On equilateral simplices in normed spaces

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Dedicated to Prof. H. Harborth on occasion of his sixtieth birthday

Abstract: It is the aim of this note to improve the lower bound for the problem of Petty on the existence of equilateral simplices in normed spaces. We show that for each k there is a $d(k)$ such that each normed space of dimension $d \geq d(k)$ contains k points at pairwise distance one, and that if the norm is sufficiently near to the euclidean norm, the maximal equilateral sets behave like their euclidean counterparts.

1. Introduction

The question whether each d-dimensional normed space contains $d+1$ points at pairwise distance one, i.e. an equilateral simplex, was first raised by Petty in 1971 [6]. This seems obvious at first, especially in the equivalent packing version: each convex body K admits a packing $(K+t_i)_{i=1}^{d+1}$ of $d+1$ pairwise touching translates. But it turned out much more difficult, as illustrated by the following near-counterexample constructed by Petty: define a norm on $\mathbb{R}^d$ by $\left\|(x_1,\ldots,x_d)\right\| := |x_1| + \sqrt{x_2^2+\cdots+x_d^2}$, so the unit ball is a double cone over a $d-1$-dimensional euclidean ball (Figure 1). Start with the two points $(0,0,\ldots,0)$ and $(1,0,\ldots,0)$ (the center of that double cone and one apex). Then any further point with distance one to both these points must be of the form $(\frac{1}{2},x_2,\ldots,x_d)$ with $\sqrt{x_2^2+\cdots+x_d^2}=\frac{1}{2}$. So all possible extensions of these two starting points to larger equilateral sets lie on a $d-1$-dimensional euclidean sphere with radius $\frac{1}{2}$, which admits at most two points with pairwise distance one. So there are norms in $\mathbb{R}^d$ for which there exist nonextendable equilateral sets of four points. Petty also showed that each normed space of dimension at least three contains four points at pairwise distance one; in fact, each equilateral set of less than four points can be extended to a four-point set. He conjectured that each normed space contains $d+1$ points at pairwise distance one; this conjecture occurs also in the book of Thompson [9, problem 4.1.1], but no progress was made beyond the lower bound of four ([3],[5]). There are, of course, normed spaces that admit much larger equilateral sets, the upper bound is 2^d, as reached by the maximum norm. For further material on equilateral and few-distance sets in normed spaces see [8], for combinatorial distance problems also [1]. In this note, we show:

Theorem 1: For each k there is a $d(k)$ such that each normed space of dimension $d \geq d(k)$ contains k points at pairwise distance one.

This follows by an application of Dvoretzky's theorem from

Theorem 2: For each dimension d there is a $\varepsilon_d^* > 0$ such that if $(V, \|\cdot\|)$ is a d-dimensional normed space with

$$(1-\varepsilon_d^*)\|x\|_{\text{euclidean}} \le \|x\| \le (1+\varepsilon_d^*)\|x\|_{\text{euclidean}} \qquad \text{for all } x \in V, \qquad (*)$$

then each equilateral set in V can be extended to an equilateral set of $d+1$ points.

So if the norm is sufficiently near to a euclidean norm, then the equilateral sets behave like euclidean equilateral sets: they can be freely rotated, without 'forbidden directions' as in Petty's double-cone example.

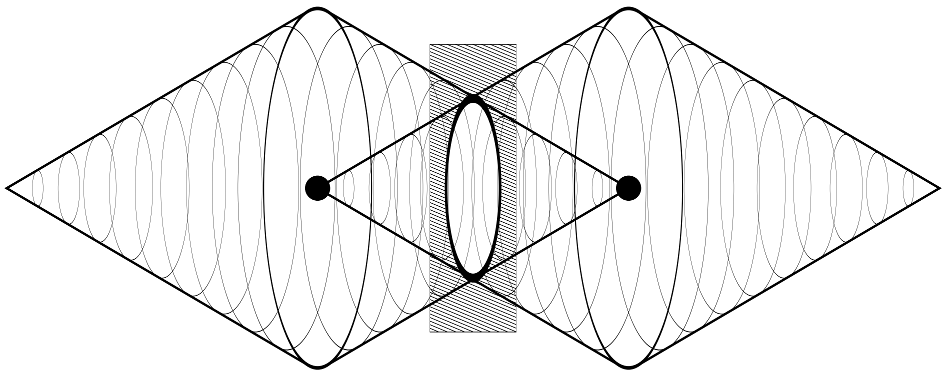

Figure 1.

2. Proof of the theorems

We need the following lemma, which states that we can prescribe arbitrary distances 'near' a regular simplex, and still find a realization in the same euclidean space, but not in a space of smaller dimension.

Lemma: For each dimension d there is a $\varepsilon_d > \frac{1}{4}(d+2)^{-\frac{3}{2}}$ such that

(1) each metric space of $d+1$ points whose distances are all between $1-\varepsilon_d$ and $1+\varepsilon_d$ can be realized in euclidean d-dimensional space.

(2) each metric space of $d+2$ points whose distances are all between $1-\varepsilon_d$ and $1+\varepsilon_d$ cannot be realized in euclidean d-dimensional space.

We note that the bound for ε_d is certainly not best possible for either property, but it is probably difficult to determine the optimal bounds. The second property is equivalent to the minimum diameter of a packing of $d+2$ unit balls in dimension d. The related planar problem of the minimum diameter packing of n unit disks is a known difficult problem by Erdős; and for higher dimensions already the minimum diameter of a packing of five unit balls in dimension three seems to be unknown.

Using this Lemma, we now prove Theorem 2. Let $(V, \|\cdot\|)$ be a d-dimensional normed space with property $(*)$ for the $\varepsilon_d^* := \frac{1}{2}\varepsilon_d$ of the Lemma. Let $p_1, \ldots, p_k$ be a set of points in V with pairwise distance one with respect to that norm. We first note that k is at most $d+1$; for otherwise we had a set of $d+2$ points in euclidean d-dimensional space with pairwise distances between $(1+\varepsilon_d^*)^{-1}$ and $(1-\varepsilon_d^*)^{-1}$, contradicting the second assertion of the Lemma.

To prove Theorem 2, we have to show that for $k \leq d$ there is an extension point p_{k+1} that also has distance one to $p_1, \ldots, p_k$. For this we select a k-dimensional linear subspace $V_k \subseteq V$ that contains $p_2 - p_1, \ldots, p_k - p_1$ and one further dimension, and a halfspace $\mathcal{H}$ in the affine space $p_1 + V_k$ that is bounded by the hyperplane through $p_1, \ldots, p_k$.

The points $p_1, \ldots, p_k$ have pairwise distances one with respect to the norm, so their pairwise euclidean distances are in the interval $[(1+\varepsilon_d^*)^{-1}, (1-\varepsilon_d^*)^{-1}] \subset [1-\varepsilon_d, 1+\varepsilon_d] =: I_d$. By the Lemma we can prescribe arbitrary euclidean distances $d_1, \ldots, d_k$ in the interval I_d from a further point x to the points $p_1, \ldots, p_k$, and always find a euclidean realization. This realization is made unique by choosing the point x from the halfspace $\mathcal{H}$. So we can apply these distances as coordinates for a well-defined point $p(d_1, \ldots, d_k)$; this defines a continuous mapping from I_d^d into $\mathcal{H}$. For this point $p(d_1, \ldots, d_k)$ we can again determine the norm distances to $p_1, \ldots, p_k$; by property $(*)$ we have

$$\left\| p(d_1, \ldots, d_k) - p_i \right\| \in [(1-\varepsilon_d^*)d_i, (1+\varepsilon_d^*)d_i] \,,$$

and we search a point for which each of these norm distances is one.

We now consider the mapping $\phi\colon (x_1, \ldots, x_d) \mapsto (y_1, \ldots, y_d)$ defined by

$$y_i := x_i + \Big(1 - \|p(x_1, \ldots, x_k) - p_i\|\Big) \ \text{ for } i = 1, \ldots, k.$$

This is a continuous mapping which maps the compact set I_d^d into itself, for

$$\begin{aligned} 1 + \varepsilon_d &\geq 1 + \varepsilon_d^*(1+\varepsilon_d) \\ &\geq 1 + \varepsilon_d^* x_i = 1 + x_i - (1-\varepsilon_d^*)x_i \\ &\geq 1 + x_i - \left\| p(x_1, \ldots, x_k) - p_i \right\| =: y_i \\ &\geq 1 + x_i - (1+\varepsilon_d^*)x_i = 1 - \varepsilon_d^* x_i \\ &\geq 1 - \varepsilon_d^*(1+\varepsilon_d) \\ &\geq 1 - \varepsilon_d \,. \end{aligned}$$

By Brouwer's Fixed-point Theorem this mapping has a fixed point $(x_1, \ldots, x_d) \in I_d^d$; for this point the correction terms in each coordinate vanish, so $\left\| p(x_1, \ldots, x_k) - p_i \right\| = 1$ for each i. Therefore $p_{k+1} := p(x_1, \ldots, x_k)$ is the point extending $p_1, \ldots, p_k$ to a bigger set of points with pairwise distance one. This completes the proof of Theorem 2.

Theorem 1 follows from Theorem 2 by application of a theorem of Dvoretzky ([2],[10]) which states that for each dimension d and each ε there is a d' such that each normed space of dimension at least d' has a subspace of dimension d that is ε-near to a euclidean space in the sense required by Theorem 2.

It remains to prove the Lemma. Let $\mathrm{CMD}(p_1, \ldots, p_k)$ denote the Cayley-Menger-determinant of $p_1, \ldots, p_k$, that is the determinant of the $(k+1) \times (k+1)$-matrix with 0's in the main diagonal, 1's in the first column and first row, and the squared distance $d(p_i, p_j)^2$ at position $(i+1), (j+1)$. We use a theorem of Menger ([4], [7]) characterizing the metric spaces embeddable into a d-dimensional euclidean space.

Theorem (*Menger*): A metric space $(M, d(\cdot,\cdot))$ is realizable in euclidean d-dimensional space if and only if one of the following conditions is satisfied:

(1) $|M| \leq d$ and M is realizable in $d-1$-dimensional space.

(2) $|M| = d+1$, $(-1)^{d+1}\,\mathrm{CMD}(M) \geq 0$ and each subset of d points of M is realizable in $d-1$-dimensional space.

(3) $|M| = d+2$, $\mathrm{CMD}(M) = 0$ and each subset of $d+1$ points of M is realizable in d-dimensional space.

(4) $|M| = d+3$, $\mathrm{CMD}(M) = 0$ and each subset of $d+2$ points of M is realizable in d-dimensional space.

(5) $|M| \geq d+4$ and each subset of $d+2$ points of M is realizable in d-dimensional space.

To prove the Lemma, we have to show that the determinant of a matrix

$$\det \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1+\delta_{12} & \cdots & 1+\delta_{1k} \\ 1 & 1+\delta_{21} & 0 & \cdots & 1+\delta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1+\delta_{k1} & 1+\delta_{k2} & \cdots & 0 \end{vmatrix} \qquad (**)$$

in which $|\delta_{ij}| < 2\varepsilon_d + \varepsilon_d^2$ for all i, j has the same sign as the determinant of the same matrix without the δ_{ij}, which is $(-1)^k k$ for a $(k+1)\times(k+1)$-matrix. This gives also the second part of the Lemma, since the necessary condition for embeddability of $d+2$-point sets is that this determinant vanishes. Elementary transformations show

$$\det \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1+\delta_{12} & 1+\delta_{13} & \cdots & 1+\delta_{1k} \\ 1 & 1+\delta_{21} & 0 & 1+\delta_{23} & \cdots & 1+\delta_{2k} \\ 1 & 1+\delta_{31} & 1+\delta_{32} & 0 & \cdots & 1+\delta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1+\delta_{k1} & 1+\delta_{k2} & 1+\delta_{k3} & \cdots & 0 \end{vmatrix} = \det \begin{vmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & \delta_{12} & \delta_{13} & \cdots & \delta_{1k} \\ 1 & \delta_{21} & -1 & \delta_{23} & \cdots & \delta_{2k} \\ 1 & \delta_{31} & \delta_{32} & -1 & \cdots & \delta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \delta_{k1} & \delta_{k2} & \delta_{k3} & \cdots & -1 \end{vmatrix}$$

$$= \det \begin{vmatrix} k & \sum_i \delta_{i1} & \sum_i \delta_{i2} & \sum_i \delta_{i3} & \cdots & \sum_i \delta_{ik} \\ 1 & -1 & \delta_{12} & \delta_{13} & \cdots & \delta_{1k} \\ 1 & \delta_{21} & -1 & \delta_{23} & \cdots & \delta_{2k} \\ 1 & \delta_{31} & \delta_{32} & -1 & \cdots & \delta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \delta_{k1} & \delta_{k2} & \delta_{k3} & \cdots & -1 \end{vmatrix} = \det \begin{vmatrix} k + \sum\limits_{ij} \delta_{ij} & \sum\limits_i \delta_{i1} & \sum\limits_i \delta_{i2} & \cdots & \sum\limits_i \delta_{ik} \\ \sum\limits_j \delta_{1j} & -1 & \delta_{12} & \cdots & \delta_{1k} \\ \sum\limits_j \delta_{2j} & \delta_{21} & -1 & \cdots & \delta_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum\limits_j \delta_{kj} & \delta_{k1} & \delta_{k2} & \cdots & -1 \end{vmatrix}$$

$$= \det \begin{vmatrix} \sum_{ij} \delta_{ij} & \sum_i \delta_{i1} & \sum_i \delta_{i2} & \sum_i \delta_{i3} & \cdots & \sum_i \delta_{ik} \\ \sum_j \delta_{1j} & -1 & \delta_{12} & \delta_{13} & \cdots & \delta_{1k} \\ \sum_j \delta_{2j} & \delta_{21} & -1 & \delta_{23} & \cdots & \delta_{2k} \\ \sum_j \delta_{3j} & \delta_{31} & \delta_{32} & -1 & \cdots & \delta_{3k} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_j \delta_{kj} & \delta_{k1} & \delta_{k2} & \delta_{k3} & \cdots & -1 \end{vmatrix} + k \det \begin{vmatrix} -1 & \delta_{12} & \delta_{13} & \cdots & \delta_{1k} \\ \delta_{21} & -1 & \delta_{23} & \cdots & \delta_{2k} \\ \delta_{31} & \delta_{32} & -1 & \cdots & \delta_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{k1} & \delta_{k2} & \delta_{k3} & \cdots & -1 \end{vmatrix}.$$

Let $\delta := \sup_{ij} |\delta_{ij}|$. The first summand of the last line may be bounded directly using

Hadamard's inequality, which gives an upper bound of

$$\left(\left(k(k-1)\delta\right)^2 + k\left((k-1)\delta\right)^2\right)^{\frac{1}{2}} \left(1 + \left((k-1)\delta\right)^2 + (k-1)\delta^2\right)^{\frac{k}{2}} < \delta k^2\left(1 + k^2\delta^2\right)^{\frac{k}{2}}$$

for the absolute value of the determinant. The second determinant is decomposed in such a way that we have have an isolated δ_{ij}-column in each matrix but one:

$$\det\begin{vmatrix} 0 & \delta_{12} & \delta_{13} & \dots & \delta_{1k} \\ \delta_{21} & -1 & \delta_{23} & \dots & \delta_{2k} \\ \delta_{31} & \delta_{32} & -1 & \dots & \delta_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \delta_{k1} & \delta_{k2} & \delta_{k3} & \dots & -1 \end{vmatrix} + \det\begin{vmatrix} -1 & \delta_{12} & \delta_{13} & \dots & \delta_{1k} \\ 0 & 0 & \delta_{23} & \dots & \delta_{2k} \\ 0 & \delta_{32} & -1 & \dots & \delta_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \delta_{k2} & \delta_{k3} & \dots & -1 \end{vmatrix} + \cdots$$

$$+\det\begin{vmatrix} -1 & 0 & 0 & \dots & \delta_{1k} \\ 0 & -1 & 0 & \dots & \delta_{2k} \\ 0 & 0 & -1 & \dots & \delta_{3k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{vmatrix} + \det\begin{vmatrix} -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \end{vmatrix}.$$

The value of the last determinant is $(-1)^k$, the other k summands are each smaller in absolute value than $\sqrt{(k-1)\delta^2}\left(1 + (k-1)\delta^2\right)^{\frac{k-1}{2}}$ (again Hadamard's inequality). So it is sufficient for the determinant $(**)$ to have the correct sign that

$$\delta k^2\left(1 + k^2\delta^2\right)^{\frac{k}{2}} + k\delta\sqrt{k-1}\left(1 + (k-1)\delta^2\right)^{\frac{k-1}{2}} < k.$$

This condition is satisfied in the case needed by the Lemma, that is $k = d+1$ or $k = d+2$, and $\delta < 2\varepsilon_d + \varepsilon_d^2$ with $\varepsilon_d = \frac{1}{4}(d+2)^{-\frac{3}{2}}$. This completes the proof.

3. References

[**1**] P. Brass: Erdős distance problems in normed spaces, Computational Geometry — Theory and Applications **6** (1996) 195–214.

[**2**] A. Dvoretzky: Some results on convex bodies and Banach spaces, Proc. Symp. on Linear Spaces (1961) 123–160.

[**3**] G. Lawlor and F. Morgan: Paired calibrations applied to soap films, immiscible fluids, and surfaces or networks minimizing other norms, Pacific J. Math. **166** (1994) 55–83

[**4**] K. Menger: Untersuchungen über allgemeine Metrik, Math. Annalen **100** (1928), 75–163.

[**5**] F. Morgan: Minimal surfaces, crystals, shortest networks and undergraduate research, Math. Intelligencer **14/3** (1992) 37–44

[**6**] C.M. Petty: Equilateral sets in Minkowski spaces Proc. Amer. Math. Soc. **29** (1971), 369-374.

[**7**] L. Piepmeyer: Punktmengen mit minimaler Anzahl verschiedener Abstände, Dissertation, TU Braunschweig 1993.

[**8**] K.J. Swanepoel: Cardinalities of k-distance sets in Minkowski spaces, Preprint 1997.

[**9**] A.C. Thompson: Minkowski geometry, Encyclopedia of Mathematics and its Applications Vol. 63, Cambridge Univ. Press 1996.

[**10**] C. Zong: Strange phenomena in convex and discrete geometry, Springer-Verlag 1996.