The Art Gallery Theorem for Rectilinear Polygons with Holes

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B 94–22
December 1994

Abstract

It is proved that any rectilinear polygon on n vertices, possibly with holes, can be partitioned into at most $\left\lfloor \frac{n}{4} \right\rfloor$ rectilinear stars each of size at most 12.

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1 Introduction

One of the well-known visibility problems in Computational Geometry is the Art Gallery Problem originally raised by V. Klee in 1973. It asks how to station a number of *watchmen* in a polygon (*art gallery*) such that they can see every point of the polygon. This is clearly equivalent to covering the polygon by stars. A star in a polygon P is the union of a family of convex regions all contained in P with a non-empty common intersection. Klee's problem has been solved by V. Chvátal [3] in 1976 proving that $\lfloor \frac{n}{3} \rfloor$ watchmen are always sufficient and sometimes necessary to guard a simple polygon on n vertices. Since then many variations of the problem and its algorithmic aspects have been studied. The interested reader is referred to the excellent monograph of J. O'Rourke [14] (see also [16]) and a recent survey by T. Shermer [17] containing almost all material on this subject.
Here we are studying the problem for rectilinear polygons with holes. In 1983 J. Kahn, M. Klawe, and D. Kleitman [12] showed that for simply connected rectilinear polygons on n vertices $\lfloor \frac{n}{4} \rfloor$ is the tight bound. Later J. O'Rourke [15] and E. Györi [5] gave much simpler proofs of this result. Especially they showed that the watchmen can be chosen in such a way that each has to watch a rectangle or an L–shaped region only, i.e. a rectilinear star of size ≤ 6. The main result of the present paper (which is a detailed and revised version of a conference paper [7]and partly of [8]) shows that the $\lfloor \frac{n}{4} \rfloor$–bound also holds for rectilinear polygons with an arbitrary number of holes (Conjecture 5.2 in [14]). The case of rectilinear polygons with at most 2 holes has previously been solved by A. Aggarwal in his thesis [1]. Moreover, we can show that our solution corresponds to partitioning the polygon into rectilinear stars each of size at most 12. While in the 1-connected case the guards can be chosen to sit in vertices of the polygon (*vertex guards*) we have now to allow them to sit in any point of the polygon (*point guards*). Figure 1a provides an example of a polygon on 14 vertices which requires $\lfloor \frac{14}{4} \rfloor = 3$ point guards but 4 vertex guards. Iterating this example as shown in Figure 1b proves a $\lfloor \frac{2n}{7} \rfloor$–lower bound hence disproving Conjecture 5.3 in [14] due to Aggarwal. We mention that size 14 is the smallest possible to show that using vertex guards only one cannot prove an $\lfloor \frac{n}{4} \rfloor$ upper bound. So we conjecture the $\lfloor \frac{2n}{7} \rfloor$–bound to be optimal. The best known upper bound for the vertex guard model is $\lfloor \frac{n}{3} \rfloor$, see [10]. However, the example in Figure 1 is consistent with Shermer's $\lfloor \frac{n+h}{4} \rfloor$–conjecture [14] for the vertex guard number in rectilinear polygons with h holes.

As to the discussion of the main result we recall that it was the first tight bound for the guard number in polygons with holes. In a certain sense it is exceptional since it states that the general case with holes is not harder (ignoring the increased star size) than the simply connected case. Recently there has been proved a tight $\lfloor \frac{n+h}{3} \rfloor$–bound for the number of point guards in general polygons with holes [9], [2] and a tight $\lfloor \frac{3n+4h+4}{16} \rfloor$–bound for the number of line guards (the watchmen are allowed to patrol along line segments) in rectilinear polygons [6]. So we see that both for vertex guards and line guards in rectilinear polygons as well as for the point guard

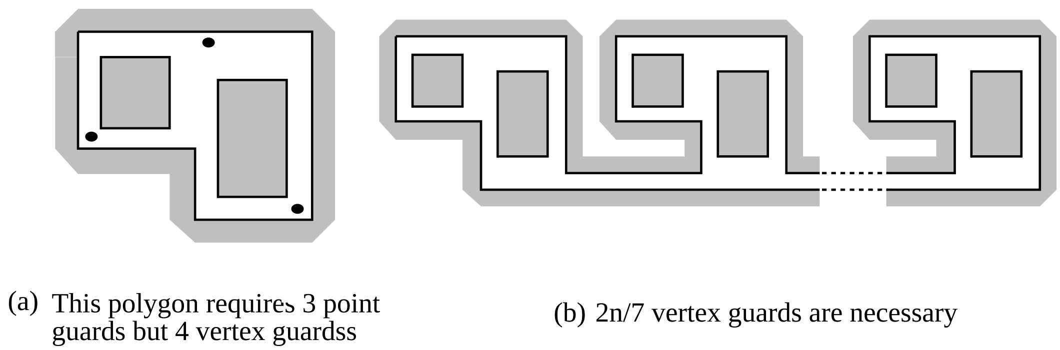


Figure 1: A lower bound for the vertex guard model

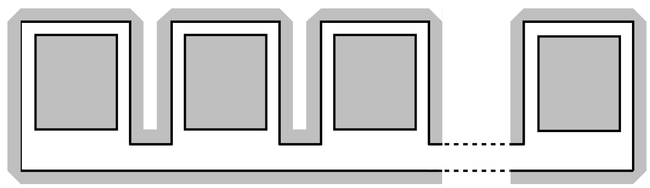


Figure 2: n/4 point guards are sometime necessary

model in general polygons the number of holes influences the bounds. The reader is referred to [6] for a more detailed discussion on the relationship between the number of holes and bounds for other guard types in rectilinear polygons.

The paper is organized as follows. In the second section we give all the necessary technical prerequisites. In Section 3 we describe problem reductions and prepare a translation of a reduced problem into a graph–theoretic formulation which will be dealt with in Section 4. Finally we briefly discuss algorithmic aspects and we conclude with a more detailed discussion of the result and related open problems.

2 Preliminaries

Let P be a rectilinear polygon [1] possibly with holes, i.e. it is bounded by horizontal and vertical edges only. We denote by $bd(P)$ the boundary of P which consists of the boundary of all holes and the outer periphery and by $int(P)$ the interior of P. Throughout this paper by the term polygon we always mean the union of its boundary and interior. So we can think of such a polygon as follows. We start from a simple rectilinear polygon and cut out from it a certain number of holes.

[1] Rectilinear polygons have also been called orthogonal and isothetic.

An (n,h)–polygon is a rectilinear polygon on n vertices with h holes. We define the following partial ordering for polygons: An (n,h)–polygon P is smaller than an (n',h')–polygon P' denoted by $P \leq P'$ iff $n \leq n'$ and $h \leq h'$.

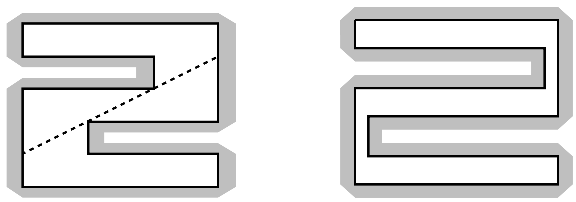


Figure 3: 2 and 3 stars are necessary

Let $Q^{(1)}(x),\ldots,Q^{(4)}(x)$ denote the four quadrants of the plane with respect to a point x. For points x and y in the plane we denote by $[x,y]$ the corresponding connecting closed line segment and by (x,y) the open line segment. The two coordinates of a point x are denoted by x_1 and x_2. Further, let $x\#y$ be the point with first coordinate x_1 and second coordinate y_2. Using this notation we define the rectangle $R[x,y]$ to be spanned by $x, y, x\#y, y\#x$ and set $R(x,y) = int(R[x,y])$. We say that two points $x,y \in P$ *see* each other, denoted by $x \leftrightarrow y$, iff $[x,y] \subset P$. In this paper, however, we use the following stronger visibility notion [13]: The points x and y are *rectangularly visible* to each other, denoted by $x\Box y$, iff $R[x,y] \subseteq P$. An edge $[y,z]$ is rectangularly visible from x if $x\Box y$ and $x\Box z$.

Now we can formulate the main problem we are interested in. For $x \in P$ we define $V(x) = \{y \in P | x\Box y\}$. Clearly this set forms a *rectilinear star* S, i.e. it is a union of rectangles all contained in P with a non–empty common intersection (the *kernel* $ker(S)$). Further we set $V^{(i)}(x) = V(x) \cap Q^{(i)}(x)$, for $i = 1,\ldots 4$.
We say that a family $\{x_i\}_{i\in I}$ of points (*guards*) covers the polygon P iff $P = \bigcup_{i\in I} V(x_i)$. For a given polygon we look for a minimal covering, that is we want to determine the minimal cardinality $r(P)$ of a point family covering P. Finally, let $r(n,h) = max\{r(P) | P\ is\ an\ (n,h)-polygon\}$.

The aim of this paper is to prove that $r(n,h) = \left\lfloor \frac{n}{4} \right\rfloor$. First of all it is trivial to show that $\left\lfloor \frac{n}{4} \right\rfloor$ is a lower bound also in the presence of holes, compare with Figure 2. Observe, this lower bound holds for either of the above visibility notions, whereas we prove the upper bound for the restricted model. We remark that for a concrete P the usual visibility notion based guard number is very sensitive to small changes of the polygon (see Figure 3) in contrast to a covering by rectilinear stars. This will be detailed out later. In the following the term visible always stands for rectangularly visible, stars are rectilinear stars, and polygon means rectilinear polygon.

We give a few more technical notations. Let D denote the set of the main compass directions $D = \{n, e, s, w\}$ and let $\phi : n \to e \to s \to w$ be a cyclic permutation on D. We consider the holes of a polygon to have boundaries with clockwise orientation whereas we fix the outer periphery to have counterclockwise orientation, see Figure 4. Hence, walking along a boundary the interior of the polygon is always on the left hand side. In this notation a vertex $x \in P$ is a *concave* (*convex*) corner of the polygon if the boundary passing through this vertex is labelled $r\phi(r)$ (resp. $\phi(r)r$) for some $r \in D$. The direction inverse to $r \in D$ is denoted by r^{-1}. An rr'–*stair* (with $r' \neq r^{-1}$) is a boundary path in a polygon labelled alternatingly r and r'.

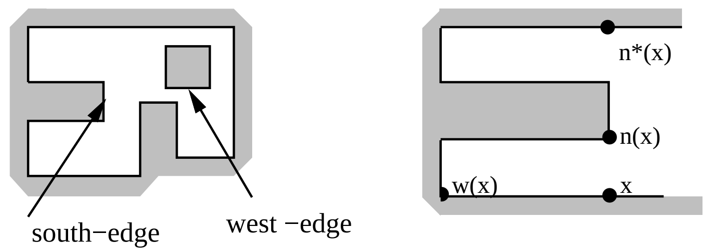


Figure 4: Orientation of edges and the neighbor relation

Finally, we define two neighbors of a point $x \in P$ associated with an $r \in D$. The neighbor $r^*(x)$ is the point which lies in direction r of x such that $[x, r^*(x)]$ has maximal length and is completely contained in P. Further, the neighbor $r(x)$ of $x \in int(P)$ is that point in direction r on the boundary of P for which $(x, r(x)) \subset int(P)$. If x itself is on a boundary then $r(x)$ is either the first polygon vertex in direction r different from x such that $[x, r(x)] \subset bd(P)$ or, as before, the first boundary point we hit shooting a ray in direction r. If there is no point in direction r of x reachable in P we set $r(x) = r^*(x) = x$. We write $rr'(x)$ to denote the point $r(r'(x))$, for $r, r' \in D$.

3 Problem Reductions

Problem reduction is one of the basic ideas used frequently in this paper. In general terms, we say that the guarding problem for an (n, h)–polygon P can be reduced to the guarding problems for a set of smaller polygons P_i, if there are solutions for the P_i which imply an $\lfloor \frac{n}{4} \rfloor$–solution for the original polygon P. In this situation we say that P is reducible. The first lemma gives a trivial example of such a reduction, see also [15], [12].

Lemma 3.1: If in a polygon P there are concave vertices x and y such that $x = r(y)$ for an $r \in P$ and $(x, y) \subset int(P)$, then P is reducible.

Proof: One adds (x, y) as a new 'wall' of 0–width to $bd(P)$. This does not change the total number of vertices. But either this wall decreases the number of holes by 1 or it dissects the polygon into two smaller ones. In both cases the reduction property is trivially fulfilled. $\square$

The main aim of this section is to describe two other local visibility configurations which imply the possibility to reduce the problem. These are *empty convex corners* and *X–shapes*. In both situations the reduction will be of the following pattern. We choose a point v in P and a star S with $v \in ker(S)$ and $S \subset P$. The removal of S from P defines a partition of $P \setminus S$ into a set of smaller (n_i, h_i)–polygons $P_i \subset P$ such that $\sum_i n_i \leq n-4$. The reduction property is fulfilled since $\sum_i \left\lfloor \frac{n_i}{4} \right\rfloor \leq \left\lfloor \frac{n}{4} \right\rfloor - 1$ and we need only one guard for S.

Remark: To reduce a polygon can be interpreted in the following way. We will see later on that stationing guards is not a local problem. That means, in general one cannot decide whether a given point x can be chosen as a guard position knowing only $V(x)$ and not the whole polygon. However, in all three reductions described in this section this is possible.

Let v be a convex vertex of in a polygon defined by boundary edges $u \rightarrow v$ and $v \rightarrow x$ labelled $\phi(r)$ and $r \in D$, respectively. We say that this vertex v is an *empty convex corner* if $R[x, u] \subset P$.

Theorem 3.2 (Reduction B): If a rectilinear polygon P contains an empty convex corner, then P is reducible.

We start with a lemma which covers a special case of the statement above, namely, if there are two consecutive empty corners in P.

Lemma 3.3 : Assume there is in P a boundary path p, say from x to y, which is a stair of length 4. If there are 2 convex corners on p from x to y seperated by a concave corner and if p is completely visible from some point $z \in P \setminus R(x, y)$, then P is reducible.

Proof: W.l.o.g. let this path from be a se–stair. We know that $(y\#x)\Box p$. Observe that $y\#x \in bd(P)$ is possible. The star S defined by the stair p and the edges $[x, y\#x]$, $[y, y\#x]$ has size 6 and is completely contained in P. Clearly, it can be watched by 1 guard. But $P \setminus S$ is a smaller polygon P' of size $n-4$ with the same number of holes as P. So the reduction property is fulfilled. Figure 5 illustrates the situation. Observe, that this construction works independently of whether x and y are convex or concave corners. $\square$

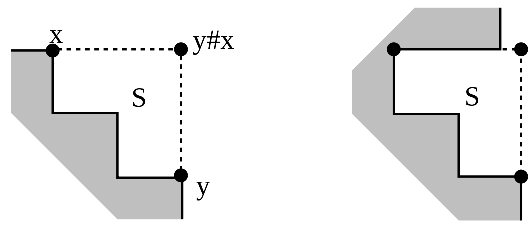


Figure 5: Reducing consecutive empty convex corners

Proof: (Theorem 3.2) Let v be the empty convex corner and without restriction of generality r be the east direction. Define $u = n(v), x = e(v)$ and $z = x\#u$.
The proof will be accomplished by a rather long case inspection according to whether u,x are convex/concave and to completely visible edges in $V^{(i)} = V^{(i)}(z)$, for $i = 1, 2, 4$. In the formulation of the (sub–) cases we use the following convention. For each case we assume the properties explicetely stated plus the negated assumptions of all cases already discussed.

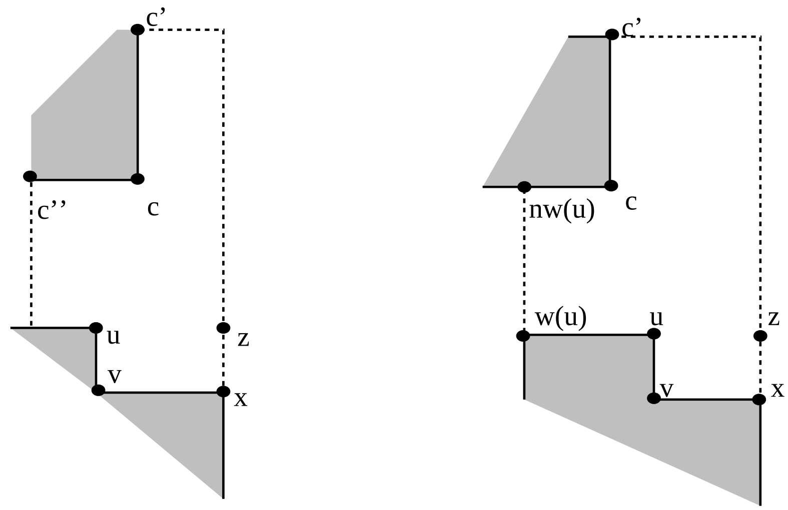
Case 1: Both u and x are concave vertices.

Let us order all z–visible edges counterclockwise. We know, all complete edges in $V^{(2)}$ must be w–edges or s–edges.

Case 1.1: In $V^{(2)}$ there are 2 consecutive polygon edges $[c', c], [c, c'']$ such that c is concave.
Observe that by assumption the rectangles $R[z, v], R[z, c''], R[z, c']$ all are completely contained in P. Their union forms a star S of size 8. By cutting out S from P we decrease the problem size by 4, so the reduction property is fulfilled. Figure 6a illustrates the situation. We remark that c' and c'' can be convex.

Case 1.2: The last complete edge in $V^{(2)}$ is an s–edge.
Let $[c', c]$ be this edge. First we observe that under this assumption $w(u)$ can be assumed to be concave. Otherwise $[nw(u), w(u)]$ would have to be the edge $[c', c]$ and we could apply Lemma 3.3.
Again, we find a star as shown in Figure 6b that can be cut out to reduce the polygon.

So far we have seen that if there are (complete !) edges in $V^{(2)}$ then all can be assumed to be w–edges. This follows from the fact that the last edge in $V^{(2)}$ cannot be an s–edge and if an s–edge is followed by a w–edge we are in case 1.1. Symmetrically, all edges in $V^{(4)}$ can be assumed to be n–edges.



(a) (b)

Figure 6: Illustration of case 1.1 and 1.2

Case 1.3: $z \Box wn(z)$
This assumption applies for example if in $V^{(2)}$ there are (if any) only w–edges. An analogous construction works if $z \Box se(z)$.

Case 1.3.1: $z \Box en(z)$
We move the segment $[n(z), en(z)]$ south until we hit either a ws–corner c in $V^{(4)}$ or we reach the horizontal line starting in $s(x)$. So we can delete either the star $R[wn(z), z] \cup R[v, z] \cup R[n(z), sen(z)] \cup (R[z, s(c)] \cap R[z, es(x)])$ or, in the latter case, $R[wn(z), z] \cup R[v, z] \cup R[ne(z), s(x)]$, see Figure 7a.

Case 1.3.2: $z \not\Box en(z)$ and $z \not\Box ne(z)$.
Then there must be a pair of consecutive polygon edges, say $[c', c], [c, c'']$ in $V^{(1)}$ which forms an wn–vertex. We shift the horizontal edge of $R[z, c']$ to the south and proceed in $V^{(4)}$ analogously to subcase 1.3.1. From $V^{(1)} \cup V^{(2)}$ we add $R[wn(z)\#c'', c\#z]$. Together with $R[v, z]$ we obtain a star of size at most 12. Figure 7b illustrates the situation.

Case 1.3.3: $z \not\Box en(z)$ and $z \Box ne(z)$
We can assume that $z \not\Box se(z)$. Otherwise, we would have the (symmetric) assumptions of case 1.3.1. This implies that $ne(z) \Box s(x)$. We shift the segment $[ne(z), e(z)]$ to the west until we reach the horizontal line through vertex $w(u)$ or the horizontal line of a sw–corner in $V^{(2)}$. Further we proceed like in case 1.3.1.(symmetric version) and add from $V^{(4)}$ the rectangle $R[e(z), s(x)]$ to the star.

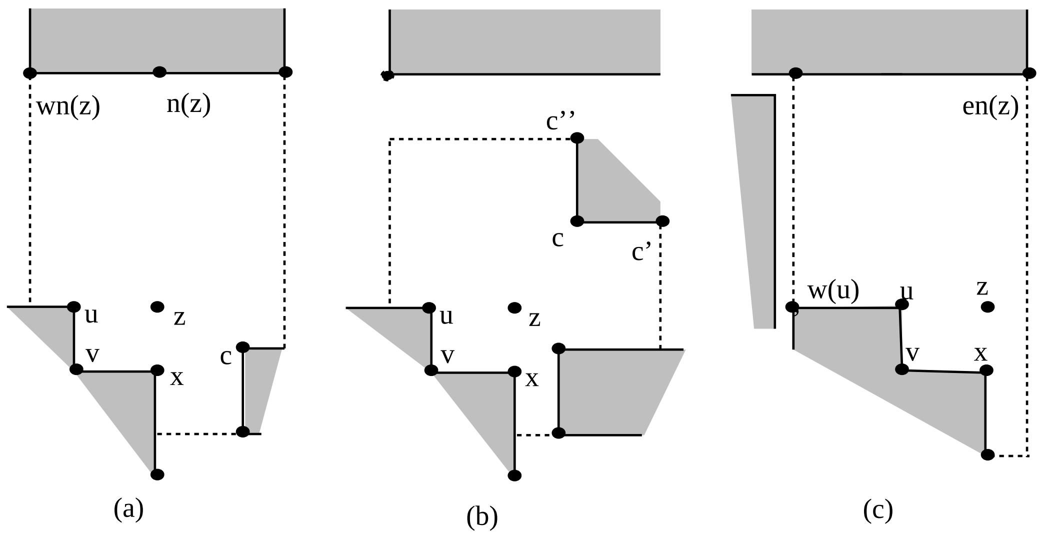


Figure 7: Illustration of case 1.3 and 1.4

Case 1.4: $z \not\square wn(z)$ and $z \not\square se(z)$

Case 1.4.1: $z \square en(z)$

We can cut out the star $R[z, nw(u)] \cup R[z, v] \cup R[en(z), s(x)]$, compare with Figure 7c.

Case 1.4.2: If additionally $z \not\square en(z)$ and $z \not\square ne(z)$

We can proceed as in case 1.3.2.

Case 1.4.3: $z \not\square en(z)$ and $z \square ne(z)$

Either there is an e–edge in $V^{(4)}$ (a case which we have dealt with) or we have $ne(z) \square s(x)$. Here $R[ne(z), w(u)] \cup R[z, v] \cup R[ne(z), s(x)]$ is the star to be cut out.

Case 2: u is a convex, x is concave.

Case 2.1: $e(u)$ is left of z.

We can treat the edge $[u, e(u)]$ like a last edge in $V^{(2)}$) and repeat the case inspection from above.

Case 2.2: $e(u)$ is right of z.

This can be treated like Case 1.3.1.

Case 3: Both u and x are convex corners.

The subcases that $e(u) \square n(x)$ or that $(z \square ne(u)$ are trivial and can be excluded, compare with Figure 8a. Observe that here it can happen that z is not in the kernel

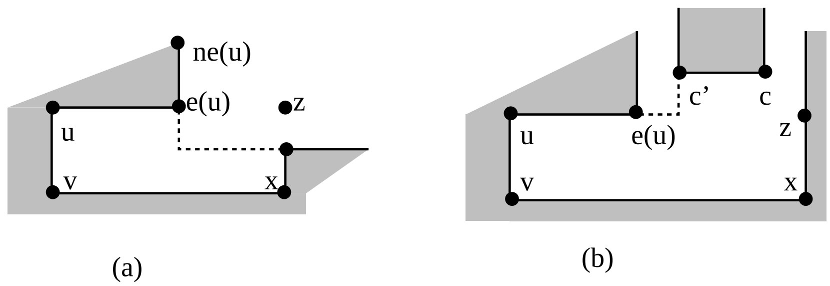


Figure 8: Illustrations of case 3

of the star we cut out.
We can assume that there is an w–edge in $V^{(2)}$, take the first such edge, say $[c, c']$. We move the horizontal segment $[c\#u, z]$ to the north until it hits either $n(x)$ or $n(c)$ and cut out the star $R[z, v] \cup R[z, c'] \cup (R[z, n(c)] \cap R[c\#u, n(x)])$, see Figure 8b.

This completes the proof of Theorem 3.2. $\square$

There is another local situation, called X–shape, where a reduction is always possible. It is defined as follows. Consider a point $z \in P$ and an z–visible edge $[x, x']$, i.e. $z \Box x$ and $z \Box x'$. Assume further that for an $i \in \{1, 2\}$ we have $z_i \leq x_i < x'_i$ or $z_i \geq x_i > x'_i$. In this case we associate with $[x, x']$ the compass direction from x_i to x'_i. We say, that z is the center of an *X–shape* if there are four z–visible edges representing all four main compass directions. To avoid confusion with the orientation of an edge defined before let us remark that an edge representing 'north' in an X–shape with center z is either an n–edge in $V^{(1)}(z)$ or a s edge in $V^{(2)}(z)$ where $V^{(i)}(z)$is again the i–th quadrant in $V(z)$.
Observe, that the vertex of such an edge which is closer to z is concave (provided that there are no empty convex corners). Moreover, two edges may share such a vertex. Figure 9 shows (up to symmetries/rotations) all possible modifications of X–shapes under the assumption that P has no empty convex corners. The dashed lines in the figure indicate the visibility assumptions.

Theorem 3.2 (Reduction C): If a polygon P contains an X–shape, then P is reducible.

Proof: We may assume that P is already reduced with respect to reduction types A and B, i.e. P is in general position and it has no empty convex corners. The proof now is very similar to that of Theorem 3.1. We construct explicitely the star

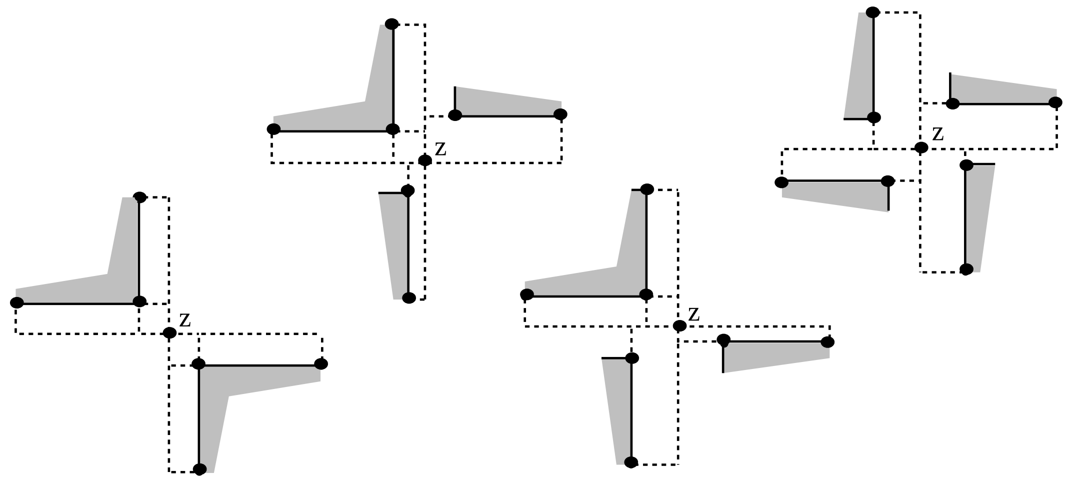


Figure 9: Possible X-shape configurations

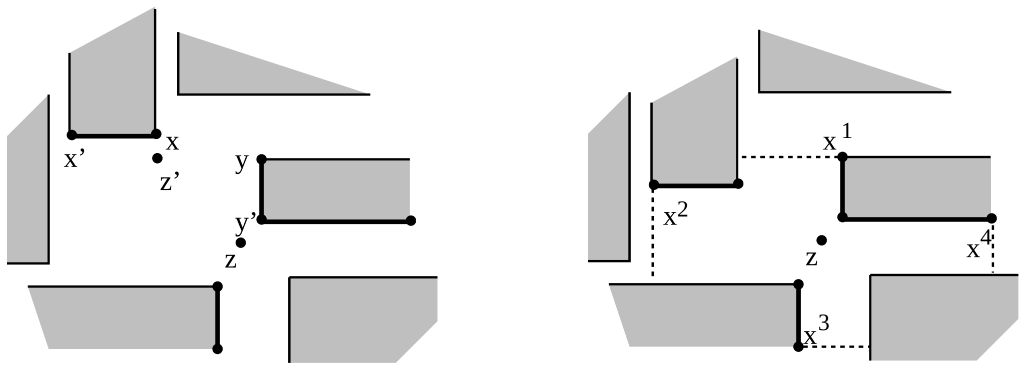
cut out.
Let z be the center of an X–shape. We can assume w.l.o.g. that among all z–visible edges representing the north–direction in an X–shape with center z we choose the edge $[y, y']$ with smallest y'_2. Analogously we choose the other edges. Let $[x, x']$ be the edge representing west. We can assume that $x_2 < y_2$. Otherwise, we consider the X–shape with center $z' = x\#y$. Observe that both x and y are concave, see Figure 10a for illustration. Similar assumptions can be made for all pairs of neighboring compass directions.
With this assumption the construction now is simple. Let $x^1, \ldots, x^4$ be the endpoints (i.e the points which have greater distance from z) of the four edges in the X–shape in counterclockwise order . The star S implying the reduction is defined by

$$S = \bigcup_{i=1}^{4} (R[x^i, x^{i+1}] \cup R[z, x^i]) \cap V(z)$$

where x^5 is set to be x^1. By construction each $(R[x^i, x^{i+1}] \cup R[z, x^i]) \cap V(z)$ contains at most one polygon corner not lying on the edges defining the X–shape. Hence the size of S is bounded by 12. Figure 10b shows a typical situation. ◻

4 The Corridor Graph

Throughout this section let P be a reduced (n, h)–polygon. We will associate with it a *corridor graph* $C(P)$ which represents the main features of our visibility problem.



(a) (b)

Figure 10: Reducing an X–shape

Its nodes correspond to pairs of polygon vertices each defining a special local shape. The edges represent straight 'corridors' connecting these local situations.
Let us consider the partition of P induced by all edge extensions. Here, by an edge extension we mean the prolongation of a polygon edge through a concave corner until it hits the boundary. A horizontal (resp. vertical) *corridor* in P is a maximal rectangle contained in P whose horizontal (resp. vertical) edges are contained in polygon edges whereas both vertical (resp. horizontal) edges are polygon edge extensions. Each corridor contains exactly two concave polygon corners which we call *corridor corners.*
Let x be a convex $\phi(r)r$–corner with neighbors x', x'' and let y be a concave $r^{-1}\phi(r^{-1})$–corner. We say, that the pair $\{x, y\}$ defines an *L–shape* if

1. $y \in R[x', x'']$
2. $x \Box y$
3. y is corner of two corridors.

There is another local configuration which we call *T–shape.*
Let x be a concave es–corner and y a concave ne–corner. We say that the pair $\{x, y\}$ defines a T–shape of north–orientation if

1. $(x \Box n(y)) \wedge (y \Box n(x))$
2. x and y are corners of horizontal corridors.
3. Either $e(x) \in (y, s(y))$ or $w(y) \in (x, s(x))$.
4. Either x or y is corner of a vertical corridor.
5. $(n(x), n(y)) \subset bd(P)$.

Analogously, we define T–shapes with east–, south–, and west–orientation. Figure 11 shows an L– and a T–shape with north orientation.

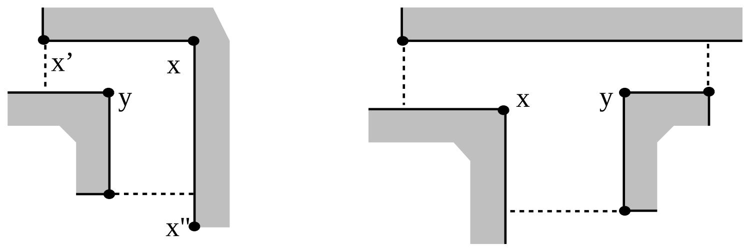


Figure 11: L–shape and T–shape

Our aim is to show that each reduced polygon can be decomposed into such L– , T–shapes, and corridors. It is sufficient to show that there is a perfect matching between all polygon vertices such that each matched pair of vertices defines an L– or a T–shape.

Proposition 4.1: Let x be a convex, say a nw–corner in a reduced polygon P. Then there is exactly one concave corner y such that the pair $\{x, y\}$ defines an L–shape.

Proof: Since x is not an empty convex corner we know that there are vertices in $R[x', x'']$ fulfilling the first two conditions of an L–shape. If there is exactly one such vertex then this vertex also satisfies the third condition. So, we can assume that there are at least two concave es–corners. Let y' be the first and y'' the last in counterclockwise order. If $w(y')\square x$ then y' is corner of two corridors. So we can assume that this does not hold. This means it is followed by an s–edge visible from x. Analogously, we can assume that $x \not\square s(y'')$. So there must be an es–corner y such that $s(y)\square x$ and $w(y)\square x$, see Figure 12a. Finally, if there are two corners y, z each defining with x an L-shape and y preceeds z then the point $z\#y$ is the center of an X–shape and we have a contradiction to our assumption that P was reduced, see Figure 12 for illustration. $\square$

Let us consider all concave corners in P which are not matched with a convex corner. We define a graph $T(P)$ over this vertex set with an edge between two vertices if they form a T–shape. The next theorem shows that $T(P)$ has a perfect matching.

Theorem 4.2: A connected component of the graph $T(P)$ is a single edge, a simple path of length 3, or a cycle of length 4.

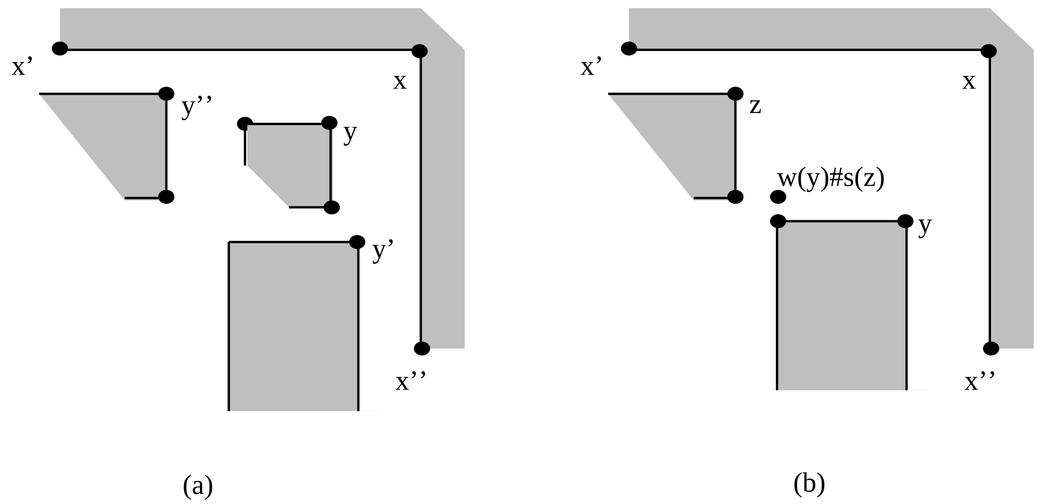


Figure 12: Illustrating Proposition 4.1

Proof: W.l.o.g. let x be a concave es–corner with neighbors $x' = w(x)$ and $x'' = s(x)$. First of all it is easy to see that x can be part of at most two T–shapes, one with north the other with east orientation. We find two candidate partners as follows. We shift the segment $[x', x]$ to the north and consider the east end of the boundary edge it meets first. This will be our first candidate z unless there is another x–visible w–edge (take the first in counterclockwise order) with an incident x–visible s–edge. In this case let z be the east end of that w–edge. Further, let y be the analogously defined candidate partner of x east of $[x, x'']$. We can assume that $w(z)_1 < x_1$ and $s(y)_2 < x_2$.
The proof will be a case inspection according to the mutual position of x, y, and z. In each of the cases we will either show that an arrangement is not possible because P is reduced or we exibit a T–shape defined by x and some other vertex (not necessarily y or z. We start with two trivial cases.

Case 1: $y, z \in Q^{(1)}(x)$ and either both are visible or both are not visible from x.
If both are visible and $y \neq z$ then $u = y\#z \in V(x)$ and u is center of an X–shape. If $y = z$ then $\{x, y\}$ defines an L–shape.In the case that both are not visible then there must be a concave wn–corner u such that $(u \square x') \wedge (u \square x'')$. But then u is the center of an X–shape.

Case 2: $z \in V^{(2)}(x), y \in V^{(1)}(x)$
We can assume that $x \square y$ and that y is a concave ne–corner since otherwise we would find an X–shape. If $y_2 < z_2$ then there cannot be a vertical edge d' north of x visible from some point in $R[x, y]$. With such an edge we would have a point from which simultaneously d', $[x, x']$, $[x, x'']$, and $[y, e(y)]$ are visible. Hence, especially $n(x)$ and $n(y)$ are points on the same horizontal edge d, the left end v of d is visible from x,

and y is the corner of a horizontal corridor. So $\{x, y\}$ define a T–shape and neither x nor y can be part of a second T–shape.
Assume now $y_2 > z_2$. If $z \Box y$ then $\{x, z\}$ forms a T–shape. Otherwise we find a center of an X–shape on the segment $[x\#z, e(z)]$.

Case 3: $z \in V^{(2)}(x), y \in V^{(4)}(x)$

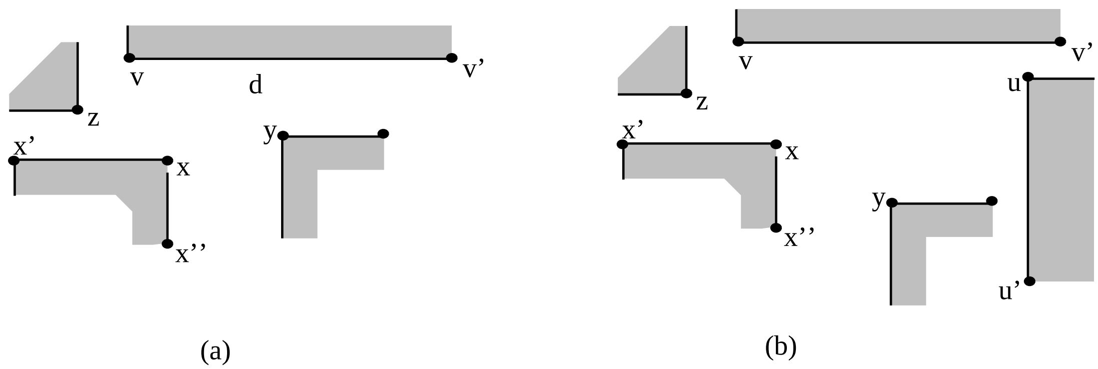


Figure 13: Illustration of Case 2 and Case 3

We observe that both y and z are concave and that $R[x, y\#z] \subset P$. Let us consider the following two conditions:

(V) There is a point on $[y, w(y)]$ which sees a vertical edge north of it.
(H) There is a point on $[z, s(z)]$ which sees a horizontal edge east of it.

We already know that these conditions cannot hold simultaneously since this would imply the existence of an X–shape.

Case 3.1: Neither (V) nor (H) hold.
Define $v = wn(x)$, $v' = en(x)$, $u = ne(x)$, $u' = se(x)$, see Figure **??**b.

Case 3.1.1: $x \Box v$ and $x \Box v'$
Here $\{x, y\}$ forms a T-shape. Analogously, if $x \Box u$ and $x \Box u'$, then $\{x, z\}$ defines a T–shape.

Case 3.1.2: $x \Box v$ but $x \not\Box v'$
Under this assumption $x \Box u$. On the other hand we know that $x \Box u'$ or $x \Box e(y)$. Hence $\{x, z\}$ or $\{x, y\}$ form a T–shape.

Case 3.1.3: $x \not\Box v$, $x \not\Box v'$, and $x \not\Box u'$
Here, x is the center of an X–shape.

Case 3.1.4: $x \not\square v, x \not\square v', x \square u'$
In this situation we know that $x \square n(z)$ and $x \square u$. If $z_2 > u_2$ then $x \# u$ is center of an X–shape. Otherwise, $\{x, z\}$ defines a T–shape.

Case 3.2: Condition (V) holds. Let's denote this edge by $[t, n(t)]$.

Case 3.2.1: t is a wn–corner. That implies $x_1 < t_1$.
We can assume $z_2 < u_2$. To avoid an X–shape with center x we have $x \square u'$, $x \square u$. Now either $z \square u$ implying that $\{x, z\}$ is a T–shape or $[v, v']$ is x–visible and $\{x, v'\}$ forms a T–shape.

Case 3.2.2: t is sw–corner.
As in the previous case, the only possibility to avoid reducibility is that $\{x, t\}$ forms a T–shape.

Case 4: $z \in V^{(2)}(x), y \in V^{(1)}(x)$
Again, either $z \square y$ and $\{x, z\}$ is a T–shape or the east end of the lowest horizontal edge in $R[z, y]$ defines with x a T–shape.

Case 5: $z \in V^{(1)}(x), y \in Q^{(1)}(x) \setminus V^{(1)}(x)$
Clearly, we can assume $R[z, y] \subset P$. But then $\{x, z\}$ is an edge in $T(P)$.

This concludes the case inspection. So far we have seen that in a reduced polygon a concave corner is either part of an L–shape or there is a concave corner y such that x and y define a T–shape. Now assume that a corner x has two neighbors y, z in $T(P)$. We know $x \square v$, v is concave, and $y \not\square n(z)$. If $\{z, u\}$ defines a T–shape then $\{y, v\}$ is a T–shape, too. So we get a 4–cycle. Otherwise, if $z \not\square n(v)$ and $v \not\square n(z)$ we find north of $[z, x \# z]$ a concave corner z' defining with z a T–shape. Therefore x is on a path of length 3 in $T(P)$. Compare with Figure 14. An analysis of all cases shows that Case 4 is the only possibility for x to form with two other vertices T–shapes. $\square$

Corollary 4.3: The graph $T(P)$ has a perfect matching.

We are now going to define the *corridor graph* $C(P)$ of a reduced polygon P. First we fix a perfect matching of $T(P)$, say M. The vertex set of $C(P)$ consists of all pairs $\{x, x'\}$ of vertices from P, either defining an L–shape or being an element of M. So $C(P)$ has $n/2$ vertices. Two vertices $\{x, x'\}, \{y, y'\}$ are joint by an edge in $C(P)$ if there are corners $u \in \{x, x'\}, v \in \{y, y'\}$ belonging to the same corridor. The edges are labelled by the orientation of the corresponding corridors.

Proposition 4.4: A corridor graph $C(P)$ has the following properties.

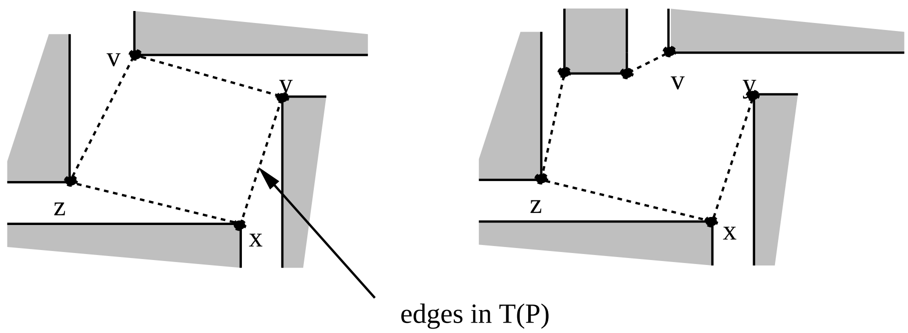


Figure 14: A 4–cycle and a length 3 path of T–shapes

(i) $C(P)$ is a planar rectilinear graph, i.e. it has a straight line, orientation preserving embedding into the integer grid. All nodes have degree 2 or 3.
(ii) If $C(P)$ is a bipartite graph, then $\lfloor \frac{n}{4} \rfloor$ point guards are sufficient to solve the Art Gallery Problem for P.

Proof: (i) The planarity follows from the characterization of planar rectilinear graphs given in [11].
(ii) Let $a = \{x, x'\}$ be a vertex of $C(P)$. If a represents an L–shape then define $R_a = R[x, x']$. Otherwise, if a corresponds to a T–shape of r–orientation we set $R_a = R[x, r(x')] \cap R[x', r(x)]$. A guard stationed in any point of R_a can simultaneously watch all incident corridors and all R_b, for all edges (a, b) in $C(P)$. So, we can take a 2–coloring of $C(P)$ and station guards in all R_a, where a is in the smaller color class of size $\leq \lfloor \frac{n}{4} \rfloor$. $\square$

We remark that the proposition above implies for example the Art Gallery Theorem for all rectilinear polygons with at most one hole, since their corridor graph is either empty or a bipartite cycle of L–shapes.
However, the polygon from Figure 1a has a non–bipartite corridor graph. We first sketch the idea how to deal with this special graph, see Figure 15.
If a graph is not bipartite then it contains an odd cycle. In our context this means that there are two consecutive corridors $\{a, b\}, \{b, c\}$ in $C(P)$ with the same orientation, say horizontal. Assume that all their other incident edges are vertical. We know, the corridors have different vertical width, say $\{a, b\}$ is wider than $\{b, c\}$. Assume further that we delete the latter from the graph and that, moreover, the resulting graph is bipartite. This implies, that either a or b is in the eventually chosen smaller color class. Say, this is a. But still we can choose a guard position in R_a from which the complete corridor corresponding to $\{b, c\}$ and R_c can be seen. Of course the width of the $\{b, c\}$–corridor imposes a restriction on the vertical (not on the horizontal!) position of a guard in R_a. Let's call the deleted edge *dummy*. This aproach suggests to delete all dummy edges, i.e. all edges whose corresponding

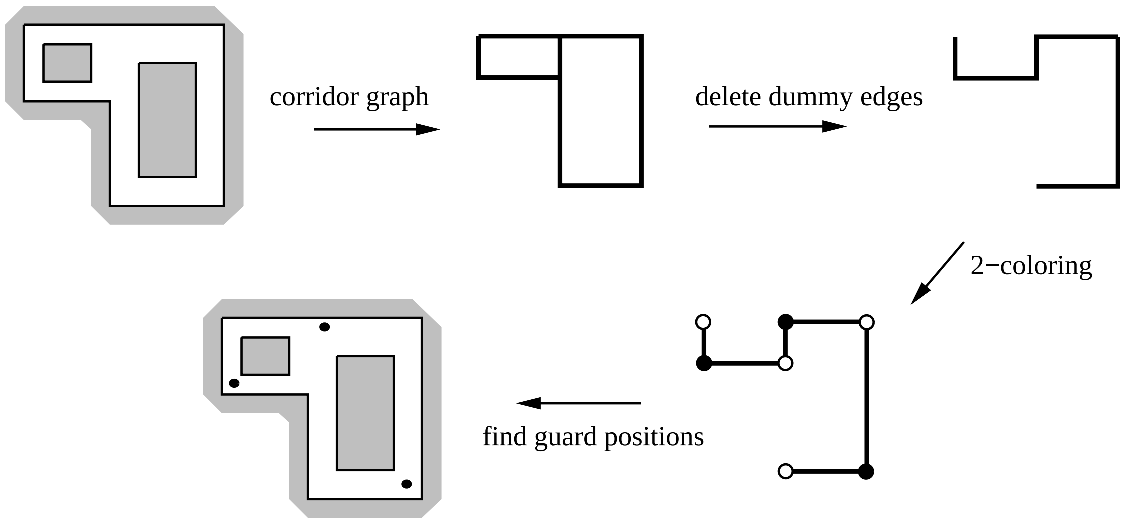


Figure 15: Guarding the lower bound example

corridor has a neighbor of the same orientation but with greater width. Clearly, the remaining edges form a bipartite graph and we could apply our coloring argument. Figure 15 shows how this works for our lower bound polygon.

Now, consider the example in Figure 16. Here, in a path of length 3 a south–T–shape is followed by a north–T–shape. Deleting the first and the last edge from the corridor graph can lead to a situation where only a is in the chosen color class. But there is no way to watch simultaneously from R_a all three horizontal corridors. How to overcome this deadlock? The idea is to insert a new horizontal edge into the graph connecting d and c. This guaranties that, provided the new graph is bipartite, either d or e is in the chosen color class. Moreover we can watch all three horizontal corridors as indicated in Figure 16.

In sum, defining the corridor graph $C(P)$ we lost certain geometric information about the polygon, especially about corridor widths. In fact, if this graph is bipartite we do not need this information at all. Otherwise, we use it to define a bipartite graph $C'(P)$ on at most $n/2$ vertices derived from $C(P)$ by deleting and adding edges such that a coloring of this new graph still implies a solution for the original guarding problem in P.

Consider a straight horizontal path ω in $C(P)$. Let's denote by $\pi(e)$ the image of the corridor corresponding to an edge $e \in \omega$ under the projection on the y–axis. Observe, that for two neighboring edges e, e' we have either $\pi(e) \subset \pi(e')$ or $\pi(e') \subset \pi(e)$. Moreover, the symmetric difference of $\pi(e)$ and $\pi(e')$ is connected.

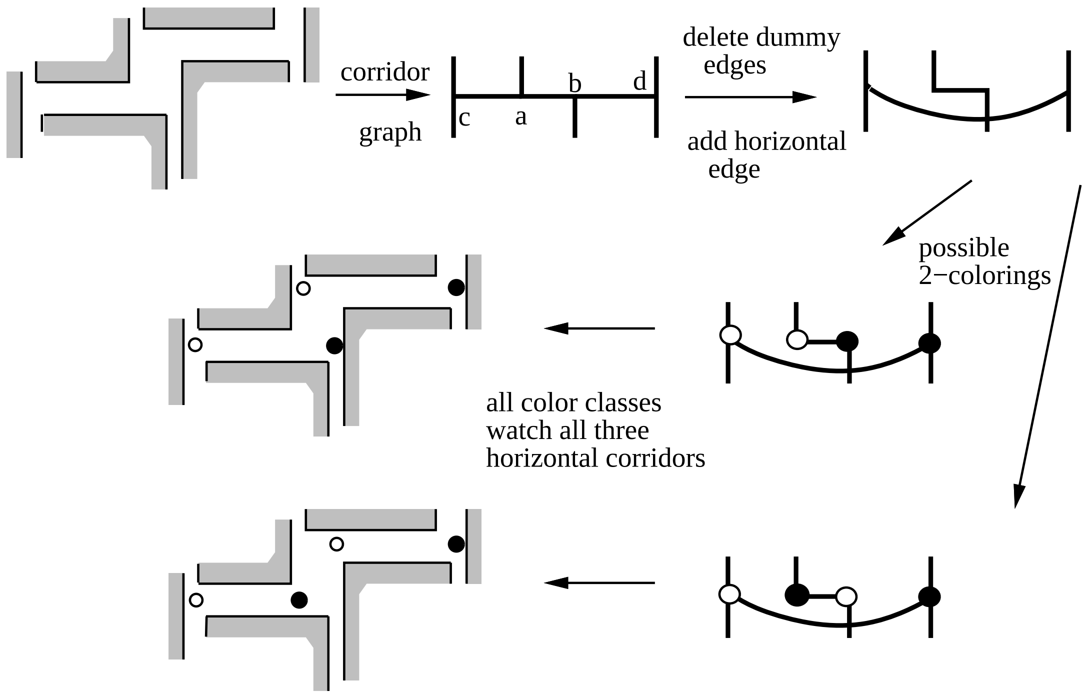


Figure 16: Adding new edges

We define that an edge e belongs to the set $Max(\omega)$ (resp. $Min(\omega)$ if $\pi(e)$ contains (resp. is contained in) the images $\pi(e')$ of all its neighboring edges e'.
Remark: Neighbor always means direct neighbor, i.e. an edge has at most 2 neighbors.

Moreover, we define the following edge sets:

$$\begin{array}{lcl} LMax(\omega) & = & \{e | e \textit{ is left neighbor of an element from } Max(\omega) \\ & & \textit{and either } e \notin Min(\omega) \textit{ or } e \textit{ is leftmost edge in } \omega.\} \\ RMax(\omega) & = & \{e | e \textit{ is right neighbor of an element from } Max(\omega) \\ & & \textit{and either } e \notin Min(\omega) \textit{ or } e \textit{ is rightmost edge in } \omega.\} \\ Char(\omega) & = & LMax(\omega) \cup RMax(\omega) \textit{ with left to right order} \\ New(\omega) & = & \{f = (x,y) | x \textit{ is left end of some } e \in LMax(\omega), \\ & & y \textit{ is right end of some } e' \in RMax(\omega) \textit{ and} \\ & & e' \textit{ is immediate successor of } e \textit{ in } Char(\omega)\}. \end{array}$$

Analogously we define these sets for vertical paths in $C(P)$. Now, we define the modified corridor graph $C'(P)$ as follows. It is spanned by the edge set

$$\bigcup_{\omega \in \Omega} Max(\omega) \cup New(\omega)$$

where Ω is the set of all vertical paths and all horizontal paths in $C(P)$.

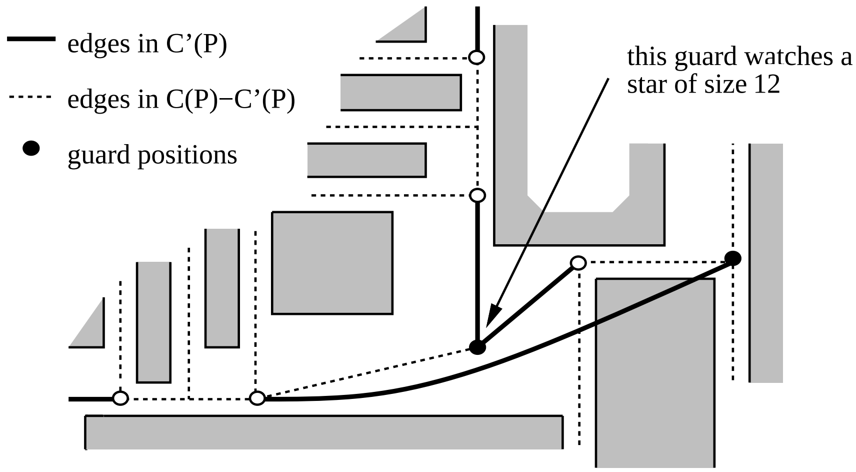


Figure 17: Illustrating the graph $C'(P)$

Theorem 4.5: The graph $C'(P)$ has the following properties.
(1) $C'(P)$ is bipartite.
(2) Stationing guards in P according to the smaller color class in a 2–coloring of $C'(P)$ solves the watchman problem for P.

Proof:
(1) This follows almost trivially from the definition. The only thing to check is that a vertex x cannot be simultaniously left end of an edge $(x, y) \in LMax(\omega)$ and right end of $(z, x) \in RMax(\omega)$. But this is impossible since one of these edges would be in $Min(\omega)$.
(2) Consider a horizontal path ω in $C(P)$. We have to show that each color class of the proper 2–coloring of vertices in $Max(\omega) \cup New(\omega)$ defines guard positions to watch all horizontal corridors defining ω.
Firstly, $Max(\omega)$ is not empty. Assume further that $New(\omega)$ is empty, i.e. in $Char(\omega)$ there is no element from $LMax(\omega)$ preceeding an element from $RMax(\omega)$. This applies for example if one of these sets is empty. Especially, if $LMax(\omega)$ is empty we know that the leftmost edge in ω is in $Max(\omega)$ and the left neighbors of all other maximal edges are in $Min(\omega)$. But then all edges between two succesive maximal edges can be watched from the left maximal edge independent of whether the guard is placed in the left or right end of this edge.
The case that $RMax(\omega)$ is empty is symmetric, hence we can assume that both sets are non–empty.
If in $Char(\omega)$ the set $RMax(\omega)$ is left of all elements from $LMax(\omega)$ (i.e., especially $New(\omega)$ is empty) we can apply the same argument as before.

So we can assume, that $New(\omega)$ is not empty. Consider an $e = (x, y) \in New(\omega)$. Let $e_1, ..., e_m$ be the edges from $Max(\omega)$ which are between x and y. It is clear that between some e_i and e_{i+1} there is exactly one edge in $C(P)$, say e_i'. Now, if the guard sits in x, then this guard can watch all corridors left of e_1 up to (and inclusively) the next minimal edge, say e''. Moreover, each e_i gets a guard which has to watch e_i and e_i'. Observe, that this is possible even when he is placed in the left end of e_i. Finally, we have to remark that the guard stationed on e_m watches all corridors on the right of e_m up to the next minimal edge say e^*. Eventually, we can remove the segment ω' which starts with e'' and ending on e^* from ω and iterate the procedure. ◻

With the above definition of $C'(P)$ we cannot bound yet the size of a star a single watchman has to guard. But this can be now easily accomplished by adding to $C'(P)$ as many as possible edges without destroying the bipartiteness of the graph.
Let ω be a straight path in $C(P)$. Denote by $rest(\omega)$ the subgraph of ω spanned by all vertices not incident to an edge in $C'(P)$. First, we add to $C'(P)$ a maximal matching in $rest(\omega)$. Then we traverse the remaining isolated vertices from left to right connecting each consecutive pair of them by a new edge which we add to $C'(P)$. Finally, for each ω there is at most one isolated vertex left.

Corollary 4.6: The watchman problem for a reduced polygon can be solved by at most $\lfloor \frac{n}{4} \rfloor$ point guards each of which has to guard a rectilinear star of size at most 12.

Proof: After augmenting edges to $C'(P)$ for each $\omega \in \Omega$ as described above each guard has to watch at most 3 horizontal (vertical) and 2 vertical (horizontal) corridors. ◻

Figure 17 shows an example of a configuration that yields a star of size 12.

5 Concluding Remarks

1. We have shown that a rectilinear polygon (possibly with holes) on n vertices can be partitioned into at most $\lfloor \frac{n}{4} \rfloor$ rectilinear stars each of size at most 12. The presented self–contained proof has two parts: a problem reduction and the graph–theoretic treatment of reduced polygons. In both parts stars of size 12 occur rather naturally. So, we don't think that the maximal star size can be decreased within the presented approach.

2. A lower bound for the maximal star size is 10 as proved by the unique watchman solution for the example in Figure 1. It remains open whether 10 or 12 is the right answer. The same example shows that one needs point guards to

prove an $\lfloor \frac{n}{4} \rfloor$ upper bound. If we are restricted to vertex guards the exact bound is not known yet. There is some strong evidence that one can prove our $\lfloor \frac{2n}{7} \rfloor$–lower bound to be tight using the multicoloring technique introduced in [10], where the best known upper bound of $\lfloor \frac{n}{3} \rfloor$. Expressed as a function of n and h the correct bound for the vertex guard model is also not known; the current lower bound is $\lfloor \frac{n+h}{4} \rfloor$, the best upper bound is still the trivial $\lfloor \frac{n+2h}{4} \rfloor$, see [14].

3. The presented solution (via partitioning into rectilinear stars) of the Art Gallery Problem is invariant under the following *stretching* operations. Consider a horizontal (or vertical) cut through the polygon by a line L that does not meet any vertical (horizontal) edge and scale all edges hit by a constant factor. Since a stretching applied to a rectilinear star gives a rectilinear star and the guards can be placed on vertices or crossings of edge prolongations only we have that the image of guard positions under a stretching defines a valid guarding set. This is not true if one starts with a partition into general stars as demonstrated by Figure 3.

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