The Linear-Extension-Diameter of a Poset

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Abstract. The *distance* between two permutations of the same set X is the number of pairs of elements being in different order in the two permutations. Given a poset $P = (X, \leq)$, a pair L_1, L_2 of linear extensions is called a *diametral pair* if it maximizes the distance among all pairs of linear extensions of P. The maximal distance will be called the *linear extension diameter* of P and is denoted $led(P)$. Alternatively $led(P)$ is the maximum number of incompararable pairs of a two-dimensional extension of P. In the first part of the paper we discuss upper and lower bounds for $led(P)$. These bounds relate $led(P)$ to well studied parameters like dimension and height. We prove that $led(P)$ is a comparability invariant and determine the linear extension diameter for the class of generalized crowns. For the Boolean lattices we have partial results.

A diametral pair generates a minimal two-dimensional extension of P or equivalently a maximal interval in the graph of linear extensions of P. Studies of such intervals lead to the definition of new classes of linear extensions. We give three characterizations of the class of *extremal linear extensions* which contains the greedy linear extensions. With *complementary linear extensions* we introduce a class contained in the set of super-greedy linear extensions. The complementary linear extension of L is the linear extension L^* obtained by taking the reverse of L as priority list in the generic algorithm for linear extensions. A *complementary pair* is a pairs L, M of linear extensions with $M = L^*$ and $L = M^*$. Iterations of the complementary mapping starting from an arbitrary linear extension eventually leads to a complementary pair.

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1 Introduction and Alternate Formulations

The *distance* between permutations π, σ of the same set X, denoted $dist(\pi, \sigma)$, is the number of pairs of elements being in different order in the two permutations. Given a poset $P = (X, \leq)$, a pair L_1, L_2 of linear extensions is called a *diametral pair* if it maximizes the distance among all pairs of linear extensions of P. The maximal distance will be called the *linear extension diameter* of P and is denoted $led(P)$. In [Reu96b] the *linear extension graph* $G(P)$ was defined as the graph with vertices the linear extensions of P and two vertices connected by an edge if the linear extensions differ by an adjacent transposition only. Figure 1 shows the six element poset called chevron and its linear extension graph. An

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easy fact about $G(P)$ is that any pair L_1, L_2 of linear extensions is connected in $G(P)$ by a path whose length equals the distance between L_1 and L_2. Hence, $led(P)$ is exactly the graph diameter of the linear extension graph $G(P)$.

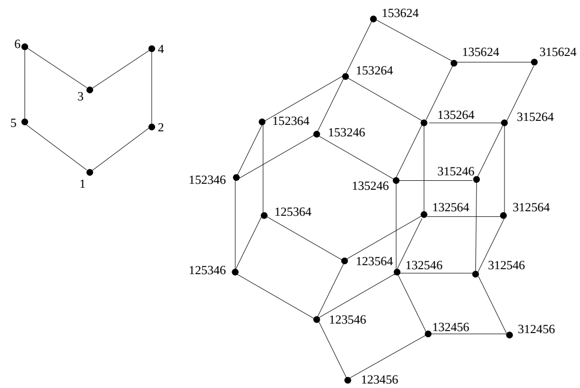

Figure 1: The chevron and its linear extension graph.
This poset has linear extension diameter 6.

The intersection of a collection $A = \{L_1, \ldots, L_k\}$ of linear extensions of P is a poset P_A which is an extension of P. The graph $G(P_A)$ is an induced subgraph of $G(P)$. Interestingly subgraphs of $G(P)$ corresponding to extensions of P are exactly the convex subgraphs of $G(P)$ (see [BW91] or [Reu96b]).

Let $inc(P)$ denote the number of incomparable pairs of P. If L_1, L_2 is a diametral pair for P then $P_{\{L_1,L_2\}}$ is a two-dimensional extension of P and L_1, L_2 is a diametral pair for $P_{\{L_1,L_2\}}$, i.e., $led(P_{\{L_1,L_2\}}) = led(P)$. The incomparable pairs of $P_{\{L_1,L_2\}}$ are exactly the pairs being in different order in L_1 and L_2, therefore, $led(P_{\{L_1,L_2\}}) = inc(P_{\{L_1,L_2\}}) = dist(L_1, L_2)$, where $inc(P)$ denotes the number of incomparable pairs of P.

We call a two-dimensional extension Q of P a *minimum two-dimensional extension* of P if Q has a minimal number of comparable pairs that are incomparable in P. Dually, a minimum two-dimensional extension maximizes $inc(P_{\{L_1,L_2\}})$. Together with the previous paragraph this proves the following Theorem.

Theorem 1 *The linear extension diameter of P equals the number of incomparable pairs of a minimum two-dimensional extension of P.*

By definition $inc(Q) \leq inc(P)$ for every extension Q of P. As a consequence of the theorem we have the general bound

$$led(P) \leq inc(P). \tag{1}$$

Equality in inequality (1) is a characterization of two-dimensional posets:

Theorem 2 *For a poset P the following two statements are equivalent:*
$$dim(P) \leq 2 \quad \text{and} \quad led(P) = inc(P).$$

Proof. We have already seen that $led(P) = inc(P)$ for two-dimensional posets. If P is one-dimensional then $led(P) = 0 = inc(P)$.

For the converse suppose $led(P) = inc(P)$ and let L_1, L_2 be a diametral pair. The number of pairs being in different order in L_1 and L_2 is $inc(P)$. Therefore, P is the intersection of L_1 and L_2 which proves $dim(P) \leq 2$. $\square$

Inequality (1) is only sharp for two-dimensional posets but as shown with the standard examples the following inequality may be sharp in any dimension

$$led(P) \leq inc(P) - (dim(P) - 2). \tag{2}$$

Proof. Take a diametral pair L_1, L_2 and add one by one linear extensions such that $\bigcap_{i=1}^{j} L_i \supset \bigcap_{i=1}^{j+1} L_i$ until $\{L_1, \ldots, L_k\}$ is a realizer of P. Since $k \geq dim(P)$ and each L_j contributes a new incomparability to the intersection the poset $P_{\{L_1, L_2\}}$ has at most $inc(P) - (dim(P) - 2)$ incomparable pairs. $\square$

In the next section we give several lower bounds on the linear extension diameter. These bounds relate the new parameter to width, dimension and fractional dimension of the poset. In Section 3 we investigate the effect of small changes at the poset on its linear extension diameter. We also show that *led* is a comparability invariant. In Section 4 we deal with special classes of posets. In particular we determine the linear extension diameter of generalized crowns. Section 5 introduces the concept of complementary linear extensions as a heuristic for finding pairs of linear extensions of large distance. We prove some properties of complementary linear extensions that seem to be interesting in their own right.

2 Lower Bounds on the Linear Extension Diameter

Given a poset $P = (X, \leq)$ and disjoint subsets $A, B \subset X$ we say A *is over* B and write A/B in a linear extension L if $a > b$ in L for all incomparable pairs $a||b$ with $a \in A$ and $b \in B$. It is well known (see e.g. [Tro92, p. 19]) that for every chain C there exist linear extensions with C/X and X/C. Such a pair of linear extensions has distance at least $\sum_{x \in C} inc(x)$ where $inc(x)$ denotes the number of elements incomparable to x. Generalizing notation by defining $inc(C) = \sum_{x \in C} inc(x)$ for every chain C we have proven our first lower bound

$$\max_{C \text{ chain}} inc(C) \leq led(P). \tag{3}$$

Equality holds for the chevron and for all width two posets. The value of this lower bound is easily computable by a maximum weighted chain algorithm. Consider a chain partition $C_1, ..., C_w$ of P. Obviously $width(P)(\max_C inc(C)) \geq \sum_{i=1}^{w} inc(C_i) = 2inc(P)$. Hence our upper and lower bounds on *led* in (1) and (3) are only apart by a factor depending on the width of P,

$$\lceil \frac{2inc(P)}{width(P)} \rceil \leq led(P) \leq inc(P). \tag{4}$$

Another lower bound relates the linear extension diameter to the dimension $dim(P)$. Take a realizer $R = \{L_1, \ldots, L_d\}$ with $d = dim(P)$ for P. Choose at random a pair S_1, S_2 of different linear extensions from R, the probability that an incomparable pair $x||y$ is incomparable in $S_1 \cap S_2$ is at least $(d-1)/\binom{d}{2}$. Therefore, the expected number of incomparable pairs in $S_1 \cap S_2$ is at least $2inc(P)/d$. This proves the bound

$$\lceil \frac{2inc(P)}{dim(P)} \rceil \leq led(P). \tag{5}$$

Since $dim(P) \leq width(P)$ this bound (5) implies (4). Brightwell and Scheinerman [BS92] introduced the *fractional dimension* of a poset ($fdim(P)$) as the least rational number d_f such that there is a m and a multiset realizer $M = \{L_1, \ldots, L_m\}$ of P, such that for every incomparable pair x, y we have $x < y$ in L_i for at least m/d_f of the linear extensions. If we choose at random a pair S_1, S_2 of linear extensions from M the probability that an incomparable pair $x||y$ is incomparable in $S_1 \cap S_2$ is at least $m/d_f(m-(m/d_f))/\binom{m}{2} = 2(m(d_f-1)/((m-1)d_f^2) \geq 2(d_f-1)/(d_f^2)$. Since fractional dimension can be substantially smaller than dimension the next bound seems worth to be stated

$$\lceil \frac{2(fdim(P)-1)inc(P)}{fdim(P)^2} \rceil \leq led(P). \tag{6}$$

A class of orders where dimension and fractional dimension get far apart are the interval orders. The dimension of interval orders grows unbounded (see e.g., [Tro92]) but the fractional dimension is bounded by 4 (see [BS92]). In fact, as shown recently by Trotter and Winkler [TW96] the fractional dimension of interval orders can be arbitrarily close to 4. From the above bound we thus obtain that $led(I) \geq (3/8)inc(I)$ for every interval order I. However, we can easily do better. It was shown by Rabinovich ([Tro92, page 196]), that an interval order $I = (X, \leq)$ has a linear extension with $A/(X \setminus A)$ for every subset A of X. Choose a random subset A of X and consider two linear extensions with $A/(X \setminus A)$ and $(X \setminus A)/A$. The expected number of incomparabilities in the intersection of the two linear extensions is at least $(1/2)inc(I)$. Hence for every interval order I

$$(1/2)inc(I) \leq led(I). \tag{7}$$

The next bound relates $inc(P)$ and the height $h = height(P)$. Let $A_1, \ldots, A_h$ be an antichain partition of P and let $a_i = |A_i|$. The weak order with A_i as ith level is a two-dimensional extension of P. The number of incomparabilities is $\sum_i \binom{a_i}{2}$ which is at least $h\binom{n/h}{2}$, hence, $led(P) \geq n(n-h)/2h$. For $inc(P)$ we have the obvious bound $inc(P) \leq \binom{n}{2} - \binom{h}{2}$. Therefore $inc(P) \leq n^2/2 - h^2/2 = n^2 - (1/2)(n^2 + h^2) \leq n^2 - nh$. Comparing the two inequalities we obtain

$$\lceil \frac{inc(P)}{2height(P)} \rceil \leq led(P). \tag{8}$$

The bounds of this section compare $led(P)$ to certain fractions of $inc(P)$. Graham Brightwell (personal communication) suggested a family P_n of random posets showing that the gap between $inc(P)$ an $led(P)$ can indeed be large. Formally, $led(P_n) = o(1)inc(P_n)$.

3 Removals and Substitutions

Consider the removal of a point x from P. Let L_1, L_2 be a diametral pair for $P - x$, there exist linear extensions L'_i of P such that removing x gives L_i for $i = 1, 2$. The distance of L'_1, L'_2 is at least as large as the distance of L_1 and L_2, hence $led(P - x) \leq led(P)$. For a lower bound on $led(P - x)$ consider a two-dimensional extension Q of P such that $inc(Q) = led(P)$. $Q - x$ is a two-dimensional extension of $P - x$ and the incomparabilities of Q are those of $Q - x$ plus those containing element x. The incomparabilities of Q containing x are at most as many as the incomparabilities of P containing x, i.e. $inc(x)$. Hence, $led(P - x) + inc(x) \geq led(P)$.

Theorem 3 *$led(P) \geq led(P - x) \geq led(P) - inc(x)$ and both inequalities can be sharp.*

Proof. It remains to show that equality may occur. Equality on both sides happens if $inc(x) = 0$. However, there are less trivial examples. On the left side take as x one of the minimal elements of $\mathbf{C}$ or $\mathbf{D}$ (these are posets from the list of 3-irreducible posets (see e.g. [Tro92, p. 62]), $\mathbf{D}$ is the chevron). On the right side equality is attained for every two-dimensional P. □

Abusing notation we write $P - r$ for the poset resulting from P after removal of a single covering relation r. $P - r$ has more linear extensions then P, more precisely, $G(P)$ is a subgraph of $G(P - r)$. Hence, $led(P) \leq led(P - r)$. Equality is again possible: let P be the chevron augmented by the comparability $r = (1 < 3)$ (see Figure 1). A lower bound for $led(P - r)$ can be obtained from the lower bound for point removal: Let r be a relation involving x, then $led(P) \geq led(P - x) = led((P - r) - x) \geq led(P - r) - (inc(x) + 1)$. The example of the crown $\mathbf{A}_n$ shows (see Section 4) that removing r can increase led by as much as $(1/2)(inc(x) + 1)$.

Theorem 4 *Let $r = (x < y)$ be a covering relation of P, then $led(P) \leq led(P - r) \leq led(P) + \min(inc(x), inc(y)) + 1$.*

Let $P = (X, \leq_P)$ and $Q = (Y, \leq_Q)$ be posets on disjoint sets. Standard constructions are the parallel composition $P + Q = (X \cup Y, \leq_P \cup \leq_Q)$ and the series composition $P * Q = (X \cup Y, \leq_P \cup \leq_Q \cup (X \times Y))$. In both cases the led of the composition is easily determined by the components.

- $led(P + Q) = led(P) + led(Q) + |X||Y|$.
- $led(P * Q) = led(P) + led(Q)$.

Let x be an element of P and let P^Q_x be the poset obtained by substituting Q for x in P. To be more specific, $P^Q_x = ((X - x) \cup Y, \leq)$ with $a \leq b$ iff $a, b \in X - x$ and $a \leq_P b$ or $a, b \in Y$ and $a \leq_Y b$ or $a \in X - x, b \in Y$ and $a \leq_P x$ or $a \in Y, b \in X - x$ and $x \leq_P b$.

Theorem 5 *$led(P) + led(Q) + (led(P) - led(P - x))(|Q| - 1) \leq led(P^Q_x) \leq led(P) + led(Q) + inc(x)(|Q| - 1)$.*

Proof. Let L_1, L_2 be a diametral pair for P and N_1, N_2 be a diametral pair for Q. Consider the linear extensions $(L_1)_x^{N_1}$ and $(L_2)_x^{N_2}$. Compute the distance between $(L_1)_x^{N_1}$ and $(L_2)_x^{N_2}$ as the number of adjacent transpositions necessary to change $(L_1)_x^{N_1}$ into $(L_2)_x^{N_2}$ and note that changing L_1 into L_2 requires at least $led(P) - led(P-x)$ adjacent transpositions involving element x. This leads to the lower bound on $led(P_x^Q)$.

For the upper bound select an element $y \in Y$ and count the incomparabilities of a two-dimensional extension of $led(P_x^Q)$ in three parts. There are at most $led(P)$ incomparabilities between two elements in $X - x + y$, there are at most $led(Q)$ incomparabilities between two elements in Y and, finally, there are at most $inc(x)(|Q|-1)$ incomparabilities between elements of $X - x$ and elements of $Y - y$. □

Another interesting aspect of *led* is the question of comparability invariance. Reuter [Reu96a] observed that the linear extension graph $G(P)$ is not a comparability invariant. Nevertheless, as will be shown next the linear extension diameter is a comparability invariant. The proof is based on the following lemma.

Lemma 6 *The linear extension diameter of P_x^Q is attained by a pair L_1, L_2 of linear extensions in both of which the elements of Q appear consecutively.*

Proof. Let L_1, L_2 be a diametral pair of P_x^Q. Let $Q = (Y, \leq_Q)$ and choose $y \in Y$ such that in $P_{\{L_1,L_2\}}$ element y is incomparable to the maximal number of elements $z \notin Y$. Let L_1' be obtained from L_1 by first removing the elements of Y from L_1 and then reinserting them at the original position of y so that their internal order remains unchanged. Let L_2' be obtained from L_2 by the same procedure. From the choice of y it follows that the distance of L_1' and L_2' is at least as large as the distance of L_1 and L_2. Therefore, L_1', L_2' is a diametral pair and the elements of Q appear consecutively in L_1' and in L_2'. □

Theorem 7 *Linear extension diameter is a comparability invariant.*

Proof. A consequence of Gallai's work [Gal67], made explicit in [DPW85], is a simple scheme for proving the comparability invariance of a property. It has only to be shown that for all posets P and Q and elements x of P the property is unable to distinguish between P_x^Q and $P_x^{Q^d}$ where Q^d denotes the dual of Q, i.e., $y \leq y'$ in Q^d iff $y' \leq y$ in Q.

Given a linear extension of P_x^Q in which the elements of Q appear consecutively we obtain a linear extension of $P_x^{Q^d}$ by reversing the order of the elements of Q. Hence, if L_1, L_2 is a diametral pair linear extensions of P_x^Q as in Lemma 6 we obtain a pair attaining the same distance for $P_x^{Q^d}$. Since the converse also works the linear extension diameters of P_x^Q and $P_x^{Q^d}$ are equal. □

4 Generalized Crowns and Boolean Lattices

In this section we first deal with a class of posets where we can determine the linear extension diameter exactly. Trotter defines generalized crowns as a class of posets that interpolates between the 3-irreducible crowns $\mathbf{A}_n$ and the standard examples $\mathbf{S}_n$. For $n \geq k \geq 2$ define $\mathbf{C}_n^k$ as the height two poset with minimal elements $\{0, 1, \ldots, (n-1)\}$ and maximal elements $\{0', 1', \ldots, (n-1)'\}$. Element i' is larger then the elements $\{i - \lfloor (k-1)/2 \rfloor, i - \lfloor (k-1)/2 \rfloor + 1, \ldots, i + \lfloor k/2 \rfloor\}$ where indices are taken modulo n.

Lemma 8 can be found in [Tro92, p. 35], for the translation note that $\mathbf{C}_n^k$ equals Trotter's $\mathbf{S}_{k+1}^{n-k-1}$. In particular $\mathbf{C}_n^2 = \mathbf{A}_n$, $\mathbf{C}_n^{n-1} = \mathbf{S}_n$ and $\mathbf{C}_n^k$ is k regular.

Lemma 8 *A linear extension L of a generalized crown $\mathbf{C}_n^k$ can have $i' < j$ in L for at most $\binom{n-k+1}{2}$ pairs (i', j).*

Consider a pair L_1, L_2 of linear extensions of $\mathbf{C}_n^k$. Since each linear extension is reversing at most $\binom{n-k+1}{2}$ of the (i', j) pairs, the poset $P_{\{L_1, L_2\}}$ has at most $(n-k+1)(n-k)$ incomparable pairs $i' || j$. Adding the min/min and the max/max pairs we obtain $(n-k+1)(n-k) + n(n-1)$ as an upper bound on $led(\mathbf{C}_n^k)$. This upper bound can be attained. For L_1 take the minimal elements of $\mathbf{C}_n^k$ in the order $0, 1, -1, 2, -2, \ldots$ and sort in the maximal elements as early as possible. When all minimal elements have been used there are k maximal elements left, depending on the parity of k we have taken the maximal elements in the order $0', 1', -1', 2', \ldots$ (k odd) or in the order $0', -1', 1', -2', \ldots$ (k even) continue this pattern for the remaining maximal elements. For L_2 begin with the reverse ordering on the minimal elements and again sort in the maximal elements as early as possible. The final k maximal elements are taken in the reverse of their order in L_1. Figure 2 illustrates the drawings of generalized crowns resulting from this process.

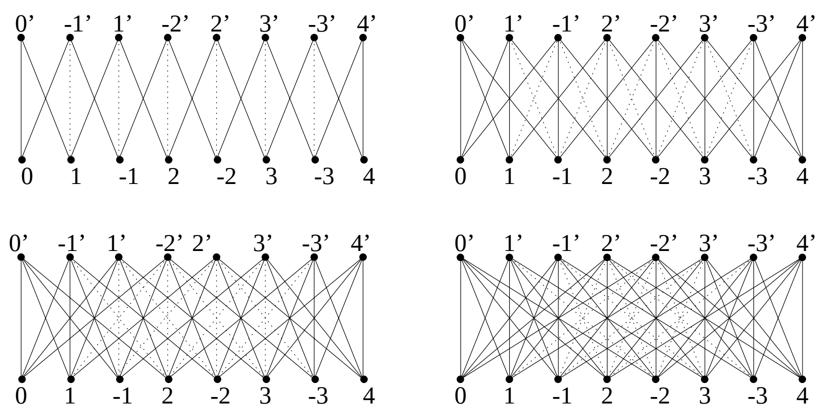

Figure 2: Drawings of the generalized crowns $\mathbf{C}_8^2, \mathbf{C}_8^3, \mathbf{C}_8^4$ and $\mathbf{C}_8^5$.
Dotted lines indicate comparabilities of minimum two-dimensional extensions.

Remark. A nice way of visualizing the construction is to use the diametral linear extensions as the row and column indices for the bipartite adjacency matrix of the $\mathbf{C}_n^k$. The results for $\mathbf{C}_n^3$ and $\mathbf{C}_n^4$ are displayed next. An entry $*$ at position (i, j') indicates that $i || j'$ in the crown but $i < j'$ in the two-dimensional extension.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} 0' & 1' & -1' & 2' & -2' & 3' & -3' & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ -1 \\ 2 \\ -2 \\ 3 \\ \vdots \end{array} &
\begin{pmatrix}
1 & 1 & 1 & & & & & \cdots \\
1 & 1 & * & 1 & & & & \cdots \\
1 & * & 1 & * & 1 & & & \cdots \\
 & 1 & * & 1 & * & 1 & & \cdots \\
 & & 1 & * & 1 & * & 1 & \cdots \\
 & & & 1 & * & 1 & * & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{cccccccc} 0' & -1' & 1' & -2' & 2' & -3' & 3' & \cdots \end{array} \\
\begin{array}{c} 0 \\ 1 \\ -1 \\ 2 \\ -2 \\ 3 \\ \vdots \end{array} &
\begin{pmatrix}
1 & 1 & 1 & 1 & & & & \cdots \\
1 & 1 & 1 & * & 1 & & & \cdots \\
1 & 1 & * & 1 & * & 1 & & \cdots \\
1 & * & 1 & * & 1 & * & 1 & \cdots \\
 & 1 & * & 1 & * & 1 & * & \cdots \\
 & & 1 & * & 1 & * & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\end{array}
$$

Theorem 9 *For each $n \geq k \geq 2$ the linear extension diameter of the generalized crown $\mathbf{C}_n^k$ is given by:*

$$led(\mathbf{C}_n^k) = 2n(n-k) + k(k-1).$$

Proof. We have shown that $(n-k+1)(n-k)+n(n-1) = 2n(n-k)+k(k-1)$ is an upper bound on $led(\mathbf{C}_n^k)$. As for the lower bound we have described a pair L_1, L_2 of linear extensions. From the above matrices it is easy to see that these two linear extensions have distance $(n-k+1)(n-k)+n(n-1)$. $\square$

Corollary 10 *For the crown $\mathbf{A}_n$ and the standard example $\mathbf{S}_n$ this gives*

- $led(\mathbf{A}_n) = 2(n-1)^2 = inc(\mathbf{A}_n) - (n-2)$ *and*
- $led(\mathbf{S}_n) = n^2 - (n-2) = inc(\mathbf{S}_n) - (n-2)$.

We now turn to the Boolean lattices. Unfortunately, we only have partial results for this seemingly simple class of posets. The goal of our investigations was a proof of the following conjecture.

Conjecture 1 *The linear extension diameter of the Boolean lattice B_n is*

$$led(B_n) = 2^{2n-2} - (n+1)2^{n-2}.$$

Proposition 11 $led(B_n) \geq 2^{2n-2} - (n+1)2^{n-2}$.

Proof. Let L be the reverse lexicographic order on the subsets of $[n]$, i.e., $A <_L B$ if the smallest element of the symmetric difference of A and B is in B. Clearly, L is a linear extension of B_n. Now revert the order on $1, .., n$ and let L' be the corresponding lexicographic order, L' is sometimes called the reverse antilexicographic order and can be described by$A <_{L'} B$ if the largest element of the symmetric difference is in B. Reverse lexicographic and antilexicographic order are hereditary, i.e., if $X \subset [n]$ then L restricted to the subsets of X is the reverse lexicographic order of these sets.

Let X be the first half of elements of L', i.e., the set of subsets of $[n]$ not containing n. and let Y be the complement of X. We count the incomparable pairs of $P_{L,L'}$ in three parts. The number of incomparable pairs (A, B) with $A \in X$ and $B \in X$ is $led(B_{n-1}) = 2^{2n-4} - n2^{n-3}$ by induction. The same is true for the pairs (A, B) with $A \in Y$ and $B \in Y$. It remains to count the incomparable pairs (A, B) with $A \in X$ and $B \in Y$, since A precedes B in L' we count pairs A, B with $n \notin A$, $n \in B$ and $B <_L A$. This number is $\binom{2^{n-1}}{2}$ since $A <_L B$ iff $A <_L B - n$. $\square$

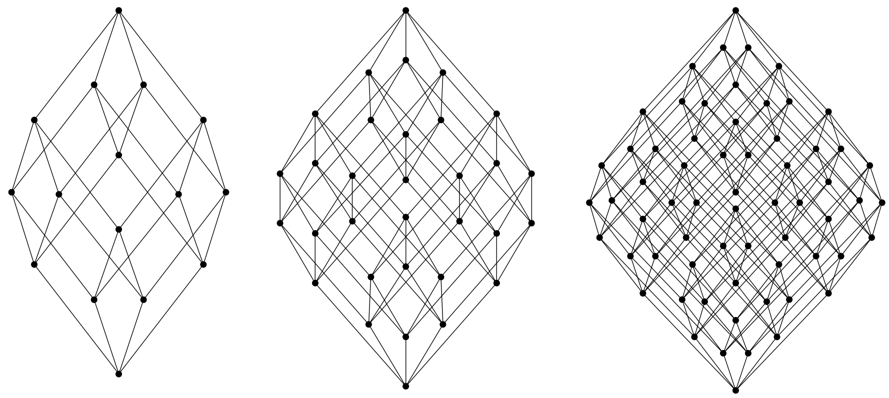

Figure 3: The drawing of B_4, B_5 and B_6 obtained from reverse lexicographic and reverse antilexicographic linear extensions.

Lemma 12 *Reverse lexicographic and reverse antilexicographic linear extensions are a diametral pair of B_n for $n \leq 4$.*

Proof. For $n \leq 3$ this is trivial. Let $n = 4$ we know that at least two of the incomparabilities of the standard example S_4 contained in B_4 are comparable in the two-dimensional poset corresponding to a diametral pair. In the standard labeling of B_4 with binary vectors we may assume that these two relations are $(0100) < (1011)$ and $(0010) < (1101)$. Let $\widetilde{B_4}$ denote the poset after addition of these two relations.

Consider the following nine induced subposets of $\widetilde{B_4}$: The first is the subposet induced by $(0001), (1000), (0110), (1001), (1110), (0111)$. The other eight are denoted $Q_{i,j}$ and are obtained by inserting i at position j in each of the vectors $(001), (010), (001), (110), (101), (011)$ for $i \in \{0, 1\}$ and $j = 1, 2, 3, 4$. Each of these 9 posets is a 3-crown and it is easily checked that no two of these crowns have a critical pair in common. It follows that in any two-dimensional extension of B_4 at least one of the 3 critical pairs of each 3-crown is comparable. This gives a total of $2 + 9$ additional comparabilities in any two-dimensional extension of B_4, i.e., $led(B_4) \leq inc(B_4) - 11 = 44$. The construction of Proposition 11 gives a two-dimensional extension of B_4 with 44 incomparabilities which is thus optimal. $\square$

We have not been able to generalize the proof of the previous lemma to the general case. There is, however, an easy property that should be true for diametral pairs that would imply the Conjecture 1. We first state the property as a conjecture. Then we prove the implication in Lemma 13. A more detailed discussion of properties of diametral pairs will be subject of the next section.

Conjecture 2 *Let L, L' be a diametral pair of a poset P then at least one of the two linear extensions L, L' reverts a critical pair of P.*

Lemma 13 *Conjecture 2 implies Conjecture 1.*

Proof. Let L, L' be a diametral pair for B_n. We may assume (Conjecture 2) that L' reverts the critical pair $(\{1, .., n-1\}, \{n\})$. As in the construction we let X and Y be the sets of the first and second half of L'. Again X is the set of subsets of $[n]$ not containing n. The number of incomparable pairs (A, B) in $P_{L,L'}$ with $A \in X$ and $B \in X$ is at most $led(B_{n-1})$. The same holds for pairs with $A \in Y$ and $B \in Y$.

It remains to estimate the number of incomparable pairs (A, B) with $A \in X$ and $B \in Y$ that are reversed by L, i.e., pairs (A, B) with $n \notin A$, $n \in B$ and $B <_L A$. Let (A, B) be such a pair and let $mate(A, B) = (B - n, A + n)$, note that $B - n \in X$ and $A + n \in Y$. Since *mate* is an involution *mate* defines a pairing of the pairs $(A, B) \in X \times Y$. At most one of (A, B) and $mate(A, B)$ can be reversed by L, otherwise, $B <_L A <_L A + n <_L B - n <_L B$ a contradiction. A pair $((A, B), mate(A, B))$ that may contribute a reversal is characterized by $A, B - n$ and these are different subsets of $[n-1]$. Therefore, the number of reversals contributed by pairs $(A, B) \in X \times Y$ is at most $\binom{|X|}{2} = \binom{2^{n-1}}{2}$. Putting things together

$$led(B_n) \leq 2led(B_{n-1}) + \binom{2^{n-1}}{2}.$$

Induction completes the proof. □

5 Intervals in $G(P)$ and Diametral Pairs

For two linear extensions M, N of P let the interval $[M, N]$ in $G(P)$ consist of all linear extensions on shortest path between M and N, put differently it is the set of linear extensions of $P_{\{M,N\}}$. We call M, N an *extremal pair* if there is no interval $[M', N']$ properly containing $[M, N]$. Note that $[M', N'] \supseteq [M, N]$ implies $dist(M', N') \geq dist(M, N)$. Hence, diametral pairs are extremal. A *locally extremal pair* is a pair M, N such that $[M, N]$ is not properly contained in $[M', N']$ with M' a neighbor of M or $M' = M$ and N' a neighbor of N or $N' = N$. Figure 4 illustrates the definitions. It is immediate that for pairs M, N of linear extensions the following implications hold

$$\textit{diametral} \Longrightarrow \textit{extremal} \Longrightarrow \textit{locally extremal}.$$

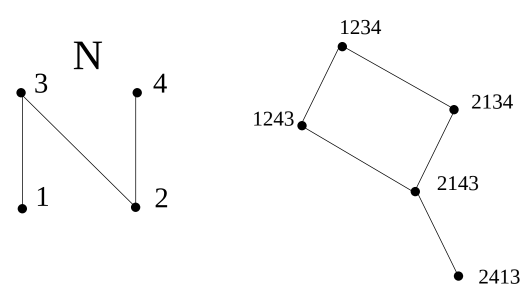

Figure 4: The N and its linear extension graph. The pair (1243,2134) is locally extremal, the unique extremal pair is (1234,2413).

Those diametral pairs we understand best are the minimal realizers of two-dimensional posets. Kierstead and Trotter [KT89] observed that the linear extensions of such a 2-realizer are super-greedy. The definition of greedy and super-greedy can be based on the following generic algorithm for linear extensions.

```
LINEAR EXTENSION
for i = 1 to n do
    choose x_i ∈ MIN(P − {x_1, .., x_{i−1}})
output x_1, x_2, ..., x_n
```

- For *greedy* linear extensions x_i is chosen from $\textsc{Min}(P - \{x_1, .., x_{i-1}\}) \cap \mathit{succ}(x_{i-1})$ whenever this set is nonempty.
- For *super-greedy* linear extensions x_i is chosen from $\textsc{Min}(P - \{x_1, .., x_{i-1}\}) \cap \mathit{succ}(x_j)$ where $j < i$ is maximal such that this set is nonempty.

Lemma 14 *Let P be a poset and L a super-greedy linear extension. Either P is a chain or L reverses a critical pair.*

Proof. We may assume that P has more then one minimal element. Let x_i be the minimal element of P that comes last in $L = x_1, \dots, x_n$. Since L is super-greedy $P - \{x_1, .., x_i\} = \mathit{succ}(x_i)$ and, hence, $\mathit{succ}(x_{i-1}) \subseteq \mathit{succ}(x_i)$. Since $\mathit{pred}(x_i) = \emptyset \subseteq \mathit{pred}(x_{i-1})$ the pair (x_i, x_{i-1}) is a critical pair reversed by L. □

5.1 Extremal Linear Extensions

Call M an *extremal linear extension* if there is a linear extension N such that there is no interval $[M', N]$ properly containing $[M, N]$. Interestingly, extremal linear extension are exactly the linear extensions participating in locally extreme pairs.

Proposition 15 *For a linear extension M the following is equivalent:*

- *M is an extremal linear extension.*
- *There exists a linear extension N such that M, N is locally extremal.*

Proof. Let M be an extremal linear extension with witness N. We define a partial order on $G(P)$ with respect to a linear extension M as follows: $L \leq_M L'$ if the set of pairs of L' which are in reverse order relative to M contains the corresponding set for L. This is equivalent to saying that the interval $[M, L']$ contains the interval $[M, L]$. If we choose N' as a maximal element above N with respect to $\leq_M$, then M, N' is a locally extremal pair. M is extremal with respect to N' because $N' \leq_{N'} N \leq_{N'} M \leq_{N'} M'$ implies $[M, N] \subseteq [M', N]$. Since, N is a witness for M's extremality this requires $M = M'$. The other direction is obvious from the definitions. □

With the next proposition we characterize extremal linear extensions. Recall that a *jump* in a linear extension $L = x_1, x_2, \dots, x_n$ is a pair x_i, x_{i+1} of consecutive elements in L that are incomparable in P. If x_i, x_{i+1} are comparable in P we call the pair a *bump* of P. The *bump decomposition* of L is obtained by cutting L in each bump. This gives an ordered partition $L = \alpha_1, \alpha_2, \ldots, \alpha_k$ such that each block α_i is a maximal interval of elements $x_{i_j}, .., x_{i_{j+1}-1}$ such that consecutive elements in α_i form a jump.

Example. Let P be the chevron labeled as in Figure 1. In $M = 132456$ there are three jumps and two bump, the bumps are (24) and (56)). The bump decomposition is $\alpha_1 = 132$, $\alpha_2 = 45$, $\alpha_3 = 6$.

Proposition 16 *A linear extension L of P is extremal iff every block α_i of the bump decomposition $\alpha_1, \alpha_2, \ldots, \alpha_k$ of L induces an antichain in P.*

Proof. Let N be such that L, N is a locally extremal pair. Assume that some block α_i does not induce an antichain and let $x, y \in \alpha_i$ with $x < y$ in P. Not all the adjacent pairs of α_i can be in reverse order to N, because this would imply $y < x$ in N. Hence some adjacent pair can be switched in α_i to increase the distance to N, a contradiction.

In order to prove the other direction let N be the word resulting from L by reversing every block of the bump decomposition of P. If all blocks induce antichains in P, then N is a linear extension of P. Moreover, L is extremal with respect to N, since only the switch of an adjacent pair of some block yields a neighboring linear extension of L. But such a linear extension is closer to N as L is. $\square$

Corollary 17 *Every greedy linear extension is extremal.*

Proof. If L is not extremal, then there exist x, y in some block α_i of L with x being covered by y in P. Observe that x and y cannot be adjacent in α_i. Now, L is not greedy, since y is a candidate to be chosen right after x. $\square$

In general, however, the class of extremal linear extensions contains non-greedy linear extensions. Even both linear extensions of a locally extremal pair may be non-greedy. Take for example the 3-crown $\mathbf{C}_3^2$ on $\{0,1,2,0',1',2'\}$ (element i' is larger then $i, i-1$) the pair $(2,1,0,0',2',1')$, $(0,1,2,1',2',0')$ is extremal but neither is greedy. Due to their vast amount extremal pairs seem to be rather useless for heuristics or approximations of the linear extension diameter. In the next subsection we discuss a much stronger property.

5.2 Complementary Linear Extensions

Let L be a linear extension of P and specify the choice function in Algorithm Linear Extension so that in each round x_i is the last element of Min(P) in L, i.e., take the reverse of L as preference list for the construction of a new linear extension M. We call M the *complementary linear extension of* L and denote the complementary mapping by $*$, i.e., $* : L \to M = L^*$. The k fold iterated complementary map of L is L^{*k}.

Example. Let P be the chevron labeled as in Figure 1. If $L = 132456$ then $L^* = 315624$.

The intuition is that L^* tends to have many pairs in the reverse order of L, hence, the distance from L to L^* should be large.

Proposition 18 *Complementary linear extensions are super-greedy.*

Proof. Let $y_1, .., y_t$ be an initial segment of L^*. For element $x \in \textsc{Min}(P - \{y_1, .., y_t\})$ let $i(x) = \max(i : x > y_i)$. We have to prove that y_{t+1} is an element x' with $i(x')$ maximal. Suppose not, $y_{t+1} = x'$ but $i(x') = r < i(x) = s$. The choice of x' implies that $x <_M x'$. Consider the situation when y_s was chosen and note that at this time x' was available. Since $y_s < x$ we have $y_s <_M x'$ contradicting the choice of y_s. $\square$

Corollary 19 *For linear extensions the following implications hold*

$$\textit{complementary} \Longrightarrow \textit{super-greedy} \Longrightarrow \textit{greedy} \Longrightarrow \textit{extremal.}$$

As it is the case with super-greedy linear extensions complementary linear extensions may be constructed by an algorithm based on a stack. To construct the complementary linear extension of L begin with an empty stack S. Push the elements of $\textsc{Min}(P)$ onto S in the order induced by L on this set. For $i = 1, .., n$ repeat: $x_i \leftarrow \text{pop}(S)$ and push the new minimal elements, i.e., the elements of the set $C_i = \textsc{Min}(P - \{x_1, .., x_i\}) - \textsc{Min}(P - \{x_1, .., x_{i-1}\})$ onto S. The order in which elements of C_i are pushed is again the order induced by L on this set. The complementary linear extension L^* of L is $x_1, \ldots, x_n$, i.e., the elements ordered by the time of their pop. The formal proof that the stack algorithm applied to L constructs the complementary linear extension L^* is very similar to the proof of Proposition 18.

We illustrate the two procedures for complementary linear extensions with the following example (Table 1). Let P be the chevron with the labeling of Figure 1 and let $L = 132456$. In the left column of the table we have L with elements already used for L^* removed. Underlined elements are the elements of $\textsc{Min}(P - \{x_1, .., x_{i-1}\})$ and bold are the elements of C_i, i.e., the new minimal elements. The next three columns correspond to the stack based construction and explain themselves. Finally, there is a column with the growing L^*. We like to remark that yet another way of interpreting the construction of L^* is as a certain depth-first-search on the diagram of P with a least element 0 added. The corresponding spanning tree consist of the edges (x_i, y) for $y \in C_i$.

L	Stack	pop	C_i	L^*
$\underline{\mathbf{1}}\,\underline{\mathbf{3}}\,2\,4\,5\,6$	13	3	$\emptyset$	3
$\underline{1}\;\;2\,4\,5\,6$	1	1	$\{2, 5\}$	3 1
$\underline{\mathbf{2}}\,4\,\underline{\mathbf{5}}\,6$	25	5	$\{6\}$	3 1 5
$\underline{2}\,4\;\;\underline{\mathbf{6}}$	26	6	$\emptyset$	3 1 5 6
$\underline{2}\,4$	2	2	$\{4\}$	3 1 5 6 2
$\underline{\mathbf{4}}$	4	4	$\emptyset$	3 1 5 6 2 4

Table 1: Demonstrating the construction of a complementary linear extension.

A *complementary pair* is a pair L, M of linear extensions with $M = L^*$ and $L = M^*$. Continuing with the example $L = 132456$ we saw $L^* = 315624$ and compute $L^2 = 125346$ and $L^{*3} = 315624$. Since $L^{*3} = L^*$ the pair L^*, L^{*2} is a complementary pair. In this case it is a diametral pair as well.

Proposition 20 *A realizer L, L' of a two-dimensional poset is a complementary pair.*

Proof. In L' the elements of $\textsc{Min}(P)$ are in the reverse of their order in L. Therefore, L' and L^* are equal in the first element x. Since $L^* = x + (L - x)^*$ and $L - x, L' - x$ is a realizer of $P - x$ induction shows $L' = L^*$. □

From the definition it is not obvious that every poset has a complementary pair this, however, is an immediate consequence of the following 'convergence' theorem.

Theorem 21 *Let P be a poset of height h and L be a linear extension then $L^{*2h-1} = L^{*2h+1}$, in other words L^{*2h-1}, L^{*2h} is a complementary pair of P.*

The proof of the theorem will be based on two lemmas.

Lemma 22 *Let I be a down-set of P. The complementary linear extension of the restriction of L to the suborder induced by P on I equals the restriction of L^* to I. With $L|X$ denoting the restriction of L to a subset X of P this can be written as $(L|I)^* = L^*|I$.*

Proof. The proof is by induction on $n = |P|$. Let x be the last minimal element of P in L and note that x is the first element of L^*. Consider $P - x$. With $M = L|(P - x)$ we have $L^* = xM^*$.

If $x \notin I$ then $M|I = L|I$ and

$$L^*|I = M^*|I = (M|I)^* = (L|I)^*$$

with the second equality being the induction hypothesis. Else, if $x \in I$ then

$$L^*|I = xM^*|(I - x) = x(M|(I - x))^* = (L|I)^*$$

with the second equality being the induction hypothesis. □

Lemma 23 *Let P be a poset, $A \subseteq \textsc{Max}(P)$ and $Q = P - A$. If L is a linear extension of P with $L^*|Q = L^{*3}|Q$ then $L^{*3} = L^{*5}$.*

Proof. For $t \geq 1$ let $L^{*t} = x_1^t, x_2^t, \ldots, x_n^t$ and use the superscript t to denote structures involved in the stack based construction of L^{*t}. For example the elements of the set $C_i^t = \textsc{Min}(P - \{x_1^t, .., x_i^t\}) - \textsc{Min}(P - \{x_1^t, .., x_{i-1}^t\})$ are the elements pushed onto stack S^t after the pop of x_i^t.

By Lemma 22 $L^*|Q = L^{*3}|Q$ implies that $L^*|Q, L^{*2}|Q$ is a complementary pair for Q. If $x_i^t \notin Q$ then obviously $C_i^t = \emptyset$. Hence, for t, t' of the same parity (both odd or both even) the same sets are pushed in the same order onto the stacks S^t and $S^{t'}$. More formally, if q_i^t denotes the index of the ith element of Q in L^{*t} then $C^t_{q_i^t} = C^{t'}_{q_i^{t'}}$ for $t = t' \bmod 2$ and $1 \leq i \leq |Q|$. Using the simplified notation $\mathcal{C}_i^t = C^t_{q_i^t}$ (with calligraphic $\mathcal{C}$) we restate this fact.

FACT. $\mathcal{C}_i^t = \mathcal{C}_i^{t'}$ for $t = t' \bmod 2$ and $1 \leq i \leq |Q|$.

The linear extension L^{*t} is completely determined by the evolution of the stack S^t. From $\mathcal{C}_i^t = \mathcal{C}_i^{t'}$ we could conclude that L^{*t} only depends on the parity of t if the order in which the elements of $\mathcal{C}_i^t$ are pushed onto S^t remained unchanged or equivalently if the order of the elements of $\mathcal{C}_i^t$ in L^{*t} remained unchanged. This will be proved for $t \geq 3$.

Let $D_{ij} = \mathcal{C}_i^1 \cap \mathcal{C}_j^2 = \mathcal{C}_i^o \cap \mathcal{C}_j^e$ for o odd and e even and note that there is an order α_{ij} of the elements of D_{ij} such that in the sequence L^{*t} the order of these elements alternates between α_{ij} for t odd and the reverse of α_{ij} for t even.

CLAIM. Let $j < k$ and $y \in D_{ij}$, $x \in D_{ik}$. For $t \geq 3$, t odd, x precedes y in L^{*t}.

Proof of Claim. Assume the existence of $o \geq 3$ odd such that y precedes x in L^{*o}, we shorten notation writing $y <_o x$ for this fact. Since $x, y \in \mathcal{C}_i^o$ we conclude that $x <_{o-1} y$. Let $e = o - 1$ and recall $j < k$ and $y \in \mathcal{C}_j^e$ and $x \in \mathcal{C}_k^e$. Hence, y was pushed onto stack S^e earlier then x and since $x <_e y$ element y was still buried in S^e when x was pushed. Inspection shows that there was a $z \in \mathcal{C}_j^e$ with $z < x$ and z was pushed after y onto S^e. It follows that the order of x, y, z in L^{*e-1} is $y <_{e-1} z <_{e-1} x$.

From $x, y \in \mathcal{C}_i^{e-1} = \mathcal{C}_i^o$ and $y <_{e-1} x$ we obtain that x was pushed before y onto S^{e-1}. Since $z < x$ element z was pushed onto S^{e-1} before x and y.

To obtain $y <_{e-1} z <_{e-1} x$ the stack S^{e-1} would thus get the elements pushed in order z, x, y and pop them off in order y, z, x. This, however, corresponds to a 3-element permutation that cannot be realized with a stack. This contradiction concludes the proof of the claim. △

It follows that for $t \geq 3$, t odd the order of the elements of $\mathcal{C}_i^o$ in L^{*t} is $\alpha_{i,n-1} <_t \alpha_{i,n-2} <_t \ldots <_t \alpha_{i,1}$. This completely determines the evolution of the stack, hence, $L^{*3} = L^{*5} = L^{*7} \ldots$. □

Proof (Theorem 21). Let $A_1, A_2, \ldots, A_h$ be the canonical antichain partition of P with *height*$(P) = h$, i.e., $A_{i+1} = \text{MIN}(P - A_1 - \ldots - A_i)$ and $\bigcup_1^h A_i = P$. Let $A_{\leq k} = A_1 \cup A_2 \cup \ldots \cup A_k$ and note that $A_{\leq k}$ is a down-set.

CLAIM. $L^{*2k-1}|A_{\leq k} = L^{*2k+1}|A_{\leq k}$ for $k = 1, \ldots, h$.

Proof of Claim. By Lemma 22 it suffices to prove $(L|A_{\leq k})^{*2k-1} = (L|A_{\leq k})^{*2k+1}$.

For $k = 1$ this is trivially true. Since $A_k \subseteq \text{MAX}(A_{\leq k})$ we can use Lemma 23 with $L = L^{*2k-4}|A_{\leq k}$ for the induction step. △

Since $A_{\leq h} = P$ this implies the theorem. □

Proposition 24 *If M, N is a complementary pair, then the interval $[M, N]$ is locally extreme in $G(P)$.*

Proof. Assume that there is neighbor N' of N such that $[M, N] \subset [M, N']$. Let (x, y) be the unique pair with $x <_N y$ and $y <_{N'} x$. Since $N = M^*$ and both x and y were minimal elements when x was chosen we find that $y <_M x$.

This implies that N' is on a shortest path from M to N, a contradiction to $[M,N] \subset [M,N']$. Similar arguments disprove the other cases. $\square$

A diametral pair need not be a complementary pair. An example is given in Figure 5.

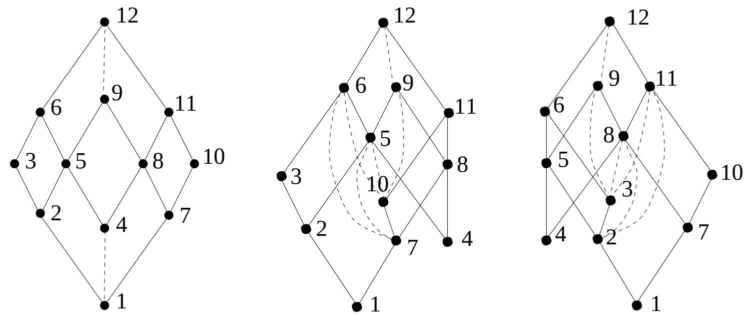

Figure 5: Left: P and its unique minimum two-dimensional extension.
Middle and right: The two complementary two-dimensional extensions of P.

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