DIMENSION, GRAPH AND HYPERGRAPH COLORING

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ABSTRACT. There is a natural way to associate with a poset $\mathbf{P}$ a hypergraph $\mathbf{H}_\mathbf{P}$, called the hypergraph of incomparable pairs, so that the dimension of $\mathbf{P}$ is the chromatic number of $\mathbf{H}_\mathbf{P}$. The ordinary graph $\mathbf{G}_\mathbf{P}$ of incomparable pairs determined by the edges in $\mathbf{H}_\mathbf{P}$ of size 2 can have chromatic number substantially less than $\mathbf{H}_\mathbf{P}$. We give a new proof of the fact that the dimension of $\mathbf{P}$ is 2 if and only if $\mathbf{G}_\mathbf{P}$ is bipartite. We also show that for each $t \geq 2$, there exists a poset $\mathbf{P}$ for which the chromatic number of the graph of incomparable pairs is t, but the dimension of $\mathbf{P}$ is at least $(3/2)^{t-1}$. However, it is not known whether there is a function $f : \mathbb{R} \to \mathbb{R}$ so that if $\mathbf{P}$ is a poset and the graph of incomparable pairs has chromatic number at most t, then the dimension of $\mathbf{P}$ is at most $f(t)$.

1. INTRODUCTION

There are many interesting analogies between dimension theory for finite partially ordered sets (posets) and chromatic number for finite graphs. In addition, researchers have quite frequently applied results and techniques from graph theory to research problems for posets. For example, the fact that there exist graphs with large girth and large chromatic number has been used to show that there exist posets with large dimension and large girth. As a second example, the dimension of interval orders is closely linked to the chromatic number of double shift graphs (see Füredi, Hajnal, Rödl and Trotter [3]). As a third example, Yannakakis [9] used a connection with graph coloring to show that the question of determining whether the dimension of a poset is at most t is NP-complete for every $t \geq 3$.

In this paper, we study a very natural connection between dimension and chromatic number. With a finite poset $\mathbf{P}$, we will associate a hypergraph $\mathbf{H}_\mathbf{P}$ so that the dimension of $\mathbf{P}$ is equal to the chromatic number of $\mathbf{H}_\mathbf{P}$. This hypergraph is called the *hypergraph of incomparable pairs*. The edges of size 2 in $\mathbf{H}_\mathbf{P}$ determine an ordinary graph $\mathbf{G}_\mathbf{P}$, which is called the *graph of incomparable pairs*.

It is natural to ask whether there is any relationship between the dimension of a poset and the chromatic number of its graph of incomparable pairs. The answer is yes—at least when the graph is bipartite. The following theorem was first proved by Doignon, Ducamp and Falmagne [1] using a variant of dimension based on the concept of Ferrer's relations. In Section 5, we will give a new proof of this result using only familiar concepts in dimension theory.

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Theorem 1.1. *Let* $\mathbf{G_P}$ *be the graph of incomparable pairs of a poset* $\mathbf{P}$ *which is not a total order. Then the dimension of* P *is* 2 *if and only if the chromatic number of* $\mathbf{G_P}$ *is* 2.

When the graph of incomparable pairs of a poset $\mathbf{P}$ is not bipartite, the dimension of $\mathbf{P}$ can be much larger. In Section 4, we will construct for each $t \geq 2$ a poset $\mathbf{P}_t$ for which the chromatic number of the graph of incomparable pairs is t. However, the dimension of $\mathbf{P}$ will be at least $(3/2)^{t-1}$.

As a consequence, it is natural to pose the following question.

Question 1.2. *Does there exist a function* $f : \mathbb{R} \to \mathbb{R}$ *so that if* $\mathbf{P}$ *is a poset and the graph of incomparable pairs has chromatic number at most* t*, then the dimension of* $\mathbf{P}$ *is at most* $f(t)$.

If such a function exists, then our example shows that it must grow fairly rapidly, at least exponentially. However, we tend to believe that there is no such function. In particular, we believe that there exist posets of arbitrarily large dimension for which the graph of incomparable pairs is 3-colorable.

2. Notation and Background Material

Throughout this paper, we consider a *partially ordered set* (or *poset*) $\mathbf{P} = (X, P)$ as a structure consisting of a set X and a reflexive, antisymmetric and transitive binary relation P on X. We call X the *ground set* of the poset $\mathbf{P}$, and we call P a *partial order* on X. The notations $x \leq y$ in P, $y \geq x$ in P and $(x, y) \in P$ are used interchangeably, and the reference to the partial order P is often dropped when its definition is fixed throughout the discussion. We write $x < y$ in P and $y > x$ in P when $x \leq y$ in P and $x \neq y$. When $x, y \in X$, $(x, y) \notin P$ and $(y, x) \notin P$, we say x and y are incomparable and write $x \| y$ in P. When $\mathbf{P} = (X, P)$ is a poset, we call the partial order $P^d = \{(y, x) : (x, y) \in P\}$ the *dual* of P and we let $\mathbf{P}^d = (X, P^d)$.

A partial order P on a set X is called a *linear order* (also, a *total order*) when no two distinct points of X are incomparable. If P and Q are partial orders on the same ground set, we say Q is an *extension* of P if $P \subseteq Q$, and we call Q a *linear extension* of P if Q is a linear order and it is also an extension of P.

If $\mathcal{R}$ is a family of linear extensions of P, we call $\mathcal{R}$ a *realizer* of P if $P = \cap\mathcal{R}$, i.e., for all $x, y \in X$, $x \leq y$ in P if and only if $x \leq y$ in L for every $L \in \mathcal{R}$. The *dimension* of the poset $\mathbf{P} = (X, P)$, denoted $\dim(\mathbf{P})$ or $\dim(X, P)$, is the least positive integer t so that P has a realizer $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of cardinality t. In this article, we will need only a few basic facts about dimension, but the interested reader is referred to Trotter's monograph [4] and survey articles [5], [6] and [7] for additional information.

Assuming some basic familiarity with concepts for posets such as chains, antichains, cartesian products and disjoint sums, we summarize some elementary properties of dimension in the following propositions, referring the reader to [4] for proofs and references.

Proposition 2.1. *Let* $\mathbf{P} = (X, P)$ *and* $\mathbf{Q} = (Y, Q)$ *be posets. Then:*

1. $\dim(\mathbf{P} + \mathbf{Q}) = \max\{2, \dim(\mathbf{P}), \dim(\mathbf{Q})\}$.
2. $\dim(\mathbf{P} \times \mathbf{Q}) \leq \dim(\mathbf{P}) + \dim(\mathbf{Q})$*, with equality holding if* $\mathbf{P}$ *and* $\mathbf{Q}$ *have greatest and least elements.*
3. *The removal of a point from* $\mathbf{P}$ *decreases* $\dim(\mathbf{P})$ *by at most one.*

4. *If A is a maximum antichain in* $\mathbf{P}$*, then* $\dim(\mathbf{P}) \leq |A|$ *and* $\dim(\mathbf{P}) \leq \max\{2, |X - A|\}$.
5. *If A is a maximal antichain in* $\mathbf{P}$ *and* $X - A \neq \emptyset$*, then* $\dim(\mathbf{P}) \leq 1 + 2\,\text{width}\big(X - A, P(X - A)\big)$.
6. *If A is the set of maximal elements of* $\mathbf{P}$ *and* $X - A \neq \emptyset$*, then* $\dim(\mathbf{P}) \leq 1 + \text{width}\big(X - A, P(X - A)\big)$.
7. $\dim(\mathbf{P}) = \dim(\mathbf{P}^d)$. □

Let $\mathbf{P} = (X, P)$ be a poset, and let $\mathcal{F} = \{\mathbf{Q}_x = (Y_x, Q_x) : x \in X\}$ be a family of posets indexed by the elements of X. Define the *lexicographic sum* of $\mathcal{F}$ over $\mathbf{P}$, denoted $\sum_{x \in \mathbf{P}} \mathcal{F}$, as the poset $\mathbf{Q} = (Y, Q)$ where $Y = \{(x, y) : x \in X, y \in Y_x\}$ and $(x_1, y_1) < (x_2, y_2)$ in Q if and only if $x_1 < x_2$ in P, or if both $x_1 = x_2$ and $y_1 < y_2$ in Q_{x_1}. With this definition, a disjoint sum is just a lexicographic sum over a two-element antichain.

Here is the general formula for dimension and lexicographic sums (see [4]).

Proposition 2.2. *Let* $\mathbf{P} = (X, P)$ *be a poset, and let* $\mathcal{F} = \{\mathbf{Q}_x = (Y_x, P_x) : x \in X\}$ *be a family of posets. Then*

$$\dim\Big(\sum_{x \in \mathbf{P}} \mathcal{F}\Big) = \max\{\dim(\mathbf{P}), \max\{\dim(\mathbf{Q}_x) : x \in X\}\}. \tag{1}$$

□

A lexicographic sum $\sum_{x \in \mathbf{P}} \mathcal{F}$ is *trivial* if either $\mathbf{P}$ has only one point, or every poset in $\mathcal{F}$ is a one point poset; otherwise the sum is *non-trivial*. A poset is *decomposable* if it is isomorphic to a non-trivial lexicographic sum; otherwise it is *indecomposable*. A poset is *t-irreducible* if it has dimension t but the removal of any point leaves a subposet of dimenson $t - 1$ (this is the analogue of a critical graph). Finally, a poset is *irreducible* if it is t-irreducible for some $t \geq 2$. Evidently, every irreducible poset is indecomposable, a fact which will be exploited later.

Given a poset $\mathbf{P} = (X, P)$, let $\text{inc}(\mathbf{P}) = \{(x, y) \in X \times X : x \| y \text{ in } P\}$. Then let L be a linear extension of P. We say L *reverses* the incomparable pair (x, y) when $x > y$ in L. Let $S \subset \text{inc}(\mathbf{P})$. We say that L *reverses* S when $x > y$ in L, for every $(x, y) \in S$. Finally, if $\mathcal{R}$ is a family of linear extensions of P and $S \subset \text{inc}(\mathbf{P})$, we say $\mathcal{R}$ *reverses* S if each pair of S is reversed by some L in $\mathcal{R}$.

Note that a family $\mathcal{R}$ of linear extensions of P is a realizer of P if and only if for every $(x, y) \in \text{inc}(\mathbf{P})$, there exists $L \in \mathcal{R}$ so that $x > y$ in L, i.e., $\mathcal{R}$ is a realizer of P if and only if it reverses the set of all incomparable pairs. For this reason, it is convenient to have a test which determines whether there is a linear extension reversing a given subset $S \subset \text{inc}(\mathbf{P})$.

For an integer $k \geq 2$, a subset $S = \{(x_i, y_i) : 1 \leq i \leq k\} \subset \text{inc}(\mathbf{P})$ is called an *alternating cycle* when $x_i \leq y_{i+1}$ in P, for all $i = 1, 2, \ldots, k$. In this last definition, the subscripts are interpreted cyclically, i.e., $y_{k+1} = y_1$. An alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ is *strict* if $x_i \leq y_j$ in P if and only if $j = i + 1$, for all $i, j = 1, 2, \ldots, k$. When an alternating cycle is strict, the following three statements hold:

1. The elements in $\{x_1, x_2, \ldots, x_k\}$ form a k–element antichain.
2. The elements in $\{y_1, y_2, \ldots, y_k\}$ form a k–element antichain.
3. If $i, j \in [k]$ and x_i is comparable to y_j, then $j = i + 1$.

In Figure 2, we show an alternating cycle of length 4 while Figure 3 illustrates a strict alternating cycle of length 3. The following elementary result is due to

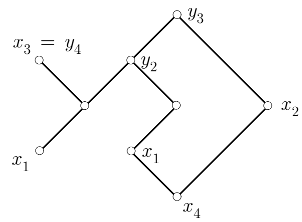

FIGURE 1. An Alternating Cycle of Length 4.

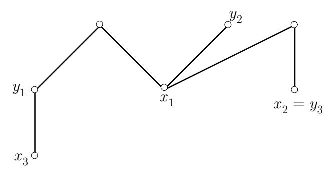

FIGURE 2. A Strict Alternating Cycle of Length 3.

Trotter and Moore [8]. See [4] for a short proof and a number of applications.

Theorem 2.3. *Let* $\mathbf{P} = (X, P)$ *be a poset and let* $S \subseteq \mathrm{inc}(\mathbf{P})$. *Then the following statements are equivalent.*

1. *There exists a linear extension* L *of* P *which reverses* S.
2. S *does not contain an alternating cycle.*
3. S *does not contain a strict alternating cycle.*

□

3. GRAPHS, HYPERGRAPHS AND CRITICAL PAIRS

Evidently, a poset has dimension 1 if and only if it is a linear order, so it makes sense to restrict our attention to posets which are not linear orders. Let $\mathbf{P} = (X, P)$ be any such poset. Then we associate with $\mathbf{P}$ a hypergraph $\mathbf{H_P}$, called the *hypergraph of incomparable pairs*, defined as follows. The vertices of $\mathbf{H_P}$ are the incomparable pairs in the poset $\mathbf{P}$. The edges of $\mathbf{H_P}$ are those sets S of incomparable pairs satisfying:

1. No linear extension of P reverses all incomparable pairs in S.
2. If T is a proper subset of S, then there is a linear extension of P which reverses all incomparable pairs in T.

Note that the edges of the hypergraph $\mathbf{H_P}$ correspond to strict alternating cycles. Then let $\mathbf{G_P}$ denote the ordinary graph determined by all edges of size 2 in $\mathbf{H_P}$. The following proposition is immediate.

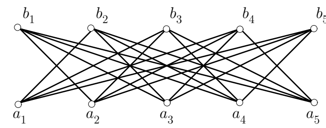

FIGURE 3. The Standard Example $\mathbf{S}_5$

Proposition 3.1. *Let $\mathbf{P} = (X, P)$ be a poset and let $\mathbf{H_P}$ and $\mathbf{G_P}$ denote the hypergraph and graph of incomparable pairs, respectively. Then*

$$\dim(\mathbf{P}) = \chi(\mathbf{H_P}) \geq \chi(\mathbf{G_P}).$$

□

Call a pair $(x, y) \in \mathrm{inc}(\mathbf{P})$ a *critical pair* if $u < x$ in P implies $u < y$ in P and $v > y$ in P implies $v > x$ in P, for all $u, v \in X$. Then let $\mathrm{crit}(\mathbf{P})$ denote the set of all critical pairs. The following elementary proposition serves to explain why the concept of a critical pair is important to the study of realizers.

Proposition 3.2. *Let $\mathcal{R}$ be a family of linear extensions of a partial order P on a ground set X. Then $\mathcal{R}$ is a realizer of P if and only if for every $(x, y) \in \mathrm{crit}(\mathbf{P})$, there exists some $L \in \mathcal{R}$ so that $x > y$ in L.* □

In other words, a family $\mathcal{R}$ of linear extensions is a realizer if and only if it reverses the set of critical pairs, and the dimension of P is just the minimum size of a family of linear extensions reversing all critical pairs. Accordingly, it makes sense to define the *hypergraph of critical pairs* $\mathbf{H^c_P}$ as the subhypergraph of $\mathbf{H_P}$ induced by the critical pairs. Similarly, we define the *graph of critical pairs* $\mathbf{G^c_P}$ as the subgraph of $\mathbf{G_P}$ induced by the critical pairs. The following lemma follows easily from Proposition 3.2.

Lemma 3.3. *For every poset $\mathbf{P} = (X, P)$,*

$$\dim(\mathbf{P}) = \chi(\mathbf{H_P}) = \chi(\mathbf{H^c_P}) \geq \chi(\mathbf{G_P}) = \chi(\mathbf{G^c_P}).$$

□

For those readers who are not familiar with posets and dimension, we present four examples to illustrate the properties of the graphs and hypergraphs we have introduced in this section.

For an integer $n \geq 3$, let $\mathbf{S}_n$ denote the poset of height two with n minimal elements $a_1, a_2, \ldots, a_n$, n maximal elements $b_1, b_2, \ldots, b_n$ and ordering $a_i < b_j$ if and only if $i \neq j$. We call $\mathbf{S}_n$ the *standard example* of an n-dimensional poset. The diagram for $\mathbf{S}_5$ is shown in Figure 3.

Example 3.4. *The hypergraph of critical pairs of the standard example $\mathbf{S}_n$ is just an ordinary graph, namely the complete graph on n vertices.* □

Example 3.5. *In Figure 3, we show a 3-dimensional poset called the "chevron." For this poset, the hypergraph of critical pairs is again an ordinary graph—a cycle on 5 vertices.* □

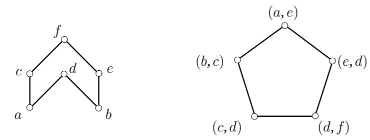

FIGURE 4. The Chevron and its Hypergraph of Critical Pairs

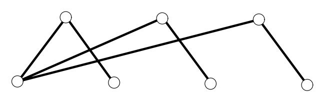

FIGURE 5. The Spider, A 3-dimensional Poset

Example 3.6. *A poset known as the "spider" is shown in Figure 3. The hypergraph of critical pairs contains two edges of size* 3*. However, the graph of critical pairs for the spider is an odd cycle on* 9 *vertices.* □

4. The Role of the Hypergraph Edges

In this section, we present an example which serves to illustrate the essential role of the hypergraph edges (those of size at least 3) in determining the dimension of a poset.

Example 4.1. *For each integer* $t \geq 2$*, we construct a poset* $\mathbf{P}_t$ *for which the chromatic number of the graph of incomparable pairs is* t*. However, the dimension of* $\mathbf{P}$ *will be at least* $(3/2)^{t-1}$*.*

We proceed by induction on t. For $t = 2$, we take $\mathbf{P}_2$ as the height 2 poset having three minimal elements x_1, x_2 and x_3; three maximal elements y_1, y_2 and y_3; with comparabilities $x_1 < y_2$, $x_2 < y_3$ and $x_3 < y_1$.

$\mathbf{P}_2$ has 6 critical pairs. Set

$$V_1 = \{(x_1, y_1), (x_2, y_2), (x_3, y_3)\} \quad \text{and} \quad V_2 = \{(x_1, y_3), (x_2, y_1), (x_3, y_2)\}.$$

Then

1. $\text{crit}(\mathbf{P}_2) = V_1 \cup V_2$,
2. V_1 and V_2 are independent in the graph of critical pairs, and
3. V_1 and V_2 are strict alternating cycles in the hypergraph of critical pairs.

As a consequence, the chromatic number of the graph of critical pairs is 2. Furthermore, the graph of critical pairs contains a complete subgraph of size 2, namely the edge between the pairs (x_1, y_1) and (x_3, y_2).

Now the dimension of $\mathbf{P}_2$ is also 2, but in order to set up the induction, we note that there are 3 critical pairs in V_1 and no linear extension can reverse more than 2 of them. This shows that the dimension of $\mathbf{P}_2$ is at least $(3/2)^1$. We say that the critical pairs in V_1 are *vertical* while the critical pairs in V_2 are *slanted*.

Now suppose that we have constructed $\mathbf{P}_t$ for some $t \geq 2$. For inductive purposes, we suppose that the chromatic number of the graph of critical pairs is t and that the graph of critical pairs contains a complete graph of size t. We suppose further that all critical pairs are min-max pairs; the chromatic number of the graph of critical pairs of $\mathbf{P}_t$ is t; and there is a subset of 3^{t-1} vertical critical pairs so that no linear extension reverses more than 2^{t-1} of these pairs.

We then construct $\mathbf{P}_{t+1}$ by starting with three disjoint copies $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$ each isomorphic to $\mathbf{P}_t$. Then add comparabilities to make each minimal element of $\mathbf{Q}_i$ less than each maximal element of $\mathbf{Q}_{i+1}$ (cyclically). The vertical pairs in $\mathbf{P}_{t+1}$ are just those which are vertical in one of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$, so that $\mathbf{P}_{t+1}$ has $3(3^{t-1}) = 3^t$ vertical critical pairs as desired. Furthermore, any linear extension reverses critical pairs from at most two of $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$, and at most 2^{t-1} pairs in any one copy of $\mathbf{P}_t$. Thus any linear extension of $\mathbf{P}_{t+1}$ reverses at most $2(2^{t-1}) = 2^t$ vertical critical pairs in $\mathbf{P}_{t+1}$. This shows that the dimension of $\mathbf{P}_{t+1}$ is at least $(3/2)^t$.

We next show that the graph of critical pairs of $\mathbf{P}_{t+1}$ is $t+1$. To show that it at most $t+1$, color the critical pairs in each $\mathbf{Q}_i$ just as in $\mathbf{P}_t$. This is allowable since no critical pair in $\mathbf{Q}_i$ is adjacent to a critical pair in $\mathbf{Q}_j$ when $i \neq j$. Then color all critical pairs of the form (x, y) where x is a minimal element in $\mathbf{Q}_{i+1}$ and y is maximal in $\mathbf{Q}_i$ with a new color.

On the other hand, note that if x is minimal in $\mathbf{Q}_3$ and y is maximal in $\mathbf{Q}_2$, then (x, y) is adjacent to all critical pairs in $\mathbf{Q}_1$ in the graph of critical pairs. This shows that the chromatic number of the graph of critical pairs of $\mathbf{P}_{t+1}$ is $t+1$. It also shows that the graph contains a complete subgraph of size $t+1$.

5. Proof of Theorem 1

Let $\mathbf{P} = (X, P)$ be a poset which is not a linear order. If $\dim(\mathbf{P}) = 2$, then it follows trivially that the chromatic number of both graphs $\mathbf{G_P}$ and $\mathbf{G_P^c}$ is 2.

Now suppose that $\chi(\mathbf{G_P}) = \chi(\mathbf{G_P^c}) = 2$. We show that $\dim(\mathbf{P}) = 2$. We argue by contradiction. Suppose this statement is false. Of all counterexamples, choose one for which the cardinality of X is minimum. Then it follows that $\mathbf{P}$ is 3-irreducible. In turn, this implies that $\mathbf{P}$ is indecomposable.

Now let ϕ be any proper 2-coloring of the the graph $\mathbf{G_P}$ of incomparable pairs of $\mathbf{P}$, say using the colors in $\{1, 2\}$. For each $i = 1, 2$, let S_i denote the set of critical pairs which are assigned color i by ϕ. Since $\dim(\mathbf{P}) = 3$, one of S_1 and S_2 contains a strict alternating cycle. Of all strict alternating cycles contained in one of the color classes, consider those of minimum length and let this minimum length be k. For each strict alternating cycle $S = \{(x_i, y_i) : 1 \leq i \leq k\}$ contained in a color class, let $f(S)$ count the number of points in

$$\cup_{i=1}^{k} \{u : x_i \leq u \leq y_{i+1}\}.$$

We then choose a strict alternating cycle S of length k contained in a single color class for which $f(S)$ is as large as possible. Without loss of generality, we may assume that S is contained in color class 1.

Claim 1. The length k of the alternating cycle S is 3.

Proof. First note that $k \geq 3$, for if $k = 2$, then the vertices in S are adjacent in both $\mathbf{G_P}$ and $\mathbf{H_P}$. It follows that for each $i = 1, 2, \ldots, k$, x_i is incomparable with both y_{i-1} and y_{i+2}. So we may choose critical pairs (u_i, v_i) and (w_i, z_i) with $u_i \leq x_i$,

$w_i \le x_i$, $y_{i-1} \le v_i$ and $y_{i+2} \le z_i$. For each $i = 1, 2, \ldots, k$, note that (w_i, z_i) is adjacent to (x_{i+1}, y_{i+1}) so each (w_i, z_i) is assigned color 2.

We claim that for each $i = 1, 2, \ldots, k$, the critical pair (u_i, v_i) is assigned color 2. For suppose that some (u_i, v_i) is assigned color 1. Then $\{(u_i, v_i)\} \cup \{(x_j, y_j) : 1 \le j \le k, j \ne i, i-1\}$ forms an alternating cycle of length $k-1$. Any minimal length alternating cycle among these $k-1$ pairs is strict, thus contradicting the choice of k. So we conclude that each pair (u_i, v_i) is assigned color 2.

Then observe that for each $i = 1, 2, \ldots, k$, $\{(u_i, v_i), (u_{i+1}, v_{i+1}), (w_{i-1}, z_{i-1})\}$ is an alternating cycle of length 3 and all three pairs are assigned color 2. This shows $k = 3$, as claimed. □

Claim 2. For each $i = 1, 2, 3$, the incomparable pair (x_i, y_{i-1}) is a critical pair.

Proof. $S' = \{(u_i, v_i) : 1 \le i \le 3\}$ is a strict alternating cycle and $f(S') \ge f(S)$. Furthermore, $f(S') > f(S)$ unless $u_i = x_i$ and $y_{i-1} = v_i$ for $i = 1, 2, 3$. □

Now consider the subposet $\mathbf{Q}$ induced by the points in the strict alternating cycle S. We observe that $\mathbf{Q}$ is a disjoint sum of three connected subposets $\mathbf{Q}_1$, $\mathbf{Q}_2$ and $\mathbf{Q}_3$, each of height at most 2. Furthermore, we may label these three subposets so that:

1. For each $i = 1, 2, 3$, if a is minimal in $\mathbf{Q}_i$ and b is maximal in $\mathbf{Q}_{i+1}$, then (a, b) is a critical pair assigned color 1 by ϕ.
2. For each $i = 1, 2, 3$, if a is minimal in $\mathbf{Q}_i$ and b is maximal in $\mathbf{Q}_{i-1}$, then (a, b) is a critical pair assigned color 2 by ϕ.

Now let $\mathbf{Q}_0$ be the largest subposet of $\mathbf{P}$ consisting of three non-empty connected components $\mathbf{Q}_1$, $\mathbf{Q}_2$, $\mathbf{Q}_3$, each of height at most 2, satisfying conditions (1) and (2) as given above. Then let Y consist of all points in the ground set X which are not in the subposet $\mathbf{Q}_0$. Since $\mathbf{P}$ is indecomposable, we know that $\mathbf{Q}_0$ is a proper subposet of $\mathbf{P}$, i.e., $Y \ne \emptyset$. Furthermore, there exists some point $d \in Y$ which is comparable to some but not all points of $\mathbf{Q}_0$.

Claim 3 Any point in Y which is less than some minimal point in $\mathbf{Q}_0$ is less than all points of $\mathbf{Q}_0$. Dually, any point in Y which is greater than any maximal point in $\mathbf{Q}_0$ is greater than all points of $\mathbf{Q}_0$.

Proof of the Claim. Suppose that $y \in Y$ and that y is less than some minimal point of $\mathbf{Q}_0$. Without loss of generality, we may assume that $y < a_1$ for some minimal point a_1 of the connected subposet $\mathbf{Q}_1$ of $\mathbf{Q}_0$. We show that $y < a_2$ for every minimal element a_2 of $\mathbf{Q}_2$. Suppose to the contrary that there is some minimal element a_2 of $\mathbf{Q}_2$ for which $y \| a_2$.

Let b_3 be any maximal point in $\mathbf{Q}_3$. Then we know that (a_2, b_3) is a critical pair assigned color 1 by ϕ. Also, since (a_1, b_3) is critical and $y < a_1$, we know that $y < b_3$.

Now choose a maximal point b_1 in $\mathbf{Q}_1$ with $a_1 \le b_1$. Then we know that (a_2, b_1) is critical and is assigned color 2 by ϕ. It follows that the incomparable pair (y, a_2) is adjacent to both (a_2, b_3) and (a_2, b_1) in $\mathbf{G_P}$, i.e,. (y, a_2) is adjacent to vertices in each of the two color classes, which is impossible. The contradiction completes the proof of the assertion that y is less than every minimal point in $\mathbf{Q}_2$. But this argument is cyclic, so we may conclude that y is less than all minimal elements in all three components. In turn, it follows that y is less than all elements of $\mathbf{Q}_0$ as claimed. △

We are now ready to complete the proof of our theorem. Choose a point $y \in Y$ which is comparable with some but not all points in $\mathbf{Q}_0$. Without loss of generality, we may assume that

1. y is incomparable with all minimal points of $\mathbf{Q}_0$.
2. There is a maximal point b_1 in $\mathbf{Q}_1$ so that $y < b_1$.
3. Any point less than y is comparable with all points of $\mathbf{Q}_0$.

We will complete the proof by showing that the subposet $\mathbf{Q}_0$ is not maximal. To accomplish this, we show that

- y is incomparable with all points in $\mathbf{Q}_2$ and $\mathbf{Q}_3$.
- For each maximal point b_2 in $\mathbf{Q}_2$, the incomparable pair (y, b_2) is critical and assigned color 1 by ϕ.
- For each maximal point b_3 in $\mathbf{Q}_2$, the incomparable pair (y, b_3) is critical and assigned color 2 by ϕ.

Suppose first that y is comparable with maximal points in all three components of $\mathbf{Q}_0$. Then none of the maximal points comparable to y can also be a minimal point. It follows that $\mathbf{P}$ contains the 3-dimensional spider (see Figure 3) and thus $\chi(\mathbf{G_P}) \geq 3)$. This is a contradiction.

Now suppose that y is comparable with maximal points in exactly two of the three components of $\mathbf{Q}_0$, say $\mathbf{Q}_1$ and $\mathbf{Q}_2$. Choose a maximal point b_2 in $\mathbf{Q}_2$ with $y < b_2$. Then let a_3 be any minimal element of $\mathbf{Q}_3$. It follows that the incomparable pair (y, a_3) is adjacent to both (a_3, b_1) and (a_3, b_2) in $\mathbf{G_P}$, but ϕ assigns different colors to these two critical pairs. The contradiction shows that y is comparable only with points from $\mathbf{Q}_1$ and incomparable with all points in $\mathbf{Q}_2$ and $\mathbf{Q}_3$.

We next show that for each maximal point b_2 in $\mathbf{Q}_2$, the incomparable pair (y, b_2) is critical and assigned color 1 by ϕ. Let $u' < y$. Then u' is less than all points of $\mathbf{Q}_0$ by property (3) above. In particular, this shows $u' < b_2$. On the other hand, let $b > b_2$. Then by Claim 3, we know that b is greater than all points of $\mathbf{Q}_0$. Thus $b > b_1 > y$ and $b > y$. Thus (y, b_2) is critical. Now let a_2 be any minimal element of $\mathbf{Q}_2$ with $a_2 < b_2$. Then (a_2, b_1) is critical and assigned color 2 by ϕ. Since (a_2, b_1) and (y, b_2) are adjacent, we conclude that ϕ assigns color 1 to (y, b_2).

The argument to show that for each maximal point b_3 of $\mathbf{Q}_3$, the incomparable pair (y, b_3) is critical and assigned color 2 by ϕ is dual. We conclude that we can add y to $\mathbf{Q}_1$ which contradicts the assumption that the cardinality of $\mathbf{Q}_0$ is maximum. With this remark, the proof of Theorem 1.1 is complete. □

6. Some Open Problems

Originally, we thought that with just a little attention to detail, we could modify the construction presented in Section 4 to settle Question 1.2 in the negative. After spending some time on this effort, we feel that it may take a new idea. We still think it would be quite surprising should this question have an affirmative answer.

Among the several interesting open problems relating graph coloring and posets, we want to mention one very interesting problem involving planar graphs and a combinatorial connection discussed briefly in Section 1. With a graph $G = (V, E)$, we associate a poset $\mathbf{A_G}$, called the *adjacency poset* of $\mathbf{G}$, and defined as follows. $\mathbf{A_G}$ is a height 2 poset contain an incomparable min-max pair (x', x'') for every vertex $x \in V$. For each edge $e = \{x, y\}$, the poset $\mathbf{A_G}$ contains the order relations $x' < y''$ and $y' < x''$. It is straightforward to verify that $\chi(\mathbf{G}) \leq \dim(\mathbf{A_G})$.

The dimension of the incidence poset of a graph can be bounded from above by a function of the chromatic number of the graph. However, this is not true for adjacency posets. For example, the adjacency poset of a bipartite graph can have arbitrarily large dimension—consider the cover graphs of standard examples. Also, since there exist graphs with large girth and large chromatic number, taking the adjacency poset, we see that there exist posets with large dimension for which the comparability graph has large girth.

Here is one interesting class of graphs for which the dimension of adjacency posets is bounded. The proof of the following theorem is given in [2].

Theorem 6.1. *If* $\mathbf{A_G}$ *is the adjacency poset of a planar graph, then* $\dim(\mathbf{A_G}) \leq 10$. □

From below, we can show that there exists a planar poset whose adjacency poset has dimension 5. Perhaps this is the right upper bound for Theorem 6.1.

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