

# L- and M-structure in lush spaces

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## Abstract

Let  $X$  be a Banach space which is lush. It is shown that if a subspace of  $X$  is either an L-summand or an M-ideal then it is also lush.

## Introduction

Toeplitz defined (1918) the *numerical range* of a matrix  $A \in \mathbb{F}^{n \times n}$  to be the set

$$W(A) = \{\langle Ax, x \rangle : \|x\| = 1, x \in \mathbb{F}^n\},$$

which easily extends to operators on Hilbert spaces. In the 1960s, Lumer (1961) and Bauer (1962) independently extended this notion to arbitrary Banach spaces. For a Banach space  $X$  whose unit sphere we denote by  $S_X$  and an operator  $T \in B(X)$  we thus call

$$V(T) = \{x^*(Tx) : x^*(x) = 1, x^* \in S_{X^*}, x \in S_X\} \quad \text{and} \quad v(T) = \sup\{|\lambda| : \lambda \in V(T)\}$$

the *numerical range* and *radius* of  $T$ , respectively. By construction, we have  $v(T) \leq \|T\|$  for all  $T \in B(X)$ . The greatest number  $m \geq 0$  that satisfies

$$m\|T\| \leq v(T) \quad \text{for every } T \in B(X)$$

is called the *numerical index* of  $X$  and denoted by  $n(X)$ . A summary of what is and what is not known about the numerical index can be found in Martín (2000) and Kadets et al. (2006). In the special case  $n(X) = 1$  the operator norm and the numerical radius coincide on  $B(X)$ .

Several attempts have been made to characterise the spaces with numerical index one among all Banach spaces geometrically, one of them in Boyko et al. (2007). We denote by

$$S(B_X, x^*, \alpha) := \{x \in B_X : \operatorname{Re} x^*(x) > 1 - \alpha\}$$

for any  $x^* \in S_{X^*}$  and  $\alpha > 0$  an open slice of the unit ball. Setting  $\mathbb{T} := \{\omega \in \mathbb{F} : |\omega| = 1\}$  allows us to write the absolutely convex hull of a subset  $F \subseteq X$  as  $\operatorname{co}(\mathbb{T}F)$ .

**Definition.** Let  $X$  be a Banach space. If for every two points  $u, v \in S_X$  and  $\varepsilon > 0$  there is a functional  $x^* \in S_{X^*}$  that satisfies

$$u \in S(B_X, x^*, \varepsilon) \quad \text{and} \quad \operatorname{dist}(v, \operatorname{co}(\mathbb{T}S(B_X, x^*, \varepsilon))) < \varepsilon,$$

the space  $X$  is said to be *lush*.

Unfortunately, whilst lush spaces do have numerical index one, spaces with numerical index one need not be lush (Kadets et al., 2009, Remark 4.2). Lushness has proved invaluable in constructing a Banach space whose dual has strictly smaller numerical index — answering a question that up until then had been open for decades. Consequently, the property deserves attention.

Let us recall some results about sums of Banach spaces.

**Proposition** (Martín and Payá (2000, Proposition 1)). *Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces. Then*

$$n(c_0((X_n)_{n \in \mathbb{N}})) = n(\ell^1((X_n)_{n \in \mathbb{N}})) = n(\ell^\infty((X_n)_{n \in \mathbb{N}})) = \inf_{n \in \mathbb{N}} n(X_n).$$

*In particular, the following statements are equivalent:*

- (i) *Every  $X_n$  has numerical index one,*
- (ii) *the space  $c_0((X_n)_{n \in \mathbb{N}})$  has numerical index one,*
- (iii) *the space  $\ell^1((X_n)_{n \in \mathbb{N}})$  has numerical index one, and*
- (iv) *the space  $\ell^\infty((X_n)_{n \in \mathbb{N}})$  has numerical index one.*

A notion that has been introduced in Fullerton (1961) is that of a CL space. Originally defined for real spaces, it has proven inappropriate for complex spaces. Thus we will deal with a weakening introduced in Lima (1978) that had previously been used by Lindenstrauss but remained unnamed.

**Definition.** Let  $X$  be a Banach space. If for every convex subset  $F \subseteq S_X$  that is maximal in  $S_X$  with respect to convexity,  $\overline{\text{co}}(\mathbb{T}F) = B_X$  holds, then  $X$  is called an *almost-CL* space.

Almost-CL spaces are easily seen to be lush spaces but the converse does not hold (Boyko et al., 2007, Example 3.4(c)). With regard to sums, the following result has been obtained.

**Proposition** (Martín and Payá (2004, Proposition 8 & 9)). *Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces. Then the following are equivalent:*

- (i) *Every  $X_n$  is an almost-CL space,*
- (ii) *the space  $c_0((X_n)_{n \in \mathbb{N}})$  is almost-CL, and*
- (iii) *the space  $\ell^1((X_n)_{n \in \mathbb{N}})$  is almost-CL.*

For the recently introduced lushness property, however, only part of the corresponding equivalence has been shown.

**Proposition** (Boyko et al. (2009, Proposition 5.3)). *Let  $(X_n)_{n \in \mathbb{N}}$  be a sequence of Banach spaces. If every  $X_n$  is lush, then so are the spaces*

$$c_0((X_n)_{n \in \mathbb{N}}), \quad \ell^1((X_n)_{n \in \mathbb{N}}), \quad \text{and} \quad \ell^\infty((X_n)_{n \in \mathbb{N}}).$$

We seek to improve this result, bringing it up to par with what has been proved for almost-CL spaces and spaces with numerical index one.

## Inheritance of lushness

To this end we will show that if  $X$  and  $Y$  are arbitrary Banach spaces and one of the two spaces  $X \oplus_1 Y$  or  $X \oplus_\infty Y$  is lush, then  $X$  and  $Y$  are lush themselves.

Such a relation between the spaces  $X$ ,  $Y$ , and their sum can also be expressed in terms of projections.

**Definition.** Let  $Z$  be a Banach space and  $P: Z \rightarrow Z$  a linear projection that satisfies  $\|z\| = \max\{\|Pz\|, \|z - Pz\|\}$  for every  $z \in Z$ . Then  $P$  and  $\text{ran } P$  are called an *M-projection* and an *M-summand*, respectively.

**Definition.** Let  $Z$  be a Banach space and  $P: Z \rightarrow Z$  a linear projection that satisfies  $\|z\| = \|Pz\| + \|z - Pz\|$  for every  $z \in Z$ . Then  $P$  and  $\text{ran } P$  are called an *L-projection* and an *L-summand*, respectively.

Basic results of L- and M-structure theory that will be used from here on can be found in Harmand et al. (1993, Section I.1). If a subspace  $X \subseteq Z$  is an M-summand, its annihilator  $X^\perp$  is an L-summand in  $Z^*$ . However, an L-summand of  $Z^*$  need not be the annihilator of any space  $X \subseteq Z$ , nor must subspaces  $X \subseteq Z$  for which  $X^\perp$  is an L-summand in  $Z^*$  be M-summands. Subspaces  $X \subseteq Z$  for which  $X^\perp$  is an L-summand in  $Z^*$  are referred to as *M-ideals*.

### M-summands

We can now proceed to show that M-summands inherit lushness.

**Proposition 1.** *Let  $X$  be an M-summand in a lush space  $Z$ . Then  $X$  is lush.*

*Proof.* Let  $u, v \in S_X$  and  $\varepsilon \in (0, 1)$  be arbitrary. Since  $X$  is an M-summand there is an M-projection  $P: Z \rightarrow Z$  with  $\text{ran}(P) = X$ . Because  $Z$  is lush there is a functional  $z^* \in S_{Z^*}$  satisfying  $u \in S(B_Z, z^*, \varepsilon/2)$  and

$$\text{dist}(v, \text{co}(\mathbb{T} S(B_Z, z^*, \varepsilon/2))) < \varepsilon/2.$$

Hence there are points  $z_1, \dots, z_n \in S(B_Z, z^*, \varepsilon/2)$  and corresponding  $\theta_1, \dots, \theta_n \in \mathbb{F}$  that satisfy  $\sum_{k=1}^n |\theta_k| \leq 1$  such that  $\|\sum_{k=1}^n \theta_k z_k - v\| < \varepsilon/2$  holds. The projection  $P$  allows us to split these points up into

$$x_k := Pz_k \quad \text{and} \quad y_k := Px_k - x_k,$$

of which the  $x_k$  appear to approximate  $v$  mostly by themselves:

$$\left\| \sum_{k=1}^n \theta_k z_k - v \right\| = \max \left\{ \left\| \sum_{k=1}^n \theta_k y_k \right\|, \left\| \sum_{k=1}^n \theta_k x_k - v \right\| \right\}.$$

By  $\text{Re } z^*(x) > 1 - \varepsilon/2$  and  $\|z^*\| = 1$  we clearly have  $\text{Re } z^*(y_k) \leq \varepsilon/2 \|x_k\| \leq \varepsilon/2$  for every  $k$  and thus

$$\text{Re } z^*(x_k) = \text{Re } z^*(z_k) - \text{Re } z^*(y_k) > 1 - \varepsilon,$$

leaving us with  $x_k \in S(B_X, z^*, \varepsilon)$ , and therefore

$$\text{dist}(v, \text{co}(\mathbb{T} S(B_X, z^*, \varepsilon))) < \varepsilon.$$

By restricting  $z^*$  to  $X$  and normalising the restriction, we obtain the desired functional.  $\square$

## M-ideals

The celebrated principle of local reflexivity due to Lindenstrauss and Rosenthal (1969) can be used to extend this observation to M-ideals. More precisely we require a refined statement.

**Theorem** (Johnson et al. (1971, Section 3)). *Let  $X$  be a Banach space,  $E \subseteq X^{**}$  and  $F \subseteq X^*$  finite dimensional and  $\varepsilon > 0$  arbitrary. Then there is an operator  $T: E \rightarrow X$  with  $\|T\|\|T^{-1}\| \leq 1 + \varepsilon$  that satisfies  $(T \circ i_X)(x) = x$  for every  $x \in X$  with  $i_X(x) \in E$  and  $x^{**}(x^*) = x^*(Tx^{**})$  for every  $x^* \in F$ ,  $x^{**} \in E$ .*

An elementary proof is given in Martínez-Abejón (1999, Theorem 2).

*Remark 1.* An  $\varepsilon$ -isometry  $T$  as in the above theorem clearly satisfies

$$1 - \varepsilon \leq \|Tz^{**}\| \leq 1 + \varepsilon \quad \text{for every } z^{**} \in S_E.$$

With that in mind extending Proposition 1 to M-ideals is straightforward.

**Theorem 2.** *Let  $X$  be an M-ideal in a lush space  $Z$ . Then  $X$  is lush as well.*

*Proof.* Let the points  $u, v \in S_X$  be arbitrary and  $\varepsilon > 0$ . The lushness of  $Z$  now guarantees that there is a functional  $z^* \in S_{Z^*}$  with  $u \in S(B_Z, z^*, \varepsilon/2)$  as well as an absolutely convex combination of points  $z_1, \dots, z_n \in S(B_Z, z^*, \varepsilon/2)$  and corresponding scalars  $\theta_1, \dots, \theta_n \in \mathbb{F}$  such that  $\|\sum_{k=1}^n \theta_k z_k - v\| < \varepsilon/2$  and  $\sum_{k=1}^n |\theta_k| \leq 1$ . We observe  $Z^{**} = X^{\perp\perp} \oplus_{\infty} M$  for some subspace  $M \subseteq Z^{**}$ . For  $k \in \{1, \dots, n\}$  we can now find a decomposition  $i_Z(z_k) = x_k^{**} + y_k^{**}$  with  $x_k^{**} \in X^{\perp\perp}$  and  $y_k^{**} \in M$ . By

$$\operatorname{Re}(i_{Z^*}(z^*))(i_Z(u)) = \operatorname{Re} z^*(u) > 1 - \varepsilon/2,$$

we clearly have

$$|y^{**}(z^*)| \leq \varepsilon/2 \quad \text{for every } y^{**} \in S_M.$$

The functionals  $x_k^{**}$  satisfy

$$\operatorname{Re} x_k^{**}(z^*) = \operatorname{Re} z^*(z_k) - \operatorname{Re} y_k^{**}(z^*) > 1 - \varepsilon$$

and in particular

$$1 - \varepsilon \leq \|x_k^{**}\| \leq \|z_k\| = 1.$$

We also remark

$$\left\| \sum_{k=1}^n \theta_k z_k - v \right\| = \max \left\{ \left\| \sum_{k=1}^n \theta_k y_k^{**} \right\|, \left\| \sum_{k=1}^n \theta_k x_k^{**} - i_Z(v) \right\| \right\}.$$

Since  $X^{\perp\perp}$  and  $X^{**}$  can be identified, we have shown that the functionals  $x_k^{**}$  meet the requirements of lushness for  $i_X(u)$  and  $i_X(v)$  in  $X^{**}$ .

In applying the principle of local reflexivity to the finite dimensional subspace  $E := \operatorname{lin}\{x_1^{**}, \dots, x_n^{**}, i_Z(v)\} \subseteq X^{**}$ , we obtain an operator  $T: E \rightarrow X$  that satisfies

- $(T \circ i_X)x = x$  for every  $x \in X$  with  $i_X(x) \in E$ ,
- $z^*(Tx^{**}) = z^*(z^*)$  for  $z^{**} \in E$  and
- $1 - \varepsilon/2 \leq \|Tx^{**}\| \leq 1 + \varepsilon/2$  for  $z^{**} \in S_E$  (as per Remark 1).

We can now project  $x_k^{**}$  onto  $X$  with any relevant structure preserved. For  $x_k := Tx_k^{**} \in X$  we observe

$$\left\| \sum_{k=1}^n \theta_k x_k - v \right\| = \left\| \sum_{k=1}^n \theta_k Tx_k^{**} - (T \circ i_Z)v \right\| \leq (1 + \varepsilon/2) \left\| \sum_{k=1}^n \theta_k x_k^{**} - i_Z(v) \right\| < \varepsilon$$

and  $\operatorname{Re} z^*(x_k) = \operatorname{Re} x_k^{**}(z^*) > 1 - \varepsilon$ . What remains to be done is normalising. We thus continue to set  $\tilde{x}_k := x_k / \|x_k\|$  and obtain

$$\begin{aligned} \|x_k - \tilde{x}_k\| &= \left| \|x_k\| - 1 \right| \\ &\leq \left| \|x_k\| - \|x_k^{**}\| \right| + \left| \|x_k^{**}\| - 1 \right| \\ &\leq \left| \|Tx_k^{**}\| - \|x_k^{**}\| \right| + \varepsilon/2 \\ &= \varepsilon \|x_k^{**}\|/2 + \varepsilon/2 \\ &\leq \varepsilon, \end{aligned}$$

and therefore

$$\left\| \sum_{k=1}^n \theta_k \tilde{x}_k - v \right\| \leq \left\| \sum_{k=1}^n \theta_k (x_k - \tilde{x}_k) \right\| + \left\| \sum_{k=1}^n \theta_k x_k - v \right\| \leq \max_{k \leq n} \|x_k - \tilde{x}_k\| + \varepsilon \leq 2\varepsilon$$

as well as

$$\operatorname{Re} z^*(\tilde{x}_k) \geq \operatorname{Re} z^*(x_k) - \|x_k - \tilde{x}_k\| > 1 - 2\varepsilon.$$

□

## L-summands

Lushness is also inherited by L-summands. To see this we replace the complementary parts  $y_k$  of  $z_k$  with elements  $\xi_k \in X$  on which the functional  $z^*$  nearly attains its norm, such that the  $\theta_k \xi_k$  nearly add up to zero.

**Theorem 3.** *Let  $X$  be an L-summand of a lush space  $Z$ . Then  $X$  is lush.*

*Proof.* Let  $u, v \in S_X$  and  $\varepsilon > 0$  be arbitrary. Since  $Z$  is lush, for any  $\eta > 0$  there is a functional  $z^* \in S_{Z^*}$  as well as  $z_1, \dots, z_n \in S(B_Z, z^*, \eta)$  and  $\theta_1, \dots, \theta_n \in \mathbb{F}$  with  $\sum_{k=1}^n |\theta_k| \leq 1$  satisfying  $u \in S(B_Z, z^*, \eta)$  and  $\left\| \sum_{k=1}^n \theta_k z_k - v \right\| < \eta$ . Let  $P$  be the L-projection onto  $X$ . We set  $x_k := Pz_k$ ,  $y_k := z_k - x_k$  and note

$$\left\| \sum_{k=1}^n \theta_k z_k - v \right\| = \left\| \sum_{k=1}^n \theta_k x_k - v \right\| + \left\| \sum_{k=1}^n \theta_k y_k \right\|.$$

In particular, this gives  $\left\| \sum_{k=1}^n \theta_k x_k - v \right\| < \eta$  and  $\left\| \sum_{k=1}^n \theta_k y_k \right\| < \eta$ . Replacing  $y_k$  with  $\xi_k := \|y_k\|/u$  by setting  $\tilde{x}_k := x_k + \xi_k$  yields  $\|\tilde{x}_k\| \leq \|z_k\| \leq 1$  and

$$\begin{aligned} \operatorname{Re} z^*(\tilde{x}_k) &= \operatorname{Re} z^*(z_k - y_k + \xi_k) \\ &> (1 - \eta) - \|y_k\| + (1 - \eta)\|y_k\| \\ &= 1 - \eta - \eta\|y_k\| \\ &\geq 1 - 2\eta. \end{aligned}$$

If  $y_k$  satisfies  $\|y_k\| \leq \eta$ , the inequality

$$(\operatorname{Im} z^*(y_k))^2 \leq 2\|y_k\|\eta$$

can easily be seen to hold. If  $y_k$  does not satisfy  $\|y_k\| \leq \eta$ , the above inequality can be obtained nonetheless:

$$\begin{aligned} (\operatorname{Im} z^*(y_k))^2 &\leq (\operatorname{Re} z^*(y_k))^2 + (\operatorname{Im} z^*(y_k))^2 - (\|y_k\| - \eta)^2 \\ &= |z^*(y_k)|^2 - \|y_k\|^2 + 2\|y_k\|\eta - \eta^2 \\ &\leq 2\|y_k\|\eta - \eta^2 \\ &< 2\|y_k\|\eta. \end{aligned}$$

We therefore have

$$\begin{aligned} \left| \sum_{k=1}^n \theta_k \operatorname{Re} z^*(y_k) \right| &= \left| \sum_{k=1}^n \theta_k z^*(y_k) - i \sum_{k=1}^n \theta_k \operatorname{Im} z^*(y_k) \right| \\ &\leq \left\| \sum_{k=1}^n \theta_k y_k \right\| + \max_{k \leq n} |\operatorname{Im} z^*(y_k)| \\ &\leq \eta + \max_{k \leq n} \sqrt{2\|y_k\|\eta} \\ &\leq \eta + 2\sqrt{\eta}. \end{aligned}$$

Since for  $\delta_k := \|y_k\| - \operatorname{Re} z^*(y_k)$  the inequality  $|\delta_k| \leq \eta$  holds because of

$$\operatorname{Re} z^*(y_k) = \operatorname{Re} z^*(z_k) - \operatorname{Re} z^*(x_k) \geq (1 - \eta) - \|x_k\| \geq \|y_k\| - \eta,$$

we conclude

$$\begin{aligned} \left\| \sum_{k=1}^n \theta_k \xi_k \right\| &\leq \left| \sum_{k=1}^n \theta_k \operatorname{Re} z^*(y_k) \right| + \left| \sum_{k=1}^n \theta_k \delta_k \right| \\ &\leq 2\eta + 2\sqrt{\eta} \end{aligned}$$

and thus

$$\begin{aligned} \left\| \sum_{k=1}^n \theta_k \tilde{x}_k - v \right\| &= \left\| \sum_{k=1}^n \theta_k (x_k + \xi_k) - v \right\| \\ &\leq \left\| \sum_{k=1}^n \theta_k x_k - v \right\| + \left\| \sum_{k=1}^n \theta_k \xi_k \right\| \\ &\leq 3\eta + 2\sqrt{\eta}. \end{aligned}$$

Going back and choosing  $\eta$  such that  $3\eta + 2\sqrt{\eta} < \varepsilon$  and  $2\eta < \varepsilon$  are satisfied yields

$$\operatorname{Re} z^*(\tilde{x}_k) > 1 - \varepsilon \quad \text{for every } k \in \{1, \dots, n\}$$

and

$$\operatorname{dist}(v, \operatorname{co}(\mathbb{T} S(B_X, z^*, \varepsilon))) < \varepsilon$$

as desired.  $\square$

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