1)).$$

Indeed, a morphism $f \in \mathrm{Hom}_{\mathrm{TCoh}}(E_1 \overset{\varphi}{\to} E_2, i_*(F_1))$ consists of a commutative diagram

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "\varphi"] & F_1 \arrow[d] \\ E_2 \arrow[r] & 0. \end{tikzcd}$$

Then, f is uniquely determined by $f_1 \in \mathrm{Hom}_{\mathrm{Coh}}(i^*(E_1 \overset{\varphi}{\to} E_2), F_1)$.

Since this functor is exact, we extend it to the derived categories and we will keep the same notations.

Finally, to prove that ${}^{\perp}\mathcal{C}_1 = \mathcal{C}_2$ we have to see that

$${}^{\perp}\mathcal{C}_1 = \{E \in \mathcal{T}_C \mid \mathrm{Hom}_{\mathcal{T}_C}(E, i_*\mathcal{C}) = 0\} = \mathcal{C}_2.$$

Indeed, for any $E_1 \overset{\varphi}{\to} E_2 \in \mathcal{T}_C$ and any $F \in \mathcal{C}$ we have

$$\mathrm{Hom}_{\mathcal{T}_C}(E_1 \overset{\varphi}{\to} E_2, i_*(F)) = \mathrm{Hom}_{\mathcal{C}}(i^*(E_1 \overset{\varphi}{\to} E_2), F_1),$$

by adjointness. This means that $\mathrm{Hom}_{\mathcal{C}}(E_1, F) = 0$ for all $F \in \mathcal{C}$. This happens if and only if $E_1 = 0$. □

Remark 2.7.7. The last step of the proof of Proposition 2.7.6, i.e. the proof of the equality $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, can be proven equivalently by showing that $\mathcal{C}_2$ is right-admissible and $\mathcal{C}_2^{\perp} = \mathcal{C}_1$.

It follows by defining in the abelian categories TCoh and Coh the following exact functors

$$\begin{aligned} j_*\colon \mathrm{Coh} &\to \mathrm{TCoh} & E &\mapsto 0 \to E \\ j^!\colon \mathrm{TCoh} &\to \mathrm{Coh} & E_1 \overset{\varphi}{\to} E_2 &\mapsto E_2 \end{aligned}$$

Remark 2.7.8. In particular, Proposition 2.7.6 states that $\mathcal{C}_i$ is admissible for $i = 1, 2, 3$. This manifests that we have a very special situation because the triangulated category $\mathcal{T}_C$ admits three semiorthogonal decompositions combining the three different ways we can see $\mathcal{C}$ as subcategory of $\mathcal{T}_C$.

Remark 2.7.9. In the course of the proof of Proposition 2.7.6 we omitted the following important observations:

1. Definition of the functors on morphisms. For example given $E, F \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(E, F)$, we have $i_*(f) \in \mathrm{Hom}_{\mathcal{T}_C}(i_*(E), i_*(F))$ as

$$\begin{tikzcd} E \arrow[r, "f"] \arrow[d] & F \arrow[d] \\ 0 \arrow[r, "0"] & 0. \end{tikzcd}$$

resp. given $E_1 \overset{\varphi_E}{\to} E_2, F_1 \overset{\varphi_F}{\to} F_2 \in \mathcal{T}_C$ and $(f_1, f_2) \in \mathrm{Hom}_{\mathcal{T}_C}(E_1 \overset{\varphi_E}{\to} E_2, F_1 \overset{\varphi_F}{\to} F_2)$ as

$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "\varphi_E"] & F_1 \arrow[d, "\varphi_F"] \\ E_2 \arrow[r, "f_2"] & F_2. \end{tikzcd}$$

we have $i^*((f_1, f_2)) = f_1 \in \mathrm{Hom}_{\mathcal{C}}(i^*(E_1 \overset{\varphi_E}{\to} E_2), i^*(F_1 \overset{\varphi_F}{\to} F_2))$. The functor properties are satisfied trivially. It can be easily checked for all the functors defined in Proposition 2.7.6.

2. Functoriality of the cones. In the construction of the semiorthogonal decompositions we have carefully avoided any direct reference to the cones, i.e. we have used only the exact functors defined on the abelian categories. Therefore, the cone $C(\varphi)$ appearing in the distinguished triangles in Remark 2.7.8 is embedded as an object in the corresponding category $\mathcal{C}_i$ for $i = 1, 2, 3$ and functoriality follows from the definition of semiorthogonal decompositions and Lemma 2.7.2.

3. Fully faithfulness of the inclusion functors i_*, j_* and l_*. After seeing the adjointness relations, this is easy to show by applying the functor isomorphisms in Remark 2.1.6, i).

2.7.2 Serre functor

Let $\mathcal{D}$ be a $\mathbb{C}$-linear triangulated category.

Definition 2.7.10. A *Serre functor* on $\mathcal{D}$ is an exact autoequivalence $S\colon \mathcal{D} \to \mathcal{D}$ such that for any $E, F \in \mathcal{D}$, there is an isomorphism

$$\eta_{E,F}\colon \mathrm{Hom}_{\mathcal{D}}(E, F) \to \mathrm{Hom}_{\mathcal{D}}(F, S(E))^*$$

(of $\mathbb{C}$-vector spaces) which is functorial on E and F.

Remark 2.7.11. For $\mathcal{D}$ of finite type (i.e. with finite dimensional $\mathrm{Hom}_{\mathcal{D}}$'s), a Serre functor, if it exists, is unique up to isomorphism. Moreover, they commute with equivalences, i.e. for $F : \mathcal{D} \to \mathcal{D}'$ an equivalence, $S_{\mathcal{D}'} \circ F \cong F \circ S_{\mathcal{D}}$. Furthermore, given an admissible subcategory $\mathcal{X} \subset \mathcal{D}$ it is easy to see that, by Serre duality, $S_{\mathcal{D}}$ sends ${}^{\perp}\mathcal{X}$ to $\mathcal{X}^{\perp}$ and $S_{\mathcal{D}}^{-1}$ sends $\mathcal{X}^{\perp}$ to ${}^{\perp}\mathcal{X}$.

Example 2.7.12. Let X be a smooth projective variety defined over k, then the autoequivalence

$$\begin{aligned} S_X\colon\ D^b(X) &\longrightarrow D^b(X) \\ E^\bullet &\longmapsto E^\bullet \otimes \omega_X[\dim X] \end{aligned}$$

where ω_X is the dualizing line bundle, is a Serre functor on $D^b(X)$. In particular, if X is a Calabi-Yau variety, its Serre functor is simply the shift functor $S_X = [\dim X]$.

Definition 2.7.13. Let $n \in \mathbb{Z}$. A triangulated category $\mathcal{D}$ is an *n-Calabi-Yau category* if it has a Serre functor $S_{\mathcal{D}}$ and $S_{\mathcal{D}} \cong [n]$. The integer n is called the *CY-dimension* of $\mathcal{D}$.

Definition 2.7.14. A triangulated category $\mathcal{D}$ is a *fractional Calabi-Yau category* if it has a Serre functor $S_{\mathcal{D}}$ and there are integers p and $q \neq 0$ such that $S_{\mathcal{D}}^q \cong [p]$. In this case we say that $\mathcal{D}$ has *(CY-)fractional dimension p/q*.

The following result implies that we will have a Serre functor.

Theorem 2.7.15 ([12, Proposition 3.8])**.** *Let $\mathcal{D}$ be a triangulated category and $\mathcal{B} \subset \mathcal{D}$ an admissible full triangulated subcategory with $\mathcal{C} := \mathcal{B}^{\perp}$ admissible. If $\mathcal{B}$ and $\mathcal{C}$ have Serre functors, then there exists a Serre functor on $\mathcal{D}$.*

Sketch of the BK-Construction of the Serre functor with $\mathcal{D} = \mathcal{T}_C$. We sketch the construction of the Serre functor given in the theorem. We consider $\mathcal{B} = \mathcal{C}_2$ and $\mathcal{C} = \mathcal{C}_1$ and see that all functors of the form $h_X := \mathrm{Hom}_{\mathcal{T}_C}(X, .)^*$ and $h^X := \mathrm{Hom}_{\mathcal{T}_C}(., X)^*$ are representable. First, we construct a representing object for h_X, denoted by $S_{\mathcal{T}_C}(X)$, i.e. satisfying

$$h_X(D) = \mathrm{Hom}_{\mathcal{T}_C}(X, D)^* = \mathrm{Hom}_{\mathcal{T}_C}(D, S_{\mathcal{T}_C}(X))$$

for all $D \in \mathcal{T}_C$. We need a representing object E for $h_X|_{\mathcal{C}_2}$ and representing objects for $h_X|_{\mathcal{C}_1}$ and $\mathrm{Hom}_{\mathcal{T}_C}(., E)|_{\mathcal{C}_1}$. To construct a representing object E for $h_X|_{\mathcal{C}_2}$, decompose X into a distinguished triangle

$$X' \longrightarrow X \longrightarrow X_2 \longrightarrow X'[1]$$

with $X' \in \mathcal{C}_3 = {}^{\perp}\mathcal{C}_2$ and $X_2 \in \mathcal{C}_2$. By adjunction, we have

$$\mathrm{Hom}_{\mathcal{T}_C}(X, D_2) = \mathrm{Hom}_{\mathcal{C}_2}(X_2, D_2)$$

for all $D_2 \in \mathcal{C}_2$. Then, $h_X|_{\mathcal{C}_2}$ is representable by $E = S_{\mathcal{C}_2}(X_2)$. To construct a representing object F for $h_X|_{\mathcal{C}_1}$, decompose X into a distinguished triangle

$$Y_2 \longrightarrow X \longrightarrow Y_1 \longrightarrow Y_2[1]$$

with $Y_2 \in \mathcal{C}_2$ and $Y_1 \in \mathcal{C}_1$. By adjunction,

$$\mathrm{Hom}_{\mathcal{T}_C}(X, D_1) = \mathrm{Hom}_{\mathcal{C}_1}(Y_1, D_1)$$

for all $D_1 \in \mathcal{C}_1$. Then, $h_X|_{\mathcal{C}_1}$ is representable by $F = S_{\mathcal{C}_1}(Y_1)$. Finally, to construct a representing object F' for $\mathrm{Hom}_{\mathcal{T}_C}(., E)|_{\mathcal{C}_1}$, decompose E into a distinguished triangle

$$W_1 \longrightarrow E \longrightarrow W' \longrightarrow W_1[1]$$

with $W_1 \in \mathcal{C}_1$ and $W' \in \mathcal{C}_3$. By adjunction,

$$\mathrm{Hom}_{\mathcal{T}_C}(D_1, E) = \mathrm{Hom}_{\mathcal{C}_1}(D_1, W_1), \tag{2.7.3}$$

for all $D_1 \in \mathcal{C}_1$. Then, $\mathrm{Hom}_{\mathcal{T}_C}(.,E)|_{\mathcal{C}_1}$ is representable by $F' = W_1$. Now, consider $D_1 = F'$ in (2.7.3) and let $\gamma \in \mathrm{Hom}_{\mathcal{T}_C}(F', E)$ be the morphism corresponding to the identity. Consider $h_X(\gamma)$ and take $\varphi \in h_X(F') = \mathrm{Hom}_{\mathcal{C}_1}(F', F)$ corresponding to the identity in $h_X(E) = \mathrm{Hom}_{\mathcal{C}_2}(E, E)$. Take the cone $C(\varphi)$ of φ, i.e. complete $F' \xrightarrow{\varphi} F$ to a distinguished triangle

$$F' \xrightarrow{\varphi} F \xrightarrow{\tau_\varphi} C(\varphi) \xrightarrow{\pi_\varphi} F'[1]$$

and let δ denote the composition $\gamma[1] \circ \pi_\varphi$. Let $S \in \mathcal{T}_C$ be the element fitting in the distinguished triangle

$$E \longrightarrow S \longrightarrow C(\varphi) \xrightarrow{\delta} E[1].$$

Bondal and Kapranov [12] proved that $S = S_{\mathcal{T}_C}(X)$, i.e.

$$h_X(T) = \mathrm{Hom}_{\mathcal{T}_C}(T, S_{\mathcal{T}_C}(X))$$

for all $T \in \mathcal{T}_C$.

The Serre functor is a very powerful tool, so we finish this section by showing nice properties of the Serre functor for arbitrary triangulated categories that we will use in our constructions. The following lemma follows easily from the definitions.

Lemma 2.7.16 ([12]). *Let $\mathcal{D}$ be a triangulated category with a Serre functor $S_{\mathcal{D}}$ and let $\mathcal{A} \subset \mathcal{D}$ be a full triangulated subcategory. We have $S_{\mathcal{D}}({}^{\perp}\mathcal{A}) = \mathcal{A}^{\perp}$ and $S_{\mathcal{D}}^{-1}(\mathcal{A}^{\perp}) = {}^{\perp}\mathcal{A}$. Moreover, if $\mathcal{A}$ is admissible, then $\mathcal{A}$ admits a Serre functor $S_{\mathcal{A}}$.*

Remark 2.7.17. The previous result implies that $S_{\mathcal{D}}(\mathcal{A}) = \mathcal{A}^{\perp\perp}$.

The next proposition shows that the Serre functor can be used to construct adjoint functors between triangulated categories.

Proposition 2.7.18. *Let $F\colon \mathcal{D} \to \mathcal{D}'$ be a functor between triangulated categories. Assume that both $\mathcal{D}$ and $\mathcal{D}'$ have Serre functors $S_{\mathcal{D}}$ and $S_{\mathcal{D}'}$ respectively.*

1. *If F admits a left adjoint $G\colon \mathcal{D}' \to \mathcal{D}$, then it also admits a right adjoint $H\colon \mathcal{D}' \to \mathcal{D}$, given as*
$$H = S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}.$$

2. *If F admits a right adjoint $H\colon \mathcal{D}' \to \mathcal{D}$, then it also admits a left adjoint $G\colon \mathcal{D}' \to \mathcal{D}$, given as*
$$G = S_{\mathcal{D}}^{-1} \circ H \circ S_{\mathcal{D}'}.$$

Proof. Let us see (1.). By definition, $\mathrm{Hom}_{\mathcal{D}}(X, H(Y)) = \mathrm{Hom}_{\mathcal{D}}(X, S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}(Y))$. By Serre duality, $\mathrm{Hom}_{\mathcal{D}}(X, S_{\mathcal{D}} \circ G \circ S_{\mathcal{D}'}^{-1}(Y)) = \mathrm{Hom}_{\mathcal{D}}(G \circ S_{\mathcal{D}'}^{-1}(Y), X)^*$. Now, since G is left adjoint to F, $\mathrm{Hom}_{\mathcal{D}}(G \circ S_{\mathcal{D}'}^{-1}(Y), X)^* = \mathrm{Hom}_{\mathcal{D}'}(S_{\mathcal{D}'}^{-1}(Y), F(X))^*$ and by Serre duality, $\mathrm{Hom}_{\mathcal{D}'}(S_{\mathcal{D}'}^{-1}(Y), F(X))^* = \mathrm{Hom}_{\mathcal{D}'}(F(X), Y)$.

The proof of (2.) is analogous. $\square$

Corollary 2.7.19. *If $\mathcal{A}$ is a left (resp. right) admissible full triangulated subcategory of a triangulated category $\mathcal{D}$, such that both $\mathcal{A}$ and $\mathcal{D}$ have Serre functors, then $\mathcal{A}$ is admissible.*

Proposition 2.7.20 ([12]). *Let $\mathcal{D}$ be a triangulated category with a Serre functor $S_{\mathcal{D}}$ and let $\mathcal{A} \subset \mathcal{D}$ be an admissible full triangulated subcategory. Then $\mathcal{A}$ admits a Serre functor $S_{\mathcal{A}}$ and all iterated right and left orthogonals are admissible.*

Remark 2.7.21. By exactness of the Serre functor together with the unicity of the triangles of a semiorthogonal decomposition, the Serre functor maps a semiorthogonal decomposition into another semiorthogonal decomposition.

Using the previous properties, we have the following technical lemma.

Lemma 2.7.22. *Given some $n \in \mathbb{Z}$ and $\theta \in [0,1)$, the following equalities hold:*

1. $S_{\mathcal{T}_C}(\mathrm{Coh}_1^\theta[n]) = j_*(S_{\mathcal{C}}(\mathrm{Coh}^\theta[n]))[1]$.

2. $S_{\mathcal{T}_C}(\mathrm{Coh}_2^\theta[n]) = l_*(S_{\mathcal{C}}(\mathrm{Coh}^\theta[n]))$.

3. $S_{\mathcal{T}_C}(\mathrm{Coh}_3^\theta[n]) = i_*(S_{\mathcal{C}}(\mathrm{Coh}^\theta[n]))$.

Proof. Case 1. By definition of Coh_1, it follows from Proposition 2.7.18

$$\begin{aligned} S_{\mathcal{T}_C}(\mathrm{Coh}_1^\theta[n]) &= S_{\mathcal{T}_C}(i_* \mathrm{Coh}^\theta[n]) \\ &= j_* \left(S_{\mathcal{C}}(\mathrm{Coh}^\theta[n+1]) \right). \end{aligned}$$

Cases 2 and 3 follow analogously. □

After the existence theorem, i.e. Theorem 2.7.15, we obtain a Serre functor $S_{\mathcal{T}_C}$ on $\mathcal{T}_C$ which acts on objects by

$$S_{\mathcal{T}_C}(E_1^\bullet \xrightarrow{\varphi} E_2^\bullet) = E_2^\bullet \otimes \omega_C[1] \xrightarrow{S_{\mathcal{C}}(\varphi)} C(\varphi) \otimes \omega_C[1]$$

with inverse

$$S_{\mathcal{T}_C}^{-1}(E_1^\bullet \xrightarrow{\varphi} E_2^\bullet) = C(\varphi) \otimes \omega_C^{-1}[-2] \xrightarrow{S_{\mathcal{C}}^{-1}(\varphi)} E_1^\bullet \otimes \omega_C^{-1}[-1].$$

See [47] for the details of the proof.

Remark 2.7.23. Note that if $C = E$ is an elliptic curve, then $S^3 = [4]$. This implies that $\mathcal{T}_C = D^b(\mathrm{TCoh}(E))$ is a fractional Calabi-Yau category of fractional dimension 4/3.

Proposition 2.7.24. *The category* $\mathrm{TCoh}(C)$ *has (finite) homological dimension 2.*

Proof. We have to show that $\mathrm{Hom}_{\mathcal{T}_C}(T, T'[i]) = 0$ for all $T, T' \in \mathrm{TCoh}(C)$ and all $i \in \mathbb{Z}$, $i > 2$. Denote $T = (E_1, E_2, \varphi_1)$ and $T' = (F_1, F_2, \varphi_2)$. Applying the functor $\mathrm{Hom}_{\mathcal{T}_C}(\cdot, T')$ to the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & E_1 \arrow[r, equal] \arrow[d, "\varphi_1"] & E_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & E_2 \arrow[r, equal] & E_2 \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

one obtains a long exact sequence

$$\begin{aligned} \cdots \quad & \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), T'[i]) \to \mathrm{Hom}_{\mathcal{T}_C}(T, T'[i]) \to \mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), T'[i]) \\ & \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), T'[i+1]) \to \cdots \end{aligned} \tag{2.7.4}$$

On the other hand, we also have a short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] \arrow[d] & F_1 \arrow[r, equal] \arrow[d, "\varphi_2"] & F_1 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & F_2 \arrow[r, equal] & F_2 \arrow[r] & 0 \arrow[r] & 0. \end{tikzcd} \tag{2.7.5}$$

Apply the functor $\mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), \cdot)$ to (2.7.5) and obtain a long exact sequence

$$\begin{aligned} \cdots \quad & \to \mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), j_*(F_2)[i]) \to \mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), T'[i]) \\ & \to \mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), i_*(F_1)[i]) \to \mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), j_*(F_2)[i+1]) \to \cdots . \end{aligned}$$

Since $\mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), j_*(F_2)[i]) = 0$ for all $i \geq 2$ and $\mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), i_*(F_1)[i]) = 0$ for all $i \in \mathbb{Z}$, we have

$$\mathrm{Hom}_{\mathcal{T}_C}(j_*(E_2), T'[i]) = 0$$

for all $i \geq 2$. Apply now the functor $\mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), \cdot)$ to (2.7.5) and obtain a long exact sequence

$$\begin{aligned} \cdots \quad & \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), i_*(F_1)[i-1]) \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), j_*(F_2)[i]) \\ & \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), T'[i]) \to \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), i_*(F_1)[i]) \to \cdots . \end{aligned}$$

Note that $\mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), j_*(E_1)[i]) = \mathrm{Hom}_{\mathcal{C}}(E_1, F_2[i-1]) = 0$ for all $i > 2$. Hence,

$$\mathrm{Hom}_{\mathcal{T}_C}(i_*(E_1), T'[i]) = 0$$

for all $i > 2$. Applying it to the long exact sequence (2.7.4) we have that for all $i > 2$

$$\mathrm{Hom}_{\mathcal{T}_C}(T, T'[i]) = \mathrm{Hom}_{\mathcal{T}_C}(i_*(E_2), T'[i]) = 0.$$

Finally, note that for $T, T' \in \mathrm{TCoh}(C)$ with $T = i_*(E_1)$ and $T' = j_*(F_2)$, by Serre duality and Lemma 2.7.22,

$$\mathrm{Hom}_{\mathcal{T}_C}(T, T'[2]) = \mathrm{Hom}_{\mathcal{T}_C}(T'[2], S_{\mathcal{T}_C}(T))^* = \mathrm{Hom}_{\mathcal{T}_C}(j_*(F_2), j_*(E_1) \otimes \omega_C)^*[2]$$

and if we take for example $E_1 = \mathcal{O}_C$ and $E_2 = \omega_C$, then

$$\mathrm{Hom}_{\mathcal{T}_C}(j_*(F_2), j_*(E_1) \otimes \omega_C) \neq 0.$$

$\square$

2.8 Gluing hearts

Now we know how to decompose $\mathcal{T}_C$ into semiorthogonal decompositions and we want to use the precise structure of the stability manifold of $D^b(\mathrm{Coh}(C))$ to construct stability conditions of $\mathcal{T}_C$. The first step towards it is the construction of hearts.

2.8.1 Recollement

The construction of hearts in triangulated categories coming from semiorthogonal decompositions was first introduced in [10].

Definition 2.8.1. Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated categories. $\mathcal{D}$ is said to be a *recollement* of $\mathcal{X}$ and $\mathcal{Y}$ if there are six triangulated functors as in the following diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "j_* = j_!"] & \mathcal{D} \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "i^* = i^!"] & \mathcal{Y} \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"]
\end{tikzcd}$$

such that

1. (i^*, i_*), $(i_!, i^!)$, (j^*, j_*), $(j_!, j^!)$ are adjoint pairs;
2. j_*, i_*, $i_!$ are full embeddings;
3. $j^! \circ i_* = 0$ (and thus also $i^* \circ j_* = 0$ and $j^* \circ i_! = 0$);
4. for every $T \in \mathcal{D}$ there are triangles

$$\begin{tikzcd}
i_! i^! T \arrow[r] & T \arrow[r] & j_* j^* T \arrow[r] & i_! i^! T[1] \; . \\
j_! j^! T \arrow[r] & T \arrow[r] & i_* i^* T \arrow[r] & j_! j^! T[1]
\end{tikzcd}$$

Note that the functors of the definition of recollement satisfy the following properties as a consequence of the vanishing condition (3.).

- ${}^\perp\mathcal{X} = \ker(j^*)$ and $j^* \circ i_! = 0$ implies that $i_!$ embeds the category $\mathcal{Y}$ as ${}^\perp\mathcal{X}$.
- $\mathcal{X}^\perp = \ker(j^!)$ and $j^! \circ i_* = 0$ implies that i_* embeds the category $\mathcal{Y}$ as $\mathcal{X}^\perp$.

Hence, if i_* denotes the natural embedding of $\mathcal{X}$ in $\mathcal{D}$, we have $\mathcal{Y} = \mathcal{X}^\perp$. In fact, the definition of recollement gives the two semiorthogonal decompositions

$$\begin{aligned} \mathcal{D} &= \langle \mathcal{X}, {}^\perp\mathcal{X} \rangle \\ &= \langle \mathcal{X}^\perp, \mathcal{X} \rangle \end{aligned}$$

associated to an admissible full subcategory $\mathcal{X} \subset \mathcal{D}$. This is a well-known fact in the literature but we want show the proof to make it more transparent.

Proposition 2.8.2. *Let $\mathcal{D}$ be a triangulated category and let $\mathcal{X} \subset \mathcal{D}$ be a full triangulated subcategory. Then, $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{X}^\perp$ if and only if $\mathcal{X}$ is (left and right) admissible.*

Proof. If $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{X}^\perp$, then $\mathcal{X}$ is admissible trivially (just by the existence of the adjoint pairs of functors (j^*, j_*) and $(j_*, j^!)$).

Conversely, let us assume that $\mathcal{X}$ is admissible and check the conditions of recollement. We have six triangulated functors as in the following diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "j_* = j_!"] & \mathcal{D} \arrow[l, bend right=60, "j^*"'] \arrow[l, bend left=60, "j^!"] \arrow[r, "i^* = i^!"] & \mathcal{X}^\perp \arrow[l, bend right=60, "i_!"'] \arrow[l, bend left=60, "i_*"]
\end{tikzcd}$$

where the j's are the functors given by the admissibility of $\mathcal{X}$, i_* denotes the natural embedding of $\mathcal{X}^\perp$ in $\mathcal{D}$, i^* is its left-adjoint and we can define $i_!$ to be the composition $i'_* \circ G$ where i'_* denotes denotes the natural embedding of ${}^\perp\mathcal{X}$ in $\mathcal{D}$ and $G \colon \mathcal{X} \perp\!\to {}^\perp\mathcal{X}$ is the functor constructed in Lemma 2.7.5 which gave the equivalence between $\mathcal{X}^\perp$ and ${}^\perp\mathcal{X}$. Left adjointness $(i_!, i^*)$ is then direct. Indeed, let $X \in \mathcal{X}^\perp$ and $T \in \mathcal{D}$. Applying $\mathrm{Hom}_{\mathcal{D}}(i_!(X), \cdot)$ to

$$j_* j^! T \longrightarrow T \longrightarrow i_* i^* T \longrightarrow j_! j^! T[1]$$

we find that

$$\mathrm{Hom}_{\mathcal{D}}(i_!(X), T) \cong \mathrm{Hom}_{\mathcal{D}}(i_!(X), i_* i^* T).$$

On the other hand, we apply $\mathrm{Hom}_{\mathcal{D}}(\cdot, i_* i^* T)$ to

$$i'_* G(X) \longrightarrow i_* X \longrightarrow X' \longrightarrow i'_* G(X)[1]$$

which it is the triangle that we used in Lemma 2.7.5 to define the functor G. Then, it follows that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{D}}(i_!(X), i_* i^* T) &\cong \mathrm{Hom}_{\mathcal{D}}(i_*(X), i_* i^* T) \\
&\cong \mathrm{Hom}_{\mathcal{X}^\perp}(X, i^* T)
\end{aligned}$$

which concludes the proof of the left adjointness $(i_!, i^*)$.

All these functors satisfy the conditions of the definition of recollement. Indeed, conditions 1, 2 and 3 are straightforward. For condition 4 simply notice that for every $T \in \mathcal{D}$ there are triangles

$$i_! i^! T \longrightarrow T \longrightarrow j_* j^* T \longrightarrow i_! i^! T[1]$$

corresponding to $\langle \mathcal{X}, {}^\perp\mathcal{X} \rangle$ and

$$j_! j^! T \longrightarrow T \longrightarrow i_* i^* T \longrightarrow j_! j^! T[1]$$

corresponding to $\langle \mathcal{X}^\perp, \mathcal{X} \rangle$. $\square$

The following theorem shows how to construct t-structures from t-structures in the smaller subcategories.

Theorem 2.8.3 ([10, Theorem 1.4.10])**.** *Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated categories such that $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{Y}$ and assume the notation of Definition 2.8.1. Let $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ and $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ be t-structures in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then there is a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ in $\mathcal{D}$ defined by:*

$$\begin{aligned}\mathcal{D}^{\leq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\leq 0}, j^*T \in \mathcal{X}^{\leq 0}\}\\ \mathcal{D}^{\geq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\geq 0}, j^!T \in \mathcal{X}^{\geq 0}\}.\end{aligned}$$

If we write $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}_\mathcal{Y}$ for the corresponding hearts in $\mathcal{X}$ and $\mathcal{Y}$ respectively, we denote by $\operatorname{rec}(\mathcal{A}_\mathcal{Y}, \mathcal{A}_\mathcal{X}) := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$.

Definition 2.8.4. Given triangulated categories $\mathcal{D}$ and $\mathcal{D}'$ endowed with t-structures $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ and $(\mathcal{D}'^{\leq 0}, \mathcal{D}'^{\geq 0})$, a functor $F\colon \mathcal{D} \to \mathcal{D}'$ is called *right (resp. left) t-exact* if $F(\mathcal{D}^{\leq 0}) \subset \mathcal{D}'^{\leq 0}$ (resp. $F(\mathcal{D}^{\geq 0}) \subset \mathcal{D}'^{\geq 0}$). We say that F is *t-exact* if it is left and right t-exact.

Corollary 2.8.5. *In the situation of Theorem 2.8.3, we have the following.*

- *The functors $i_!$ and j^* are right t-exact.*
- *The functors i^* and j_* are t-exact.*
- *The functors i_* and $j^!$ are left t-exact.*

Since we know already from Proposition 2.8.2 that all three subcategories $\mathcal{C}_i$ of $\mathcal{T}_C$ are admissible, the next theorem shows the explicit structure of $\mathcal{T}_C$ as recollement.

Theorem 2.8.6. *The triangulated category $\mathcal{T}_C = D^b(\mathrm{TCoh}(C))$ is a recollement of $\mathcal{C}$ and $\mathcal{C}$ in three different ways, which we will refer as*

1. *$\mathcal{C}_2$ and $\mathcal{C}_1$, with diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "j_*"] & \mathcal{T}_C \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j^!"] \arrow[r, "i^*"] & \mathcal{C} \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \end{tikzcd}$$

with the following morphisms:

$$\begin{aligned} &j_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_2 \longmapsto 0 \to E_2; && i^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_1;\\ &j^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto C(\varphi); && i_!\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_1 \longmapsto E_1 \xrightarrow{\mathrm{id}} E_1;\\ &j^!\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_2; && i_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_1 \longmapsto E_1 \to 0. \end{aligned}$$

2. $\mathcal{C}_1$ and $\mathcal{C}_3$, with diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "i_*"] & \mathcal{T}_C \arrow[l, bend right=50, "i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "l^*"] & \mathcal{C} \arrow[l, bend right=50, "l_!"'] \arrow[l, bend left=50, "l_*"] \end{tikzcd}$$

with the following morphisms:

$$\begin{aligned} &i_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_1 \longmapsto E_1 \to 0; && l^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_2;\\ &i^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_1; && l_!\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E \longmapsto 0 \to E;\\ &i^!\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto C(\varphi)[-1]; && l_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E \longmapsto E \xrightarrow{\mathrm{id}} E. \end{aligned}$$

3. $\mathcal{C}_3$ and $\mathcal{C}_2$, with diagram diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "l_*"] & \mathcal{T}_C \arrow[l, bend right=50, "l^*"'] \arrow[l, bend left=50, "l^!"] \arrow[r, "j^*"] & \mathcal{C} \arrow[l, bend right=50, "j_!"'] \arrow[l, bend left=50, "j_*"] \end{tikzcd}$$

with the following morphisms:

$$\begin{aligned} &l_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E \longmapsto E \xrightarrow{\mathrm{id}} E; && j^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto C(\varphi);\\ &l^*\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_2; && j_!\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_2 \longmapsto E_2[-1] \to 0;\\ &l^!\colon \mathcal{T}_C \longrightarrow \mathcal{C},\ E_1 \xrightarrow{\varphi} E_2 \longmapsto E_1; && j_*\colon \mathcal{C} \longrightarrow \mathcal{T}_C,\ E_2 \longmapsto 0 \to E_2. \end{aligned}$$

Remark 2.8.7. We want to point out the following adjunction relations between the previous functors:

$$j_! \dashv j^* \dashv j_* \dashv j^!, \quad l_! \dashv l^* \dashv l_* \dashv l^! \quad \text{and} \quad i_! \dashv i^* \dashv i_* \dashv i^!.$$

Moreover, we keep the notations with different characters according to the recollement one is working on but the special structure of our decompositions makes that some of them actually agree. To make it more clear, we have the following table where rows i) and ii) contain our functors and row iii) contains the "classical"notations for these functors. Rows from left to right represent left adjointness and columns represent the same functors (up to isomorphism):

i)	$j_!$	j^*	j_*	$j^!$	l_*	$l^!$	i_*	$i^!$
ii)	$i_*[-1]$	$i^![1]$	$l_!$	l^*	$i_!$	i^*	$j_![1]$	$j^*[-1]$
iii)	$i_1[-1]$	$K[1]$	i_2	ρ_2	Δ	λ_1	i_1	K

For $ij \in \{12, 31, 23\}$, We will denote by $\mathrm{rec}_{ij}(\mathcal{A}_i, \mathcal{A}_j)$ the heart obtained by applying Theorem 2.8.3 to the hearts $\mathcal{A}_i \subset \mathcal{C}_i$ and $\mathcal{A}_j \subset \mathcal{C}_j$.

2.8.2 CP-Gluing

Collins and Polishuck introduced in [25] (and Collins in [24]) a way to construct hearts from semiorthogonal decompositions which later allows to define stability conditions on $\mathcal{D}$ in a natural way.

Let $\mathcal{D}$ be a triangulated category equipped with a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. As before, let $\rho_2 \colon \mathcal{D} \to \mathcal{D}_2$ be the right adjoint functor to the full embedding $i_2 \colon \mathcal{D}_2 \to \mathcal{D}$ and let $\lambda_1 \colon \mathcal{D} \to \mathcal{D}_1$ be the left adjoint functor to the full embedding $i_1 \colon \mathcal{D}_1 \to \mathcal{D}$.

Proposition 2.8.8 ([25, Lemma 2.1])**.** *With the above notations, assume that we have t-structures $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 0})$ with hearts $\mathcal{A}_i$ on $\mathcal{D}_i$, for $i = 1, 2$, such that*

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(i_1\mathcal{A}_1, i_2\mathcal{A}_2) = 0. \tag{2.8.1}$$

Then there is a t-structure on $\mathcal{D}$ with the heart

$$\mathrm{gl}(i_1\mathcal{A}_1, i_2\mathcal{A}_2) = \{E \in \mathcal{D} \mid \rho_2 E \in \mathcal{A}_2, \lambda_1 E \in \mathcal{A}_1\}. \tag{2.8.2}$$

With respect to this t-structure on $\mathcal{D}$ the functors λ_1 and ρ_2 are t-exact.

Definition 2.8.9. We will refer to hearts of the form (2.8.2) as hearts obtained by *CP-gluing.*

Remark 2.8.10. T-exactness of the functors λ_1 and ρ_2 implies that

$$i_k\mathcal{A}_k \subset \mathcal{A} \coloneqq \mathrm{gl}(i_1\mathcal{A}_1, i_2\mathcal{A}_2)$$

for $k = 1, 2$ and this gives automatically that $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(i_1\mathcal{A}_1, i_2\mathcal{A}_2) = 0$, because of the definition of heart of a bounded t-structure (see Lemma 2.2.5 i)).

Lemma 2.8.11. *Let $\mathcal{D}$ be a triangulated category equipped with a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Let $\mathcal{A}_i$ be a heart of a bounded t-structure $(\mathcal{D}_i^{\leq 0}, \mathcal{D}_i^{\geq 1})$ of $\mathcal{D}_i$ for $i = 1, 2$. Then, they satisfy* (2.8.1) *if and only if* $\mathrm{Hom}_{\mathcal{D}}(i_1\mathcal{D}_1^{\leq 0}, i_2\mathcal{D}_2^{\geq 0}) = 0$.

Proof. First, if $\mathrm{Hom}_{\mathcal{D}}(i_1\mathcal{D}_1^{\leq 0}, i_2\mathcal{D}_2^{\geq 0}) = 0$, it implies $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(i_1\mathcal{A}_1, i_2\mathcal{A}_2) = 0$ since $\mathcal{A}_1 \subset \mathcal{D}_1^{\leq 0}$ and

$$\mathcal{A}_2[k] \subset \mathcal{D}_2^{\geq 0}[k] \subset \mathcal{D}_2^{\geq 0}$$

for every $k \leq 0$.

Conversely, assume $\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(i_1\mathcal{A}_1, i_2\mathcal{A}_2) = 0$. Recall that $\mathcal{D}_1^{\leq 0}$ (resp. $\mathcal{D}_2^{\geq 0}$) is the extension-closed subcategory generated by the subcategories $\mathcal{A}_1[k_1]$ (resp. $\mathcal{A}_2[k_2]$) for integers $k_1 \geq 0$ (resp. $k_2 \leq 0$). Then, for every $k_1 \geq 0$ and every $k_2 \leq 0$, we have

$$\mathrm{Hom}_{\mathcal{D}}(i_1\mathcal{A}_1[k_1], i_2\mathcal{A}_2[k_2]) = \mathrm{Hom}_{\mathcal{D}}(i_1\mathcal{A}_1, i_2\mathcal{A}_2[k_2 - k_1])$$

which is zero by assumption. Thus, $\mathrm{Hom}_{\mathcal{D}}(i_1\mathcal{D}_1^{\leq 0}, i_2\mathcal{D}_2^{\geq 0}) = 0$. □

Now we want to point out the relation between hearts obtained by CP-gluing and the ones obtained by recollement. Although the following result seems very natural from the constructions in [25] and [24], we show a complete proof for the sake of completeness.

Proposition 2.8.12. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{D}$ be triangulated categories such that $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{Y}$ and assume the notation of Definition 2.8.1. Let $(\mathcal{X}^{\leq 0}, \mathcal{X}^{\geq 0})$ and $(\mathcal{Y}^{\leq 0}, \mathcal{Y}^{\geq 0})$ be t-structures with hearts $\mathcal{A}_{\mathcal{X}}$ and $\mathcal{A}_{\mathcal{Y}}$ in $\mathcal{X}$ and $\mathcal{Y}$ respectively. Then we have two semiorthogonal decompositions with notation as in Proposition 2.8.2. Then,*

1. *If* $\operatorname{Hom}_{\mathcal{D}}^{\leq 0}(i_*\mathcal{A}_{\mathcal{Y}}, j_*\mathcal{A}_{\mathcal{X}}) = 0$*, then* $\operatorname{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}}) = \operatorname{gl}(i_*\mathcal{A}_{\mathcal{Y}}, j_*\mathcal{A}_{\mathcal{X}})$.
2. *If* $\operatorname{Hom}_{\mathcal{D}}^{\leq 0}(j_*\mathcal{A}_{\mathcal{X}}, i_!\mathcal{A}_{\mathcal{Y}}) = 0$*, then* $\operatorname{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}}) = \operatorname{gl}(j_*\mathcal{A}_{\mathcal{X}}, i_!\mathcal{A}_{\mathcal{Y}})$.

Proof. 1. Assume that $\operatorname{Hom}_{\mathcal{D}}^{\leq 0}(i_*\mathcal{A}_{\mathcal{Y}}, j_*\mathcal{A}_{\mathcal{X}}) = 0$. The heart $\operatorname{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}})$ given by recollement of $\mathcal{X}$ and $\mathcal{Y}$, is defined as in Theorem 2.8.3 by the following t-structure:

$$\begin{aligned} \mathcal{D}^{\leq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\leq 0}, j^*T \in \mathcal{X}^{\leq 0}\} \\ \mathcal{D}^{\geq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\geq 0}, j^!T \in \mathcal{X}^{\geq 0}\}. \end{aligned} \tag{2.8.3}$$

On the other hand, the heart $\operatorname{gl}(i_*\mathcal{A}_{\mathcal{Y}}, j_*\mathcal{A}_{\mathcal{X}})$ given by the semiorthogonal decomposition of $\mathcal{D}$ by $\mathcal{X}$ and $\mathcal{Y}$, is defined as in (2.8.2) by the following t-structure:

$$\begin{aligned} \mathcal{D}'^{\leq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\leq 0}, j^!T \in \mathcal{X}^{\leq 0}\} \\ \mathcal{D}'^{\geq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\geq 0}, j^!T \in \mathcal{X}^{\geq 0}\}. \end{aligned}$$

Hence, we need to see that $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{D}'^{\leq 0} \cap \mathcal{D}'^{\geq 0}$. In particular, since $\mathcal{D}^{\geq 0} = \mathcal{D}'^{\geq 0}$, we are going to see that $j^!T \in \mathcal{X}^{\leq 0}$ if and only if $j^*T \in \mathcal{X}^{\leq 0}$. For that, we take $D \in \mathcal{X}^{\geq 1}$ arbitrary and we want to see that $\operatorname{Hom}_{\mathcal{D}}(j_*j^*T, j_*D) = 0$ if and only if $\operatorname{Hom}_{\mathcal{D}}(j_*j^!T, j_*D) = 0$.

Indeed, let us consider the two long exact sequences induced by both semiorthogonal decompositions from the recollement:

- For $\mathcal{D} = \langle \mathcal{X}, {}^{\perp}\mathcal{X} \rangle$, apply $\operatorname{Hom}_{\mathcal{D}}(\cdot, j_*D)$ to

  $$i_!i^*T \longrightarrow T \longrightarrow j_*j^*T \longrightarrow i_!i^!T[1]$$

  and obtain for $k \in \mathbb{Z}$,

  $$\begin{aligned} \cdots &\to \operatorname{Hom}_{\mathcal{D}}^k(j_*j^*T, j_*D) \to \operatorname{Hom}_{\mathcal{D}}^k(T, j_*D) \to \operatorname{Hom}_{\mathcal{D}}^k(i_!i^*T, j_*D) \\ &\to \operatorname{Hom}_{\mathcal{D}}^{k+1}(j_*j^*T, j_*D) \to \cdots \end{aligned} \tag{2.8.4}$$

- For $\mathcal{D} = \langle \mathcal{Y}, \mathcal{X} \rangle$, apply $\operatorname{Hom}_{\mathcal{D}}(\cdot, j_*D)$ to

  $$j_*j^!T \longrightarrow T \longrightarrow i_*i^*T \longrightarrow j_!j^!T[1]$$

  and obtain for $k \in \mathbb{Z}$,

  $$\begin{aligned} \cdots &\to \operatorname{Hom}_{\mathcal{D}}^k(i_*i^*T, j_*D) \to \operatorname{Hom}_{\mathcal{D}}^k(T, j_*D) \to \operatorname{Hom}_{\mathcal{D}}^k(j_*j^!T, j_*D) \\ &\to \operatorname{Hom}_{\mathcal{D}}^{k+1}(i_*i^*T, j_*D) \to \cdots \end{aligned} \tag{2.8.5}$$

Note that $\mathrm{Hom}^k_{\mathcal{D}}(i_! i^* T, j_* D) = 0$ for every k, since $D \in \mathcal{X}$ and $i_!$ maps $i^*T \in \mathcal{Y}$ as an element from ${}^{\perp}\mathcal{X}$. Using this in (2.8.4), implies that

$$\mathrm{Hom}^k_{\mathcal{D}}(j_* j^* T, j_* D) \cong \mathrm{Hom}^k_{\mathcal{D}}(T, j_* D)$$

for all k. On the other hand, we know that for any $T \in \mathrm{gl}(i_*\mathcal{A}_{\mathcal{Y}}, j_*\mathcal{A}_{\mathcal{X}})$ (and any $T \in \mathrm{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}})$), $i^*T \in \mathcal{A}_{\mathcal{Y}} = \mathcal{Y}^{\leq 0} \cap \mathcal{Y}^{\geq 0} \subset \mathcal{Y}^{\leq 0}$ and by Lemma 2.8.11, we have $\mathrm{Hom}^k_{\mathcal{D}}(i_* i^* T, j_* D) = 0$ for all $k \leq 1$. Using this in (2.8.5) implies that

$$\mathrm{Hom}^k_{\mathcal{D}}(T, j_* D) \cong \mathrm{Hom}^k_{\mathcal{D}}(j_* j^! T, j_* D)$$

for all $k \leq 0$. In particular,

$$\mathrm{Hom}_{\mathcal{D}}(j_* j^* T, j_* D) \cong \mathrm{Hom}_{\mathcal{D}}(j_* j^! T, j_* D).$$

Therefore, $j^*T \in \mathcal{X}^{\leq 0}$ if and only if $j^!T \in \mathcal{X}^{\leq 0}$.

2. Assume that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(j_*\mathcal{A}_{\mathcal{X}}, i_!\mathcal{A}_{\mathcal{Y}}) = 0$. The heart $\mathrm{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}})$ is the same as before (2.8.3).

On the other hand, the heart $\mathrm{gl}(j_*\mathcal{A}_{\mathcal{X}}, i_!\mathcal{A}_{\mathcal{Y}})$ given by the semiorthogonal decomposition of $\mathcal{D}$ by ${}^{\perp}\mathcal{X}$ and $\mathcal{X}$, is defined as in (2.8.2) by the following t-structure:

$$\begin{aligned} \mathcal{D}''^{\leq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\leq 0}, j^*T \in \mathcal{X}^{\leq 0}\} \\ \mathcal{D}''^{\geq 0} &:= \{T \in \mathcal{D} \mid i^*T \in \mathcal{Y}^{\geq 0}, j^*T \in \mathcal{X}^{\geq 0}\}. \end{aligned}$$

Hence, we need to see that $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0} = \mathcal{D}''^{\leq 0} \cap \mathcal{D}''^{\geq 0}$. Since $\mathcal{D}^{\leq 0} = \mathcal{D}''^{\leq 0}$, we are going to see that $j^!T \in \mathcal{X}^{\geq 0}$ if and only if $j^*T \in \mathcal{X}^{\geq 0}$. For that, we take $D \in \mathcal{X}^{\leq 0}$ arbitrary and we want to see that $\mathrm{Hom}_{\mathcal{D}}(j_*D, j_*j^*T[-1]) = 0$ if and only if $\mathrm{Hom}_{\mathcal{D}}(j_*D, j_*j^!T[-1]) = 0$.

Indeed, let us consider the two long exact sequences induced by both semiorthogonal decompositions from the recollement:

- For $\mathcal{D} = \langle \mathcal{X}, {}^{\perp}\mathcal{X} \rangle$, apply $\mathrm{Hom}_{\mathcal{D}}(j_*D, \cdot)$ to

$$i_! i^* T \longrightarrow T \longrightarrow j_* j^* T \longrightarrow i_! i^! T[1]$$

  and obtain for $k \in \mathbb{Z}$,

$$\begin{aligned} \cdots &\to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, i_! i^* T) \to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, T) \to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, j_*j^*T) \\ &\to \mathrm{Hom}^{k+1}_{\mathcal{D}}(j_*D, i_! i^* T) \to \cdots \end{aligned} \tag{2.8.6}$$

- For $\mathcal{D} = \langle \mathcal{Y}, \mathcal{X} \rangle$, apply $\mathrm{Hom}_{\mathcal{D}}(j_*D, \cdot)$ to

$$j_* j^! T \longrightarrow T \longrightarrow i_* i^* T \longrightarrow j_! j^! T[1]$$

  and obtain for $k \in \mathbb{Z}$,

$$\begin{aligned} \cdots &\to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, j_*j^!T) \to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, T) \to \mathrm{Hom}^k_{\mathcal{D}}(j_*D, i_*i^*T) \\ &\to \mathrm{Hom}^{k+1}_{\mathcal{D}}(j_*D, j_*j^!T) \to \cdots \end{aligned} \tag{2.8.7}$$

Note that $\mathrm{Hom}^k_{\mathcal{D}}(j_*D, i_*i^*T) = 0$ for every k, since $D \in \mathcal{X}$ and $i^*T \in \mathcal{Y} = \mathcal{X}^\perp$. Using this in (2.8.7), implies that

$$\mathrm{Hom}^k_{\mathcal{D}}(j_*D, j_*j^!T) \cong \mathrm{Hom}^k_{\mathcal{D}}(j_*D, T)$$

for all k. On the other hand, we know that for any $T \in \mathrm{gl}(j_*\mathcal{A}_{\mathcal{X}}, i_!\mathcal{A}_{\mathcal{Y}})$ (and any $T \in \mathrm{rec}(\mathcal{A}_{\mathcal{Y}}, \mathcal{A}_{\mathcal{X}})$), $i^*T \in \mathcal{A}_{\mathcal{Y}} = \mathcal{Y}^{\leq 0} \cap \mathcal{Y}^{\geq 0} \subset \mathcal{Y}^{\geq 0}$ and by Lemma 2.8.11, we have $\mathrm{Hom}^k_{\mathcal{D}}(j_*D, i_!i^*T) = 0$ for all $k \leq 0$. Using this in (2.8.7) implies that

$$\mathrm{Hom}^k_{\mathcal{D}}(j_*D, T) \cong \mathrm{Hom}^k_{\mathcal{D}}(j_*D, j_*j^*T)$$

for all $k \leq -1$. In particular,

$$\mathrm{Hom}^{-1}_{\mathcal{D}}(j_*D, j_*j^!T) \cong \mathrm{Hom}^{-1}_{\mathcal{D}}(j_*D, j_*j^*T).$$

Therefore, $j^*T \in \mathcal{X}^{\geq 0}$ if and only if $j^!T \in \mathcal{X}^{\geq 0}$. □

We have seen in Proposition 2.7.6 that we have three semiorthogonal decompositions of $\mathcal{T}_C = D^b(\mathrm{TCoh}(C))$ and we use the following notation:

- Case $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, recall the functors defined in Theorem 2.8.6 1. $\lambda_1 = i^*$ and $\rho_2 = j^!$. A heart in $\mathcal{T}_C$ glued from $\mathcal{A}_k$ in $\mathcal{C}_k$, for $k = 1, 2$, will be denoted by $\mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$.
- Case $\mathcal{T}_C = \langle \mathcal{C}_3, \mathcal{C}_1 \rangle$, recall the functors defined in Theorem 2.8.6 2. $\lambda_1 = l^*$ and $\rho_2 = i^!$. A heart in $\mathcal{T}_C$ glued from $\mathcal{A}_k$ in $\mathcal{C}_k$, for $k = 3, 1$, will be denoted by $\mathrm{gl}_{31}(\mathcal{A}_3, \mathcal{A}_1)$.
- Case $\mathcal{T}_C = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, recall the functors defined in Theorem 2.8.6 3. $\lambda_1 = j^*$ and $\rho_2 = l^!$. A heart in $\mathcal{T}_C$ glued from $\mathcal{A}_k$ in $\mathcal{C}_k$, for $k = 2, 3$, will be denoted by $\mathrm{gl}_{23}(\mathcal{A}_2, \mathcal{A}_3)$.

Recall that each triangulated subcategory $\mathcal{C}_i$ is equivalent to $D^b(\mathrm{Coh}(C))$ and in the latter, by Theorem 2.5.3 we know that all hearts giving stability conditions are of the form $\mathrm{Coh}^\theta[n]$ for some $n \in \mathbb{Z}$ and $\theta \in [0, 1)$.

Now, given hearts in the smaller categories, we want to check first the gluing condition (2.8.1) for the hearts.

Proposition 2.8.13. *We distinguish 3 cases according to the semiorthogonal decomposition of $\mathcal{T}_C$.*

1. *Case $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, we have*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_1^{\theta_1}[n_1], \mathrm{Coh}_2^{\theta_2}[n_2]) = 0$$

if and only if $n_1 + \theta_1 \geq n_2 + \theta_2$.

2. *Case $\mathcal{T}_C = \langle \mathcal{C}_3, \mathcal{C}_1 \rangle$, we have*

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3}[n_3], \mathrm{Coh}_1^{\theta_1}[n_1]) = 0$$

if and only if $n_3 + \theta_3 \geq n_1 + \theta_1 + 1$.

3. Case $\mathcal{T}_C = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, we have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_2^{\theta_2}[n_2], \mathrm{Coh}_3^{\theta_3}[n_3]) = 0$$

if and only if $n_2 + \theta_2 \geq n_3 + \theta_3 + 1$.

Proof. First, we want to see under which conditions

$$\mathrm{Hom}_{\mathcal{T}_C}(\mathrm{Coh}_1^{\theta_1}[n_1], \mathrm{Coh}_2^{\theta_2}[n_2 + i]) = 0$$

for every $i \leq 0$. Assume $n_2 = 0$, up to shifting by $-n_2$. For a fixed i, take $0 \to E_2 \in \mathrm{Coh}_2^{\theta_2}$ and $E_1 \to 0 \in \mathrm{Coh}_1^{\theta_1}$. By Serre duality we have

$$\mathrm{Hom}_{\mathcal{T}_C}(E_1[n_1] \to 0, 0 \to E_2[i]) = \mathrm{Hom}_{\mathcal{T}_C}(0 \to E_2, S_{\mathcal{T}_C}(E_1 \to 0)[n_1 - i])^*. \quad (2.8.8)$$

By Lemma 2.7.22, $S_{\mathcal{T}_C}(E_1 \to 0)[n_1] \in j_*(S_{\mathcal{C}}(\mathrm{Coh}^{\theta_1}[n_1]))[1]$, so (2.8.8) vanishes for all $i \leq 0$ if and only if $n_1 \geq 0$ and if $n_1 = 0$, we see that we need that $\theta_1 \geq \theta_2$. Indeed, if $n_1 = 0$, remember that each heart $\mathrm{Coh}_i^{\theta_i}$ was defined by tilting $\mathrm{Coh}_i^{\theta_i} = \langle F_i^{\theta_i}[1], T_i^{\theta_i} \rangle$, for $i = 1, 2$. By the previous argument with the Serre functor, the only restriction appears when $E_1 \to 0 \in T_1^{\theta_1}$ and $0 \to E_2[1] \in F_2^{\theta_2}[1]$. Here, if $\theta_1 \geq \theta_2$, we have $T_1^{\theta_1} \subset T_2^{\theta_2}$. Use Serre duality for $\mathcal{C}$, so that

$$\mathrm{Hom}_{\mathcal{C}}(E_2[1], E_1 \otimes \omega_C[2 - i]) = \mathrm{Hom}_{\mathcal{C}}(E_1[1 - i], E_2[1])^*$$

and it vanishes for all $i \leq 0$, since $(T_2^{\theta_2}, F_2^{\theta_2})$ is a torsion pair.

2. We want to see under which conditions

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3}[n_3], \mathrm{Coh}_1^{\theta_1}[n_1]) = 0.$$

Note that if we apply the Serre functor, by Lemma 2.7.22, we get that

$$\begin{aligned}\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3}[n_3], \mathrm{Coh}_1^{\theta_1}[n_1]) &= \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(S_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3})[n_3], S_{\mathcal{T}_C}(\mathrm{Coh}_1^{\theta_1})[n_1]) \\ &= \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(i_*(S_{\mathcal{C}}(\mathrm{Coh}^{\theta_3}[n_3])), j_*(S_{\mathcal{C}}(\mathrm{Coh}^{\theta_1}[n_1 + 1]))).\end{aligned}$$

Then, after applying the autoequivalence $\cdot \otimes \omega_C^*$ we can use the previous condition to conclude $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3}[n_3], \mathrm{Coh}_1^{\theta_1}[n_1]) = 0$ if and only if $n_3 + \theta_3 \geq n_1 + \theta_1 + 1$.

3. We want to see under which conditions

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_2^{\theta_2}[n_2], \mathrm{Coh}_3^{\theta_3}[n_3]) = 0.$$

Note that if we apply the inverse of the Serre functor, after Lemma 2.7.22, we get that

$$\begin{aligned}\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}_2^{\theta_2}[n_2], \mathrm{Coh}_3^{\theta_3}[n_3]) &= \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(S^{-1}_{\mathcal{T}_C}(\mathrm{Coh}_2^{\theta_2})[n_2], S^{-1}_{\mathcal{T}_C}(\mathrm{Coh}_3^{\theta_3})[n_3]) \\ &= \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(i_*(S^{-1}_{\mathcal{C}}(\mathrm{Coh}^{\theta_2}[n_2 - 1])), j_*(S^{-1}_{\mathcal{C}}(\mathrm{Coh}^{\theta_3}[n_3]))).\end{aligned}$$

Then, after applying the autoequivalence $\cdot \otimes \omega_C^*$ we can use the previous condition to conclude $\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(\mathrm{Coh}^{\theta_2}[n_2], \mathrm{Coh}^{\theta_3}[n_3]) = 0$ if and only if $n_2 + \theta_2 \geq n_3 + \theta_3 + 1$. $\square$

Therefore, by Proposition 2.8.12 if any of the CP-gluing conditions of Proposition 2.8.13 hold, then both hearts agree, i.e.

$$\mathrm{rec}_{ij}(\mathcal{A}_i, \mathcal{A}_j) = \mathrm{gl}_{ij}(\mathcal{A}_i, \mathcal{A}_j)$$

for $ij \in \{12, 31, 23\}$. By definition of CP-gluing, all hearts constructed by CP-gluing contain the original hearts, i.e.

$$\mathcal{A}_i, \mathcal{A}_j \subset \mathrm{gl}_{ij}(\mathcal{A}_i, \mathcal{A}_j)$$

for $ij \in \{12, 31, 23\}$. The next proposition shows under which conditions there is even a third heart inside the glued heart.

Proposition 2.8.14. *We distinguish 3 cases according to the semiorthogonal decomposition of $\mathcal{T}_C$.*

1. *Case $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, we have*

$$\mathrm{Coh}_3^{\theta_1}[n_1] \subset \mathrm{gl}_{12}(\mathrm{Coh}_1^{\theta_1}[n_1], \mathrm{Coh}_2^{\theta_2}[n_2])$$

if and only if $n_1 + \theta_1 = n_2 + \theta_2$.

2. *Case $\mathcal{T}_C = \langle \mathcal{C}_3, \mathcal{C}_1 \rangle$, we have*

$$\mathrm{Coh}_2^{\theta_3}[n_3] \subset \mathrm{gl}_{31}(\mathrm{Coh}_3^{\theta_3}[n_3], \mathrm{Coh}_1^{\theta_1}[n_1])$$

if and only if $n_3 + \theta_3 = n_1 + \theta_1 + 1$.

3. *Case $\mathcal{T}_C = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$, we have*

$$\mathrm{Coh}_1^{\theta_2}[n_2 - 1] \subset \mathrm{gl}_{23}(\mathrm{Coh}_2^{\theta_2}[n_2], \mathrm{Coh}_3^{\theta_3}[n_3])$$

if and only if $n_2 + \theta_2 = n_3 + \theta_3 + 1$.

Proof. 1. Put $\mathcal{A}_i := \mathrm{Coh}_i^{\theta_i}[n_i]$ for $i = 1, 2$. We will see that $i_! \mathcal{A}_1 \subset \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2)$ if and only if $\mathcal{A}_1 = \mathcal{A}_2$. Indeed, let us take an object $E \in \mathcal{A}_1$. Since $i^* i_! E = E$, by definition of glued heart, $i_! E \in \mathrm{gl}(\mathcal{A}_1, \mathcal{A}_2)$ if and only if $j^! i_! E \in \mathcal{A}_2$. We conclude by observing that $j^! i_! E = l^* l_* E = E$.

2. Put $\mathcal{A}_i := \mathrm{Coh}_i^{\theta_i}[n_i]$ for $i = 3, 1$. We will see that $l_! \mathcal{A}_3 \subset \mathrm{gl}_{31}(l_* \mathcal{A}_3, i_* \mathcal{A}_1)$ if and only if $\mathcal{A}_1 = \mathcal{A}_3[-1]$. Indeed, let us take an object $E \in \mathcal{A}_3$. Since $l^* l_! E = E$, by definition of glued heart, $l_! E \in \mathrm{gl}_{31}(l_* \mathcal{A}_3, i_* \mathcal{A}_1)$ if and only if $i^! l_! E \in \mathcal{A}_1$. We conclude by observing that $i^! l_! E = E[-1]$.

3. Put $\mathcal{A}_i := \mathrm{Coh}_i^{\theta_i}[n_i]$ for $i = 2, 3$. We will see that $j_! \mathcal{A}_2 \subset \mathrm{gl}_{23}(\mathcal{A}_2, \mathcal{A}_3)$ if and only if $\mathcal{A}_3 = \mathcal{A}_2[-1]$. Indeed, let us take an object $E \in \mathcal{A}_2$. Since $j^* j_! E = E$, by definition of glued heart, $j_! E \in \mathrm{gl}_{23}(\mathcal{A}_2, \mathcal{A}_3)$ if and only if $l^! j_! E \in \mathcal{A}_3$. We conclude by observing that $l^! j_! E = i^* i_*[-1] E = E[-1]$. □

Remark 2.8.15. In general, for an arbitrary triangulated category $\mathcal{D}$, consider the notion of recollement as in Definition 2.8.1. If we have hearts $\mathcal{A}_\mathcal{X}$ and $\mathcal{A}_\mathcal{Y}$ satisfying CP-gluing condition 2.8.1, to ask whether $i_! \mathcal{A}_\mathcal{Y} \subset \mathrm{rec}(\mathcal{A}_\mathcal{Y}, \mathcal{A}_\mathcal{X})$ only makes sense when all three triangulated subcategories are equivalent, i.e. $\mathcal{X} \cong {}^\perp\mathcal{X} \cong \mathcal{X}^\perp$.

The next proposition reveals the numerical conditions to construct a heart $\mathcal{A}$ in $\mathcal{T}_C$ by recollement of hearts $\mathcal{A}_i$ in $\mathcal{C}_i$ that agree with the CP-gluing condition.

Proposition 2.8.16. *We distinguish three cases according to the type of recollement:*

1. *Case 12. Given hearts $\mathcal{A}_1 \cong \mathrm{Coh}^{\theta_1}[n_1]$ and $\mathcal{A}_2 \cong \mathrm{Coh}^{\theta_2}[n_2]$ in $\mathcal{C}_1$ and in $\mathcal{C}_2$ respectively, then*
$$\mathrm{rec}_{12}(\mathcal{A}_1, \mathcal{A}_2) = \begin{cases} \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2), & \text{if } n_1 + \theta_1 \geq n_2 + \theta_2 \\ \mathrm{gl}_{23}(\mathcal{A}_2, \mathrm{Coh}_3^{\theta_1}[n_1]), & \text{if } n_1 + \theta_1 \leq n_2 + \theta_2 - 1. \end{cases}$$

2. *Case 31. Given hearts $\mathcal{A}_3 \cong \mathrm{Coh}^{\theta_3}[n_3]$ and $\mathcal{A}_1 \cong \mathrm{Coh}^{\theta_1}[n_1]$ in $\mathcal{C}_3$ and in $\mathcal{C}_1$ respectively, then*
$$\mathrm{rec}_{31}(\mathcal{A}_3, \mathcal{A}_1) = \begin{cases} \mathrm{gl}_{31}(\mathcal{A}_3, \mathcal{A}_1), & \text{if } n_3 + \theta_3 \geq n_1 + \theta_1 + 1 \\ \mathrm{gl}_{12}(\mathcal{A}_1, \mathrm{Coh}_2^{\theta_3}[n_3]), & \text{if } n_3 + \theta_3 \leq n_1 + \theta_1. \end{cases}$$

3. *Case 23. Given hearts $\mathcal{A}_2 \cong \mathrm{Coh}^{\theta_2}[n_2]$ and $\mathcal{A}_3 \cong \mathrm{Coh}^{\theta_3}[n_3]$ in $\mathcal{C}_2$ and in $\mathcal{C}_3$ respectively, then*
$$\mathrm{rec}_{23}(\mathcal{A}_2, \mathcal{A}_3) = \begin{cases} \mathrm{gl}_{23}(\mathcal{A}_2, \mathcal{A}_3), & \text{if } n_2 + \theta_2 \geq n_3 + \theta_3 + 1 \\ \mathrm{gl}_{31}(\mathcal{A}_3, \mathrm{Coh}_1^{\theta_2}[n_2 - 1]), & \text{if } n_2 + \theta_2 \leq n_3 + \theta_3. \end{cases}$$

Proof. Follows directly from the CP-gluing conditions of Proposition 2.8.13. together with Proposition 2.8.12. □

2.9 Construction of stability conditions

2.9.1 Gluing stability conditions

We begin by describing stability conditions glued as in [25]. Let $\mathcal{D}$ be a triangulated category equipped with a semiorthogonal decomposition $\mathcal{D} = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$. Denote by ρ_2 the right adjoint functor to the inclusion $\mathcal{D}_2 \to \mathcal{D}$ and by λ_1 the left adjoint to the inclusion $\mathcal{D}_1 \to \mathcal{D}$.

Definition 2.9.1 ([25])**.** Let $\sigma_i = (Z_i, \mathcal{A}_i)$ be stability conditions on $\mathcal{D}_i$ for $i = 1, 2$, such that the hearts $\mathcal{A}_i$ satisfy (2.8.1). Then we say that a stability condition $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$ is *CP-glued* from σ_1 and σ_2 if the heart $\mathcal{A}$ is given by (2.8.2) and $Z\colon K(\mathcal{A}) \to \mathbb{C}$ is given by
$$Z = Z_1 \circ \lambda_1 + Z_2 \circ \rho_2. \tag{2.9.1}$$

Remark 2.9.2. Note that this CP-glued stability condition is uniquely determined by σ_1 and σ_2. It exists if and only if the Harder-Narasimhan property holds for the stability function Z on the glued heart $\mathcal{A}$. We will check this property separately later.

Lemma 2.9.3 ([25, Proposition 2.2]). *1. A stability condition $\sigma = (Z, \mathcal{A})$ on $\mathcal{D}$ is glued from $\sigma_1 = (Z_1, \mathcal{A}_1)$ on $\mathcal{D}_1$ and $\sigma_2 = (Z_2, \mathcal{A}_2)$ on $\mathcal{D}_2$ if and only if $Z_i = Z|_{\mathcal{D}_i}$ for $i = 1, 2$, $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}_1, \mathcal{A}_2) = 0$ and $\mathcal{A}_i \subset \mathcal{A}$ for $i = 1, 2$.*

2. Let $\sigma = (Z, \mathcal{A})$ be a stability condition on $\mathcal{D}$. Assume that the heart $\mathcal{A}$ is glued from hearts $\mathcal{A}_1 \subset \mathcal{D}_1$ and $\mathcal{A}_2 \subset \mathcal{D}_2$, with $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}_1, \mathcal{A}_2) = 0$, such that that (2.8.2) holds. Then, there exists a stability condition $\sigma_i = (Z_i = Z|_{\mathcal{D}_i}, \mathcal{A}_i)$ on $\mathcal{D}_i$, for $i = 1, 2$, such that σ is glued from σ_1 and σ_2.

3. If $\sigma = (Z, \mathcal{P})$ is glued from $\sigma_1 = (Z_1, \mathcal{P}_1)$ and $\sigma_2 = (Z_2, \mathcal{P}_2)$, then $\mathcal{P}_1(\phi) \subset \mathcal{P}(\phi)$ and $\mathcal{P}_2(\phi) \subset \mathcal{P}(\phi)$ for every $\phi \in \mathbb{R}$.

Example 2.9.4. 1. The stability conditions on $\mathcal{T}_C$ with heart TCoh described in Theorem 2.6.12, are examples of stability conditions obtained by CP-gluing of two stability conditions on $\mathcal{C}$ with heart Coh, i.e.

$$\begin{aligned} \mathrm{TCoh} &= \mathrm{gl}_{12}(\mathrm{Coh}_1, \mathrm{Coh}_2) \\ &= \mathrm{gl}_{12}(i_* \mathrm{Coh}, j_* \mathrm{Coh}) \end{aligned}$$

and

$$Z(r_1, d_1, r_2, d_2) := -A_1 d_1 - A_2 d_2 + B_1 r_1 + B_2 r_2 + i(C_1 r_1 + C_2 r_2)$$

where $A_i, B_i, C_i \in \mathbb{R}$ are such that $A_i, C_i > 0$, for $i = 1, 2$.

2. In this second example we show how we can obtain stability conditions on $\mathcal{T}_C$ that are not given by CP-gluing by applying the action of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ on CP-gluing ones.

In our usual setting we consider the stability condition on $\mathcal{T}_C$ obtained by CP-gluing σ_1 and σ_2, where $\sigma_1 := (Z_\mu, \mathrm{Coh})$ is the standard stability structure on $\mathcal{C}_1$ and take $\sigma_2 := \sigma_1 \circ \overline{g}$, in $\mathcal{C}_2$, where $\overline{g} = (N, f)$ denotes the following element of $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$

$$N = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$$

and f is the unique associated compatible strictly increasing map, which satisfies $f(0) = 0$. We then have $\mathcal{P}_2(t) = \mathcal{P}_1(f(t))$.

Note that we have $t < f(t) < 1$ for all $t \in (0, 1)$. Indeed, let $E \in \mathcal{P}_1(t)$ be an element of phase $t \in (0, 1)$. In particular, the rank r of E is strictly positive. Now, since $E \in \mathcal{P}_1(t) = \mathcal{P}_2(f^{-1}(t))$, together with the fact that $\Im Z_2(E) = \Im Z_1(E) = r$ and $\Re Z_2(E) = -d + r > \Re Z_1(E)$, we have $f^{-1}(t) < t$. The first inequality $t < f(t)$ follows because f is strictly increasing. The second inequality follows simply because $t < 1$, the function f is strictly increasing and $f(1) = 1$.

Now, for each $a \in (0, 1)$ consider

$$\begin{aligned} \mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(i_* \mathcal{P}_1(a, a+1], j_* \mathcal{P}_2(a, a+1]) &= \mathrm{Hom}^{\leq 0}_{\mathcal{C}}(\mathcal{P}_1(a, a+1], \mathcal{P}_2(a, a+1][-1]) \\ &= \mathrm{Hom}^{\leq 0}_{\mathcal{C}}(\mathcal{P}_1(a, a+1], \mathcal{P}_1(f(a) - 1, f(a)]) \end{aligned}$$

but this is nonzero because we can find an $\epsilon > 0$ such that $\mathcal{P}_1(f(a-\epsilon))$ contains a non-zero object and $a < f(a-\epsilon) < 1$.

To make it clearer, note that for $a \in (0,1)$, we have

$$\mathcal{P}_1((a, a+1]) = \mathrm{Coh}^a$$

and

$$\begin{aligned}\mathcal{P}_2((a, a+1]) &= \mathcal{P}_1((f(a), f(a)+1]) \\ &= \mathrm{Coh}^{f(a)}\end{aligned}$$

(i.e. we interpret the action of $a \in (0,1)$ as a rotation by πa) and they don't satisfy gluing conditions since $a < f(a) < 1$.

Remark 2.9.5. Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$ such that it is CP-glued from stability conditions $\sigma_i = (Z_i, \mathcal{P}_i)$ on $\mathcal{D}_i$ for $i = 1, 2$. Then, for any $(T, f) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R})$, recall that it acts by $\sigma.(T, f) = (Z', \mathcal{P}')$ with $Z' = T^{-1}Z$ and $\mathcal{P}'(\phi) = \mathcal{P}(f(\phi))$.

We write $\mathcal{A} := \mathcal{P}((0,1|)$ and $\mathcal{A}_i := \mathcal{P}_i((0,1|)$ for $i = 1, 2$ (resp. $\mathcal{A}' := \mathcal{P}'((0,1|)$ and $\mathcal{A}'_i := \mathcal{P}'_i((0,1|)$ for $i = 1, 2$) to denote the corresponding hearts to σ and σ_i for $i = 1, 2$ (resp. σ' and σ'_i for $i = 1, 2$).

Note that by property number 1. in Lemma 2.9.3, we have $Z'_i = Z'|_{\mathcal{D}_i}$ for $i = 1, 2$, $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}'_1, \mathcal{A}'_2) = 0$ and $\mathcal{A}'_i \subset \mathcal{A}'$ for $i = 1, 2$. In fact, we have

$$\begin{aligned}Z'_i &= T^{-1}Z_i \\ &= T^{-1}Z|_{\mathcal{D}_i} \\ &= Z'|_{\mathcal{D}_i}\end{aligned}$$

for $i = 1, 2$ and on the other hand,

$$\begin{aligned}\mathcal{A}'_i &= \mathcal{P}_i((f(0), f(1)]) \\ &\subset \mathcal{P}((f(0), f(1)]) = \mathcal{A}'.\end{aligned}$$

So the only condition that remains to check is the gluing condition on the hearts, i.e.

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}'_1, \mathcal{A}'_2) = 0$$

but this will not always be true, as we have seen in Example 2.9.4, 2.

A natural question would be whether stability conditions on $\mathcal{T}_C$ that are obtained as in Example 2.9.4, 2. have hearts that can be constructed as recollement of hearts that do not satisfy CP-gluing conditions. The following proposition shows that this is not possible.

Lemma 2.9.6 (Jealousy Lemma)**.** *Let $\mathcal{A} \subset \mathcal{T}_C$ be a heart constructed by recollement of hearts $\mathcal{A}_i \subset \mathcal{C}_i$, $\mathcal{A}_j \subset \mathcal{C}_j$ which do not satisfy CP-gluing conditions. Then, $\mathcal{A}$ does not accept a stability function defined on $K(\mathcal{A})$, i.e. $Z(\mathcal{A}) \not\subset \overline{\mathbb{H}}$ for every $Z\colon K(\mathcal{A}) \to \mathbb{C}$.*

Proof. We give the proof for case 12 and the other cases will follow by acting with the Serre functor $S_{\mathcal{T}_C}$ (or its inverse) on σ.

Let $\sigma = (Z, \mathcal{A}_{12})$ be a stability condition on $\mathcal{T}_C$ such that $\mathcal{A}_{12} := \mathrm{rec}_{12}(\mathcal{A}_1, \mathcal{A}_2)$ is a heart in $\mathcal{T}_C$ defined by recollement from given hearts $\mathcal{A}_i := \mathrm{Coh}^{\theta_i}[n_i]$ in $\mathcal{C}_i$, with $n_i \in \mathbb{Z}$ and $\theta_i \in [0,1)$, for $i = 1, 2$, such that

$$n_2 + \theta_2 - 1 < n_1 + \theta_1 < n_2 + \theta_2. \tag{2.9.2}$$

First of all, we claim that the hearts $i_*\mathcal{A}_1$, $j_*\mathcal{A}_2$ and $l_*\mathcal{A}_1$ are in $\mathcal{A}_{12}$. Indeed, it follows from the definitions. Recall from the isomorphism of Theorem 2.5.3 that we can identify $\mathcal{A}_i = \mathcal{P}(n_i + \theta_i, n_i + \theta_i + 1]$ for $i = 1, 2$, where $\mathcal{P}(0, 1] = \mathrm{Coh}(C)$. Therefore, $j_*\mathcal{P}(n_2+\theta_2, n_2+\theta_2+1] \subset \mathcal{A}_{12}$ since $i^*j_* = 0$ and by adjunction $j^*j_* = \mathrm{id}$ and $j^!j_* = \mathrm{id}$. Also $i_*\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \subset \mathcal{A}_{12}$, since

$$\begin{aligned}
i^*i_*\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] &= \mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \\
&= \mathcal{A}_1, \\
j^*i_*\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] &= j^*j_![1]\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \\
&= \mathcal{P}(n_1 + \theta_1 + 1, n_1 + \theta_1 + 2] \\
&\subset \mathcal{P}(n_2 + \theta_2, \infty) \\
&= \mathcal{A}_2^{\leq 0}
\end{aligned}$$

and $j^!i_* = 0$. Similarly, $l_*\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \subset \mathcal{A}_{12}$, since $l_* = i_!$,

$$\begin{aligned}
i^*i_!\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] &= \mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \\
&= \mathcal{A}_1, \\
j^!i_!\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] &= l^*l_*\mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \\
&= \mathcal{P}(n_1 + \theta_1, n_1 + \theta_1 + 1] \\
&\subset \mathcal{P}(-\infty, n_2 + \theta_2 + 1] \\
&= \mathcal{A}_2^{\geq 0}
\end{aligned}$$

and $j^*i_! = 0$.

We assume n_2 to be (up to shift) equal to 0. Note that equation (2.9.2) implies that either $n_1 = n_2$ and $\theta_1 < \theta_2 < \theta_1 + 1$ or $n_1 = n_2 - 1$ and $\theta_2 < \theta_1 < \theta_2 + 1$.

If we look closely to the imaginary part of Z, it has the form

$$\Im Z(r_1, d_1, r_2, d_2) = D_1 d_1 + D_2 d_2 + C_1 r_1 + C_2 r_2$$

with $C_i, D_i \in \mathbb{R}$, for $i = 1, 2$. The restrictions to the previous hearts are

$$\begin{aligned}
&\Im Z \mid_{i_*\mathcal{P}(n_1+\theta_1, n_1+\theta_1+1]} = D_1 d + C_1 r \\
&\Im Z \mid_{j_*\mathcal{P}(n_2+\theta_2, n_2+\theta_2+1]} = D_2 d + C_2 r \\
&\Im Z \mid_{l_*\mathcal{P}(n_1+\theta_1, n_1+\theta_1+1]} = (D_1 + D_2) d + (C_1 + C_2) r
\end{aligned}$$

for $d, r \in \mathbb{Z}$ with $r \geq 0$. We recall that if $Cr + Dd$ is the imaginary part of a stability function on Coh^θ, then the value θ is determined by the quotient D/C which implies that θ_1 is determined by the quotients D_1/C_1 and $(D_2 + D_1)/(C_1 + C_2)$. But these two quotients cannot determine the same θ_1 unless $\theta_1 = \theta_2$, which contradicts the assumption (2.9.2). $\square$

2.9.2 Harder-Narasimhan and support property

Now that we have hearts in $\mathcal{T}_C$ with the corresponding stability functions, we have to check that they satisfy the Harder-Narasimhan property and the support property. We begin by the Harder-Narasimhan property along the lines of [25].

Theorem 2.9.7 ([25, Theorem 3.6])**.** *Let $(\mathcal{D}_1, \mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose $\sigma_i = (Z_i, \mathcal{P}_i)$ is a stability condition on $\mathcal{D}_i$ for $i = 1, 2$ and let $a \in (0, 1)$ be a real number. Assume the following conditions hold:*

1. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(i_1\mathcal{P}_1(0, 1], i_2\mathcal{P}_2(0, 1]) = 0$
2. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(i_1\mathcal{P}_1(a, a + 1], i_2\mathcal{P}_2(a, a + 1]) = 0.$

Then, there exists a locally finite pre-stability condition σ glued from σ_1 and σ_2.

Definition 2.9.8. For a real number $a \in (0, 1)$, we define the subset $S(a)$ as the subset of pairs of stability conditions $(\sigma_1, \sigma_2) \in \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ satisfying

1. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(i_1\mathcal{P}_1(0, 1], i_2\mathcal{P}_2(0, 1]) = 0$
2. $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(i_1\mathcal{P}_1(a, a + 1], i_2\mathcal{P}_2(a, a + 1]) = 0.$

Theorem 2.9.9 ([25, Theorem 4.3])**.** *Let* $\mathrm{gl}\colon \mathrm{S(a)} \to \mathrm{Stab}(\mathcal{D})$ *be the map associating to $(\sigma_1, \sigma_2) \in \mathrm{Stab}(\mathcal{D}_1) \times \mathrm{Stab}(\mathcal{D}_2)$ the corresponding glued pre-stability condition σ on $\mathcal{D}$ (defined by Theorem 2.9.7). Then, the map* gl *is continuous on $S(a)$.*

For $a \in (0, 1)$, we have a precise description of the sets $S(a)$ for the semiorthogonal decomposition $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$.

Proposition 2.9.10. *Consider the semiorthogonal decomposition $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. For $a \in (0, 1)$, we have that $S(a)$ is isomorphic to*

$$\left\{((T_1, f_1), (T_2, f_2)) \in \widetilde{\mathrm{GL}}^+(2, \mathbb{R}) \times \widetilde{\mathrm{GL}}^+(2, \mathbb{R})\colon f_1(0) \geq f_2(0) \text{ and } f_1(a) \geq f_2(a)\right\}.$$

Proof. Suppose $\sigma_i = (Z_i, \mathrm{Coh}_i^{\theta_i}[n_i])$ is a stability condition on $\mathcal{C}_i$ with $\theta_i \in [0, 1)$ and $n_i \in \mathbb{Z}$, for $i = 1, 2$. Assume that these stability conditions satisfy the gluing condition, i.e. $n_1 + \theta_1 \geq n_2 + \theta_2$. Let (T_i, f_i) be the elements in $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$ corresponding to σ_i under the equivalence in Theorem 2.5.3 for $i = 1, 2$. Note that $f_i(0) = n_i + \theta_i$ for $i = 1, 2$, so the condition 1 in Theorem 2.9.7 is equivalent to $f_1(0) \geq f_2(0)$. We end the proof by showing that condition 2 is equivalent to $f_1(a) \geq f_2(a)$. Indeed we will have

$$\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(i_*\mathcal{P}_1(a, a + 1], j_*\mathcal{P}_2(a, a + 1]) = 0$$

if and only if the stability condition σ' obtained from σ, acting by rotation of angle a satisfies gluing property (recall what we explained in Remark 2.4.24 and Example

2.9.4). Hence, if we denote $\mathcal{P}(0,1] = \mathrm{Coh}(C)$ the standard heart associated to slope-stability Z_μ, then

$$\begin{aligned}\mathcal{P}_i(a,a+1] &= \mathcal{P}(f_i(a), f_i(a)+1] \\ &= \mathrm{Coh}^{f_i(a)}\end{aligned}$$

for $i = 1,2$ and they will satisfy the gluing condition if and only if $f_1(a) \geq f_2(a)$. $\square$

Example 2.9.11. Consider the semiorthogonal decomposition $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$. Suppose $\sigma_i = (Z_i, \mathrm{Coh}_i^{\theta_i}[n_i])$ is a stability condition on $\mathcal{C}_i$ with $\theta_i \in [0,1)$ and $n_i \in \mathbb{Z}$, for $i = 1,2$.

1. If $n_1 + \theta_1 \geq n_2 + \theta_2 + 1$, i.e. $f_1(0) \geq f_2(0) + 1$, then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in (0,1)$.

2. If $n_1 = n_2$ and $\theta_1 > \theta_2$ then there always exists some $a \in (0,1)$ such that $f_2(a) < f_1(0)$ and $f_1(0) < f_1(a)$. Then, $(\sigma_1, \sigma_2) \in S(a)$ for every such a.

3. Recall that in Example 2.9.4 2, we had $\sigma_1 = \sigma_\mu$ and $\sigma_2 = (N_x^{-1} Z_\mu, \mathrm{Coh})$ where
$$N_x = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$
with $x < 0$, then the pair of stability conditions (σ_1, σ_2) does not belong to $S(a)$ for any $a \in (0,1)$, since they fulfill the condition 1 in Theorem 2.9.7 but not 2.

4. Take N_x with $x \geq 0$, then the construction in Example 2.9.4 provides concrete examples of stability conditions on $\mathcal{C}_i$ for $i = 1,2$ that belong to $S(a)$ for all $a \in (0,1)$.

5. Take
$$A_r = \begin{pmatrix} r & 0 \\ 0 & 1/r \end{pmatrix}$$
with $r \in \mathbb{R}_{>0}$. Then the construction in Example 2.9.4 provides concrete examples of stability conditions on $\mathcal{C}_i$ for $i = 1,2$ that belong to $S(a)$ for some $a \in (0,1)$:
    - if $r > 1$, then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in (0, 1/2]$.
    - if $r < 1$, then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in [1/2, 1)$.
    - if $r = 1$ then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in (0,1)$, since this case agrees with Example 4 above with $x = 0$.

6. Finally we analyze examples 3 and 5 together. We consider $\sigma_1 = \sigma_\mu$ and now we take $\sigma_2 = ((A_r N_x)^{-1} Z, \mathrm{Coh})$ with $r, x \in \mathbb{R}$, $r > 0$ and $x < 0$. We show that again they are a concrete example of stability conditions on $\mathcal{C}_i$ for $i = 1,2$ that belong $S(a)$ for some $a \in (0,1)$. Then, we have the following cases:

- if $r > 1$, then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in (\phi_{xr}, 1/2)$, where ϕ_{rx} denotes the phase of the complex number $rx + i(1/r - r)$.
- if $r < 1$, then $(\sigma_1, \sigma_2) \in S(a)$ for every $a \in (\phi_{xr}, 1)$, where ϕ_{rx} denotes the phase of the complex number $-rx + i(r - 1/r)$.
- if $r = 1$ then $(\sigma_1, \sigma_2) \notin S(a)$ for every $a \in (0, 1)$, since this case agrees with Example 3 above.

For small hearts that fulfill the condition 1 in Theorem 2.9.7 but not 2, we may need a different strategy. Let $\langle \mathcal{D}_1, \mathcal{D}_2 \rangle$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$ and let $\sigma_i = (Z_i, \mathcal{A}_i)$ be a locally finite pre-stability condition on $\mathcal{D}_i$ for $i = 1, 2$. Assume that $\mathrm{Hom}^{\leq 0}_{\mathcal{D}}(\mathcal{A}_1, \mathcal{A}_2) = 0$ and let $\mathcal{A}$ be the heart in $\mathcal{D}$ glued from $\mathcal{A}_1$ and $\mathcal{A}_2$. Consider the stability function $Z = Z_1 \circ \lambda_1 + Z_2 \circ \rho_2$ on $\mathcal{A}$.

Lemma 2.9.12 ([25, Proposition 3.5])**.** *If* 0 *is an isolated point of* $\Im Z_i(\mathcal{A}_i) \subset \mathbb{R}_{\geq 0}$ *for* $i = 1, 2$, *then* Z *has the HN-property on* $\mathcal{A}$.

Proposition 2.9.13 (HN-property for $\mathbb{Q}$)**.** *Let*

$$\mathrm{gl}_{12}(\sigma_1, \sigma_2) = (Z_{12} = Z_1 \circ i^* + Z_2 \circ j^!, \mathcal{A}_{12} = \mathrm{gl}_{12}(\mathcal{A}_1, \mathcal{A}_2))$$

be a candidate stability condition on $\mathcal{T}_C$ *obtained by CP-gluing the stability conditions* $\sigma_i = (Z_i, \mathcal{A}_i)$ *on* $\mathcal{C}_i$ *with* $\mathcal{A}_i = \mathrm{Coh}_i^{\theta_i}$ *for* $\theta_i \in [0, 1)$ *for* $i = 1, 2$. *If* $\tan(\pi\theta_i) \in \mathbb{Q}$, *for all* i, *then* Z_{12} *has the HN-property.*

Proof. Under the previous notations, we will show that Z_{12} has the HN-property using Lemma 2.9.12. So we need to check that 0 is an isolated point of $\Im Z_i(\mathcal{A}_i) \subset \mathbb{R}_{\geq 0}$ for $i = 1, 2$. Without lost of generality, we will show it for $i = 1$.

We know that Z_1 is of the form

$$Z_1(r, d) = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} \cos \pi\theta & \sin \pi\theta \\ -\sin \pi\theta & \cos \pi\theta \end{pmatrix} \begin{pmatrix} -d \\ r \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$ and $a, c > 0$. Therefore, $\Im Z_1(r, d) = c(d \sin \pi\theta + r \cos \pi\theta)$. We take the unit vector $(u, v) := (-\sin \pi\theta, \cos \pi\theta)$, such that

$$(u, v) \cdot \begin{pmatrix} -d \\ r \end{pmatrix} = \frac{1}{c} \Im Z_1(r, d).$$

We see that 0 is an isolated point of the set $\{vr - ud \mid r, d \in \mathbb{Z}\}$. Since by assumption $u/v \in \mathbb{Q}$, consider $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$ such that $u/v = m/n$. Then,

$$vr - ud = \frac{v}{n}(nr - md)$$

and the claim follows since v/n is constant and $nr - md \in \mathbb{Z}$. $\square$

We obtain as a corollary that the CP-gluing of the stability conditions of Example 2.9.11 3. are pre-stability conditions.

Corollary 2.9.14. *Let*

$$\mathrm{gl}_{12}(\sigma_1,\sigma_2) = (Z_{12} = Z_1 \circ i^* + Z_2 \circ j^!, \mathcal{A}_{12} = \mathrm{gl}_{12}(\mathcal{A}_1,\mathcal{A}_2))$$

be a candidate stability condition on $\mathcal{T}_C$ obtained by CP-gluing the stability conditions $\sigma_i = (Z_i, \mathcal{A}_i)$ on $\mathcal{C}_i$ with $\mathcal{A}_i = \mathrm{Coh}_i^{\theta_i}$ for $\theta_i \in [0,1)$ for $i = 1,2$, such that $\theta_1 \geq \theta_2$ but it does not belong to $S(a)$ for any $a \in (0,1)$. If $\tan(\pi\theta_i) \in \mathbb{Q}$, for all i, then $\mathrm{gl}_{12}(\sigma_1,\sigma_2)$ is a pre-stability condition.

Proof. Let $\mathrm{gl}_{12}(\sigma_1,\sigma_2)$ be a candidate stability condition on $\mathcal{T}_C$ obtained by CP-gluing with the notation above. It requires the numerical condition $\theta_1 \geq \theta_2$ by Proposition 2.8.13, which states it as the gluing condition for the case $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2\rangle$.

Then, if $\tan(\pi\theta_i) \in \mathbb{Q}$, for all i, then $\mathrm{gl}_{12}(\sigma_1,\sigma_2)$ is a pre-stability condition by Proposition 2.9.13.

□

The strategy now is to extend it to the rest by means of the support property. The next proposition shows that in general every CP-glued stability condition will be close to satisfy the support property.

Proposition 2.9.15. *Let $(\mathcal{D}_1,\mathcal{D}_2)$ be a semiorthogonal decomposition of a triangulated category $\mathcal{D}$. Suppose that σ is obtained from CP-gluing of pre-stability conditions $\sigma_i = (Z_i, \mathcal{P}_i)$ on $\mathcal{D}_i$ for $i = 1,2$. If σ is a pre-stability condition on $\mathcal{D}$, then there exists a quadratic form Q such that*

a) for every σ-semistable object $E \in \mathcal{P}(\phi)$, we have $Q(v(E)) \geq 0$.

b) Q is negative semi-definite with respect to the kernel of Z.

Proof. Assume we have a stability condition on $\mathcal{D}$ of the form

$$\sigma_{12} = (Z_{12}, \mathcal{A}_{12}) \coloneqq \mathrm{gl}_{12}(\sigma_1,\sigma_2)$$

where $\sigma_i \coloneqq (Z_i, \mathcal{A}_i)$ denotes a pre-stability condition on $\mathcal{D}_i$ for $i = 1,2$. Note that the stability function Z_{12} is of the form

$$Z_{12} = Z_1 \circ \lambda_1 + Z_2 \circ \rho_2.$$

In the rest of the proof Z_1 (resp. Z_2) will denote the composition $Z_1 \circ \lambda_1$ (resp. $Z_2 \circ \rho_2$) and $Z(E)$ (resp. $Q(E)$) will denote $Z(v(E))$ (resp. $Q(v(E))$).

We will show that the following quadratic form

$$Q = \Im Z_1 \Re Z_2 - \Im Z_2 \Re Z_1 + \Im Z_1 \Im Z_2, \tag{2.9.3}$$

where we write $\Im Z_i$ (resp. $\Re Z_i$) to denote the imaginary (resp. real) part of Z_i with $i = 1,2$, does the job.

First of all, we show that for every σ_{12}-semistable object $E \in \mathcal{P}(\phi)$ we have $Q(E) \geq 0$. Indeed, recall the short exact sequence in $\mathcal{A}_{12}$

$$0 \longrightarrow i_2\rho_2(E) \longrightarrow E \longrightarrow i_1\lambda_1(E) \longrightarrow 0 \tag{2.9.4}$$

and note that σ_{12}-semistability of E implies the following inequality of phases:

$$\phi(Z_2(E)) \leq \phi \leq \phi(Z_1(E)). \tag{2.9.5}$$

Assuming that $\phi \in (0,1]$, we have E is such that $\Im Z_i(E) \geq 0$ for all $i = 1,2$. Assume first $\Im Z_1(E)\Im Z_2(E) > 0$. We have $Q(E) > 0$ by hypothesis plus the inequality (2.9.5). Now, if $\Im Z_1(E) = 0$, then $\Re Z_1(E) < 0$ and

$$Q(E) = -\Re Z_1(E)\Im Z_2(E) \geq 0.$$

On the other hand, if $\Im Z_2(E) = 0$, then $\phi_{\sigma_{12}}(i_2\rho_2(E)) = 1$. Moreover, since $\phi \in (0,1]$, the short exact sequence (2.9.4) implies that all 3 elements belong to the heart. This means that the inequalities (2.9.5) are in fact equalities

$$\phi(Z_2(E)) = \phi = \phi(Z_1(E)) = 1.$$

This implies that in this case we also have $\Im Z_1(E) = 0$. Therefore, $Q(E) = 0$.

We end by showing that Q is negative semi-definite on the kernel of Z_{12}. Note that $E \in \ker Z_{12}$ if and only if $\Im Z_1(E) = -\Im Z_2(E)$ and $\Re Z_1(E) = -\Re Z_2(E)$. If we plug this in equation (2.9.3), we obtain that

$$Q(E) = -\left(\Im Z_2(E)\right)^2 \leq 0. \qquad \square$$

The only discrepancy between the conditions satisfied in Proposition 2.9.15 is that Q is negative semi-definite with respect to the kernel of Z but not necessarily negative definite. Since the kernels of the stability functions Z_i $i = 1,2$ are trivial, we might improve the quadratic form Q (2.9.3) by adding the term $\Re Z_1 \Re Z_2$. Now the formula reads

$$Q' = \Im Z_1 \Re Z_2 - \Im Z_2 \Re Z_1 + \Im Z_1 \Im Z_2 + \Re Z_1 \Re Z_2 \tag{2.9.6}$$

and for $0 \neq E \in \ker Z_{12}$, we have

$$Q'(E) = -\left(\Im Z_2(E)\right)^2 - \left(\Re Z_2(E)\right)^2 < 0. \tag{2.9.7}$$

Remark 2.9.16. Let σ_{12} be a pre-stability condition on $\mathcal{D}$ of the form

$$\sigma_{12} = \mathrm{gl}_{12}(\sigma_1, \sigma_2)$$

where σ_i denotes a pre-stability condition on $\mathcal{D}_i$ for $i = 1,2$. Notice that σ_{12} will satisfy the conditions of Proposition 2.9.15 independently of whether the former pre-stability conditions σ_1, σ_2 satisfy similar conditions or not.

Proposition 2.9.17. *Consider a pre-stability condition $\sigma := (Z, \mathcal{A})$ where $\mathcal{A}$ is a CP-glued heart of the form $\mathrm{gl}_{12}(\mathrm{Coh}_1^\theta(C), \mathrm{Coh}_2^\theta(C))$ and*

$$Z = Z_1 \circ \lambda_1 + Z_2 \circ \rho_2$$

with $\theta \in [0,1)$ and $Z_1 = Z_2$. Then, σ satisfies the support property with respect to the quadratic form Q' (2.9.6).

Proof. We assume that $\sigma = (Z, \mathrm{TCoh}(C))$ with

$$Z = Z_\mu \circ \lambda_1 + Z_\mu \circ \rho_2.$$

This can be done without loss of generality, up to $\widetilde{\mathrm{GL}}^+(2, \mathbb{R})$-action.

In what follows, we will denote $Z_1 \coloneqq Z_\mu \circ \lambda_1$ and $Z_2 \coloneqq Z_\mu \circ \rho_2$ and $Z(E)$ (resp. $Q(E)$) will denote $Z(v(E))$ (resp. $Q(v(E))$).

First of all, Q' is negative definite on the kernel of Z by the arguments above. Then, we have to see that $Q'(E) \geq 0$ for every σ-semistable object

$$E \coloneqq (E_1 \xrightarrow{\varphi} E_2) \in \mathcal{P}(\phi).$$

Indeed, recall the short exact sequence in $\mathcal{A}$

$$0 \longrightarrow i_2 E_2 \longrightarrow E \longrightarrow i_1 E_1 \longrightarrow 0 \tag{2.9.8}$$

and note that σ-semistability of E implies the following inequality of phases:

$$\phi(Z_2(E)) \leq \phi \leq \phi(Z_1(E)). \tag{2.9.9}$$

Assuming that $\phi \in (0, 1]$, we have E is such that $\Im Z_i(E) \geq 0$ for all $i = 1, 2$.

If $E_2 \neq 0$ and $\Im Z_2(E) = 0$, then $\phi(Z_2(E)) = 1$. Moreover, since $\phi \in (0, 1]$, the short exact sequence (2.9.8) implies that all 3 elements belong to the heart. This means that the inequalities (2.9.9) are in fact equalities

$$\phi(Z_2(E)) = \phi = \phi(Z_1(E)) = 1.$$

This implies that in this case we also have $\Im Z_1(E) = 0$. Therefore,

$$Q'(E) = \Re Z_1(E) \Re Z_2(E) > 0.$$

On the other hand, if $E_1 \neq 0$, $\Im Z_1(E) = 0$ and $\Im Z_2(E) \neq 0$ then we claim that there is no σ-semistable triple E with these data. Let us assume that E is a σ-semistable triple satisfying the previous conditions. Then, we note that E_2 is torsion-free. Indeed, if it is not torsion-free, let us consider T_2 to be the torsion subsheaf of E_2. Then we have a subtriple $i_2(T_2)$ of E with phase $\phi(Z_2(i_2(T))) = 1$ and together with the short exact sequence (2.9.8) it contradicts the σ-semistability of E. Therefore, E_2 is torsion-free. This implies that the morphism φ is 0. Therefore $i_1(E_1)$ is a non-trivial subtriple of E of phase 1 which by the previous argument contradicts the σ-semistability of E and the claim follows.

We end by showing the case $\Im Z_1(E) \Im Z_2(E) > 0$. Indeed, note that by hypothesis E_1 and E_2 have to be torsion-free. Otherwise, if we denote by T_i the corresponding torsion subsheaf of E_i for $i = 1, 2$, then we would have a subtriple $T = T_1 \to T_2$ of E with $\phi(T) = 1$ contradicting the σ-semistability assumption. Then, by Remark 2.6.7 with $\alpha = 0$, we know that $\phi(Z_1(E)) = \phi(Z_2(E))$, which implies $Q'(E) > 0$. □

The support property of the stability conditions above extends to the connected component containing them and therefore we obtain the following theorem.

Theorem 2.9.18. *All the pre-stability conditions on $\mathcal{T}_C$ given as in Proposition 2.9.17 are stability conditions on $\mathcal{T}_C$.*

Remark 2.9.19. The previous result stresses the connection between the stability conditions we constructed and the classical results for holomorphic triples by Bradlow, García-Prada et al. The other cases are worked out in [47] where we need to construct a generalization of the inequalities of Proposition 2.6.6 in our setting.

Finally we make an important remark about the existence of Harder-Narasimhan filtrations.

As in previous constructions, we focus on the case of the semiorthogonal decomposition $\mathcal{T}_C = \langle \mathcal{C}_1, \mathcal{C}_2 \rangle$, up to the action of the Serre functor of $\mathcal{T}_C$. Furthermore, assume we have a candidate stability condition

$$\sigma_{12} = (Z_{12}, \mathcal{A}_{12}) \coloneqq \mathrm{gl}_{12}(\sigma_1, \sigma_2)$$

defined, up to shift, by CP-gluing of a stability condition on $\mathcal{C}_1$ of the form $\sigma_1 \coloneqq (Z_1, \mathrm{Coh}_1^{\theta_1}[n])$ with $n \in \mathbb{Z}_{\geq 0}$ and $\theta_1 \in [0,1)$ and a stability condition on $\mathcal{C}_2$ of the form $\sigma_2 = (Z_2, \mathrm{Coh}_2^{\theta_2})$ with $\theta_2 \in [0,1)$, satisfying CP-gluing condition as in Proposition 2.8.13, i.e. $n + \theta_1 \geq \theta_2$. Note that the stability function Z_{12} is of the form

$$Z_{12}(r_1, d_1, r_2, d_2) = A_1 d_1 + B_1 r_1 + A_2 d_2 + B_2 r_2 + i(C_1 r_1 + D_1 d_1 + C_2 r_2 + D_2 d_2)$$

for some real numbers A_i, B_i, C_i and D_i with $D_i/C_i = \tan(\pi\theta_i)$, for $i = 1, 2$.

We claim that all such σ_{12} will be locally finite stability conditions on $\mathcal{T}_C$, provided that they all satisfy the HN-property, by extending Theorem 2.9.18.

The remaining case to verify the HN-property is when (σ_1, σ_2) does not belong to any $S(a)$ for any $a \in (0,1)$ and $\tan(\pi\theta_i) \in \mathbb{R}\backslash\mathbb{Q}$ for $i = 1$ or $i = 2$.

Let us consider that $\tan(\pi\theta_1) \in \mathbb{R}\backslash\mathbb{Q}$. By Bridgeland's deformation theorem (recall Theorem 2.4.22), If we show that for any such Z_{12} there exists a (pre-)stability condition σ'_{12} on $\mathcal{T}_C$ of the form

$$\sigma'_{12} \coloneqq (Z'_{12} = Z'_1 \circ i^* + Z_2 \circ j^!, \mathcal{A}'_{12} \coloneqq \mathrm{gl}_{12}(\mathrm{Coh}_1^{\theta'_1}, \mathrm{Coh}_2^{\theta'_2}))$$

for $\theta'_i \in (0,1)$ with $\tan(\pi\theta'_i) \in \mathbb{Q}$ for $i = 1, 2$ satisfying

$$|Z_{12}(T) - Z'_{12}(T)| < \sin(\pi/8)|Z'_{12}(T)| \tag{2.9.10}$$

for all $T \in \mathcal{T}_C$ that are σ'_{12}-stable, then there exists a unique locally-finite stability condition $\tau = (Z_{12}, \mathcal{A})$ on $\mathcal{T}_C$ with distance between the corresponding slices satisfying

$$d_S(\mathcal{P}_{\mathcal{A}'_{12}}, \mathcal{P}_{\mathcal{A}}) < 1/8.$$

The problem is to compare both hearts $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{P}_{\mathcal{A}_{12}}$, since the definition of distance requires a notion of HN-filtration.

Nevertheless, due to the existence of the stability conditions shown in this thesis and the good behavior of the deformation properties of the stability manifold in general, we have that all gluing cases are going to define stability conditions on $\mathcal{T}_C$.

Theorem 2.9.20. *[47] All gluing of stability conditions $\sigma_i = (Z_i, \mathcal{P}_i)$ on $\mathcal{C}_i$ for $i = 1, 2, 3$ give stability conditions on $\mathcal{T}_C$.*

2.9.3 Scope

In this section sketch the description of the stability manifold, $\mathrm{Stab}(\mathcal{T}_C)$. The full description will be found in [47]. The following lemma is an analogous result to Lemma 2.5.1 for holomorphic triples.

Lemma 2.9.21. *Given a distinguished triangle in $\mathcal{T}_C$ of the form*

$$E \longrightarrow i_*(X) \longrightarrow A \longrightarrow E[1]$$

i.e.

$$\begin{tikzcd} E_1 \arrow[r] \arrow[d, "\varphi_E"'] & X \arrow[r] \arrow[d] & A_1 \arrow[r] \arrow[d, "\varphi_A"] & E_1[1] \arrow[d, "\varphi_E[1]"] \\ E_2 \arrow[r] & 0 \arrow[r] & A_2 \arrow[r] & E_2[1] \end{tikzcd} \tag{2.9.11}$$

with $X \in \mathrm{Coh}(C)$ and

$$\mathrm{Hom}^{\leq 0}_{\mathcal{T}_C}(E, A) = 0, \tag{2.9.12}$$

then, $E_1, A_1 \in \mathrm{Coh}(C)$.

Using Lemma 2.9.21, we can prove the following result about semistability of skyscraper sheaves and line bundles in $\mathcal{T}_C$.

Proposition 2.9.22. *Let X be either a skyscraper sheaf $\mathbb{C}(x)$ of a point $x \in C$ or $\mathcal{L}$, a line bundle on C. For any stability condition $\sigma \in \mathrm{Stab}(\mathcal{T}_C)$, if $i_*(X)$ is not σ-semistable, then $j_*(X)$ and $l_*(X)$ are σ-stable.*

Remark 2.9.23. In particular, we have found that if $X = i_*(\mathbb{C}(x))$ (resp. $i_*(\mathcal{L})$) is not σ-semistable with respect to some (arbitrary!) stability condition $\sigma \in \mathrm{Stab}(\mathcal{T}_C)$, then its HN-filtration is precisely of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\varphi"'] & X \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "\varphi[1]"] \\ X \arrow[r] & 0 \arrow[r] & X[1] \arrow[r] & X[1] \end{tikzcd}$$

which resembles the (unique) decomposition of $i_*(\mathbb{C}(x))$ with respect to the semiorthogonal decomposition $\mathcal{T}_C = \langle \mathcal{C}_2, \mathcal{C}_3 \rangle$.

The proof of Proposition 2.9.22 shows that the result works for an arbitrary choice among the elements $i_*(\mathbb{C}(x))$, $j_*(\mathbb{C}(x))$ and $l_*(\mathbb{C}(x))$ (resp. $i_*(\mathcal{L})$, $j_*(\mathcal{L})$ and $l_*(\mathcal{L})$), meaning that if one of them is not semistable, the remaining two are semistable.

The following proposition shows that there is no stability condition with mixed-type stable elements.

Proposition 2.9.24. *Consider the following set of pairs of elements of $\mathcal{T}_C$:*

$$\left\{ \{i_*(X)\}_{X=\mathbb{C}(x),\mathcal{O}_C}, \{j_*(X)\}_{X=\mathbb{C}(x),\mathcal{O}_C}, \{l_*(X)\}_{X=\mathbb{C}(x),\mathcal{O}_C} \right\}. \tag{2.9.13}$$

Then, for every stability condition $\sigma \in \mathrm{Stab}(\mathcal{T}_C)$, at least 2 out of the 3 pairs in (2.9.13) are σ-stable.

Finally, we define the following subsets of $\mathrm{Stab}(\mathcal{T}_C)$:

- $\Theta_{12} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), i_*(\mathcal{O}_C), j_*(\mathbb{C}(x))$ and $j_*(\mathcal{O}_C)$ are σ-stable$\}$
- $\Theta_{31} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid i_*(\mathbb{C}(x)), i_*(\mathcal{O}_C), l_*(\mathbb{C}(x))$ and $l_*(\mathcal{O}_C)$ are σ-stable$\}$
- $\Theta_{23} \coloneqq \{\sigma \in \mathrm{Stab}(\mathcal{T}_C) \mid j_*(\mathbb{C}(x)), j_*(\mathcal{O}_C), l_*(\mathbb{C}(x))$ and $l_*(\mathcal{O}_C)$ are σ-stable$\}$

Theorem 2.9.25.

$$\mathrm{Stab}(\mathcal{T}_C) = \Theta_{12} \cup \Theta_{23} \cup \Theta_{13}.$$

Proof. Follows immediately from Proposition 2.9.24 □

After Theorem 2.9.25, we know that we can describe the structure of the whole stability manifold of $\mathcal{T}_C$. However, the HN-property in general for small CP-glued hearts and the description of the topology of the stability manifold $\mathrm{Stab}(\mathcal{T}_C)$ will be given in our join preprint [47].

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Summary

Part I - *Arakelov bundles over arithmetic curves.* In Chapter I we compile the basics about Arakelov geometry that we briefly described above. We define Arakelov vector bundles on arithmetic curves and we explore the relationship of nefness and the tensor product problem as evidence of the pathologies of the Arakelov setting. Chapter II reproduces Behrend's construction of complementary polyhedra for stability of group schemes and the later adaptation to Arakelov geometry by Harder and Stuhler. The main results of this part are contained in chapter III, where we define Arakelov principal bundles. We provide a notion of stability and prove that our definition agrees with all the previous constructions.

Part II - *Bridgeland stability conditions on holomorphic triples over curves.* Chapter IV gathers basic facts about triangulated and derived categories. In Chapter V we introduce the general definition of Bridgeland stability conditions and explore few examples of constructions of stability conditions that are interesting for our constructions. Finally, chapter VI contains all our constructions of Bridgeland stability conditions on holomorphic triples over curves. First we describe the bounded derived category of holomorphic triples on curves $\mathcal{T}_C$ as semiorthogonal decomposition of the bounded derived category of coherent sheaves on the curve and we construct the Serre functor $S_{\mathcal{T}_C}$. Next, we compare recollement and CP-gluing to construct hearts via semiorthogonal decompositions, by gluing hearts in the smaller categories and we compute the necessary numerical conditions for triples. Finally, we construct stability conditions on $\mathcal{T}_C$ by gluing stability conditions from $\mathrm{Stab}(C)$. We study the Harder-Narasimhan and the support properties of glued stability conditions in general and for triples. The very last section shows the sketch of how we finally come up with the full description of the stability manifold $\mathrm{Stab}(\mathcal{T}_C)$.

The results contained in chapter VI will appear soon in the co-authored paper *Bridgeland stability conditions on holomorphic triples over curves* as a preprint on the Mathematics ArXiv, [47]. The sections reproduced here are those that existed in similar form in my research before the paper was finished. The proofs in the final section of that chapter have not been included as they will be presented in the co-author's PhD theses.

Zusammenfassung

Teil I - Arakelov-Bündel über arithmetischen Kurven. In Kapitel I fassen wir die Grundlagen der Arakelov-Geometrie zusammen, die wir oben kurz beschrieben haben. Wir definieren Arakelov-Vektorbündel auf arithmetischen Kurven und untersuchen die Beziehung von Nefness und dem Tensorproduktproblem als Beweis für die Pathologien der Arakelov-Einstellung. Kapitel II reproduziert Behrends Konstruktion komplementärer Polyeder für die Stabilität von Gruppenschemata und die spätere Anpassung an die Arakelov-Geometrie von Harder und Stühler. Die Hauptergebnisse dieses Teils sind in Kapitel III zu finden, in dem wir Arakelov-Hauptbündel definieren. Wir stellen einen Begriff von Stabilität vor und beweisen, dass unsere Definition mit allen früheren Konstruktionen übereinstimmt.

Teil II - Bridgeland Stabilitätsbedingungen auf holomorphen Tripeln über Kurven. Kapitel IV trägt grundlegende Fakten über triangulierte und abgeleitete Kategorien zusammen. In Kapitel V führen wir die allgemeine Definition der Bridgeland-Stabilitätsbedingungen ein und untersuchen einige Beispiele, die für unsere Konstruktionen interessant sind. Schließlich enthält Kapitel VI alle unsere Konstruktionen von Bridgeland-Stabilitätsbedingungen für holomorphe Tripel über Kurven. Zuerst beschreiben wir die beschränkte abgeleitete Kategorie von holomorphen Tripeln in den Kurven $\mathcal{T}_C$ als semiorthogonale Zerlegung der beschränkten abgeleiteten Kategorie von kohärenten Garben auf der Kurve und konstruieren den Serre-Funktor $S_{\mathcal{T}_C}$. Als nächstes vergleichen wir die Rekollement und CP-Verklebung, um Herzen über semiorthogonale Zerlegung zu konstruieren, indem wir Herzen in die kleineren Kategorien kleben und die notwendigen numerischen Bedingungen für Tripel berechnen. Schließlich konstruieren wir Stabilitätsbedingungen für $\mathcal{T}_C$, indem wir Stabilitätsbedingungen aus $\mathrm{Stab}(C)$ verkleben. Wir untersuchen die Harder-Narasimhan- und die Stützeigenschaften von geklebten Stabilitätsbedingungen im Allgemeinen und für Tripel. Im letzten Abschnitt wird skizziert, wie wir zu einer vollständigen Beschreibung der Stabilitäts-Mannigfaltigkeit $\mathrm{Stab}(\mathcal{T}_C)$ gelangen.

Die Ergebnisse in Kapitel VI werden bald in dem gemeinsam verfassten Artikel *Bridgeland stability conditions on holomorphic triples over curves* als ein Vorabdruck auf dem Mathematics ArXiv [47] erscheinen. Die hier wiedergegebenen Abschnitte sind diejenigen, die in ähnlicher Form in meinen Forschungen vor der Fertig-stellung des Artikels existierten. Die Beweise im letzten Abschnitt dieses Kapitels wurden nicht berücksichtigt, da sie in der Doktorarbeit des Co-Autors präsentiert werden.