Polygonal Chains with pairwise identical Hausdorff Distance

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Abstract. The complexity of the Voronoi diagram for n polygonal chains and the Hausdorff distance is shown to be worst case $2^n - 1$. Some bounds are given on the number of chains with limited size and fixed distance, and the relation between the sizes of two polygonal chains and their distance is examined.

Keywords: Polygonal chains, Hausdorff distance, Voronoi Diagram, pairwise distances

1. Introduction

The necessity of comparing polygonal chains arises in proximity problems, as in nearest neighbour searches, the determination of closest pairs or clustering problems. The application might go as far as to approximate shapes, such as roads or rivers, by polygonal shapes. The aim mainly is to construct a data structure, in which to manage the chains efficiently.

As one of the first attempts to calculate the Hausdorff distance under translation for polygons, [ABB92] gives an algorithm which is not optimal in it's running time, but uses the fact that in the optimal placement of the polygons the distance actually occurs three times between different point pairs within the polygons. Only the decomposition of the polygons in line segments is needed for the algorithm to operate. It might also be used to calculate the distance between arbitrary sets of line segments. As is shown in [AST92], the running time might be improved. The given algorithm is based upon parametric search. In the most current proposals, point sets are examined and the aim is to present approximate solutions, as in [FCI99] or data

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structures are adapted, as in [BK01]. Not much of the geometric properties of polygonal chains are considered.

This paper tries to cover some of the basic properties of the underlying metric space, exploiting some geometric properties of its building blocks, that is the polygonal chains.

For $p \in \mathbf{R}^2$ and compact $P, Q \subset \mathbf{R}^2$ let

$$\vec{d}(p,Q) = \min_{q \in Q} \| p - q \|$$
$$\vec{d}(P,Q) = \max_{p \in P} \vec{d}(p,Q)$$
$$d_H(P,Q) = \max\{\vec{d}(P,Q), \vec{d}(Q,P)\}$$
$$d(P,Q) = \min_{t \in \mathbf{R}^2} d_H(P + t, Q)$$

where $\| \cdot \|$ is the Euclidean distance in $\mathbf{R}^2$. $\vec{d}$ is called the single sided Hausdorff distance, d_H the Hausdorff distance, and d the Hausdorff distance under translation.

This paper is aimed primarily at the subspace where the points are polygonal chains and the Hausdorff distance under translation, although some of the results apply to a more general case.

For a given compact set $P \subset \mathbf{R}^2$ denote by K_P the smallest enclosing circle around P with radius r_P and centre point m_P, by $\overline{K}_P$ the corresponding closed disk. The radius of the smallest enclosing circle is a natural measure for the size of a set, for this is the Hausdorff distance under translation to a single point.

In any metric space, the worst case complexity of the Voronoi diagram is at most $2^n - 1$, that is any Voronoi diagram representing n points has at most that many different nonempty Voronoi cells. In the second section, a simple example is constructed to show that the Voronoi diagram of polygonal chains can be of that order. So, if an algorithm tries to locate the nearest neighbours to a given query chain out of a set of chains, it should even with logarithmic search time, turn out to be linear in the total number of chains. The drawback of the example is that the larger n, the larger the chains grow in size, or the smaller the difference between them gets. This aspect is investigated in section four.

Section three is of a somewhat different nature and gives boundaries on the distance of two chains, with respect to their size.

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2. The complexity of the Voronoi diagram

Let P, together with a metric d, be a metric space and let $\boldsymbol{P} \subset \mathsf{P}$. For any nonempty subset $\boldsymbol{R}$ of $\boldsymbol{P}$, the *Voronoi cell* of $\boldsymbol{R}$ is the set $V(\boldsymbol{R}) \subset \mathsf{P}$ of points equidistant from all elements of $\boldsymbol{R}$ and further apart from the remaining points in $\boldsymbol{P}$:

$$V(\boldsymbol{R}) = \{P \in \mathsf{P} \mid d(P, R_i) = d(P, R_j) \text{ and } d(P, R_i) < d(P, Q) \text{ for all } R_i, R_j \in \boldsymbol{R}, Q \in \boldsymbol{P} \setminus \boldsymbol{R}\}$$

For convenience, let $V(\varnothing) = \varnothing$.

The *Voronoi diagram* of $\boldsymbol{P}$ is the collection of all nonempty Voronoi cells $V(\boldsymbol{R})$, $\boldsymbol{R}$ a subset of $\boldsymbol{P}$:

$$Vor(\boldsymbol{P}) = \{ V(\boldsymbol{R}) \mid \boldsymbol{R} \subseteq \boldsymbol{P}, V(\boldsymbol{R}) \neq \varnothing \}$$

The *combinatorial complexity* of the Voronoi diagram is the number of cells in $Vor(\boldsymbol{P})$.

Clearly, $V(\boldsymbol{R}) \cap V(\boldsymbol{R}') = \varnothing$ for $\boldsymbol{R} \neq \boldsymbol{R}'$. Further observe that the combinatorial complexity of a Voronoi diagram of n points of a metric space is trivially bounded by the number of nonempty subsets of the points, that is $2^n - 1$.

Theorem 1. *In the space of polygonal chains, together with the Hausdorff metric under translation, there exists a set of n polygonal chains, such that the corresponding Voronoi diagram has combinatorial complexity* $2^n - 1$.

Proof. The n polygonal chains $P_1, .., P_n$ are constructed as follows: For $l > 0$, take a segment of length l and divide it into $n+1$ equal parts, thus producing n marks on it. P_i now has the segment as its main shape, except a steep peak of height t at the i-th mark (see Figure 1).
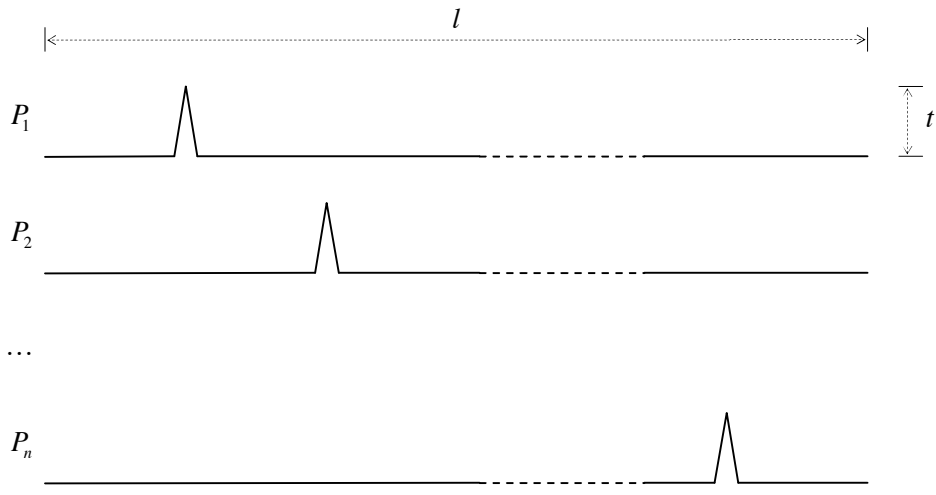
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Figure 1. The set of polygonal chains

Provided that the height t of the peaks is relatively small compared to distances of the marks, we have $d(P_i, P_j) = t$ for $i \neq j$, which will be shown next.

Aligning two different segments P_i and P_j at their leftmost points (see Figure 2) will result in a placement where $d_H(P_i, P_j) = t$, and this distance is assumed from the peak points towards the "main segments". We will try to find a movement which brings the peak point s of P_i closer to P_j as the only chance to reduce the overall distance.

Assume wlog. that $i < j$, that is the peak point of P_i in the original placement is to the left of the peak point of P_j. A vertical move of P_i might bring s closer towards the "main segment" of P_j, but will also increase the distance from the peak point of P_j to P_i, thus increasing the total distance $d_H(P_i, P_j)$. So the only chance to find a closer point for s is to move P_i in a way such that s is closer to the peak of P_j. But this will move P_i such a distance to the right, that there is no point of P_i within distance t for the leftmost point of P_j, so the overall distance $d_H(P_i, P_j)$ will again be larger than t.

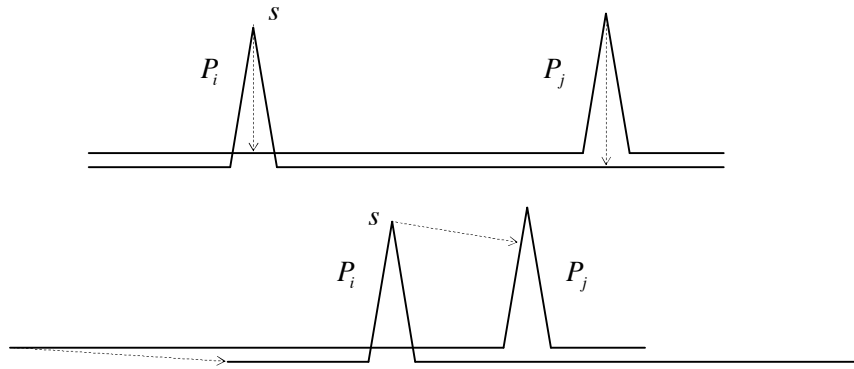This shows that actually $d(P_i, P_j) = t$.

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Figure 2. Determining the Hausdorff Distance under Translation

Now take a subset P from $\{P_1,..,P_n\}$, consisting of at least two different polygonal chains, and construct the chain $Q \in V(P)$ as follows: Q also consists of the segment of length l, but has a peak in position i for every $P_i \in P$, each of height $t/2$. Aligning any $P_i \in P$ with Q at their leftmost points results in a distance of $t/2$, which is optimal; for the distance from the peak point of P_i to the corresponding peak of Q is exactly $t/2$, as is the distance from any other peak point of Q to the "main segment" of P_i (Figure 3a). Again, a motion argument will show that this position is optimal.
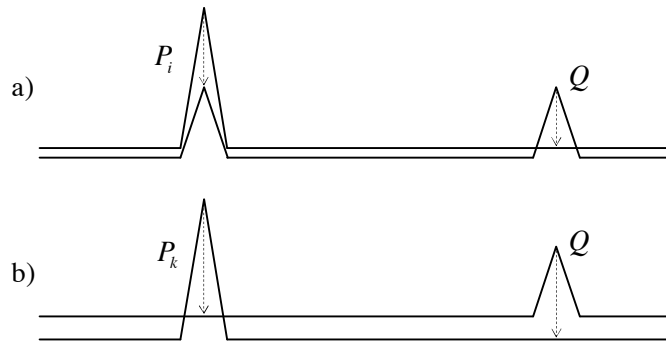

Figure 3. Comparing Q to the chains

Last, comparing Q to any P_k not in P, will result in a distance larger than $t/2$ (Figure 3b): Bringing the two leftmost points together will result in a distance of t from the peak of P_k to Q, but this distance can be decreased by moving P_k down

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towards the "main segment" of Q. But while moving, the distance from any other peak of Q, which originally was $t/2$, will increase. Again, there is no way to bring different peaks together, so the distance is larger than $t/2$ (in fact, the smallest distance is $3t/4$). So what is shown is that $Q \in V(P)$, so $V(P) \neq \varnothing$. Of course, $V(P_i) \neq \varnothing$ for every i, since $P_i \in V(P_i)$, that is $V(P) \neq \varnothing$ for any non empty subset of $\{P_1,..,P_n\}$, and the theorem is proven.

As a result from the theorem, a nearest neighbour search seams to be very difficult from the combinatorial point of view. But, in the given example, with growing n, either l has to grow as well, making the chains "larger", or t has to be reduced, so the distance between the chains becomes smaller. However, in a "real live application", the size of the polygonal chains might be limited and the distance between them negligible when being small. Some bounds on the number of chains in dependence of the size and the pairwise distances is given in section 4.

3. Size and distances

When comparing two sets with respect to the Hausdorff metric under translation, it is quite obvious that the larger the sets become in diameter, the larger the distance between them might become. On the one hand, when one chain is much larger than the other, this size difference will make up the major part of the distance. On the other hand, when their sizes are almost the same, their distance might be small, when they are almost identical, or quite large, when they are not - this section answers the question on how large or small it then might be. Although the results seem to be quite obvious, it is not quite clear how to construct an example with the maximum distance: Many deformations of the one chain which increase the distance will be cancelled by a translation of the other.

For a given compact set $P \subset \mathbf{R}^2$ denote by K_P the smallest enclosing circle around P with radius r_P and centre point m_P, by $\overline{K}_P$ the corresponding closed disk. In the following, let P and Q be two compact, connected subsets of $\mathbf{R}^2$ (e.g. polygonal chains). For the notions of d, d_H and $\vec{d}$, refer to section 1.

Lemma 1. $|r_P - r_Q| \leq d(P,Q)$.

Proof. Let without loss of generality $r_P \geq r_Q$, and let P and Q be aligned such that m_P and m_Q are identical. Denote with $\overline{P} = P \cap K_P$ the points of P which lie on K_P.
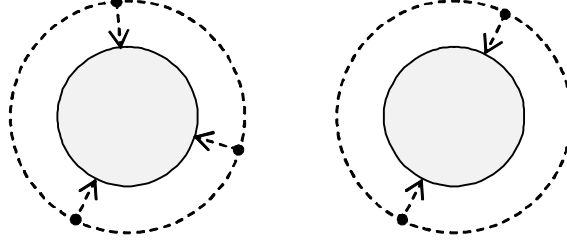
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Figure 4. Determining $\vec{d}(\overline{P} + t, \overline{K}_Q)$

Although the details might be weary, it is quite apparent from Figure 4, that $\vec{d}(\overline{P}, \overline{K}_Q) = r_P - r_Q$ and $\vec{d}(\overline{P} + t, \overline{K}_Q) \geq r_P - r_Q$ for any $t \in \mathbf{R}^2$. But, as $\overline{P} \subseteq P$ and $Q \subseteq \overline{K}_Q$, we have

$$r_P - r_Q \leq \vec{d}(\overline{P} + t, \overline{K}_Q) \leq \vec{d}(P + t, \overline{K}_Q) \leq \vec{d}(P + t, Q)$$

for every $t \in \mathbf{R}^2$, from which $r_P - r_Q \leq d(P,Q)$ and hence the claim easily follows.

Lemma 2. *If P and Q are compact subsets of* $\mathbf{R}^2$*, then* $d(P,Q) \leq \sqrt{r_P^2 + r_Q^2}$.

Proof. Translate the sets such that the centres of the enclosing circles lie in the origin, that is $m_P = m_Q = 0$. Pick any point $p \in P$. If $p = 0$, then obviously there exists a point $q \in Q$ with $d(p,q) \leq r_Q < \sqrt{r_P^2 + r_Q^2}$. So let $p \neq 0$. Since K_Q is the smallest enclosing circle around Q, there must be a point $q \in Q$ lying on K_Q within the closed half space bounded by the line through the origin and perpendicular to p, that is $\| q \|^2 = r_Q^2$ and $p \cdot q \geq 0$ (see Figure 5). We have

$$\| p - q \|^2 = \| p \|^2 - 2pq + \| q \|^2 \leq \| p \|^2 + \| q \|^2 \leq r_P^2 + r_Q^2$$

so $d_H(p,Q) \leq \sqrt{r_P^2 + r_Q^2}$ for each $p \in P$; symmetrically, $d_H(q,P) \leq \sqrt{r_Q^2 + r_P^2}$ for each $q \in Q$, and the claim follows.
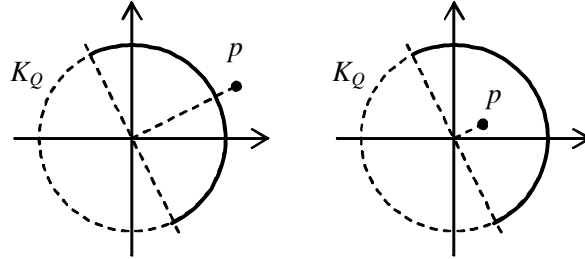
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Figure 5. q is on the bold arc.

In order to improve the result for connected subsets, we need the following

Observation. *If P and Q are compact and connected and $r_P \geq r_Q$, then there exists a translation t with $\overline{K}_{Q+t} \subseteq \overline{K}_P$ and $(Q+t) \cap P \neq \varnothing$:*

First consider the case $r_P = r_Q$. There is only one translation with $\overline{K}_{Q+t} = \overline{K}_P$, namely that which maps m_Q to m_P. For simplicity, assume $m_P = m_Q$ and hence $K_P = K_Q$, and for the sake of contradiction assume further that the intersection of P and Q is empty.

If K_P can be spanned by only two points $p_1, p_2 \in P$: Since P is connected and the points p_1, p_2 lie upon a line l through m_P, the only points of Q which lie on $K_Q = K_P$ must lie on the same side of l, which is a contradiction, for K_Q is the smallest enclosing circle of Q (note that $P \cap Q = \varnothing$, so $p_1, p_2 \notin Q$).

So there are three points $p_1, p_2, p_3 \in P$ lying on K_P with the property, that the line through one of these points and m_P has the remaining two points on different sides, or else K_P would not be the minimal circle. Now Q has points on K_P, let one of these points lie on the arc between p_i and p_j not containing the third point p_k. Then all points of Q lying on K_P also lie between p_i and p_j, for P is connected. So all border points of Q lie on the same side of the line l_i through p_i and m_P, which again yields the contradiction.
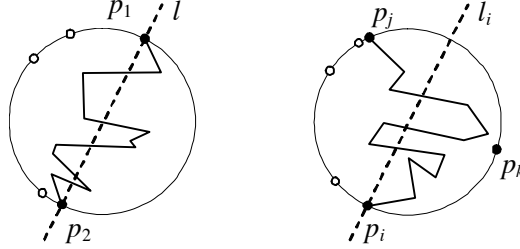
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Figure 6. K_P in the cases $r_P = r_Q$

So the statement is true if $r_P = r_Q$. If now $r_P > r_Q$, blow up Q until it has the same size as P, find the intersection point and shrink Q back to its original size while leaving the intersection point invariant. More specifically, translate P and Q such that $m_P = m_Q$ lie in the origin. Let $Q' = (r_P / r_Q)Q$. Then $K_P = K_{Q'}$, so P and Q' intersect in a point q. Let

$$Q'' = (r_Q / r_P)Q' + (1 - r_Q / r_P)q = Q + (1 - r_Q / r_P)q .$$

Since $q \in Q'$, we have

$$q = (r_Q / r_P)q + (1 - r_Q / r_P)q \in Q'' ,$$

and since $K_{Q''} = (r_Q / r_P)K_{Q'} + (1 - r_Q / r_P)q = (r_Q / r_P)K_P + (1 - r_Q / r_P)q$, for any $x \in K_{Q''}$ there exists an $y \in K_P$ such that $x = (r_Q / r_P)y + (1 - r_Q / r_P)q$ and

$$\| x \| \leq (r_Q / r_P) \| y \| + (1 - r_Q / r_P) \| q \| \leq (r_Q / r_P)r_P + (1 - r_Q / r_P)r_P = r_Q + r_P - r_Q = r_P ,$$

so $K_{Q''} \subseteq \bar{K}_P$ and hence $\bar{K}_{Q''} \subseteq \bar{K}_P$, which shows the claim.

Lemma 3. *If P and Q are connected and compact subsets of $\mathbf{R}^2$, then $d(P,Q) \leq \max\{r_P , r_Q\}$.*

Proof. Without loss of generality assume $r_P \geq r_Q$, further translate P such that $m_P = 0$. If $-P = \{-p \mid p \in P\}$, then obviously $-P$ is compact and connected and further $r_{-P} = r_P$ and $m_{-P} = 0$. Due to the previous observation, it is possible to translate Q in such a way that

$$Q \cap -P \neq \varnothing \quad \text{and} \quad \bar{K}_Q \subseteq \bar{K}_{-P} .$$

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Choose $s \in Q \cap -P$. We will show $d_H(P+s,Q) \le r_P$:

    (i) $\vec{d}(P+s,Q) \le r_P$: Let $p \in P$. Since $m_P = 0$ and $p \in \overline{K}_P$, we have $\| p + s - s \| = \| p \| \le r_P$, which shows $\vec{d}(p+s,Q) \le r_P$, as $s \in Q$.

    (ii) $\vec{d}(Q,P+s) \le r_P$: Let $q \in Q$. Since $q \in \overline{K}_Q \subseteq \overline{K}_{-P}$ and $m_{-P} = m_P = 0$, we have $\| q \| \le r_{-P} = r_P$, and since $s \in -P$, we have $-s \in P$, and so $\vec{d}(q,P+s) \le \| q - (-s+s) \| = \| q \| \le r_P$.

    Now $d(P,Q) \le d_H(P+s,Q) \le r_P$, and the Lemma is proven.

The connectedness of *both* sets is necessary to achieve the better bound of Lemma 3. Figure 7 shows an example for a connected P and an unconnected Q, where $d(P,Q)$ is near to $\sqrt{r_P^2 + r_Q^2}$. It should be noted that in the example r_P and r_Q are of roughly the same size. The example kind of collapses if $r_Q \le r_P/2$ or $r_P \le r_Q/2$. The relation between the ratio of the radii in unconnected or partially unconnected settings might be further explored.
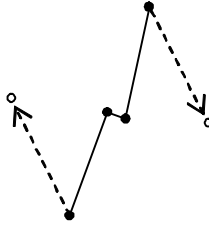

Figure 7. The rough boundary for unconnected sets is sharp.

Of course, in each of the three Lemmas, the given bounds are sharp.

4. A tight packing

In the Euclidean space of dimension d there exist $d+1$ points with pairwise same distances. As observed in the second section, in the space of polygonal chains with the Hausdorff distance, there exists no such limit. This problem has influences on the design of a nearest neighbour search algorithm, for if n points $p_1,..,p_n$ are given with $| p_i - p_j | = t$ for $i \ne j$ and, for a given point q, $| q - p_k | \le t/2$, then p_k is a nearest neighbour, for $| q - p_l | < t/2$ yields $| p_k - p_l | \le | p_k - q | + | q - p_l | < t/2 + t/2 = t$, so $p_k = p_l$.

    In $\mathbf{R}^d$, the possible number of points with pairwise same distance does neither depend on the position of the points in the space, nor on the actual value of the distance. With polygonal chains, things look different: There is only a finite number of "small" chains with pairwise same distance, and their number is also limited by

the actual value of the distance. The size of a polygonal chain will again be measured by the radius of the smallest enclosing circle.

To start with, let $P_1, P_2, ..., P_n$ be polygonal chains with $d(P_i, P_j) \geq t$ for a fixed $t \in \mathbf{R}$ and all $i \neq j$, denote with $r_i = r(P_i)$ the radius of the smallest enclosing circle around P_i, or the size of P_i for short. Without loss of generality, let $r_1 \leq r_2 \leq .. \leq r_n = r$. By Lemmas 1 and 3, we have $t \leq d(P_i, P_j) \leq \max\{r_i, r_j\}$, and so $t \leq r_i$ for all $i > 1$.

Lemma 4. *If $P_1, P_2, ..., P_n$ are polygonal chains with $d(P_i, P_j) \geq t$ for all $i \neq j$ and $r = \max r(P_i)$, then $n \leq 2^{2\pi(\sqrt{2}r/t+1)^2}$*

Proof. Translate each P_i such that the midpoint of the corresponding smallest enclosing circle lies in the origin. Then, $d_H(P_i, P_j) \geq d(P_i, P_j) \geq t$, and all points of all P_i lie within the circle C_r of radius r around the origin.

From the $\frac{t}{2} \times \frac{t}{2}$-grid, take every cell that is intersected by $\overline{C}_r$. Since every such cell lies within the circle of radius $r + \sqrt{2} \cdot t/2$, there are at most $\pi(r + \sqrt{2} \cdot t/2)^2 / (t/2)^2 = 2\pi(\sqrt{2}r/t + 1)^2$ such cells. For a given P_i, let S_i be the set of midpoints of these squares, that are intersected by P_i; we then have $d_H(P_i, S_i) \leq t/\sqrt{8}$, since for every $p \in P_i$ there was an $s \in S_i$ chosen, and every $s \in S_i$ was chosen because there is a point in P_i with a corresponding distance.
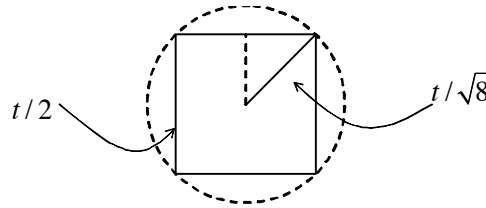

$t/2$ $t/\sqrt{8}$

Figure 8.

Note that, if $S_i = S_j$, then $d(P_i, P_j) \leq d(P_i, S_i) + d(P_j, S_j) \leq t/\sqrt{8} + t/\sqrt{8}$ $t/\sqrt{2}$ t and hence $P_i = P_j$. Since there are only a finite number of possibilities to construct S_i, namely $2^{2\pi(\sqrt{2}t/r+1)^2}$ by choosing squares, this number also limits n.

Lemma 5. *The maximum possible number n of polygonal chains $P_1, P_2, ..., P_n$ with $r(P_i) \leq r$ for all i and $d(P_i, P_j) = t$ for all $i \neq j$ is $2^{\Theta(r^2/t^2)}$.*

Proof. The upper bound is given by Lemma 4. An example is constructed to proof the lower bound: Take the $t \times t$-grid points contained in the circle of radius r around the origin. The "base polygonal chain" starts at the up most, left point, moving right,

going down three rows, moving left, down again three rows and so on, till the circle is filled (cf. Figure 9). Upon all but the first horizontal segment, collect every third point, starting from the third from the right on left moves and from the third from left on right moves as candidate points. If such a candidate point is "marked", instead of running through the point, the chain will take a detour just to the grid point above it (see Figure 10).
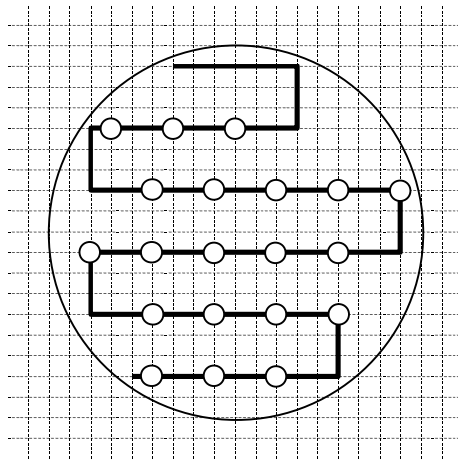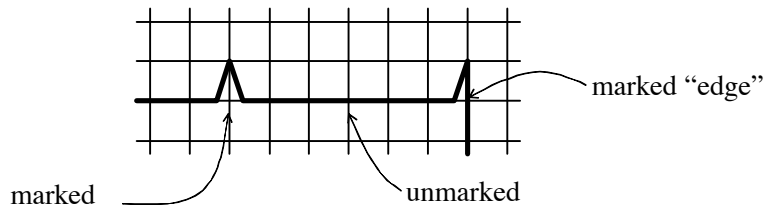

Figure 9. Zigzag through the circle.

Figure 10. Taking a detour (with a variation on edge points).

If one such chain P has a marked candidate point, whereas another chain Q has not, but Q has a marked candidate point which is unmarked in P, those chains will have a distance of exactly t: Let $p \in P$ respectively $q \in Q$ be the "peak points". The only points of Q within distance of $2t$ from p are the points on the segment below p, so P must be moved down to decrease the total distance. But this will increase the distance of q, and hence of Q, to P, for the same argument holds for q. Finally, any

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move larger than t will leave one of the bottom most points unmatched within a distance of t. The original distance t cannot be improved.

Now, about a third of the roughly $\pi r^2 / t^2$ many grid points within the circle are touched by a horizontal segment and again asymptotically a ninth of them are chosen as candidate points, that is there are $\Theta(r^2 / t^2)$ candidate points. To finally construct the set of polygonal chains, divide the candidate points in pairs; there are still $\Theta(r^2 / t^2)$ such pairs. For every pair, choose one of the combinations "marked/unmarked" or "unmarked/marked", resulting in $2^{\Theta(r^2/t^2)}$ different polygonal chains. When selecting any two of them, they will have the property that one has a marked candidate point where the other has not and vice versa, and so they have a distance of exactly t, which concludes the construction.

Corollary. *The maximum number of polygonal chains of size maximal r and a pairwise distance of at least t is* $2^{\Theta(r^2/t^2)}$.

Proof. If true for a distance of exactly t, as implied by Lemma 5, the statement will hold by Lemma 4 for the weaker assumptions.

The above construction leads to a constant in the exponent of below $\pi / 19 \approx 0.1745$, or a lower bound of approximately $1.1286^{r^2/t^2}$. When constructing a set of many chains with a pairwise distance of at least t, it seams natural to have as many chains as possible close together, giving a larger total number. Assumingly, taking obvious extensions into account (scaling, rotations), the number of chains arising from the example will increase the asymptotic growth only marginally.

So far, the results of this section can be applied to compact and connected sets, for polygonal chains are compact and connected, and any compact and connected set can be approximated with respect to the Hausdorff distance up to an arbitrary distance by a polygonal chain. But, as might be a drawback in the proof of Lemma 5, the polygonal chain then might be composed of numerous small segments.

The next Lemma takes the number of segments as another grade of complexity into account, giving a bound which is smaller, when the number of segments is limited.

Lemma 6. *Let* $P_1, P_2, .., P_n$ *be polygonal chains with a pairwise distance of at least t and of size at most r, each consisting of at most k points (or, alternatively, k-1 segments). Then* $n < (2\pi)^k (\sqrt{2}r/t + 1)^{2k}$.

Proof. As above in the proof of Lemma 4, move each P_i such that the centre of its smallest enclosing circle lies in the origin. Now every vertex point p of P_i resides inside a $\frac{t}{2} \times \frac{t}{2}$-grid cell; map p to the corresponding centre point p' of the cell

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(taking any centre point in case of ties). The chain $P = (p_1, p_2, ..., p_m)$ itself will be mapped to P', the union of the segments (or points) $p_i' p_{i+1}'$. As above, $|p_i - p_i'| \le t/\sqrt{8}$ for all i. But since the single sided Hausdorff distance from one segment to another will be assumed at an endpoint, it follows that

$$\vec{d}_H(p_i p_{i+1}, P') \le \vec{d}_H(p_i p_{i+1}, p_i' p_{i+1}') = \max\{ \vec{d}(p_i, p_i' p_{i+1}'), \vec{d}(p_{i+1}, p_i' p_{i+1}') \} \le$$
$$\le \max\{ \| p_i - p_i' \|, \| p_{i+1} - p_{i+1}' \| \} \le t/\sqrt{8}$$

and hence $\vec{d}_H(P, P') = \max \vec{d}_H(p_i p_{i+1}, P') \le t/\sqrt{8}$. Since the same argument holds in the other direction, we have $d_H(P, P') \le t/\sqrt{8}$, and thus again $P_i' \ne P_j'$ for all $i \ne j$.

Now all that remains is to count the possible number of different P': Any grid cell of size $\frac{t}{2} \times \frac{t}{2}$ that is touched by the circle of radius r lies within the circle of radius $r + \sqrt{2}t/2$ and so within an area of $\pi(r + \sqrt{2}t/2)^2$. As each cell has an area of $t^2/4$, there are less than $2\pi(\sqrt{2}r/t + 1)^2$ grid cells. Choosing a sequence of k of these cells gives the upper bound $(2\pi)^k(\sqrt{2}r/t + 1)^{2k}$. Not every sequence is a feasible projection of a polygonal chain due to the translation towards the origin, and some sequences correspond to the same chains P', but the upper bound is proven to be correct.

Remark. The boundaries of Lemmas 4 and 6 can be simplified: Any $\frac{t}{2} \times \frac{t}{2}$-grid cell touched by the circle of radius r lies completely inside a square with side length $2r$, so there are no more than $(2r)^2/(t/2)^2 = 16(r/t)^2$ such cells. So, there are no more than $2^{16^k(t/r)^{2k}}$ polygonal chains of size at most r with pairwise distance at least t, and no more than $16^k(r/t)^{2k}$ of them when they only consist of k points.

Acknowledgment

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