Triangles of extremal area or perimeter in a finite planar point set

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Abstract

We show the following two results on a set of n points in the plane, thus answering questions posed by Erdős and Purdy (1971).

1. The maximum number of triangles of maximum area (or of maximum perimeter) in a set of n points in the plane is exactly n.
2. The maximum possible number of triangles of minimum positive area in a set of n points in the plane is $\Theta(n^2)$.

1 Introduction

A classical problem of combinatorial geometry, first raised by Erdős in 1946, and still far from solution, is to bound the maximum number of occurrences of the same distance among n points in the plane. Numerous variants of this problem were considered, such as special distances (largest, smallest,...), special sets of points (convex position, general position,...), other metrics and higher dimensions.

Erdős and Purdy studied the related problem of the maximum number of occurrences of the same area among the triangles determined by n points in the plane, and as a common generalization the number of occurrences of the same k-dimensional measure among the k-dimensional simplices determined by n points in d-dimensional space [10, 20, 11, 12, 13].

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Here the low-dimensional problems are especially interesting, since in dimensions of at least 4, Lenz-type constructions (points on orthogonal concentric circles) give lower bounds large enough that relatively weak combinatorial structures are sufficient to obtain good upper bounds. In the plane (and 3-dimensional space) such constructions are not possible. For the unit distance as well as the unit area problem in the plane, sections of a square or triangular lattice are the asymptotically best known constructions, giving $\Omega(ne^{c\log n/\log\log n})$ unit distances and $\Omega(n^2 \log\log n)$ unit area triangles among n points. The corresponding best known upper bounds are $O(n^{\frac{4}{3}})$ and $O(n^{\frac{7}{3}})$, respectively. Erdős and Purdy also asked for the maximum number of maximum area and minimum area triangles [10]. Here the corresponding distance problems are much simpler; there are at most n maximum [15, 21] and $\lfloor 3n - \sqrt{12n-3} \rfloor$ minimum [14] distances among n points in the plane. Erdős and Purdy remarks: 'Unfortunately we have only trivial results': $O(n^2)$ and $\Omega(n)$ for the number of maximum area triangles.

We show that the number of maximum area (or maximum perimeter) triangles is indeed similar to the number of maximum distances, and can be determined exactly.

Theorem 1 *A set of n points in the plane, not all collinear, determines at most n triangles of maximum area. This bound is sharp.*

Theorem 2 *A set of n points in the plane determines at most n triangles of maximum perimeter. This bound is sharp.*

The proof of both theorems is almost the same, and is given in Section 2. It uses an alternation property that was already discovered in algorithmic studies of the same question: to determine the maximum area (or maximum perimeter) k-gon inscribed in a convex n-gon. This can be done in $O(n)$ time [3, 5, 6]. This algorithmic question was also studied without the convexity assumption (finding the maximum area triangle contained in a simple n-gon), where it becomes much harder [16]. The perimeter result also holds for any strictly convex norm, since we use only the triangle inequality.

The maximum number of (non-collinear) minimum area triangles is quite unlike its distance counterpart, as can be seen by a $\Omega(n^2)$ lower bound. Two examples that both give this quadratic number of minimum area triangles among n points are

- an $\sqrt{n} \times \sqrt{n}$ square lattice section, and
- two groups of $\frac{n}{2}$ equidistant points on two parallel lines

It should be noted that we here ask for triangles of minimum *positive* area, otherwise $\binom{n}{3}$ triangles of zero area are obviously possible.

These two constructions give, up to a constant factor, the maximum number of minimum area triangles:

Theorem 3 *A set of n points in the plane determines at most $O(n^2)$ (non-collinear) triangles of minimum area. This bound is sharp.*

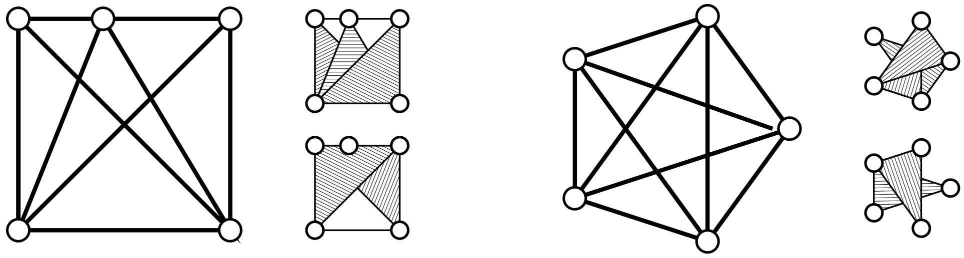

Figure 1: Constructing a set of n points with n maximum area triangles

The proof is in Section 3. Note again that we are only counting triangles of positive area. Thus the algorithm of Edelsbrunner [7, Chapter 12.4] for determining the minimum area triangles treats an essentially different problem, since he assumes there are no three points on a line, reporting otherwise 0 as the minimum triangle area. But if no three points are on a line, the $O(n^2)$ bound becomes trivial, since any two points can be extended only in at most four ways to a triangle of given area.

Triangles of minimum perimeter behave again like smallest distances, their maximum number is $O(n)$, but unfortunately we are unable to find the right multiplicative constant. A section of the square lattice gives a construction with $4n - O(\sqrt{n})$ minimum perimeter triangles, which we conjecture to be extremal. A linear upper bound can be proved by the following argument: If δ is the minimum perimeter, then for each point p of the set, the disk of radius $\frac{1}{2}\delta$ around p contains all triangles of minimum perimeter having p as vertex. Since any sufficiently small disk (radius $< \frac{1}{3\sqrt{3}}\delta$) contains at most two points of the set, the $\frac{1}{2}\delta$-disk around p contains only a bounded number of points, so p belongs to at most a bounded number of minimum perimeter triangles.

2 Triangles of maximum area and perimeter

Consider the vertices $v_1, \ldots, v_n$ of a regular n-gon (n not divisible by 3). Then the n triangles $v_i v_{\lfloor n/3 \rfloor + i} v_{\lfloor 2n/3 \rfloor + i}$ are of maximum area and maximum perimeter. To this basic construction we may add new points, creating new maximal triangles as follows. For the case of maximum area, we may add any point p on the segment $v_i v_{i-1}$, and then $p v_{\lfloor n/3 \rfloor + i} v_{\lfloor 2n/3 \rfloor + i}$ is a new triangle of maximum area. For the case of maximum perimeter, we add p to the arc from v_i to v_{i-1} of the ellipse with foci $v_{\lfloor n/3 \rfloor + i}$ and $v_{\lfloor 2n/3 \rfloor + i}$. See Figures 1 and 2 for maximum area and perimeter examples, respectively. There are, however, also some quite different extremal sets; the question for all extremal sets seems to be difficult (see Figure 3).

The proof of Theorems 1 and 2 uses a special case of Lemma 2.2 from [3], which captures the geometric content of the problem. For the sake of completeness, we prove this lemma at the end of the section.

Lemma 1 *If Δ, Δ^* are two maximum area (or maximum perimeter) triangles determined by a set of points in convex position, then each edge of Δ^* has a point in common with some edge of Δ.*

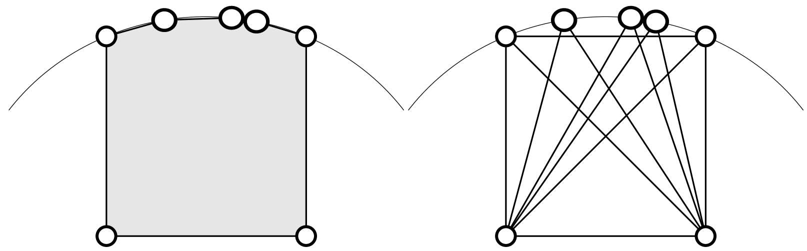

Figure 2: Constructing a set of n points with n maximum perimeter triangles

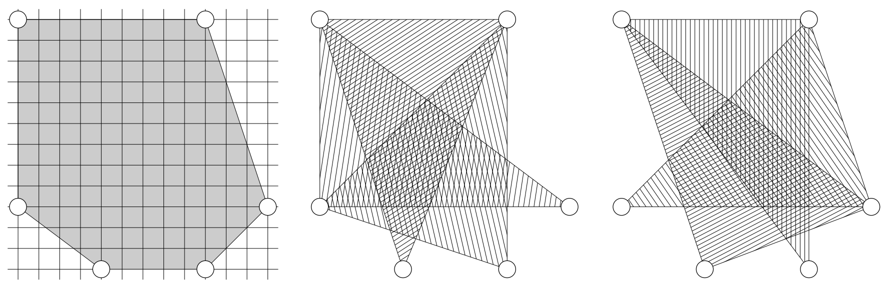

Figure 3: A nontrivial example with many maximum area triangles

Proof of Theorems 1 and 2. In the case of maximum perimeter the set of n points must be the vertex set of a convex n-gon. In the case of maximum area, the set must be in convex position. However, it is then still possible for a point to be between two vertices of the convex hull. Such a point can belong to at most one triangle of maximum area, so we may remove this point and use induction.

We may therefore assume without loss of generality that the n points are the vertices of a convex n-gon. We label these vertices in their natural order $\ldots, v_{-2}, v_{-1}, v_0, v_1, v_2, \ldots$, using all integers as subscripts (thus $v_i = v_{i+n}$ for all $i \in \mathbb{Z}$). From now on we consider both cases of maximum area and perimeter together, speaking only of *maximal triangles*. We say that (i,j,k) is a *proper triple* if $i < j < k < i+n$ and $v_i v_j v_k$ is a maximal triangle. We partially order $\mathbb{Z}^3$ by

$$(i,j,k) \leq (i',j',k') \iff i \leq i', j \leq j', k \leq k'.$$

It follows from Lemma 1 that any two proper triples are comparable in this order. Thus the set of all proper triples forms a chain $\mathcal{C}$. Fix a proper triple (a,b,c). Note that for any maximal triangle, the subchain

$$\mathcal{C}' = \{(i,j,k) \in \mathcal{C} : (a,b,c) \leq (i,j,k) < (a+n, b+n, c+n)\}$$

contains three proper triples corresponding to this maximal triangle. Finally, any chain in $\mathbb{Z}^3$ with minimum element (a,b,c) and maximum element $(a+n, b+n, c+n)$ contains at most $3n+1$ triples. It follows that there are at most n maximal triangles. ■

This proof was inspired by a similar circular numbering scheme used in [18, 19].

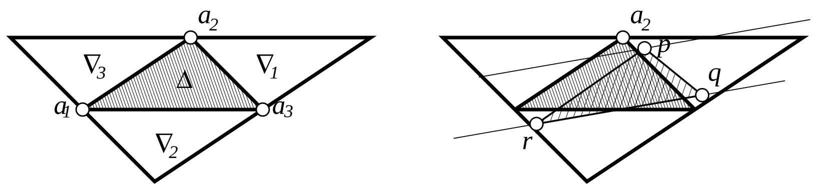

Figure 4:

Proof of Lemma 1. Let $\Delta = a_1a_2a_3$ and $\Delta^* = pqr$. We first consider the case of maximum area. Each point of a set in which Δ is a maximum area triangle must be in the triangle bounded by the three lines through a_i parallel to $a_{i+1}a_{i+2}$, $i = 1, 2, 3$: any point outside this region will generate together with two vertices of Δ a triangle with area larger than that of Δ. This large triangle is cut by Δ into three translates of $-\Delta$; we denote them by $\nabla_1, \nabla_2, \nabla_3$. The vertices of Δ^* are contained in $\nabla_1 \cup \nabla_2 \cup \nabla_3$. If all vertices of Δ^* are in the same ∇_i, then Δ^* coincides with ∇_i, since they have the same area, and the claim of the Lemma is satisfied. If each ∇_i contains one vertex of Δ^*, then the edges of Δ^* intersect Δ, and again our claim is satisfied. In the remaining case two vertices are in one ∇_i, and the third in another; we can assume $\Delta^* = pqr$ with $p, q \in \nabla_1$, $p, q \notin \Delta$, and $r \in \nabla_2$ (Figure 4). Then the line through p parallel to qr separates a_2 from qr, so the triangle a_2qr has an area larger than that of Δ^*, violating the assumption of the Lemma.

We now consider the case of maximum perimeter. If the conclusion of the Lemma is false, then some two vertices of Δ^* are strictly between two vertices of Δ in the natural ordering of $\{a_1, a_2, a_3, p, q, r\}$. We may assume without loss of generality that p and q are strictly between a_2 and a_3, and by symmetry that r is not in the part between a_1 and p ($r \neq a_2$, and not between a_1 and a_2 or between a_2 and p). Thus the possible vertex orderings are '$a_1a_2pqra_3$', '$a_1a_2pq(r=a_3)$', '$a_1a_2pqa_3r$' and '$(a_1=r)a_2pqa_3$'. In each of these cases the segments $\overline{a_2r}$ and $\overline{a_1p}$ intersect in some point x, and the segments $\overline{a_2q}$ and $\overline{a_3p}$ intersect in some point y, and we have

$$\begin{aligned} & \text{perimeter}(a_1pa_3) + \text{perimeter}(a_2qr) \\ = \;& \big(|a_1x| + |xp| + |py| + |ya_3| + |a_3a_1|\big) + \big(|a_2y| + |yq| + |qr| + |rx| + |xa_2|\big) \\ = \;& \big(|a_1x| + |xa_2| + |a_2y| + |ya_3| + |a_3a_1|\big) + \big(|py| + |yq| + |qr| + |rx| + |xp|\big) \\ > \;& \text{perimeter}(a_1a_2a_3) + \text{perimeter}(pqr)\,, \end{aligned}$$

where the last inequality is strict unless a_1xa_2 are collinear and pyq are collinear. Since $\{p, q\}$ are by assumption distinct from $\{a_1, a_2, a_3\}$, and sides of maximum perimeter triangles cannot partially overlap, these collinearities cannot occur. But the strict inequality contradicts our assumption of maximum perimeter of $a_1a_2a_3$ and pqr. ∎

3 Triangles of minimum area

The following proof of Theorem 3 is similar to the Pach-Sharir proof [17] of the sharp upper bound for the number of congruent angles in a set of n points in the plane.

Proof of Theorem 3. Let S be the set of n points in the plane and let $\mathcal{L}$ be the set of lines passing through at least two points of S. Also let $\mathcal{T}$ be the set of all minimum area triangles in S, $\mathcal{T}_1$ be the subset of all $abc \in \mathcal{T}$ for which there is a point $d \in S$ such that ab and cd are parallel ($ab, cd \in \mathcal{L}$), and $\mathcal{T}_2 = \mathcal{T} \setminus \mathcal{T}_1$. Then $\mathcal{T}_2$ contains $O(n^2)$ triangles, since each pair of points ab can extended by at most two points $c_1 c_2$ to a triangle in $\mathcal{T}_2$ without generating a line $c_i c_j$ parallel to ab. To count the $\mathcal{T}_1$-triangles, we classify them according to the directions of their sides. Let $\mathcal{L}^\theta = \{\ell_1^\theta, \dots, \ell_k^\theta\}$ be the lines in $\mathcal{L}$ with direction θ, consecutively numbered. By the minimum area property, a $\mathcal{T}_1$-triangle with a side of direction θ has two points on one line ℓ_i^θ, and the remaining point either on ℓ_{i-1}^θ or on ℓ_{i+1}^θ, and the two points on ℓ_i^θ are consecutive points on that line. Thus if there are a_i^θ points of S on ℓ_i^θ (with $a_0^\theta = a_{k+1}^\theta = 0$), then the number of triangles in $\mathcal{T}_1$ with a side of direction θ is at most

$$\sum_{i=1}^{k} (a_i^\theta - 1)(a_{i-1}^\theta + a_{i+1}^\theta) < 2 \sum_{i=1}^{k-1} a_i^\theta a_{i+1}^\theta \le 2 \sum_{i=1}^{k} (a_i^\theta)^2 = 2 \sum_{\ell \in \mathcal{L}^\theta} |\ell \cap S|^2.$$

Taking now the sum over all directions θ, we find

$$|\mathcal{T}_1| \le 2 \sum_{\ell \in \mathcal{L}} |\ell \cap S|^2 .$$

Let now b_r be the number of lines in $\mathcal{L}$ containing at least r points ($b_{n+1} = 0$). Since there are at most $O(m^{\frac{2}{3}} n^{\frac{2}{3}} + m + n)$ incidences between n points and m lines [22] we have

$$b_r \le c \max\left\{ \frac{n^2}{r^3}, \frac{n}{r} \right\} .$$

This gives the claimed bound since $|\mathcal{T}| = |\mathcal{T}_1| + |\mathcal{T}_2| \le |\mathcal{T}_1| + O(n^2)$ and

$$\begin{aligned}
|\mathcal{T}_1| &\le 2 \sum_{\ell \in \mathcal{L}} |\ell \cap S|^2 \\
&= 2 \sum_{r=2}^{n} r^2 (b_r - b_{r+1}) = 2b_2 + 2 \sum_{r=2}^{n} \left(r^2 - (r-1)^2\right) b_r \\
&\le 2\binom{n}{2} + 2 \sum_{r=2}^{n} (2r-1) b_r \le n^2 + 4 \sum_{r=2}^{\sqrt{n}} r b_r + 4 \sum_{r=\sqrt{n}}^{n} r b_r \\
&\le n^2 + 4 \sum_{r=2}^{\sqrt{n}} r \frac{cn^2}{r^3} + 4 \sum_{r=\sqrt{n}}^{n} r \frac{cn}{r} < n^2 + 4cn^2 \sum_{r=2}^{\infty} \frac{1}{r^2} + 4cn \sum_{r=1}^{n} 1 \\
&\le \left(1 + c\frac{2\pi^2}{3}\right) n^2 .
\end{aligned}$$

■

4 Related Problems

Let $a_d(n)$ denote the maximum number of unit area triangles among n points in $\mathbb{R}^d$. The best currently known bounds are

$d = 2$: $a_2(n) = \Omega(n^2 \log\log n)$ [10] and $ua_2(n) = O(n^{\frac{7}{3}})$ [17], the lower bound by lattice sections;

$d = 3$: $a_3(n) = O(n^{\frac{8}{3}})$ [10], $a_3(n) \geq a_2(n) = \Omega(n^2 \log\log n)$ (no better construction known);

$d = 4$: $a_4(n) \leq a_5(n)$ and $a_4(n) \geq a_2(n)$ (no better bounds known);

$d = 5$: $a_5(n) = O(n^{3-\epsilon})$ for some $\epsilon > 0$ [20], and $a_5 = \Omega(n^{\frac{7}{3}})$ as can be seen by taking $\frac{n}{2}$ points with $\Omega(n^{\frac{4}{3}})$ unit distances on a sphere in a 3-dimensional subspace [9], and $\frac{n}{2}$ points on a circle in an orthogonal plane;

$d \geq 6$: $a_d(n) = \Theta(n^3)$ by a Lenz-type construction: $\frac{n}{3}$ points on three concentric circles in pairwise orthogonal planes [20].

Analogous to the distance problems one can also ask for the minimum number of distinct areas or k-volumes of k-simplices determined by n points in $\mathbb{R}^d$ [12]; here the conjectured value is $\lfloor \frac{n-1}{d} \rfloor$, which is reached by n points distributed as cartesian product of a regular $(d-1)$-simplex and an arithmetic progression in the orthogonal 1-dimensional subspace. In the plane this gives an upper bound of $\lfloor \frac{n-1}{2} \rfloor$ for the minimum number of distinct areas determined by n points not all on a line; the best lower bound is $\frac{1}{2}(1 - \frac{1}{3+2\sqrt{2}})n - O(1) \approx 0.4142n - O(1)$ which can be obtained by combining the method of [4] and the result of [23], improving previous bounds in [4] and [12].
Another generalization of the unit distances problem is the question for the maximum number of congruent triangles determined by n points in $\mathbb{R}^d$ [10]. In the plane this is the same as the unit distances problem, since each unit distance can belong to at most four triangles; the upper bounds are $O(n^{\frac{19}{9}})$ in three-dimensional space [11], $O(n^{\frac{65}{23}})$ in dimension four [2], and $\Theta(n^3)$ for dimension $d \geq 6$. Further related problems are the question for the maximum number of similar triangles, treated in [8] for arbitrary triangles and in [1] for equilateral triangles.

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