SERIE B — INFORMATIK

Isoperimetric Inequalities for Densities of Lattice-periodic Sets

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B 97-05
June 1997

Abstract

The minimum boundary length density of a lattice-periodic set with given period-lattice and area density is determined, together with the extremal sets, and a conjecture on the higher-dimensional analogue is made. This improves previous results of Hadwiger for d-dimensional sets with integer period lattice and of Schnell and Wills for twodimensional sets with arbitrary period lattice.

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by

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Abstract. The minimum boundary length density of a lattice-periodic set with given period lattice and area density is determined, together with the extremal sets, and a conjecture on the higher-dimensional analogue is made. This improves previous results of Hadwiger for Z^d-periodic d-dimensional sets and of Schnell and Wills on twodimensional sets with arbitrary period-lattice.

1. Introduction

A set $X \subset \mathbb{R}^d$ is lattice-periodic with period lattice Γ iff the lattice translations from Γ are symmetries of X, that is $X + \gamma = X$ for each $\gamma \in \Gamma$. Figure 1 shows a periodic set and the fundamental parallelograms of its period lattice.

Any nonempty such set X is necessarily unbounded, but if it is reasonably well-formed (e.g. is locally the limit of polyhedral approximations) we can define its volume density $v(X)$ and perimeter density $p(X)$ as the average amounts of X and ∂X per unit volume of $\mathbb{R}^d$. The isoperimetric problem for lattice-periodic sets then is to determine the minimum perimeter density of a set with given period lattice and volume density. Since we may exchange X and its complement without changing the perimeter density, the minimum perimeter density is the same for the volume densities v and $1 - v$.

This problem was first studied by Hadwiger [1] for the case of Z^d-periodic sets; he proved the lower bound $p(X) \geq 8v(X)\big(1 - v(X)\big)$, which remarkably holds independent of the dimension d. This bound is sharp only for $v(X) \in \{0, \frac{1}{2}, 1\}$ and holds only for the special period lattice Z^d.

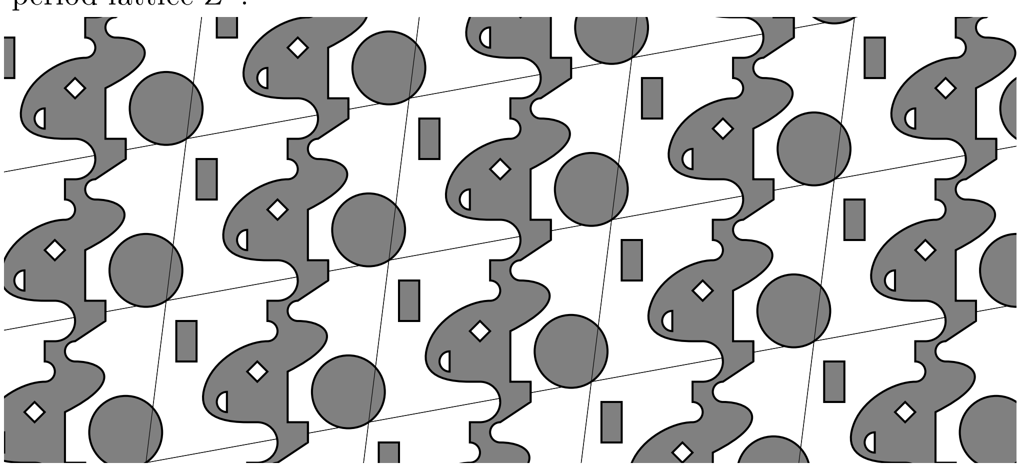

Figure 1.

1991 Mathematics Subject Classification: 51M16

Key words: isoperimetric inequalities, periodic sets

In the two-dimensional case Schnell and Wills [4] extended this bound to arbitrary period lattices $\Gamma \subset \mathbb{R}^2$, proving

$$p(X) \geq 8 \, \frac{\min\Big\{\|\gamma\| \mid \gamma \neq 0, \gamma \in \Gamma\Big\}}{\det(\Gamma)} \; v(X)\Big(1 - v(X)\Big).$$

In this two-dimensional case $p(X)$ denotes the boundary length density and $v(X)$ the area density of X. Again the bound is sharp only for $v(X) \in \{0, \frac{1}{2}, 1\}$. For higher dimensions d and arbitrary period lattices $\Gamma \subset \mathbb{R}^d$ Schnell [3] proved

$$p(X) \geq 2d^{-\frac{3}{2}} \, \frac{\min\Big\{\det(\hat{\Gamma}) \mid \begin{array}{l}\hat{\Gamma} \text{ is a } (d-1)\text{-dimensional} \\ \text{sublattice of } \Gamma\end{array}\Big\}}{\det(\Gamma)} \; v(X)\Big(1 - v(X)\Big).$$

Using the last successive minimum $\lambda_d(\Gamma)$ of Γ instead of the minimal subdeterminants Schnell [2] found the bound

$$p(X) \geq \frac{8}{\det(\Gamma)\lambda_d(\Gamma)} \; v(X)\Big(1 - v(X)\Big).$$

2. Results and Conjectures

Let Γ denote a lattice in $\mathbb{R}^d$ and $X \subset \mathbb{R}^d$ a Γ-periodic set. If X is well-formed in the sense that volume and surface area are defined for some finite section (e.g. with a ball) which is big enough to contain a fundamental parallelotope of Γ, then we can define volume and perimeter densities of X. For this we can use the limits of the intersection with large balls $v(X) := \lim_{r\to\infty} \frac{\mathrm{vol}(X\cap B_r)}{\mathrm{vol}(B_r)}$ and $p(X) := \lim_{r\to\infty} \frac{\mathrm{area}(X\cap B_r)}{\mathrm{vol}(B_r)}$. Equivalently we may select a fundamental parallelotope P such that the intersection of the boundaries $\partial P \cap \partial X$ has $(d-1)$-dimensional measure zero and define $v(X) := \frac{\mathrm{vol}(X\cap P)}{\mathrm{vol}(P)}$ and $p(X) := \frac{\mathrm{area}(\partial X\cap P)}{\mathrm{vol}(P)}$. In the two-dimensional case considered in the theorem v and p denote the area- and boundary-length densities of X.

Theorem : Let $X \subset \mathbb{R}^2$ be a Γ-periodic set for which $v(X)$ and $p(X)$ are defined. Then

$$p(X) \geq 2 \min\left(\sqrt{\tfrac{\pi}{\det(\Gamma)} v(X)}\,,\; \frac{\min\{\|\gamma\| \mid \gamma\neq 0, \gamma\in\Gamma\}}{\det(\Gamma)}\,,\; \sqrt{\tfrac{\pi}{\det(\Gamma)}\Big(1 - v(X)\Big)}\right)$$

This lower bound is reached for each given v by one of the following sets

1) $X = B_r + \Gamma$ with $r = \sqrt{\frac{\det(\Gamma)}{\pi} v}$, that is the union of circular discs, one per fundamental domain,
2) $X = \mathbb{R}\gamma + \{t\hat{\gamma} \mid t \in \mathbb{R}, (t - \lfloor t \rfloor) \leq v\}$, where γ is the minimum-norm nonzero vector of Γ and $\hat{\gamma}$ another element of Γ such that $\gamma, \hat{\gamma}$ generate a fundamental parallelogramm,
3) $X = \mathbb{R}^2 \setminus \Big(B_r + \Gamma\Big)$ with $r = \sqrt{\frac{\det(\Gamma)}{\pi}(1 - v)}$, that is the plane with circular holes, one per fundamental domain.

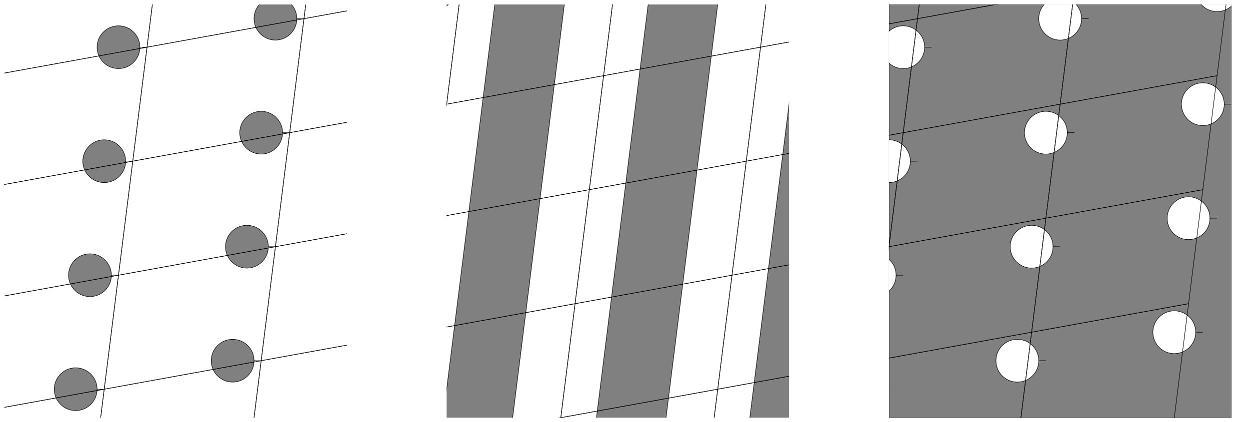

Figure 2.

We believe that this structure of the extremal sets holds also in higher dimensions. Using κ_i for the volume of the i-dimensional ball of radius 1 and $\det_i(\Gamma)$ for the minimum determinants of i-dimensional sublattices of Γ we can state the

Conjecture: Let $X \subset \mathbb{R}^d$ be a Γ-periodic set for which $v(X)$ and $p(X)$ are defined. Then

$$p(X) \geq \min_{i=1,\dots,d} i \left(\frac{\det_{d-i}(\Gamma)\,\kappa_i}{\det_d(\Gamma)}\right)^{\frac{1}{i}} \Big(\min(v(X), 1 - v(X))\Big)^{1-\frac{1}{i}}.$$

Probably the same structure holds even if the period lattice is not fulldimensional: The sets of minimal perimeter with given volume per fundamental domain consist of i-dimensional cylinders over $d-i$-dimensional balls, the cylinders taken in subspaces where the determinant of the sublattice is minimal, or the complement of such a set.

3. Proof

Let X be a set with the given period lattice Γ and volume density v which is of minimum perimeter density among all such sets. We may assume X to be closed. Let $\{X_\iota\}_{\iota\in I}$ be the connected components of $X = \bigcup_{\iota\in I} X_\iota$, and let $\Lambda(X_\iota)$ be the maximal sublattice of Γ that leaves X_ι invariant ($X_\iota + \Lambda(X_\iota) = X_\iota$).

If one of the $\Lambda(X_\iota)$ is nontrivial, i.e. there is a nonzero vector $t \in \Lambda(X_\iota)$, then X_ι contains with each point $p \in X_\iota$ the whole one-dimensional point-lattice $(p + zt)_{z\in Z}$. Since X_ι is connected, there is a (shortest) arc α_p joining p and $p+t$ in X, and this arc stays within a bounded distance to the line through p and $p+t$. We can extend this arc periodically to a set $\bigcup_{z\in Z}(\alpha_p + zt) \subset X_\iota$ which is connected, contains all the lattice points $(p+zt)_{z\in Z}$, and stays within a bounded distance to the line $(p+rt)_{r\in\mathbb{R}}$. So there is a pseudoline $\alpha_\iota \subset X_\iota$ which stays within a bounded distance to the line $(p+rt)_{r\in\mathbb{R}}$ and which cuts the plane in two halves. If there is another connected component X_j which also has nontrivial $\Lambda(X_j)$, then the associated lines may not intersect (being in distinct connected components); so each nonzero vector $s \in \Lambda(X_j)$ must be collinear with each nonzero vector $t \in \Lambda(X_\iota)$.

Therefore either there is one $\Lambda(X_\iota)$ that is two-dimensional and all other X_j have a trivial $\Lambda(X_j)$, or all nontrivial $\Lambda(X_\iota)$ generate the same 1-dimensional subspace (i.e. are collinear). Since X is Γ-periodic, we have $\Gamma/\Lambda(X_\iota)$ copies of connected component X_ι in X. So if there is an X_ι with two-dimensional $\Lambda(X_\iota)$, we have $\Lambda(X_\iota) = \Gamma$, for otherwise there would be further connected components X_j with twodimensional $\Lambda(X_j)$. And if $\Lambda(X_\iota)$ is one-dimensional, it must be generated by a primitive lattice vector of Γ, for otherwise there are several collinear copies of X_ι, which contain alternating points of the same one-dimensional sublattice, and which are translates of each other, so they intersect.

So there are three possible cases: either all connected components have trivial Λ, or there are connected components with a one-dimensional Λ which is generated by a primitive vector of Γ, or there is one component with $\Lambda = \Gamma$, and the complement consists of bounded sets with trivial Γ.

If there is a component X_ι with one-dimensional Λ which is generated by $v \in \Gamma$, then $p(X) \geq 2\frac{\|v\|}{\det(\Gamma)}$. For let $w \in \Gamma$ be a vector such that $\{v, w\}$ generates Γ (this exists, since v is primitive). Then $(zw + X_\iota)_{z\in\mathbf{Z}}$ are further connected components of X. For each $m, n \in \mathbb{N}$ the parallelogram $0, mv, mv + nw, nw$ is intersected by $n + O(1)$ copies of X_ι, each of which (with the exception of the first and last $O(1)$ copies) has a boundary length of at least $2m\|v\|$ within this parallelogram (which is of area $mn\det(\Gamma)$. So the density contributed by the translates of X_ι is at least $2\frac{\|v\|}{\det(\Gamma)}$ for some nonzero $v \in \Gamma$. Since the set has minimal $p(X)$, it must be at least as good as the parallel strips construction of the theorem, so v is a vector of minimum length, and there are no other connected components in X (which could only increase $p(X)$).

So we may restrict us to the case that no X_ι has one-dimensional Λ. Exchanging X and $\mathbb{R}^2 \setminus X$ one sees that the other two cases are symmetric. Therefore we consider only the first case, i.e. all connected components have trivial Λ. Then we can partition the connected components into equivalence classes by Γ; we select one element $(\hat{X}_\kappa)_{\kappa\in K}$ of each equivalence class. Since there is one copy of each $\hat{X}_\kappa$ per each fundamental domain of Γ, we have $v(X) = \frac{1}{\det(\Gamma)}\sum_{\kappa\in K}\mathrm{vol}(\hat{X}_\kappa)$ and $p(X) = \frac{1}{\det(\Gamma)}\sum_{\kappa\in K} p(\hat{X}_\kappa)$. For each $\hat{X}_\kappa$ we may apply the standard isoperimetric inequality, giving $p(\hat{X}_\kappa) \geq 2\sqrt{\pi\,\mathrm{vol}(\hat{X}_\kappa)}$, so we get

$$p(X) \geq \frac{1}{\det(\Gamma)}\sum_{\kappa\in K} 2\sqrt{\pi\,\mathrm{vol}(\hat{X}_\kappa)} \geq \frac{2}{\det(\Gamma)}\sqrt{\pi\sum_{\kappa\in K}\mathrm{vol}(\hat{X}_\kappa)} = 2\sqrt{\frac{\pi}{\det(\Gamma)}v(X)}.$$

The same lower bound, with $1 - v(X)$ instead of $v(X)$, holds in the last case (one X_ι with $\Lambda(X_\iota) = \Gamma$), in which we exchanged X and $\mathbb{R}^2 \setminus X$. The lower bound of the theorem is now the minimum of the three possibilities for the extremal sets. This proves the theorem.

4. References

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