Chapter 5

Closed Solution of Several Specific Cases

5.1 Doubly-Periodic Homogeneous Cylindrical Inlay CPS Problem

In this section, as a practical application we deal with the doubly-periodic cylindrical inlay CPS problem. After referring to the transformation introduced in Chapter 1, the boundary value problems are transfered into doubly quasi-periodic boundary value problems, then, by employing the solutions of the doubly quasi-periodic boundary value problem we obtain the general solution in closed form. For an illustrating example of practical interest, e.g. the doubly-periodic circular cylindrical inlay CPS problem, the exact solution is obtained. And when we fix one of its periods, say $\omega_1 = a\pi$ and let $|\omega_2| \to \infty$ in such a way that $Im(\frac{\omega_2}{\omega_1} > 0)$, as a by-product we get the exact solution of the singly-periodic case with primitive period $a\pi$, furthermore, when we let $|\omega_1| \to \infty$ and $|\omega_2| \to \infty$ in the doubly-periodic case, or $a \to \infty$ in the singly-periodic case, we have immediately the solution of the non-periodic case, which is identical with the classical one.

Let there be given an elastic body with cylindrical holes, which distrbuted doubly-periodically on the x_1, x_2 plane, and let solid cylindrical inlays of the same material be inserted into these holes. It will be supposed that the interfaces of the inserted inlays and of the corresponding holes are brought into contact without any gaps. Then, there are doubly-periodic gaskets on the x_1, x_2 transverse cross section (see Figure 5.1). The primitive periods will be $2\omega_1$, $2\omega_2$ such that $Im\left(\frac{\omega_2}{\omega_1}\right) > 0$, $Im(\omega_1) = 0$. The doubly periodic fundamental parallelogram on the x_1, x_2 plane of the elastic body will be denoted by P_{00}. Its vertices are 0, $2\omega_1$, $2\omega_1 + 2\omega_2$ and $2\omega_2$. The boundary of P_{00} will be denoted by $\mathit{\Gamma} = \bigcup\limits_{i=1}^{4} \mathit{\Gamma}_i$ with the positive direction taken to be anticlockwise, $L = \bigcup\limits_{j=0}^{m-1} L_j$ being the interface of the elastic gaskets in P_{00}, the positive direction of L is taken to be clockwise. In the doubly periodic fundamental parallelogram P_{00}, the elastic regions located on the left-hand and right-hand sides of L are denoted by S_0^+ and S_0^-, respectively (see Figure 5.1), the aggregate of $S_0^\pm$ and all its congruent regions will be denoted by $S^\pm$, respectively. It will be assumed that S^+ and S^- are made up of the same material. The displacement discontinuity function on L

$$g(t) = [u^+(t) + iv^+(t)] - [u^-(t) + iv^-(t)]$$

and $e_3 = constant$ are given, and assume $g'(t) \in H(L)$. In addition, the stress resultant principal vectors F_k on $\mathit{\Gamma}_k$, k=1, 2, on the x_1, x_2 plane, and the shear stress $T_k, k = 1, 2$, in the x_3 direction are given, too.

In this case, proceeding in an analogous manner, one finds, in the former notation, the boundary conditions

$$\phi^+(t) + t\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\overline{\phi'^-(t)} + \overline{\psi^-(t)}, t \in \mathcal{L}(m,n), \qquad (5.1)$$

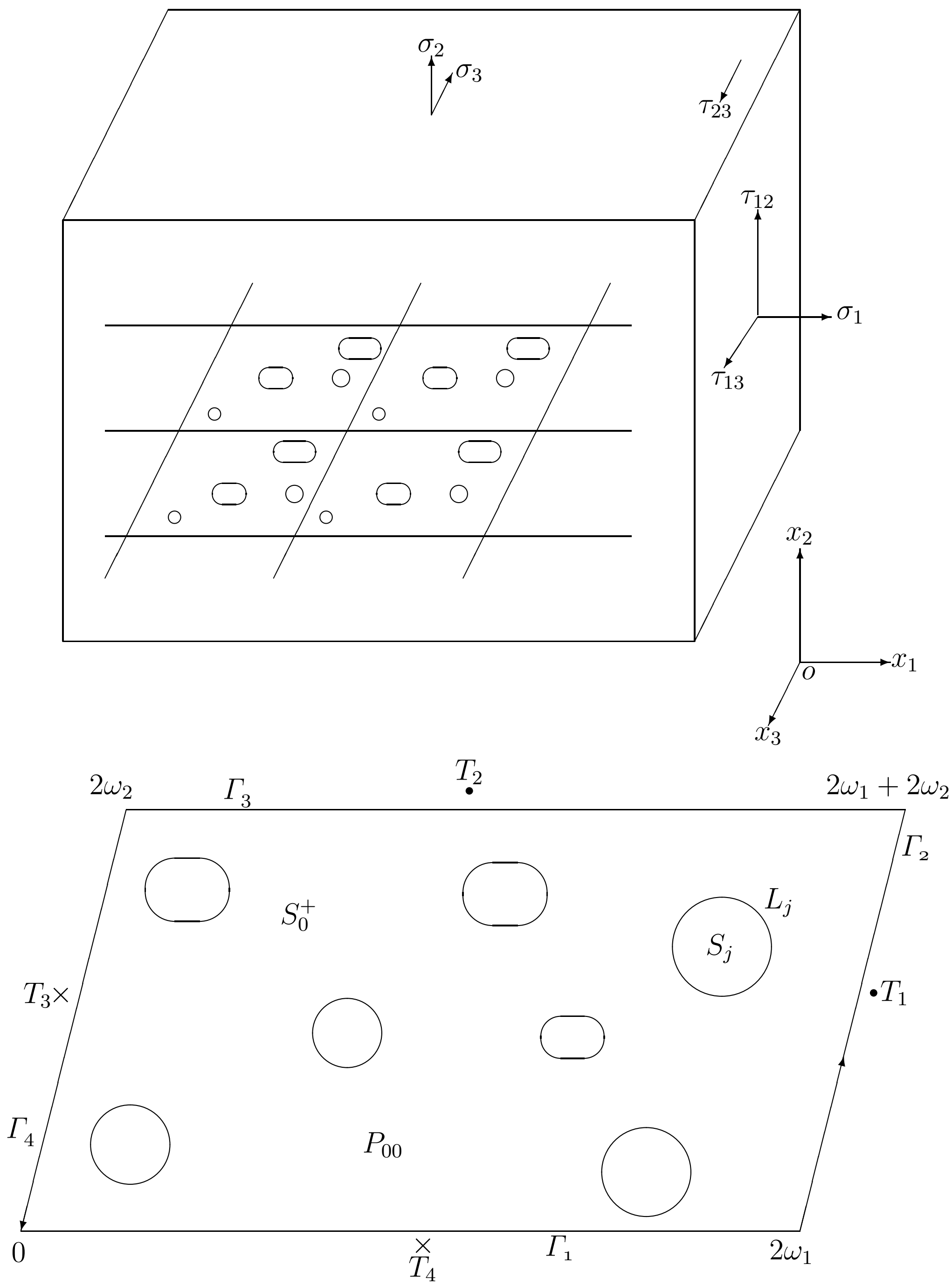

Figure 5.1: Model of the doubly-periodic cylindrical inlay CPS problem

$$\kappa\phi^+(t)-t\overline{\phi'^+(t)}-\overline{\psi^+(t)} = \kappa\phi^-(t)-t\overline{\phi'^-(t)}-\overline{\psi^-(t)}+2\mu g(t), t \in \mathcal{L}(m,n), \quad (5.2)$$

$$\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_{\Gamma_k} = iF_k, k = 1,2, \quad (5.3)$$

$$F^+(t) + \overline{F^+(t)} = F^-(t) + \overline{F^-(t)}, t \in \mathcal{L}(m,n), \quad (5.4)$$

$$\left[F^+(t) - \overline{F^+(t)}\right] = \left[F^-(t) - \overline{F^-(t)}\right], t \in \mathcal{L}(m,n), \quad (5.5)$$

$$\mu\left[F(z) - \overline{F(z)}\right]_{\Gamma_k} = |\omega_k|T_k, k = 1,2. \quad (5.6)$$

After adding boundary conditions (5.1) and (5.2), and considering the conjugation of condition (5.1), we obtain the Riemann boundary value jump problems

$$\phi^+(t) - \phi^-(t) = \frac{2\mu}{\kappa+1}g(t), t \in \mathcal{L}(m,n), \quad (5.7)$$

$$\begin{aligned}\psi^+(t) - \psi^-(t) &= -\left[\overline{\phi^+(t)} - \overline{\phi^-(t)}\right] - \bar{t}\frac{d}{dt}\left[\phi^+(t) - \phi^-(t)\right] \\ &= \frac{2\mu}{\kappa+1}h(t), t \in \mathcal{L}(m,n), \end{aligned} \quad (5.8)$$

where

$$h(t) = -\overline{g(t)} - \bar{t}g'(t).$$

For the doubly-periodic plane elasticity problem, taking Lemma 1.1.2 into account, for obtaining a doubly-periodic solution, we introduce a transformation

$$\begin{cases} \phi(z) = \phi_0(z), \\ \psi(z) = D(z)\phi_0'(z) + \psi_0(z). \end{cases} \quad (5.9)$$

Then, $\phi_0(z)$ and $\psi_0(z)$ both are doubly quasi-periodic functions. Here, the notation $D(z)$ is given by (1.48).

Substituting the transformation (5.9) into the boundary conditions (5.7) and (5.8) we get the simplest doubly quasi-periodic Riemann boundary value problems

$$\phi_0^+(t) - \phi_0^-(t) = \frac{2\mu}{\kappa+1}g(t), t \in \mathcal{L}(m,n), \quad (5.10)$$

$$\begin{aligned}\psi_0^+(t)-\psi_0^-(t) &= \frac{2\mu}{\kappa+1}h(t)-D(t)\frac{d}{dt}\left[\phi_0^+(t)-\phi_0^-(t)\right] \\ &= \frac{2\mu}{\kappa+1}h_0(t), t\in\mathcal{L}(m,n),\end{aligned} \tag{5.11}$$

where

$$h_0(t)=-\overline{g(t)}-\overline{m(t)}g'(t), \tag{5.12}$$

and $D(z)$ is given by (1.64).

Substituting the transformation (5.9) into boundary conditions (5.3), due to the fact that $m(z)$ and $\phi_0'(z)$ both are doubly-periodic, we have

$$\left[\phi_0(z)+\overline{\psi_0(z)}\right]_{\Gamma_k}=iF_k, k=1,2. \tag{5.13}$$

By Corollary 1.2.1, it is easy to obtain the general solutions of (5.10) and (5.11),

$$\phi_0(z)=\begin{cases}\dfrac{\mu}{(\kappa+1)\pi i}\displaystyle\int_L g(t)\zeta(t-z)dt+C_1z+C+mg_1+ng_2, \\ \text{if } z=[z]_0+\Omega_{mn}\in S^+, \\ \dfrac{\mu}{(\kappa+1)\pi i}\displaystyle\int_L g(t)\zeta(t-z)dt+C_1z+C, \text{ if } z\in S^-,\end{cases} \tag{5.14}$$

$$\psi_0(z)=\begin{cases}\dfrac{\mu}{(\kappa+1)\pi i}\displaystyle\int_L h_0(t)\zeta(t-z)dt+C_2z+C+mg_1+ng_2, \\ \text{if } z=[z]_0+\Omega_{mn}\in S^+, \\ \dfrac{\mu}{(\kappa+1)\pi i}\displaystyle\int_L h_0(t)\zeta(t-z)dt+C_2z+C, \text{ if } z\in S^-,\end{cases} \tag{5.15}$$

where $[z]_0$ indicates the congruent point in P_{00} of z, and C may be an arbitrarily fixed constant.

Substituting (5.14) and (5.15) into the boundary conditions (5.13) we obtain

$$C_1=\frac{1}{iS}\left[\frac{\mu\delta_2}{\kappa+1}\int_L g(t)dt-\frac{\mu}{2(\kappa+1)}\int_L\overline{h_0(t)dt}+i(\overline{\omega_1}F_2-\overline{\omega_2}F_1)\right], \tag{5.16}$$

$$C_2=\frac{1}{S}\left[\frac{i\mu}{2(\kappa+1)}\int_L\overline{g(t)dt}-\frac{i\mu\delta_2}{\kappa+1}\int_L h_0(t)dt+(\overline{\omega_2F_1}-\overline{\omega_1F_2})\right], \tag{5.17}$$

where S is the area of the fundamental periodic-parallelogram P_{00}.

Then, we get $\phi(z)$ and $\psi(z)$ immediately by (5.14), (5.15) and (5.9).

Smilarly, after adding boundary conditions (5.4) and (5.5) we get

$$F^+(t) = F^-(t), t \in \mathcal{L}(m,n). \tag{5.18}$$

Then, we obtain its solution immediately by Corollary 1.2.1 as

$$F(z) = \begin{cases} C_3 z + C + mg_1 + ng_2, \ z = [z]_0 + \Omega_{mn} \in S^+, \\ C_3 z + C, \ z \in S^-. \end{cases} \tag{5.19}$$

Substituting (5.19) into the boundary conditions (5.6) we have

$$C_3 = \frac{\overline{\omega_2}|\omega_1|}{\mu} T_1 - \frac{\overline{\omega_1}|\omega_2|}{\mu} T_2. \tag{5.20}$$

Now we consider a special case, e.g. the doubly-periodic circular cylindrical inlay CPS problem. We assume that there is only one circular cylindrical inlay in the doubly-periodic fundamental parallelogram, namely one gasket in P_{00} on the x_1, x_2 transverse cross section. L_0, the boundary of the gasket, is a circle $|z-b| = r$ (see Figure 5.2), and

$$g(t) = -\epsilon e^{i\theta} = -\frac{\epsilon(t-b)}{r}, \ t = b + re^{i\theta} \in L_0. \tag{5.21}$$

Hence, from (5.12) and (5.21) we have

$$h_0(t) = \frac{2r\epsilon}{t-b} + \left[\overline{b} + D(t)\right] \frac{\epsilon}{r}. \tag{5.22}$$

By (5.14), (5.15) and (5.9) we get the solution of this case after omitting the trival translation constants

$$\phi(z) = \begin{cases} C_1^0 z + mg_1 + ng_2, \ z = [z]_0 + \Omega_{mn} \in S^+, \\ -\frac{2\mu\epsilon}{(\kappa+1)r}(z-b) + C_1^0 z, \ z \in S^-, \end{cases} \tag{5.23}$$

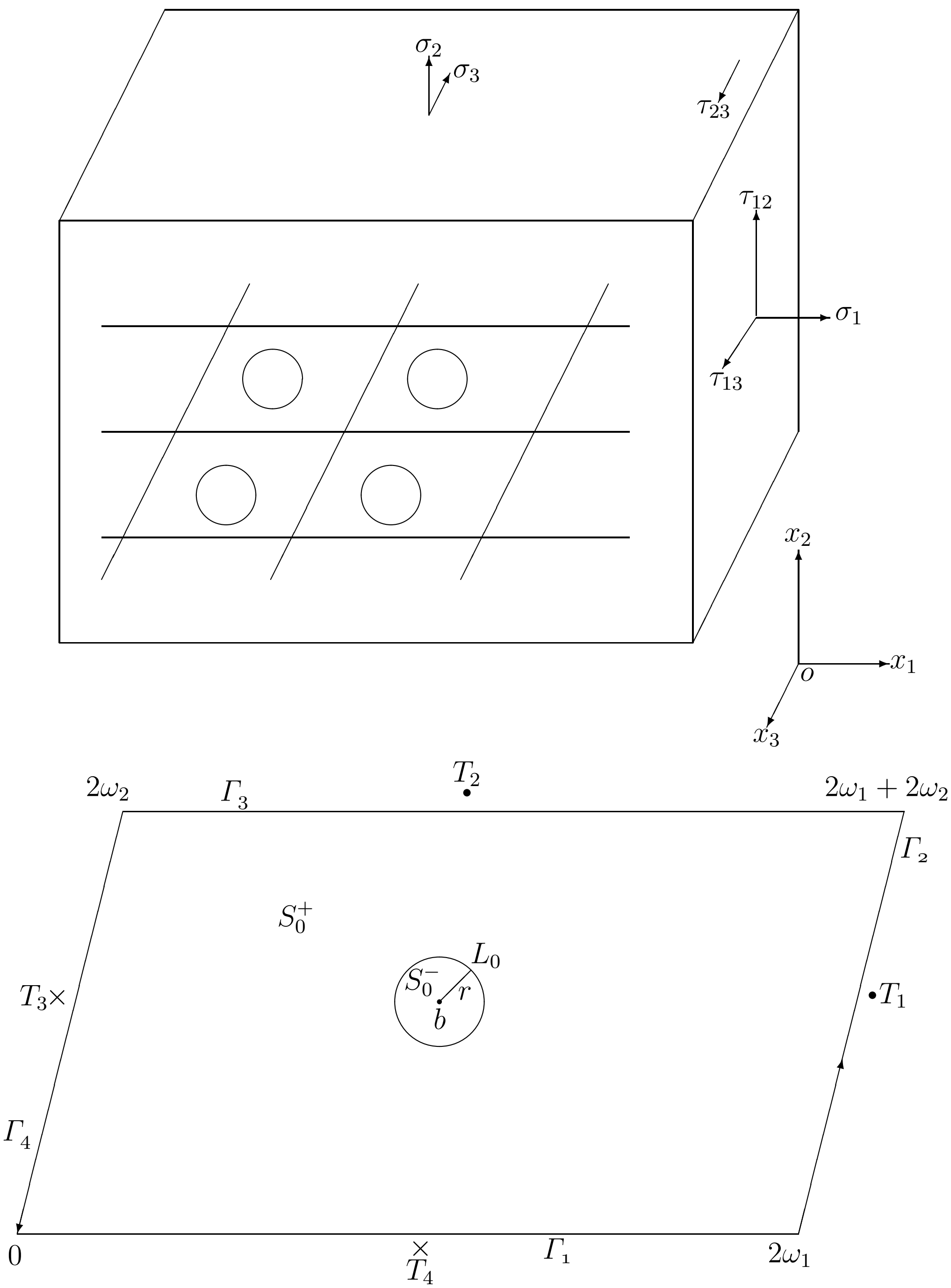

Figure 5.2: Doubly-periodic circular cylindrical inlay CPS problem

$$\psi(z) = \begin{cases} -\dfrac{4\mu\epsilon r}{\kappa+1}\zeta(b-z) + C_2^0 z + mg_1 + ng_2, \ z = [z]_0 + \Omega_{mn} \in S^+, \\ \dfrac{4\mu\epsilon r}{\kappa+1}\left[\dfrac{1}{z-b} + \zeta(b-z)\right] + C_2^0 z, \ z \in S^-, \end{cases} \tag{5.24}$$

where

$$C_1^0 = \frac{\mu}{(\kappa+1)S}\left[2\pi r\epsilon + i(\overline{\omega_1}F_2 - \overline{\omega_2}F_1)\right],$$

$$C_2^0 = \frac{\mu}{(\kappa+1)S}\left[(\overline{\omega_2 F_1} - \overline{\omega_1 F_2}) - 4\pi\delta_2 r\epsilon\right].$$

and $F(z)$ is obtained by (5.19) and (5.20).

For contrasting with the classical results in non-periodic cases of [49], we consider a special case of the doubly-periodic circular cylindrical inlay problem, namely the doubly-periodic circular gasket problem in two dimensional elasticity theory, say, $T_k = 0$, $k = 1, 2$, and $e_3 = 0$, therefore, the boundary conditions (5.4)-(5.6) disappear in this case, due to $\zeta(-z) = -\zeta(z)$, we get the solution in P_{00} immediately,

$$\phi(z) = \begin{cases} C_1^0 z, \ z \in S_0^+, \\ -\dfrac{2\mu\epsilon}{(\kappa+1)r}(z-b) + C_1^0 z, \ z \in S_0^-, \end{cases} \tag{5.25}$$

$$\psi(z) = \begin{cases} -\dfrac{4\mu\epsilon r}{(\kappa+1)}\zeta(z-b) + C_2^0 z, \ z \in S_0^+, \\ \dfrac{4\mu\epsilon r}{\kappa+1}\left[\dfrac{1}{z-b} - \zeta(z-b)\right] + C_2^0 z, \ z \in S_0^-, \end{cases} \tag{5.26}$$

Now, we investigate the limiting case. When $\omega_1 = a\pi$, $\omega_2 \to \infty$, the domain S_0^+ in the doubly-periodic fundamental parallelogramP_{00} will be extended to the domain in periodic strip except S_0^- in the singly-periodic case. When $\omega_1 \to \infty$, $\omega_2 \to \infty$, the domain S_0^+ in the doubly-periodic fundamental parallelogramP_{00} will be extended to the whole plane except S_0^- in the non-periodic case. In the limiting case, when we let $F_k = 0$, $k = 1, 2$, say, the stresses vanish at infinity. Taking the following limiting functions into account, namely [5]

$$\lim_{\substack{\omega_1 = a\pi \\ |\omega_2| \to \infty}} \zeta(z) = \frac{1}{3a^2} z + \frac{1}{a} \cot\left(\frac{z}{a}\right), \tag{5.27}$$

$$\lim_{\substack{|\omega_1| \to \infty \\ |\omega_2| \to \infty}} \zeta(z) = \frac{1}{z}, \tag{5.28}$$

and (1.43) and (1.74) we get

$$\lim_{\substack{\omega_1 = a\pi \\ |\omega_2| \to \infty}} C_1^0 = \lim_{\substack{\omega_1 = a\pi \\ |\omega_2| \to \infty}} C_2^0 = 0, \tag{5.29}$$

$$\lim_{\substack{|\omega_1| \to \infty \\ |\omega_2| \to \infty}} C_1^0 = \lim_{\substack{|\omega_1| \to \infty \\ |\omega_2| \to \infty}} C_2^0 = 0, \tag{5.30}$$

Furthermore, we obtain the solutions

$$\phi(z) = \begin{cases} 0, \ z \in S_0^+, \\ -\dfrac{2\mu\epsilon}{(\kappa+1)r}(z-b), \ z \in S_0^-, \end{cases} \tag{5.31}$$

$$\psi(z) = \begin{cases} -\dfrac{4\mu\epsilon r}{(\kappa+1)a}\left[\dfrac{1}{3a}(z-b) + \cot\left(\dfrac{z-b}{a}\right)\right], \ z \in S_0^+, \\ \dfrac{4\mu\epsilon r}{\kappa+1}\left\{\dfrac{1}{z-b} - \dfrac{1}{a}\left[\dfrac{1}{3a}(z-b) + \cot\left(\dfrac{z-b}{a}\right)\right]\right\}, \ z \in S_0^-, \end{cases} \tag{5.32}$$

for the singly-periodic circular gasket problem, and

$$\phi(z) = \begin{cases} 0, \ z \in S_0^+, \\ -\dfrac{2\mu\epsilon}{(\kappa+1)r}(z-b), \ z \in S_0^-, \end{cases} \tag{5.33}$$

$$\psi(z) = \begin{cases} -\dfrac{4\mu\epsilon r}{\kappa+1}\dfrac{1}{z-b}, \ z \in S_0^+, \\ o, \ z \in S_0^-, \end{cases} \tag{5.34}$$

for the non-periodic circular gasket problem, which is identical with the classical solution (see [49]).

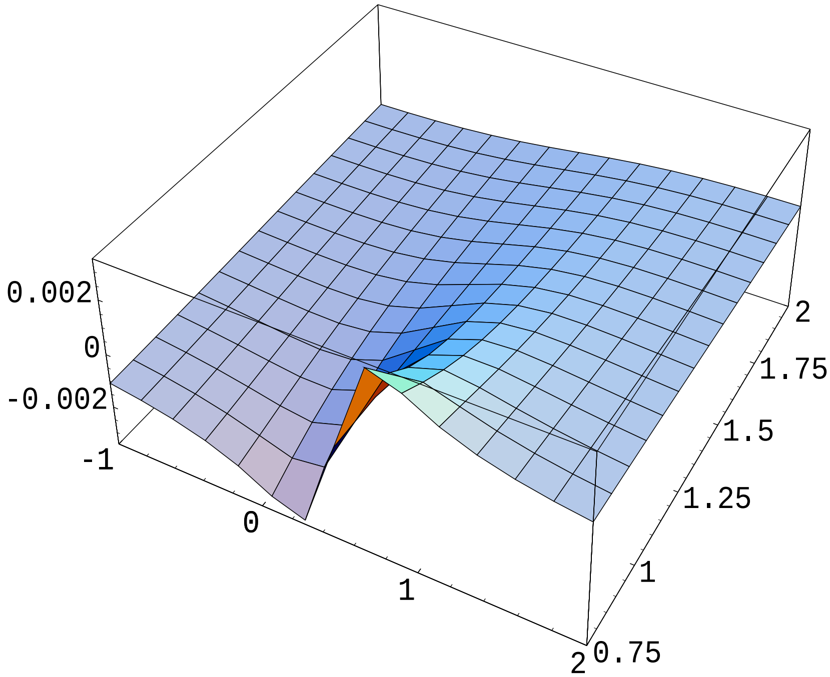

Figure 5.3: Displacement u for $-1 < x < 2$, $0.75 < y < 2$

When $|\omega_1|$, $|\omega_2|$ and a are large enough, or ϵ is small enough, we get the same displacement distribution plots shown in Fig. 5.3,Fig. 5.4, Fig. 5.5 and Fig. 5.6 for the above three cases in some region when $\kappa = 2$, $\mu = 0.5$, $\epsilon = 0.01$, $b = \frac{1}{2} + \frac{1}{2}i$ and $r = \frac{1}{4}$.

5.2 Effect of Homogeneous Cylindrical Inlay on Cracks in the Doubly-Periodic CPS Problem

Effect of homogeneous cylindrical inlay on cracks in the doubly-periodic CPS problem is investigated in this section. By employing the solutions of doubly quasi-periodic and doubly-periodic Riemann boundary value problem we

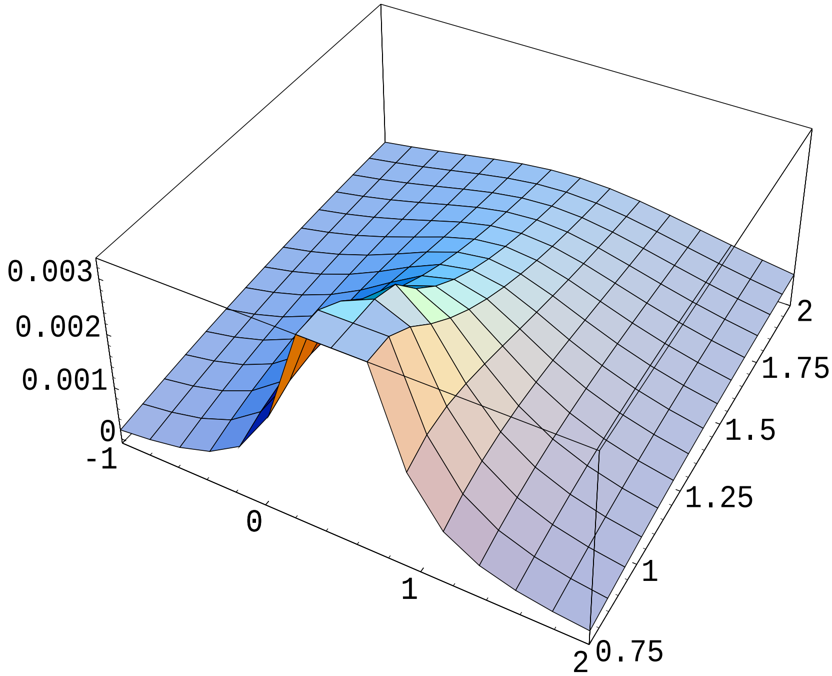

Figure 5.4: Displacement v for $-1 < x < 2$, $0.75 < y < 2$

obtain the general solution in closed form.

Let there be given a three-dimensional elastic body with cylindrical holes and cracks, which distributed doubly-periodically on the x_1, x_2 transverse cross section, and let solid cylindrical of the same material be inserted into these holes. The primitive periods will be $2\omega_1 = a - ib$, $2\omega_2 = a + ib$ with $Im\left(\frac{\omega_2}{\omega_1}\right) > 0$. The doubly-periodic fundamental parallelogram P_{00} on the x_1, x_2 plane of the elastic body has the shape of a rhombus. Its vertices are 0, $2\omega_1$, $2\omega_1 + 2\omega_2$ and $2\omega_2$. The boundary of P_{00} will be denoted by Γ with the positive direction taken to be anticlockwise. Inside P_{00} there are one hole with the centre a (hence, one elastic gasket in P_{00}) and two cracks γ_1 and γ_2, of the same length situated along a diagonal symmetrically with respect to the centre of P_{00} (see Fig. 5.7). We denote $\gamma = \gamma_1 \bigcup \gamma_2$, and let a_1, b_1, a_2, b_2 be the co-ordinates of

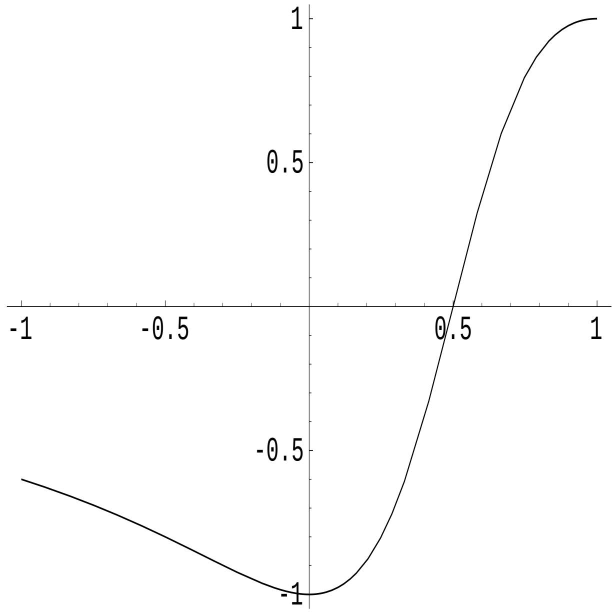

Figure 5.5: Displacement u for $-1 < x < 1$, $y = 0$

the ends of the cracks:

$$a_2 = 2\omega_1 + 2\omega_2 - b_1, \quad b_2 = 2\omega_1 + 2\omega_2 - a_1.$$

The positive direction of L is taken to be clockwise. The elastic regions located on the left-hand and right-hand sides of L are denoted by S_0^+ and S_0^-, respectively. The displacement discontinuity of the two sides of the interface L

$$g(t) = [u^+(t) + iv^+(t)] - [u^-(t) + iv^-(t)] \tag{5.35}$$

is given, strain $e_3 = constant$, and normal load $p(t)$ is given at the edges of the cracks, and the shearing stresses are zero. In addition, the stress resultant principal vectors F_k on $\mathit{\Gamma}_k$, k=1, 2, and the shear stress $T_k, k = 1, 2$, in the x_3 direction are given, too. Then,

$$\sigma_2(x, \pm 0) - i\tau_{12}(x, \pm 0) = -p(x), x \in \gamma, \tag{5.36}$$

$$2\mu(u^+ + iv^+) - (u^- + iv^-) = g(t), t \in L, \tag{5.37}$$

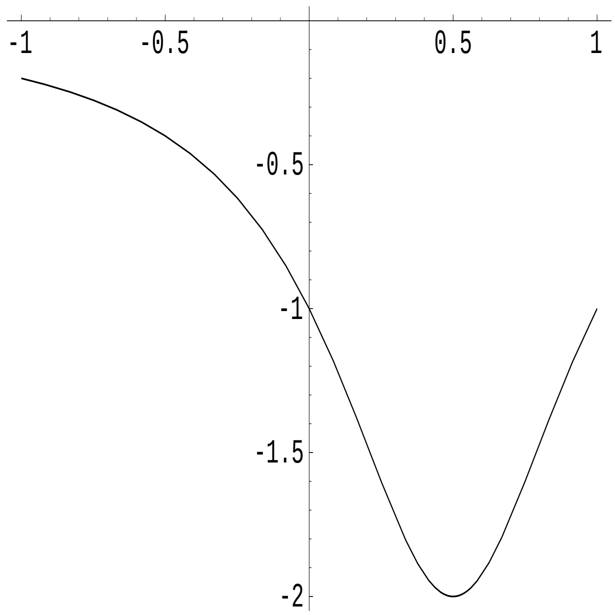

Figure 5.6: Displacement v for $-1 < x < 1$, $y = 0$

$p(t)$ ($\in H$) is a known function on γ, $g(t)$ is a known function on L, and we assume $g'(t) \in H$.

The combinations of stresses $\sigma_1 + i\tau_{12}$ and $\sigma_2 - i\tau_{12}$ expressed in terms of $\Phi(z)$ and $\Psi(z)$ is of the form [49],

$$\begin{cases} \sigma_1 + i\tau_{12} = \Phi(z) + \overline{\Phi(z)} - z\overline{\Phi'(z)} - \overline{\Psi(z)}, \\ \sigma_2 - i\tau_{12} = \Phi(z) + \overline{\Phi(z)} + z\overline{\Phi'(z)} + \overline{\Psi(z)}. \end{cases} \tag{5.38}$$

Under these conditions one has the following boundary conditions at first,

$$\phi^+(t) + t\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\overline{\phi'^-(t)} + \overline{\psi^-(t)}, t \in L \bigcup \gamma, \tag{5.39}$$

$$\kappa\phi^+(t) - t\overline{\phi'^+(t)} - \overline{\psi^+(t)} = \kappa\phi^-(t) - t\overline{\phi'^-(t)} - \overline{\psi^-(t)} + 2\mu G(t), t \in L \bigcup \gamma, \tag{5.40}$$

$$\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_{\Gamma_k} = iF_k, k = 1, 2. \tag{5.41}$$

$$F^+(t) + \overline{F^+(t)} = F^-(t) + \overline{F^-(t)}, t \in L \bigcup \gamma, \tag{5.42}$$

$$\left[F^+(t) - \overline{F^+(t)}\right] = \left[F^-(t) - \overline{F^-(t)}\right], t \in L \bigcup \gamma, \tag{5.43}$$

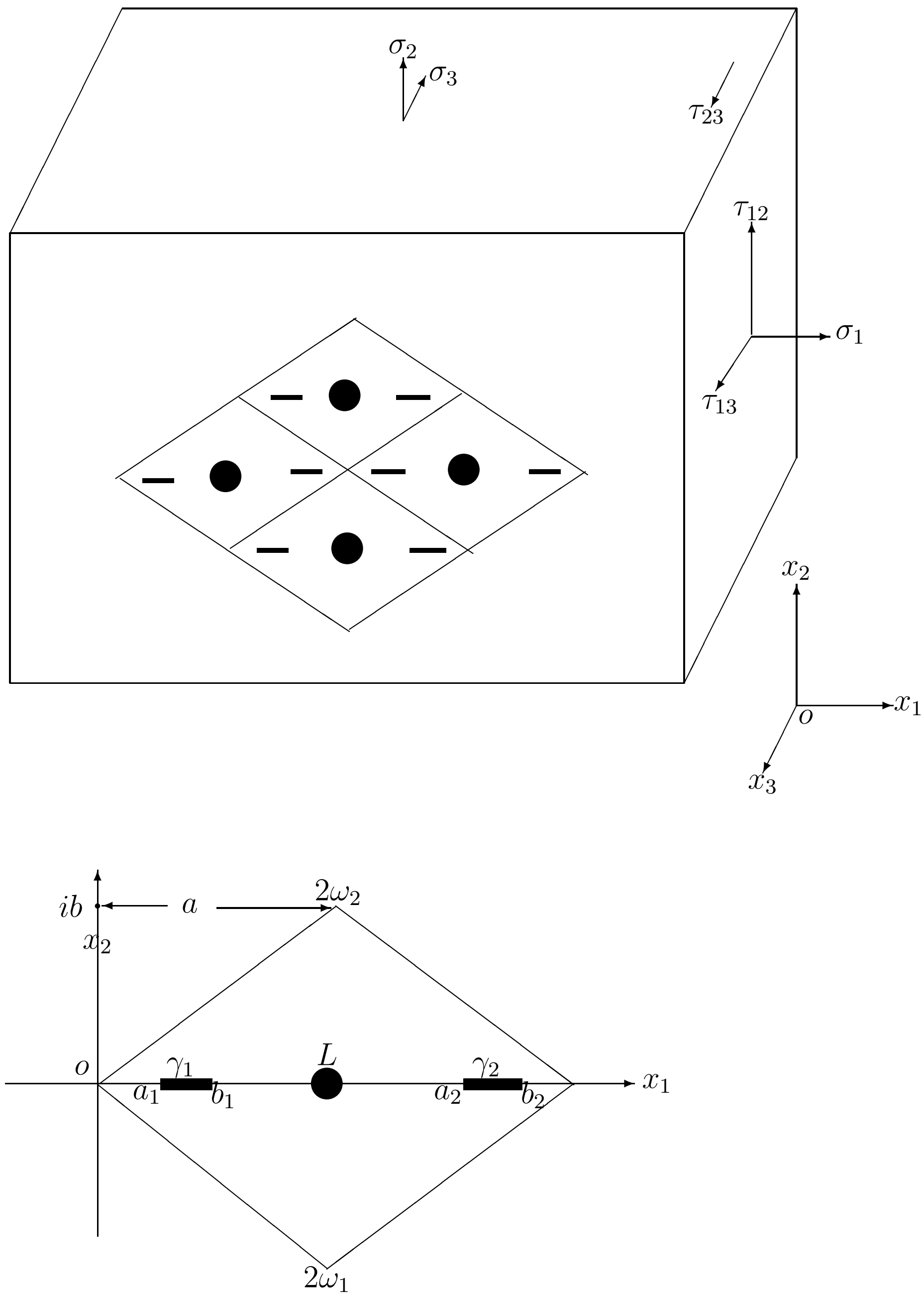

Figure 5.7: Model of the doubly-periodic inlay and cracks CPS problem

$$\mu\left[F(z)-\overline{F(z)}\right]_{\Gamma_k}=|\omega_k|T_k, k=1,2, \tag{5.44}$$

where

$$G(t)=2\mu[u^+(t)+iv^+(t)]-[u^-(t)+iv^-(t)]=\begin{cases} g(t),\ t\in L, \\ \omega(t),\ t\in\gamma. \end{cases} \tag{5.45}$$

Here $\omega(t)$ is an unknown displacement discontinuity function on γ

From (5.39) and (5.40), after simple evaluation, we have the Riemann boundary value problems

$$\phi^+(t)-\phi^-(t)=\frac{2\mu}{\kappa+1}G(t), t\in L\bigcup\gamma, \tag{5.46}$$

$$\psi^+(t)-\psi^-(t)=\frac{2\mu}{\kappa+1}h(t), t\in L\bigcup\gamma, \tag{5.47}$$

where

$$h(t)=\begin{cases} -\overline{g(t)}-\bar{t}g'(t),\ t\in L, \\ -\overline{\omega(t)}-\bar{t}\omega'(t),\ t\in\gamma. \end{cases} \tag{5.48}$$

For obtaining a doubly quasi-periodic boundary value problems, considering the double periodicity of the stress components, we still use the transformation (5.9). Substituting transformation (5.9) into conditions (5.46) and (5.47) we get the simplest doubly quasi-periodic Riemann boundary value problems

$$\phi_0^+(t)-\phi_0^-(t)=\frac{2\mu}{\kappa+1}G(t), t\in L\bigcup\gamma, \tag{5.49}$$

$$\begin{aligned}\psi_0^+(t)-\psi_0^-(t) &= -\left\{\left[\overline{\phi_0^+(t)}-\overline{\phi_0^-(t)}\right]+\overline{m(t)}\frac{d}{dt}\left[\phi_0^+(t)-\phi_0^-(t)\right]\right\} \\ &= \frac{2\mu}{\kappa+1}h_0(t), t\in L\bigcup\gamma,\end{aligned} \tag{5.50}$$

where $\phi_0(z)$ and $\psi_0(z)$ both are doubly quasi-periodic functions, and

$$h_0(t)=\begin{cases} h_1(t)=-\overline{\omega(t)}-\overline{m(t)}\omega'(t),\ t\in\gamma, \\ h_2(t)=-\overline{g(t)}-\overline{m(t)}g'(t),\ t\in L. \end{cases} \tag{5.51}$$

We obtain the solutions of (5.49) and (5.50) as follows,

$$\phi_0(z) = \phi_1(z) + \phi_2(z), \tag{5.52}$$

$$\psi_0(z) = \psi_1(z) + \psi_2(z), \tag{5.53}$$

where

$$\phi_1(z) = \frac{\mu}{(\kappa+1)\pi i}\int_\gamma \omega(t)\zeta(t-z)dt, \tag{5.54}$$

$$\phi_2(z) = \frac{\mu}{(\kappa+1)\pi i}\int_L g(t)\zeta(t-z)dt, \tag{5.55}$$

$$\psi_1(z) = \frac{\mu}{(\kappa+1)\pi i}\int_\gamma h_1(t)\zeta(t-z)dt, \tag{5.56}$$

$$\psi_2(z) = \frac{\mu}{(\kappa+1)\pi i}\int_L h_2(t)\zeta(t-z)dt. \tag{5.57}$$

Substituting formulae (5.52)-(5.57) into the condition (5.36), taking the single-valued condition of displacement, we get

$$\Phi_1^+(x) + \Phi_1^-(x) = p(x) - q(x), x \in \gamma, \tag{5.58}$$

where

$$q(x) = \Phi_2(x) + m'(x)\overline{\Phi_2(x)} + m(x)\overline{\Phi_2'(x)} - \overline{\Psi_2(x)}. \tag{5.59}$$

Here $\Phi_2(z) = \phi_2'(z)$ and $\Psi_2(z) = \psi_2'(z)$ have been known functions by virtue of formulae (5.55) and (5.57).

Referring to (5.54) we know $\phi_1(z)$ has been an odd doubly quasi-periodic function, then $\Phi_1(z) = \phi_1'(z)$ is an even doubly-periodic function. Therefore, (5.58) is an even doubly-periodic Riemann boundary value problem. We obtain the solution as follows[6],

$$\begin{aligned}\Phi_1(z) = {} & \frac{1}{2\pi i}X(z)\int_L \frac{[p(t)-q(t)]\wp'(t)dt}{X^+(t)[\wp(t)-\wp(z)]} \\ & +\frac{1}{2}X(z)\left[\iota_1 + \iota_2\wp(z)\right],\end{aligned} \tag{5.60}$$

where

$$X(z) = \{[\wp(a_1) - \wp(z)] [\wp(a_1) - \wp(z)]\}^{\frac{1}{2}}, \tag{5.61}$$

$$\begin{cases} X^+(x) = i \{[\wp(a_1) - \wp(x)] [\wp(a_1) - \wp(x)]\}^{\frac{1}{2}}, \ a_1 < x < b_1, \\ X^+(x) = -i \{[\wp(a_1) - \wp(x)] [\wp(a_1) - \wp(x)]\}^{\frac{1}{2}}, \ a_2 < x < b_2. \end{cases} \tag{5.62}$$

Due to the conditions of mirror symmetry about the x_1, x_2 axes and $\wp(z) = \overline{\wp(z)}$, ι_1 and ι_2 are real undetermined constants.

To ensure the double periodicity of the stress components, additional conditions (5.41) must be satisfied, substituting (5.9) into (5.41), we have

$$\left[\phi_0(z) + \overline{\psi_0(z)}\right]_{\Gamma_k} = iF_k, k = 1, 2. \tag{5.63}$$

By integration of $\Phi_1(z)$ we get $\phi_1(z)$, from (5.52) and (5.50) we obtain $\phi_0(z)$ and $\psi_0(z)$, respectively, then, we find constants ι_1 and ι_2 from (5.63). As in Section 5.1, one finds that the solution of (5.42)-(5.44) is still obtained by (5.19). When there is no any holes and gaskets in P_{00} and $e_3 = 0$, $F_k = 0$, $T_k = 0$, $k = 1, 2$, and constant load p applied at the edges of the crack, we have the simpler expression of $\Phi_1(z)$,

$$\begin{aligned} \Phi_1(z) = \frac{p}{2} - \frac{pX(z)}{4} [\wp(a_1) + \wp(b_1) - 2\wp(z)] \\ + \frac{X(z)}{2} [\iota_1 + \iota_2 \wp(z)], \end{aligned} \tag{5.64}$$

and the approximate expression of the stress intensity factor at the point a_1,

$$\begin{aligned} K_I &= \lim_{x_1 \to a} \sqrt{2\pi(a_1 - x_1)} \sigma_2(x_1, 0)|_{x_1 < a_1} \\ &\approx 2 \lim_{x_1 \to a} \sqrt{2\pi(a_1 - x_1)} \Phi(x_1) = 2 \lim_{x_1 \to a} \sqrt{2\pi(a_1 - x_1)} \Phi_1(x_1) \\ &\approx -\sqrt{\frac{\pi}{2}} \frac{p[\wp(a_1) - \wp(b_1)] - 2\iota_1 - 2\iota_2 \wp(a_1)}{\sqrt{-\wp'(a_1)[\wp(a_1) - \wp(b_1)]}}, \end{aligned} \tag{5.65}$$

where

$$\iota_1^0 = \frac{1}{2} p \left\{ \frac{\delta_1^0 ImI_2 - i\delta_2^0 ReI_2}{ImI_1 ReI_2 - ReI_1 ImI_2} - [\wp(a_1) - \wp(b_1)] \right\} \tag{5.66}$$

$$\iota_2^0 = p - \frac{1}{2} p \frac{\delta_1^0 ReI_1 - i\delta_2^0 ImI_1}{ImI_1 ReI_2 - ReI_1 ImI_2}, \tag{5.67}$$

$$\delta_1^0 = (\omega_1 + \omega_2), \quad \delta_2^0 = (\omega_1 - \omega_2),$$

$$I_1 = \int_0^{\omega_1} X(z)dz, \quad I_2 = \int_0^{\omega_1} \wp(z)X(z)dz,$$

which is identical with the known results in [53].

5.3 CPS Problem of a Nonhomogeneous Body with a Doubly-Periodic Set of Cylindrical Inlay

In this section, we study the CPS problem of a nonhomogeneous body with a doubly-periodic set of cylindrical inlay, We obtain the general solution in closed form. For an illustrating example, e.g. the doubly-periodic circular cylindrical inlay problem, the exact solution is obtained. When we fix one of its periods, say $\omega_1 = a\pi$ and let $|\omega_2| \to \infty$ in such a way that $Im(\frac{\omega_2}{\omega_1} > 0)$, we get the exact solution of the singly-periodic case with the primitive period $a\pi$, this solution is identical with the known solution (see [44]), furthermore, when we let $|\omega_1| \to \infty$ and $|\omega_2| \to \infty$ in the doubly-periodic case, or $a \to \infty$ in the singly-periodic case, we have immediately the exact solution of the non-periodic case.

Let there be given an elastic body with a doubly-periodic set of holes, and let solid cylindrical inlays of a different isotropic material, but with the same modulus of elasticity μ, be inserted into these holes. All the notations are the same as in Section 5.1.

In this case, one gets the boundary conditions.

$$\phi^+(t) + t\overline{\phi'^+(t)} + \overline{\psi^+(t)} = \phi^-(t) + t\overline{\phi'^-(t)} + \overline{\psi^-(t)}, t \in \mathcal{L}(m,n), \tag{5.68}$$

$$\kappa^+\phi^+(t) - t\overline{\phi'^+(t)} - \overline{\psi^+(t)} = \kappa^-\phi^-(t) - t\overline{\phi'^-(t)} - \overline{\psi^-(t)} + 2\mu\left[g(t) + (\nu^+ - \nu^-)t\right], t \in \mathcal{L}(m,n), \tag{5.69}$$

$$\left[\phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right]_{\Gamma_k} = iF_k, k = 1,2. \tag{5.70}$$

$$F^+(t) + \overline{F^+(t)} = F^-(t) + \overline{F^-(t)}, t \in \mathcal{L}(m,n), \tag{5.71}$$

$$\left[F^+(t) - \overline{F^+(t)}\right] = \left[F^-(t) - \overline{F^-(t)}\right], t \in \mathcal{L}(m,n), \tag{5.72}$$

$$\mu\left[F(z) - \overline{F(z)}\right]_{\Gamma_k} = |\omega_k|T_k, k = 1,2. \tag{5.73}$$

Similarly to Section 5.1 from the boundary conditions (5.68) and (5.69), we have the Riemann boundary value problems

$$\phi^+(t) = \frac{\kappa^- + 1}{\kappa^+ + 1}\phi^-(t) + \frac{2\mu}{\kappa^+ + 1}\left[g(t) + (\nu^+ - \nu^-)t\right], t \in \mathcal{L}(m,n), \tag{5.74}$$

$$\psi^+(t) - \psi^-(t) = -\left\{\left[\overline{\phi^+(t)} - \overline{\phi^-(t)}\right] + \bar{t}\frac{d}{dt}\left[\phi^+(t) - \phi^-(t)\right]\right\}, \tag{5.75}$$

Substituting the transformation (5.9) into the boundary conditions (5.74) and (5.75) we get the doubly quasi-periodic Riemann boundary value problems

$$\phi_0^+(t) = \frac{\kappa^- + 1}{\kappa^+ + 1}\phi_0^-(t) + \frac{2\mu}{\kappa^+ + 1}\left[g(t) + (\nu^+ - \nu^-)t\right], \tag{5.76}$$

$$\psi_0^+(t) - \psi_0^-(t) = h_1(t), t \in \mathcal{L}(m,n), \tag{5.77}$$

where

$$h_1(t) = -\left\{\left[\overline{\phi_0^+(t)} - \overline{\phi_0^-(t)}\right] + \overline{m(t)}\frac{d}{dt}\left[\phi_0^+(t) - \phi_0^-(t)\right]\right\}. \tag{5.78}$$

By Corollary 1.2.1, one gets the general solutions of (5.76) and (5.77),

$$\phi_0(z) = \begin{cases} \dfrac{\mu}{(\kappa^+ + 1)\pi i}\displaystyle\int_L g(t)\zeta(t-z)dt + C_4 z + C + mg_1 + ng_2, \\ \text{if } z = [z]_0 + \Omega_{mn} \in S^+, \\ \dfrac{\mu}{(\kappa^- + 1)\pi i}\displaystyle\int_L g(t)\zeta(t-z)dt + e_3(\nu^+ - \nu^-)z + C_5 z + C, \\ \text{if } z \in S^-, \end{cases} \tag{5.79}$$

$$\psi_0(z) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_L h_1(t)\zeta(t-z)dt + C_6 z + C + mg_1 + ng_2, \\ \text{if } z = [z]_0 + \Omega_{mn} \in S^+, \\ \dfrac{1}{2\pi i}\displaystyle\int_L h_1(t)\zeta(t-z)dt + C_7 z + C, \text{if } z \in S^-, \end{cases} \tag{5.80}$$

Substituting (5.79) and (5.80) into the boundary conditions (5.13) we have

$$C_4 = \frac{1}{iS}\left[\frac{\mu\delta_2}{\kappa^+ + 1}\int_L g(t)dt - \frac{1}{2}\int_L \overline{h_1(t)dt} + i(\overline{\omega_1}F_2 - \overline{\omega_2}F_1)\right], \tag{5.81}$$

$$C_5 = \frac{1}{iS}\left[\frac{\mu\delta_2}{\kappa^- + 1}\int_L g(t)dt - \frac{1}{2}\int_L \overline{h_1(t)dt} + i(\overline{\omega_1}F_2 - \overline{\omega_2}F_1)\right], \tag{5.82}$$

$$C_6 = \frac{1}{S}\left[\frac{i\mu}{2(\kappa^+ + 1)}\int_L \overline{g(t)dt} - i\delta_2\int_L h_1(t)dt + (\overline{\omega_2 F_1} - \overline{\omega_1 F_2})\right], \tag{5.83}$$

$$C_7 = \frac{1}{S}\left[\frac{i\mu}{2(\kappa^- + 1)}\int_L \overline{g(t)dt} - i\delta_2\int_L h_1(t)dt + (\overline{\omega_2 F_1} - \overline{\omega_1 F_2})\right]. \tag{5.84}$$

where S is given by (1.74) Then, we can obtain $\phi(z)$ and $\psi(z)$ immediately by (5.79), (5.80) and (5.9).

Smilarly, we get the solution of (5.71) and (5.72) by the formulae (5.19) and (5.20).

For example, in the plane elasticity problem, namely $e_3 = 0$, when

$$\int_L g(t)\zeta(t-z)dt = 0,\ z \in S^-, \tag{5.85}$$

then $\phi_0^-(z) \equiv 0$. From (5.76) we have

$$\phi_0^+(t) = \frac{2\mu}{\kappa^+ + 1}g(t), \tag{5.86}$$

therefore,

$$h_1(t) = \frac{2\mu}{\kappa^+ + 1}\left[-\overline{g(t)} + \overline{m(t)}g'(t)\right] = \frac{2\mu}{\kappa^+ + 1}h_0(t). \tag{5.87}$$

It is interesting that we observe the fact, after comparing equations (5.14) and (5.15) with (5.79) and (5.80), that similar results hold for the solution of (5.10)-(5.11) with the solution of (5.76)-(5.77) in the case that $g(t)$ satisfies (5.85) when $z \in S^-$ (or $z \in S^+$), only κ has to be replaced by κ^+ (or κ^-) for the latter.

Now, we consider the doubly-periodic circular cylindrical inlay CPS problem. Let L_0 be the circle $|z-b| = r$, and $g(t)$ be given by (5.21). Hence, from (5.12) and (5.21) we have

$$h_1(t) = \frac{2\mu}{\kappa^+ + 1} \left\{ \frac{2r\epsilon}{t-b} + \left[\overline{b} + D(t) \right] \frac{\epsilon}{r} \right\}, \tag{5.88}$$

and condition (5.85) is automatically satisfied, thus, we get the solution after omitting the trival translation constants.

$$\phi(z) = \begin{cases} C_1^0 z + mg_1 + ng_2, \ z = [z]_0 + \Omega_{mn} \in S^+, \\ -\dfrac{2\mu\epsilon}{(\kappa^+ + 1)r}(z-b) + C_1^0 z, \ z \in S^-, \end{cases} \tag{5.89}$$

$$\psi(z) = \begin{cases} -\dfrac{4\mu\epsilon r}{\kappa^+ + 1}\zeta(b-z) + C_2^0 z + mg_1 + ng_2, \ z = [z]_0 + \Omega_{mn} \in S^+, \\ \dfrac{4\mu\epsilon r}{\kappa^+ + 1}\left[\dfrac{1}{z-b} + \zeta(b-z)\right] + C_2^0 z, \ z \in S^-, \end{cases} \tag{5.90}$$

where

$$C_1^0 = \frac{\mu}{(\kappa^+ + 1)S}\left[2\pi r\epsilon + +i(\overline{\omega_1}F_2 - \overline{\omega_2}F_1)\right],$$
$$C_2^0 = \frac{\mu}{(\kappa^+ + 1)S}\left[(\overline{\omega_2 F_1} - \overline{\omega_1 F_2}) - 4\pi\delta_2 r\epsilon\right].$$

and $F(z)$ can be obtained by (5.19) and (5.20).

Similarly, to the doubly-periodic circular gasket problem in two dimensional elasticity theory, say, $T_k = 0$,$k = 1, 2$, and $e_3 = 0$, we get the solution in P_{00} immediately,

$$\phi(z) = \begin{cases} C_1^0 z, \ z \in S_0^+, \\ -\dfrac{2\mu\epsilon}{(\kappa^+ + 1)r}(z-b) + C_1^0 z, \ z \in S_0^-, \end{cases} \tag{5.91}$$

$$\psi(z)=\begin{cases} -\dfrac{4\mu\epsilon r}{(\kappa^{+}+1)}\zeta(z-b)+C_2^0 z,\ z\in S_0^{+}, \\ \dfrac{4\mu\epsilon r}{\kappa^{+}+1}\left[\dfrac{1}{z-b}-\zeta(z-b)\right]+C_2^0 z,\ z\in S_0^{-}, \end{cases} \tag{5.92}$$

and solutions in the limiting cases, e.g. when $\omega_1=a\pi$, $\omega_2\to\infty$, we obtain

$$\phi(z)=\begin{cases} 0,\ z\in S_0^{+}, \\ -\dfrac{2\mu\epsilon}{(\kappa^{+}+1)r}(z-b),\ z\in S_0^{-}, \end{cases} \tag{5.93}$$

$$\psi(z)=\begin{cases} -\dfrac{4\mu\epsilon r}{(\kappa^{+}+1)a}\left[\dfrac{1}{3a}(z-b)+\cot\left(\dfrac{z-b}{a}\right)\right],\ z\in S_0^{+}, \\ \dfrac{4\mu\epsilon r}{\kappa^{+}+1}\left\{\dfrac{1}{z-b}-\dfrac{1}{a}\left[\dfrac{1}{3a}(z-b)+\cot\left(\dfrac{z-b}{a}\right)\right]\right\},\ z\in S_0^{-}, \end{cases} \tag{5.94}$$

for the singly-periodic circular gasket problem, which is identical with the known solution (see [44]). When $\omega_1\to\infty$, $\omega_2\to\infty$, we have

$$\phi(z)=\begin{cases} 0,\ z\in S_0^{+}, \\ -\dfrac{2\mu\epsilon}{(\kappa^{+}+1)r}(z-b),\ z\in S_0^{-}, \end{cases} \tag{5.95}$$

$$\psi(z)=\begin{cases} -\dfrac{4\mu\epsilon r}{\kappa^{+}+1}\dfrac{1}{z-b},\ z\in S_0^{+}, \\ o,\ z\in S_0^{-}, \end{cases} \tag{5.96}$$

for non-periodic circular gasket problem.