Chapter 4

Modified Doubly-Periodic Second Fundamental CPS Problems with Relative Displacements

4.1 Three Formulations of MDPP

A new formulation on the second fundamental problem with relative displacements has been posed for plane elasticity problem in the non-periodic case by [42]. In this chapter we pose three formulations of the modified doubly-periodic second fundamental CPS problem with relative displacements. A three-dimensional elastic body with cylindrical holes is considered. It will be assumed that there exists a system of m holes with smooth and non-intersecting boundaries $\mathcal{L}_j \equiv L_j \; (mod\, 2\omega_1, 2\omega_2)$, $(j = 0, 1, \cdots, m-1)$ inside of each periodic parallelogram on the $x_1, \; x_2$ plane, and $L_j \; (j = 0, 1, \cdots, m-1)$, the boundaries of holes in the doubly periodic fundamental parallelogram P_{00}, are denoted by $L = \bigcup_{j=0}^{m-1} L_j$. The elastic regions located on the left-hand and right-hand sides of L in P_{00} on the $x_1,\ x_2$ plane are denoted by S_0 and S_j^-, respectively (see Figure 4.1). The origin will be chosen inside S_0. The so-called second fundamental CPS problem with relative displacement is, when the displacements $g_j(t) = u_j(t) + iv_j(t)$ $(j = 0, 1, \cdots, m-1)$ given on L_j are relative to certain rigid motions which are different to each other for different contours, to determine the elastic equilibrium. It means the undetermined real displacements should be $g_j(t) + i\alpha_j t + c'_j$, $(j = 0, 1, \cdots, m-1)$, here α_j $(j = 0, 1, \cdots, m-1)$ will be undetermined real constants, while the c'_j are complex. And the displacement $w(t)$ on L_j in x_3-direction with cyclic increments w_k $(k = 1, 2)$ is also given. The strain e_3 is *constant*.

In addition, for the unique existence of solution in the present case, the external resultant principal vector $X_{1j} + iX_{2j}$ and moment M_j of the tractions exerted on L_j must be given. To be specific, we assume $\alpha_0 = 0$.

The stress functions $\phi(z)$ and $\psi(z)$ have the following expressions in this case,

$$\phi(z) = \frac{-1}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} + iX_{2j}) \log \sigma(z - z_j) + \phi_0(z), \tag{4.1}$$

$$\psi(z) = \frac{\kappa}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} - iX_{2j}) \log \sigma(z - z_j) + \psi_0(z), \tag{4.2}$$

where z_j are points arbitrarily situated inside S_j, $\phi_0(z)$ and $\psi_0(z)$ are holomorphic functions and hence single-valued in S_0. We have the derivatives of $\phi(z)$ and $\psi(z)$ as follows,

$$\Phi(z) = \frac{-1}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} + iX_{2j}) \, \zeta(z - z_j) + \Phi_0(z), \tag{4.3}$$

$$\Psi(z) = \frac{\kappa}{2\pi(\kappa+1)} \sum_{j=0}^{m-1} (X_{1j} - iX_{2j}) \, \zeta(z - z_j) + \Psi_0(z). \tag{4.4}$$

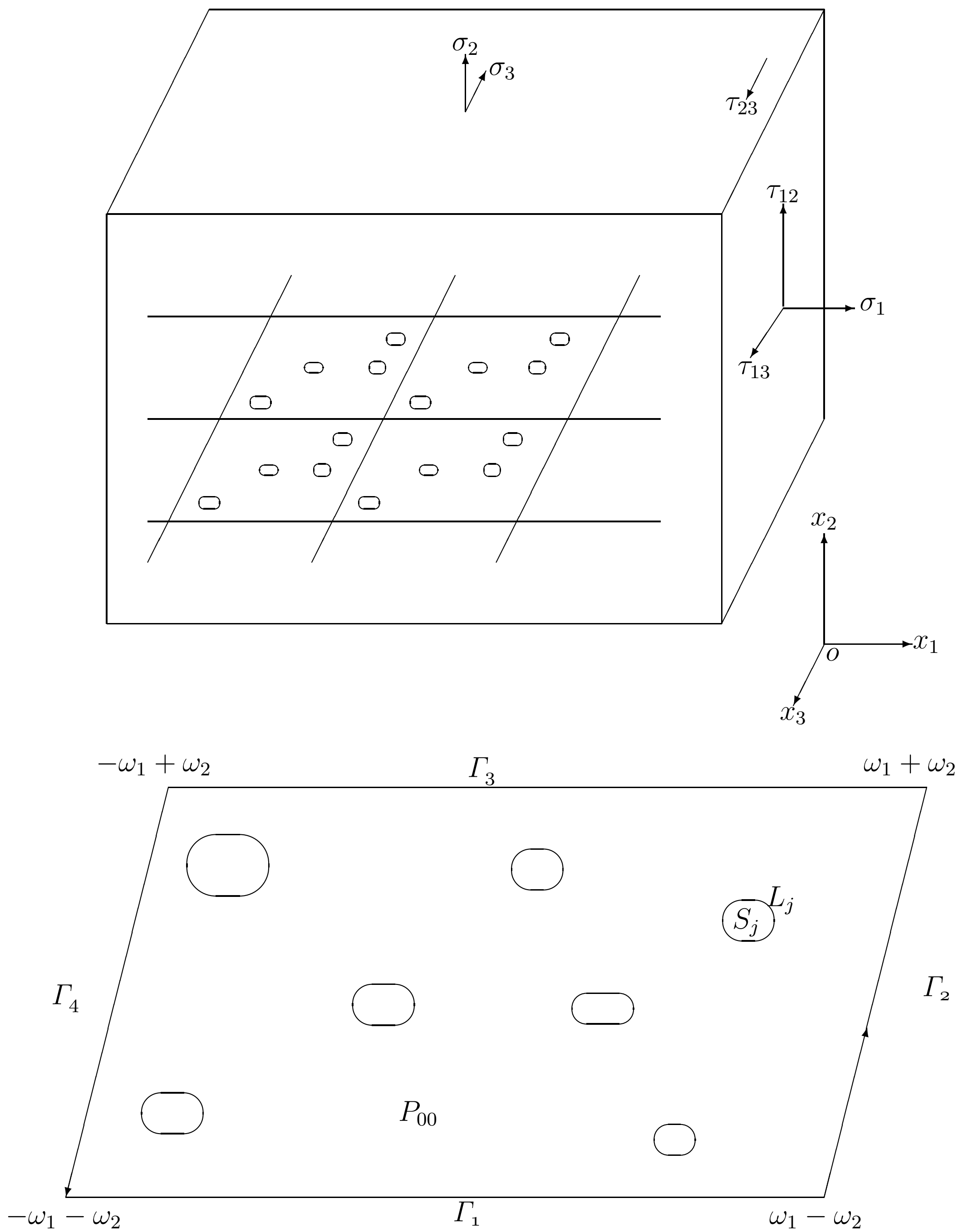

Figure 4.1: Model of an elastic body with a doubly-periodic set of holes

From (1.10), (4.1) and (4.2) one has immediately

$$\begin{aligned}2\mu(u+iv) = &-\frac{\kappa}{\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})log|\sigma(z-z_j)| \\ &+\frac{z}{2\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})\overline{log\zeta(z-z_j)} \\ &+\kappa\phi_0(z)-z\overline{\phi_0'(z)}-\overline{\psi_0(z)}-2\mu e_3\nu z. \end{aligned} \tag{4.5}$$

Then we have the displacement conditions on the boundaries of the holes

$$\kappa\phi_0(t)-t\overline{\phi_0'(t)}-\overline{\psi_0(t)}] = 2\mu g^*(t)+i\beta_j t+c_j, t\in L_j, \tag{4.6}$$

where

$$\begin{aligned}2\mu g^*(t) = &\, 2\mu\left[g(t)+e_3\nu t\right]+\frac{\kappa}{\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})log|\sigma(z-z_j)| \\ &-\frac{t}{2\pi(1+\kappa)}\sum_{j=1}^{m}(X_{1j}+iX_{2j})\overline{log\zeta(z-z_j)}, \end{aligned} \tag{4.7}$$

$$g(t) = g_j(t)+i\alpha_j t+c_j', t\in L_j$$

$\beta_j = 2\mu\alpha_j,\ c_j = 2\mu c_j',\ j = 0,1,\cdots,m-1.$

We can simplify the formula of moment conditions. In fact, the moment of the tractions exerted on an arc $\widehat{AB}$ (from positive side) can be expressed as [49]

$$M_{\widehat{AB}} = Re\left[\chi(z)-z\psi(z)+|z|^2\phi'(z)\right]_A^B, \tag{4.8}$$

when the arc $\widehat{AB}$ be choosen as closed contour C in S_0, because $\phi'(z)$ is single-valued in S_0, we have immediately,

$$\begin{aligned}M_C &= Re\left[\chi(z)-z\psi(z)\right]_C \\ &= Re\int_C \frac{d}{dz}\left[\chi(z)-z\psi(z)\right]dz \\ &= -Re\int_C z\psi'(z)dz. \end{aligned} \tag{4.9}$$

As we are interested in the moment of the tractions exerted on L_j. Let $C = L_j$ $(j = 0, 1, \cdots, m-1)$, in equation (4.9). Substituting (4.4) into equation (4.9), we get

$$Re\left\{\int_{L_j} t\left[\frac{\kappa}{2\pi(1+\kappa)}\sum_{j=0}^{m-1}(X_{1j}-iX_{2j})\zeta(t-z_j)+\psi_0'(t)\right]dt\right\} = -M_j. \quad (4.10)$$

Because $\psi_0(z)$ is single-valued in the region S_0, by integration by parts, hence,

$$\int_{L_j} t\psi_0'(t)dt = -\int_{L_j}\psi_0(t)dt.$$

Furthermore, from (4.10),

$$Re\int_{L_j}\psi_0(t)dt = M_j + Re\left[\frac{\kappa}{2\pi(1+\kappa)}\sum_{j=0}^{m-1}\int_{L_j}(X_{1j}-iX_{2j})t\zeta(t-z_j)dt\right]. \quad (4.11)$$

Because as known

$$\zeta(z-z_j) = \frac{1}{z-z_j} + \zeta_0(z, z_j),$$

where $\zeta_0(z, z_j)$ is a holomophic function, so, one gets

$$Re\int_{L_j}\psi_0(t)dt = M_j + \frac{\kappa}{\kappa+1}Im[z_j(X_{1j}-iX_{2j})]. \quad (4.12)$$

Thus, we may give three formulations of the modified doubly-periodic second fundamental CPS problem with relative displacements (**MDPP**), say, (**MDPP1**), (**MDPP2**) and (**MDPP3**), respectively.

The first formulation is, except the above formulated conditions, given the external resultant principal vectors F_k on Γ_k $(k = 1, 2)$, to find the state of elastic equilibrium. In this case, proceeding in an analogous manner, one finds, in the former notation, the boundary conditions

$$
(\mathbf{MDPP1}) \quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\phi_0(z) + z\overline{\phi_0'(z)} + \overline{\psi_0(z)}\right]_{\Gamma_k} = iF_k, k = 1, 2, \\ Re\left[\int_{L_j} \psi_0(t)dt\right] = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{4.13}
$$

If the displacement's cyclic increment g_1 and the external resultant principal vectors F_2 are given, we get the boundary conditions for the second formulation as

$$
(\mathbf{MDPP2}) \quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)}\right]_{\Gamma_1} = 2\mu g_1, \\ \left[\phi_0(z) + z\overline{\phi_0'(z)} + \overline{\psi_0(z)}\right]_{\Gamma_2} = iF_2, \\ Re\int_{L_j} \psi_0(t)dt = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{4.14}
$$

If the displacement cyclic increments g_1 and g_2 are given, we get the boundary conditions for the third formulation in the form

$$
(\mathbf{MDPP3}) \quad \begin{cases} \kappa\phi_0(t) - t\overline{\phi_0'(t)} - \overline{\psi_0(t)} = 2\mu g^*(t) + i\beta_j t + c_j, t \in L_j, \\ \left[\kappa\phi_0(z) - z\overline{\phi_0'(z)} - \overline{\psi_0(z)}\right]_{\Gamma_k} = 2\mu g_k, k = 1, 2, \\ Re\int_{L_j} \psi_0(t)dt = M_j, j = 0, 1, \cdots, m-1, \\ F(t) + \overline{F(t)} = w(t), t \in L, \\ \left[F(z) + \overline{F(z)}\right]_{\Gamma_k} = w_k, k = 1, 2. \end{cases} \tag{4.15}
$$

4.2 Solution of MDPP

Because the multi-valued parts of $\phi(z)$ and $\psi(z)$ have been separeted out, for convenience, we can assume $X_{1j} + iX_{2j} = 0, j = 0, 1, \cdots, m-1$. At first, let

$\beta_j = 0, j = 0, 1, \cdots, m-1$. We consider

$$\kappa\phi_{00}(t) - t\overline{\phi'_{00}(t)} - \overline{\psi_{00}(t)} = 2\mu g^*(t) + c_{j0}, t \in L_j. \qquad (4.16)$$

This is the boundary condition of the second fundament problem with relative displacements to translation. In fact, this is a special case of a problem treated in Chapter 3. Then, the Sherman transform takes the simpler form,

$$\phi_{00}(z) = \frac{1}{2\pi i}\int_L \omega_0(t)\left[\zeta(t-z) - \zeta(t)\right] dt + \sum_{j=0}^{m-1} b_j \zeta(z - z_j) + Az, \qquad (4.17)$$

$$\begin{aligned}\psi_{00}(z) = &\frac{1}{2\pi i}\int_L \left\{[(\overline{\omega_0(t)}dt + \omega_0(t)d\overline{t}]\left[\zeta(t-z) - \zeta(t)\right]\right\} \\ &-\frac{1}{2\pi i}\int_L \omega_0(t)\left[\overline{t}\wp(t-z) - \rho_1(t-z)\right] dt \\ &+\sum_{j=0}^{m-1} b_j\left[\zeta(z-z_j) + \rho_1(z-z_j)\right] + Bz, \end{aligned} \qquad (4.18)$$

$$b_j = \frac{1}{2\pi i}\int_{L_j}\left[\omega_0(t)d\overline{t} - \overline{\omega_0(t)}dt\right], j = 0, 1, \cdots, m-1. \qquad (4.19)$$

Then, considering the boundary value problem for fixed k, $k = 1, \cdots, m-1$, respectively,

$$\kappa\phi_k(t) - t\overline{\phi'_k(t)} - \overline{\psi_k(t)} = i\delta_{kj}t + c_{jk}, t \in L_j, j = 0, 1, \cdots, m-1, \qquad (4.20)$$

where δ_{kj} is the Kronecker symbol,

$$\delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

This is also a special case from Chapter 3. The representations used for the functions $\phi_k(z)$ and $\psi_k(z)$ in the present case are

$$\phi_k(z) = \frac{1}{2\pi i}\int_L \omega_k(t)\left[\zeta(t-z) - \zeta(t)\right] dt + \sum_{j=0}^{m-1} b_j\zeta(z - z_j), \qquad (4.21)$$

$$
\begin{aligned}
\psi_k(z) = & \frac{1}{2\pi i}\int_L \left\{\left[(\overline{\omega_k(t)}dt + \omega_k(t)d\bar{t}\right]\left[\zeta(t-z)-\zeta(t)\right]\right\} \\
& -\frac{1}{2\pi i}\int_L \omega_k(t)\left[\bar{t}\wp(t-z)-\rho_1(t-z)\right]dt \\
& +\sum_{j=0}^{m-1} b_j\left[\zeta(z-z_j)+\rho_1(z-z_j)\right]. \qquad (4.22)
\end{aligned}
$$

We know that if (4.17) and (4.18) are solution of equation (4.16), (4.21) and (4.22) are solution of equation (4.20), then

$$
\begin{cases}
\phi_0(z) = \phi_{00}(z) + \sum\limits_{k=1}^{m-1}\beta_k\phi_k(z), \\
\psi_0(z) = \psi_{00}(z) + \sum\limits_{k=1}^{m-1}\beta_k\psi_k(z), \\
C_j = C_{j0} + \sum\limits_{k=1}^{m-1} C_{jk}, j = 1,\cdots,m-1,
\end{cases} \qquad (4.23)
$$

will be a solution of equation (4.6).

As an example, the solution method for (**MDPP2**) will be given.

From (4.14), (4.23) we have

$$
\left\{\kappa\phi_{00}(z) - z\overline{\phi'_{00}} - \overline{\psi_{00}(z)} + \sum_{k=1}^{m-1}\beta_k\left[\kappa\phi_k(z) - z\overline{\phi'_k} - \overline{\psi_k(z)}\right]\right\}_{\Gamma_1} = 2\mu g_1, \quad (4.24)
$$

$$
\left\{\phi_{00}(z) + z\overline{\phi'_{00}} + \overline{\psi_{00}(z)} + \sum_{k=1}^{m-1}\beta_k\left[\phi_k(z) + z\overline{\phi'_k} + \overline{\psi_k(z)}\right]\right\}_{\Gamma_2} = iF_2, \qquad (4.25)
$$

Subsituting (4.17), (4.18), (4.21) and (4.22) into equation (4.24) and (4.25), we get

$$
\begin{cases}
(\kappa A - \overline{A})\omega_1 - \overline{B}\overline{\omega_1} = \delta_1 + \mu g_1, \\
(A + \overline{A})\omega_2 + \overline{B}\overline{\omega_2} = \delta_2 + \frac{1}{2}F_2 i,
\end{cases} \qquad (4.26)
$$

where

$$
\delta_1 = \kappa b_0\eta_1 + \overline{a_0\eta_1} - \overline{b_0 r_1} + \sum_{k=1}^{m-1}\beta_k(\kappa b_k\eta_1 + \overline{a_k\eta_1} - \overline{b_k r_1}),
$$

$$
\delta_2 = b_0\eta_2 - \overline{a_0\eta_2} + b_0\overline{r_2} + \sum_{k=1}^{m-1}\beta_k(b_k\eta_2 - \overline{a_k\eta_2} + b_k\overline{r_2}),
$$

$$
a_0 = \frac{1}{\pi i}\int_L \overline{\omega_0(t)}dt, \quad a_k = \frac{1}{\pi i}\int_L \overline{\omega_k(t)}dt,
$$

$$b_0 = \frac{1}{2\pi i}\int_L \left[\omega_0(t)d\overline{t} - \overline{\omega_0(t)}dt\right],$$
$$b_k = \frac{1}{2\pi i}\int_L \left[\omega_k(t)d\overline{t} - \overline{\omega_k(t)}dt\right].$$

Taking the complex conjugate of the system of equations (4.26), we obtain a system of equations with unknown A, $\overline{A}$, B and $\overline{B}$,

$$\begin{cases} \kappa\omega_1 A - \omega_1\overline{A} - \overline{\omega_1}\overline{B} = \delta_1 + \mu g_1, \\ \omega_2 A + \omega_2\overline{A} + \overline{\omega_2}\overline{B} = \delta_2 + \frac{1}{2}F_2 i, \\ -\overline{\omega_1}A + \kappa\overline{\omega_1}\overline{A} - \omega_1 B = \overline{\delta_1} + \mu\overline{g_1}, \\ \overline{\omega_2}A + \overline{\omega_2}\overline{A} + \omega_2 B = \overline{\delta_2} - \frac{1}{2}\overline{F_2}i. \end{cases} \tag{4.27}$$

The determinant of the system of equations (4.27) is

$$\begin{vmatrix} \kappa\omega_1 & -\omega_1 & 0 & -\overline{\omega_1} \\ \omega_2 & \omega_2 & 0 & \overline{\omega_2} \\ -\overline{\omega_1} & \kappa\overline{\omega_1} & -\omega_1 & 0 \\ \overline{\omega_2} & \overline{\omega_2} & \omega_2 & 0 \end{vmatrix} = -i\kappa S Re(\omega_1\overline{\omega_2}) \neq 0, \tag{4.28}$$

Hence, we can obtain A and B uniquely,

$$A = \frac{i\omega_1\overline{\omega_2}^2(\delta_1 + \mu g_1) - \kappa\omega_2\overline{\omega_1}^2(i\delta_2 - \frac{1}{2}F_2)}{\kappa S Re(\omega_1\overline{\omega_2})}, \tag{4.29}$$

$$B = \frac{\kappa\omega_1^2\overline{\omega_2}(\frac{1}{2}\overline{F_2} + i\overline{\delta_2}) - i\omega_2^2\overline{\omega_1}(\overline{\delta_1} + \mu\overline{g_1})}{\kappa S Re(\omega_1\overline{\omega_2})}. \tag{4.30}$$

Letting $z \to t_0 \in L$ and substituting (4.17) and (4.18) into equation (4.16), by employing the modified Plemelj formulae one obtains

$$\kappa\omega_0(t_0) + \frac{\kappa}{2\pi i}\int_L \omega_0(t)d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]$$
$$+\frac{1}{2\pi i}\int_L \overline{\omega_0(t)}d\left[\overline{\zeta(t-t_0)} - (t-t_0)\overline{\zeta(t-t_0)}\right] \tag{4.31}$$
$$+M_7[\omega_0(t), t_0] = N_7(t_0), \tag{4.32}$$

where

$$M_7[\omega_0(t),t_0] = \frac{1}{2\pi i}\int_L \overline{\omega_0(t)}\left[t\overline{\wp(t-t_0)} - \overline{\rho(t-t_0)}\right] d\bar{t}$$
$$+\sum_{j=0}^{m-1} b_j \left[2Re\overline{\zeta(t_0-z_j)} + \overline{\rho_1(t_0-z_j)} - t_0\overline{\wp(t_0-z_j)} + (\kappa-1)\zeta(t_0-z_j)\right]$$
$$+\int_{L_j}\omega_0(t)ds + \kappa A t_0 - \overline{A}t_0 + \overline{Bt_0}, t_0 \in L,$$

$$N_7(t_0) = 2\mu\left[g(t_0) + e_3\nu t_0\right],$$

$$\zeta_0'(z) = -\rho(z), \zeta_0(0) = 0,$$

$$g(t_0) = g_j(t_0), t_0 \in L_j.$$

Substituting formulae (4.21) and (4.22) into equation (4.20) we get

$$\kappa\omega_k(t_0) + \frac{\kappa}{2\pi i}\int_L \omega_k(t) d\left[\log\frac{\sigma(t-t_0)\overline{\sigma(t)}}{\overline{\sigma(t-t_0)}\sigma(t)}\right]$$
$$+\frac{1}{2\pi i}\int_L \overline{\omega_0(t)} d\left[\overline{\zeta(t-t_0)} - (t-t_0)\overline{\zeta(t-t_0)}\right] \tag{4.33}$$
$$+M_8[\omega_k(t),t_0] = N_8(t_0), \tag{4.34}$$

where

$$M_8[\omega_k(t),t_0] = \frac{1}{2\pi i}\int_L \overline{\omega_k(t)}\left[t\overline{\wp(t-t_0)} - \overline{\rho(t-t_0)}\right] d\bar{t}$$
$$+\sum_{j=0}^{m-1} b_j \left[2Re\overline{\zeta(t_0-z_j)} + \overline{\rho_1(t_0-z_j)} - t_0\overline{\wp(t_0-z_j)} + (\kappa-1)\zeta(t_0-z_j)\right]$$
$$+\int_{L_j}\omega_k(t)ds + \kappa A t_0 - \overline{A}t_0 + \overline{Bt_0}, t_0 \in L,$$

$$N_8(t_0) = i\delta_{kj}t_0.$$

As a special case of equation (3.28), after choosing $c_{00} = 0$ and

$$c_{j0} = -\int_{L_j}\omega_0(t)ds, j = 1, \cdots, m-1, \tag{4.35}$$

equation (4.31) has a unique solution $\omega_0^0(t)$. Then by the formulae (4.17), (4.18) and (4.35), one gets

$$\begin{cases} \phi_{00}(z) = \phi_{00}^0(z) + Az, \\ \psi_{00}(z) = \psi_{00}^0(z) + Bz, \\ c_{j0} = c_{j0}^0, \end{cases} \tag{4.36}$$

where

$$\phi_{00}^0(z) = \frac{1}{2\pi i}\int_L \omega_0^0(t)\left[\zeta(t-z) - \zeta(t)\right]dt + \sum_{j=0}^{m-1} b_j^0 \zeta(z - z_j), \tag{4.37}$$

$$\begin{aligned} \psi_{00}(z) = & \frac{1}{2\pi i}\int_L \left\{\left[(\overline{\omega_0^0(t)}dt + \omega_0^0(t)d\bar{t}\right]\left[\zeta(t-z) - \zeta(t)\right]\right\} \\ & -\frac{1}{2\pi i}\int_L \omega_0^0(t)\left[\bar{t}\wp(t-z) - \rho_1(t-z)\right]dt \\ & + \sum_{j=0}^{m-1} b_j^0\left[\zeta(z-z_j) + \rho_1(z-z_j)\right], \end{aligned} \tag{4.38}$$

$$b_j^0 = \frac{1}{2\pi i}\int_{L_j}\left[\omega_0^0(t)d\bar{t} - \overline{\omega_0^0(t)dt}\right], j = 0, 1, \cdots, m-1. \tag{4.39}$$

Similarly, after choosing $c_{0k} = 0$ and

$$c_{jk} = -\int_{L_j} \omega_k(t)ds, j = 1, \cdots, m-1, \tag{4.40}$$

equation (4.33) has a unique solution $\omega_k^0(t)$.

By formulae (4.21), (4.22) and (4.40), one has

$$\phi_k(z) = \phi_k^0(z), \quad \psi_k(z) = \psi_k^0(z), \quad c_{jk} = c_{jk}^0. \tag{4.41}$$

Substituting (4.23) into (4.12), taking $X_{1j} = X_{2j} = 0$ into account, one has immediately

$$Re\int_{L_j}\left[\psi_{00}(t) + \sum_{k=1}^{m-1}\beta_k\psi_k(t)dt\right] = M_j, j = 1, \cdots, m-1, \tag{4.42}$$

or rewritten as

$$\sum_{k=1}^{m-1} A_{jk}\beta_k = M_j - B_j, j = 1, \cdots, m-1, \tag{4.43}$$

where

$$A_{jk} = Re\int_{L_j} \psi_k(t)dt,\ \ j,k = 1,\cdots,m-1, \tag{4.44}$$

$$B_j = Re\int_{L_j} \psi_{00}(t)dt,\ \ j = 1,\cdots,m-1. \tag{4.45}$$

Because

$$\int_{L_j} Btdt = 0, j = 1,\cdots,m-1, \tag{4.46}$$

then, all A_{jk}, B_j are known constants. Then (4.43) will be a system of linear algbraic equations with unknowns β_k. Now, we prove that (4.43) will be uniquely solvable. In fact, we only need to prove that the metrix (A_{jk}) is non-singular. For this purpose, we consider the homogeneous condition: $g_j(t) = 0$, $X_{1j}(t) + iX_{2j}(t) = 0$, $M_j = 0$, $\ \ j = 0, 1,\cdots,m-1$. In this case, the real displacement is $i\alpha_j t + c'_j$. As in [49], consider the integral

$$J = \int_L (X_{1n}u + X_{2n}v)\, ds, \tag{4.47}$$

where

$$\begin{cases} X_{1n} = \sigma_1 cos(n,x_1) + \sigma_{12} cos(n,x_2), \\ X_{2n} = \sigma_{21} cos(n,x_1) + \sigma_2 cos(n,x_2). \end{cases} \tag{4.48}$$

By Green's theorem, we know the classical formula [49]

$$J = \iint\limits_{S_0} \left[\lambda(e_1+e_2)^2 + 2\mu(e_1^2 + 2e_{12}^2 + e_2^2)\right] dx_1 dx_2. \tag{4.49}$$

In our case,

$$\begin{aligned} J &= \int_L (X_{1n}u + X_{2n}v)\, ds \\ &= \sum_{j=0}^{m-1} \int_{L_j} \left\{\left[-\alpha_j x_2 + Re(c'_j)\right] X_{1n} + \left[\alpha_j x_1 + Im(c'_j)\right] X_{2n}\right\} ds \\ &= \sum_{j=0}^{m-1} \left[-\alpha_j \int_{L_j} (x_1X_{2n} - x_2X_{1n})ds + Re(c'_j)\int_{L_j} X_{1n}ds + Im(c'_j)\int_{L_j} X_{2n}ds\right] \\ &= \sum_{j=0}^{m-1} \left[-\alpha_j M_j + Re(c'_j)X_{1j} + Im(c'_j)X_{2j}\right]. \end{aligned} \tag{4.50}$$

Taking the above homogeneous conditions we have

$$J = 0. \tag{4.51}$$

Because the integrand on the right-hand side of (4.49) is a positive definite quadratic form, we have immediately,

$$e_1 = e_2 = e_{12} = 0. \tag{4.52}$$

Thus, there is only a rigid body displacement now, due to the fact that we have assumed $\alpha_0 = 0$. So there is no rotation any more. Therefore, all $\alpha_j = 0,\ j = 0, 1, \cdots, m-1$, furthermore, all $\beta_j = 2\mu\alpha_j = 0,\ j = 0, 1, \cdots, m-1$. This means the matrix (A_{jk}) is nonsingular, so we can obtain a unique solution β_j $(j = 0, 1, \cdots, m-1)$ from the system of equations (4.43).

By representing $F(z)$ as in (3.31), only the integral curve $L \bigcup \gamma$ being replaced by L, we may solve the last two boundary value problems of (**MDPP2**) in the same way used in Chapter 3.