Noncommutative Deformations of Toric Varieties

Matej Filip

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Selbständigkeitserklärung

Hiermit versichere ich, dass ich alle Hilfsmittel und Hilfen angeben habe und auf dieser Grundlage die Arbeit selbständig verfasst habe. Meine Arbeit ist nicht schon einmal in einem früheren Promotionsverfahren eingereicht worden.

Matej Filip, Berlin, 23. November 2017.

1. Gutachter: Prof. Dr. Klaus Altmann
2. Gutachter: Prof. Dr. Arne B. Sletsjøe

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Abstract

For an affine toric variety Spec(A) we give a convex geometric description of the Hodge decomposition of its Hochschild cohomology. Under certain assumptions we compute the dimensions of the Hodge summands $T^1_{(i)}(A)$, generalizing the existing results about the André-Quillen cohomology group $T^1_{(1)}(A)$. We prove that every Poisson structure on a possibly singular affine toric variety can be quantized in the sense of deformation quantization. Furthermore, we give a convex geometric description of the Harrison cup product formula $T^1_{(1)}(A) \times T^1_{(1)}(A) \to T^2_{(1)}(A)$, which gives us the quadratic equations of the versal base space. Moreover, a differential graded Lie algebra $\mathfrak{g}$ controlling Poisson deformations of an arbitrary affine variety is constructed. In the toric case we simplify the computation of the Poisson cohomology groups $H^k(\mathfrak{g})$.

Contents

1 Introduction

Deformation theory appeared as the investigation of how many complex structures may appear on a fixed compact manifold. In 19th century Riemann [60] already mentioned $3g - 3$ moduli determining the complex structure of an algebraic curve of genus $g \geq 2$.

Following Gerstenhaber's approach [31] we consider deformations of algebras. Let k be a field of characteristic 0 and let A be a k-algebra. A deformation of A over an Artin ring B is a pair (A', π), where A' is a B-algebra and $\pi : A' \otimes_B k \to A$ is an isomorphism of k-algebras. Two such deformations (A', π_1) and (A'', π_2) are equivalent if there exists an isomorphism of B-algebras $\phi : A' \to A''$ such that it is compatible with π_1 and π_2, i.e., such that $\pi_1 = \pi_2 \circ (\phi \otimes_B k)$.

Let us additionally assume that A is equipped with a Poisson structure. Deforming the product in the direction of the chosen Poisson structure on A leads us to the problem of deformation quantization, which has been appearing in the literature for many years and was established by Bayen, Flato, Frønsdal, Lichnerowicz and Sternheimer in [9]. A major result, concerning the existence of deformation quantization is Kontsevich's formality theorem [40, Theorem 4.6.2], which implies that every Poisson structure on a real manifold can be quantized, i.e., admits a star product. Kontsevich [39] also extended the notion of deformation quantization into the algebro-geometric setting. From Yekutieli's results [71], [72] it follows that on a smooth algebraic variety X (under certain cohomological restrictions) every Poisson structure admits a star product. As in Kontsevich's case, the construction is canonical and induces a bijection between the set of formal Poisson structures up to gauge equivalence and the set of star products up to gauge equivalence.

When $X = \mathrm{Spec}(A)$ is a smooth affine variety, we use the following formality theorem: there exists an L_∞-quasi-isomorphism between the Hochschild differential graded Lie algebra $C^\bullet(A)[1]$ and its cohomology complex $H^\bullet(A)[1]$, extending the Hochschild-Kostant-Rosenberg quasi-isomorphism of these complexes. Dolgushev, Tamarkin and Tsygan [22] proved an even stronger statement by showing that the Hochschild complex $C^\bullet(A)$ is formal as a homotopy Gerstenhaber algebra. Consequently, every Poisson structure on a smooth affine variety can be quantized.

In this thesis we drop the smoothness assumption and consider the deformation quantization problem for possibly singular affine toric varieties. In the singular case the Hochschild-Konstant-Rosenberg map is no longer a quasi-isomorphism and thus also the n-th Hochschild cohomology group is no longer isomorphic to the Hodge summand $H^n_{(n)}(A) \cong \mathrm{Hom}_A(\Omega^n_{A|k}, A)$. Therefore, other components of the Hodge decomposition come into play, making the problem of deformation quantization interesting from the cohomological point of view. For arbitrary singularities, many parts of the Hodge decomposition are still unknown. The case of complete intersections has been settled in [30], where Frønsdal and Kontsevich also motivated the problem of deformation quantization on singular varieties. In the toric case Altmann and Sletsjøe [6] computed the Harrison parts of the Hodge decomposition.

Deformation quantization of singular Poisson algebras does not exist in general; see Mathieu [47] for counterexamples. For known results about quantizing singular Poisson algebras we refer the reader to [63] and references therein. The associative deformation theory for complex analytic spaces was developed by Palamodov in [56] and [57]. For recent developments concerning the problem of deformation quantization in derived geometry, see [15].

Studying noncommutative deformations (also called quantizations) of toric varieties is important for constructing and enumerating noncommutative instantons (see [17], [18]), which is closely related to the computation of Donaldson-Thomas invariants on toric threefolds (see [37], [16]).

Considering only commutative deformations of algebras, the whole information about the singularity of A is encoded in the so called versal base space. In the case of complete intersection singularities, the versal base space is obtained by certain perturbations of the defining equations. As soon as we leave the class of complete intersections, computing the versal base space becomes a challenging problem.

For toric surfaces Kollár and Shepherd-Barron [38] showed that there is a correspondence between certain partial resolutions (P-resolutions) and reduced versal base components. Moreover, Arndt [7] obtained equations for the versal base space. Furthermore, in [21] and [67] Christophersen and Stevens give a simpler set of equations for each reduced component of the versal base space. Altmann [4] constructed the versal family for isolated toric Gorenstein singularities.

In [5] Altmann also constructed infinitely many one-parameter deformations for non-isolated three-dimensional toric Gorenstein singularities and explained that the answers to the following questions would provide important information about three-dimensional flips.

1. Which sets of one-parameter families belong to a common irreducible component of the base space?

2. How can those families be combined to find a general fiber of this component?

Note that if $X = \mathrm{Spec}(A)$ is not an isolated singularity, the versal base space is infinite dimensional. However, as long as $T^2_{(1)}(A) < \infty$, we can still present the versal base space as an analytic set of finite definition (see e.g. [69]).

In order to better understand the commutative deformation theory of X, we need to understand the cup product $T^1_{(1)}(A) \times T^1_{(1)}(A) \to T^2_{(1)}(A)$, which will also give us quadratic equations of the versal base space and thus provide a partial answer to the first question above. A formula for computing the cup product for toric varieties that are smooth in codimension 2 was obtained in [3]. Since this formula is especially simple in the case of three-dimensional isolated toric Gorenstein singularities, it helped Altmann to construct the versal base space in [4]. The cup product of toric varieties was also analyzed by Sletsjøe [65].

In recent years there has been a lot of interest in Poisson deformations, i.e., in deformations of a pair consisting of a variety and a Poisson structure on it (see [28], [33], [52], [53], [54]).

1.1 Main results

We now provide an overview of the thesis and state our main results. Some parts of this dissertation have appeared in [27]. We expect that the reader is familiar with the language of algebraic and toric geometry on the level of [34] and [20].

In Chapter 2 we recall definitions and some techniques for computing Hochschild cohomology. Let $H^n(A)$ denote the n-th Hochschild cohomology group of A and let

$$H^n(A) \cong H^n_{(1)}(A) \oplus \cdots \oplus H^n_{(n)}(A)$$

be its Hodge decomposition. The higher André-Quillen cohomology groups $T^{n-i}_{(i)}(A)$ are isomorphic to $H^n_{(i)}(A)$ for $i = 1, ..., n$. Analyzing the Künneth spectral sequence and using Michler's results in [49], [50], give us the following.

Main result 1 (Proposition 2.4.5, Theorem 2.5.9): *Let $X = \mathrm{Spec}(A)$ be smooth in codimension d. For each $i \geq 1$ and $0 \leq j \leq d+1$, we have $T^j_{(i)}(A) \cong \mathrm{Ext}^j_A(\Omega^i_{A|k}, A)$. For reduced isolated hypersurfaces in $\mathbb{A}^N$ of dimension ≥ 2 we obtain that*

$$H^n(A) \cong \begin{cases} \mathrm{Hom}_A(\Omega^n_{A|k}, A) \oplus A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}) & \text{if } n < N \\ A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}) & \text{if } n \geq N. \end{cases}$$

In Chapter 3 we compute the Hochschild cohomology for affine toric varieties. Let $X_\sigma = \mathrm{Spec}(A)$ be an affine toric variety given by a cone $\sigma = \langle a_1, ..., a_N \rangle \subset N_{\mathbb{R}}$. We have $A = k[\sigma^\vee \cap M]$, where M is the dual lattice of the lattice N, and k is a field of characteristic 0. For an element $R \in M$, let $T^{k,R}_{(i)}(A) \cong H^{k+i,R}_{(i)}(A)$ denote the degree R part of the k-th higher André-Quillen cohomology group $T^k_{(i)}(A)$. The results describing $T^{k,R}_{(i)}(A)$ are obtained using spectral sequence arguments on the double complex defined in Section 3.3.

Main result 2 (Theorem 3.4.3): *Let $X_\sigma = \mathrm{Spec}(A)$ be an affine toric variety that is smooth in codimension d. Let $i \geq 1$ be a fixed integer. Then the k-th cohomology group of the complex*

$$0 \to \bar{C}^i_{(i)}(M_k; k) \to \oplus_{j=1}^N \bar{C}^i_{(i)}(\mathrm{Span}_k\, E^R_j; k) \to \cdots \to \oplus_{\tau \leq \sigma, \dim \tau = d+1} \bar{C}^i_{(i)}(\mathrm{Span}_k\, E^R_\tau; k) \qquad (1.1)$$

is isomorphic to $T^{k,-R}_{(i)}(A)$ for $k = 0, ..., d$ ($\bar{C}^i_{(i)}(M_k; k)$ is the degree 0 term). Moreover, if X is an isolated singularity (i.e. $\dim(X) = d+1$), then

$$T^{k,-R}_{(i)}(A) \cong \begin{cases} \mathrm{Coker}\left(\oplus_{\tau \leq \sigma, \dim \tau = d} \bar{C}^i_{(i)}(K^R_\tau; k) \to \bar{C}^i_{(i)}(K^R_\sigma; k)\right) & \text{if } k = \dim(X) \\ H^{k-\dim(X)+i}_{(i)}(K^R_\sigma; k) & \text{if } k \geq \dim(X) + 1. \end{cases}$$

Analyzing the complex (1.1) for $d = 1$ gives us a formula for $T^1_{(i)}(A)$ in the case of toric surfaces (see Section 3.5). For higher dimensional toric varieties we obtain the following. Let

$$\mathbb{A}(R) := [R = 1] = \{a \in N_{\mathbb{R}} \mid \langle a, R \rangle = 1\} \subset N_{\mathbb{R}}$$

be an affine space. We define the cross-cut of σ in degree R to be the polyhedron $Q(R) := \sigma \cap [R = 1] \subset \mathbb{A}(R)$.

Main result 3 (Proposition 3.6.2, Theorem 3.6.7): *If $Q(R)$ lies in a two-dimensional affine space, we have*

$$\dim_k T^{1,-R}_{(i)}(A) = \max\Big\{0, \sum_{j=1}^N V^i_j(R) - \sum_{d_{jk} \in Q(R)} Q^i_{jk}(R) - \binom{n}{i} + s^i_{Q(R)}\Big\}. \qquad (1.2)$$

Moreover, if $X = \mathrm{Spec}(A)$ is an n-dimensional affine cone over a smooth toric Fano variety ($n \geq 3$), then

$$T^1_{(i)}(A) = \begin{cases} N - n & \text{if } i = n-1 \\ 0 & \text{otherwise}. \end{cases}$$

The numbers $V_j^i(R)$, $Q_{jk}^i(R)$ and $s_{Q(R)}^i$ are easily computed and thus the equation (1.2) gives us an explicit formula for $T^1_{(i)}(-R)$ in the case of three-dimensional affine toric varieties (see Subsection 3.6.1).

In Chapter 4 we consider the problem of deformation quantization on singular affine toric varieties. We assume additionally that our field k is also algebraically closed. Using some of the results from Chapter 3, together with Maurer-Cartan formalism, Kontsevich's formality theorem (or more precisely its Corollary 4.3.3) and the GIT quotient construction for an affine toric variety without torus factors, we obtain the following.

Main result 4 (Theorem 4.4.4): *Every Poisson structure on an affine toric variety can be quantized.*

In Chapter 5 we analyze commutative deformations of affine toric varieties. They are controlled by the Harrison differential graded Lie algebra, which has cohomology groups isomorphic to $T^k_{(1)}(A)$, $k \geq 0$.

In particular, we are interested in affine Gorenstein toric varieties, which are obtained by putting a lattice polytope $Q \subset \mathbb{A}$ into the affine hyperplane $\mathbb{A} \times \{1\} \subset \mathbb{A} \times \mathbb{R} =: N_{\mathbb{R}}$ and defining $\sigma := \operatorname{Cone}(Q)$, the cone over Q. Then the canonical degree R^* equals $(\underline{0}, 1)$. Focusing on three-dimensional Gorenstein toric varieties, we arrange the rays $a_1, ..., a_N$ of σ in a cycle and we define $a_{N+1} := a_1$ and $d_j := a_{j+1} - a_j$. Altmann [4] showed that $T^1_{(1)}(-R^*) \cong V/k \cdot \underline{1}$, where $V := \{\underline{t} = (t_1, ..., t_N) \in k^N \mid \sum_{j=1}^N t_j d_j = 0\}$ denotes the set of (generalized) Minkowski summands. The complex (1.1) for $i = 1$ and $R = 2R^*$ is in the case of three-dimensional Gorenstein singularities equal to

$$0 \to N_k \xrightarrow{\psi} N_k^N \xrightarrow{\delta} \oplus_{j=1}^N (N_k/\delta_j d_j) \xrightarrow{\eta} (\operatorname{Span}_k R^*)^* \to 0, \tag{1.3}$$

where $\psi(x) = (x, ..., x)$, $\delta(b_1, ..., b_N) = (b_1 - b_2, b_2 - b_3, ..., b_N - b_1)$, $\eta(q_1, ..., q_N) = \sum_{j=1}^N q_j$ and

$$\delta_j := \begin{cases} 0 & \text{if the 2-face } \langle a_j, a_{j+1} \rangle \text{ is smooth} \\ 1 & \text{if the 2-face } \langle a_j, a_{j+1} \rangle \text{ is not smooth.} \end{cases}$$

Main result 5 (Theorem 5.1.5, Theorem 5.2.3): *Let $X_\sigma = \operatorname{Spec}(A)$ be an arbitrary toric variety and let $R, S \in M$. We give a convex geometric description of the Harrison cup product $T^{1,-R}_{(1)}(A) \times T^{1,-S}_{(1)}(A) \to T^{2,-R-S}_{(1)}(A)$. Focusing on three-dimensional toric Gorenstein singularities, the cup product $T^{1,-R^*}_{(1)}(A) \times T^{1,-2R^*}_{(1)}(A) \to T^{2,-R^*}_{(1)}(A)$ equals the bilinear map*

$$V/(k \cdot \underline{1}) \times V/(k \cdot \underline{1}) \mapsto \ker \eta / \operatorname{im} \delta \tag{1.4}$$

$$(\underline{t}, \underline{s}) \mapsto (s_1 t_1 d_1, ..., s_N t_N d_N).$$

In particular, we show that for three-dimensional Gorenstein isolated singularities our cup product formula agrees with Altmann's formula in [3], which was obtained with different methods.

In Section 5.4, using the cup product formula (1.4) and following Altmann's construction in [4], we conjecture a set of equations of the versal base space of Gorenstein toric singularities in degree $-R^*$. In Section 5.5 we construct a differential graded Lie algebra on the complex (1.3), which extends the cup product formula (1.4).

In Chapter 6 we study Poisson deformations, i.e., deformations of a pair consisting of a variety and a Poisson structure on it.

Main result 6 (Theorem 6.1.3, Proposition 6.2.2): *We construct a differential graded Lie algebra $\mathfrak{g}$ controlling the Poisson deformations. Focusing on toric varieties we also simplify the computation of the Poisson cohomology groups $H^k(\mathfrak{g})$ and the cup product of the Hochschild differential graded Lie algebra $H^2(A) \times H^2(A) \to H^3(A)$.*

2 Differential graded Lie algebras and deformation theory

In this chapter we study differential graded Lie algebras and their applications to deformation theory. In Section 2.1 we recall formal deformation theory, where one of the most important results is Schlessinger's criterion for a functor to have a hull or to be prorepresentable. In Section 2.2 we use the language of differential graded Lie algebras to define the cotangent complex, which is essential for studying deformations of affine varieties. In Section 2.3 we construct the Hochschild differential graded Lie algebra and prove that it controls the deformations of associative algebras. Section 2.4 relates Hochschild cohomology groups in the case of normal affine varieties with Ext groups. Finally, in Section 2.5 we provide an explicit calculation of the Hochschild (co)-homology groups in the case of reduced isolated hypersurfaces.

2.1 Formal deformation theory

Let k be a field of characteristic 0 and let X be a variety, i.e., an integral scheme over k, such that the structure morphism $X \to \mathrm{Spec}(k)$ is separated and of finite type.

Definition 1. A *local deformation* of X is a cartesian diagram

$$\begin{array}{ccc} X & \overset{i}{\hookrightarrow} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \mathrm{Spec}(k) & \longrightarrow & S \end{array}$$

where π is a flat morphism and $S = \mathrm{Spec}(B)$ where B is a local k-algebra with residue field k, and X is identified with the fibre over the closed point.

If $B = k[t]/t^2$ is the ring of dual numbers, then we speak of *a first order deformation.* Given two local deformations of X

$$\begin{array}{ccc} X & \overset{i_1}{\hookrightarrow} & \mathcal{X} \\ \downarrow & & \downarrow \pi_1 \\ \mathrm{Spec}(k) & \longrightarrow & S \end{array} \qquad \begin{array}{ccc} X & \overset{i_2}{\hookrightarrow} & \mathcal{X}' \\ \downarrow & & \downarrow \pi_2 \\ \mathrm{Spec}(k) & \longrightarrow & S \end{array}$$

parametrised by the same base $S = \mathrm{Spec}(B)$, an *isomorphism of local deformations* is defined to be a morphism $f : \mathcal{X} \to \mathcal{X}'$ of schemes over S inducing the identity on the closed fibre, i.e., such that the diagram in Figure 2.1 is commutative.

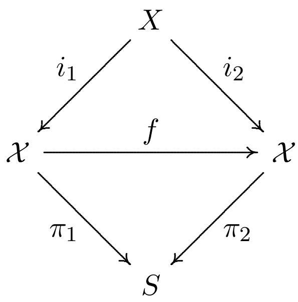


Figure 2.1: A commutative diagram of local deformations

Before we define the above construction as a functor, we need the following definitions. An *Artin ring* is a ring A in which every descending sequence of ideals

$$\cdots \subset I_3 \subset I_2 \subset I_1 \subset A$$

stabilizes, i.e., there exists n such that we have $I_m = I_n$ for all $m \geq n$. Let (R, m_R) and (S, m_S) be local rings. A morphism $f : R \to S$ is a *local morphism* if $f(m_R) \subset m_S$.

Definition 2. Let $\mathcal{A}$ be the category of local Artin k-algebras with the residue field k (with local homomorphisms as morphisms).

Definition 3. The *completion* $\hat{R}$ of a local ring (R, m_R) is the inverse limit of the factor rings

$$\hat{R} := \varprojlim_{n \in \mathbb{N}} (R/m_R^n).$$

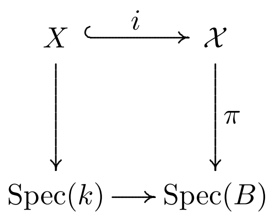
We say that R is complete if the natural morphism $R \to \hat{R}$ is an isomorphism.

Definition 4. Let $\hat{\mathcal{A}}$ be the category of complete noetherian local k-algebras R such that $R_n = R/m_R^n$ is in $\mathcal{A}$ for all $n \in \mathbb{N}$. Note that $\mathcal{A}$ is a subcategory of $\hat{\mathcal{A}}$.

Let $\mathcal{S}$ denote the category of sets.

Definition 5. We define the covariant functor $\hat{\mathrm{Def}}_X : \hat{\mathcal{A}} \to \mathcal{S}$ of local deformations up to isomorphism.

We want to know if this functor is representable, i.e., if there exists a noetherian local k-algebra B and a local deformation



which is universal, i.e., such that any other local deformation (over a base $\mathrm{Spec}(A)$) is obtained by pulling back under a unique $\mathrm{Spec}(A) \to \mathrm{Spec}(B)$.

We first analyze the restriction of $\hat{\mathrm{Def}}_X$ to $\mathcal{A}$.

Definition 6. Let us denote by Def_X the restriction of the functor $\hat{\mathrm{Def}}_X$ to $\mathcal{A}$. We call Def_X the *deformation functor of* X.

We consider a covariant functor $F : \mathcal{A} \to \mathcal{S}$, such that $F(k)$ is a set that contains just one element (we denote this set with $*$).

Definition 7. A covariant functor $F : \mathcal{A} \to \mathcal{S}$ (with $F(k) = *$) is called a *functor of Artin rings.* To every complete local k-algebra R we can associate a functor of Artin rings h_R by

$$h_R(A) := \mathrm{Hom}(R, A).$$

A functor that is isomorphic to h_R for some R is called *prorepresentable.*

Remark 1. Let $R \in \hat{\mathcal{A}}$ and let $A \in \mathcal{A}$ with $m_A^n = 0$ for some n. It holds that $\mathrm{Hom}(R, A) = \mathrm{Hom}(R/m_R^n, A)$.

One of the most important results of classical formal deformation theory is Schlessinger's criterion for a functor F to be pro-representable. Before we recall this criterion we need some definitions.

Note that the category $\mathcal{A}$ has fibered direct products. If $A' \to A$ and $A'' \to A$ are morphisms in $\mathcal{A}$, we take $A' \times_A A''$ to be the set-theoretic fibered product

$$\{(a', a'') \mid a' \text{ and } a'' \text{ have the same image in } A\}.$$

The ring operations extend naturally, giving another object of $\mathcal{A}$, and this object is also the categorical fibered direct product in $\mathcal{A}$.

By ϵ and ϵ_i we will always mean indeterminates annihilated by the maximal ideal, and in particular of square zero (e.g., the algebra $k[\epsilon]$ has dimension 2 and $k[\epsilon_1, \epsilon_2]$ has dimension 3 as a k-vector space).

Definition 8. We call $F(k[\epsilon])$ the *tangent space* of F.

The tangent space of a functor h_R is equal to the dual vector space of m_R/m_R^2.

There is a bijection between the set $\hat{F}(R) := \varprojlim_{n\in\mathbb{N}} F(R/m_R^n)$ and the set of morphisms $\mathrm{Hom}(h_R, F)$ (see [35, Chapter 15]).

Definition 9. Let $R \in \hat{\mathcal{A}}$ and choose $\xi \in \hat{F}(R)$. By above ξ corresponds to a morphism $h_R \to F$. We call such a pair (R, ξ) a *pro-couple.*

If F is pro-representable and (R, ξ) is a pro-couple corresponding to the isomorphism $h_R \to F$, then we say that the pro-couple (R, ξ) *pro-represents* the functor F.

For every $f : R \to S$ we denote

$$\hat{F}(f) : \hat{F}(R) \to \hat{F}(S)$$

to be the map induced by the maps $F(R/m_R^n) \to F(S/m_S^n)$, $n \geq 1$.

Definition 10. A morphism $F \to G$ is called *smooth* if for any surjective morphism $A \to B$ in $\mathcal{A}$, the map

$$F(A) \to F(B) \times_{G(B)} G(A)$$

is surjective.

Definition 11. A functor F is *smooth* if the morphism $F \to *$ is smooth, i.e., if $F(A) \to F(B)$ is surjective for every surjective morphism $A \to B$.

Definition 12. Let (R, ξ) be a pro-couple for $F : \mathcal{A} \to \mathcal{S}$ corresponding to a morphism $h_R \to F$. Then (R, ξ) is called a *hull* of F if the corresponding map $h_R \to F$ is smooth and the induced map

$$\mathrm{Hom}(R, k[\epsilon]) \to F(k[\epsilon])$$

on tangent spaces is bijective.

Definition 13. A *small extension* in $\mathcal{A}$ (resp. $\hat{\mathcal{A}}$) is a short exact sequence

$$e : 0 \to M \to B \to A \to 0,$$

where $B \to A$ is in $\mathcal{A}$ (resp. $\hat{\mathcal{A}}$) and $m_B M = 0$. Thus M is a B/m_B-vector space. A small extension is called *principal* if $\dim_{B/m_B}(M) = 1$.

Definition 14. Given a functor $F : \mathcal{A} \to \mathcal{S}$ and morphisms $f : A' \to A$, $g : A'' \to A$ in $\mathcal{A}$, let $f * g$ be the natural map

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{2.1}$$

Let us introduce Schlessinger's conditions (H_1), (H_2), (H_3) and (H_4).

Definition 15. (H_1) The map (2.1) is surjective if $g : A'' \to A$ is a principal small extension.

(H_2) The map (2.1) is bijective if $A'' = k[\epsilon]$ and $A = k$.

(H_3) Conditions (H_1) and (H_2) hold (which implies that $F(k[\epsilon])$ is a k-vector space) and $F(k[\epsilon])$ is a finite dimensional k-vector space.

(H_4) The map (2.1) is bijective if $g : A'' \to A$ is a principal small extension.

We now present Schlessinger's criterion.

Theorem 2.1.1. *Let $F : \mathcal{A} \to \mathcal{S}$ be a functor of Artin rings. Then F has a hull if and only if F satisfies $(H_1), (H_2)$ and (H_3). Furthermore, F is pro-representable if and only if in addition F satisfies (H_4).*

Proof. See Hartshorne [35, Theorem 16.2]. □

Theorem 2.1.2. *Let X be a scheme over k. Then the functor Def_X (see Definition 6) has a hull under either of the following hypothesis:*

- *X is affine with isolated singularities,*
- *X is projective.*

Proof. See Hartshorne [35, Theorem 18.1]. □

Definition 16. If the pro-couple (R, ξ) of a functor F is a hull, then we say that (R, ξ) is a *versal family* for F.

Remark 2. Note that our definition of a versal family is the same as the definition of a miniversal family in [35] and a semiuniversal couple in [64].

A functor which has infinite-dimensional tangent space does not have a versal family.

Definition 17. Let (R, ξ) and (S, η) be two pro-couples of a functor F. A *morphism of pro-couples*

$$f : (R, \xi) \to (S, \eta)$$

is a morphism $f : R \to S$ in $\hat{\mathcal{A}}$ such that $\hat{F}(f)(\xi) = \eta$. We call f an *isomorphism of pro-couples* if in addition $f : R \to S$ is an isomorphism.

Proposition 2.1.3. *If (R, ξ) and (S, μ) are versal families for F, there exists an isomorphism of versal families $(R, \xi) \cong (S, \mu)$ which is not necessarily unique.*

Proof. See [64, Proposition 2.2.7]. □

Definition 18. If the deformation functor Def_X has a versal family (R, ξ), then R is by Proposition 2.1.3 uniquely determined up to isomorphism and we call R the *versal base space*.

2.2 Differential graded (Lie) algebras and the cotangent complex

In the last thirty years differential graded Lie algebras have become a very important tool in deformation theory. Using the language of differential graded Lie algebras we define the cotangent complex, which plays a crucial role in the deformation theory of affine varieties. We will follow Manetti [45], [46].

2.2.1 Differential graded (Lie) algebras

Definition 19. We denote by $\mathcal{G}$ the category of *$\mathbb{Z}$-graded k-vector spaces*. Objects of $\mathcal{G}$ are k-vector spaces endowed with a $\mathbb{Z}$-graded direct sum decomposition ($V \in \mathcal{G} \implies V = \oplus_{i \in \mathbb{Z}} V_i$). If $a \in V_i \subset V$ for some i, we say that a has degree i and we denote it by $|a| = i$. Morphisms in $\mathcal{G}$ are degree-preserving linear maps.

Given two graded vector spaces $V, W \in G$ we denote by $\mathrm{Hom}_k^n(V, W)$ the vector space of k-linear maps $f : V \to W$ such that $f(V_i) \subset W_{i+n}$ for every $i \in \mathbb{Z}$. Observe that $\mathrm{Hom}_k^0(V, W) = \mathrm{Hom}_{\mathcal{G}}(V, W)$ is the space of morphisms in the category $\mathcal{G}$. Given $V, W \in \mathcal{G}$ we set

$$V \otimes W := \oplus_{i \in \mathbb{Z}} (V \otimes W)_i, \quad \text{where } (V \otimes W)_i = \oplus_{j \in \mathbb{Z}} V_j \otimes W_{i-j},$$
$$\mathrm{Hom}^*(V, W) := \oplus_n \mathrm{Hom}_k^n(V, W).$$

Definition 20. We denote by $\mathcal{DG}$ the category of *$\mathbb{Z}$-graded differential k-vector spaces* (also called complexes of vector spaces). The objects of $\mathcal{DG}$ are pairs (V, d), where $V = \oplus_{i \in \mathbb{Z}} V_i$ is an object of $\mathcal{G}$ and $d : V \to V$ is a linear map called the *differential*, such that $d(V_i) \subset V_{i+1}$ and $d^2 = d \circ d = 0$. Morphisms in $\mathcal{DG}$ are degree-preserving linear maps commuting with the differentials.

We will often consider $\mathcal{G}$ as the full subcategory of $\mathcal{DG}$ whose objects are the complexes $(V, 0)$ with trivial differential.

Given (V, d) in $\mathcal{DG}$ we define $Z^i(V) := \ker(d : V^i \to V^{i+1})$, $B^i(V) := \mathrm{im}(d : V^{i-1} \to V^i)$ and we call $H^i(V) := Z^i(V)/B^i(V)$ the i-th cohomology group of V.

Definition 21. A morphism in $\mathcal{DG}$ is called *quasi-isomorphism* if it induces an isomorphism in cohomology. A differential graded vector space (V, d) is called *acyclic* if

$$H(V) := \oplus_{i \in \mathbb{Z}} H^i(V) = 0.$$

For every integer $n \in \mathbb{Z}$ we denote by $k[n] \in \mathcal{G} \subset \mathcal{DG}$ the object with homogenous components equal to

$$k[n]_i := \begin{cases} k & \text{if } i + n = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Definition 22. Given $n \in \mathbb{Z}$, the shift functor $[n] : \mathcal{DG} \to \mathcal{DG}$ is defined by setting $V[n] = k[n] \otimes V$, $V \in \mathcal{DG}$, $f[n] = Id_{k[n]} \otimes f$, $f \in \mathrm{Mor}_{\mathcal{DG}}$. More informally, the complex $V[n]$ is the complex V with degrees shifted by n, i.e., $V[n]_i = V_{i+n}$, and differentials multiplied by $(-1)^n$.

Definition 23. An *associative graded algebra* is a $\mathbb{Z}$-graded vector space $A = \oplus A_i \in \mathcal{G}$ endowed with a product $A_i \times A_j \to A_{i+j}$ satisfying the properties:

1. $a(bc) = (ab)c$,
2. $a(b + c) = ab + ac, (a + b)c = ac + bc$,
3. $ab = (-1)^{|a||b|} ba$ for a, b homogeneous (Koszul sign convention).

Definition 24. A *differential graded algebra* (dg-algebra for short) is the data of an associative graded algebra A and a k-linear map $d : A \to A$, called *differential*, with the properties:

1. $d(A_n) \subset A_{n+1}$, $d^2 = 0$,
2. $d(ab) = d(a)b + (-1)^{|a|} ad(b)$ (graded Leibnitz rule).

Definition 25. A *differential graded Lie algebra* (dgla for short) is the data of a $\mathbb{Z}$-graded differential vector space $(\mathfrak{g}, d)$ together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ (called bracket) of degree 0 such that

1. $[a, b] = -(-1)^{|a||b|}[b, a]$ (graded skewsymmetry),
2. $[a, [b, c]] = [[a, b], c] + (-1)^{|a||b|}[b, [a, c]]$ (graded Jacobi identity),
3. $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$ (graded Leibnitz rule).

A morphism $f : (\mathfrak{g}, d_A) \to (\mathfrak{h}, d_B)$ of differential graded algebras is a morphism of graded algebras commuting with differentials (i.e. $d_A f_n = f_{n+1} d_B$ for every n).

2.2.2 The Maurer-Cartan equation and gauge equivalence

Definition 26. For a dgla $\mathfrak{g}$ we define the functor of Artin rings $\mathrm{MC}_{\mathfrak{g}} : \mathcal{A} \to \mathcal{S}$ by

$$B \mapsto \big\{ x \in \mathfrak{g}^1 \otimes m_B \mid d(x) + \frac{1}{2}[x, x] = 0 \big\}.$$

$\mathrm{MC}_{\mathfrak{g}}$ is said to be the Maurer-Cartan functor associated to $\mathfrak{g}$. Elements of $\mathrm{MC}_{\mathfrak{g}}(B)$ are the Maurer-Cartan elements of the dgla $\mathfrak{g} \otimes m_B$.

Definition 27. Let $\mathcal{G}$ denote the category of groups and let $\mathfrak{g}$ be a dgla. We define the functor $\mathrm{G}_{\mathfrak{g}} : \mathcal{A} \to \mathcal{G}$ given by

$$B \mapsto \exp(\mathfrak{g}^0 \otimes m_B),$$

where exp is the standard exponential functor on Lie algebras. $\mathrm{G}_{\mathfrak{g}}$ is said to be the *gauge functor* associated to $\mathfrak{g}$.

Remark 3. Note that the functor $\mathrm{G}_{\mathfrak{g}}$ is well defined since $m_B^n = 0$ for some $n \in \mathbb{N}$.

The gauge functor $\mathrm{G}_{\mathfrak{g}}$ acts naturally on the Maurer-Cartan functor $\mathrm{MC}_{\mathfrak{g}}$ by the formula

$$\mathrm{G}_{\mathfrak{g}}(B) \times \mathrm{MC}_{\mathfrak{g}}(B) \to \mathrm{MC}_{\mathfrak{g}}(B)$$

$$(e^b, x) \mapsto x + \sum_{n=0}^{\infty} \frac{[b, \cdot]^n}{(n+1)!}([b, x] - d(b)).$$

This action is called the *gauge action.* Note that the image is indeed an element in $\mathrm{MC}_{\mathfrak{g}}(B)$ (see e.g. Manetti [45]).

Definition 28. Let $\mathfrak{g}$ be a dgla. The *deformation functor of* $\mathfrak{g}$ is the functor of Artin rings $\mathrm{Def}_{\mathfrak{g}} : \mathcal{A} \to \mathcal{S}$ given by

$$B \mapsto \frac{\mathrm{MC}_{\mathfrak{g}}(B)}{G_{\mathfrak{g}}(B)}.$$

We say that a dgla $\mathfrak{g}$ controls a functor F, if $\mathrm{Def}_{\mathfrak{g}} \cong F$ holds.

Example 1. Let $\mathfrak{g}$ be a dgla with $H^1(\mathfrak{g}) < \infty$. To find the solution space of the MC equation for $\mathfrak{g}$ we use the following procedure (also called the power series Ansatz; see [66, pp. 64]). We choose a basis $t_1, ..., t_n$ of $H^1(\mathfrak{g})$ and representatives $\varphi_1, ..., \varphi_n \in \mathfrak{g}^1$ of this basis. We construct the local ring R of the solution space of the MC equation as a quotient of $k[[t_1, ..., t_n]]$. Let $m = (t_1, ..., t_n)$ denote the maximal ideal of $k[[t_1, ..., t_n]]$. Over $R_1 := k[[t_1, ..., t_n]]/m^2$ we have the solution $\sum_{i=1}^{n} t_i \varphi_i$. To find higher order terms we write

$$\varphi = \sum_{|\alpha|>1} t^\alpha \varphi_\alpha,$$

where we use multi-variable power series and multi-index notation ($t = (t_1, ..., t_n)$). The primary obstruction comes from

$$\sum_{|\alpha|=2} t^\alpha d\varphi_\alpha + \frac{1}{2} \sum_{|i|=|j|=1} t^i t^j [\varphi_i, \varphi_j] = 0. \tag{2.2}$$

We can express the class of $[\varphi_i, \varphi_j]$ in $H^2(\mathfrak{g})$ in terms of a basis $\Omega_1, ..., \Omega_s$ as $\sum_k c_{ij}^k \Omega_k$. The equation (2.2) is solvable if and only if

$$g_2^{(k)} := \frac{1}{2} \sum_{|i|=|j|=1} c_{ij}^k t^i t^j = 0,$$

for all $k = 1, ..., s$. Set $R_2 := k[[t_1, ..., t_n]]/(g_2 + m^3)$, where $g_2 = (g_2^{(1)}, ..., g_2^{(s)})$ and continue this procedure as in [66]. In many examples (especially when we are considering deformations of toric varieties) we can find the local ring R of the solution space of the MC equation after finitely many steps.

2.2.3 Differential graded modules

Definition 29. Let (A, d) be a dg-algebra. An *A-dg module* is a differential graded vector space (M, d), together with two associative distributive multiplication maps $A \times M \to M$, $M \times A \to M$ with the properties:

1. $A_i M_j \subset M_{i+j},\ M_i A_j \subset M_{i+j}$,
2. $am = (-1)^{|a||m|} ma$, for homogenous $a \in A$, $m \in M$,
3. $d(am) = d(a)m + (-1)^{|a|} a d(m)$.

Let (A, d_A), (N, d_N) and (M, d_M) be dg-algebras. The tensor product $N \otimes_A M$ is defined as the quotient of $N \otimes_k M$ by the graded submodules generated by all elements $na \otimes m - n \otimes am$. The *tensor product* $N \otimes_A M$ has a natural structure of an A-dg-module with $a(n \otimes m) := an \otimes m$ and the differential

$$d(n \otimes m) = d_N(x) \otimes y + (-1)^q x \otimes d_M(y),$$

with $x \in N$, $|x| = q$, $y \in M$.

Given two A-dg modules (M, d_M), (N, d_N) we denote

$$\mathrm{Hom}^n_A(M, N) := \{f \in \mathrm{Hom}^n_k(M, N) \mid f(am) = f(m)a, m \in M, a \in A\},$$
$$\mathrm{Hom}^*_A(M, N) := \oplus_{n \in \mathbb{Z}} \mathrm{Hom}^n_A(M, N).$$

The graded vector space $\mathrm{Hom}^*_A(M, N)$ has a natural structure of an A-dg-module with $(af)(m) := af(m)$ and the differential

$$d : \mathrm{Hom}^n_A(M, N) \to \mathrm{Hom}^{n+1}_A(M, N), \quad df = d_N \circ f - (-1)^n f \circ d_M.$$

Note that $f \in \mathrm{Hom}^0_A(M, N)$ is a morphism of A-dg-modules if and only if $df = 0$.

Definition 30. A *homotopy* between two morphisms of dg-modules $f, g : M \to N$ is an element $h \in \mathrm{Hom}^{-1}_A(M, N)$ such that $f - g = dh = d_N h + h d_M$. We also say that f is *homotopic* to g.

The relation f is homotopic to g is an equivalence relation.

Definition 31. We say that dg-modules M and N are *homotopically equivalent* if there exist maps $f : M \to N$ and $g : N \to M$ such that $f \circ g$ is homotopic to id_N and $g \circ f$ is homotopic to id_M.

Given a morphism of dg-algebras $B \to A$ and an A-dg-module M we set:

$$\mathrm{Der}^n_B(A, M) := \{\phi \in \mathrm{Hom}^n_k(A, M) \mid \phi(ab) = \phi(a)b + (-1)^{n|a|} a\phi(b), \quad \phi(B) = 0\},$$

$$\mathrm{Der}^*_B(A, M) := \oplus_{n \in \mathbb{Z}} \mathrm{Der}^n_B(A, M).$$

As in the case of Hom^*, there exists a structure of an A-dg-module on $\mathrm{Der}^*_B(A, M)$ with $(a\phi)(b) := a\phi(b)$ and the differential

$$d : \mathrm{Der}^n_B(A, M) \to \mathrm{Der}^{n+1}_B(A, M), \quad d\phi = d_M \phi - (-1)^n \phi d_A.$$

Given $\phi \in \mathrm{Der}^n_B(A, M)$ and $f \in \mathrm{Hom}^m_A(M, N)$ their composition $f\phi$ belongs to $\mathrm{Der}^{n+m}_B(A, N)$.

Proposition 2.2.1. *Let $B \to A$ be a morphism of dg-algebras: there exists an A-dg module $\Omega_{A|B}$ together with a closed derivation $d : A \to \Omega_{A|B}$ (i.e. $\delta d = 0$, where δ is the differential of $\Omega_{A|B}$) of degree 0, such that for every A-dg module M the composition with d gives an isomorphism*

$$\mathrm{Hom}^*_A(\Omega_{A|B}, M) \cong \mathrm{Der}^*_B(A, M).$$

Proof. The construction is similar to the case of algebras. We define the graded vector space

$$F_A = \oplus A dx,$$

where the direct sum runs through homogenous elements $x \in A$. We define $|dx| = |x|$. F_A is an A-dg-module with $a(bdx) := abdx$ and the differential

$$\delta(adx) = \delta adx + (-1)^{|a|} ad(\delta x),$$

where we also denote by δ the differential of A. Note that in particular $\delta(dx) = d(\delta x)$. Let $I \subset F_A$ be the homogenous submodule generated by the elements

$$d(x+y) - dx - dy, \quad d(xy) - x(dy) - (-1)^{|x||y|} y(dx), \quad d(b) \text{ for } b \in B.$$

Since $\delta(I) \subset I$, the quotient $\Omega_{A|B} := F_A/I$ is still an A-dg-module. □

Definition 32. The module $\Omega_{A|B}$ is called the *module of relative Kähler differentials of A over B*.

For basic properties of the module of Kähler differentials in the case of algebras see Matsumura [48].

2.2.4 The cotangent complex

In this subsection we define the cotangent complex using differential graded algebras and their semifree resolutions. Note that original idea by Quillen [61] was to define it using simplicial algebras and free simplicial resolutions. Palamodov [55] used the Tyurina resolution.

Definition 33. A dg-algebra (R, s) with differential s is called *semifree* if:

1. The underlying graded algebra R is a polynomial algebra over k: $k[x_i \mid i \in I]$.
2. There exists a filtration $\emptyset = I(0) \subset I(1) \subset \cdots, \cup_{n \in \mathbb{N}} I(n) = I$, such that $s(x_i) \in R(n)$ for every $i \in I(n+1)$, where by definition $R(n) = \mathbb{K}[x_i \mid i \in I(n)]$.

Note that $R(0) = k$ and $R = \cup R(n)$.

Definition 34. A *semifree resolution* of a dg-algebra A is a surjective quasi-isomorphism $R \to A$ where R is a semifree dg-algebra.

Theorem 2.2.2. *Every dg-algebra A admits a semifree resolution.*

Proof. We prove it just in the case of algebras (i.e. A has only one non-zero degree A_0); for a general proof see Manetti [46]. We can find a surjective map $P_0 := k[x_{i_0} \mid i_0 \in I_0] \to A$, for some index set I_0 (mapping x_{i_0} to the generators of A). Now we take generators a_{i_1}, $i \in I_1$ of the kernel of the above map, and we define $P_{-1} := k[x_{i_1} \mid i_1 \in I_1] \to P_0$, mapping x_{i_1} to a_{i_1}. We continue with this procedure and we get that the complex

$$\cdots P_{-2} \to P_{-1} \to P_0 \to 0$$

is quasi-isomorphic to $A = A_0$. Moreover, we see that $P_\bullet \to A$ is a semifree resolution with the filtration $I(n+1) := I_{i_0} \cup \cdots \cup I_{i_n}$, for $n \geq 0$. □

Proposition 2.2.3. *Let $R \to A$ be a semifree resolution of A. The homotopy class of the A-dg module $\mathbb{L}_{A|k} := \Omega_{R|k} \otimes_R A$ is independent from the choice of the resolution.*

Proof. See Manetti [46]. □

Definition 35. We call $\mathbb{L}_{A|k}$ the *cotangent complex* of A.

It is important to choose a semifree resolution as we will see in the following example.

Example 2. Let A be the dg-algebra $k[x] \xrightarrow{x} k[x]$, which is non-zero in degrees -1 and 0. There exists a surjective quasi-isomorphism between A and the dg-algebra that have k in degree 0 as the only non-zero degree. We have $H^0(\Omega_{A|k}) \neq 0$ since $\delta(dx) = d(\delta x)$ holds and thus we can not get dx in the image. Thus we obtain that $\Omega_{k|k} = 0$ is not in the same homotopy class as $\Omega_{A|k}$. The problem is that A is not a semifree resolution of k.

In the next example we compute the cotangent complex in the case of reduced hypersurfaces.

Example 3. Let $X = \mathrm{Spec}(A)$ be a reduced hypersurface, where

$$A = k[x_1, ..., x_N]/(f(x_1, ..., x_N)).$$

A semifree resolution of A is given by $R = k[x_1, ..., x_n, y]$, where y has degree -1 and x_i have degree 0 for all i. The differential s is given by $s(y) = f$. We have

$$\Omega_{R|k} \cong Rdx_1 \oplus \cdots \oplus Rdx_n \oplus Rdy$$

and

$$s(dy) = d(s(y)) = d(f) = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

The cotangent complex $\mathbb{L}_{A|k}$ is therefore

$$0 \to Ady \xrightarrow{s} \oplus_{i=1}^n Adx_i \to 0.$$

Definition 36. Let $X = \mathrm{Spec}(A)$ and we look on the cotangent complex $\mathbb{L}_{A|k}$ as a chain complex (terms with degree $-i$ become terms with degree i). The n-th homology group of the cotangent complex $\mathbb{L}_{A|k}$ is called the n-th *Andre-Quillen homology group* and denoted by

$$T_n(A) := H_n(\mathbb{L}_{A|k}).$$

The n-th *Andre-Quillen cohomology group* is the n-th cohomology group of $\mathrm{Hom}_A(\mathbb{L}_{A|k}, A)$, denoted by

$$T^n(A) := H^n(\mathrm{Hom}_A(\mathbb{L}_{A|k}, A)).$$

Remark 4. For $i = 0, 1, 2$, $T^i(A)$ has the same meaning as in books of Hartshorne [35] and Sernesi [64] (they use the notation $T^i(A|k, A)$).

2.3 The Hochschild differential graded Lie algebra

In this section we will obtain modules $T^i(A)$ as a cohomology groups of another complex, called the Harrison complex, which is quasi-isomorphic to the cotangent complex. Moreover, the Harrison complex is the subcomplex of the Hochschild complex. We will see their role in deformation theory.

2.3.1 The Hochschild complex

Let A be an associative algebra. Consider the A-module $C_n(A) := A \otimes A^{\otimes n}$ (where $\otimes = \otimes_k$ and $A^{\otimes n} = A \otimes \cdots \otimes A$, n factors). It is an A-module through multiplication on the left A factor.

Definition 37. The *Hochschild boundary* is the k-linear map $\partial : C_n = A \otimes A^{\otimes n} \to C_{n-1} = A \otimes A^{\otimes(n-1)}$, given by the formula

$$\partial(a, a_1, ..., a_n) := \sum_{i=0}^{n} (-1)^i d_i(a, a_1, ..., a_n),$$

where

$$\begin{aligned} d_0(a, a_1, ..., a_n) &:= (aa_1, a_2, ..., a_n) \\ d_i(a, a_1, ..., a_n) &:= (a, a_1, ..., a_i a_{i+1}, ..., a_n) \quad \text{for} \quad 1 \leq i < n, \\ d_n(a, a_1, ..., a_n) &:= (a_n a, a_1, ..., a_{n-1}). \end{aligned}$$

It holds that $\partial \circ \partial = 0$ and thus we get the complex $C_\bullet(A)$ that is called the *Hochschild chain complex.* The corresponding homology groups are called *Hochschild homology groups* and denoted by $H_\bullet(A)$. The complex $C^\bullet(A)$, where $C^n(A)$ is the space of k-linear maps $f : A^{\otimes n} \to A$, is called the *Hochschild (cochain) complex.* Note that every element $\phi \in \mathrm{Hom}_A(C_n, A)$ is completely determined by the k-linear map $f : A^{\otimes n} \to A$:

$$\phi(a, a_1, ..., a_n) = af(a_1, ..., a_n).$$

The differential is given by

$$\begin{aligned} (df)(a_1 \otimes \cdots \otimes a_n) := \quad & a_1 f(a_2 \otimes \cdots \otimes a_n) + \\ & \textstyle\sum_{i=1}^{n-1} (-1)^i f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) + \\ & (-1)^n f(a_1 \otimes \cdots \otimes a_{n-1}) a_n. \end{aligned}$$

The corresponding cohomology groups are called *Hochschild cohomology groups* and denoted by $H^\bullet(A)$.

Definition 38. The *circle product* of Hochschild cochains $f \in C^m(A)$, $g \in C^n(A)$ is the element $f \circ g \in C^{m+n-1}(A)$ given by

$$f \circ g(a_1 \otimes \cdots \otimes a_{m+n-1}) := \sum_{i=1}^{m} (-1)^{(i-1)(n+1)} f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}).$$

Definition 39. The *Gerstenhaber bracket* $[f, g]$ of $f \in C^m(A)$, $g \in C^n(A)$ is

$$[f, g] := f \circ g - (-1)^{(m+1)(n+1)} g \circ f.$$

Proposition 2.3.1. *It holds that* $d[f,g] = [f, dg] + (-1)^{n+1}[df, g]$.

Proof. See Gerstenhaber [31]. □

Lemma 2.3.2. *The Gerstenhaber bracket defines a dgla structure on the shifted complex* $\mathfrak{g} := C^\bullet(A)[1]$.

Proof. Since we are shifting the complex, we also change the differential (by Definition 22). It turns out that the shifted differential $d_\mathfrak{g}$ is equal to $d_\mathfrak{g} = [m, \cdot]$, where $m \in C^2(A)$ belongs to algebra multiplication ($m(a,b) = a \cdot b$). Simple computation shows that if $f \in C^n(A)$ with n odd, we have $d_\mathfrak{g}(f) = [m, f] = d(f)$. If n is even, then we have $d_\mathfrak{g}(f) = [m, f] = -d(f)$. Note that $[\cdot,\cdot]$ defines a graded Lie algebra structure on $\mathfrak{g}$ (see Schedler [63, Remark 4.1.4]) and that the condition $d_\mathfrak{g}[f,g] = [d_\mathfrak{g} f, g] + (-1)^{|f|}[f, d_\mathfrak{g} g]$ is equivalent to the graded Jacobi identity, which is satisfied. □

2.3.2 The Hodge decomposition of the Hochschild complex

We will now recall the construction of the decomposition of the Hochschild complex from Gerstenhaber-Schack [32].

In the group ring $\mathbb{Q}[S_n]$ of the permutation group S_n one defines the shuffle $s_{i,n-i}$ to be $\sum(\operatorname{sgn}\pi)\pi$, where the sum is taken over those permutations $\pi \in S_n$ such that

$$\pi(1) < \pi(2) < \cdots < \pi(i)$$

and

$$\pi(i+1) < \pi(i+2) < \cdots < \pi(n).$$

We assume that $0 < i < n$, setting $s_{0,n} = s_{n,0} = 0$. We denote $a_1 \otimes \cdots \otimes a_n \in A^{\otimes n}$ by $(a_1, ..., a_n)$ and define an action of the permutations group S_n on $A^{\otimes n}$ as follows: $\pi(a_1, ..., a_n) = (a_{\pi^{-1}1}, ..., a_{\pi^{-1}n})$, $\pi \in S_n$. With this action we can consider $A^{\otimes n}$ as a $\mathbb{Q}[S_n]$-module.

Theorem 2.3.3. *There are canonical decompositions*

$$H_n(A) \cong H_n^{(1)}(A) \oplus \cdots \oplus H_n^{(n)}(A),$$

$$H^n(A) \cong H^n_{(1)}(A) \oplus \cdots \oplus H^n_{(n)}(A),$$

which are also known as the Hodge decompositions of the Hochschild (co-)homology.

We sketch the proof following [32], where they use Barr's theorem (see [10]): let $s_n := \sum_{i=1}^{n-1} s_{i,n-i}$, then $\partial s_n = s_{n-1}\partial$ holds.

An element of a finite-dimensional algebra over a field must be a root of some monic polynomial with coefficients in that field. The polynomial of the lowest degree is called the *minimal polynomial.* The next proposition describes the minimal polynomial of s_n as an element of $\mathbb{Q}$-algebra $\mathbb{Q}[S_n]$.

Proposition 2.3.4. *The minimal polynomial of* s_n *is*

$$\mu_n(x) = \prod_{i=1}^{n}(x - (2^i - 2)) = (x - (2^n - 2))\mu_{n-1}(x).$$

Proof. See [32]. □

Thus μ_n has the form $(x-\lambda_1)\cdots(x-\lambda_n)$, where $\lambda_i = 2^i - 2$. Let $e_n(j)$ be the j-th Lagrange interpolation polynomial evaluated at s_n, i.e.,

$$e_n(j) = \big(\prod_{i\neq j} \lambda_j - \lambda_i\big)^{-1} \prod_{i\neq j}(s_n - \lambda_i).$$

Proposition 2.3.5. *The $e_n(j)$ are mutually orthogonal idempotents whose sum is the unit element. Moreover, in $\mathbb{Q}[S_n]$ it holds that*

$$\lambda_1 e_n(1) + \lambda_2 e_n(2) + \cdots + \lambda_n e_n(n) = s_n. \tag{2.3}$$

Proof. Following [32, Theorem 1.2]: multiplication by s_n is an operator on the n-dimensional subspace of $\mathbb{Q}[S_n]$ spanned by $1, s_n, s_n^2, ..., s_n^{n-1}$. It has n distinct eigenvalues $\lambda_1, ..., \lambda_n$. With the choice of an eigenvector basis is this multiplication representable by the $n \times n$ matrix with λ_i on the diagonal. The proof easily follows. □

From Proposition 2.3.5 we get the decomposition $C_n(A) = \oplus_j e_n(j) C_n(A)$ and since $\partial e_n(j) = e_{n-1}(j)\partial$ holds (see [32, Theorem 1.3]), we also obtain the decomposition of $H_\bullet(A)$. We denote $C_n^{(j)}(A) := e_n(j)C_n(A)$ and the corresponding homology groups of the complex $C_\bullet^{(j)}$ by $H_n^{(j)}(A)$. We define $C^n_{(j)}(A) := \{f \in C^n(A) \mid f \circ s_n = (2^j - 2)f\}$. From Proposition 2.3.5 we obtain that $C^n(A) = C^n_{(1)}(A) \oplus \cdots \oplus C^n_{(n)}(A)$. The Hochschild differential d respects this decomposition and we denote the cohomology groups of the subcomplex $C^\bullet_{(j)}(A)$ by $H^n_{(j)}(A)$. We obtain the decomposition of $H^\bullet(A)$ and thus conclude the proof of Theorem 2.3.3.

Example 4. $C^2_{(1)}(A) = \{f \in C^2(A) \mid f(a,b) = f(b,a)\}$, since $s_2 = s_{1,1} = id - (12)$ and thus $f \circ s_2(a,b) = 0$ means $f(a,b) - f(b,a) = 0$. $C^2_{(2)}(A) = \{f \in C^2(A) \mid f(a,b) = -f(b,a)\}$, since $f \circ s_2(a,b) = 2f(a,b)$ means $f(a,b) = -f(b,a)$.

The following result is classical.

Proposition 2.3.6. *It holds that $H_n^{(n)}(A) \cong \Omega^n_{A|k}$, the n-th exterior power of the module of Kähler differentials. If $X = \operatorname{Spec}(A)$ is smooth, then $H_n(A) \cong H_n^{(n)}(A)$.*

Proof. See Loday [43, Theorem 4.5.12] and Weibel [70, Section 9.4]. □

Definition 40. The complex $C^\bullet_{(1)}(A)$ is called the *Harrison complex* and its cohomology groups are called the *Harrison cohomology groups*, denoted by

$$\mathrm{Har}^n(A) := H^n_{(1)}(A).$$

Remark 5. Note that the Gerstenhaber bracket is not graded product with respect to the Hodge decomposition, i.e., in general it does not hold that $[\cdot,\cdot] : H^{n+1}_{(j+1)}(A) \times H^{m+1}_{(k+1)}(A) \to H^{n+m+1}_{(j+k+1)}$. The image can be bigger (Bergeron and Wolfgang [11] gave counterexamples). On the other hand for $j = k = 0$ we have $[\cdot,\cdot] : H^{n+1}_{(1)}(A) \times H^{m+1}_{(1)}(A) \to H^{n+m+1}_{(1)}(A)$, which give us an important differential graded Lie algebra as we will see in the following.

Proposition 2.3.7. *The Gerstenhaber bracket induces the dgla structure on the complex $C^\bullet_{(1)}(A)[1]$.*

Proof. See [10]. □

Definition 41. The dgla from Proposition 2.3.7 is called the *Harrison dgla* and denoted by $C^\bullet_{(1)}(A)[1]$.

2.3.3 Relations between the Hochschild and cotangent complex

The following proposition relates the cotangent and Harrison complex.

Proposition 2.3.8. *The complex $\mathbb{L}_{A|k}[1]$ is quasi-isomorphic to the Harrison chain complex $C_\bullet^{(1)}(A)$.*

Proof. See Quillen [61] or Loday [43, Proposition 4.5.13]. □

There exist some "operations" on the cotangent complex to get complexes that are quasi-isomorphic to $C^\bullet_{(i)}(A)$ for $i > 1$. This can be done using the *derived exterior powers* $\wedge^i\mathbb{L}_{A|k}$ of a cotangent complex $\mathbb{L}_{A|k}$ (see Illusie [36], Loday [43, Section 3.5.4] or Buchweitz-Flenner [13], [14] for definitions). We only define the derived exterior power of a complex with two non-zero terms (by Example 3 we know that this is the case for $\mathbb{L}_{A|k}$, where A is the algebra of regular functions of a reduced hypersurface).

Definition 42. Let $d : L \to E$ be a morphism of locally free $\mathcal{O}_X$-modules on a scheme X, where L has rank 1 and E has finite rank. Let $\mathcal{K} : L \to E$ be the chain complex, with E lying in the degree 0. We define the *derived exterior power* $\wedge^q(\mathcal{K})$ of the complex $\mathcal{K}$ to be the chain complex

$$L^{\otimes q} \to E \otimes L^{\otimes q-1} \to \cdots \to \wedge^{q-n}E \otimes L^{\otimes n} \to \cdots \to \wedge^q E,$$

with the differentials $d_n(x_0 \otimes x_1^{\otimes n}) = (x_0 \wedge dx_1) \otimes x_1^{\otimes(n-1)}$, where $\wedge^q E$ is degree 0 term.

Proposition 2.3.9. *Definition 42 agrees with the general definition of the derived exterior power given in [36].*

Proof. See Saito [62, Chapter 4]. □

We define *higher Andre-Quillen homology groups* $T_n^{(i)}(A)$ for $i \geq 1$ by putting

$$T_n^{(i)}(A) := H_n(\wedge^i(\mathbb{L}_{A|k})).$$

We also define *higher Andre-Quillen cohomology groups*

$$T^n_{(i)}(A) := H^n(\mathrm{Hom}_A(\wedge^i\mathbb{L}_{A|k}, A)).$$

With our notation $T_n(A) = T_n^{(1)}(A)$ and $T^n(A) = T^n_{(1)}(A)$ hold.

Theorem 2.3.10. *The complexes $\wedge^i(\mathbb{L}_{A|k})[i]$ and $C_\bullet^{(i)}(A)$ are quasi-isomorphic for each $i = 1, ..., n$ and it holds that*

$$\mathrm{Har}^n(A) = H^n_{(1)}(A) \cong T^{n-1}_{(1)}(A)$$

or more generally

$$H^n_{(i)}(A) \cong T^{n-i}_{(i)}(A).$$

Proof. See Quillen [61] or Loday [43, Proposition 4.5.13]. □

2.3.4 The Hochschild cohomology and deformation theory

This subsection is very classical. We follow [63]. Recall that a Hochschild two-cocycle is an element $\gamma \in \mathrm{Hom}_k(A \otimes A, A)$, satisfying

$$a\gamma(b \otimes c) - \gamma(ab \otimes c) + \gamma(a \otimes bc) - \gamma(a \otimes b)c = 0. \tag{2.4}$$

This has a nice interpretation in terms of infinitesimal deformations.

Definition 43. An infinitesimal deformation of an associative algebra A is an algebra $A_\epsilon := (A[\epsilon]/(\epsilon^2), *)$ such that $a * b \cong ab \pmod{\epsilon}$.

We say that two infinitesimal deformations γ_1, γ_2 are equivalent if there is a $k[\epsilon]/(\epsilon^2)$-module automorphism of A_ϵ which is the identity modulo ϵ and maps γ_1 to γ_2. Such a map has the form $\phi := \mathrm{id} + \epsilon \cdot \phi_1$ for some linear map $\phi_1 : A \to A$, i.e., $\phi_1 \in C^1(A)$. It holds that

$$\phi^{-1}(\phi(a) *_\gamma \phi(b)) = a *_{\gamma + d\phi_1} b.$$

Proposition 2.3.11. $H^2(A)$ *is the vector space of equivalence classes of infinitesimal deformations of* A.

Proof. An infinitesimal deformation is given by a linear map $\gamma : A \otimes A \to A$, by the formula

$$a *_\gamma b = ab + \epsilon\gamma(a \otimes b).$$

Then the associativity condition of $*_\gamma$ in $(A[\epsilon]/(\epsilon^2), *_\gamma)$ is exactly (2.4). The above computation also shows us that equivalence classes agree. □

Remark 6. Starting with a Harrison cocycle gives us commutativity of the above star product.

Definition 44. A *one-parameter formal deformation* of an associative algebra B is an associative algebra $B_\hbar = (B[[\hbar]], *)$, such that

$$a * b = ab \pmod{\hbar},$$

for each $a, b \in B$. We require that $*$ is associative, $k[[\hbar]]$-bilinear and continuous, which means that

$$\Big(\sum_{m \geq 0} b_m \hbar^m\Big) * \Big(\sum_{n \geq 0} c_n \hbar^n\Big) = \sum_{m,n \geq 0} (b_m * c_n)\hbar^{m+n}.$$

Suppose now that we have an infinitesimal deformation given by $\gamma_1 : A \otimes A \to A$. To extend this to a second-order deformation, we require $\gamma_2 : A \otimes A \to A$, such that

$$a * b := ab + \epsilon\gamma_1(a \otimes b) + \epsilon^2\gamma_2(a \otimes b)$$

defines an associative product on $A \otimes k[\epsilon]/\epsilon^3$.

Looking at the new equation in second degree, this can be written as

$$a\gamma_2(b \otimes c) - \gamma_2(ab \otimes c) + \gamma_2(a \otimes bc) - \gamma_2(a \otimes b)c = \gamma_1(\gamma_1(a \otimes b) \otimes c) - \gamma_1(a \otimes \gamma_1(b \otimes c)). \tag{2.5}$$

The LHS is $d\gamma_2(a \otimes b \otimes c)$, so the condition for γ_2 to exists is exactly that the RHS is a Hochshild coboundary. Moreover, the RHS is equal to $\frac{1}{2}[\gamma_1, \gamma_1]$. So this element defines a class of $H^3(A)$ which is the *obstruction* to extending the above infinitesimal deformation to a second-order deformation.

Remark 7. More generally we can consider n-th order deformation, i.e., a deformation over $k[\epsilon]/(\epsilon^{n+1})$. We can show that the obstruction to extending an n-th order deformation $\sum_{i=1}^{n} \epsilon^i \gamma_i$ (where here $\epsilon^{n+1} = 0$) to an $(n+1)$-st order deformation $\sum_{i=1}^{n+1} \epsilon^i \gamma_i$ (now setting $\epsilon^{n+2} = 0$), is also a class in $H^3(A)$.

Corollary 2.3.12. *If $H^3(A) = 0$, then all first-order deformations extend to a one-parameter formal deformation.*

2.3.5 Deformations of associative algebras

We consider the following deformation problem.

Definition 45. A *deformation* of A over an Artin ring B is a pair (A', π), where A' is a B-algebra and $\pi : A' \otimes_B k \to A$ is an isomorphism of k-algebras. Two such deformations (A', π_1) and (A'', π_2) are *equivalent* if there exists an isomorphism of B-algebras $\phi : A' \to A''$ such that it is compatible with π_1 and π_2, i.e., such that $\pi_1 = \pi_2 \circ (\phi \otimes_B k)$.

A functor that encodes this deformation problem is

$$\mathrm{Def}_A : \mathcal{A} \to \mathcal{S}$$

$$B \mapsto \{\text{deformations of } A \text{ over } B\}/\sim .$$

It is a well known fact that this deformation problem is controlled by the Hochschild dgla. In the following we will give a complete proof. Some ideas are taken from [63, Sections 4.3,4.4] and [45].

Lemma 2.3.13. *Let $\mathfrak{g}$ be a dgla and let $\xi \in \mathrm{MC}(\mathfrak{g})$. The map $d^\xi : y \mapsto dy + [\xi, y]$ defines a new differential on $\mathfrak{g}$. Moreover, $(\mathfrak{g}, d^\xi, [\cdot, \cdot])$ is also a dgla.*

Proof. An explicit verification, see Schedler [63, Proposition 4.2.3]. □

Definition 46. We call the dgla $(\mathfrak{g}, d^\xi, [\cdot, \cdot])$ given in Lemma 2.3.13 the *twist by* ξ, and denote it by $\mathfrak{g}^\xi$.

Lemma 2.3.14. *Maurer-Cartan elements of $\mathfrak{g}$ are in bijection with Maurer-Cartan elements of $\mathfrak{g}^\xi$ by the correspondence*

$$\xi + \eta \in \mathfrak{g} \leftrightarrow \eta \in \mathfrak{g}^\xi.$$

Proof. We immediately see that $d^\xi(\eta) + \frac{1}{2}[\eta, \eta] = d(\xi+\eta) + \frac{1}{2}[\xi+\eta, \xi+\eta]$, using that $d\xi + \frac{1}{2}[\xi, \xi] = 0$. □

Definition 47. Let V be a vector space. We denote by $C^n(V)$ the space of k-linear maps $V^{\otimes n} \to V$. The $C^\bullet(V)[1]$ is a dgla with Gerstenhaber bracket and zero differential.

Lemma 2.3.15. *Let V be a vector space. Giving an associate product on V is the same as giving an element $\mu \in C^2(V)$ satisfying $\frac{1}{2}[\mu, \mu] = 0$, which is the MC equation for the dgla $C^\bullet(V)[1]$.*

Proof. We define the multiplication on V by $a \cdot b := \mu(a, b)$. It holds that $(ab)c - a(bc) = \frac{1}{2}[\mu, \mu]$. The dgla $C^\bullet(V)[1]$ has trivial differential and thus $\frac{1}{2}[\mu, \mu]$ is the MC equation. □

Lemma 2.3.16. *Let A be an algebra. We set A_0 to be as a vector space equal to A but viewed as an algebra with trivial multiplication. Let $\mu \in C^2(A_0)$ represent the multiplication on A. It holds that $C^\bullet(A)[1] = C^\bullet(A_0)[1]^\mu$.*

Proof. It follows from Lemma 2.3.2, since the differential on A is given by $d = [\mu, \cdot]$. □

Lemma 2.3.17. *Let B be an Artin ring. MC elements of $C^\bullet(A \otimes m_B)[1]$ are in bijection with associative products of the vector space $A_0 \otimes B$, giving the known product on A modulo m_B.*

Proof. Let $\mu \in C^2(A_0)$ represent the multiplication on A. Then associative products of the vector space $A_0 \otimes B$, giving the known product on A modulo m_B are given by

$$[\mu + \xi, \mu + \xi] = 0 \tag{2.6}$$

for $\xi \in C^2(A \otimes m_B)$. Since $[\mu, \mu] = 0$ and the differential on $C^\bullet(A \otimes m_B)[1]$ is given by $[\mu, \cdot]$, we see that equation (2.6) give us an MC element ξ. We can also reverse this proof. □

Proposition 2.3.18. *The Hochschild dgla $C^\bullet(A)[1]$ controls the functor Def_A, i.e., the deformation functor of $C^\bullet(A)[1]$ is isomorphic to Def_A.*

Proof. Let us for short denote $\mathfrak{g} := C^\bullet(A)[1]$. Elements of $\mathrm{MC}_\mathfrak{g}(B)$ are the Maurer-Cartan elements of the dgla $\mathfrak{g} \otimes m_B$. By Lemma 2.3.17 there exists a bijection between elements of $\mathrm{MC}_\mathfrak{g}(B)$ and associative products of the vector space $A_0 \otimes B$, giving the known product on A modulo m_B.

To conclude the proof we need to show that two products $*$ and $*'$ on $A_0 \otimes B$ are equivalent (in the sense of Definition 45) if and only if the corresponding elements $\gamma, \gamma' \in \mathrm{MC}_\mathfrak{g}(B)$ are gauge equivalent. If the products are equivalent we can easily see that there exists $\alpha \in \mathfrak{g}^0 \otimes m_B$ such that

$$a *' b = \exp(\alpha)(\exp(-\alpha)(a) * \exp(-\alpha)(b)). \tag{2.7}$$

As before let $\mu \in C^2(A_0)$ denote the multiplication on A.

Rewriting (2.7) gives us

$$(\mu + \gamma')(a \otimes b) = \exp(\alpha)(\exp(-\alpha)(a) * \exp(-\alpha)(b)) = \exp(\mathrm{ad}\,\alpha)(\mu + \gamma)(a \otimes b),$$

where the last equality follows from basic theory of Lie groups (see [63, Section 4.4]).

Thus it follows that

$$\begin{gathered}(\mu + \gamma') = \exp(\mathrm{ad}\,\alpha)(\mu + \gamma) = \\ \mu + \gamma + \sum_{i=0}^{\infty} \frac{(\mathrm{ad}\,\alpha)^n}{(n+1)!}([\alpha, \mu + \gamma]) = \\ \mu + \gamma + \sum_{i=0}^{\infty} \frac{(\mathrm{ad}\,\alpha)^n}{(n+1)!}([\alpha, \gamma] - d\alpha),\end{gathered}$$

where we used that $d\alpha = [\mu, \alpha] = -[\alpha, \mu]$. We see that γ and γ' are gauge equivalent. We can also reverse the argument and show the other direction. □

2.3.6 Deformations of commutative algebras

Consider the following deformation problem. Let A be a commutative algebra (by that we always mean a commutative and associative algebra). A *commutative deformation* of A over an Artin ring B is a pair (A', π), where A' is a commutative B-algebra, such that the natural map $m_B \otimes_B A' \to A'$ is injective and $\pi : A' \otimes_B k \to A$ is an isomorphism of k-algebras. Two such deformations (A', π_1) and (A'', π_2) are *equivalent* if there exists an isomorphism of B-algebras $\phi : A' \to A''$ such that it is compatible with π_1 and π_2, i.e., such that $\pi_1 = \pi_2 \circ (\phi \otimes_B k)$. A functor that encodes this deformation problem is

$$\mathrm{CDef}_A : \mathcal{A} \to \mathcal{S}$$

$$B \mapsto \{\text{commutative deformations of } A \text{ over } B\}/\sim .$$

Lemma 2.3.19. *A' is flat over B.*

Proof. It is enough to prove that $\mathrm{Tor}_1^B(k, A') = 0$ by [23, Theorem 6.8]. After tensoring the exact sequence

$$0 \to m_B \to B \to k \to 0$$

with A' we obtain

$$0 \to \mathrm{Tor}_1^B(k, A') \to m_B \otimes_B A' \to B \otimes_B A' \to k \otimes_B A' \to 0.$$

By the assumption the map $m_B \otimes_B A' \to B \otimes_B A'$ is injective and thus $\mathrm{Tor}_1^B(k, A') = 0$. □

Corollary 2.3.20. *Let $X = \mathrm{Spec}(A)$. Functors CDef_A and Def_X are isomorphic.*

Proposition 2.3.21. *The Harrison dgla $C^\bullet_{(1)}(A)[1]$ controls the functor CDef_A, i.e., the deformation functor of $C^\bullet_{(1)}(A)[1]$ is isomorphic to CDef_A.*

Proof. The proof is very similar to the proof of Proposition 2.3.18. Commutativity we get by restricting $C^2(A)$ to $C^2_{(1)}(A)$ (see Example 4). Other steps are the same. □

2.4 The Hochschild cohomology of normal affine varieties

Not much is known for the groups $H^n_{(i)}(A)$ in the case when $i \neq 1, n$. In this subsection we show that when A is normal (i.e. the algebra of regular functions on a normal variety) we can say more about other parts.

Lemma 2.4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be an exact covariant functor. If (A, d) is a chain (resp. cochain) complex in $\mathcal{A}$, then*

$$H_\bullet F(A) \cong F(H_\bullet(A)),$$

respectively

$$H^\bullet F(A) \cong F(H^\bullet(A)).$$

A similar statement holds also for a contravariant functor.

Proposition 2.4.2. *Let $P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \cdots$ be a complex of projective A-modules. Then there is a first-quadrant spectral sequence*

$$E_2^{p,q} = \operatorname{Ext}_A^p(H_q(P_\bullet), A) \Rightarrow H^{p+q}(\operatorname{Hom}(P_\bullet, A)),$$

with differentials $d_2 : E_2^{p,q} \to E_2^{p-1,q+2}$.

Proof. Let $A \to Q_\bullet$ be an injective resolution and consider the first-quadrant double complex $\operatorname{Hom}(P_\bullet, Q_\bullet)$. We have two spectral sequences. First one gives us

$$E_1^{pq} = H^q(\operatorname{Hom}(P_p, Q_\bullet)) = \operatorname{Hom}(P_p, H_q(Q_\bullet)),$$

where we used the projectivity of P_p and Lemma 2.4.1. Thus we get $E_\infty^{pq} = E_2^{pq} = H^p(\operatorname{Hom}(P_\bullet), A)$ if $q = 0$ and 0 otherwise. From this we see that $H^n(\operatorname{tot}^\bullet(\operatorname{Hom}(P_\bullet, Q_\bullet))) = H^n(\operatorname{Hom}(P_\bullet), A)$.

The second spectral sequence gives us

$$E_1^{pq} = H^q(\operatorname{Hom}(P_\bullet, Q_p)) = \operatorname{Hom}(H_q(P_\bullet), Q_p),$$

where we used the injectivity of Q_p and Lemma 2.4.1. Hence

$$E_2^{pq} = \operatorname{Ext}_A^p(H_q(P_\bullet), A) \Rightarrow H^{p+q}(\operatorname{tot}^\bullet(\operatorname{Hom}(P_\bullet, Q_\bullet))) = H^{p+q}(\operatorname{Hom}(P_\bullet), A)$$

and thus we finish the proof. □

Definition 48. The spectral sequence from Proposition 2.4.2 is called the *Künneth spectral sequence.*

Proposition 2.4.3. *Let R be a ring and let M and N be finitely generated R-modules. If $\operatorname{ann} M + \operatorname{ann} N = R$ then $\operatorname{Ext}_R^r(M, N) = 0$ for every r. Otherwise $\operatorname{depth}(\operatorname{ann} M, N)$ is the smallest number r such that $\operatorname{Ext}_R^r(M, N) \neq 0$.*

Proof. See Eisenbud [23, Proposition 18.4]. □

Proposition 2.4.4. *For R Cohen-Macaulay it holds that $\operatorname{gr}(M) := \operatorname{depth}(\operatorname{ann} M, R) = \dim R - \dim M$, where $\dim M := \dim R/\operatorname{ann} M$.*

Proof. From Eisenbud [23, Theorem 18.7] we know that for every proper ideal I in a Cohen-Macaulay ring R we have $\operatorname{depth}(I, R) = \dim R - \dim R/I$. Using $I = \operatorname{ann} M$ we get our result that $\operatorname{gr}(M) := \operatorname{depth}(\operatorname{ann} M, R) = \dim R - \dim M$. □

Proposition 2.4.5. *Let $X = \operatorname{Spec}(A)$ be smooth in codimension d. For each $i \geq 1$ and $0 \leq j \leq d + 1$, we have $T_{(i)}^j(A) \cong \operatorname{Ext}_A^j(\Omega_{A|k}^i, A)$.*

Proof. Since each term of $\wedge^i \mathbb{L}_{A|k}$ is a projective A-module for each $i \geq 1$, we have a Künneth spectral sequence:

$$E_2^{p,q} = \operatorname{Ext}_A^p(T_q^{(i)}(A), A) \Rightarrow T_{(i)}^{p+q}(A).$$

Modules $T_q^{(i)}(A)$ have support on the singular locus for $q \geq 1$ by Proposition 2.3.6. Since A is smooth in codimension d, we have $\operatorname{Ext}_A^p(T_q^{(i)}(A), A) = 0$ for $q \geq 1$ and $p = 0, 1, ..., d$; here we used Proposition 2.4.4: since for $q \geq 1$ it holds that $\dim \left(T_q^{(i)}(A)\right) \leq \dim A - d - 1$, we have $\operatorname{gr}(T_q^{(i)}(A)) \geq d + 1$. If $q \geq 1$ it follows that $E_2^{p,q} = 0$. Thus we have

$$E_2^{p,q} = \operatorname{Ext}_A^p(T_0^{(i)}(A), A) = E_\infty^{p,q} = T_{(i)}^p(A).$$

We conclude the proof using $T_0^{(i)}(A) = \Omega_{A|k}^i$, which holds by Theorem 2.3.6. □

Corollary 2.4.6. *Let A be a coordinate ring of a normal variety. We have the Hodge decompositions*

$$H^2(A) = \mathrm{Ext}^1_A(\Omega^1_{A|k}, A) \oplus \mathrm{Hom}_A(\Omega^2_{A|k}, A)$$

$$H^3(A) = \mathrm{Ext}^2_A(\Omega^1_{A|k}, A) \oplus \mathrm{Ext}^1_A(\Omega^2_{A|k}, A) \oplus \mathrm{Hom}_A(\Omega^3_{A|k}, A).$$

Moreover, for each $i \in \mathbb{N}$ we have $T^1_{(i)}(A) \cong \mathrm{Ext}^1_A(\Omega^i_{A|k}, A)$ and $T^2_{(i)}(A) \cong \mathrm{Ext}^2_A(\Omega^i_{A|k}, A)$.

Proof. We use Proposition 2.4.5 for $d = 1$ and the Hodge decomposition. □

2.5 The Hochschild (co-)homology of affine hypersurfaces

In this section we will compute Hochschild (co-)homology of a reduced affine hypersurface $X \subset \mathbb{A}^N$. The results describing the Hochschild homology were already obtained by Michler [49], [50]. Here we obtain the results in a little bit different way. The main part of this section is the computation of the Hochschild cohomology. The main result of this section is Theorem 2.5.9, which will also give us a more complete view on the results that we will obtain in the next chapter (see Example 8).

2.5.1 The Hochschild homology of reduced affine hypersurfaces

Let $X = \mathrm{Spec}(A)$, where $A = k[x_1, ..., x_N]/(f(x_1, ..., x_N))$. We write for short $A = P/f$ and Ω_P for $\Omega_{k[x_1,...,x_N]|k}$.

Proposition 2.5.1. *The derived exterior power $\wedge^i \mathbb{L}_{A|k}$ is isomorphic to the chain complex*

$$0 \to A \xrightarrow{\wedge df} \Omega^1_P \otimes_P A \xrightarrow{\wedge df} \cdots \xrightarrow{\wedge df} \Omega^i_P \otimes_P A \to 0, \tag{2.8}$$

where $\Omega^i_P \otimes_P A$ is degree 0 term.

Proof. From Example 3 we know that $\mathbb{L}_{A|k}$ is isomorphic to

$$0 \to Ady \xrightarrow{s} \oplus_{i=1}^N Adx_i \to 0.$$

We can use Definition 42 with $L = Ady$ and $E = \oplus_{i=1}^n Adx_i$ and thus we get that $\wedge^q \mathbb{L}_{A|k}$ is isomorphic to

$$L^{\otimes q} \to E \otimes L^{\otimes q-1} \to \cdots \to \wedge^{q-n} E \otimes L^{\otimes n} \to \cdots \to \wedge^q E,$$

where $\wedge^{q-n} E \otimes L^{\otimes n} \cong \oplus_{1 \leq p_1 < \cdots < p_{q-n} < n} A(dx_{p_1} \wedge \cdots \wedge dx_{p_{q-n}}) \otimes Ady \cong \Omega^{q-n}_P \otimes_P A$ and differentials agree since $s(dy) = df$. □

Lemma 2.5.2. *The cokernel of the map $\Omega^{k-1}_P \otimes_P A \xrightarrow{\wedge df} \Omega^k_P \otimes_P A$ is equal to $\Omega^k_{A|k}$.*

Proof. See [51, Lemma 3]. □

Corollary 2.5.3. $H_0(\wedge^i \mathbb{L}_{A|k}) \cong \Omega^i_{A|k}$ *(this we already know by Theorem 2.3.6). From definition of differentials $\wedge df$ of the complex (2.8) we have*

$$H_0(\wedge^N \mathbb{L}_{A|k}) \cong \Omega^N_{A|k} \cong A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}).$$

Lemma 2.5.4. *The k-the homology (for $0 < k < i$) of the chain complex* (2.8) *is equal to* $\mathrm{tors}(\Omega^{i-k}_{A|k})$.

Proof. See [44, Lemma 4.11] or [19, pp. 7]. □

Let us focus now in the case when A is an isolated hypersurface singularity.

Lemma 2.5.5. *If A has an isolated singularity at the origin as the only singular point, then we have* $\mathrm{tors}(\Omega^{N-1}_{A|k}) \cong \Omega^N_{A|k}$ *(an isomorphism of A-modules is given by the exterior derivative* $\Omega^{N-1}_{A|k} \to \Omega^N_{A|k}$*) and* $\mathrm{tors}(\Omega^i_{A|k}) = 0$ *for $i < N-1$ and $i > N$.*

Proof. See Michler [51, Theorem 2] or [49, Proposition 3]. □

Corollary 2.5.6. *Let A be a hypersurface in $\mathbb{A}^N$ with an isolated singularity at the origin. We have*

$$H_n(\wedge^N \mathbb{L}_{A|k}) \cong \begin{cases} \Omega^N_{A|k} & \textit{if } n = 0,1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proposition 2.5.7. *Let A be a hypersurface in $\mathbb{A}^N$ with an isolated singularity at the origin. For $n \geq N$ we have*

$$H^{(i)}_n(A) \cong \begin{cases} \Omega^N_{A|k} & \textit{if } 2i - n = N-1, N \\ 0 & \textit{otherwise.} \end{cases}$$

For $n < N$ we have

$$H^{(i)}_n(A) \cong \begin{cases} \Omega^n_{A|k} & \textit{if } i = n, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. It follows from the results above, see also [50]. □

Corollary 2.5.8. *For $n \geq N$ it holds that*

$$\dim_k H_n(A) = \dim_k \oplus_{i=1}^n H^{(i)}_n(A) = \dim_k(\Omega^N_{A|k}) = \dim_k \big(A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N})\big),$$

which is the Tyurina number *of the hypersurface.*

Proof. It follows from the Hodge decomposition and Proposition 2.5.7. □

Example 5. Let $X = \mathrm{Spec}(A)$ be the Gorenstein toric surface defined by the polynomial

$$p(x, y, z) = xy - z^{r+1}.$$

For $n \geq 3$ we have $\dim_k H_n(A) = r$, the Milnor number of the surface.

2.5.2 The Hochschild cohomology of isolated hypersurface singularities

In this subsection we compute the Hochschild cohomology for reduced isolated hypersurface singularities.

Theorem 2.5.9. *Let A be a reduced isolated hypersurface singularity in $\mathbb{A}^N$, $N \geq 3$. We have*

$$H^n(A) \cong \begin{cases} \mathrm{Hom}_A(\Omega^n_{A|k}, A) \oplus A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}) & \textit{if } n < N \\ A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}) & \textit{if } n \geq N. \end{cases}$$

Proof. If $n < N$, then by Proposition 2.4.5 it follows that

$$H^n(A) \cong \mathrm{Hom}_A(\Omega^n_{A|k}, A) \oplus \mathrm{Ext}^1_A(\Omega^{n-1}_{A|k}, A) \oplus \cdots \oplus \mathrm{Ext}^{n-1}_A(\Omega^1_{A|k}, A).$$

We denote by $C_\bullet := \wedge^N \mathbb{L}_{A|k}$ the complex

$$0 \to A \xrightarrow{\wedge df} \Omega^1_P \otimes_P A \xrightarrow{\wedge df} \cdots \xrightarrow{\wedge df} \Omega^N_P \otimes_P A \to 0.$$

A perfect pairing $\Omega^k_P \otimes_P \Omega^{N-k}_P \to \Omega^N_P \cong P$ induces a perfect pairing

$$C_k \otimes_A C_{N-k} \to C_N \cong A,$$

where C_k is degree k term of the complex $C_\bullet$. From this we get that the complex

$$0 \to \mathrm{Hom}_A(C_N, A) \to \mathrm{Hom}_A(C_{N-1}, A) \to \cdots \to \mathrm{Hom}_A(C_0, A) \to 0 \tag{2.9}$$

is isomorphic to $C_\bullet$. Looking on the complex (2.9) as a cochain complex $\mathrm{Hom}_A(C_\bullet, A)$ with $\mathrm{Hom}_A(C_N, A)$ of degree 0, we see that

$$H^n(\mathrm{Hom}_A(C_\bullet, A)) \cong H_{N-n}(C_\bullet).$$

Using Corollary 2.5.6 we thus obtain

$$H^n(\mathrm{Hom}_A(C_\bullet, A)) \cong H_{N-n}(C_\bullet) \cong \begin{cases} \Omega^N_{A|k} & \text{if } n = N-1, N \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

Note that we have $\Omega^j_P = 0$ for $j \geq N+1$ since Ω_P is a free module of rank N. Using Lemma 2.5.4 and Lemma 2.5.5 we thus see that for $i \geq N$ we have

$$T^j_{(i)}(A) \cong \begin{cases} A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N}) & \text{if } j = i-1, i \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, using again Lemma 2.5.4 and Lemma 2.5.5 we see that

$$0 \to A \xrightarrow{\wedge df} \Omega^1_P \otimes_P A \xrightarrow{\wedge df} \cdots \xrightarrow{\wedge df} \Omega^k_P \otimes_P A \to 0$$

is quasi-isomorphic to $\Omega^k_{A|k}$ for $k \leq N-1$. From the equation (2.10) it follows that

$$\mathrm{Ext}^j_A(\Omega^k_{A|k}, A) = 0,$$

if $j \neq 0, k-1, k$ ($k \leq N-1$). Thus we see that in the decomposition

$$\mathrm{Ext}^1_A(\Omega^{n-1}_{A|k}, A) \oplus \cdots \oplus \mathrm{Ext}^{n-1}_A(\Omega^1_{A|k}, A)$$

only one direct summand is nonzero and isomorphic to $\Omega^N_{A|k} \cong A/(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, ..., \frac{\partial f}{\partial x_N})$. The Hodge decomposition concludes the proof. □

Example 6. Let $X = \mathrm{Spec}(A)$ be the Gorenstein toric surface defined by the polynomial

$$p(x, y, z) = xy - z^{r+1}.$$

We obtain that for $n \geq 3$ we have $\dim_k H^n(A) = r$, the Milnor number of the surface.

3 The Hochschild cohomology of toric varieties

In Section 3.1 we briefly recall basic definitions of toric geometry. We analyze the Hochschild complex in the case of toric varieties in Section 3.2. Section 3.3 contains a construction of an important double complex of convex sets. Using the spectral sequence arguments we are able to give a convex geometric description of the Hodge decomposition of the Hochschild cohomology for affine toric varieties in Section 3.4 (see Theorem 3.4.3). As an application we explicitly calculate $T^1_{(i)}(A)$, for all $i \in \mathbb{N}$, in the case of two and three-dimensional toric varieties (see Proposition 3.5.2 and Proposition 3.6.2). The two-dimensional case is considered in Section 3.5 and the three-dimensional case is considered in Section 3.6, where we also compute $T^1_{(i)}(A)$ for affine cones over smooth toric Fano varieties in arbitrary dimensions (see Theorem 3.6.7).

3.1 Toric geometry

Let k be our field of characteristic 0. Let M, N be mutually dual, finitely generated, free Abelian groups; we denote by $M_{\mathbb{R}}$, $N_{\mathbb{R}}$ the associated real vector spaces obtained via base change with $\mathbb{R}$. Assume we are given a rational, polyhedral cone $\sigma = \langle a_1, ..., a_N \rangle \subset N_{\mathbb{R}}$ with apex in 0 and with $a_1, ..., a_N \in N$ denoting its primitive fundamental generators (i.e. none of the a_i is a proper multiple of an element of N). We define the dual cone $\sigma^\vee := \{r \in M_{\mathbb{R}} \mid \langle \sigma, r \rangle \geq 0\} \subset M_{\mathbb{R}}$ and denote by $\Lambda := \sigma^\vee \cap M$ the resulting semi-group of lattice points. Its spectrum $\operatorname{Spec}(k[\Lambda])$ is called an affine toric variety. For $\lambda \in \Lambda$ we denote by x^λ the monomial corresponding to λ. Since Λ is saturated, $\operatorname{Spec}(k[\Lambda])$ is normal (see e.g. [20, Theorem 1.3.5]).

Definition 49. A variety X is called $\mathbb{Q}$-*Gorenstein* if the double dual of some tensor product of ω_X is an invertible sheaf on X.

The following facts about toric $\mathbb{Q}$-Gorenstein varieties can be found in [2, Section 6.1]. For an affine toric variety given by a cone $\sigma = \langle a_1, ..., a_N \rangle$ we have that X is $\mathbb{Q}$-Gorenstein if and only if there exists a primitive element $R^* \in M$ and a natural number $g \in \mathbb{N}$ such that $\langle a_j, R^* \rangle = g$ for each $j = 1, ..., N$. X is Gorenstein if and only if additionally $g = 1$. In particular, toric $\mathbb{Q}$-Gorenstein singularities are obtained by putting a lattice polytope $P \subset \mathbb{A}$ into the affine hyperplane $\mathbb{A} \times \{g\} \subset N_{\mathbb{R}} := \mathbb{A} \times \mathbb{R}$ and defining $\sigma := \operatorname{Cone}(P)$, the cone over P. Then the canonical degree R^* equals $(\underline{0}, 1)$.

From now on we will try to simplify the results obtained in the previous chapter using the lattice grading that comes with toric varieties.

3.2 Grading of the Hochschild cohomology

Definitions and statements in this subsection already appeared in [6] for $i = 1$. We give a generalization for arbitrary $i \geq 1$.

Let $A = \oplus_{i\in\mathbb{Z}} A_i$ be a graded k-algebra. If $a_0, ..., a_p$ are homogenous elements, define the *weight* of $a_0 \otimes \cdots \otimes a_p \in A^{\otimes p+1}$ to be $w = \sum |a_i|$, where $|a_i| = j$ means that $a_i \in A_j$. This makes the tensor product $A^{\otimes p+1}$ into a graded k-module. Since differentials preserve the weight, this equip both $H_p(A)$ and $H^p(A)$ with the structure of graded k-modules.

In the case when $\mathrm{Spec}(A)$ is an affine toric variety there exists M-grading on A. Let $A = k[\Lambda] = k[\sigma^\vee \cap M]$.

Definition 50. We say that an element $f \in C^n(A)$ has degree $R \in M$ if f maps an element with weight w to an element of degree $R+w$ in A. This means that f is of the form $f(x^{\lambda_1}\otimes\cdots\otimes x^{\lambda_n}) = f_0(\lambda_1, .., \lambda_n)x^{R+\lambda_1+\cdots+\lambda_n}$. We need to take care that the expression is well defined, i.e., that $f_0(\lambda_1, ..., \lambda_n) = 0$ for $R + \lambda_1 + \cdots \lambda_n \notin \Lambda$ (in the following we will also use $R + \lambda_1 + \cdots \lambda_n \ngeq 0$ since we can look on M as a partially order set where positive elements lie in the cone Λ). Let $C^{n,R}(A)$ denote the degree R elements of $C^n(A)$ and let $C^{n,R}_{(i)}(A)$ denote the degree R elements of $C^n_{(i)}(A)$.

We would like to understand the space $C^{n,R}(A)$ better and the following definition will be useful.

Definition 51. $L \subset \Lambda$ is said to be *monoid-like* if for all elements $\lambda_1, \lambda_2 \in L$ the relation $\lambda_1 - \lambda_2 \in \Lambda$ implies $\lambda_1 - \lambda_2 \in L$. Moreover, a subset $L_0 \subset L$ of a monoid-like set is called *full* if $(L_0 + \Lambda) \cap L = L_0$.

For any subset $P \subset \Lambda$ and $n \geq 1$ we introduce $S^n(P) := \{(\lambda_1, ..., \lambda_n) \in P^n \mid \sum_{v=1}^n \lambda_v \in P\}$. If $L_0 \subset L$ are as in the previous definition, then this gives rise to the following vector spaces ($1 \leq i \leq n$):

$$C^n_{(i)}(L, L \setminus L_0; k) := \{\varphi : S^n(L) \to k \mid \varphi \circ s_n = (2^i - 2)\varphi,\ \varphi \text{ vanishes on } S^n(L \setminus L_0)\},$$

which turn into a complex with the differential

$$d^n : C^{n-1}_{(i)}(L, L \setminus L_0; k) \to C^n_{(i)}(L, L \setminus L_0; k),$$

$$(d^n\varphi)(\lambda_1, ..., \lambda_n) :=$$

$$\varphi(\lambda_2, ..., \lambda_n) + \sum_{i=1}^{n-1}(-1)^i\varphi(\lambda_1, ..., \lambda_i + \lambda_{i+1}, ..., \lambda_n) + (-1)^n\varphi(\lambda_1, ..., \lambda_{n-1}).$$

Definition 52. By $H^n_{(i)}(L, L\setminus L_0; k)$ we denote the *Hochschild cohomology groups* of the above complex $C^\bullet_{(i)}(L, L \setminus L_0; k)$.

Lemma 3.2.1. *For all $R \in M$ it holds that*

$$C^{n,-R}_{(i)}(A) \cong C^n_{(i)}(\Lambda, \Lambda \setminus (R + \Lambda); k).$$

Proof. For $f \in C^{n,-R}_{(i)}(A)$, we have $f(x^{\lambda_1} \otimes \cdots \otimes x^{\lambda_n}) = f_0(\lambda_1, .., \lambda_n)x^{\lambda_1+\cdots+\lambda_n-R}$ and then the isomorphism is given by $f \mapsto f_0$. □

It is a trivial check that Hochschild differentials respect the grading given by the degrees $R \in M$. Thus we get the Hochschild subcomplex $C^{\bullet,-R}_{(i)}$ and we denote the corresponding cohomology groups by $H^{n,-R}_{(i)}(A) \cong T^{n-i,-R}_{(i)}(A)$. When the ring A will be clear from the context, we will also write $H^n_{(i)}(-R) \cong T^{n-i}_{(i)}(-R)$.

From definitions it follows that $C^n_{(i)}(A) = \oplus_R C^{n,-R}_{(i)}(A)$, $C^n(A) = \oplus_R C^{n,-R}(A)$ and $H^n_{(i)}(A) = \oplus_R H^{n,-R}_{(i)}(A)$, $H^n(A) = \oplus_R H^{n,-R}(A)$.

Proposition 3.2.2. *Let $R \in M$ and let $A = k[\Lambda]$. We have*

$$T^{n-i,-R}_{(i)}(A) \cong H^n_{(i)}(\Lambda, \Lambda \setminus (R + \Lambda); k). \tag{3.1}$$

Proof. We use Lemma 3.2.1 and the decomposition of the Hochschild cohomology. □

Remark 8. In next chapters we will also use the positive grading

$$T^{n-i,R}_{(i)}(A) \cong H^n_{(i)}(\Lambda, \Lambda \setminus (-R + \Lambda); k).$$

Poisson structures lie in $T^0_{(2)}(A)$, which is non-zero for positive degrees ($R \in \Lambda$).

3.3 A double complex of convex sets

In this section we follow the paper [6] verbatim. Arguments mentioned in [6] in the case $i = 1$ works also for arbitrary $i \geq 1$ using the definitions from Section 3.2.

Let $\sigma = \langle a_1, ..., a_N \rangle$. For $\tau \subset \sigma$ let us define the convex sets introduced in [6]:

$$K^R_\tau := \Lambda \cap (R - \operatorname{int} \tau^\vee). \tag{3.2}$$

The above convex sets admit the following properties:

- $K^R_0 = \Lambda$ and $K^R_{a_j} = \{r \in \Lambda \mid \langle a_j, r \rangle < \langle a_j, R \rangle\}$ for $j = 1, ..., N$.
- For $\tau \neq 0$ the equality $K^R_\tau = \cap_{a_j \in \tau} K^R_{a_j}$ holds.
- $\Lambda \setminus (R + \Lambda) = \cup^N_{j=1} K^R_{a_j}$.

We have the following double complexes $C^\bullet_{(i)}(K^R_\bullet; k)$ for each $i \geq 1$ (see Figure 3.1). We define $C^q_{(i)}(K^R_\tau; k) := C^q_{(i)}(K^R_\tau, \emptyset; k)$ and

$$C^q_{(i)}(K^R_p; k) := \oplus_{\tau \leq \sigma, \dim \tau = p} C^q_{(i)}(K^R_\tau; k) \quad (0 \leq p \leq \dim \sigma).$$

The differentials $\delta^p : C^q_{(i)}(K^R_p) \to C^q_{(i)}(K^R_{p+1}; k)$ are defined in the following way: we are summing (up to a sign) the images of the restriction map $C^q_{(i)}(K^R_\tau; k) \to C^q_{(i)}(K^R_{\tau'}; k)$, for any pair $\tau \leq \tau'$ of p and $(p+1)$-dimensional faces, respectively. The sign arises from the comparison of the (pre-fixed) orientations of τ and τ' (see also [20, pp. 580] for more details).

Example 7. The map $\delta : \oplus^N_{j=1} C^q_{(i)}(K^R_{a_j}; k) \to \oplus_{\langle a_j, a_k \rangle \leq \sigma} C^q_{(i)}(K^R_{a_j} \cap K^R_{a_k}; k)$ is simply given by: $(f_1, ..., f_N)$ gets mapped to $f_j - f_k \in C^q_{(i)}(K^R_{a_j} \cap K^R_{a_k}; k)$.

The following results (obtained in [6] for $i = 1$) can also be generalized to $i > 1$:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^3_{(i)}(\Lambda;k) & \xrightarrow{\delta} & C^3_{(i)}(K^R_1;k) & \xrightarrow{\delta} & C^3_{(i)}(K^R_2;k) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^2_{(i)}(\Lambda;k) & \xrightarrow{\delta} & C^2_{(i)}(K^R_1;k) & \xrightarrow{\delta} & C^2_{(i)}(K^R_2;k) & \xrightarrow{\delta} & \cdots \\
\uparrow d & & \uparrow d & & \uparrow d & & \\
C^1_{(i)}(\Lambda;k) & \xrightarrow{\delta} & C^1_{(i)}(K^R_1;k) & \xrightarrow{\delta} & C^1_{(i)}(K^R_2;k) & \xrightarrow{\delta} & \cdots
\end{array}
$$

Figure 3.1: The double complex $C^\bullet_{(i)}(K^R_\bullet;k)$

Lemma 3.3.1. *The canonical k-linear map $C^q_{(i)}(\Lambda, \Lambda \setminus (R+\Lambda);k) \to C^q_{(i)}(K^R_\bullet;k)$ is a quasi-isomorphism, i.e., a resolution of the first vector space.*

Proof. For $r \in \Lambda \subset M$ we define the k-vector space

$$V^q_{(i)}(r) := \{\varphi : \{\underline{\lambda} \in \Lambda^q \mid \sum_v \lambda_v = r\} \to k \mid \varphi \circ s_n = (2^i - 2)\varphi\}.$$

Then our complex $C^q_{(i)}(K^R_\bullet;k)$ splits into a direct product over $r \in \Lambda$. Its homogenous factors equal

$$0 \to V^q_{(i)}(r) \to V^q_{(i)}(r)^{\{j \mid r \in K^R_{a_j}\}} \to V^q_{(i)}(r)^{\{\tau \leq \sigma \mid \dim \tau = 2;\ r \in K^R_\tau\}} \to \cdots.$$

On the other hand, denoting $H^+_{r,R} := \{a \in N_\mathbb{R} \mid \langle a, r\rangle < \langle a, R\rangle\} \subset N_\mathbb{R}$, the relation $r \in K^R_\tau$ is equivalent to $\tau \setminus \{0\} \subset H^+_{r,R}$. Hence, the complex for computing the reduced cohomology of the topological space

$$\bigcup_{\tau \setminus \{0\} \subset H^+_{r,R}} (\tau \setminus \{0\}) \subset \sigma$$

equals

$$0 \to k \to k^{\{j \mid r \in K^R_{a_j}\}} \to k^{\{\tau \leq \sigma \mid \dim \tau = 2;\ r \in K^R_\tau\}} \to \cdots$$

if $\sigma \cap H^+_{r,R} \neq \emptyset$ (i.e. if $r \in \cup_j K^R_{a_j}$) and it is trivial otherwise. Since $\cup_{\tau \setminus \{0\} \subset H^+_{r,R}} (\tau \setminus \{0\})$ is contractible, this complex is always exact. Thus, $C^q_{(i)}(K^R_\bullet; k) = \prod_{r \in \Lambda} V^q_{(i)}(r)^{\{\tau \leq \sigma \mid \dim \tau = \bullet;\; r \in K^R_\tau\}}$ has $\prod_{r \in \Lambda \setminus (\cup_j K^R_{a_j})} V^q_{(i)}(r) = C^q_{(i)}(\Lambda \setminus (R + \Lambda), \Lambda; k)$ as cohomology in 0, and it is exact elsewhere. □

Corollary 3.3.2. *Let $i \geq 1$ be a fixed integer. For $q \geq i$ and $p \geq 0$ there is a spectral sequence*

$$E_1^{p,q} = \oplus_{\dim \tau = p} H^q_{(i)}(K^R_\tau; k) \Rightarrow T^{p+q-i,-R}_{(i)}(A) = H^{p+q,-R}_{(i)}(A).$$

Proof. We use first the differentials δ^p and then the differentials d^n. □

Proposition 3.3.3. $T^{n-i,-R}_{(i)}(A) = H^n\big(\operatorname{tot}^\bullet(C^\bullet_{(i)}(K^R_\bullet; k))\big)$ *for* $1 \leq i \leq n$.

Proof. We use first the differentials d^n and Lemma 3.3.1 and then the differentials δ^p. □

Proposition 3.3.4. *If $\tau \leq \sigma$ is a smooth face, then $H^q_{(i)}(K^R_\tau; k) = 0$ for $q \geq i + 1$.*

Proof. We proceed by induction on $\dim \tau$, i.e., we may assume that the vanishing holds for all proper faces of τ. Let $r(\tau)$ be an arbitrary element of $\operatorname{int}(\sigma^\vee \cap \tau^\perp) \cap M$, i.e., $\tau = \sigma \cap [r(\tau)]^\perp$. Then, via $R_g := R - g \cdot r(\tau)$ with $g \in \mathbb{Z}$, one obtains an infinite (if $\tau \neq \sigma$) series of degrees admitting the following two properties:

- $K^{R_g}_\tau = K^R_\tau$ for any $g \in \mathbb{Z}$ (since $R_g = R$ on τ), and
- $K^{R_g}_{\tau'} \neq \emptyset$ implies $\tau' \leq \tau$ for any face $\tau' \leq \sigma$ and $g \gg 0$ (since $\langle a_j, R_g \rangle \leq 0$ if $a_j \notin \tau$).

In particular, in degree $-R_g$ with $g \gg 0$ the first level of our spectral sequence is shaped as follows:

- For $p < \dim \tau$ only $H^q_{(i)}(K^R_{\tau'}; k)$ with $\tau' \leq \tau$ appear as summands of $E_1^{p,q}$. By induction hypothesis they vanish for $q \geq i + 1$ and by definition they vanish for $q < i$.
- For $p = \dim \tau$ it follows that $E_1^{p,q} = H^q_{(i)}(K^R_\tau; k)$.
- All vector spaces $E_1^{p,q}$ vanish beyond the $[p = \dim \tau]$-line.

Hence, the differential $d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ are trivial for $r \geq 1$, $i - 1 \leq q$ and $r \geq 1$, $q \geq i + 1$ and we obtain

$$T^{q+\dim \tau - i}_{(i)}(-R_g) = H^q_{(i)}(K^R_\tau; k) \text{ for } g \gg 0, q \geq i + 1. \tag{3.3}$$

Let $T^n_{(i)}(\tau) := T^n_{(i)}(\operatorname{Spec}(k[\tau^\vee \cap M]))$ and similarly $T^n_{(i)}(\sigma) := T^n_{(i)}(A)$. We have

$$T^n_{(i)}(\sigma) \otimes_{k[\sigma^\vee \cap M]} k[\sigma^\vee \cap M]_{x^{r(\tau)}} = T^n_{(i)}(\tau) = 0 \text{ for } n \geq 1, \tag{3.4}$$

since $k[\tau^\vee \cap M]$ equals the localization of $k[\sigma^\vee \cap M]$ by the element $x^{r(\tau)}$. The last equality holds by Proposition 2.3.6 since τ is a smooth face. From (3.4) we see that any element of $T^{q+\dim \tau - i}_{(i)}(-R_g) \subset T^{q+\dim \tau - i}_{(i)}$ will be killed by some power of $x^{r(\tau)}$, which implies that $H^q_{(i)}(K^R_\tau; k) = 0$ by (3.3). □

3.4 The Hochshild cohomology in degree $-R \in M$

The main result in this section is Theorem 3.4.3. The results in this subsection do not follow immediately from [6] as in Section 3.3.

The first reason that computations of $T^n_{(i)}(-R)$ become more challenging for $i > 1$ is that it is not immediately clear how to generalize an easy description of $H^1_{(1)}(K^R_\tau; k)$ to $H^i_{(i)}(K^R_\tau; k)$.

Definition 53. We say that $f \in C^n_{(n)}(L, L \setminus L_0; k)$ is multi-additive if it is additive on every component, provided that the sum of all entries lies in L. Being additive in the first component means $f(a+b, \lambda_2, ..., \lambda_n) = f(a, \lambda_2, ..., \lambda_n) + f(b, \lambda_2, ..., \lambda_n)$, with $a+b+\lambda_1+\cdots+\lambda_n \in L$. We denote

$$\bar{C}^n_{(n)}(L, L \setminus L_0; k) := \{f \in C^n_{(n)}(L, L \setminus L_0; k) \mid f \text{ is multi-additive}\}.$$

In the case $n = 1$ it holds trivially that $H^1_{(1)}(L, L \setminus L_0; k)$ equals $\bar{C}^1_{(1)}(L, L \setminus L_0; k)$. Some additional effort is necessary to show this for $n > 1$. Note that computations of $H^n_{(n)}(K^R_\tau; k)$ are still easier than computations of $H^n_{(i)}(K^R_\tau; k)$, $i \neq n$, because in the case $i = n$ we do not have coboundaries.

Proposition 3.4.1. *We have*

$$H^n_{(n)}(L, L \setminus L_0; k) = \bar{C}^n_{(n)}(L, L \setminus L_0; k)$$

for all $n \geq 1$.

Proof. That every multi-additive function $f \in C^n_{(n)}(L, L \setminus L_0; k)$ satisfies $df = 0$ is obvious by definition of d. For the other direction we use the following computation (similarly as in the proof of Loday [43, Proposition 1.3.12]):

we have

$$\begin{gathered}\textstyle\sum_\sigma df(\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n+1)}) = \\ n!\big(f(\lambda_1, \lambda_3, \lambda_4, ..., \lambda_{n+1}) + f(\lambda_2, \lambda_3, \lambda_4, ..., \lambda_{n+1}) - f(\lambda_1+\lambda_2, \lambda_3, \lambda_4, ..., \lambda_{n+1})\big),\end{gathered} \tag{3.5}$$

where the sum is taken over all permutations $\sigma \in S_{n+1}$ such that $\sigma(1) < \sigma(2)$.

The proof of (3.5) for $n = 1$ is trivial, let us prove it for $n = 2$:

$$\begin{aligned}&df(\lambda_1, \lambda_2, \lambda_3) - df(\lambda_1, \lambda_3, \lambda_2) + df(\lambda_3, \lambda_1, \lambda_2) = \\ &f(\lambda_2, \lambda_3) - f(\lambda_1+\lambda_2, \lambda_3) + f(\lambda_1, \lambda_2+\lambda_3) - f(\lambda_1, \lambda_2) \\ &-\big(f(\lambda_3, \lambda_2) - f(\lambda_1+\lambda_3, \lambda_2) + f(\lambda_1, \lambda_3+\lambda_2) - f(\lambda_1, \lambda_3)\big) \\ &+f(\lambda_1, \lambda_2) - f(\lambda_3+\lambda_1, \lambda_2) + f(\lambda_3, \lambda_1+\lambda_2) - f(\lambda_3, \lambda_1) = \\ &2f(\lambda_2, \lambda_3) + 2f(\lambda_1, \lambda_3) - 2f(\lambda_1+\lambda_2, \lambda_3).\end{aligned}$$

Let us prove (3.5) for general n: we first sum over all permutations $\sigma \in S_{n+1}$ such that $\sigma(1) < \sigma(2)$ with additional condition $\sigma(1) = 1$. In this sum we have the summand $n!f(\lambda_2, \lambda_3, \lambda_4, ..., \lambda_{n+1})$:

$$\sum_{\sigma \in S_{n+1} \ \mid \ \sigma(1)<\sigma(2), \sigma(1)=1} df(\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n+1)}) = n!f(\lambda_2, \lambda_3, \lambda_4, ..., \lambda_{n+1}) + \cdots.$$

Then we sum with additional condition $\sigma(2) = n+1$, where the summand $n!f(\lambda_1, \lambda_3, \lambda_4, ..., \lambda_{n+1})$ appears:

$$\sum_{\sigma \in S_{n+1} \ \mid \ \sigma(1)<\sigma(2), \sigma(2)=n+1} df(\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n+1)}) = n!f(\lambda_1, \lambda_3, \lambda_4, ..., \lambda_{n+1}) + \cdots.$$

Finally, we restrict the sum on the condition $\sigma(2) = \sigma(1) + 1$ where we get the summand $-n!f(\lambda_1 + \lambda_2, \lambda_3, ..., \lambda_{n+1})$:

$$\sum_{\sigma \in S_{n+1} \ | \ \sigma(2)=\sigma(1)+1} df(\lambda_{\sigma^{-1}(1)}, ..., \lambda_{\sigma^{-1}(n+1)}) = -n \cdot (n-1)!f(\lambda_1 + \lambda_2, \lambda_3, ..., \lambda_{n+1}) + \cdots .$$

From the above we can easily show the equality (3.5). □

The next proposition will give us very useful formulas for $H^n_{(n)}(K^R_\tau; k)$.

Proposition 3.4.2. *Let $\tau \leq \sigma$ be a smooth face. The injections $\bar{C}^n_{(n)}(\operatorname{Span}_k K^R_\tau; k) \to \bar{C}^n_{(n)}(K^R_\tau; k)$ are isomorphisms. Moreover,* $\operatorname{Span}_k K^R_\tau = \cap_{a_j \in \tau} \operatorname{Span}_k K^R_{a_j}$*, and we have*

$$\operatorname{Span}_k K^R_{a_j} = \begin{cases} 0 & \textit{if } \langle a_j, R\rangle \leq 0 \\ (a_j)^\perp & \textit{if } \langle a_j, R\rangle = 1 \\ M \otimes_{\mathbb{Z}} k & \textit{if } \langle a_j, R\rangle \geq 2. \end{cases}$$

Proof. We will prove the case $n = 2$, generalization to other n is then immediate. Let $f \in \bar{C}^2_{(2)}(K^R_\tau; k)$. We want to show that $f \in \bar{C}^2_{(2)}(\operatorname{Span}_k K^R_\tau; k)$.

Without loss of generality we can assume that $\tau = \langle a_1, ..., a_m\rangle$, with $\langle a_i, R\rangle \geq 2$ for $i = 1, ..., l$ and $\langle a_j, R\rangle = 1$ for $j = l+1, ..., m$, since if R was non-positive on any of the generators of τ, then K^R_τ would be empty.

By smoothness of τ there exist elements $r_1, ..., r_l$ such that $\langle r_i, a_k\rangle = \delta_{ik}$ for $1 \leq i \leq l$ and $1 \leq k \leq m$. Hence for elements $s_v, s_w \in K^R_\tau$ it holds that

$$f(s_v, s_w) = \sum_{i=1}^{l}\sum_{u=1}^{l} \langle a_i, s_v\rangle\langle a_u, s_w\rangle f(r_i, r_u) + f(p_v, p_w),$$

with $p_v := s_v - \sum_{i=1}^l \langle a_i, s_v\rangle r_i \in \tau^\perp \cap M$ and $p_w := s_w - \sum_{i=1}^l \langle a_i, s_w\rangle r_i \in \tau^\perp \cap M$. We can easily show that $\sum_v \sum_w f(s_v, s_w)$ does depend only on $s_1 := \sum_v s_v$ and $s_2 := \sum_w s_w$, and not on the summands themselves:

$$\sum_v \sum_w f(s_v, s_w) = \sum_v \big(\sum_{i,j} \langle a_i, s_v\rangle\langle a_j, s_2\rangle f(r_i, r_j) + f(p_v, s_2 - \sum_i \langle a_i, s_2\rangle r_i)\big) =$$
$$= \sum_{i,j} \langle a_i, s_1\rangle\langle a_j, s_2\rangle f(r_i, r_j) + f(s_1 - \sum_i \langle a_i, s_1\rangle r_i, s_2 - \sum_i \langle a_i, s_2\rangle r_i).$$

Then, $f(s_1, s_2)$ may be defined as this value. The second claim follows as in [6] by

$$\cap_{a_i \in \tau} \operatorname{Span}_k K^R_{a_i} = \cap^k_{j=l+1}(a_j)^\perp = \operatorname{Span}_k(\tau^\perp, r_1, ..., r_l) = \operatorname{Span}_k K^R_\tau.$$

□

To shorten notation we write M_k (resp. N_k) instead of $M \otimes_{\mathbb{Z}} k$ (resp. $N \otimes_{\mathbb{Z}} k$).

Remark 9. Note that 0 and 1-dimensional faces are always smooth. For $\tau = 0$ we obtain that $\bar{C}^i_{(i)}(\Lambda; k) \cong \bar{C}^i_{(i)}(\operatorname{Span}_k \Lambda; k) \cong \bar{C}^i_{(i)}(M_k; k)$. Thus if $\sigma = \langle a_1, ..., a_N\rangle \subset M_k \cong k^n$, then $f \in \bar{C}^i_{(i)}(\Lambda; k)$ is completely determined by the values $f(s_{k_1}, ..., s_{k_i})$, for $1 \leq k_1 < \cdots < k_i \leq n$, where $s_1, ..., s_n \in \Lambda$ are linearly independent (k-basis in k^n).

Let E be a minimal set that generates the semigroup $\Lambda := \sigma^\vee \cap M$. E is called a *Hilbert basis.* We write $E_j^R := E \cap K_{a_j}^R$, $E_{jk}^R := E \cap K_{a_j}^R \cap K_{a_k}^R$ for a 2-face $\langle a_j, a_k\rangle \leq \sigma$ and $E_\tau^R := \cap_{a_j \in \tau} E_j^R$ for faces $\tau \leq \sigma$.

Theorem 3.4.3. *Let $X_\sigma = \mathrm{Spec}(A)$ be an affine toric variety that is smooth in codimension d. Let $i \geq 1$ be a fixed integer. Then the k-th cohomology group of the complex*

$$0 \to \bar{C}^i_{(i)}(M_k; k) \to \oplus_j \bar{C}^i_{(i)}(\mathrm{Span}_k\, E_j^R; k) \to \cdots \to \oplus_{\tau \leq \sigma, \dim \tau = d+1} \bar{C}^i_{(i)}(\mathrm{Span}_k\, E_\tau^R; k) \tag{3.6}$$

is isomorphic to $T^{k,-R}_{(i)}(A)$ for $k = 0, ..., d$ ($\bar{C}^i_{(i)}(M_k; k)$ is the degree 0 term).

Moreover, if X is an isolated singularity (i.e. $\dim(X) = d+1$), then

$$T^{k,-R}_{(i)}(A) \cong \begin{cases} \mathrm{Coker}\left(\oplus_{\tau \leq \sigma, \dim \tau = d} \bar{C}^i_{(i)}(K_\tau^R; k) \to \bar{C}^i_{(i)}(K_\sigma^R; k)\right) & \text{if } k = \dim(X) \\ H^{k-\dim(X)+i}_{(i)}(K_\sigma^R; k) & \text{if } k \geq \dim(X)+1. \end{cases}$$

Proof. By Corollary 3.3.2 we have

$$E_1^{p,q} = \oplus_{\tau \leq \sigma, \dim \tau = p} H^q_{(i)}(K_\tau^R; k) \Rightarrow T^{p+q-i,-R}_{(i)}(A) = H^{p+q,-R}_{(i)}(A),$$

for $q \geq i$ and $p \geq 0$. By the assumption j-dimensional faces are smooth for $j \leq d$. From Proposition 3.3.4 it follows that $E_1^{0,q} = E_1^{1,q} = \cdots = E_1^{d,q} = 0$, for $q \geq i+1$. Thus $E_2^{p,i} = E_\infty^{p,i} = \oplus_{\tau \leq \sigma, \dim \tau = p} H^i_{(i)}(K_\tau^R; k)$ for $d+1 \geq p \geq 1$. It follows that $T^{k,-R}_{(i)}(A)$ is isomorphic to the k-th cohomology group of the complex

$$H^i_{(i)}(\Lambda; k) \to \oplus_j H^i_{(i)}(K_{a_j}^R; k) \to \cdots \to \oplus_{\tau \leq \sigma, \dim \tau = d+1} H^i_{(i)}(K_\tau^R; k).$$

We conclude the first part using Proposition 3.4.1 and Proposition 3.4.2.

If X is an isolated singularity, then we also have $E_1^{p,q} = 0$ for $p \geq d+2$. Thus $E_2^{d+1,q} = E_\infty^{d+1,q} = H^q_{(i)}(K_\sigma^R; k)$ for $q \geq i+1$, which finishes the proof. □

Corollary 3.4.4. *Since toric varieties are smooth in codimension 1, we obtain that $T^1_{(i)}(-R)$ is isomorphic to the cohomology group of the complex*

$$\bar{C}^i_{(i)}(M_k; k) \to \oplus_j \bar{C}^i_{(i)}(\mathrm{Span}_k\, E_j^R; k) \to \oplus_{\langle a_j, a_k\rangle < \sigma} \bar{C}^i_{(i)}(\mathrm{Span}_k\, E_{jk}^R; k). \tag{3.7}$$

3.5 The Hochschild cohomology of toric surfaces

In this section we compute $\dim_k T^{1,-R}_{(i)}(A)$ for all $i \in \mathbb{N}$ in the case when A is a two-dimensional affine toric variety (a two-dimensional cyclic quotient singularity). Let $X(n,q)$ denote the quotient by the $\mathbb{Z}/n\mathbb{Z}$-action $\xi \to \begin{pmatrix} \xi & 0 \\ 0 & \xi^q \end{pmatrix}$, $(\xi = \sqrt[n]{1})$. $X(n,q)$ is given by the cone $\sigma = \langle a_1, a_2\rangle = \langle (1,0); (-q,n)\rangle$. We can develop $\frac{n}{n-q}$ into a continued fraction

$$\frac{n}{n-q} = b_1 + \cfrac{1}{b_2 + \cfrac{1}{\ddots + \cfrac{1}{b_r}}}$$

$(b_i \geq 2)$. Then E is given as the set $E = \{w^0, ..., w^{r+1}\}$, with elements $w^i \in \mathbb{Z}^2$ and

1. $w^0 = (0, 1)$, $w^1 = (1, 1)$, $w^{r+1} = (n, q)$,
2. $w^{i-1} + w^{i+1} = b_i \cdot w^i$ $(i = 1, ..., r)$.

We now compute $T^{1,-R}_{(i)}(A)$ for toric surfaces $A = A(n, q) = k[\Lambda := \langle w^0, w^{r+1}\rangle \cap M]$.

Proposition 3.5.1. *For $i > 2$ we have $T^{1,-R}_{(i)}(A) = 0$. Otherwise we have*

$$\dim_k T^{1,-R}_{(i)}(A) =$$

$$\max\{0, \dim_k \bar{C}^i_{(i)}(\mathrm{Span}_k\, E^R_1; k) + \dim_k \bar{C}^i_{(i)}(\mathrm{Span}_k\, E^R_2; k) - \dim_k \bar{C}^i_{(i)}(\mathrm{Span}_k\, E^R_{12}; k) - c_i\},$$

where

$$c_i := \begin{cases} 2 = \dim_k \bar{C}^1_{(1)}(M_k; k) & \text{if } i = 1 \\ 1 = \dim_k \bar{C}^2_{(2)}(M_k; k) & \text{if } i = 2. \end{cases}$$

Proof. It follows immediately from (3.7): the map $f : \oplus_j \bar{C}^i_{(i)}(K^R_{a_j}; k) \to \bar{C}^i_{(i)}(K^R_{a_1} \cap K^R_{a_2}; k)$ give us $\ker f = \bar{C}^i_{(i)}(K^R_{a_1}; k) + \bar{C}^i_{(i)}(K^R_{a_2}; k) - \dim(\mathrm{im}\, f)$, where $\dim(\mathrm{im}\, f) = \dim \bar{C}^i_{(i)}(K^R_{a_1} \cap K^R_{a_2}; k)$ since f is surjective. The number c_i is the dimension of $\bar{C}^i_{(i)}(\Lambda; k)$ since the map

$$\bar{C}^i_{(i)}(\Lambda; k) \to \oplus_j \bar{C}^i_{(i)}(K^R_{a_j}; k)$$

is injective. □

We obtain the following corollaries:

Corollary 3.5.2. *Focusing on $T^{1,-R}_{(2)}(A)$ we can easily check that*

$$h^2_{(2)}(\Lambda; k) := \dim_k H^2_{(2)}(\Lambda; k) = \dim_k \bar{C}^2_{(2)}(\Lambda; k) = 1$$

and that $h^2_{(2)}(K^R_{a_i}; k) := \dim_k H^2_{(2)}(K^R_{a_i}; k) \leq 1$ for $i = 1, 2$. We consider four different cases for the multidegree $R \in M \cong \mathbb{Z}^2$:

- *$R = w^1$ (or analogously $R = w^r$). We obtain $E_1 = \{w^0\}$ and $E_2 = \{w^2, ..., w^{r+1}\}$. We have*
$$\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_1; k) = \dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_{12}; k) = 0$$
and thus Proposition 3.5.1 yields $T^{1,-R}_{(2)}(A) = 0$.

- *$R = w^i$ $(2 \leq i \leq r - 1)$. We obtain $E_1 = \{w^0, ..., w^{i-1}\}$ and $E_2 = \{w^{i+1}, ..., w^{r+1}\}$. We have $\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_{12}; k) = 0$,*
$$\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_1; k) = \dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_2; k) = 1$$
and thus Proposition 3.5.1 yields $\dim_k T^{1,-R}_{(2)}(A) = 1$.

- *$R = l \cdot w^i(1 \leq i \leq r, 2 \leq l \leq b_i$ for $r \geq 2$, or $i = 1, 2 \leq l \leq b_1$ for $r = 1)$. We obtain $E_1 = \{w^0, ..., w^i\}$ and $E_2 = \{w^i, ..., w^{r+1}\}$. We have $\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_{12}; k) = 0$,*
$$\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_1; k) = \dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_2; k) = 1$$
and thus Proposition 3.5.1 yields $\dim_k T^{1,-R}_{(2)} = 1$.

- *For the remaining $R \in M$, either $E_1 \subset E_2$ or $E_2 \subset E_1$ or $\#(E_1 \cap E_2) \geq 2$. In these cases hold either* $\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_j; k) = 0$ *for some* j, *or we have* $\dim_k \bar{C}^2_{(2)}(\mathrm{Span}_k\, E^R_{12}; k) \neq 0$. *Thus in all these cases Proposition 3.5.1 yields* $T^{1,-R}_{(2)}(A) = 0$.

Corollary 3.5.3. *Results for $T^{1,-R}(A)$ (already appeared in [59]):*

- $R = w^1$ *(or analogously $R = w^r$). We obtain* $\dim_k T^1(-R) = 1$*(or* 0 *if* $r = 1$*).*
- $R = w^i$ $(2 \leq i \leq r-1)$*. We obtain* $\dim_k T^1(-R) = 2$.
- $R = l \cdot w^i (1 \leq i \leq r, 2 \leq l \leq a_i)$ *for* $r \geq 2$*, or* $i = 1, 2 \leq l \leq a_1$ *for* $r = 1$*). We obtain* $\dim_k T^1(-R) = 1$.
- *For every other degree R, we obtain that* $T^1(-R) = 0$.

The following example shows that in the case of Gorenstein toric surfaces the computations in this chapter agree with the computations in the previous chapter.

Example 8. Let $X_{\sigma_n} = \mathrm{Spec}(A_n)$ be the Gorenstein toric surface, given by the polynomial $f(x,y,z) = xy - z^{n+1}$ in $\mathbb{A}^3$. From Theorem 2.5.9 we know that $H^3(A_n) \cong A_n/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$, which has dimension as a k-vector space equal to n (the Milnor number of the hypersurface). From the Hodge decomposition and Corollary 2.4.6 we have

$$H^3(A_n) \cong T^0_{(3)}(A_n) \oplus T^1_{(2)}(A_n) \oplus T^2_{(1)}(A_n) \cong \oplus_{i=0}^2 \mathrm{Ext}^i_{A_n}(\Omega^{3-i}_{A_n|k}, A_n).$$

Using Corollary 3.5.2 we can be even more precise: the cone σ_n is given by

$$\sigma_n = \langle (1,0), (-n, n+1) \rangle.$$

Its continued fraction has $r = 1$, $b_1 = n+1$ and thus we have $\dim_k T^{1,-R}_{(2)}(A_n) = 1$ for the degrees $R = (2,2), ..., (n+1, n+1)$ and $\dim_k T^{1,-R}_{(2)}(A_n) = 0$ for other degrees. Thus we proved that

$$H^3(A_n) \cong T^1_{(2)}(A_n) \cong \mathrm{Ext}^1(\Omega^2_{A_n|k}, A_n) \cong A_n/(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$$

and they have dimension n as k-vector spaces. In particular, it holds that $\mathrm{Ext}^2_{A_n}(\Omega_{A_n|k}, A_n) = \mathrm{Hom}(\Omega^3_{A_n|k}, A_n) = 0$, which can also be easily checked using the results from Section 2.5.

3.6 The Hochschild cohomology of higher dimensional toric varieties

In this section we compute $T^{1,-R}_{(i)}(A)$ for higher dimensional toric varieties. Altmann [4], [5] described a relation between the computation of $T^1_{(1)}(-R)$ and the convex geometry of $Q(R)$ (using Minkowski summands of $Q(R)$). We will develop another approach that will also allow us to compute $T^1_{(i)}(-R)$ for $i > 1$. At the end we will obtain explicit formulas for 3-dimensional toric varieties (see Proposition 3.6.2). As far as we know the techniques that we use to obtain these calculations are new even in the case $i = 1$. In this section we also obtain a formula for $T^1_{(i)}(-R)$ for affine cones over smooth toric Fano varieties in arbitrary dimension (see Theorem 3.6.7).

Let a cone $\sigma = \langle a_1, ..., a_N \rangle$ represent an n-dimensional toric variety, $n \geq 3$. For $R \in M$ we define an affine space

$$\mathbb{A}(R) := [R = 1] = \{a \in N_{\mathbb{R}} \mid \langle a, R \rangle = 1\} \subset N_{\mathbb{R}}.$$

The *cross-cut* of σ in degree R is the polyhedron

$$Q(R) := \sigma \cap [R = 1] \subset \mathbb{A}(R).$$

The compact part of $Q(R)$ is generated by its vertices $\bar{a}_j := a_j / \langle a_j, R \rangle$ for j satisfying $\langle a_j, R \rangle \geq 1$. We write $d_1, ..., d_K \in R^{\perp}$ for the compact edges of $Q(R)$. For each compact 2-face $\epsilon < Q(R)$ we define its *sign vector* $\underline{\epsilon} \in \{0, \pm 1\}^K$ to be

$$\epsilon_i := \begin{cases} \pm 1 & \text{if } d_i \text{ is an edge of } \epsilon \\ 0 & \text{otherwise,} \end{cases}$$

where the signs are chosen so that the oriented edges $\epsilon_i d_i$ fit into a cycle along the boundary of ϵ. In particular, $\sum_i \epsilon_i d_i = 0$.

Let us recall Altmann's construction. It can be divided into three steps (see [5]):

- Step 1: $T^1(-R)$ equals the complexified (in our case $\mathbb{C}$ will be replaced by a field k) cohomology of the complex

$$N_{\mathbb{R}} \to \oplus_j (\mathrm{Span}_{\mathbb{R}} E_j^R)^* \to \oplus_{\langle a_j, a_k \rangle < \sigma} (\mathrm{Span}_{\mathbb{R}} E_{jk}^R)^*. \tag{3.8}$$

- Step 2:

  We can represent an element of $\oplus_j (\mathrm{Span}_{\mathbb{R}} E_j^R)^*$ by a family of elements

$$\begin{cases} b_j \in N_{\mathbb{R}} & \text{if} \langle a_j, R \rangle \geq 2, \\ b_j \in N_{\mathbb{R}} / \mathbb{R} \cdot a_j & \text{if} \langle a_j, R \rangle = 1. \end{cases}$$

  We choose now "new coordinates"

$$\bar{b}_j := b_j - \langle b_j, R \rangle \bar{a}_j \in R^{\perp}, \text{ which is well-defined even in the case } \langle a_j, R \rangle = 1;$$

$$s_j := -\langle b_j, R \rangle \text{ for } j \text{ meeting } \langle a_j, R \rangle \geq 2 \text{ (inducing an element of W(R) defined below).}$$

  We can relate this coordinates with Minkowski summands of a polytope $Q(R)$ and thus we obtain that $T^1(-R) \subset V_{\mathbb{C}}(R) \oplus W_{\mathbb{C}}(R) / (\underline{1}, \underline{1})$,

  where

$$V(R) := \{(t_1, ..., t_K) \in \mathbb{R}^K \mid \sum_i t_i \epsilon_i d_i = 0\} \text{ for every compact 2-face } \epsilon < Q(R)\},$$

$$W(R) := \mathbb{R}^{\#\{\text{vertices of } Q(R) \text{ not in } N\}}.$$

- Step 3:

  We describe the relations between elements $(\underline{t}, \underline{s}) \in V(R) \oplus W(R)$.

We already generalized Step 1 (see Corollary 3.4.4). Now we use another approach that will also give us explicit formulas for all i (we also do not know how to generalize Step 2 and Step 3). In the three-dimensional case we obtain a formula for $T^1_{(i)}(-R)$ for all i that can be easily computed and depends only on basic combinatorial properties of the cone (see Proposition 3.6.2). In particular, we obtain explicit formulas also in the case $i = 1$ and we will see that for isolated and Gorenstein singularities our formula agrees with Altmann's formula obtained with Minkowski summands (see Corollary 3.6.3 and Corollary 3.6.4).

Lemma 3.6.1. *Let Y be a toric surface given by $\sigma = \langle a_1, a_2\rangle \subset N_{\mathbb{R}} \cong \mathbb{R}^2$. We have $\dim_k \operatorname{Span}_k E^R_{12} = \max\{0, W_1(R) + W_2(R) - 2 - \dim_k T^{1,-R}_{(1)}(Y)\}$, where*

$$W_j(R) := \begin{cases} 2 & \text{if } \langle a_j, R\rangle > 1 \\ 1 & \text{if } \langle a_j, R\rangle = 1 \\ 0 & \text{if } \langle a_j, R\rangle \le 0. \end{cases}$$

Proof. It follows immediately by Proposition 3.5.1. □

Remark 10. $W_j(R)$ is a number and is not related to Altmann's notation of $W(R)$ defined above. The same for $V^i_j(R)$ defined below.

Let $d_{jk} := \overline{\bar{a}_j\bar{a}_k}$ denote the compact edges of $Q(R)$ (for $\langle a_j, a_k\rangle \le \sigma$, $\langle a_j, R\rangle \ge 1$, $\langle a_k, R\rangle \ge 1$). We denote the lattice $N \cap \operatorname{Span}_k\langle a_j, a_k\rangle$ by $\bar{N}_{jk}$ and its dual by $\bar{M}_{jk}$. Let $\bar{R}_{jk}$ denote the projection of R to $\bar{M}_{jk}$.

Proposition 3.6.2. *If the compact part of $Q(R)$ lies in a two-dimensional affine space we have*

$$\dim_k T^1_{(i)}(-R) = \max\Big\{0, \sum_{j=1}^{N} V^i_j(R) - \sum_{d_{jk}\in Q(R)} Q^i_{jk}(R) - \binom{n}{i} + s^i_{Q(R)}\Big\},$$

where

$$V^i_j(R) := \begin{cases} \binom{n}{i} & \text{if } \langle a_j, R\rangle > 1 \\ \binom{n-1}{i} & \text{if } \langle a_j, R\rangle = 1 \\ 0 & \text{if } \langle a_j, R\rangle \le 0, \end{cases}$$

$$Q^i_{jk}(R) := \begin{cases} \binom{W_j(R)+W_k(R)+n-4-\dim_k T^1_{\langle a_j,a_k\rangle}(-\bar{R}_{jk})}{i} & \text{if } \langle a_j, R\rangle, \langle a_k, R\rangle \neq 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$s^i_{Q(R)} := \begin{cases} \dim_k \wedge^i \big(\bigcap_{d_{jk}\in Q(R)} \operatorname{Span}_k E^R_{jk}\big) & \text{if } Q(R) \text{ is compact} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. From Theorem 3.4.3 we know that $T^1_{(i)}(-R)$ is the cohomology group of the complex

$$\bar{C}^i_{(i)}(M_k; k) \to \oplus_j \bar{C}^i_{(i)}(\operatorname{Span}_k E^R_j; k) \to \oplus_{\langle a_j, a_k\rangle \le \sigma} \bar{C}^i_{(i)}(\operatorname{Span}_k(E^R_{jk}); k).$$

Let $f := (f_1, ..., f_N) \in \oplus_j \bar{C}^i_{(i)}(\operatorname{Span}_k E^R_j)$. We see that $V^i_j(R) = \dim_k(\wedge^i \operatorname{Span}_k E^R_j)$. Assume now that $\operatorname{Span}_k E^R_j$, $\operatorname{Span}_k E^R_k \neq \emptyset$, otherwise we have $\operatorname{Span}_k E^R_{jk} = \emptyset$. We can easily verify that $Q^i_{jk}(R) = \dim_k(\wedge^i \operatorname{Span}_k E^R_{jk})$: we have $\dim_k(\operatorname{Span}_k E^R_{jk}) = n - 2 + \dim_k(\operatorname{Span}_k \bar{E}^{\bar{R}_{jk}}_{jk})$, where

$\bar{E}_{jk}$ is a generating set of $\langle a_j, a_k\rangle^\vee \cap \bar{M}_{jk}$. From Lemma 3.6.1 we know that $\dim_k(\mathrm{Span}_k \bar{E}^{\bar{R}}_{jk}) = \max\{0, W_j(R) + W_k(R) - 2 - \dim_k T^1_{\langle a_j,a_k\rangle}(-\bar{R}_{jk})\}$. Thus we have

$$\dim_k T^1_{(i)}(-R) = \max\big\{0, \sum_{j=1}^{N} V^i_j(R) - \sum_{d_{jk}} Q^i_{jk}(R) - \binom{n}{i} + s^i\big\},$$

where s^i equals the dimension of the domain of restrictions (that we get with restricting $f_j = f_k$ on $\mathrm{Span}_k\, E^R_{jk}$) that repeats. We can easily verify that $s^i = s^i_{Q(R)}$. □

3.6.1 Computations of $T^1_{(i)}(A)$ for three-dimensional toric varieties

Using Proposition 3.6.2 we can easily compute $T^1_{(i)}(-R)$ for three-dimensional affine toric varieties. From straightforward computation of the formula in Proposition 3.6.2 we obtain the following corollary.

Corollary 3.6.3. *Let X be an isolated 3-dimensional toric singularity. Without loss of generality we can assume that generators $a_1, ..., a_N$ are arranged in a cycle (we define $a_{N+1} := a_1$). We have the following formulas:*

$$\dim_k T^1_{(1)}(-R) = \begin{cases} \max\big\{0, \#\{\bar{a}_j \mid \bar{a}_j \in N,\ \textit{i.e.},\ \langle a_j, R\rangle = 1\} - 3\big\} & \textit{if } R > 0 \\ \#\{\bar{a}_j \mid \bar{a}_j \in N,\ \textit{not contained in a noncompact edge}\} & \textit{if } R \not> 0, \end{cases}$$
$$\dim_k T^1_{(2)}(-R) = \begin{cases} \max\big\{0, \#\{\bar{a}_j \mid \bar{a}^j \in N\} + C(R) - 3\big\} & \textit{if } R > 0 \\ \max\big\{0, \#\{\bar{a}_j \mid \bar{a}^j \in N\} + C(R) - 2\big\} & \textit{if } R \not> 0, \end{cases}$$
$$\dim_k T^1_{(3)}(-R) = \max\{0, C(R) - 1\},$$
$$\dim_k T^1_{(i)}(-R) = 0 \textit{ for } i \geq 4,$$

where $C(R) := \#\{$chambers with $\langle a_j, R\rangle > 1\}$ and a chamber with $\langle a_j, R\rangle > 1$ means $\langle a_j, R\rangle > 1$ for $j = j_0, j_0+1, ..., j_0+k$ for some $j_0, k \in \mathbb{N}$ and $\langle a_j, R\rangle \leq 1$ for $j = j_0 - 1$ and $j = j_0 + k + 1$.

Proof. We use Theorem 3.6.2 with $n = 3$. We also have $T^1_{\langle a_j,a_{j+1}\rangle}(-\bar{R}_{j,j+1}) = 0$ for all j since X is smooth in codimension 2. Let m_1 be a number of a_j with $\langle a_j, R\rangle = 1$ (i.e. m_1 is the number of lattice vertices of the polytope $Q(R)$) and m_2 be a number of vertices a_j with $\langle a_j, R\rangle > 1$.

If $R > 0$ we have $N = m_1 + m_2$ and thus we can easily compute that

$$s^i_{Q(R)} = \dim_k \wedge^i \bigcap_j \mathrm{Span}_k E^R_{j,j+1} = \binom{\max\{0, 3 - m_1\}}{i}.$$

For $i = 1$ we have $\sum_{j=1}^N V^1_j(R) = 3m_2 + 2m_1$, $\sum_{j=1}^N W_j(R) = 2m_1 + m_2$ and thus

$$\sum_{d_j} Q^1_{j,j+1}(R) = 2\sum_{j=1}^{N}(W_j(R)) - N = 4m_2 + 2m_1 - m_1 - m_2 = 3m_2 + m_1.$$

Thus we see that $T^1_{(1)}(-R) = \max\{0, m_1 - 3\}$.

For $i = 2$ we have

$$Q^2_{j,j+1}(R) = \begin{cases} 3 & \text{if } V^2_j(R) = V^2_{j+1}(R) = 3 \\ 1 & \text{if } V^2_j(R) = 2, V^2_{j+1}(R) = 3 \text{ or } V^2_j(R) = 3, V^2_{j+1}(R) = 2 \\ 0 & \text{otherwise} \end{cases}$$

and thus

$$V_j^2(R) - Q_{j,j+1}^2(R) = \begin{cases} 1 & \text{if } \langle a_j, R\rangle = 1 \text{ and } \langle a_{j+1}, R\rangle = 1 \\ 0 & \text{if } \langle a_j, R\rangle = 1 \text{ and } \langle a_{j+1}, R\rangle = 2 \\ 2 & \text{if } \langle a_j, R\rangle = 2 \text{ and } \langle a_{j+1}, R\rangle = 1 \\ 0 & \text{if } \langle a_j, R\rangle = 2 \text{ and } \langle a_{j+1}, R\rangle = 2 \\ 0 & \text{otherwise,} \end{cases}$$

from which we easily obtain the formula that we want.

For $i = 3$ we have $\sum_{j=1}^N V_j^3(R) = m_2$,

$$Q_{j,j+1}^3(R) = \begin{cases} 1 & \text{if } V_j^3(R) = V_{j+1}^3(R) = 3 \\ 0 & \text{otherwise} \end{cases}$$

and the formula follows.

If $R \not> 0$ we do not have any compact 2-faces in $Q(R)$. We define the index set S for vertices that do not lie on unbounded edges of $Q(R)$ (note that $|S| = m_1 + m_2 - 2$). We denote two vertices that lie on unbounded edges by k and l.

For $i = 1$ Theorem 3.6.2 gives us (since $W_j(R) = V_j^1(R) - 1$ if $\langle a_j, R\rangle > 0$) the following:

$$\begin{aligned} &\sum_{j=1}^N V_j^1(R) - \sum_{j=1}^N Q_{j,j+1}^1(R) - 3 = \\ &= V_k^1(R) + V_l^1(R) + \big(\sum_{j\in S} V_j^1(R)\big) - \big(2(\sum_{j\in S} V_j^1(R)) + V_k^1(R) + V_l^1(R) - 3(m_1 + m_2 - 1)\big) - 3 = \\ &= -\big(\sum_{j\in S} V_j^1(R)\big) + 3(m_1 + m_2 - 2), \end{aligned}$$

which equals $\#\{\bar a_j \mid \bar a_j \in N, \text{ not contained in a noncompact edge}\}$.

In the following we denote for short $C := C(R)$. We consider the case $i = 2$. Let $\bar a_l$ denote the vertex that lies on an unbounded edge and has the highest index l (recall that generators a_j are arranged in a cycle). We consider two cases: first if $\langle a_l, R\rangle = 1$, then we compute that $\sum_{j=1}^{l-1}(V_j^2(R) - Q_{j,j+1}^2(R)) = C + m_1$, thus to get $\dim_k T_{(2)}^1(-R)$ we also need to add $V_l^2(R) - 3 = -2$. If $\langle a_l, R\rangle > 1$ we see that

$$\sum_{j=1}^{l-1}(V_j^2(R) - Q_{j,j+1}^2(R)) = C + m_1 - 2.$$

Note that we get -2 because in $C + m_1$ we count also the last chamber and the last vertex with $\langle a_j, R\rangle = 1$ and thus we need to subtract 2. We also need to add $V_l^2(R) - 3 = 0$. In both cases (if $\langle a_l, R\rangle = 1$ or if $\langle a_l, R\rangle > 1$) we obtain the same formula, i.e.,

$$\dim_k T_{(2)}^1(-R) = C + m_1 - 2.$$

For $i = 3$ we again consider two cases: first if $\langle a_l, R\rangle = 1$, then we compute that

$$\sum_{j=1}^{l-1}(V_j^3(R) - Q_{j,j+1}^3(R)) = \max\{0, C - 1\}.$$

If $\langle a_l, R\rangle > 1$, then we see that

$$\sum_{j=1}^{l-1}(V_j^3(R) - Q_{j,j+1}^3(R)) = \max\{0, C-1\} - 1.$$

In both cases we obtain the same formula

$$\dim_k T^1_{(3)}(-R) = \max\{0, C-1\}.$$

□

Remark 11. Note that in the case $i = 1$ we obtain the same formula as Altmann in [1].

Let X be a three-dimensional toric Gorenstein singularity given by a cone $\sigma = \langle a_1, ..., a_N\rangle$, where $a_1, ..., a_N$ are arranged in a cycle. Let $s_1, ..., s_N$ be the fundamental generators of the dual cone $\sigma^\vee$, labelled so that $\sigma \cap (s_j)^\perp$ equals the face spanned by $a_j, a_{j+1} \in \sigma$. Let R^* denote the degree such that $\langle R^*, a_i\rangle = 1$ for all i (R^* exists for Gorenstein toric varieties). With $\ell(j)$ we denote the length of the edge d_j. With P we denote the polytope $\sigma \cap [R^* = 1]$. The following corollary is also obtained with a straightforward computation of the formula in Proposition 3.6.2.

Corollary 3.6.4. *Let X be a three-dimensional toric Gorenstein singularity given by a cone $\sigma = \langle a_1, ..., a_N\rangle$, where $a_1, ..., a_N$ are arranged in a cycle. It holds that $T^1_{(1)}(-R)$ is non-trivial in the following cases:*

- *$R = R^*$ with $\dim_k T^1_{(1)}(-R) = N - 3$,*
- *$R = qR^*$ (for $q \geq 2$) with $\dim_k T^1_{(1)}(-R) = \max\{0, \#\{j \mid q \leq \ell(j)\} - 2\}$,*
- *$R = qR^* - ps_j$ with $2 \leq q \leq \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin int(\sigma^\vee)$. In this case $\dim_k T^1_{(1)}(-R) = 1$.*

Additional degrees exist only in the following two (overlapping) exceptional cases:

- *P contains a pair of parallel edges d_j, d_k, both longer than every other edge. Then $\dim_k T^1_{(1)}(-qR^*) = 1$ for q in the range*

$$\max\{\ell(l) \mid l \neq j, k\} < q \leq \min\{\ell(j), \ell(k)\}\},$$

- *P contains a pair of parallel edges d_j, d_k with distance d ($d := \langle a_j, s_k\rangle = \langle a_k, s_j\rangle$) and it holds that $\ell(k) > d \geq \max\{\ell(l) \mid l \neq j, k\}$. In this case $\dim_k T^1_{(1)}(-R) = 1$ for $R = qR^* + ps_j$ with $1 \leq q \leq \ell(j)$ and $1 \leq p \leq (\ell(k) - q)/d$.*

$T^1_{(2)}(-R)$ is non-trivial in the following cases:

- *$R = R^*$ with $\dim_k T^1_{(2)}(-R) = N - 3$,*
- *$R = qR^*$ (for $q \geq 2$) with $\dim_k T^1_{(2)}(-R) = \max\{0, 2 \cdot \#\{j \mid q \leq \ell(j)\} - 3\}$,*
- *$R = qR^* - ps_j$ with $2 \leq q \leq \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin int(\sigma^\vee)$. In this case $\dim_k T^1_{(2)}(-R) = 2$.*

Additional degrees exist only in the following two (overlapping) exceptional cases:

- *P contains a pair of parallel edges d_j, d_k, both longer than every other edge. Then $\dim_k T^1_{(2)}(-qR^*) = 2$ for q in the range*

$$\max\{\ell(l) \mid l \neq j,k\} < q \leq \min\{\ell(j), \ell(k)\}\},$$

- *P contains a pair of parallel edges d_j, d_k with distance $d = \langle a_j, s_k\rangle = \langle a_k, s_j\rangle$ and it holds that $\ell(k) > d \geq \max\{\ell(l) \mid l \neq j,k\}$. In this case $\dim_k T^1_{(2)}(-R) = 2$ for $R = qR^* + ps_j$ with $1 \leq q \leq \ell(j)$ and $1 \leq p \leq (\ell(k) - q)/d$.*

$T^1_{(3)}(-R)$ is non-trivial in the following cases:

- *$R = qR^*$ (for $q \geq 2$) with $\dim_k T^1_{(3)}(-R) = \max\{0, \#\{j \mid q \leq \ell(j)\} - 1\}$,*
- *$R = qR^* - ps_j$ with $2 \leq q \leq \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin int(\sigma^\vee)$. In this case $\dim_k T^1_{(3)}(-R) = 1$.*

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- *P contains a pair of parallel edges d_j, d_k with distance $d = \langle a_j, s_k\rangle = \langle a_k, s_j\rangle$ and it holds that $\ell(k) > d \geq \max\{\ell(l) \mid l \neq j,k\}$. In this case $\dim_k T^1_{(3)}(-R) = 1$ for $R = qR^* + ps_j$ with $1 \leq q \leq \ell(j)$ and $1 \leq p \leq (\ell(k) - q)/d$.*

And we have $T^1_{(i)}(-R) = 0$ for $i \geq 4$.

Proof. We distinguished the following cases.

- Let $R = R^*$.

  We see that $s^i_{Q(R^*)} = 0$ for all i. By Corollary 3.5.3 we also have $T^1_{\langle a_j, a_{j+1}\rangle}(-\bar{R}^*{}_{j,j+1}) = 0$ for all j. By Proposition 3.6.2 we have $\dim_k T^1(-R^*) = \dim_k T^1_{(2)}(-R^*) = N - 3$ and $T^1_{(i)}(-R^*) = 0$ for $i > 2$.

- Let $R = qR^*$, where $q \geq 2$.

  We have $\sum_{j=1}^N V^i_j(R) = \binom{3}{i}N$. A two face $\langle a_j, a_{j+1}\rangle \subset \bar{N}_{j,j+1} \cong \mathbb{Z}^2$ is a Gorenstein cyclic quotient singularity of type $A_{\ell(j)-1}$. Let us define $v := \#\{j \mid q \leq \ell(j)\}$.

  For $i = 1$ we have $\sum_{j=1}^N Q^1_{j,j+1}(R) = 3N - v$ (since for $q \leq \ell(j)$ we have $\dim_k T^1_{\langle a_j, a_k\rangle}(-q\bar{R}) = 1$). Thus $\dim_k T^1_{(1)}(-R) = v - 3 + s^1_{Q(R)}$ holds by Proposition 3.6.2.

  In the case $i = 2$ we have $\sum_{j=1}^N Q^2_{j,j+1}(R) = \binom{2}{2}v + \binom{3}{2}(N - v) = 3N - 2v$. Thus Proposition 3.6.2 gives us that $\dim_k T^1_{(2)}(-R) = 2v - 3 + s^2_{Q(R)}$.

For $i = 3$ we have $\sum_{j=1}^N Q^3_{j,j+1}(R) = N - v$. By Proposition 3.6.2 we have $\dim_k T^1_{(3)}(-R) = v - 1 + s^3_{Q(R)}$.

We now compute $\dim_k \cap_j \mathrm{Span}_k E^R_{j,j+1}$ (and thus $s^i_{Q(R)}$ for all i). We have

$$\dim_k \cap_j \mathrm{Span}_k E^R_{j,j+1} \geq 1$$

since $\mathrm{Span}_k\{R\} \subset \cap_j \mathrm{Span}_k E^R_{j,j+1}$ for all $j = 1, ..., N$ (note that we are in the case $R = qR^*$ for $q \geq 2$). If $\dim_k \cap_j \mathrm{Span}_k E^R_{j,j+1} = 3$, then trivially $T^1_{(i)}(-R) = 0$ for all i. Now we will show

CLAIM: For $R = qR^*$ we have $\dim_k \cap_j \mathrm{Span}_k E^R_{j,j+1} = 2$ if and only if P consists of parallel edges d_j, d_k and $\mathrm{Span}_k E^R_{l,l+1} = N_{\mathbb{R}}$ holds for all $l \in \{1, ..., N\} \setminus \{j, k\}$ (in particular, we have $\mathrm{Span}_k E^R_{j,j+1} = \mathrm{Span}_k\{a_j^\perp \cap a_{j+1}^\perp, R\}$ and $\mathrm{Span}_k\{a_k^\perp \cap a_{k+1}^\perp, R\} = \mathrm{Span}_k E^R_{k,k+1}$).

Proof: we need to show that $a \in \mathrm{Span}_k E^R_{j,j+1}$ if and only if $a \in \mathrm{Span}_k E^R_{k,k+1}$. Since $\mathrm{Span}_k\{R\} \subset \mathrm{Span}_k E^R_{j,j+1}, \mathrm{Span}_k E^R_{k,k+1}$, it is enough to show that

$$a \in a_j^\perp \cap a_{j+1}^\perp \implies a \in \mathrm{Span}_k E^R_{k,k+1}$$

and

$$a \in a_k^\perp \cap a_{k+1}^\perp \implies a \in \mathrm{Span}_k E^R_{j,j+1}.$$

Let $a \in a_j^\perp \cap a_{j+1}^\perp$. Since d_j and d_k are parallel, we have $a_{k+1} = a_k + \alpha(a_{j+1} - a_j)$, thus we see that $\langle a, a_{k+1}\rangle = \langle a, a_k\rangle$, which implies that $a \in \mathrm{Span}_k E^R_{k,k+1} = \mathrm{Span}_k\{a_k^\perp \cap a_{k+1}^\perp, R\} = \{c \in M_k \mid \langle c, a_k\rangle = \langle c, a_{k+1}\rangle\}$, since R also has a property that $\langle R, a_{k+1}\rangle = \langle R, a_k\rangle$. The same for the other direction and thus we prove the claim.

From this we immediately obtain formulas that we want (note that the exceptional cases are given when $\dim_k \cap_j \mathrm{Span}_k E^R_{j,j+1} = 2$ and $v = 2$.

- Let $R \not> 0$. We immediately see that the only possible cases for having a non-zero $T^1_{(i)}(-R)$ are when $R = qR^* - ps^j$ with $2 \leq q \leq l(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R \notin \mathrm{int}(\sigma^\vee)$. In these cases $\dim_k T^1_{(1)}(-R) = \dim_k T^1_{(3)}(-R) = 1$ and $\dim_k T^1_{(2)}(-R) = 2$.

- Let $R > 0$ and $R \neq qR^*$. We can check (as we did above) that $T^1_{(i)}(-R) = 0$ for all i, except when P contains a pair of parallel edges d_j, d_k with distance $d = \langle a_j, s_k\rangle = \langle a_k, s_j\rangle$ and it holds that $\ell(k) > d \geq \max\{\ell(l) \mid l \neq j, k\}$. In this case we have $\dim_k T^1_{(1)}(-R) = \dim_k T^1_{(3)}(-R) = 1$ and $\dim_k T^1_{(2)}(-R) = 2$ for $R = qR^* + ps_j$ with $1 \leq q \leq \ell(j)$ and $1 \leq p \leq (\ell(k) - q)/d$.

□

We see that in the case $i = 1$ our formulas agree with the ones given in [5].

3.6.2 Computations of $T^1_{(i)}(A)$ for affine cones over smooth toric Fano varieties

We start with the following observation.

Remark 12. When $Q(R)$ is not contained in a two-dimensional affine space, we can still follow the proof of Proposition 3.6.2 and we obtain that

$$\dim_k T^1_{(i)}(-R) \geq \sum_{j=1}^{N} V^i_j(R) - \sum_{d_{jk}\in Q(R)} Q^i_{jk}(R) - \binom{n}{i}. \tag{3.9}$$

The cycles in $Q(R)$ give us some repetitions on the restrictions ($f_j = f_k$ on $\mathrm{Span}_k E^R_{jk}$) and thus it is hard to obtain a formula for $\dim_k T^1_{(i)}(-R)$ in higher dimensions. For every tree T in $Q(R)$ we obtain also upper bounds:

$$\dim_k T^1_{(i)}(-R) \leq \sum_{j=1}^{N} V^i_j(R) - \sum_{d_{jk}\in T} Q^i_{jk}(R) - \binom{n}{i}, \tag{3.10}$$

since no cycles appear in T.

We focus now on higher dimensional toric varieties. Let us consider the special case of $\mathbb{Q}$-Gorenstein toric varieties that are smooth in codimension two.

Lemma 3.6.5. *Let Y be a $\mathbb{Q}$-Gorenstein variety which is smooth in codimension two. If $R \in M$ is a degree such that $\langle a_j, R\rangle \geq 2$ for some $j \in \{1, ..., N\}$, then $T^1_{(i)}(-R) = 0$ for all $i \geq 1$.*

Proof. The hyperplane $H := \{a \in N_{\mathbb{R}} \mid \langle a, gR - R^*\rangle = 0\}$ subdivides the set of generators of σ: $H^R_{\leq 0} := \{a_j \mid \langle a_j, R\rangle \leq 0\}$, $H^R_1 = \{a_j \mid \langle a_j, R\rangle = 1\}$ and $H^R_{\geq 2} = \{a_j \mid \langle a_j, R\rangle \geq 2\}$. We fix a vertex $\bar{a}_{j_0}$ of $Q(R)$ with $\langle a_{j_0}, R\rangle \geq 2$. Skipping some of the edges, we can arrange $Q(R)$ into a tree T with the main vertex $\bar{a}_{j_0}$, the set of leaves equal to H^R_1 and the set of inner vertices equal to $H^R_{\geq 2} \setminus \bar{a}_{j_0}$. From the equation (3.10) we see that $\dim_k T^1_{(i)}(-R) \leq \sum_{j=1}^{N} V^i_j(R) - \sum_{d_{jk}\in T} Q^i_{jk}(R) - \binom{n}{i}$ and we can easily verify that this is ≤ 0. □

Deformation theory of affine varieties is closely related to the Hodge theory of smooth projective varieties. We will use the following recent result.

Theorem 3.6.6. *Let $X = \mathrm{Spec}(A)$ be an affine cone over a projective variety Y. On $T^q_{(i)}(A)$ we have a natural $\mathbb{Z}$ grading and if Y is arithmetically Cohen-Macaulay and $\omega_Y \cong \mathcal{O}_Y(m)$, then*

$$T^q_{(i)}(A)_m = \begin{cases} H^{n-i,q}_{prim}(Y) & \text{if } i > q \\ H^{n-q-1,i}_{prim}(Y) & \text{if } i \leq q, \end{cases}$$

where $T^q_{(i)}(A)_m$ denotes the degree $m \in \mathbb{Z}$ elements of $T^q_{(i)}(A)$ and $H^{p,q}_{prim}(Y)$ is the primitive cohomology, namely the kernel of the Lefschetz maps

$$H^{p,q}(Y) \to H^{p+1,q+1}(Y).$$

Proof. See [25, Corollary 3.4]. □

We will apply Theorem 3.6.6 to the case of Fano toric varieties, where reflexive polytopes come into the play.

Definition 54. A full dimensional lattice polytope $P \subset M_{\mathbb{R}}$ is called *reflexive* if $0 \in \mathrm{int}(P)$ and, moreover, its dual

$$P^\vee := \{a \in N_{\mathbb{R}} \mid \langle a, P\rangle \geq -1\}$$

is also a lattice polytope. Here the expression $\langle a, P\rangle$ means the minimum over the set $\{\langle a, r\rangle \mid r \in P\}$.

Reflexive polytopes lead to interesting toric varieties that are important for mirror symmetry. There is a one-to-one correspondence between Gorenstein toric Fano varieties and reflexive polytopes (see [20, Theorem 8.3.4]).

If X is a Gorenstein affine toric variety given by $\sigma = \mathrm{Cone}(P)$, where P is a reflexive polytope, then X is an affine cone over a smooth Fano toric variety Y, embedded in some $\mathbb{P}^n$ by the anticanonical line bundle.

Theorem 3.6.7. *Let $X = \mathrm{Spec}(A)$ be an n-dimensional affine cone over a smooth toric Fano variety Y $(n \geq 3)$. Then $T^1_{(i)}(A) = 0$ for $n \geq 4$ and $i = 2, ..., n-2$. Moreover, $\dim_k T^1_{(n-1)}(A) = N - n$ and $T^1_{(k)}(A) = 0$ for $k \geq n \geq 3$. Furthermore, $\dim_k T^1_{(1)}(A) = N - 3$ for $n = 3$ and $T^1_{(1)}(A) = 0$ for $n > 3$.*

Proof. It holds that $H^{p,q}(Y) = 0$ for $p \neq q$ (see e.g. [12]) and thus also $H^{p,q}_{\mathrm{prim}}(Y) = 0$. By Theorem 3.6.6 we have $T^1_{(i)}(A)_{-1} = 0$ for $n \geq 4$ and $i = 2, ..., n-2$. Following the proof of Lemma 3.6.5, we see that if $R \neq R^* = (\underline{0}, 1)$ we have the following options:

1. there exists a_j, such that $\langle a_j, R\rangle \geq 2$, which implies that $T^{1,-R}_{(i)}(A) = 0$ for all $i \geq 1$ by Lemma 3.6.5.
2. $H^R_{\geq 2} = 0$ and $H^R_1 = \{a_j \in F\}$ for a facet F. There exists $s \in M$ such that $\langle s, a_j\rangle = 0$ for all $a_j \in F$. If $T^{1,-R}_{(i)}(A) \neq 0$ for some i, then $\dim_k T^{1,-R+\alpha s}_{(i)}(A) \neq 0$ for infinitely many $\alpha \in \mathbb{Z}$. Thus $\dim_k T^1_{(i)}(A) = \infty$, which is a contradiction since $T^1_{(i)}(A)$ is supported on the singular locus and A is an isolated singularity. Thus $T^{1,-R}_{(i)}(A) = 0$ for all $i \geq 1$.
3. $H^R_{\geq 2} = H^R_1 = 0$, which trivially implies that $T^{1,-R}_{(i)}(A) = 0$.

Now we focus in the case $i = n-1$. Above we saw that $T^{1,-R}_{(n-1)}(A) = 0$ if $R \neq R^*$. The inequality (3.9) is in the case $R = R^*$, $i = n-1$ an equality since no restrictions repeat and thus we obtain

$$\dim_k T^{1,-R^*}_{(n-1)}(A) = \max\Big\{0, \sum_{j=1}^{N} V_j^{n-1}(R^*) - \sum_{d_{jk} \in Q(R^*)} Q_{jk}^{n-1}(R^*) - \binom{n}{n-1}\Big\}.$$

Since $V_j^{n-1}(R^*) = \binom{n-1}{n-1} = 1$ and $Q_{jk}^{n-1}(R^*) = \binom{n-2}{n-1} = 0$ we obtain $T^{1,-R^*}_{(n-1)}(A) = N - n$. With the same procedure we immediately see that $T^1_{(k)}(A) = 0$ for $k \geq n$. Finally we focus on the case $i = 1$. With the same computations as above we see that $\dim_k T^1_{(1)}(A) = 0$ if $n > 3$. If $n = 3$, then $\dim_k T^1_{(1)}(A)_{-1} = \dim_k T^1_{(1)}(A)$ as above and $T^1_{(1)}(A) = H^{1,1}_{\mathrm{prim}}(Y)$ by Theorem 3.6.6. We have $\dim_k H^{1,1}_{\mathrm{prim}}(Y) = N - 3$ by [20, Theorem 9.4.11] and thus we conclude the proof. ∎

Remark 13. From Theorem 3.6.6 and Theorem 3.6.7 it follows that

$$\dim_k H^{1,1}_{\text{prim}}(Y) = N - n = \text{rk}(\text{pic}(Y)) - 1.$$

For $i = n-2$ we can generalize Theorem 3.6.7 to the following:

Proposition 3.6.8. *Let $X = \text{Spec}(A)$ be an n-dimensional $\mathbb{Q}$-Gorenstein variety given by $\sigma = \text{Cone}(P)$, where P is a simplicial polytope. Then $T^1_{(n-2)}(A) = 0$.*

Proof. The only non-clear part is when X is Gorenstein and we consider the degree $R = R^*$. From the proof of Proposition 3.6.2 we see that

$$\dim_k T^{1,-R^*}_{(n-2)}(A) = \max\big\{0, \sum_{j=1}^{N} V_j^{n-2}(R^*) - \sum_{d_{jk} \in Q(R^*)} Q_{jk}^{n-2}(R^*) - \binom{n}{n-2}\big\},$$

since no restrictions repeat. Let e denote the number of edges in $Q(R^*)$. Since $V_j^{n-2}(R^*) = \binom{n-1}{n-2} = n-1$ and $Q_{jk}^{n-2}(R^*) = \binom{n-2}{n-2} = 1$, we obtain

$$\dim_k T^1_{(n-2)}(-R^*) = \max\{0, N(n-1) - e - n(n-1)/2\}.$$

For simplicial polytopes it holds that $e \geq N(n-1) - n(n-1)/2$ by the lower bound conjecture proved in [10] and thus $\dim_k T^1_{(n-2)}(-R^*) = 0$. □

Remark 14. For $i = 1$ we can generalize Theorem 3.6.7 to the following: $\mathbb{Q}$-Gorenstein toric varieties that are smooth in codimension 2 and $\mathbb{Q}$-factorial (or equivalently simplicial) in codimension 3 are globally rigid (see [68] or [2] for the affine case).

4 Deformation quantization

In Section 4.1 we compute the Gerstenhaber bracket in the toric setting. Poisson structures from a deformation point of view are analyzed in Section 4.2. We introduce the notion of deformation quantization of a Poisson structure. In Section 4.3 we present the formality theorem, which implies that every Poisson structure on a smooth affine variety can be quantized. The formality theorem can not be generalized to singular affine varieties (see Example 10). On the other hand we manage to prove that every Poisson structure on a possibly singular affine toric variety can be quantized (see Theorem 4.4.4 in Section 4.4), which is the main result of this chapter.

For basic theory of Poisson structures we refer the reader to [41]. For motivation and known results about quantizing (singular) affine Poisson varieties we refer to [30] and [63]. In [63] it is considered the quantization problem for the nilpotent cone (the nilpotent cone $\mathrm{Nil}\mathfrak{g} \subset \mathfrak{g}^*$ is the set of elements $\phi \in \mathfrak{g}^*$ such that for some $x \in \mathfrak{g}$ we have $\mathrm{ad}(x)\phi = \phi$). In the special case when $\mathfrak{g} = \mathfrak{sl}_2$ we obtain that $\mathrm{Nil}\mathfrak{g}$ is a Gorenstein toric surface. Using deformation of Calabi-Yau algebras Etingof and Ginzburg [24] analyze quantization of affine surfaces in $\mathbb{C}^3$ and quantization of del Pezzo surfaces. For quantization of singular projective varieties see results of Palamodov [56], [57] and [58].

4.1 The Gerstenhaber bracket for toric varieties

Recall the orthogonal idempotents $e_1 := e_3(1)$, $e_2 := e_3(2)$ and $e_3 := e_3(3)$ of the group ring $\mathbb{Q}[S_3]$ from the Subsection 2.3.2.

Lemma 4.1.1. *It holds that*

$$\begin{aligned}
e_1(a,b,c) &= \frac{1}{6}\big(2(a,b,c) - 2(c,b,a) + (a,c,b) - (b,c,a) + (b,a,c) - (c,a,b)\big),\\
e_2(a,b,c) &= \frac{1}{2}\big((a,b,c) + (c,b,a)\big),\\
e_3(a,b,c) &= \frac{1}{6}\big((a,b,c) - (c,b,a) - (a,c,b) + (b,c,a) - (b,a,c) + (c,a,b)\big).
\end{aligned}$$

Proof. Elementary computations (see also [56]). □

If $A = k[\sigma^\vee \cap M] = k[\Lambda]$, we can use the grading of M to rewrite the Gerstenhaber bracket. Very important will be the formula of the Gerstenhaber bracket $[f,g]$ for $f \in C^{2,-R}(A)$ and $g \in C^{2,-S}(A)$. We can write (similarly as in the proof of Lemma 3.2.1) $[f,g] \in C^{3,-R-S}(A)$ as:

$$\begin{aligned}
[f,g](x^{\lambda_1} \otimes x^{\lambda_2} \otimes x^{\lambda_3}) &= f(g(x^{\lambda_1} \otimes x^{\lambda_2}) \otimes x^{\lambda_3}) - f(x^{\lambda_1} \otimes g(x^{\lambda_2} \otimes x^{\lambda_3})) + \\
&\quad + g(f(x^{\lambda_1} \otimes x^{\lambda_2}) \otimes x^{\lambda_3}) - g(x^{\lambda_1} \otimes f(x^{\lambda_2} \otimes x^{\lambda_3}))\\
&= \big(f_0(-S+\lambda_1+\lambda_2,\lambda_3)g_0(\lambda_1,\lambda_2) - f_0(\lambda_1,-S+\lambda_2+\lambda_3)g_0(\lambda_2,\lambda_3) + \\
&\quad + g_0(-R+\lambda_1+\lambda_2,\lambda_3)f_0(\lambda_1,\lambda_2) - g_0(\lambda_1,-R+\lambda_2+\lambda_3)f_0(\lambda_2,\lambda_3)\big)x^{\lambda-R-S},
\end{aligned}$$

where $\lambda = \lambda_1 + \lambda_2 + \lambda_3$.

In general we have the following.

Lemma 4.1.2. *Let* $A = k[\Lambda]$, $f(x^{\lambda_1}, ..., x^{\lambda_m}) = \sum_{i=0}^{p} f_i(\lambda_1, ..., \lambda_m) x^{-R_i+\lambda_1+\cdots+\lambda_m} \in C^m(A)$ *and* $g(x^{\lambda_1}, ..., x^{\lambda_n}) = \sum_{j=0}^{r} g_j(\lambda_1, ..., \lambda_n) x^{-S_j+\lambda_1+\cdots\lambda_n} \in C^n(A)$, *where*

$$f_i \in C^m(\Lambda, \Lambda \setminus (R_i + \Lambda); k),$$

for $i = 0, .., p$ *and* $g_j \in C^n(\Lambda, \Lambda \setminus (S_j + \Lambda); k)$ *for* $j = 0, ..., r$. *Then*

$$[f, g](x^{\lambda_1}, ..., x^{\lambda_{m+n-1}}) = \sum_{i,j} [f_i, g_j] x^{-R_i-S_j+\lambda_1+\cdots\lambda_{m+n-1}},$$

where

$$[f_i, g_j] := f_i \circ g_j - (-1)^{(m+1)(n+1)} g_j \circ f_i \in C^{m+n-1}(\Lambda, \Lambda \setminus (R_i + S_j + \Lambda); k),$$

where $f_i \circ g_j(\lambda_1, ..., \lambda_{m+n-1}) :=$

$$\sum_{u=1}^{m} (-1)^{(u-1)(n+1)} \cdot f_i(\lambda_1, ..., \lambda_{u-1}, -S_j + \lambda_u + \cdots + \lambda_{u+n-1}, \lambda_{u+n}, ..., \lambda_{m+n-1}) g_j(\lambda_u, ..., \lambda_{u+n-1}).$$

Proof. It follows from the isomorphism in Lemma 3.2.1. □

For defining and deforming Poisson structures in the next section, the following computations will be useful.

Lemma 4.1.3. *If* $p, q \in C^2_{(2)}(A)$, *we have*

$$[p, q] = p(q(a, b), c) - p(a, q(b, c)) + q(p(a, b), c) - q(a, p(b, c)).$$

Projecting give us:

$$\begin{aligned}
e_1[p, q] &= \frac{2}{3} p(q(a, c), b) - \frac{1}{3} p(a, q(b, c)) + \frac{1}{3} p(q(a, b), c) + \\
&+ \frac{2}{3} q(p(a, c), b) - \frac{1}{3} q(a, p(b, c)) + \frac{1}{3} q(p(a, b), c), \\
e_2[p, q] &= 0, \\
e_3[p, q] &= \frac{2}{3} \big(p(q(a, b), c) + p(q(b, c), a) + p(q(c, a), b) + \\
&+ q(p(a, b), c) + q(p(b, c), a) + q(p(c, a), b) \big).
\end{aligned}$$

If we have $p, q \in C^2_{(1)}(A)$, *then* $[p, q] = e_1[p, q]$. *If we have* $p \in C^2_{(1)}(A)$ *and* $q \in C^2_{(2)}(A)$, *then* $[p, q] = e_2[p, q]$.

In particular, when $p = q$ *we have*

$$[p, p] = 2\big(p(p(a, b), c) - p(a, p(b, c))\big),$$

with

$$e_1[p, p] = 2\big(\frac{2}{3} p(p(a, c), b) - \frac{1}{3} p(a, p(b, c)) + \frac{1}{3} p(p(a, b), c)\big)$$

$$e_2[p, p] = 0 \tag{4.1}$$

$$e_3[p, p] = \frac{4}{3} \big(p(p(a, b), c) + p(p(b, c), a) + p(p(c, a), b) \big). \tag{4.2}$$

Proof. Everything is straightforward and easy computation; see also [56]. □

4.2 Poisson structures

Definition 55. A *Poisson algebra* is a k-vector space A equipped with two mulltiplications $(F,G) \mapsto F \cdot G$ and $(F,G) \mapsto \{F,G\}$, such that

- $(A, \cdot)$ is a commutative associative algebra over k, with unit 1,
- $(A, \{\cdot,\cdot\})$ is a Lie algebra over k,
- the two multiplications are compatible in the sense that

$$\{a \cdot b, c\} = a \cdot \{b,c\} + b \cdot \{a,c\}, \tag{4.3}$$

  where a, b and c are arbitrary elements of A.

The Lie bracket $\{\cdot,\cdot\}$ is then called the *Poisson bracket* (or the *Poisson structure*).

Definition 56. Let $X = \mathrm{Spec}(A)$ be an affine variety and suppose that A is equipped with a Lie bracket $\{\cdot,\cdot\} : A \times A \to A$, which makes A into a Poisson algebra. Then we say that X is an *affine Poisson variety*, or simply a *Poisson variety*.

Definition 57. Let $(X_1, \{\cdot,\cdot\})$ and $(X_2, \{\cdot,\cdot\}_2)$ be two Poisson varieties. A morphism of varieties $\psi : X_1 \to X_2$ is called a *Poisson morphism* or a *Poisson map* if the dual morphism $\psi^* : \mathcal{O}(X_2) \to \mathcal{O}(X_1)$ is a morphism of Poisson algebras.

Proposition 4.2.1. *Let A be a coordinate ring of a variety (not necessarily smooth). An element $p \in C^2_{(2)}(A)$ such that $dp = 0$ (i.e. $p \in H^2_{(2)}(A) \cong \mathrm{Hom}_A(\Omega^2_{A|k}, A)$) and $e_3([p,p]) = 0 \in C^3_{(3)}(A)$ determines the Poisson structure and every Poisson structure on A is obtained in this way.*

Proof. Condition $dp = 0$ gives us all properties of a Poisson algebra except the Jacobi identity of $(A, \{\cdot,\cdot\})$. We now use computations from the previous section saying that $e_3([p,p]) = 0$ if and only if $p(p(a,b),c) + p(p(b,c),a) + p(p(c,a),b) = 0$ (see (4.2)), which gives us the Jacobi identity. We can also easily see that all Poisson structures come in this way since $\mathrm{Hom}_A(\Omega^2_{A|k}, A)$ is the space of skew-symmetric biderivations. □

Lemma 4.2.2. *If A is smooth (or more generally when $Har^3(A) = 0$), then the condition $e_3([p,p]) = 0$ is equivalent to the condition $[p,p] = 0 \in H^3(A)$.*

Proof. We have $e_2([p,p]) = 0$ (see (4.1)) and $e_1([p,p]) = 0 \in T^2_{(1)}(A)$ because $T^2_{(1)}(A) \cong \mathrm{Har}^3(A) = 0$. □

Proposition 4.2.3. *Every Poisson structure p on an affine toric variety $Spec(k[\Lambda])$ is of the form*

$$p(x^{\lambda_1}, x^{\lambda_2}) = \sum_{i=0}^{d} f_i(\lambda_1, \lambda_2) x^{R_i + \lambda_1 + \lambda_2}, \tag{4.4}$$

where $f_i \in \bar{C}^2_{(2)}(\Lambda, \Lambda \backslash (-R_i + \Lambda); k)$, $R_i \in M$. We call $f_i(\lambda_1, \lambda_2) x^{R_i + \lambda_1 + \lambda_2}$ the Poisson structure of degree R_i and we call p the Poisson structure of index $(R_0, ..., R_d)$.

Proof. A Poisson structure p is an element of $H^2_{(2)}(k[\Lambda];k)$ such that $e_3[p,p]=0$. From Propositions 3.2.2 and 3.4.1 we know that

$$H^{n,R}_{(n)}(k[\Lambda]) = H^n_{(n)}(\Lambda, \Lambda \setminus (-R+\Lambda); k) \cong \bar{C}^n_{(n)}(\Lambda, \Lambda \setminus (-R+\Lambda); k).$$

Thus p is of the form (4.4), and $e_3[p,p]=0$ gives us additional restrictions on f_i, $i=0,..,d$. □

Example 9. For every hypersurface given by a polynomial $g(x,y,z)$ in $\mathbb{A}^3$, we can define a Poisson structure π_g on the quotient $k[x,y,z]/g$, namely:

$$\pi_g := \partial_x(g)\partial_y \wedge \partial_z + \partial_y(g)\partial_z \wedge \partial_x + \partial_z(g)\partial_x \wedge \partial_y,$$

i.e., we contract the differential 1-form dg to $\partial_x \wedge \partial_y \wedge \partial_z$. Consider the toric surface A_n given by $g(x,y,z) = xy - z^{n+1}$. We would like to express π_g in the form (4.4). We see that $\pi_g(x,y) = -(n+1)z^n$, $\pi_g(z,x) = x$ and $\pi_g(y,z) = y$ hold.

In this case Λ is generated by $S_1 := [0,1]$, $S_2 := [1,1]$ and $S_3 := [n+1,n]$, with a relation $S_1 + S_3 = (n+1)S_2$. We would like to find f of the form (4.4), such that $f = \pi_g$. With a simple computation we see that f will be of degree S_2:

$$f(x^{\lambda_1}, x^{\lambda_2}) = f_0(\lambda_1,\lambda_2)x^{-S_2+\lambda_1+\lambda_2},$$

where $f_0(S_1,S_3) = -(n+1)$. The function f_0 is with this completely determined by skew-symmetry and bi-additivity.

Recall from Definition 44 that a one-parameter formal deformation of A is an associative algebra $(A[[\hbar]],*)$, such that

$$a * b = ab \ (\mathrm{mod}\hbar).$$

Definition 58. We say that a Poisson structure $p \in H^2_{(2)}(A)$ can be quantized if there exist γ_2, γ_3,... in $C^2(A)$, such that

$$a * b := ab + \frac{1}{2}p(a\otimes b)\hbar + \gamma_2(a\otimes b)\hbar^2 + \gamma_3(a\otimes b)\hbar^3 + \cdots$$

is a one-parameter formal deformation.

Remark 15. By Lemma 4.2.2 we know that when $X = \mathrm{Spec}(A)$ is smooth, a Poisson structure p on X can be extended to a second order deformation (i.e. γ_2 always exists (mod $\hbar^3$)). In the next section we will present the formality theorem, which implies that we can actually deform p to any order, i.e., p can be quantized. In general (when X is singular) there exist obstructions to the existence of a quantization (see Schelder [63, Remark 2.3.14] or Mathieu [47]).

Proposition 4.2.4. *One-parameter formal deformations* $(A[[\hbar]],*)$ *of an associative algebra* A *are in bijection with Maurer-Cartan elements of a dgla* $\mathfrak{g} := \big(\hbar C^\bullet(A)[1]\big)[[\hbar]]$.

Proof. Let $\gamma := \sum_{m\geq 1}\hbar^m\gamma_m \in \mathfrak{g}^1$. Here $\gamma_m \in C^2(A)$ for all m, since $\mathfrak{g}$ is shifted. To $\gamma \in \mathfrak{g}$ we associate the star product $f * g = fg + \sum_{m\geq 1}\hbar^m\gamma_m(f\otimes g)$. We need to show that $*$ is associative if and only if γ satisfies the Maurer-Cartan equation. This follows from a direct computation (see Schedler [63, Remark 4.3.2] for a more conceptual explanation):

$$
\begin{aligned}
&f*(g*h)-(f*g)*h = \\
&\sum_{m\geq 1}\hbar^m\cdot\big(f\gamma_m(g\otimes h)-\gamma_m(fg\otimes h)+\gamma_m(f\otimes gh)-\gamma_m(f\otimes g)h\big)+ \\
&+\sum_{m,n\geq 1}\hbar^{m+n}\big(\gamma_m(f\otimes\gamma_n(g\otimes h))-\gamma_m(\gamma_n(f\otimes g)\otimes h)\big)= \\
&=-d_{\mathfrak{g}}\gamma-\gamma\circ\gamma=-\big(d\gamma+\frac{1}{2}[\gamma,\gamma]\big),
\end{aligned}
$$

where we denote by $d_{\mathfrak{g}}$ the differential of $\mathfrak{g}$ (see Lemma 2.3.2). □

Thus we see that we can quantize $p=\gamma_1$ if and only if there exist γ_2, γ_3,... solving the equation

$$d_{\mathfrak{g}}\gamma+\frac{1}{2}[\gamma,\gamma]=0,$$

where $\gamma=\sum_{m\geq 1}\hbar^m\gamma_m$.

We need to solve the following system of equations:

$$
\begin{aligned}
&0=d_{\mathfrak{g}}\gamma_1 \\
&0=d_{\mathfrak{g}}\gamma_2+\frac{1}{2}[\gamma_1,\gamma_1] \\
&0=d_{\mathfrak{g}}\gamma_3+[\gamma_1,\gamma_2] \\
&\vdots \\
&0=d_{\mathfrak{g}}\gamma_n+\frac{1}{2}\sum_{i=1}^{n-1}[\gamma_i,\gamma_{n-i}] \\
&\vdots
\end{aligned}
$$

In general it is very hard to compute this equations and also the process may never end. Next section describes an alternative way to solve this equations using the formality theorem.

4.3 The formality theorem

In this section we show that there exists a quasi-isomorphism between the Hochschild complex and its cohomology complex (with zero differentials), called the Hochschild-Kostant-Rosenberg (HKR) morphism. However this does not extend to a dgla morphism on shifted complexes, since it does not preserve the Lie bracket. The idea of Kontsevich was to correct this and show that the HKR morphism actually extends to an L_∞-morphism, which we now define.

Let $\mathfrak{g}=\oplus_{l\in\mathbb{N}}\mathfrak{g}_l$ be a graded Lie algebra. For $x\in\mathfrak{g}_l$ we write $|x|=l$. Let $\bar{\mathfrak{g}}$ be a vector space $\mathfrak{g}$ with the grading d^0 defined by $d^0(x)=|x|-1$. The symmetric algebra that plays an important role in what follows is the graded commutative algebra $S^\bullet\bar{\mathfrak{g}}$. On $S^\bullet\bar{\mathfrak{g}}$ we consider the following grading:

$$d^0(x_1\cdots x_l):=\sum_{i=1}^{l}|x_i|-l=\sum_{i=1}^{l}d^0(x_i).$$

Let $\mathfrak{g} = \oplus_{l\in\mathbb{Z}}\mathfrak{g}_l$ and $\mathfrak{h} = \oplus_{l\in\mathbb{Z}}\mathfrak{h}_l$ be two differential graded Lie algebras. We will be mainly interested in graded linear maps $\Phi : S^\bullet(\mathfrak{g}[1]) \to \mathfrak{h}[1]$ of degree 0. For $n \in \mathbb{N}$ we denote by Φ_n the restriction of Φ to $S^n(g[1])$. The fact that Φ is graded of degree 0 means that Φ_n maps $\mathfrak{g}^{k_1}\cdots\mathfrak{g}^{k_n}$ to $\mathfrak{h}^{k_1+\cdots+k_n+1-n}$ for all $k_1, ..., k_n \in \mathbb{Z}$ (the restriction of Φ_n to $\mathfrak{g}^{k_1}...\mathfrak{g}^{k_n}$ we denote by $\Phi_{(k_1,...,k_n)}$). In particular, $\Phi_{(1,...,1)}$ maps $\mathfrak{g}^1\cdots\mathfrak{g}^1$ to $\mathfrak{h}^1$. This fact will be useful when we will consider solutions of the Maurer-Cartan equations associated to $\mathfrak{g}$ and $\mathfrak{h}$.

Definition 59. If there exists a map $\Phi : S^\bullet(\mathfrak{g}[1]) \to \mathfrak{h}[1]$ we call such a map *pre-L_∞-morphism* (see also [40, pp. 14-15]).

Definition 60. Using the natural isomorphism $S^n(\mathfrak{g}[1]) \cong (\wedge^n(\mathfrak{g}))[n]$ we say that a *pre-L_∞-morphism* $\mathcal{F}$ is an L_∞-morphism if and only if it satisfies the following equation for any $n = 1, 2, ...$ and homogenous elements $\gamma_i \in \mathfrak{g}$:

$$d\Phi_n(\gamma_1 \wedge \cdots \wedge \gamma_n) - \sum_{i=1}^{n} \pm\Phi_n(\gamma_1 \wedge \cdots \wedge d\gamma_i \wedge \cdots \wedge \gamma_n) =$$

$$\frac{1}{2}\sum_{k,l\geq 1, k+l=n} \frac{1}{k!l!} \sum_{\sigma\in S_n} \pm[\Phi_k(\gamma_{\sigma(1)} \wedge \cdots \wedge \gamma_{\sigma(k)}), \Phi_l(\gamma_{\sigma(k+1)} \wedge \cdots \wedge \gamma_{\sigma(n)})]$$

$$+\sum_{i<j} \pm\Phi_{n-1}([\gamma_i, \gamma_j] \wedge \gamma_1 \wedge \cdots \wedge \hat{\gamma}_i \wedge \cdots \wedge \hat{\gamma}_j \wedge \cdots \wedge \gamma_n).$$

Here are first two equations in the explicit form:

$$d\Phi_1(\gamma_1) = \Phi_1(d\gamma_1),$$

$$d\Phi_2(\gamma_1 \wedge \gamma_2) - \Phi_2(d\gamma_1 \wedge \gamma_2) - (-1)^{\overline{\gamma_1}}\Phi_2(\gamma_1 \wedge d\gamma_2) = \Phi_1([\gamma_1, \gamma_2]) - [\Phi_1(\gamma_1), \Phi_1(\gamma_2)].$$

Definition 61. Let $\mathfrak{g}$ and $\mathfrak{h}$ be two differential graded Lie algebras and let $\Phi : S^\bullet\mathfrak{g}[1] \to \mathfrak{h}[1]$ be an L_∞-morphism. Then Φ is called an *L_∞-quasi-isomorphism* if the morphism $\Phi_1^\bullet : H^\bullet\mathfrak{g} \to H^\bullet\mathfrak{h}$, induced by the restriction $\Phi_1 : \mathfrak{g} \to \mathfrak{h}$ of Φ to $\mathfrak{g}$, is an isomorphism.

Definition 62. A *pointed differential graded Lie algebra* $\mathfrak{g}$ is a dgla, with the differential given by $d_z(y) := [z, y]$, for some $z \in \mathfrak{g}$ with the property $[z, z] = 0$. The graded Jacobi identity implies $d_z \circ d_z = 0$ (see [41, pp. 373]).

We can naturally extend Φ to $S^\bullet\mathfrak{g}[1][[\hbar]]$ and thus we get a map $\tilde{\Omega}_\Phi : \hbar\mathfrak{g}[[\hbar]] \to \hbar\mathfrak{h}[[\hbar]]$ defined by:

$$\tilde{\Omega}_\Phi(x) := \sum_{k\in\mathbb{N}} \frac{1}{k!}\Phi_k(x^k).$$

Proposition 4.3.1. *Let $\mathfrak{g}$ and $\mathfrak{h}$ be two pointed differential graded Lie algebras. Let $\Phi : S^\bullet\mathfrak{g}[1] \to \mathfrak{h}[1]$ be an L_∞-morphism and let x be an MC element of $\hbar\mathfrak{g}[1][[\hbar]]$. Then $\tilde{\Omega}_\Phi(x)$ is an MC element of $\hbar\mathfrak{h}[1][[\hbar]]$.*

Proof. See [41, Proposition 13.41(1)]. □

Definition 63. A dgla $\mathfrak{g}$ is *formal* if there exists a pair of L_∞-quasi-isomorphisms of differential graded Lie algebras

$$\mathfrak{g} \leftarrow \mathfrak{f} \to \mathfrak{h}$$

with $\mathfrak{h}$ having trivial differentials.

Now we present the formality theorem (see [40], [22]):

Theorem 4.3.2. *Let $X = \mathrm{Spec}(A)$ be a smooth affine variety. There exists an L_∞-quasi-isomorphism between the Hochschild dgla $C^\bullet(A)[1]$ and the dgla $H^\bullet(A)[1]$ (i.e. the cohomology complex $H^\bullet(A)[1]$ is a graded Lie algebra with trivial differential). In particular, the Hochschild dgla $C^\bullet(A)[1]$ is formal.*

Proof. We only sketch an idea of the proof. The map $\mathrm{HKR}_n : \wedge^n_A \mathrm{Der}_k(A, A) \to C^n(A)$ defined by

$$\mathrm{HKR}_n(\xi_1 \wedge \cdots \wedge \xi_n)(a_1 \otimes \cdots \otimes a_n) := \frac{1}{m!} \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma)\xi_{\sigma(1)}(a_1) \cdots \xi_{\sigma(n)}(a_n), \tag{4.5}$$

gives an isomorphism of A-modules $\wedge^n_A \mathrm{Der}_k(A, A)$ and $H^n_{(n)}(A)$ (see Loday [43]). We have

$$H^n(A) \cong \mathrm{Hom}_A(\Omega^n_{A|k}, A) \cong H^n_{(n)}(A)$$

and by (4.5) they are also isomorphic to $\wedge^n_A \mathrm{Der}_k(A, A)$. There exists a quasi-isomorphism of complexes $H^\bullet(A)$ and $C^\bullet(A)$ also called the Hochschild-Kostant-Rosenberg quasi-isomorphism: $\mathrm{HKR} : H^\bullet(A) \to C^\bullet(A)$ is given by

$$\mathrm{HKR}(\xi_1 \wedge \cdots \wedge \xi_n)(a_1 \otimes \cdots \otimes a_n) := \mathrm{HKR}_n(\xi_1 \wedge \cdots \wedge \xi_n)(a_1 \otimes \cdots \otimes a_n).$$

As already mentioned in the introduction, HKR morphism does not extend to a dgla morphism on shifted complexes $C^\bullet(A)[1]$ and $H^\bullet(A)[1]$, since it does not preserve the Lie bracket. Kontsevich [40] manage to construct an explicit sequence of linear maps $\Phi_n : S^n(H^\bullet(A)[1]) \to C^\bullet(A)[1]$, where Φ_1 is the map HKR and other Φ_n satisfy conditions of an L_∞-morphism (see also [22] for a more general proof). Since Φ_1 is a quasi-isomorphism, we obtain an L_∞-quasi-isomorphism of differential graded Lie algebras by Definition 61. □

Corollary 4.3.3. *Every Poisson structure π on a smooth affine variety $\mathrm{Spec}(A)$ can be quantized.*

Proof. A Poisson structure π is trivially an MC element of $H^\bullet(A)[1]$. By Theorem 4.3.2 there exists an L_∞-quasi-isomorphism Φ between $C^\bullet(A)[1]$ and $H^\bullet(A)[1]$, with $\Phi_1 = \phi$, where we denote the HKR morphism by ϕ. $H^\bullet(A)[1]$ is trivially pointed dgla and $C^\bullet(A)[1]$ is pointed dgla by the proof of Lemma 2.3.2. Thus by Proposition 4.3.1 we know that $\tilde{\Omega}_\Phi(\pi)$ is an MC element of $\hbar C^\bullet(A)[1][[\hbar]]$. From Proposition 4.2.4 we know that we have a star product $a * b = ab + \Phi_1(\pi) + \cdots = ab + \frac{\pi}{2} + \cdots$ by definition of ϕ: $\Phi_1(\pi) = \phi(\pi) = \frac{\pi}{2}$. Thus π can be quantized by Definition 58. □

Remark 16. Since for singular varieties in general we have $H^n_{(i)}(A) \neq 0$ for $i \neq n$ we see that the HKR quasi-isomorphism can not be generalized to singular varieties. And thus also the formality theorem and Corollary 4.3.3 can not be generalized to singular varieties.

Recall now the functors $\mathrm{MC}_\mathfrak{g}$ and $\mathrm{Def}_\mathfrak{g}$ of a dgla $\mathfrak{g}$ from Subsection 2.2.2.

Theorem 4.3.4. *If $f : \mathfrak{g} \to \mathfrak{h}$ is an L_∞-quasi-isomorphism, then the induced maps $\mathrm{MC}_\mathfrak{g} \to \mathrm{MC}_\mathfrak{h}$ and $\mathrm{Def}_\mathfrak{g} \to \mathrm{Def}_\mathfrak{h}$ are isomorphisms.*

Proof. See Manetti [45, Chapter IX]. □

Example 10. If $\mathfrak{g}$ is formal with $\dim_k H^1(\mathfrak{g}) = n < \infty$, then by Theorem 4.3.4 the local ring of the solution space of the MC equation for $\mathfrak{g}$ (see Example 1) is isomorphic to $k[[t_1, ..., t_n]]/(g)$, where g is generated by quadratic equations. Let $X_\sigma = \mathrm{Spec}(A)$ be an isolated toric singularity, such that its versal base space can not be generated only by linear and quadratic equations (for example if $\sigma = \mathrm{Cone}(P)$, where P is an octagon or more generally when P is "thick" enough; see [4] for more details). We know that the deformation functor of the Harrison dgla $C^\bullet_{(1)}(A)[1]$ is isomorphic to the functor Def_{X_σ} by Corollary 2.3.20 and Proposition 2.3.21. Moreover, by Theorem 2.1.2 we know that $H^1(C^\bullet_{(1)}(A)[1]) < \infty$. As a corollary we obtain that in these cases the Harrison dgla $C^\bullet_{(1)}(A)[1]$ is not formal and thus also the Hochschild dgla $C^\bullet(A)[1]$ is not formal. On the other hand we will show in the next section that Corollary 4.3.3 is still satisfied for singular toric varieties.

4.4 Deformation quantization of affine toric varieties

In this section we prove that every Poisson structure on an affine toric variety can be quantized. We will use the Maurer-Cartan formalism, Kontsevich's formality theorem (or more precisely its Corollary 4.3.3) and the GIT quotient construction for an affine toric variety $\mathrm{Spec}(A)$ without torus factors: we can write $\mathrm{Spec}(A) = \mathbb{A}^N /\!\!/ G$ for some group G. This construction works over an algebraically closed field k of characteristic 0. Our proof works also in the case of affine toric varieties with torus factors.

Let X be a toric variety without torus factors, i.e., given by a full-dimensional cone $\sigma = \langle a_1, ..., a_N \rangle \subset N_\mathbb{R}$. We recall now the construction that presents X as a GIT quotient $\mathbb{A}^N /\!\!/ G$, where G is a group (see e.g. [20, Chapter 5]). We have a short exact sequence

$$0 \to M \xrightarrow{g} \mathbb{Z}^{\sigma(1)} \to \mathrm{Cl}(X) \to 0,$$

where $\mathrm{Cl}(X)$ is the class group of X, $\sigma(1) = N$ the number of ray generators and for $R \in M$ we have the injective map $g(R) = \langle R, a_1 \rangle e_1 + \cdots + \langle R, a_N \rangle e_N$, where $\{e_i \mid i = 1, ..., N\}$ is the standard basis for $\mathbb{Z}^N$. We have $X = \mathbb{A}^N /\!\!/ G$, where $G = \mathrm{Hom}_\mathbb{Z}(\mathrm{Cl}(X), k^*)$; here we need the assumption that k is also algebraically closed. Moreover, the class group is of the form

$$\mathrm{Cl}(X) \cong \mathbb{Z} \times \mathbb{Z} \times \cdots \mathbb{Z} \times \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \mathbb{Z}_{p_k}$$

and thus our group G is of the form

$$G \cong k^* \times \cdots \times k^* \times G_{p_1} \times G_{p_2} \times \cdots G_{p_k},$$

where G_{p_i} are groups of p_i-th roots of unity.

The map g induces a semi-group isomorphism between $\Lambda \subset M$ and its image $\Lambda^G := g(\Lambda)$. This map determines the isomorphism of k-algebras $G' : k[\Lambda] \to k[x_1, ..., x_N]^G$, with $G'(x^R) = x^{g(R)} := x_1^{\langle R, a_1 \rangle} \cdots x_N^{\langle R, a_N \rangle}$. Elements that lie in Λ^G are *G-invariant elements*. Thus we have $X = \mathrm{Spec}(k[x_1, ..., x_N]) /\!\!/ G = \mathrm{Spec}(k[x_1, ..., x_N]^G)$.

Example 11. Let $\sigma = \langle (1,0), (-(n-1), n) \rangle$, $\sigma^\vee = \langle (0,1), (n-1, n) \rangle$. The map $g : M \cong \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by $g(\lambda_1, \lambda_2) = \lambda_1 e_1 + (n\lambda_2 - (n-1)\lambda_1) e_2$. We see that the degrees of the generators of the ring of invariants are $g(0,1) = (0,n)$, $g(1,1) = (1,1)$ and $g(n, n-1) = (n, 0)$. Thus the ring of invariants is $k[x^n, xy, y^n]$.

Proposition 4.4.1. *For $\lambda, R \in M$ it holds that*

$$\lambda \in \cup_{j\in I} K_{a_j}^{R} \text{ if and only if } g(\lambda) \in \cup_{j\in I} K_{e_j}^{g(R)},$$

where $I = \{1, ..., N\}$ and $K_{e_j}^{g(R)}$ are the convex sets (3.2) *of the cone $\langle e_1, ..., e_N\rangle \subset \mathbb{R}^N$.*

Proof. By the definition of g we know that $\langle g(\lambda), e_j\rangle = \langle \lambda, a_j\rangle$ and $\langle g(R), e_j\rangle = \langle R, a_j\rangle$. For $g(\lambda) \in \cup_j K_{e_j}^{g(R)}$ there exists j such that $\langle g(\lambda), e_j\rangle < \langle g(R), e_j\rangle$ which means that there exists j such that $\langle \lambda, a_j\rangle < \langle R, a_j\rangle$, which is equivalent to $\lambda \in \cup_j K_{a_j}^{R}$. □

Every affine toric variety can be decomposed into the product of a torus and a toric variety without torus factors. Let $A = k[\sigma^\vee \cap M]$ and $X = \mathrm{Spec}(A)$ be a toric variety without torus factors. Let $T_k = \mathrm{Spec}(k[\mathbb{Z}^k])$ and $A_k = k[\Lambda \times \mathbb{Z}^k]$ $(A_0 \cong A)$. We denote $X_k = \mathrm{Spec}(A_k) = X \times T_k$. Let $Y_k = \mathbb{A}^N \times T_k = \mathrm{Spec}(B_k)$, where $B_k = k[\mathbb{N}_0^N \times \mathbb{Z}^k]$ and $\mathbb{N}_0$ is the set of natural numbers with 0. We define the lattices $\widetilde{M} := M \times \mathbb{Z}^k$, $\widetilde{N} := N \times \mathbb{Z}^k$ and the map $g' : \Lambda \times \mathbb{Z}^k \to \mathbb{N}_0^N \times \mathbb{Z}^k$ with

$$g'(\lambda, \mu) = (g(\lambda), \mu).$$

We now briefly recall basic definitions and propositions from Poisson geometry.

Definition 64. Let V_2 be a subvariety of an affine Poisson variety $(V_1, \{\cdot,\cdot\})$ and let $p : V_2 \to P$ be a surjetive map, where P is also an affine variety. We say that the triple (V_1, V_2, P) :

$$\begin{array}{ccc} & V_2 & \\ \swarrow & & \searrow^{p} \\ V_1 & & P \end{array}$$

is *Poisson reducible* if there exists a Poisson structure $\{\cdot,\cdot\}_P$ on P, such that for every $x \in V_2$,

$$\{F, G\}_P(p(x)) = \{\bar{F}, \bar{G}\}(x),$$

for all $F, G \in \mathcal{O}(P)$ and for all extensions $\bar{F}, \bar{G}$ of $F \circ p$ and $G \circ p$. If $V_1 = V_2$, then the Poisson structure $\{\cdot,\cdot\}_P$ is called a *reduced Poisson structure* of the Poisson structure $\{\cdot,\cdot\}$.

The next propositions are important for proving that every Poisson structure on an affine toric variety can be quantized.

Proposition 4.4.2. *Every Poisson structure p on X_k can be seen as a reduced Poisson structure of some Poisson structure P on Y_k.*

Proof. From Proposition 4.2.3 we know that every Poisson structure on X_k is of the form

$$p(x^{\lambda_1}, x^{\lambda_2}) = \sum_{i=0}^{d} f_i(\lambda_1, \lambda_2) x^{R_i + \lambda_1 + \lambda_2},$$

where $f_i \in \bar{C}^2_{(2)}(\Lambda \times \mathbb{Z}^k, (\Lambda \times \mathbb{Z}^k) \setminus (-R_i + (\Lambda \times \mathbb{Z}^k)); k)$, $R_i \in \widetilde{M}$.

We now construct a Poisson structure P on a smooth affine variety Y_k:

$$P(x^\lambda, x^\mu) = \sum_{i=0}^{d} F_i(\lambda, \mu) x^{g'(R_i) + \lambda + \mu},$$

where F_i has the property that $F_i(g'(\lambda_1), g'(\lambda_2)) = f_i(\lambda_1, \lambda_2)$, for each i.

STEP 1: Functions F_i with the above property exist for each i:
We choose $k+n$ linearly independent vectors $s_1, ..., s_{k+n} \in \Lambda \times \mathbb{Z}^k$ such that $s_1, ..., s_k \in 0 \times \mathbb{Z}^k$ and $s_{k+1}, ..., s_{k+n} \in \Lambda \times 0$. Note also that f_i are completely determined by the values $f_i(s_j, s_l)$, for $1 \leq j < l \leq k+n$ by Remark 9. Since g' is injective we can choose $F_i \in \bar{C}^2_{(2)}(\mathbb{N}_0^N \times \mathbb{Z}^k; k)$, such that $F_i(g'(s_j), g'(s_l)) = f_i(s_j, s_l)$, for $1 \leq j < l \leq k+n$.

Let $t_1, ..., t_{N-n} \in \mathbb{N}_0^N$ be chosen such that $s_{k+1}, ..., s_{k+n}, t_1, ..., t_{N-n}$ determine an $\mathbb{R}$-basis of $\mathbb{R}^N$. We choose F_i such that $F_i(t_j, t_l) = 0$ for $1 \leq j, l \leq N-n$ and $F_i(s_j, t_l) = 0$ for $j = 1, ..., k+n$ and $l = 1, ..., N-n$ (this will be important to prove the Jacobi identity for P in Step 3). We easily see that $F_i(g'(\lambda_1), g'(\lambda_2)) = f_i(\lambda_1, \lambda_2)$ holds.

STEP 2: P is well defined:
For $P(x^{\lambda_1}, x^{\lambda_2})$ to be well defined, it must hold for each i that $F_i(\lambda, \mu) = 0$ for $g'(R) + \lambda + \mu \ngeq 0$. We need to check that this agrees with the property $F_i(g'(\lambda_1), g'(\lambda_2)) = f_i(\lambda_1, \lambda_2)$: without loss of generality we assume that $\lambda_1, \lambda_2 \in \Lambda \times 0$. We have $F_i(g(\lambda_1), g(\lambda_2)) = 0$ for $g(R) + g(\lambda_1) + g(\lambda_2) \ngeq 0$ or equivalently for $g(\lambda_1 + \lambda_2) \in \mathbb{N}_0^N \setminus \mathbb{N}_0^N(-g(R)) = \cup_{j \in I} K_{e_j}^{-g(R)}$, where $I = \{1, ..., N\}$. By Proposition 4.4.1 this is equivalent to $\lambda_1 + \lambda_2 \in \cup_{j \in I} K_{a_j}^{-R}$ and we indeed have $f_i(\lambda_1, \lambda_2) = 0$ for $R + \lambda_1 + \lambda_2 \ngeq 0$.

STEP 3: P satisfies the Jacobi identity:
We have $e_3(3)([p,p])(x^{\lambda_1}, x^{\lambda_2}, x^{\lambda_3}) = 0$, since p is a Poisson structure. Using Lemma 4.1.2 and the equalities $F_i(g'(\lambda_1), g'(\lambda_2)) = f_i(\lambda_1, \lambda_2)$ from Step 1, we see that

$$e_3(3)([P,P])(x^{g'(\lambda_1)}, x^{g'(\lambda_2)}, x^{g'(\lambda_3)}) = 0.$$

Since $e_3(3)[P,P] \in H^3_{(3)}(Y_k)$ we can use Proposition 3.4.1 and thus from the construction of F_i in Step 1 ($F_i(t_j, t_l) = 0$ and $F_i(s_j, t_l) = 0$) we immediately see that $e_3(3)[P,P] = 0$. Thus the Jacobi identity is satisfied. □

Let $\mathfrak{g}$ denote the differential graded Lie algebra $\big(\hbar C^\bullet(A_k)[1]\big)[[\hbar]]$ and let $\mathfrak{h}$ denote the differential graded Lie algebra $\big(\hbar C^\bullet(B_k)[1]\big)[[\hbar]]$.

Proposition 4.4.3. *Let $\gamma(x^{\lambda_1}, x^{\lambda_2}) := \sum_{m \geq 1} \hbar^m \gamma_m(x^{\lambda_1}, x^{\lambda_2}) \in \mathfrak{h}^1$ be an MC element of the dgla $\mathfrak{h}$, where γ_1 is a Poisson structure on Y_k of index $(g'(R_0), ..., g'(R_d))$. Then γ induces an MC element $\widetilde{\gamma}(x^{\lambda_1}, x^{\lambda_2}) := \sum_{m \geq 1} \hbar^m \widetilde{\gamma}_m(x^{\lambda_1}, x^{\lambda_2}) \in \mathfrak{g}^1$ of the dgla $\mathfrak{g}$, where $\widetilde{\gamma}_1$ is a reduced Poisson structure (on X_k) of the Poisson structure γ_1 and it has index $(R_0, ..., R_d)$.*

Proof. We prove it only for $d = 0$ and $k = 0$ (i.e. for γ_1 of degree R_0 on a toric variety $X = X_0$ without torus factors). The rest follows immediately, just the notation is more tedious.

We know that $\gamma_m(x^{\lambda_1}, x^{\lambda_2}) = \gamma_{0m}(\lambda_1, \lambda_2) x^{mg(R) + \lambda_1 + \lambda_2}$, where

$$\gamma_{0m} \in C^2(\mathbb{N}_0^N, \mathbb{N}_0^N \setminus \mathbb{N}_0^N(-mg(R)); k).$$

We define $\widetilde{\gamma}_{0m}(\lambda, \mu) := \gamma_{0m}(g(\lambda), g(\mu))$ and $\widetilde{\gamma} := \sum_{m \geq 1} \hbar^m \widetilde{\gamma}_m(x^\lambda, x^\mu)$, where

$$\widetilde{\gamma}_m(x^\lambda, x^\mu) = \widetilde{\gamma}_{0m}(\lambda, \mu) x^{mR + \lambda + \mu}.$$

We first need to check that $\widetilde{\gamma}(x^\lambda, x^\mu) = \sum_{m \geq 1} \hbar^m \widetilde{\gamma}_m(x^\lambda, x^\mu)$ is well defined, i.e., if $mR + \lambda + \mu \ngeq 0$, then $\gamma_{0m}(g(\lambda), g(\mu)) = 0$. This can be done as in Step 2 of Proposition 4.4.2.

Looking only at G-invariant elements (i.e. $\lambda = g(\lambda')$ and $\mu = g(\mu')$ for some $\lambda', \mu' \in \Lambda$) in the MC equation for γ and using Lemma 4.1.2, we see that the MC equation also holds for $\widetilde{\gamma}$. □

Theorem 4.4.4. *Every Poisson structure p on an affine toric variety can be quantized.*

Proof. As above let X_k denote an arbitrary affine toric variety. By Proposition 4.2.3, p is of the form $p(x^{\lambda_1}, x^{\lambda_2}) = \sum_{i=0}^{d} f_i(\lambda_1, \lambda_2) x^{R_i+\lambda_1+\lambda_2}$ for some $R_i \in \Lambda \times \mathbb{Z}^k$. By the construction in the proof of Proposition 4.4.2 this Poisson structure can be seen as a reduced Poisson structure of P on Y_k:

$$P(x^{\lambda}, x^{\mu}) = \sum_{i=0}^{d} F_i(\lambda, \mu) x^{g'(R_i)+\lambda+\mu},$$

where the functions F_i have the property that $F_i(g'(\lambda_1), g'(\lambda_2)) = f_i(\lambda_1, \lambda_2)$. Since P is a Poisson structure on the smooth affine variety Y_k, we know by Corollary 4.3.3 that P can be quantized. In other words, there exists a one-parameter formal deformation and by Lemma 4.2.4 we know that this corresponds to an MC element $\gamma(x^{\lambda_1}, x^{\lambda_2}) := \sum_{m\geq 1} \hbar^m \gamma_m(x^{\lambda_1}, x^{\lambda_2}) \in \mathfrak{h}^1$, where γ_1 is of index $(g'(R_0), ..., g'(R_d))$. By Proposition 4.4.3 we know that this give us an MC element

$$\widetilde{\gamma}(x^{\lambda_1}, x^{\lambda_2}) := \sum_{m\geq 1} \hbar^m \widetilde{\gamma}_m(x^{\lambda_1}, x^{\lambda_2}) \in \mathfrak{g}^1,$$

where $\widetilde{\gamma}_1$ is a reduced Poisson structure on X_k of the Poisson structure γ_1 and it has index $(R_0, ..., R_d)$. By the construction we have $\widetilde{\gamma}_1 = p$. Using again Lemma 4.2.4 we see that p can be quantized. □

5 Commutative deformations of toric varieties

In Section 5.1 we give a convex geometric description of the Harrison cup product formula $T^1_{(1)}(A) \times T^1_{(1)}(A) \to T^2_{(1)}(A)$. We show that our cup product formula in the case of three-dimensional isolated Gorenstein toric varieties recovers Altmann's cup product formula obtained in [3]. This is done in Section 5.2. In Section 5.3 we analyze the cup product formula between non-negative degrees. In Section 5.4 we conjecture a set of equations defining the versal base space in degree $-R^*$ for not necessarily isolated Gorenstein singularities. In Section 5.5 we extend our cup product formula to a differential graded Lie algebra structure on Altmann's deformation complex.

5.1 The Harrison cup product formula for toric varieties

In this section we give a formula for the cup product of toric varieties, extending Altmann's cup product formula for toric varieties that are smooth in codimension 2 (see [3]). Note that Altmann obtained the cup product formula with different methods (using Laudal's cup product).

Definition 65. The Lie bracket [,] of the Harrison dgla induces a product $T^1_{(1)}(A)\times T^1_{(1)}(A) \to T^2_{(1)}(A)$ that we call the *(Harrison) cup product*.

We denote $T^n(A) := T^n_{(1)}(A)$ for $n \geq 0$. We now recall some results obtained by Sletsjøe in [65]. For $R \in M$ we have an exact sequence of complexes:

$$0 \to C^\bullet_{(1)}(\Lambda, \Lambda \setminus (R+\Lambda); k) \to C^\bullet_{(1)}(\Lambda; k) \to C^\bullet_{(1)}(\Lambda \setminus (R+\Lambda); k) \to 0.$$

Note that $H^k_{(1)}(\Lambda; k) = 0$ for $k \geq 2$ by Proposition 3.3.4. Thus we can write the corresponding long exact sequence in cohomology and we get the following.

Corollary 5.1.1. *The sequence*

$$0 \to H^1_{(1)}(\Lambda, \Lambda \setminus (R+\Lambda); k) \to H^1_{(1)}(\Lambda; k) \to H^1_{(1)}(\Lambda \setminus (R+\Lambda); k) \xrightarrow{d} H^2_{(1)}(\Lambda, \Lambda \setminus (R+\Lambda); k) \to 0$$

is exact and

$$H^n_{(1)}(\Lambda \setminus (R+\Lambda); k) \cong H^{n+1}_{(1)}(\Lambda, \Lambda \setminus (R+\Lambda); k)$$

for $n \geq 2$. *These isomorphisms are induced by the map* d.

Remark 17. Here with the map d we mean that we first extend a function from $\Lambda \setminus (R+\Lambda)$ to the whole of Λ by 0 and then we apply our differential d. Both maps we will denote by d and the meaning will be clear from the context.

Let $\sigma = \langle a_1, ..., a_N \rangle$ and $\Lambda(R) := \Lambda + R$ for $R \in M$. Let ξ be an element from $H^1_{(1)}(\Lambda \setminus \Lambda(R); k)$. We extend (not additively) ξ to the whole of Λ by 0 (i.e. $\xi(\lambda) = 0$ for $\lambda \in \Lambda(R)$). This extended function we denote by ξ^0. We have $T^{1,-R}(A) \cong H^2_{(1)}(\Lambda, \Lambda \setminus (R + \Lambda); k)$ by Proposition 3.2.2 and the surjective map

$$H^1_{(1)}(\Lambda \setminus (R + \Lambda); k) \xrightarrow{d} H^2_{(1)}(\Lambda, \Lambda \setminus (R + \Lambda); k)$$

by Corollary 5.1.1. Thus we see that every element of $T^1(-R)$ can be written as $d\xi^0$ for some $\xi \in H^1_{(1)}(\Lambda \setminus \Lambda(R); k)$.

Proposition 5.1.2. *Let $R, S \in M$ and let ξ and μ be elements from $H^1_{(1)}(\Lambda \setminus \Lambda(R); k)$ and $H^1_{(1)}(\Lambda \setminus \Lambda(S); k)$, respectively. It holds that*

$$[d\xi^0, d\mu^0] = dC,$$

where

$$C(\lambda_1, \lambda_2) := \tag{5.1}$$
$$\xi^0(\lambda_1)\mu^0(\lambda_2) + \xi^0(\lambda_2)\mu^0(\lambda_1) - d\xi^0(\lambda_1, \lambda_2)\mu^0(-R + \lambda_1 + \lambda_2) - d\mu^0(\lambda_1, \lambda_2)\xi^0(-S + \lambda_1 + \lambda_2).$$

Proof. See [65, Theorem 4.8]. □

Sletsjøe [65] claimed that Proposition 5.1.2 gives us a nice cup product formula, but unfortunately there is a mistake at the end of the paper: in [65] it was written that only the first two terms of $C(\lambda_1, \lambda_2)$ matter for the computations of the cup product formula and that the other two vanish with d. This is not correct since $d\xi^0 \notin C^2_{(1)}(\Lambda, \Lambda \setminus \Lambda(R + S); k)$, which was wrongly assumed (see [65, pp. 128]). We only have $d\xi^0 \in C^2_{(1)}(\Lambda, \Lambda \setminus \Lambda(R); k)$.

Thus we need to consider $C(\lambda_1, \lambda_2)$ with all 4 terms and we will try to simplify this using the double complex $C^\bullet_{(1)}(K^R_\bullet; k)$ (see Figure 3.1 for $i = 1$). On $K^{R+S}_{a_j}$ $(j = 1, ..., N)$ we define the function

$$h_j(\lambda) := -(\xi_j \cdot \mu_j)(\lambda) + \xi_j(-S + \lambda)\mu_j(\lambda) + \mu_j(-R + \lambda)\xi_j(\lambda),$$

where ξ_j is an additive extension (not necessarily unique) of ξ from $K^R_{a_j}$ to the whole of lattice M (note that it is possible to extend ξ by Remark 9 since the rays a_j are smooth cones). If $\langle a_j, R \rangle = 1$ holds, then we can extend it to $M \cap a_j^\perp$ and consequently to the whole of M (not uniquely). Similarly we define μ_j.

As we will see there is a close connection between dh_j and $C(\lambda_1, \lambda_2)$. For $\lambda \in \Lambda$ we define

$$\xi^0_j(\lambda) := \begin{cases} \xi(\lambda) & \text{if } \lambda \in K^R_{a_j} \\ \xi_j(\lambda) & \text{if } \lambda \in K^{R+S}_{a_j} \cap \left(\cup_{k;k \neq j} K^R_{a_k} \right) \\ 0 & \text{otherwise} \end{cases}$$

and similarly we define μ^0_j. Note that $\xi(\lambda) = \xi_j(\lambda)$ for $\lambda \in K^R_{a_j}$. Note also that on $K^{R+S}_{a_j}$ the functions $\xi^0_j(\lambda)$ and $\xi^0(\lambda)$ are in general different for $\lambda \in K^{R+S}_{a_j} \cap \left(\cup_{k;k \neq j} K^R_{a_k} \right)$. The following proposition gives us a nice interpretation of the cup product.

Proposition 5.1.3. *On $K^{R+S}_{a_j}$ (i.e. for $(\lambda_1, \lambda_2) \in \Lambda \times \Lambda$ such that $\lambda_1 + \lambda_2 \in K^{R+S}_{a_j}$) it holds that*

$$d(h_j)(\lambda_1, \lambda_2) = C^0(\lambda_1, \lambda_2) := \tag{5.2}$$
$$\xi^0_j(\lambda_1)\mu^0_j(\lambda_2) + \xi^0_j(\lambda_2)\mu^0_j(\lambda_1) - d\xi^0_j(\lambda_1, \lambda_2)\mu^0_j(-R + \lambda_1 + \lambda_2) - d\mu^0_j(\lambda_1, \lambda_2)\xi^0_j(-S + \lambda_1 + \lambda_2).$$

Proof. We have

$$d(h_j)(\lambda_1,\lambda_2) =$$
$$-\xi_j(\lambda_2)\mu_j(\lambda_2) + \xi_j(-S+\lambda_2)\mu_j(\lambda_2) + \mu_j(-R+\lambda_2)\xi_j(\lambda_2)-$$
$$\big(-\xi_j(\lambda_1+\lambda_2)\mu_j(\lambda_1+\lambda_2) + \xi_j(-S+\lambda_1+\lambda_2)\mu_j(\lambda_1+\lambda_2) + \mu_j(-R+\lambda_1+\lambda_2)\xi_j(\lambda_1+\lambda_2)\big)$$
$$-\xi_j(\lambda_1)\mu_j(\lambda_1) + \xi_j(-S+\lambda_1)\mu_j(\lambda_1) + \mu_j(-R+\lambda_1)\xi_j(\lambda_1).$$

We will compute $dh_j(\lambda_1,\lambda_2)$ in terms of ξ_j^0 and μ_j^0 (so far we compute dh_j in terms of ξ_j and μ_j) for different choices of λ_1 and λ_2. We can then simply check that in each of these cases the equality (5.2) is satisfied.

1. $\lambda_1 \not\geq R, S$ and $\lambda_2 \not\geq R, S$:
   - $\lambda_1+\lambda_2 \geq R, S$:
     We have $dh_j(\lambda_1,\lambda_2) = \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1) - \xi_j(-S+\lambda_1+\lambda_2)\big(\mu_j(\lambda_1) + \mu_j(\lambda_2)\big) - \mu_j(-R+\lambda_1+\lambda_2)\big(\xi_j(\lambda_1)+\xi_j(\lambda_2)\big)$.
   - $\lambda_1+\lambda_2 \geq R$, $\lambda_1+\lambda_2 \not\geq S$:
     $dh_j(\lambda_1,\lambda_2) = \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1) - \mu_j(-R+\lambda_1+\lambda_2)\big(\xi_j(\lambda_1)+\xi_j(\lambda_2)\big)$.
   - $\lambda_1+\lambda_2 \not\geq R$, $\lambda_1+\lambda_2 \geq S$:
     $dh_j(\lambda_1,\lambda_2) = \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1) - \xi_j(-S+\lambda_1+\lambda_2)\big(\mu_j(\lambda_1)+\mu_j(\lambda_2)\big)$.
   - $\lambda_1+\lambda_2 \not\geq R, S$
     $dh_j(\lambda_1,\lambda_2) = \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1)$.

   In all four cases above we have $dh_j = C^0$, since $\xi_j^0(\lambda_k) = \xi_j(\lambda_k)$ and $\mu_j^0(\lambda_k) = \mu_j(\lambda_k)$ hold for $k = 1, 2$.

2. $\lambda_1 \not\geq R, S$ and $\lambda_2 \geq R, S$
   We have $\xi_j(\lambda_1) = \xi_j^0(\lambda_1)$ and $\mu_j(\lambda_1) = \mu_j^0(\lambda_1)$. Note that these equalities does not necessarily hold for λ_2. We also know that $\lambda_1+\lambda_2 \geq R, S$ and thus we have

   $$dh_j(\lambda_1,\lambda_2) = \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1) - \xi_j(-S+\lambda_1+\lambda_2)\big(\mu_j(\lambda_1)+\mu_j(\lambda_2)\big)+$$
   $$\xi_j(-S+\lambda_2)\mu_j(\lambda_2) + \mu_j(-R+\lambda_2)\xi_j(\lambda_2) =$$
   $$= \xi_j(\lambda_1)\mu_j(\lambda_2) + \xi_j(\lambda_2)\mu_j(\lambda_1) - \big(\xi_j(\lambda_1)+\xi_j(-S+\lambda_2)\big)\big(\mu_j(\lambda_1)+\mu_j(\lambda_2)\big)$$
   $$-\big(\mu_j(-R+\lambda_2)+\mu_j(\lambda_1)\big)\big(\xi_j(\lambda_1)+\xi_j(\lambda_2)\big) + \xi_j(-S+\lambda_2)\mu_j(\lambda_2) + \mu_j(-R+\lambda_2)\xi_j(\lambda_2) =$$
   $$= -\mu_j(\lambda_1)\big(\xi_j(\lambda_1)+\xi_j(-S+\lambda_2)\big) - \xi_j(\lambda_1)\big(\mu_j(-R+\lambda_2)+\mu_j(\lambda_1)\big).$$

   On the other hand we have

   $$C^0(\lambda_1,\lambda_2) = -\xi_j^0(\lambda_1)\big(\mu_j^0(-R+\lambda_2)+\mu_j^0(\lambda_1)\big) - \mu_j^0(\lambda_1)\big(\xi_j^0(-S+\lambda_2)+\xi_j^0(\lambda_1)\big).$$

   Since $\lambda_2 \in K_{a_j}^{R+S}$ we have $-R+\lambda_2 \not\geq S$ and $-S+\lambda_2 \not\geq R$ and thus $\mu_j^0(-R+\lambda_2) = \mu_j(-R+\lambda_2)$ and $\xi_j^0(-S+\lambda_2) = \xi_j(-S+\lambda_2)$. It follows that the equality $dh_j = C^0$ is satisfied in this case.

3. $\lambda_1 \not\geq R$, $\lambda_2 \not\geq R$; $\lambda_1 \not\geq S$, $\lambda_2 \geq R$

- $\lambda_1 + \lambda_2 \geq S$

  We have $dh_j(\lambda_1, \lambda_2) = \xi_j(\lambda_2)\mu_j(\lambda_1) + \xi_j(\lambda_1)\mu_j(\lambda_2) + \mu_j(-R+\lambda_2)\xi_j(\lambda_2) - \xi_j(-S + \lambda_1 + \lambda_2)\big(\mu_j(\lambda_1) + \mu_j(\lambda_2)\big) - \big(\mu_j(\lambda_1) + \mu_j(-R+\lambda_2)\big)\big(\xi_j(\lambda_1) + \xi_j(\lambda_2)\big)$,

  $C^0(\lambda_1, \lambda_2) = \xi_j^0(\lambda_1)\mu_j^0(\lambda_2) - \xi_j^0(\lambda_1)(\mu_j^0(-R+\lambda_2) + \mu_j^0(\lambda_1)) - (\mu_j^0(\lambda_1) + \mu_j^0(\lambda_2))\xi_j^0(-S + \lambda_1 + \lambda_2)$ and thus the equality (5.2) is satisfied.

- $\lambda_1 + \lambda_2 \ngeq S$

  We have $dh_j(\lambda_1, \lambda_2) = \xi_j(\lambda_2)\mu_j(\lambda_1) + \xi_j(\lambda_1)\mu_j(\lambda_2) + \mu_j(-R+\lambda_2)\xi_j(\lambda_2) - \big(\mu_j(\lambda_1) + \mu_j(-R+\lambda_2)\big)\big(\xi_j(\lambda_1) + \xi_j(\lambda_2)\big)$,

  $C^0(\lambda_1, \lambda_2) = \xi_j^0(\lambda_1)\mu_j^0(\lambda_2) - \xi_j^0(\lambda_1)(\mu_j^0(-R+\lambda_2) + \mu_j^0(\lambda_1))$ and thus the equality (5.2) is satisfied.

Similarly as above we can check that the equality (5.2) is satisfied also in the remaining cases.

□

We will now explain how to use Proposition 5.1.3 in order to compute the cup product $T^1(-R) \times T^1(-S) \to T^2(-R-S)$.

From the double complex $C^\bullet_{(1)}(K^R_\bullet; k)$ in Figure 3.1 we know that $C \in \oplus_j C^2_{(1)}(K^{R+S}_{a_j}; k)$ (i.e. for each j we restrict C to $K^{R+S}_{a_j}$) represents the cup product. Note that we have $dC = \delta C = 0$. By Proposition 5.1.3 there exist functions h_j, $j = 1, ..., N$, such that $dh_j = C^0$.

Lemma 5.1.4. *For each $j = 1, ..., N$, there exist functions $F_j \in C^1_{(1)}(\Lambda \setminus \Lambda(R+S); k)$ such that $dF_j = C - dh_j$.*

Proof. It follows immediately since $H^2_{(1)}(K^{R+S}_{a_j}; k) = 0$ by Proposition 3.3.4. □

Collecting all the results gives us:

Theorem 5.1.5. *Every element of $T^1(-R)$ (resp. $T^1(-S)$) can be written as $d\xi^0$ (resp. $d\mu^0$) for some $\xi \in H^1_{(1)}(\Lambda \setminus \Lambda(R); k)$ (resp. $\mu \in H^1_{(1)}(\Lambda \setminus \Lambda(S); k)$). The cup product $[d\xi^0, d\mu^0] \in T^2(-R-S)$ is equal to*

$$\delta(F_1, ..., F_N) + \delta(h_1, ..., h_N) \in T^2(-R-S).$$

Remark 18. An element $\delta(F_1, ..., F_N) + \delta(h_1, ..., h_N) \in C^1_{(1)}(K^{R+S}_2; k)$ is mapped to zero with both maps d and δ and thus it is an element of $T^2(-R-S)$. The functions F_j can be easily constructed since the functions $C - dh_j$ have many zeros.

5.2 Deformations of three-dimensional affine Gorenstein toric varieties

In this section we apply Theorem 5.1.5 to the case of three-dimensional affine Gorenstein toric varieties.

Affine Gorenstein toric varieties are obtained by putting a lattice polytope $Q \subset \mathbb{A}$ into the affine hyperplane $\mathbb{A} \times \{1\} \subset \mathbb{A} \times \mathbb{R} =: N_\mathbb{R}$ and defining $\sigma := \mathrm{Cone}(Q)$, the cone over Q. Then the canonical degree R^* equals $(\underline{0}, 1)$.

Let X_σ be a three-dimensional affine Gorenstein toric variety given by a cone $\sigma = \langle a_1, ..., a_N \rangle$, where $a_1, ..., a_N$ are arranged in a cycle. We will also write Gorenstein singularity for singular Gorenstein variety. We define $a_{N+1} := a_1$. Let us denote $d_j := a_{j+1} - a_j$ and let

$$V := \{\underline{t} = (t_1, ..., t_N) \in k^N \mid \sum_{j=1}^{N} t_j d_j = 0\}$$

denote the set of (generalized) Minkowski summands (see [4]).

Proposition 5.2.1. *It holds that* $T^1(-R^*) \cong V/k \cdot \underline{1}$.

Proof. See [5]. □

Remark 19. Note that if X_σ is isolated, we have $T^1(-R^*) = T^1$. In general T^1 is non-zero also in other degrees (see Corollary 3.6.4).

The complex (3.6) for $i = 1$ and $R = 2R^*$ becomes

$$0 \to N_k \xrightarrow{\psi} N_k^N \xrightarrow{\delta} \oplus_j (N_k/\delta_j d_j) \xrightarrow{\eta} (\operatorname{Span}_k R^*)^* \to 0,$$

where $\psi(x) = (x, ..., x)$, $\delta(b_1, ..., b_N) = (b_1 - b_2, b_2 - b_3, ..., b_N - b_1)$, $\eta(q_1, ..., q_N) = \sum_{j=1}^{N} q_j$ and

$$\delta_j := \begin{cases} 0 & \text{if the 2-face } \langle a_j, a_{j+1} \rangle \text{ is smooth} \\ 1 & \text{if the 2-face } \langle a_j, a_{j+1} \rangle \text{ is not smooth.} \end{cases}$$

Corollary 5.2.2. *We have* $T^2(-2R^*) \cong \ker \eta / im\ \delta$ *and moreover, if* X_σ *is isolated we see that* $T^2(-2R^*) \cong (M_k/R^*)^*$ *since the complex*

$$N_k^N \xrightarrow{\delta} N_k^N \xrightarrow{\eta} N_k$$

is exact.

5.2.1 The cup product $T^1(-R^*) \times T^1(-R^*) \to T^2(-2R^*)$

In the case of isolated three-dimensional toric Gorenstein singularities Altmann [3] obtain the following cup product

$$\begin{aligned} V/(k \cdot \underline{1}) \times V/(k \cdot \underline{1}) &\mapsto (M_k/R^*)^* \\ (\underline{t}, \underline{s}) &\mapsto \sum_{j=1}^{N} s_j t_j d_j. \end{aligned} \tag{5.3}$$

We want to apply Theorem 5.1.5 to the case of three-dimensional toric Gorenstein singularities. To do that we will first show how to construct a function ξ_j (defined on $a_j^\perp$) from an element $\underline{t} \in V$. From Altmann's construction (see [5, Section 2.7]) there exist $\overline{b}_j \in R^\perp$ for $j = 1, ..., N$ such that $\forall j$ it holds that

$$\overline{b}_{j+1} - \overline{b}_j = t_j(a_{j+1} - a_j). \tag{5.4}$$

Since $\sum_{j=1}^{N} t_j d_j = 0$ we have a solution of this system of equations, namely $\overline{b}_2 = \overline{b}_1 + t_1 d_1$, $\overline{b}_3 = \overline{b}_1 + t_1 d_1 + t_2 d_2$,..., $\overline{b}_N = \overline{b}_1 + \sum_{i=1}^{N-1} t_i d_i$. Now we project $\overline{b}_j \in R^\perp$ to $a_j^\perp$ along the vector

a_j: we obtain $b_j := \overline{b}_j - \frac{\langle \overline{b}_j, a_j\rangle}{\langle a_j, a_j\rangle} a_j$. Our function $\xi_j \in (a_j)^\perp$ is defined by $\xi_j(x) = \langle b_j, x\rangle = \langle \overline{b}_j, x\rangle$. Without loss of generality we can assume $\overline{b}_1 = 0$ and thus we obtain that $\xi_j(x) = \langle \sum_{k=1}^{j-1} t_k d_k, x\rangle$. Note that we indeed have $\xi_j - \xi_{j+1} = 0$ on $a_j^\perp \cap a_{j+1}^\perp$.

Using Theorem 5.1.5 we will generalize Altmann's cup product formula to the case of not necessarily isolated toric Gorenstein singularities. Note that Altmann was using different methods (Laudal's cup product) in his proof.

Theorem 5.2.3. *The cup product $T^1(-R^*) \times T^1(-R^*) \to T^2(-2R^*)$ equals the bilinear map*

$$V/(k \cdot \underline{1}) \times V/(k \cdot \underline{1}) \mapsto \ker \eta / \operatorname{im} \delta$$

$$(\underline{t}, \underline{s}) \mapsto (s_1 t_1 d_1, ..., s_N t_N d_N).$$

We write for short $R = R^*$. We first need to compute the function

$$h_j = -\xi_j(\lambda)\mu_j(\lambda) + \xi_j(-R+\lambda)\mu_j(\lambda) + \mu_j(-R+\lambda)\xi_j(\lambda)$$

on $K_{a_j}^{2R}$ and then compute $h_j - h_{j+1}$ on $K_{j,j+1}^{2R} := K_j^{2R} \cap K_{j+1}^{2R}$. We see that $\xi_j(-R+\lambda)\mu_j(\lambda) = 0$ on $K_{a_j}^{2R}$ since $\mu_j(-R+\lambda) = 0$ for $\lambda \in a_j^\perp$ (thus either $\xi_j(-R+\lambda) = 0$ or $\mu_j(\lambda) = 0$). The same argument holds for $\mu_j(-R+\lambda)\xi_j(\lambda) = 0$.

We have $h_j = -\xi_j(\lambda)\mu_j(\lambda)$ and thus on $K_{j,j+1}^{2R}$ it holds that

$$h_j - h_{j+1} = (s_j d_j)(t_j d_j) + (s_j d_j)(\sum_{k=1}^{j-1} t_k d_k) + (t_j d_j)(\sum_{k=1}^{j-1} s_k d_k). \tag{5.5}$$

We now consider the function $(dh_j - C)(\lambda_1, \lambda_2) \in C_{(1)}^2(K_{a_j}^{2R}; k)$. Let

$$\lambda^j \in \Lambda \cap a_j^\perp \cap a_{j+1}^\perp,$$

$$\gamma^j \in \Lambda \cap a_{j-1}^\perp \cap a_j^\perp,$$

$$\lambda_1^j \in P_1^j := (K_{a_j}^{2R} \cap K_{a_{j+1}}^R) \setminus (a_j^\perp \cap a_{j+1}^\perp),$$

$$\lambda_2^j \in P_2^j := (K_{a_{j-1}}^R \cap K_{a_j}^{2R}) \setminus (a_{j-1}^\perp \cap a_j^\perp).$$

If $\langle a_j, a_{j+1}\rangle$ is smooth, then P_1^j and P_2^j are infinite sets contained in the lines parallel to $a_j^\perp \cap a_{j+1}^\perp$ and $a_{j-1}^\perp \cap a_j^\perp$, respectively. If $\langle a_j, a_{j+1}\rangle$ is not smooth, then $P_1^j = P_2^j = \emptyset$ and thus we can easily see that $dh_j = C$ holds on $K_{a_j}^{2R}$. Moreover, $K_{j,j+1}^{2R} \subset \operatorname{Span}_k(R^*, a_j^\perp \cap a_{j+1}^\perp)$ and thus $h_j - h_{j+1} = 0$ for a non-smooth $\langle a_j, a_{j+1}\rangle$ and this agrees with our cup product formula, since $t_j s_j d_j = 0$ on N_k/d_j.

We focus now on the case when $\langle a_j, a_{j+1}\rangle$ is smooth. If $\lambda \in P_1^j \cup P_2^j$, then $\langle \lambda, a_j\rangle = 1$. We want to find the functions $F_j \in C_{(1)}^1(K_{a_j}^{2R}; k)$ for which $dh_j + dF_j = C$ holds. Let

$$F_j(c) := \begin{cases} -\xi(c)s_j d_j(c) - \mu(c)t_j d_j(c) = \xi(c)s_j + \mu(c)t_j & \text{if } c \in P_1^j \\ \xi(c)s_{j-1}d_{j-1}(c) + \mu(c)t_{j-1}d_{j-1}(c) = -\xi(c)s_{j-1} - \mu(c)t_{j-1} & \text{if } c \in P_2^j \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5.2.4. *On $K_{a_j}^{2R}$ (i.e. for $(\lambda_1, \lambda_2) \in \Lambda \times \Lambda$ such that $\lambda_1 + \lambda_2 \in K_{a_j}^{2R}$) it holds that $dh_j + dF_j = C$.*

Proof. We write the proof in a way that also becomes clear how we chose the functions F_j. Recall the definitions of C and $dh_j = C^0$ (equations (5.1) and (5.2)). In order that functions F_j satisfy the equation $dh_j + dF_j = C$ on $K^{2R}_{a_j}$ the following claims need to hold.

Claim 1: It holds that

$$\begin{aligned} &F_j(\lambda^j + \lambda^j_1) = \\ &= F_j(\lambda^j) + F_j(\lambda^j_1) - \xi(\lambda^j)s_j d_j(\lambda^j_1) - \mu(\lambda^j)t_j d_j(\lambda^j_1) = \\ &= F_j(\lambda^j) + F_j(\lambda^j_1) + \xi(\lambda^j)s_j + \mu(\lambda^j)t_j \end{aligned} \tag{5.6}$$

and

$$F_j(\gamma^j + \lambda^j_2) = F_j(\gamma^j) + F_j(\lambda^j_2) - \xi(\gamma^j)s_{j-1}d_{j-1}(\lambda^i_2) - \mu(\gamma^j)t_{j-1}d_{j-1}(\lambda^j_2). \tag{5.7}$$

Indeed, $(C-dh_j)(\lambda^j, \lambda^j_1) = \xi(\lambda^j)s_j d_j(\lambda^j_1) + \mu(\lambda^j)t_j d_j(\lambda^j_1)$, since $\xi(\lambda^j) = \xi^0_j(\lambda^j)$, $\xi(\lambda^j_1) - \xi^0_j(\lambda^j_1) = t_j d_j(\lambda^j_1)$ and $d\xi(\lambda^j, \lambda^j_1) = d\xi^0_j(\lambda^j, \lambda^j_1) = 0$ (similarly also for μ). With the same procedure we obtain also the other equation and thus Claim 1 is proved.

Let $z^j_1 := \lambda^j + \gamma^j + \lambda^j_1$ and $z^j_2 := \lambda^j + \gamma^j + \lambda^j_2$, where $\lambda^j \neq 0$ and $\mu^j \neq 0$.

Claim 2: Functions F_j must also satisfy the following equations:

$$\begin{aligned} &F_j(\lambda^j + \gamma^j + \lambda^j_1) = \\ &F_j(\lambda^j + \lambda^j_1) + F_j(\gamma^j) - \xi(\gamma^j)s_j d_j(\lambda^j + \lambda^j_1) - \mu(\gamma^i)t_j d_j(\lambda^j + \lambda^j_1) \\ &+ \xi(-R + z^j_1)s_j d_j(\lambda^j + \lambda^j_1) + \mu(-R + z^j_1)t_j d_j(\lambda^j + \lambda^j_1). \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} &F_j(\lambda^j + \gamma^j + \lambda^j_2) = \\ &F_j(\gamma^j + \lambda^j_2) + F(\lambda^j) + \xi(\lambda^j)s_{j-1}d_{j-1}(\gamma^j + \lambda^j_2) + \mu(\lambda^j)t_{j-1}d_{j-1}(\gamma^j + \lambda^j_2) \\ &- \xi(-R + z^i_2)s_{j-1}d_{j-1}(\gamma^j + \lambda^j_2) - \mu(-R + z^i_2)t_{j-1}d_{j-1}(\gamma^j + \lambda^j_2). \end{aligned} \tag{5.9}$$

Proof of the Claim 2: It holds that

$$\begin{aligned} &(C - dh_j)(\lambda^j + \lambda^j_1, \gamma^j) = \\ &\xi(\gamma^j)s_j d_j(\lambda^j + \lambda^j_1) + \mu(\gamma^j)t_j d_j(\lambda^j + \lambda^j_1) + \xi^0_j(\lambda^j + \lambda^j_1)\mu^j_0(-R + z^j_1) + \\ &\mu^j_0(\lambda^j + \lambda^j_1)\xi^j_0(-R + z^j_1) - \xi(\lambda^j + \lambda^j_1)\mu(-R + z^j_1) - \mu(\lambda^j + \lambda^j_1)\xi(-R + z^j_1). \end{aligned}$$

Since $\lambda^j \neq 0$ and $\mu^j \neq 0$, we have $-R + z^j_2 \geq 0$ and $-R + z^j_1 \geq 0$. Thus $\xi^j_0(-R + z^j_1) = \xi(-R + z^j_1)$, $\xi^j_0(\lambda^j_1 + \lambda^j) - \xi(\lambda^j_1 + \lambda^j) = -t_j d_j(\lambda^j_1)$ and similarly for $(C - dh_j)(\gamma^j + \lambda^j_2, \lambda^j)$. Thus we conclude the proof of Claim 2. We can easily verify that our function F satisfies the properties (5.8), (5.9) and that $dF_j + dh_j = C$ indeed holds. □

To conclude the proof of Theorem 5.2.3, we need to show that $\delta(F_1, ..., F_N) + \delta(h_1, ..., h_N) = (t_1 s_1 d_1, ..., t_N s_N d_N)$. Recall the formula for $h_j - h_{j+1}$ on $K^{2R}_{j,j+1}$ (see the equation (5.5)). We distinguish the following cases

1. $c \in P^j_1$: it holds that $\langle a_j, c\rangle = 1$, $\langle a_{j+1}, c\rangle = 0$ and thus we have $F_{j+1}(c) = 0$, $F_j(c) = \xi(c)s_j + \mu(c)t_j$, where $\xi(c) = \sum_{i=1}^{j} t_i d_i(c)$, $\mu(c) = \sum_{i=1}^{j} s_i d_i(c)$. Using $d_j(c) = -1$ we obtain that
$$(F_j - F_{j+1})(c) + (h_j - h_{j+1})(c) = -s_j t_j.$$

2. $c \in P_2^{j+1}$: it holds that $\langle a_j, c\rangle = 0$, $\langle a_{j+1}, c\rangle = 1$ and thus we have $F_j(c) = 0$, $F_{j+1}(c) = -\xi(c)s_j - \mu(c)t_j$, where $\xi(c) = \sum_{i=1}^{j-1} t_i d_i(c)$, $\mu(c) = \sum_{i=1}^{j-1} s_i d_i(c)$. It follows that

$$(F_j - F_{j+1})(c) + (h_j - h_{j+1})(c) = t_j s_j.$$

3. $c \in a_j^{\perp} \cap a_{j+1}^{\perp}$: it holds that $(F_j - F_{j+1})(c) + (h_j - h_{j+1})(c) = 0$.

4. $c = R$: it follows that $(F_j - F_{j+1})(c) + (h_j - h_{j+1})(c) = 0$.

Thus we completely described the element $\delta(F_1, ..., F_N) + \delta(h_1, ..., h_N) \in \oplus_{j=1}^{N} H^1_{(1)}(K^{2R}_{j,j+1}; k)$. By [6, Proposition 5.4] we know that $H^1_{(1)}(\operatorname{Span}_k K^{2R}_{j,j+1}; k) \cong H^1_{(1)}(K^{2R}_{j,j+1}; k)$ since a 2-face $\langle a_j, a_{j+1}\rangle$ admits at most Gorenstein singularities. If $\langle a_j, a_{j+1}\rangle$ is smooth, then it holds that $d_j(c) = -1$ for $c \in P_1^j$ and $d_j(c) = 1$ for $c \in P_2^{j+1}$. We see that our additive function, corresponding to the $\delta(F_j) + \delta(h_j)$, is equal to $t_j s_j d_j$. If $\langle a_j, a_{j+1}\rangle$ is not smooth, then the corresponding function is equal to 0. Thus we conclude the proof of Theorem 5.2.3.

Corollary 5.2.5. *If X_σ is an isolated Gorenstein singularity, then Theorem 5.2.3 gives us Altmann's cup product* (5.3).

5.3 The cup product between non-negative degrees

Let X_σ be a non-isolated three-dimensional toric Gorenstein singularity. In this section we compute the cup product $T^1(-R) \times T^1(-S) \to T^2(-R-S)$ for $R, S \not\geq 0$. If R and S have the last entry equal to 0, then the computations in this section have implications in deformation theory of projective toric varieties.

The following notation already appeared in Subsection 3.6.1. Let $s_1, ..., s_N$ be the fundamental generators of the dual cone $\sigma^\vee$, labelled so that $\sigma \cap (s_j)^\perp$ equals the face spanned by $a_j, a_{j+1} \in \sigma$. With $\ell(j)$ we denote the length of the edge d_j. Let $R_j^{p,q} := qR^* - ps_j$ with $2 \le q \le \ell(j)$ and $p \in \mathbb{Z}$ sufficiently large such that $R_j^{p,q} \notin \operatorname{int}(\sigma^\vee)$. In this case we already know that $\dim_k T^1(-R_j^{p,q}) = 1$ by Corollary 3.6.4.

Lemma 5.3.1. *If $\#\{a_j \mid \langle a_j, R\rangle > 0\} \le 2$, then $T^2(-R) = 0$.*

Proof. If $\#\{a_j \mid \langle a_j, R\rangle > 0\} \le 1$, then the statement is trivial. Without loss of generality $\langle a_j, R\rangle > 0$ for $j = 1, 2$ and $\langle a_j, R\rangle \le 0$ for other j. Now the statement follows from the fact that $T^2 = 0$ for the Gorenstein surface $\langle a_1, a_2\rangle \subset N_\mathbb{R} \cong \mathbb{R}^2$ (see Example 8). □

Proposition 5.3.2. *Let $R_1 := R_j^{p_1,q_1}$ and $R_2 := R_k^{p_2,q_2}$, where j and k are chosen such that a_j and a_k are not neighbours (we allow $j = k$). The cup product $T^1(-R_1) \times T^1(-R_2) \to T^2(-R_1 - R_2)$ is zero.*

Proof. Let $\xi \in H^1_{(1)}(\Lambda \setminus \Lambda(R_1); k)$ and $\mu \in H^1_{(1)}(\Lambda \setminus \Lambda(R_2); k)$ represent basis elements for $T^1(-R_1)$ and $T^1(-R_2)$, respectively. We will show that the cup product $[d\xi^0, d\mu^0] \in T^2(-R_1 - R_2)$ is equal to zero. If $j = k$ the statement follows from Lemma 5.3.1. Since a_j and a_k are not neighbours, it holds that $\langle a_j, R_1 + R_2\rangle \le \langle a_j, R_1\rangle$ and $\langle a_k, R_1 + R_2\rangle \le \langle a_k, R_1\rangle$, from which it follows that $K^{R_1+R_2}_{a_i} \subset K^{R_1}_{a_i}$ for $i = j, j+1, k, k+1$. Thus we have $d(h_1, ..., h_N) = C$ and by Theorem 5.1.5 it follows that the cup product is equal to $\delta(h_1, ..., h_N) \in \oplus_{j=1}^{N} H^1_{(1)}(K^{R_1+R_2}_{j,j+1}; k)$. We can easily see that $h_i = 0$ if $i \ne j, j+1, k, k+1$ and

$$h_i(\lambda) := -\xi(\lambda)\mu(\lambda) + \xi(-R_2 + \lambda)\mu(\lambda) + \mu(-R_1 + \lambda)\xi(\lambda) \in C^1_{(1)}(K^{R_1+R_2}_{a_i}; k),$$

if $i = j, j+1, k, k+1$. We see that $\delta(h_1, ..., h_N) = 0$ and thus we conclude the proof. $\square$

The following example shows that we can also compute the cup product between the elements of degrees $R_1 := R_j^{p_1,q_1}$ and $R_2 := R_{j+1}^{p_2,q_2}$.

Example 12. A typical example of a non-isolated, three-dimensional toric Gorenstein singularity is the affine cone X_σ over the weighted projective space $\mathbb{P}(1,2,3)$. The cone σ is given by $\sigma = \langle a_1, a_2, a_3 \rangle$, where

$$a_1 = (-1,-1,1), \quad a_2 = (2,-1,1), \quad a_3 = (-1,1,1).$$

We obtain $\sigma^\vee = \langle s_1, s_2, s_3 \rangle$ with

$$s_1 = (0,1,1), \quad s_2 = (-2,-3,1), \quad s_3 = (1,0,1).$$

T^1 is non-zero in degrees $R^1_\alpha := 2R^* - \alpha s_3$, $R^2_\beta := 2R^* - \beta s_1$ and $R^3_\gamma := 2R^* - \gamma s_1$ with $\alpha \geq 1$, $\beta \geq 1$ and $\gamma \geq 2$. Let us denote the corresponding basis element of R^1_α, R^2_β and R^3_γ by z^1_α, z^2_β and z^3_γ, respectively.

We have

$$\begin{aligned} \langle a_1, R^1_\alpha \rangle = \langle a_3, R^1_\alpha \rangle = 2, &\quad \langle a_2, R^1_\alpha \rangle = 2 - 3\alpha, \\ \langle a_1, R^2_\beta \rangle = \langle a_2, R^2_\beta \rangle = 2, &\quad \langle a_3, R^2_\beta \rangle = 2 - 2\beta, \\ \langle a_1, R^3_\gamma \rangle = \langle a_2, R^3_\gamma \rangle = 3, &\quad \langle a_3, R^3_\gamma \rangle = 3 - 2\gamma. \end{aligned}$$

By Lemma 5.3.1 we know that the only possible non-zero cup products can be $[z^1_1, z^2_1]$ and $[z^1_1, z^3_2]$, since in other cases we have $T^2(R^i_j + R^k_l) = 0$. Using Theorem 5.1.5 we can easily verify that $[z^1_1, z^2_1] \neq 0$ and $[z^1_1, z^3_2] \neq 0$. In this case the equations $z^1_1 \cdot z^2_1 = z^1_1 \cdot z^3_2 = 0$ already define the whole versal base space. Stevens checked this using the computer algebra system Macaulay, see [5, Section 5.2].

5.4 The versal base space of a three-dimensional toric Gorenstein singularity

In this section we conjecture a set of equations of the versal base space of degree R^* for not necessarily isolated three-dimensional toric Gorenstein singularities. Note that in the isolated case the equations were obtained by Altmann in [4].

For $b \in \mathbb{Z}$ we define

$$b^+ := \begin{cases} b & \text{if } b \geq 0 \\ 0 & \text{otherwise,} \end{cases} \qquad b^- := \begin{cases} 0 & \text{if } b \geq 0 \\ -b & \text{otherwise.} \end{cases}$$

We define an ideal $I = (\sum_{i=1}^N b_i u_i^k \mid k \geq 1) \subset k[u_1, ..., u_N]$, where $b_i \in \mathbb{Z}$ for all $i = 1, ..., N$ and it holds that $\sum_{i=1}^N b_i = 0$. The following proposition will be very useful. In some parts of the proof we follow [4] verbatim.

Proposition 5.4.1. (1) *I is generated by polynomials from $k[u_i - u_j]$,*

(2) *$I \subset k[u_1, ..., u_N]$ is the smallest ideal that meets property* (1) *and on the other hand contains $r(\underline{u}) := \prod_{i=1}^N u_i^{b_i^+} - \prod_{i=1}^N u_i^{b_i^-}$.*

Proof. We define

$$g_k(\underline{u}) := \sum_{j=1}^{N} b_j u_j^k, \quad \text{for } k \geq 1.$$

We know that $g_1(\underline{u}) = \sum_{j=1}^{N} b_j u_j$ and since $\sum_{j=1}^{N} b_j = 0$, we have

$$g_1(\underline{u}) = b_2(u_2 - u_1) + \cdots + b_N(u_N - u_1).$$

Replacing u_j by $u_j - u_1$ as arguments in g_k yields

$$\begin{aligned} g_k(u_2 - u_1, ..., u_N - u_1) &= \sum_{j=1}^{N} b_j (u_j - u_1)^k = \sum_{j=1}^{N} b_j \big(\sum_{v=0}^{k} (-1)^v \binom{k}{v} u_1^v u_j^{k-v} \big) \\ &= \sum_{v=0}^{k} (-1)^v \binom{k}{v} u_1^v \cdot \big(\sum_{j=1}^{N} b_j u_j^{k-v} \big) = \sum_{v=0}^{k-1} (-1)^v \binom{k}{v} u_1^v g_{k-v}(\underline{u}), \end{aligned}$$

from which (1) follows.

Without loss of generality we assume that $b_1, ..., b_M \geq 0$, $b_{M+1}, ..., b_N \leq 0$. After renaming

$$u_i = x_i, \quad u_{M+j} = y_{M+j} \quad (1 \leq i \leq M, \ 1 \leq j \leq N - M),$$

we obtain

$$g_k(\underline{x}, \underline{y}) = \big(\sum_{i=1}^{M} b_i x_i^k \big) - \big(\sum_{j=M+1}^{N} b_j^- y_j^k \big),$$

$$r(\underline{x}, \underline{y}) = (x_1^{b_1} \cdots x_M^{b_M}) - (y_{M+1}^{b_{M+1}^-} \cdots y_N^{b_N^-}).$$

Let S denote the multiset

$$S := \{1, ..., 1, 2, ..., 2, ..., M, ..., M\},$$

where i has multiplicity b_i (for $i = 1, ..., M$) and thus

$$l := |S| = b_1^+ + \cdots + b_N^+ = b_1^- + \cdots + b_N^-.$$

For $A \subset S$ we write $x^A := \prod_{i \in A} x_i$, which is a monomial of degree $|A|$. We can generalize arguments with symmetric polynomials made in [4, Lemma 3.3] as follows: let

$$s_j(\underline{x}) := \sum_{A \subset S, |A| = j} x^A, \quad \text{for } j = 1, ..., l.$$

We write $g_k(\underline{x}) = \sum_{i=1}^{M} b_i x_i^k$ and $g_k(\underline{y}) = \sum_{j=M+1}^{N} b_j^- y_j^k$. Note that we have

$$g_1(\underline{x}) = s_1(\underline{x}) = \sum_{i=1}^{M} b_i x_i.$$

We can show that there exist $\tilde{b}_k \in \mathbb{Q}$, $(k = 1, ..., l)$ with $b_l \neq 0$ such that

$$(x_1^{b_1} \cdots x_M^{b_M}) = \tilde{b}_1 (b_1 x_1 + \cdots + b_M x_M)^l + \sum_{k=2}^{l-1} \tilde{b}_k g_k(\underline{x}) s_{l-k}(\underline{x}) + \tilde{b}_l g_l(\underline{x}). \tag{5.10}$$

We choose $\tilde{b}_1 := \frac{b_1! \cdots b_M!}{l!}$ and then we choose $\tilde{b}_2$ such that $\tilde{b}_1(x_1 + \cdots + x_M)^l + \tilde{b}_2 g_2(\underline{x}) s_{l-2}(\underline{x})$ does not have monomials of the form $x_i^{b_i+1} \cdot m_i(x_1, ..., \hat{x}_i, ..., x_M)$ (for each $i = 1, ..., M$), where m_i is a monomial of degree $l - b_i - 1$. Such $\tilde{b}_2$ exists and we see that we can naturally continue this procedure, i.e., we choose $\tilde{b}_3$ such that $\tilde{b}_1(x_1 + \cdots + x_M)^l + \tilde{b}_2 g_2(\underline{x}) s_{l-2}(\underline{x}) + \tilde{b}_3 g_3(\underline{x}) s_{l-3}$ does also not have monomials of the form $x_i^{b_i+2} \cdot \tilde{m}_i(x_1, ..., \hat{x}_i, ..., x_M)$ for each $i = 1, ..., M$. At the end we obtain the equation (5.10).

We have $s_1(\underline{x}) = g_1(\underline{x})$, $s_2(\underline{x}) = \frac{1}{2}\big((g_1(\underline{x})^2 - g_2(\underline{x}))\big)$ and as above we can easily verify that for a fixed j $(1 \leq j \leq l)$ there exist $\tilde{c}_i \in \mathbb{Q}$, $(i = 1, ..., j)$ with $\tilde{c}_i \neq 0$ such that

$$s_j(\underline{x}) = \tilde{c}_1(x_1 + \cdots x_M)^j + \sum_{i=2}^{j-1} \tilde{c}_i g_i(\underline{x}) s_{j-i}(\underline{x}) + \tilde{c}_j g_j(\underline{x}).$$

Thus we see that for $1 \leq k \leq l$ we can write $s_k(\underline{x}) = P_k(g_1(\underline{x}), ..., g_{k-1}(\underline{x})) + c_k g_k(\underline{x})$, $s_k(\underline{y}) = P_k(g_1(\underline{y}), ..., g_{k-1}(\underline{y})) + c_k g_k(\underline{y})$ for some polynomials $P_k \in \mathbb{Q}[z_1, ..., z_{k-1}]$ and non-vanishing rational numbers c_k. In particular, we have

$$r(\underline{x}, \underline{y}) = P_l(g_1(\underline{x}), ..., g_{l-1}(\underline{x})) - P_l(g_1(\underline{y}), ..., g_{l-1}(\underline{y})) + c_l g_l(\underline{x}) - c_l g_l(\underline{y}).$$

We can conclude the proof following [4, Lemma 3.4]: each polynomial $q(\underline{u})$ can be uniquely written as

$$q(\underline{u}) = \sum_{v \geq 0} q_v(u_2 - u_1, ..., u_N - u_1) \cdot u_1^v.$$

If $\tilde{I} \subset k[\underline{u}]$ is an ideal generated by polynomials in $\underline{u} - u_1$ only, then for each $q(\underline{u}) \in \tilde{I}$ the components q_v are automatically contained in $\tilde{I}$, too. Hence, we should look for the components of the polynomial p. In the polynomial ring $k[\underline{X}, \underline{Y}, U]$ we know that

$$r(U + \underline{X}, U + \underline{Y}) = (U + X_1)^{b_1} \cdots (U + X_M)^{b_M} - (U + Y_{M+1})^{b^-_{M+1}} \cdots (U + Y_N)^{b^-_N}$$

has $s_k(\underline{X}) - s_k(\underline{Y})$ as the coefficient of U^{l-k} $(k = 1, ..., l)$.

We obtain

$$\begin{aligned} s_k(\underline{X}) - s_k(\underline{Y}) &= P_k(g_1(\underline{X}), ..., g_{k-1}(\underline{X})) - P_k(g_1(\underline{Y}), ..., g_{k-1}(\underline{Y})) + c_k g_k(\underline{X}) - c_k g_k(\underline{Y}) \\ &= \sum_{v=1}^{k-1} q_v(\underline{X}, \underline{Y}) g_v(\underline{X}, \underline{Y}) + c_k g_k(\underline{X}, \underline{Y}) \end{aligned}$$

for some coefficients q_v. If we show that $I = (\sum_{j=1}^N u_j^k b_j \mid 1 \leq k \leq l)$, then specialization (first by $U \mapsto x_1$, $X_i \mapsto x_i - x_1$, $Y_i \mapsto y_i - x_1$, then followed by the usual one) shows that the ideal generated by the components $r_v(\underline{u} - u_1)$ of r equals I. We conclude the proof by showing that $I = (\sum_{j=1}^N u_j^k b_j \mid 1 \leq k \leq l)$: we can generalize arguments with symmetric polynomials made in [4, Lemma 3.3] as follows: for each $k > l$ there exist polynomials $P_k \in \mathbb{Q}[s_1, ..., s_l]$, such that

$$g_k(\underline{x}) - g_k(\underline{y}) = P_k(s_1(\underline{x}), ..., s_l(\underline{x})) - P_k(s_1(\underline{y}), ..., s_l(\underline{y})).$$

As in [4, Lemma 3.3] we conclude that

$$g_k(\underline{x}, \underline{y}) = \sum_{v=1}^{l} g_v(\underline{x}, \underline{y}) \cdot z_v(\underline{x}, \underline{y}),$$

for some polynomials z_v. Thus we conclude the proof. □

Recall that X_σ is not necessarily isolated three-dimensional toric Gorenstein singularity. The edges of the polytope $Q \subset \mathbb{A} := [R^* = 1]$ are $d_1, ..., d_N$ and we have $d_1 + \cdots + d_N = 0$. We have the vector space

$$V := \{\underline{t} = (t_1, ..., t_N) \in k^N \mid \sum_{j=1}^{N} t_j d_j = 0\}.$$

The lattice length of d_j is denoted by $\ell(d_j)$ for $j = 1, ..., N$. Let $V \hookrightarrow k^N$ be the standard inclusion given by $\underline{t} \to \underline{t}$. We denote the lattice $\mathbb{L} := \mathbb{A} \cap \mathbb{Z}^n$.

We define the ideal

$$J := (\sum_{j=1}^{N} t_j^k d_j \mid k \geq 1) \subset k[t_1, ..., t_N]$$

and the affine scheme

$$M := \mathrm{Spec}(k[t_1, ..., t_N]/J) \subset k^N.$$

Let us denote

$$\tilde{r}(t_1, ..., t_N) := \prod_{i=1}^{N} t_i^{d_i^+} - \prod_{i=1}^{N} t_i^{d_i^-},$$

with $\underline{d} \in \big(\ell(d_1)\mathbb{Z} \times \cdots \times \ell(d_N)\mathbb{Z}\big) \cap \mathrm{Span}_k\{(\langle d_1, c\rangle, ..., \langle d_N, c\rangle) \mid c \in \mathbb{A}^*\}$.

Denote by p the projection $p : k^N \to k^N/k(1, ..., 1)$, which on the level of regular functions corresponds to the inclusion $k[t_i - t_j \mid 1 \leq i, j \leq N] \subset k[t_1, ..., t_N]$.

The following theorem generalizes [4, Theorem 2.4].

Theorem 5.4.2. *The following holds:*

1. *J is generated by polynomials from $k[t_i - t_j]$, i.e., $M = p^{-1}(\bar{M})$ for some affine closed subscheme $\bar{M} \subset V/k(1, ..., 1)$.*
2. *$J \subset k[t_1, ..., t_N]$ is the smallest ideal that has the above property and on the other hand contains $\tilde{r}$.*

Proof. The proof follows from Proposition 5.4.1 and the fact that for every $c \in \mathbb{L}^*$ we have

$$\big(\langle d_1, c\rangle, ..., \langle d_N, c\rangle\big) \in \big(\ell(d_1)\mathbb{Z} \times \cdots \times \ell(d_N)\mathbb{Z}\big).$$

□

In [4] Altmann proved that $\bar{M}$ is the versal base space for isolated three-dimensional Gorenstein singularities and he also constructed a versal family. We conjecture that $\bar{M}$ is the versal base space in degree $-R^*$ also for not necessarily isolated Gorenstein singularities.

5.5 A differential graded Lie algebra extending the cup product

In this subsection we construct a dgla extending the cup product from Theorem 5.2.3.

Let X_σ be a three-dimensional affine Gorenstein toric variety. We define

$$\mathfrak{g}^i(-R) := \oplus_{\tau \leq \sigma; \dim \tau = i}(\mathrm{Span}_k(E_\tau^R)^*),$$

for all $R \in M$. The bracket $[\ ,\ ]$ is defined in the following way: $[b_1, b_2] = 0$ if for at least one $j \in \{1,2\}$ holds that $b_j \notin \mathfrak{g}^1(-kR^*)$ for $k \geq 1$. Let ℓ be the linear form on N with $\ell(e_1) = \ell(e_2) = \ell(e_3) = 1$.

We define $[\ ,\ ] : \mathfrak{g}^1(-kR^*) \times \mathfrak{g}^1(-lR^*) \to \mathfrak{g}^2(-(k+l)R^*)$ as $[\underline{b}, \underline{c}] := \big((\underline{b} \cup \underline{c})_1, ..., (\underline{b} \cup \underline{c})_N\big)$, where

$$(\underline{b} \cup \underline{c})_j := \frac{\ell(p(b_{j+1}) - p(b_j)) \cdot (p(c_{j+1}) - p(c_j)) + \ell(p(c_{j+1}) - p(c_j)) \cdot (p(b_{j+1}) - p(b_j))}{(k+l-1)\ell(d_j)}$$

and $p(b_j) \in N$ (resp. $p(c_j) \in N$) are defined in the following way. Note first that $\mathfrak{g}^1(-R^*) = \bigoplus_{j=1}^{N} N_{\mathbb{R}}/a_j\mathbb{R}$ and $\mathfrak{g}^1(-kR^*) = N_{\mathbb{R}}^N$, for all $k \geq 2$. Thus we have $\underline{b} = (b_1, ..., b_N)$ and b_j is either an element of $N_{\mathbb{R}}/a_j\mathbb{R}$ or $N_{\mathbb{R}}$. If $b_j \in N_{\mathbb{R}}$ we define $p(b_j) := b_j$. If $b_j \in N_{\mathbb{R}}/a_j\mathbb{R}$, then we identify b_j with $\xi_j \in (a_j^{\perp})^*$ as we did in Section 5.1. Now we project ξ_j to $R^{*\perp}$ along the vector a_j for each j. The resulting element is defined as our $p(b_j) \in R^{*\perp} \subset N$. In the same way we construct $p(c_j)$.

Differential on $\mathfrak{g}$ is coming from the complex $(\mathrm{Span}_k(E^R)^*)_\bullet$ and with the above product $[\ ,\ ]$ we give a dgla structure on $\mathfrak{g}$. We can easily verify that the dgla $\mathfrak{g}$ extends the cup product from Theorem 5.2.3.

Let $\underline{t} = (t_1, ..., t_N) \in V$ be an arbitrary point in the versal base space in degree $-R^*$ for X_σ conjectured in Section 5.4, i.e., $\underline{t} \in V$ satisfies $\sum_{j=1}^{N} t_j^k d_j = 0$ for all $k \in \mathbb{N}$. From $\underline{t}$ we can construct an MC element of the dgla $\mathfrak{g}$ as follows: we define $x^k := (0, t_1^k d_1, t_1^k d_1 + t_2^k d_2, ..., \sum_{j=1}^{N-1} t_j^k d_j) \in \mathfrak{g}^1(-kR^*)$ for $k \geq 1$. We can easily verify that

$$x = \{x^k = (x_1^k, \ldots, x_N^k) \mid k \geq 1\} \in \bigoplus_{k \geq 1} \mathfrak{g}^1(-kR^*)$$

satisfy the Maurer-Cartan equation: in degree $-kR^*$ the MC equation $dx + \frac{1}{2}[x,x]$ reduces to $dx^k + \sum_{u+v=k} \frac{1}{2}[x^u, x^v] = 0$. We have $dx^k = (-t_1^k d_1, -t_2^k d_2, .., -t_N^k d_N)$, $p(x_j^k) - p(x_{j+1}^k) = t_j^k d_j$ and thus

$$(x^u \cup x^v)_j = \frac{2t_j^k d_j}{k-1}.$$

In the sum $\sum_{u+v=k} \frac{1}{2}[x^u, x^v]$ we have $k-1$ summands from which it follows that x is an MC element.

6 Poisson deformations

Poisson deformations are deformations of a pair consisting of a variety and a Poisson structure on it. Lately there has been a lot of interest in these deformations, see for example results of Namikawa [52],[53], [54] or Kaledin and Ginzburg [33].

In Section 6.1 we construct a differential graded Lie algebra structure controlling Poisson deformations. In Section 6.2 we give a convex geometric description of the Hochschild cup product and simplify the computation of Poisson cohomology groups.

6.1 A differential graded Lie algebra controlling Poisson deformations

We consider the following deformation problem.

Definition 66. A *Poisson deformation* of a Poisson algebra A over an Artin ring B is a pair (A', π), where A' is a Poisson B-algebra and $\pi : A' \otimes_B k \to A$ is an isomorphism of Poisson k-algebras. Two such deformations (A', π_1) and (A'', π_2) are *equivalent* if there exists an isomorphism of Poisson B-algebras $\phi : A' \to A''$ such that it is compatible with π_1 and π_2, i.e., such that $\pi_1 = \pi_2 \circ (\phi \otimes_B k)$.

A functor that encodes this deformation problem is

$$\mathrm{PDef}_A : \mathcal{A} \to \mathcal{S}$$

$$B \mapsto \{\text{Poisson deformations of } A \text{ over } B\}/\sim .$$

In the following we define a dgla that controls the above deformation problem. Consider the double complex given in Figure 6.1.

The map d_p is defined as $d_p := -[\mu_p, \cdot] : C^n(A) \to C^{n+1}(A)$, where $\mu_p \in C^2_{(2)}(A)$ is a Poisson structure $\{\ ,\ \}$ of A. In the double complex in Figure 6.1 we restrict d_p on the chosen domains and codomains. Note that we have $d[\mu_p, f] = [\mu_p, df] + 0$ by Proposition 2.3.1, and thus we really obtain a double complex. We denote its total complex by $D^\bullet$.

We define the bracket $[\ ,\ ]_p$ on $D^\bullet$ as follows: let $C^n(A) = C^n_{(1)}(A) \oplus \cdots \oplus C^n_{(n)}(A)$ and define

$$[\cdot,\cdot]_p : C^m(A) \times C^n(A) \to C^{m+n-1}(A)$$

$$[(f_1, ..., f_m), (g_1, ..., g_n)]_p := ([f_1, g_1], ..., \sum_{i+j=k} [f_i, g_j], ..., [f_m, g_n]),$$

where we restrict $[f_i, g_j]$ to $C^{m+n-1}_{(i+j-1)}(A)$.

This bracket defines a dgla structure on $D^\bullet[1]$: the shifted differential $d_p[1]$ is equal to $[\mu_p, \cdot]_p$ and the shifted differential $d[1]$ is equal to $[\mu, \cdot]_p$, where μ is the commutative multiplication on A. We denote the shifted differential of $D^\bullet[1]$ by $\tilde{d}$. It is given by $\tilde{d} = [\mu + \mu_p, \cdot]_p$. We can immediately check that the bracket $[\ ,\ ]_p$ and differential $\tilde{d}$ equip $D^\bullet[1]$ with the structure of a dgla. We denote this dgla by $C^\bullet_p(A)[1]$.

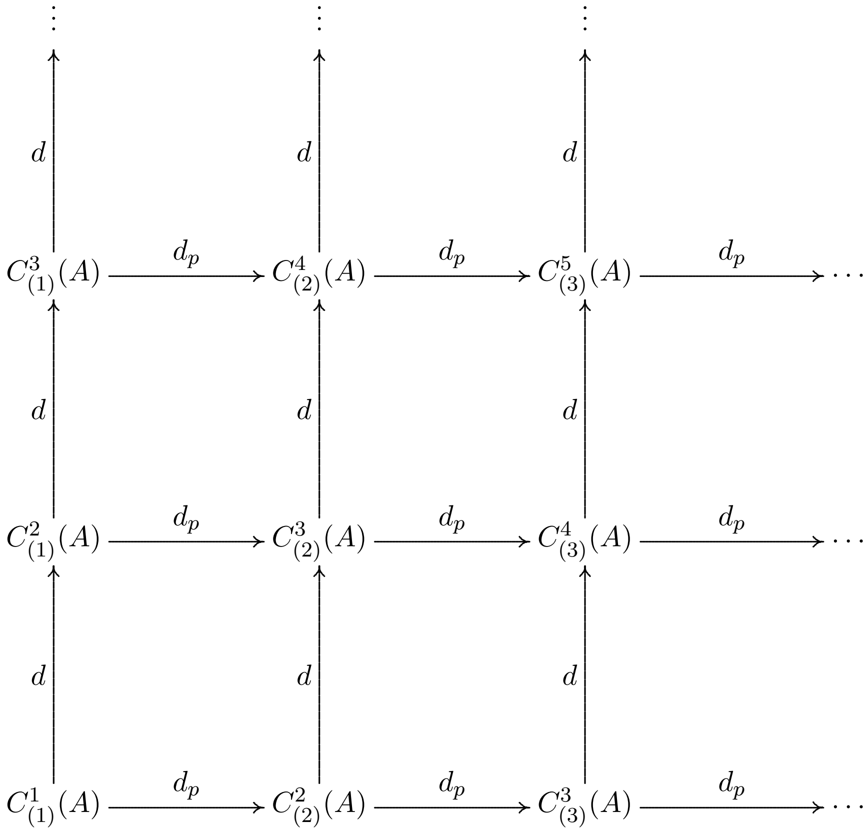

Figure 6.1: The double complex controlling the Poisson deformations

Remark 20. Note that the Gerstenhaber bracket is not graded with respect to the Hodge decomposition and thus the above product is not the Gerstenhaber bracket. By Lemma 4.1.3 we have $[\mu,\mu]_p = [\mu,\mu]$, $[\mu,\mu_p]_p = [\mu,\mu_p]$ and $[\mu_p,\mu_p]_p = e_3[\mu_p,\mu_p]$.

To show that the functor PDef_A is controlled by the dgla $C^\bullet_p(A)[1]$ we first need a few Lemmata. Some ideas are taken from [63, Sections 4.3 and 4.4]. Let V be a vector space. Recall from Definition 47 that $C^n(V)$ is the space of k-linear maps $V^{\otimes n} \to V$. Following the Hodge decomposition we define $C^n_{(i)}(V) := \{f \in C^n(V) \mid f \circ s_n = (2^i - 2)f)\}$. Thus we can define $C^\bullet_p(V)[1]$ to be a dgla with the Lie bracket $[\ ,\ ]_p$ and zero differential.

Lemma 6.1.1. *Let V be a vector space. Giving the Poisson algebra structure on V is the same as giving an element $(\mu,\mu_p) \in C^2_{(1)}(V) \oplus C^2_{(2)}(V)$ satisfying $\frac{1}{2}[\mu,\mu] = [\mu,\mu_p] = \frac{1}{2}[\mu_p,\mu_p]_p = 0$, i.e., (μ,μ_p) is an MC element of $C^\bullet_p(V)[1]$.*

Proof. Let (μ,μ_p) be an MC element of $C^\bullet_p(V)[1]$. We define the multiplication on V by $a \cdot b := \mu(a,b)$ and the Poisson structure by $\{a,b\} := \mu_p(a,b)$. The product $\cdot$ is commutative and associative if and only if $\mu \in C^2_{(1)}(V)$ and $\frac{1}{2}[\mu,\mu] = 0$ (see also Subsection 2.3.6). Now we show that μ_p defines a Poisson structure. Since $\mu_p \in C^2_{(2)}(V)$, everything except the Jacobi identity is clear. The Jacobi identity we get from $\frac{1}{2}[\mu_p,\mu_p]_p = 0$ as in Lemma 4.1.3 (note that we have $[\mu_p,\mu_p]_p = e_3[\mu_p,\mu_p]$). We now show the following claim:

$$\{a, b\cdot c\} = \{a,b\}c + \{a,c\}b \text{ (i.e. } \mu_p(a,\mu(b,c)) = \mu(\mu_p(a,b),c) + \mu(\mu_p(a,c),b))$$

holds if and only if $[\mu, \mu_p] = 0$. Assume that

$$F(a,b,c) := \mu_p(a, \mu(b,c)) - \mu(\mu_p(a,b),c) - \mu(\mu_p(a,c),b) = 0$$

holds. We have

$$\begin{aligned}
&F(a,b,c) + F(c,a,b) = \\
&\big(\mu_p(a,\mu(b,c)) - \mu(\mu_p(a,b),c) - \mu(\mu_p(a,c),b)\big) + \big(\mu_p(c,\mu(a,b)) - \mu(\mu_p(c,a),b) - \mu(\mu_p(c,b),a)\big) = \\
&-[\mu_p, \mu].
\end{aligned}$$

and thus we see one direction. For the other direction we compute

$$\begin{aligned}
&[\mu_p, \mu](a,b,c) + [\mu_p, \mu](a,c,b) - [\mu_p, \mu](b,a,c) = \\
&\big(\mu_p(ab,c) - \mu_p(a,bc) + \mu_p(a,b)c - \mu_p(b,c)a\big) + \big(\mu_p(ac,b) - \mu_p(a,cb) + \mu_p(a,c)b - \mu_p(c,b)a\big) - \\
&\big(\mu_p(ba,c) - \mu_p(b,ac) + \mu_p(b,a)c - \mu_p(a,c)b\big) = \\
&2\big(-\mu_p(a,bc) + \mu_p(a,b)c + \mu_p(a,c)b\big) = -2F(a,b,c).
\end{aligned}$$

To shorten the notation we wrote $ab = \mu(a,b)$ and similarly for other elements. Thus the claim is proved. The other direction of the proof follows immediately. □

Definition 67. The *Poisson product* on the vector space V is a pair $(\cdot, \{\ ,\ \})$, such that $(V, \cdot, \{\ ,\ \})$ is a Poisson algebra.

Lemma 6.1.2. *Let B be an Artin ring. MC elements of $C_p^\bullet(A \otimes m_B)[1]$ are in bijection with Poisson products of the vector space $A_0 \otimes B$, giving the known Poisson product on A (modulo m_B).*

Proof. Let $(\mu, \mu_p) \in C^2_{(1)}(A_0) \oplus C^2_{(2)}(A_0)$ represent the Poisson bracket on A_0. Then Poisson products on the vector space $A_0 \otimes B$, giving the known product on A (modulo m_B) are obtained by

$$[(\mu, \mu_p) + (\xi, \xi_p), (\mu, \mu_p) + (\xi, \xi_p)]_p = 0, \tag{6.1}$$

for $(\xi, \xi_p) \in C^2_{(1)}(A \otimes m_B) \oplus C^2_{(2)}(A \otimes m_B)$. Since $[(\mu, \mu_p), (\mu, \mu_p)]_p = 0$ and the differential on $C_p^\bullet(A \otimes m_B)[1]$ is given by $[(\mu, \mu_p), \cdot]$, then we see that the equation (6.1) gives us MC elements (ξ, ξ_p) of $C_p^\bullet(A \otimes m_B)[1]$. □

Theorem 6.1.3. *The functor PDef_A is controlled by the dgla $C_p^\bullet(A)[1]$.*

Proof. We write for short $\mathfrak{g} := C_p^\bullet(A)[1]$. By Lemma 6.1.2 there exists a bijection between $\mathrm{MC}_{\mathfrak{g}}(B)$ and Poisson products of the vector space $A_0 \otimes B$ giving the known Poisson product on A (modulo m_B).

To conclude the proof we show that two Poisson products $(\cdot, \{\ ,\ \})$ and $(\cdot', \{\ ,\ \}')$ on $A_0 \otimes B$ are equivalent (in the sense of Definition 66) if and only if the corresponding elements $(\gamma, \gamma_p), (\gamma', \gamma'_p) \in \mathrm{MC}_{\mathfrak{g}}(B)$ are gauge equivalent. Since the products are equivalent we can easily see that there exists $\alpha \in \mathfrak{g}^0 \otimes m_B$ such that

$$a \cdot' b = \exp(\alpha)(\exp(-\alpha)(a) \cdot \exp(-\alpha)(b)), \tag{6.2}$$

$$\{a, b\}' = \exp(\alpha)(\{\exp(-\alpha)(a), \exp(-\alpha)(b)\}'). \tag{6.3}$$

As above let $(\mu, \mu_p) \in C^2_{(1)}(A_0) \oplus C^2_{(2)}(A_0)$ represent the Poisson bracket of A.

From (6.2) we obtain

$$(\mu+\gamma')(a\otimes b)=\exp(\alpha)(\exp(-\alpha)(a)*\exp(-\alpha)(b))=\exp(\operatorname{ad}\alpha)(\mu+\gamma)(a\otimes b). \tag{6.4}$$

From (6.3) we obtain

$$(\mu_p+\gamma_p')(a\otimes b)=\exp(\alpha)(\exp(-\alpha)(a)*\exp(-\alpha)(b))=\exp(\operatorname{ad}\alpha)(\mu_p+\gamma_p)(a\otimes b). \tag{6.5}$$

Elements $(\gamma,\gamma_p)\in \mathrm{MC}_{\mathfrak{g}}(B)$ and $(\gamma',\gamma_p')\in \mathrm{MC}_{\mathfrak{g}}(B)$ are gauge equivalent if

$$(\gamma',\gamma_p')=(\gamma,\gamma_p)+\sum_{n=0}^{\infty}\frac{[\alpha,\cdot]^n}{(n+1)!}([\alpha,(\gamma,\gamma_p)]_p-\tilde{d}(\alpha)) \tag{6.6}$$

holds.

Since $\tilde{d}(\alpha)=[(\gamma,\gamma_p),\alpha]_p=-[\alpha,(\gamma,\gamma_p)]_p$, we see that (6.6) holds if and only if the equations (6.4) and (6.5) hold. □

6.2 The cup product of the Hochschild dgla and the Poisson cohomology in the toric setting

Definition 68. *The cup product of the Hochschild dgla* is the map

$$[\ ,\ ]: H^2(A)\times H^2(A)\to H^3(A).$$

In the next lemma we recall some computations from Chapter 4.

Lemma 6.2.1. *For an element $p\in H^2_{(2)}(A)$ and an element $q\in H^2_{(1)}(A)$ we have the following:*

- $e_3[p,p]=0$ *is the Jacobi identity,*
- $[p,q]=e_2[p,q]$ *and* $[q,q]=e_1[q,q]$.

Proof. The equation $e_3[p,p]=0$ is the Jacobi identity by the proof of Proposition 4.2.1. Equations $[p,q]=e_2[p,q]$ and $[q,q]=e_1[q,q]$ hold by Lemma 4.1.3. □

Using the Hodge decomposition, the isomorphism $T^1(A)\cong H^2_{(1)}(A)$ (from Theorem 2.3.10) and Lemma 6.2.1, we see that the cup product of the Hochschild dgla consists of the products $T^1(A)\times T^1(A)\to T^2(A)$, $T^1(A)\times H^2_{(2)}(A)\to H^3_{(2)}(A)$ and $H^2_{(2)}(A)\times H^2_{(2)}(A)\to H^3(A)$.

In Chapter 4 we showed that every Poisson structure $p\in H^2_{(2)}(A)$ on an affine toric variety $X_\sigma=\operatorname{Spec}(A)$ can be quantized, which implies that $[p,p]=0\in H^3(A)$. In Chapter 5 we analyzed the cup product $T^1(A)\times T^1(A)\to T^2(A)$. In this section we will analyze the product $[\ ,\ ]: T^1(A)\times H^2_{(2)}(A)\to H^3_{(2)}(A)$ in the toric setting.

From Section 5.1 we recall the following: Let $\sigma=\langle a_1,...,a_N\rangle$ and $R,S\in M$. Let μ be an element from $H^1_{(1)}(\Lambda\setminus\Lambda(S);k)$. We extend (not additively) μ to the whole of Λ by 0 (i.e. $\mu(\lambda)=0$ for $\mu\in\Lambda(S)$). This extended function we denote by μ^0. We have

$$T^{1,-S}(A)\cong H^2_{(1)}(\Lambda,\Lambda\setminus(S+\Lambda);k)$$

by Proposition 3.2.2 and the surjective map

$$H^1_{(1)}(\Lambda\setminus(S+\Lambda);k)\xrightarrow{d} H^2_{(1)}(\Lambda,\Lambda\setminus(S+\Lambda);k)$$

by Corollary 5.1.1. Thus we see that every element of $T^1(-S) \cong H^2_{(1)}(-S)$ can be written as $d\mu^0$ for some $\mu \in H^1_{(1)}(\Lambda \setminus \Lambda(S); k)$.

The following proposition simplifies the product $[\ ,\ ] : T^1(A) \times H^2_{(2)}(A) \to H^3_{(2)}(A)$ in the toric setting.

Proposition 6.2.2. *Let $\mu \in H^1_{(1)}(\Lambda \setminus \Lambda(S); k)$ and $\xi \in H^2_{(2)}(\Lambda, \Lambda \setminus \Lambda(R); k)$. Let*

$$G(\lambda_1, \lambda_2) := G_1(\lambda_1, \lambda_2) - G_2(\lambda_1, \lambda_2) \in C^2_{(2)}(\Lambda; k),$$

where

$$G_1(\lambda_1, \lambda_2) := \xi(-S + \lambda_1 + \lambda_2, \lambda_2)\mu^0(\lambda_1) + \xi(\lambda_1, -S + \lambda_1 + \lambda_2)\mu^0(\lambda_2),$$
$$G_2(\lambda_1, \lambda_2) := \xi(\lambda_1, \lambda_2)\mu^0(\lambda_1 + \lambda_2 - R).$$

Let $\lambda_{123} := \lambda_1 + \lambda_2 + \lambda_3$.

1. *If $\lambda_1 + \lambda_2 \geq S$, $\lambda_2 + \lambda_3 \geq S$ we have $dG(\lambda_1, \lambda_2, \lambda_3) = [\xi, d\mu^0](\lambda_1, \lambda_2, \lambda_3)$.*
2. *If $\lambda_1 + \lambda_2 \not\geq S$, $\lambda_2 + \lambda_3 \geq S$ we have*
$$\big(dG - [\xi, d\mu^0]\big)(\lambda_1, \lambda_2, \lambda_3) =$$
$$\mu^0(\lambda_1)\big(\xi(-S + \lambda_{123}, \lambda_2) + \xi(\lambda_2, \lambda_3)\big) + \mu^0(\lambda_2)\big(\xi(\lambda_1, -S + \lambda_{123}) - \xi(\lambda_1, \lambda_3)\big).$$
3. *If $\lambda_1 + \lambda_2 \geq S$, $\lambda_2 + \lambda_3 \not\geq S$ we have*
$$\big(dG - [\xi, d\mu^0]\big)(\lambda_1, \lambda_2, \lambda_3) =$$
$$\mu^0(\lambda_2)\big(\xi(\lambda_1, \lambda_3) - \xi(-S + \lambda_{123}, \lambda_3)\big) + \mu^0(\lambda_3)\big(\xi(-S + \lambda_{123}, \lambda_2) - \xi(\lambda_1, \lambda_2)\big).$$
4. *If $\lambda_1 + \lambda_2 \not\geq S$, $\lambda_2 + \lambda_3 \not\geq S$ we have*
$$\big(dG - [\xi, d\mu^0]\big)(\lambda_1, \lambda_2, \lambda_3) = \mu^0(\lambda_1)\big(\xi(-S + \lambda_{123}, \lambda_2) + \xi(\lambda_2, \lambda_3)\big) +$$
$$+ \mu^0(\lambda_2)\big(\xi(\lambda_1, -S + \lambda_{123}) - \xi(-S + \lambda_{123}, \lambda_3)\big) +$$
$$+ \mu^0(\lambda_3)\big(\xi(-S + \lambda_{123}, \lambda_2) - \xi(\lambda_1, \lambda_2)\big).$$

Proof. We first compute

$$
\begin{aligned}
&[\xi, d\mu^0](\lambda_1, \lambda_2, \lambda_3) = \\
&= \xi(-S + \lambda_1 + \lambda_2, \lambda_3)\big(\mu^0(\lambda_1) + \mu^0(\lambda_2) - \mu^0(\lambda_1 + \lambda_2)\big) \\
&- \xi(\lambda_1, -S + \lambda_2 + \lambda_3)\big(\mu^0(\lambda_2) + \mu^0(\lambda_3) - \mu^0(\lambda_2 + \lambda_3)\big) \\
&+ d\mu^0(-R + \lambda_1 + \lambda_2, \lambda_3)\xi(\lambda_1, \lambda_2) - d\mu^0(\lambda_1, -R + \lambda_2 + \lambda_3)\xi(\lambda_2, \lambda_3) = \\
&= \mu^0(\lambda_1)\big(\xi(-S + \lambda_1 + \lambda_2, \lambda_3) - \xi(\lambda_2, \lambda_3)\big) \\
&+ \mu^0(\lambda_2)\big(\xi(-S + \lambda_1 + \lambda_2, \lambda_3) - \xi(\lambda_1, -S + \lambda_2 + \lambda_3)\big) \\
&+ \mu^0(\lambda_3)\big(-\xi(\lambda_1, -S + \lambda_2 + \lambda_3) + \xi(\lambda_1, \lambda_2)\big) - \mu^0(\lambda_1 + \lambda_2)\xi(-S + \lambda_1 + \lambda_2, \lambda_3) \\
&+ \mu^0(\lambda_2 + \lambda_3)\xi(\lambda_1, -S + \lambda_2 + \lambda_3) - dG_2(\lambda_1, \lambda_2, \lambda_3),
\end{aligned}
$$

where we use the fact that ξ is bi-additive.

We now compute

$$\begin{aligned}
&dG_1(\lambda_1,\lambda_2,\lambda_3) = \\
&= \xi(-S+\lambda_2+\lambda_3,\lambda_3)\mu^0(\lambda_2) + \xi(\lambda_2,-S+\lambda_2+\lambda_3)\mu^0(\lambda_3) \\
&- \xi(-S+\lambda_1+\lambda_2+\lambda_3,\lambda_3)\mu^0(\lambda_1+\lambda_2) - \xi(\lambda_1+\lambda_2,-S+\lambda_1+\lambda_2+\lambda_3)\mu^0(\lambda_3) \\
&+ \xi(-S+\lambda_1+\lambda_2+\lambda_3,\lambda_2+\lambda_3)\mu^0(\lambda_1) + \xi(\lambda_1,-S+\lambda_1+\lambda_2+\lambda_3)\mu^0(\lambda_2+\lambda_3) \\
&- \xi(-S+\lambda_1+\lambda_2,\lambda_2)\mu^0(\lambda_1) - \xi(\lambda_1,-S+\lambda_1+\lambda_2)\mu^0(\lambda_2).
\end{aligned}$$

We need to consider the following cases:

1. $\lambda_1+\lambda_2 \geq S$, $\lambda_2+\lambda_3 \geq S$

   We have $\mu^0(\lambda_1+\lambda_2) = \mu^0(\lambda_2+\lambda_3) = 0$. Thus we compute

$$\begin{aligned}
&dG_1(\lambda_1,\lambda_2,\lambda_3) = \\
&= \mu^0(\lambda_1)\big(\xi(-S+\lambda_1+\lambda_2,\lambda_3) + \xi(\lambda_3,\lambda_2)\big) + \\
&+ \mu^0(\lambda_2)\big(\xi(-S+\lambda_2+\lambda_3,\lambda_3) - \xi(\lambda_1,-S+\lambda_1+\lambda_2)\big) + \\
&+ \mu^0(\lambda_3)\big(-\xi(\lambda_1,-S+\lambda_2+\lambda_3) - \xi(\lambda_2,\lambda_1)\big).
\end{aligned}$$

   It holds that

$$\begin{aligned}
&\xi(-S+\lambda_2+\lambda_3,\lambda_3) - \xi(\lambda_1,-S+\lambda_1+\lambda_2) = \\
&= \xi(-S+\lambda_1+\lambda_2+\lambda_3,\lambda_3) - \xi(\lambda_1,\lambda_3) \\
&- \xi(\lambda_1,-S+\lambda_1+\lambda_2+\lambda_3) + \xi(\lambda_1,\lambda_3) = \\
&= \xi(-S+\lambda_1+\lambda_2,\lambda_3) - \xi(\lambda_1,-S+\lambda_2+\lambda_3)
\end{aligned}$$

   and thus we see that in this case $dG(\lambda_1,\lambda_2,\lambda_3) = [\xi, d\mu^0](\lambda_1,\lambda_2,\lambda_3)$ holds.

2. $\lambda_1+\lambda_2 \not\geq S$, $\lambda_2+\lambda_3 \geq S$:

   We have $\mu^0(\lambda_2+\lambda_3) = 0$ and $\mu^0(\lambda_1+\lambda_2) = \mu^0(\lambda_1)+\mu^0(\lambda_2)$. It holds that

$$\begin{aligned}
&dG_1(\lambda_1,\lambda_2,\lambda_3) = \\
&= \mu^0(\lambda_1)\xi(-S+\lambda_1+\lambda_2+\lambda_3,\lambda_2) + \mu^0(\lambda_2)\big(\xi(-S+\lambda_2+\lambda_3,\lambda_3) - \xi(-S+\lambda_1+\lambda_2+\lambda_3,\lambda_3)\big) + \\
&+ \mu^0(\lambda_3)\big(\xi(\lambda_2,-S+\lambda_2+\lambda_3) - \xi(\lambda_1+\lambda_2,-S+\lambda_1+\lambda_2+\lambda_3)\big),
\end{aligned}$$

$$\begin{aligned}
&[\xi, d\mu](\lambda_1,\lambda_2,\lambda_3) = \\
&= \mu^0(\lambda_1)(-\xi(\lambda_2,\lambda_3)) + \mu^0(\lambda_2)(-\xi(\lambda_1,-S+\lambda_2+\lambda_3)) + \\
&+ \mu^0(\lambda_3)\big(-\xi(\lambda_1,-S+\lambda_2+\lambda_3) + \xi(\lambda_1,\lambda_2)\big).
\end{aligned}$$

   If we compute $\big(dG_1 - [\xi, d\mu^0]\big)(\lambda_1,\lambda_2,\lambda_3)$ we see that the term before $\mu^0(\lambda_3)$ vanishes because

$$\begin{aligned}
&\xi(\lambda_2,-S+\lambda_2+\lambda_3) - \xi(\lambda_1+\lambda_2,-S+\lambda_1+\lambda_2+\lambda_3) = \\
&= \xi(\lambda_2,-S+\lambda_1+\lambda_2+\lambda_3) - \xi(\lambda_2,\lambda_1) - \xi(\lambda_1+\lambda_2,-S+\lambda_1+\lambda_2+\lambda_3) = \\
&= -\xi(\lambda_1,-S+\lambda_2+\lambda_3) + \xi(\lambda_1,\lambda_2).
\end{aligned}$$

3. $\lambda_1 + \lambda_2 \geq S$, $\lambda_2 + \lambda_3 \ngeq S$:

$$dG_1(\lambda_1, \lambda_2, \lambda_3) =$$
$$= \mu^0(\lambda_1)\big(\xi(-S + \lambda_1 + \lambda_2 + \lambda_3, \lambda_2 + \lambda_3) - \xi(-S + \lambda_1 + \lambda_2, \lambda_2)\big)+$$
$$+ \mu^0(\lambda_2)\big(\xi(\lambda_1, -S + \lambda_1 + \lambda_2 + \lambda_3) - \xi(\lambda_1, -S + \lambda_1 + \lambda_2)\big)+$$
$$+ \mu^0(\lambda_3)\big(- \xi(\lambda_1 + \lambda_2, -S + \lambda_1 + \lambda_2 + \lambda_3) + \xi(\lambda_1, -S + \lambda_1 + \lambda_2 + \lambda_3)\big),$$

$$[\xi, d\mu^0](\lambda_1, \lambda_2, \lambda_3) =$$
$$= \mu^0(\lambda_1)\big(\xi(-S + \lambda_1 + \lambda_2, \lambda_3) - \xi(\lambda_2, \lambda_3)\big) + \mu^0(\lambda_2)\xi(-S + \lambda_1 + \lambda_2, \lambda_3) + \mu^0(\lambda_3)\xi(\lambda_1, \lambda_2).$$

As before we see that in $\big(dG_1 - [\xi, d\mu]\big)(\lambda_1, \lambda_2, \lambda_3)$ the term before $\mu(\lambda_1)$ vanishes.

4. $\lambda_1 + \lambda_2 \ngeq S$, $\lambda_2 + \lambda_3 \ngeq S$

In this case we have

$$\big(dG - [\xi, d\mu]\big)(\lambda_1, \lambda_2, \lambda_3) = dG_1(\lambda_1, \lambda_2, \lambda_3) =$$
$$= \mu^0(\lambda_1)\xi(-S + \lambda_1 + \lambda_2 + \lambda_3, \lambda_2)+$$
$$+ \mu^0(\lambda_2)\big(\xi(\lambda_1, -S + \lambda_1 + \lambda_2 + \lambda_3) - \xi(-S + \lambda_1 + \lambda_2 + \lambda_3, \lambda_3)\big)+$$
$$+ \mu^0(\lambda_3)\xi(-S + \lambda_1 + \lambda_2 + \lambda_3, \lambda_2).$$

□

Corollary 6.2.3. *Every element of $T^1(-S) \cong H^2_{(1)}(-S)$ can be written as $d\mu^0$ for some*

$$\mu \in H^1_{(1)}(\Lambda \setminus \Lambda(S); k).$$

Let $\xi \in H^2_{(2)}(-R) \cong H^2_{(2)}(\Lambda, \Lambda \setminus \Lambda(R); k)$. The product $[d\mu^0, \xi] \in H^3_{(2)}(-R - S)$ is equal to the cohomological class of the element

$$(\delta(G), d(G) - [d\mu^0, \xi]) \in C^2_{(2)}(K_1^{R+S}; k) \oplus C^3_{(2)}(\Lambda; k)$$

in the total complex of the complex $C^\bullet_{(2)}(K^R_\bullet; k)$ (see Section 3.3, where also the map δ is defined). Note that the map $d(G) - [d\mu^0, \xi]$ has many zeros by Proposition 6.2.2 and thus we can easily compute it (see Example 13).

After applying the differentials d on the double complex in Figure 6.1, we obtain for $j, k \geq 1$:

$$E_1^{j,k} = H^{j+k-1}_{(j)}(A) \Rightarrow H^{j+k-1}(C^\bullet_p(A)[1]),$$

where $d_1 = -[\mu_p, \cdot] : E_1^{j,k} \to E_1^{j+1,k}$. We have $E_1^{1,2} = H^2_{(1)}(A) \cong T^2(A)$ and $E_1^{2,2} = H^3_{(2)}(A)$. The map $d_1 : E_1^{1,2} \to E_1^{2,2}$ is the special case of the product analyzed in Proposition 6.2.2.

In the following example we collect some results from previous chapters in order to compute the Poisson cohomology groups of the Poisson structure defined in Example 9.

Example 13. Let $X_{\sigma_n} = \mathrm{Spec}(A_n)$ be the Gorenstein toric surface given by $g(x, y, z) = xy - z^{n+1}$. In Section 3.5 we saw that $\Lambda_n := \sigma_n^\vee \cap M$ is generated by $S_1 := (0, 1)$, $S_2 := (1, 1)$ and $S_3 := (n + 1, n)$, with the relation $S_1 + S_3 = (n + 1)S_2$. We have

$$\dim_k H^2_{(1)}(-R) = \dim_k H^3_{(2)}(-R) = \begin{cases} 1 & \text{if } R = kS_2 \text{ for } 2 \leq k \leq n+1 \\ 0 & \text{otherwise} \end{cases}$$

by Corollary 3.5.3 and Example 8. Moreover, $T^2(A_n) \cong H^3_{(1)}(A_n) = E_1^{1,3} = 0$ by Example 8. From the proof of Theorem 2.5.9 it follows that for $i \geq 3$ we have $T^k_{(i)}(A_n) = 0$ if $k \neq i - 1, i$ and

$$T^{i-1}_{(i)}(A_n) \cong T^i_{(i)}(A_n) \cong A_n/(\frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial g}{\partial x_3}).$$

The later has k-dimension equal to n. Since $T^{i-1}_{(i)}(A_n) \cong H^{2i-1}_{(1)}(A_n)$ and $T^i_{(i)}(A_n) \cong H^{2i}_{(i)}(A_n)$ we see that $E_2^{j,k} = E_\infty^{j,k}$ holds for every $j, k \geq 1$.

$$0 \xrightarrow{d_1} 0 \xrightarrow{d_1} 0 \xrightarrow{d_1} H^8_{(4)}(A_n) \xrightarrow{d_1} H^9_{(5)}(A_n)$$

$$0 \xrightarrow{d_1} 0 \xrightarrow{d_1} H^6_{(3)}(A_n) \xrightarrow{d_1} H^7_{(4)}(A_n) \xrightarrow{d_1} 0$$

$$0 \xrightarrow{d_1} H^4_{(2)}(A_n) \xrightarrow{d_1} H^5_{(3)}(A_n) \xrightarrow{d_1} 0 \xrightarrow{d_1} 0$$

$$H^2_{(1)}(A_n) \xrightarrow{d_1} H^3_{(2)}(A_n) \xrightarrow{d_1} 0 \xrightarrow{d_1} 0 \xrightarrow{d_1} 0$$

$$H^1_{(1)}(A_n) \xrightarrow{d_1} H^2_{(2)}(A_n) \xrightarrow{d_1} 0 \xrightarrow{d_1} 0 \xrightarrow{d_1} 0$$

Figure 6.2: The spectral sequence terms $E_1^{j,k}$ for $1 \leq j, k \leq 5$

We focus now on the Poisson structure π_g from Example 9. We proved that

$$\pi_g(x^{\lambda_1}, x^{\lambda_2}) = f_0(\lambda_1, \lambda_2)x^{-S_2+\lambda_1+\lambda_2},$$

where f_0 is skew-symmetric and bi-additive with $f_0(S_1, S_3) = -(n+1)$. Thus we see that $\pi_g \in H^{2,-S_2}_{(2)}(A_n)$. Let $\mathfrak{g}_n := C^{\bullet}_p(A_n)[1]$. From a straightforward computation we see that $d_1 : H^1_{(1)}(A_n) \to H^2_{(2)}(A_n)$ is surjective (we can also check this using [33, Lemma 3.1]).

Let $\{\bar{\mu}_k \in T^{1,-kS_2}(A_n) \mid 2 \leq k \leq n+1\}$ be a basis of $T^1(A_n) \cong H^2_{(1)}(A_n)$, such that $\bar{\mu}_k$ is represented by $\mu_k \in C^1_{(1)}(\Lambda \setminus \Lambda(kS_2); k)$ with

$$\mu_k(\lambda) = \begin{cases} a & \text{if } \lambda = aS_3, \text{ for } a \in \mathbb{N} \\ 0 & \text{otherwise .} \end{cases}$$

From Proposition 6.2.2 we can immediately see that $dG = [\pi_g, d\mu^0_k]$ holds (in all cases), $G(\lambda_1, \lambda_2) = 0$ for $\lambda_1 + \lambda_2 \not\geq R + S$ and thus $\delta(G) = 0$. We conclude that $[\pi_g, \bar{\mu}_k] = 0 \in H^3_{(2)}(A_n)$ for all $2 \leq k \leq n+1$ and thus $d_1 : H^2_{(1)}(A_n) \to H^3_{(2)}(A_n)$ is the zero map. Thus from the spectral sequence arguments we are able to compute the most important cohomology groups from deformation theory point of view: $H^1(\mathfrak{g}_n)$ and $H^2(\mathfrak{g}_n)$. We see that

$$H^1(\mathfrak{g}_n) \cong H^2_{(1)}(A_n) \cong T^1(A_n)$$

and

$$H^2(\mathfrak{g}_n) \cong H^3_{(2)}(A_n).$$

Thus $\dim_k H^1(\mathfrak{g}_n) = \dim_k T^1(A_n) = n$ (this was already proven with different methods in [33, Lemma 3.1]) and also $\dim_k H^2(\mathfrak{g}_n) = \dim_k H^3_{(2)}(A_n) = n$.

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Zusammenfassung

In dieser Arbeit untersuchen wir die Hochschild Kohomologiegruppen affiner torischer Varietäten und ihre Anwendung in der Deformationsquantisierung und kommutativen Deformationstheorie. Wir können die n-te Hochschild Kohomologiegruppe in die direkte Summe $T^0_{(n)}(A) \oplus T^1_{(n-1)}(A) \oplus \cdots \oplus T^{n-1}_{(1)}(A)$ zerlegen, wobei $T^k_{(i)}(A)$ die höheren André-Quillen Kohomologiegruppen sind.

Unter bestimmten Annahmen berechnen wir die Dimensionen der Hodge-Summanden $T^1_{(i)}(A)$, was existierende Resultate über André-Quillen Kohomologiegruppen $T^1_{(1)}(A)$ von Sletsjøe und Altmann aus [6] verallgemeinert. Insbesondere berechnen wir $T^1_{(i)}(A)$ für alle $i \in \mathbb{N}$ im Falle von zwei- und dreidimensionalen affinen torischen Varietäten. In höheren Dimensionen berechnen wir $T^1_{(i)}(A)$ für affine Kegel über glatten torischen Fano-Varietäten. Das Verständnis der Hochschild Kohomologie ist wichtig für die Deformationsquantisierung. Ein Hauptergebnis hinsichtlich der Existenz der Deformationsquantisierung ist Kontsevichs Formalitätssatz [40, Theorem 4.6.2], der impliziert, dass jede Poisson-Struktur auf einer reellen Mannigfaltigkeit quantisiert werden kann, d.h. ein Sternprodukt zulässt.

Kontsevich [39] erweiterte auch den Begriff der Deformationsquantisierung auf den Kontext der algebraischen Geometrie. Für singuläre Varietäten gilt Kontsevichs Formalitätstheorem nicht mehr. Wir zeigen jedoch, dass jede Poisson Struktur auf einer möglicherweise singulären affinen torischen Varietät im Sinne von Deformationsquantisierung quantisiert werden kann.

Für kommutative Deformationen torischer Varietäten geben wir eine konvex-geometrische Beschreibung der Harrison Cup-Produktformel $T^1_{(1)}(A) \times T^1_{(1)}(A) \to T^2_{(1)}(A)$. Dies ermöglicht eine Beschreibung der quadratischen Gleichungen des versellen Deformationsraums.

In dieser Arbeit erhalten wir des Weiteren einige allgemeinere Ergebnisse, die auch für Varietäten, die nicht notwendigerweise torisch sind, gelten. Beispielsweise berechnen wir die n-ten Kohomologiegruppen einer reduzierten isolierten Hyperflächensingularität. Außerdem konstruieren wir eine differentielle graduierte Lie Algebra $\mathfrak{g}$, die die Possion Deformationen einer allgemeinen affinen Varietät kontrolliert.

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