ON THE DIAMETER OF SETS WITH MAXIMUM NUMBER OF UNIT DISTANCES

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Dedicated to Professor H. Harborth on occasion of his sixtieth birthday

The problem of the maximum number of unit distances among n points in the plane is one of the best known and most intuitive problems in combinatorial geometry, but it is still far from its solution. Although the upper bound has been reduced from the initial $O(n^{\frac{3}{2}})$ [6] in several steps [1],[7] to $O(n^{\frac{4}{3}})$ [5], [8], [10], [11], the lower bound $\Omega\left(ne^{c\frac{\log n}{\log\log n}}\right)$ of the lattice section construction seems as distant as before; especially since the existence of strictly convex norms on $\mathbb{R}^2$ for which $cn^{\frac{4}{3}}$ unit distances are possible [3] implies that the current methods are exhausted by this result.

For this reason we now try to obtain structural results for the extremal sets, which may be more enlightening than the bounds on the numbers. Natural questions are the relation of the extremal sets to lattice subsets (are the lattice section examples in any way typical for extremal sets) as well as to the extremal sets for other similar problems (especially the number of distinct distances). Do extremal sets

- contain many collinear points?
- determine only few directions of unit distances?
- contain many parallelograms?
- have small rational dimension?
- contain only few distinct distances?
- Are there other frequent distances?
- Is the unit distance large or small within the set?

In [2] we showed that lattice section structure holds for sufficiently big number of points if we count unit distances only in a fixed number of directions. This behaviour starts, however, only for quite big sets, with respect to the cardinality as well as to the diameter. But the extremal sets for the maximum number of unit distances are quite small:

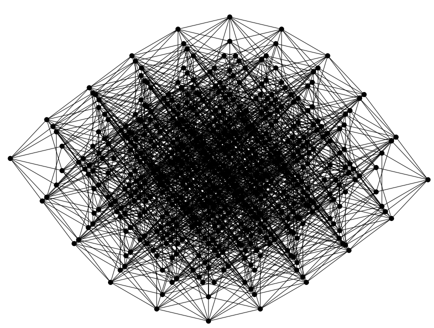
Theorem: For infinitely many n the diameter of each set of n points with maximum number of unit distances is bounded by $O\left((\log n)^{33}\right)$.

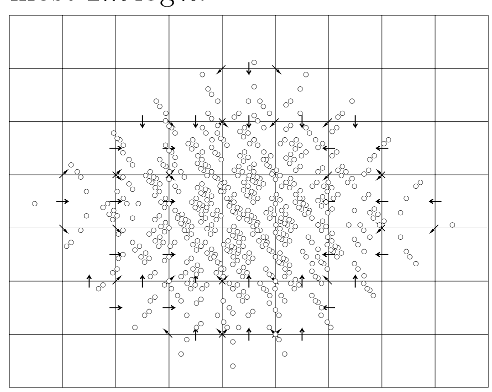
A simple application of Turán's theorem (using that the graph of distances smaller than one has independence number $O\left((\log n)^{66}\right)$) show that there are many distances smaller than the unit distance in the point set.

Corollary: For infinitely many n the number of distances smaller than one in a set of n points with maximum number of unit distances is at least $\Omega\left(\frac{n^2}{(\log n)^{132}}\right)$.

No attempt was made to find good bounds for the exponent of the $\log n$-factors; but the hypercube projections whose unit distance graph is the cube graph show that we cannot expect bounds smaller than $\log n$ by this method, since any subgraph of Q_d has a smaller average degree.

Proof: Let $u(n)$ be the maximum number of unit distances among n points in the plane, $d(n) := \max_{1 \le i \le n} \frac{2u(i)}{i}$ be the maximum average degree that can be reached in a unit distance graph with at most n points, and $\hat{n}$ a number such that $d(\hat{n}) = \frac{2u(\hat{n})}{\hat{n}}$ (since $d(2n) \geq d(n) + 1$ there is always such a number between n and $2n$). Let S be a set of $\hat{n}$ points with $u(\hat{n})$ unit distances. We subdivide this set by a unit square mesh and define a directed graph on the nonempty cells with an edge from cell Q_i to cell Q_j iff Q_j is one of the eight neighbours of Q_i and $Q_j \cap S$ contains at least twice the number of points of $Q_i \cap S$. Let m be the number of maximal cells, i.e. those cells with no directed edge going out. Each nonempty cell has distance at most $\log n$ in this graph to a maximal cell, since along a directed path the number of points in the current cell doubles with each edge. So S is covered by at most $4(\log n)^2 m$ cells, taking a metrical disc of radius $\log n$ around each of the maximal cells. Since the graph of unit distances of an extremal set is connected, this gives also a bound for the diameter which is at most $2m \log n$.





We now consider the maximal cell Q^* which contains the minimum number of points from S; let this number be k. Each of the neighbouring cells contains at most $2k$ points, so Q^* and its eight neighbours contain at most $17k$ points. Since each subgraph of the unit distance graph of S has a smaller average degree, each point has a degree at least $\frac{1}{2}d(\hat{n})$. All neighbours of points in Q^* are in Q^* or one of the neighbouring squares, so these at most

$17k$ points contain at least $\frac{1}{4}d(\hat{n})k$ edges. So we have $d(17k) \geq \frac{1}{34}d(\hat{n})$

Suppose now that for all but finitely many $\hat{n}$ we have an extremal set S whose diameter is bigger than $(\log \hat{n})^{33}$. For these we have $m(\hat{n}) \geq (\log \hat{n})^{32}$ and $k \leq \frac{\hat{n}}{m(\hat{n})} \leq \frac{\hat{n}}{(\log \hat{n})^{32}}$, so if the claim of the theorem is false, we have

$$d\left(17\frac{\hat{n}}{(\log \hat{n})^{32}}\right) \geq d(17k) \geq \frac{1}{34}d(\hat{n})$$

for all but finitely many $\hat{n}$. Since d increases monotonically and for all n there is a $\hat{n}$ between n and $2n$, this implies

$$d\left(34\frac{n}{(\log n)^{32}}\right) \geq \frac{1}{34}d(n) \quad \text{for all } n > n_0.$$

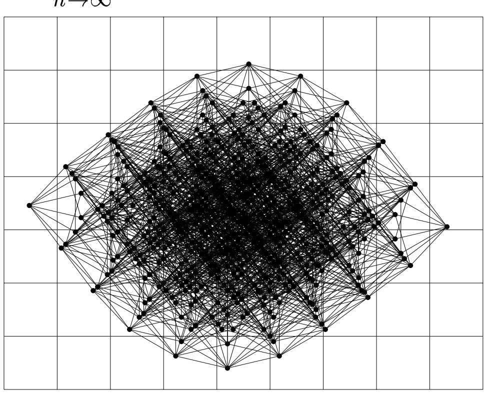
This is an upper bound on the growth rate of $d(n)$, obtained under the assumption that the diameter of the extremal sets S_n grows fast, by cutting out small (but dense) pieces of S_n. We can now compare this upper bound with the growth rate realized by the lattice section construction:

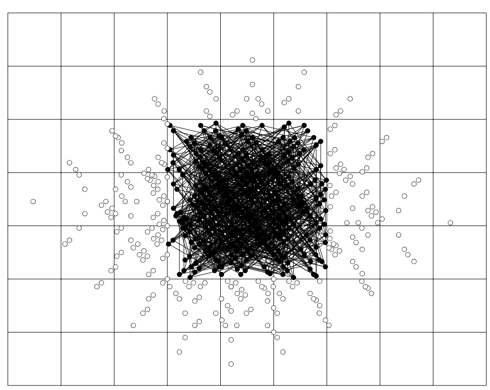
Let $d(n) =: \gamma_n e^{\frac{1}{9}\frac{\log n}{\log\log n}}$ and $p(n) := \frac{34\,n}{(\log n)^{32}}$, then

$\log\log p(n) = \log\log n - 32\frac{\log\log n}{\log n} + O\left(\frac{(\log\log n)^2}{(\log n)^2}\right)$ and

$$\begin{aligned}
\gamma_n &\leq 34\, e^{-\frac{1}{9}\frac{\log n}{\log\log n}}\, e^{\frac{1}{9}\frac{\log p(n)}{\log\log p(n)}}\, \gamma_{p(n)} \\
&= 34\, e^{\frac{1}{9}\frac{\log p(n)\log\log n - \log n \log\log p(n)}{\log\log n \log\log p(n)}}\, \gamma_{p(n)} \\
&= 34 \exp\left(\frac{1}{9}\, \frac{\log n \log\log n - 32(\log\log n)^2 + \log(34)\log\log n - \log n \log\log n + 32\log\log n - O\left(\frac{(\log\log n)^2}{\log n}\right)}{(\log\log n)^2 - O\left(\frac{(\log\log n)^2}{\log n}\right)}\right) \gamma_{p(n)} \\
&= 34 e^{\frac{1}{9}\left(-32 + O\left(\frac{1}{\log\log n}\right)\right)}\gamma_{p(n)} \\
&\leq 0.98\gamma_{p(n)} \quad \text{for all } n > n_0.
\end{aligned}$$

So $\gamma_{p^{-k}(n_0)} \leq 0.98^k \gamma_{n_0}$ and therefore $\liminf_{n\to\infty} \gamma_n = 0$. This contradicts the lower bound obtained by the lattice section construction (see [6], [9]) which is $\liminf_{n\to\infty} \gamma_n \geq 1$. This completes the proof of the theorem.





References

[**1**] J. Beck and J. Spencer: Unit distances, *J. Combinatorial Theory Ser. A* **37** (1984), 231–238

[**2**] P. Brass: On point sets with many unit distances in few directions, to appear in *Discrete & Computational Geometry*

[**3**] P. Brass: On lattice polyhedra and pseudocircle arrangements, to appear in the conference volume *Colloquium Carolus Magnus*

[**4**] P. Brass: Häufige Abstände in endlichen Punktmengen, *Jahrbuch Überblicke Mathematik* 1998, 66–75

[**5**] K.L. Clarkson, H. Edelsbrunner, L.J. Guibas, M. Sharir and E. Welzl: Combinatorial complexity bounds for arrangements of curves and spheres, *Discrete & Computational Geometry* **5** (1990), 99–160

[**6**] P. Erdős: On sets of distances of n points, *Amer. Math. Monthly* **53** (1946), 248–250

[**7**] S. Józsa and E. Szemerédi: The number of unit distances in the plane, 939–950 in *Infinite and Finite Sets* (A. Hajnal e.a., Eds.) North Holland, Amsterdam, 1975

[**8**] J. Pach and P.K. Agarwal: *Combinatorial Geometry*, Wiley, New York, 1995

[**9**] C. Schade: *Exakte Maximalzahlen gleicher Abstände*, Diplomarbeit bei H. Harborth, Universität Braunschweig, 1993

[**10**] J. Spencer, E. Szemerédi and W. Trotter: Unit distances in the euclidean plane, 293–304 in *Graph Theory and Combinatorics* (B. Bollobás, Ed.) Academic Press, London, 1984

[**11**] L.A. Székely: Crossing numbers and hard Erdős problems in discrete geometry, Preprint 1996