SERIE B — INFORMATIK

On-Line Chain Partitions of Orders

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Abstract

We analyze the on-line chain partitioning problem as a two-person game. One person builds an order one point at a time. The other person responds by making an irrevocable assignment of the new point to a chain of a chain partition. Kierstead gave a strategy showing that width k orders can be on-line chain partitioned into $(5^k-1)/4$ chains. We first prove that width two orders can be partitioned on-line into 5 chains. Secondly, we introduce a variant of the game. We impose the restriction that the new point presented by the first player has to be a maximum element in the present order. For this up-growing variant we prove matching upper and lower bounds of $\binom{k+1}{2}$ on orders of width k.

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An on-line chain partitioning algorithm receives as input an on-line order, this means the elements of the order are taken one by one from some externally determined list. With a new element the algorithm learns the comparability status of previously presented elements to the new one. Based on this knowledge the algorithm must make an irrevocable assignment of the new element to a chain. The performance of an on-line chain partitioning algorithm is measured by comparing the number of chains used with the number of chains used by an optimal off-line algorithm, i.e., with the width of the order. For order theoretic terminology we refer the reader to [4].

An on-line chain partitioning can be viewed as a two-person game. We call the players Alice and Bob. Alice represents an on-line algorithm an Bob represents an adaptive adversary. In the on-line chain partitioning game for width k orders Bob builds an on-line order of width at most k and Alice maintains a chain partition of the order. The game is played in rounds. During round i Bob introduces a new point x to the order and describes the comparabilities between x and the points from previous rounds. Alice responds by assigning x to a chain. The value of the game for width k orders is the largest integer $val(k)$ so that Bob has a strategy that forces Alice to use $val(k)$ chains. Note that by game theoretic duality we may as well define $val(k)$ as the least integer so that there is an algorithm for Alice that never uses more chains.

An argument of Szemerédi shows $val(k) \geq \binom{k+1}{2}$. On the other hand Kierstead [1] has proven that $val(k) \leq (5^k - 1)/4$. In the next section we deal with the chain partitioning game for $k = 2$. It was known that $5 \leq val(2) \leq 6$. We propose an algorithm that only needs 5 chains thus proving $val(2) = 5$. In section three we introduce a variant of the game. We restrict the legal moves of Bob by the rule that the sequence in which elements are released is a linear extension of the order, i.e, a comparability of a new element x to an older y has to be of the form $y < x$. On-line orders with this property will be called up-growing on-line orders. In this variant we are able to determine the value of the game exactly. Finally, in section four we discuss the on-line dimension problem for up-growing on-line orders.

1 On-line chain partitions for width two

Kierstead [1] proves lower and upper bounds of 5 and 6 for the value of the chain partitioning game for orders of width two and asks for the precise value. In this section we propose a strategy for Alice that never uses more then 5 chains.

Consider the serial decomposition of an order P of width two. This decomposition may be viewed as the finest partition of the ground set of P with the property that a pair of incomparable points always belongs to the same block of the partition. A component of this decomposition that contains more then one element will be called *rigid*. Note that a rigid component has two maximal and two minimal elements. These at most four elements are called the *corners* of the component, maximal elements are *top corners* and minimal elements are *bottom corners*.

Given a new point x we classify how the point operates on the components of the serial decomposition of P. We distinguish five possibilities.

(1) Element x forms a new component by its own, i.e., x is comparable to all previously introduced points.
(2) Element x together with two rigid components and possibly several singleton components form a new rigid component.
(3) Element x together with some singleton components form a new rigid component.
(4) Element x together with one rigid component and possibly several singleton components form a new rigid component with x as a corner element.
(5) Element x extends a rigid component and x is not a corner of this component.

For the description of the invariant the algorithm maintains it is convenient to identify the chains of the partition with the colors $1, 2, 3, 4$ and g. As invariant we formulate three properties:

(A) If y is a corner of a rigid component then y has an associated set $vc(y)$ of two *virtual colors* from $\{1, 2, 3, 4\}$.
(B) Every color class forms a chain, i.e., $\{ z : \mathit{color}(z) = \gamma \text{ or } \gamma \in vc(z)\}$ is a chain for $\gamma = 1, .., 4$.
(C) The sets of virtual colors used for the top corners of a rigid component R and for the bottom corners of the next rigid component above R are different. In particular if t is a top corner of R and b a bottom corner of the next component above then $|vc(t) \cap vc(b)| = 1$.

We are ready to state the rules guiding the assignment of a color to a new element x.

Case 1. If element x is of type 1 then $\mathit{color}(x) = g$. This assignment is certainly legal and the invariant remains true.

Case 2. Element x is of type 2. Let $\{t_1, t_2\}$ be the top corners of the lower and $\{b_1, b_2\}$ be the bottom corners of the higher of the two components melt by x. Note that x is comparable with exactly one of t_1, t_2 say with t_1 and with exactly one of b_1, b_2 say with b_1. In this case the unique color of $vc(t_1) \cap vc(b_1)$ is assigned to x. By invariance property B this assignment is legal and the invariant remains trivially true.

Case 3. Element x is of type 3, i.e., x generates a new rigid component. There may be a chain $x_1, .., x_h$ of g-colored points incomparable to x. From the invariance assumption it follows that up to a permutation of colors the virtual colors of the bottom corners of the rigid component above x are $\{1, 2\}, \{3, 4\}$ and the virtual colors of the top corners of the rigid component below x are $\{1, 3\}, \{2, 4\}$. Defining $vc(x) = \{1, 4\}$ and $vc(x_1) = vc(x_h) = \{2, 3\}$ it is easily that the invariant remains true. Finally, we assign to x a color from $vc(x)$.

Case 4. Suppose that x is a new corner of component R. By duality it suffices to deal with the case of x being a bottom corner. Let b_1, b_2 be the old bottom corners of R and let x and y be the bottom corners of the enlarged component. Assuming

$x < b_1$ we note that $x||b_2$. Since y is incomparable to x the relation $y < b_2$ is necessary to avoid a 3 antichain. We define $vc(x) = vc(b_1)$ and $vc(y) = vc(b_2)$ and assign to x a color from $vc(x)$. This is easily seen to be consistent with the invariance.

For the last case we need a lemma. Loosely speaking the lemma tells us that the chain partition of a rigid component is 'rigid' with respect to enlargements.

Lemma 1 *Let R be a rigid component and let C_1, C_2 be a chain partition of R. If x is a point extending R then either $C_1 + x$ or $C_2 + x$ is a chain.*

Proof. The incomparability graph of a rigid component is a connected bipartite graph. As R and $R+x$ are rigid we see that the unique bipartition of the incomparability graph of $R+x$ is obtained from the unique bipartition of the incomparability graph of R by extending one of the sides with x. $\triangle$

Case 5. Element x falls into the interior of a component R. Assume that C_1+x, C_2 is the chain partition of $R+x$. Let y be the first element below x in C_1 that was ever a corner of a rigid component and let z be the first element above x in C_1 that was ever a corner. These two elements exist since C_1 is bounded by corners of R. We claim that there is a color γ in $vc(y) \cap vc(z)$ and we may legally assign c to x. We leave it to the reader to use the above lemma and supply the proof of this claim.

This concludes the description of the rules of the algorithm. As shown these rules are applicable if the invariance properties hold and they leave the validity of these properties untouched. This proves the theorem.

Theorem 1 *An on-line order of width two can be partitioned on-line into 5 chains.* $\Box$

2 Chain partitions of up-growing orders

2.1 A strategy for Bob

As already noted there is a lower bound of $\binom{k+1}{2}$ for the value of the unrestricted on-line chain partitioning game on orders of width k. We restate the original result of Szemerédi.

Theorem 2 *For every positive integer k the value of the on-line chain partitioning game on the class of on-line orders of width at most k is at least $\binom{k+1}{2}$. This remains true if the on-line order is specified by an on-line 2-realizer.*

The on-line order constructed in the proof of this theorem as given in [2] is not up-growing. Next we proof that the same bound holds true for up-growing on-line orders.

Theorem 3 *For every positive integer k the value of the on-line chain partitioning game on the class of up-growing on-line orders of width at most k is at least $\binom{k+1}{2}$.*

Proof. As in the previous section we identify chains and colors. The chain corresponding to color γ is denoted by C_γ and $top(\gamma)$ is the maximum element of this chain. If x is a maximal element of an order partitioned into chains (colors) then $private(x)$ is the set of colors γ with $top(\gamma) \leq x$ and $top(\gamma) \not\leq y$ for all maximal elements $y \neq x$.

Claim . For every positive integer k there is a strategy $S(k)$ for Bob so that after a finite number n_k of rounds: The order P given so far is of width k and has exactly k maximal elements. Moreover, the maximal elements can be numbered $x_1, \ldots, x_k$ so that for each $i = 1, .., k$ the size of $private(x_i)$ is at least i.

As the sets $private(x_i)$ are pairwise disjoint the theorem is an immediate consequence of this claim. We show the existence of strategy $S(k+1)$ by induction. Strategy $S(1)$ is trivial. Bob exhibits as single point and any assignment of a color to this point leads to the desired situation.

Strategy $S(k+1)$ is a threefold iteration of strategy $S(k)$. We describe and analyze $S(k+1)$ as a sequence of phases.

Phase 1. Bob runs strategy $S(k)$. This phase ends with an order Q^1 with maximum elements $x_1, \ldots, x_k$ and $|private(x_i)| \geq i$ for $i = 1, .., k$.

Phase 2. Bob again runs strategy $S(k)$. This time every new element is made greater than each of $x_1, \ldots, x_{k-1}$ and their predecessors in Q^1 but incomparable to all other points of Q^1. In particular, every new element is incomparable with all elements $top(\gamma)$ for $\gamma \in private(x_k)$. The phase ends with an order Q^2 with $k+1$ maximal elements $y_1, \ldots, y_k, x_k$. Note that we have at least i colors in $private(y_i)$ for $i = 1, .., k$ and additional k colors in $private(x_k)$. At this point Bob has already forced the use of at least $\binom{k+2}{2} - 1$ colors.

Phase 3. Bob adds a new element z so that z is greater than all elements of Q^2. For the color γ assigned to z it holds $\gamma \notin private(x_k)$ or $\gamma \notin private(y_k)$. We assume that $\gamma \notin private(x_k)$, otherwise interchange the role of x_k and y_k in the remainder of the argument. The set $private(z)$ now contains the color of z and all of $private(x_k)$, these $k+1$ colors will be the final private set of $z = z_{k+1}$.

Phase 4. In this final phase Bob runs strategy $S(k)$ with all new elements greater than $y_1, \ldots, y_k$ and their predecessors. The phase ends with maximal elements $z_1, \ldots, z_k, z_{k+1} = z$ so that $private(z_i) \geq i$ for $i = 1, .., k+1$.

This completes the proof of the claim and hence of the theorem. $\square$

It would be interesting to know the value of the game if we simultaneously impose the restrictions from Theorem 2 and Theorem 3. That is, if Bob has to build an up-growing order by means of an on-line 2-realizer.

2.2 A strategy for Alice

In this section we develop a strategy for Alice showing that every up-growing online order of width k can be partitioned on-line into $\binom{k+1}{2}$ chains. It was shown by Kierstead [2] that the greedy strategy (First-Fit) may need an unbounded number of chains to partition an up-growing on-line order of width 2 into chains. Hence, we will have to develop a somewhat more sophisticated algorithm. Again the classes of the partition will be identified with colors. We assume that a set Γ of $\binom{k+1}{2}$ colors is partitioned into k classes $\Gamma_1, \ldots, \Gamma_k$ so that Γ_i has exactly i elements for $i = 1, .., k$.

Recall a classical theorem of Dilworth. The set of maximum antichains of an order P forms a lattice. The order relation of this lattice is given by $A \leq B$ for maximum antichains A, B of P iff for all $a \in A$ there is a $b \in B$ with $a \leq b$ in P. We will use the notation $\textsc{hma}(P)$ to denote the *highest maximum antichain* of P, i.e., $\textsc{hma}(P)$ is the unique maximal element of the lattice of maximum antichains of P.

During the game Alice maintains an auxiliary structure $\mathcal{S}$ depending on P and the coloring of P. When Box expands P to P^+ by adding a new maximal point x then Alice constructs the new structure $\mathcal{S}^+$ for P^+. If $\mathcal{S}^+$ is constructed a legal color for x will be read of from this structure. The invariant gives the properties of $\mathcal{S}$.

Invariant. If $width(P) = l$ structure $\mathcal{S} = \mathcal{S}(P)$ is an l-tuple $(S_l, S_{l-1}, .., S_1)$ where each S_i a triple (A_i, a_i, ϕ_i) so that

(1) $A_l = \textsc{hma}(P)$ and if $i < l$ then $A_i = \textsc{hma}(T_i)$ where T_i is the set of elements y with $y \geq a$ for some $a \in A_{i+1} - a_{i+1}$.

(2) a_i is an element of A_i for $i = 1, .., l$.

(3) $\phi_i : A_i \to \Gamma_i$ is a bijection such that $top(\phi_i(a)) \leq a$ for all $a \in A_i$ and $i = 1, .., l$.

Let $T_l = P$ then $A_l = \textsc{hma}(T_l)$. With this convention the induction of the lemma below shows that for $i = 1, .., l$ the size of A_i is indeed i as required by property (3).

Lemma 2 *If $A_i = \textsc{hma}(T_i)$ is an i-antichain and $a_i \in A_i$ then $width(T_{i-1}) = i - 1$, hence, $A_{i-1} = \textsc{hma}(T_{i-1})$ is an $(i-1)$-antichain.*

Proof. Removing a_i from A_i leaves an $(i-1)$-antichain in T_{i-1}, therefore, $width(T_{i-1}) \geq i-1$. For the converse assume that there is an i-antichain A in T_{i-1}. Since $T_{i-1} \subset T_i$ antichain A is an i-antichain in T_i. Obviously, $A \geq \textsc{hma}(T_i)$ but $a_i \notin A$, hence, $A > \textsc{hma}(T_i)$ a contradiction. □

Let x be a new element, recall that x is a maximal element and denote the new order $P + x$ by P^+. The main task of the algorithm is the update of the auxiliary structure, i.e, the definition of the new set $\mathcal{S}^+$ of triples (A_i^+, a_i^+, ϕ_i^+) satisfying invariance conditions (1), (2) and (3). The color for element x is then chosen to be $\phi_i^+(x)$ where i is the unique index, so that, in $\mathcal{S}^+$ the i-th triple contains x as special element, i.e., $S_i^+ = (A_i^+, x, \phi_i^+)$.

Below we give an algorithm for the construction of $\mathcal{S}^+$. The sequence $\mathcal{S}^+ = (S_l^+, S_{l-1}^+, .., S_1^+)$ is constructed term by term. Therefore, when it comes to the definition of S_i^+ the set T_i^+ and, hence, also $A_i^+ = \textsc{hma}(T_i^+)$ is already known.

Let j be the size of a maximum antichain containing x. For all $i > j$ we leave S_i unchanged, i.e., $S_i^+ = S_i$, this corresponds to Case A in the algorithm. The highest maximum j-antichain A_j^+ in $T_j^+ = T_j + x$ contains x and is higher then A_j. Continuing there may be some indices $i \geq j$ with $\{x, a_i\} \subseteq A_i^+$ and A_i^+ is higher then

A_i, this is Case B, we let $a_i^+ = a_i$ and define ϕ_i^+ by pushing ϕ_i up along a matching between A_i and A_i^+. After iterating in Case B there will be a unique index i_c with the situation of Case C. This is $x \in A_{i_c}^+$ but $a_{i_c} \notin A_{i_c}^+$, we let $a_{i_c}^+ = x$ and define $\phi_{i_c}^+$ as in Case B. It can be shown that in Case C $A_{i_c}^+ = A_{i_c - 1} + x$. Hence for $i_c - 1$ and all remaining indices i we again leave S_i unchanged.

Step 1
if $width(P^+) = l + 1$ **then**
  let ϕ_{l+1}^+ be an arbitrary bijection $\phi_{l+1}^+ \colon A_l + x \to \Gamma_{l+1}$,
  $S_{l+1}^+ = (A_l + x, x, \phi_{l+1}^+)$
  $T_l^+ = \{\, y \in P : \text{ there is an } a \in A_l \text{ with } y \geq a\}$
else
  $T_l^+ = P^+$
Step 2
for $i = l$ **downto** 1 **do**
  switch to case
    Case A $T_i^+ = T_i + x$ and $x \notin \textsc{hma}(T_i + x)$
      then $S_i^+ = S_i$
    Case B $\textsc{hma}(T_i^+) = \textsc{hma}(T_i + x)$ and
        $x \in \textsc{hma}(T_i^+)$ and $a_i \in \textsc{hma}(T_i^+)$
      then $S_i^+ = (\textsc{hma}(T_i^+), a_i, \phi_i^+)$
    Case C $\textsc{hma}(T_i^+) = \textsc{hma}(T_i + x)$ and
        $x \in \textsc{hma}(T_i^+)$ and $a_i \notin \textsc{hma}(T_i^+)$
      then $S_i^+ = (\textsc{hma}(T_i^+), x, \phi_i^+)$
    Case D $T_i^+ \subseteq T_i$
      then $S_i^+ = S_i$
  $T_{i-1}^+ = \{\, y \in T_i^+ : \text{ there is an } a \in A_i^+ - a_i^+ \text{ with } y \geq a\}$
endfor

It remains to specify how to choose the bijection ϕ_i^+ Cases B and C. Let Q_i denote the order induced by $A_i^+ \cup A_i$. Assume that for all $i = 1, .., l$ the following two claims hold.

Claim 1. The width of Q_i is i.

Claim 2. $A_i^+ \geq A_i$ in the lattice of maximum antichains of Q_i.

These two claims will be proved later. Claim 1 implies that a minimum chain partition of Q_i defines a bijection between A_i^+ and A_i. Let $\psi_i : A_i^+ \to A_i$ be such a bijection and define $\phi_i^+ = \psi_i \circ \phi_i$. Clearly $\phi_i^+ \colon A_i^+ \to \Gamma_i$ is a bijection.

Claim 3. $top(\phi_i^+(a)) \leq a$ for all $a \in A_i^+$.

Proof. Rephrasing Claim 2 we obtain $\psi_i(a) \leq a$ for all $a \in A_i^+$. By induction $top(\phi_i(a')) \leq a'$ for all $a' \in A_i$. The claim follows from a combination of the two inequalities. $\triangle$

Hence, property (3) from the invariant holds for $\mathcal{S}^+$ and the assignment of color $\phi_i^+(x)$ to x is a legal move.

Lemma 3 *Let A and B be maximal antichains of an order T and let $A \geq B$ in the lattice of antichains. If x is a new maximal element and $B+x$ is an antichain in $T+x$ then $A+x$ is an antichain in $T+x$ and $A+x \geq B+x$.*

Proof. Suppose x is comparable with an $a \in A$ since x is maximal $a < x$. By the maximality of B there is a $b \in B$ comparable with a. From $A \geq B$ it follows that $a \geq b$, hence, $x > b$ a contradiction. $\square$

We now come to an analysis of the algorithm for the construction of $\mathcal{S}^+$. With induction from l to 1 we are going to show Claims 1 and 2. Note that if $x \in T_i^+$ then $\text{HMA}(T_i^+) = \text{HMA}(T_i + x)$ immediately implies $\text{HMA}(T_i^+) \geq \text{HMA}(T_i)$, i.e, Claim 2. Claim 1 follows if we additionally have $width(T_i + x) = i$. Hence, as long as $x \in T_i^+$ we assume for the induction that $width(T_i + x) = i$ and $\text{HMA}(T_i^+) = \text{HMA}(T_i + x)$ and show that the same holds for $i-1$. Actually we prove some more details that will be necessary for the proof of invariance properties (1) and (2).

Fact A. If S_i^+ is determined by Case A then $A_i = \text{HMA}(T_i^+)$ and $T_{i-1}^+ = T_{i-1} + x$.
Proof. From $x \notin \text{HMA}(T_i + x)$ we obtain with Lemma 3 that there is no maximum antichain of $T_i + x$ containing x. Therefore, the lattices of maximum antichains of T_i and $T_i + x$ coincide. This proves $\text{HMA}(T_i^+) = \text{HMA}(T_i) = A_i$.

Since x is a maximal element we obtain from $x \notin \text{HMA}(T_i)$ that x is greater than at least two elements of $\text{HMA}(T_i)$. This shows $x \in T_{i-1}^+$ and hence $T_{i-1}^+ = T_{i-1} + x$. $\triangle$

Fact B. If S_i^+ is determined by Case B then $x \in \text{HMA}(T_{i-1}^+) = \text{HMA}(T_{i-1} + x)$.
Proof. We fist note that $width(T_{i-1} + x) = i-1$: The existence of an i-antichain A in $T_{i-1} + x$ would contradict the assumption $a_i \in \text{HMA}(T_i^+) = \text{HMA}(T_i + x)$.

By the inductive assumption $A_i^+ \geq A_i$. Since $a_i^+ = a_i$ we obtain $T_{i-1}^+ \subseteq T_{i-1} + x$. Observe that T_{i-1}^+ and $T_{i-1} + x$ are upward closed sets of width $i-1$ and both contain the $(i-1)$-antichain $A - a_i$ hence their respective highest $(i-1)$-antichains coincide. Since $x \in A - a_i$ the element x is contained in the highest $(i-1)$-antichain of T_{i-1}^+ by Lemma 3. $\triangle$

Fact C. If S_i^+ is determined by Case C then $\text{HMA}(T_i^+) = A_{i-1} + x$, consequently $A_{i-1} \subseteq T_{i-1}^+ \subseteq T_{i-1}$.
Proof. Let $A = \text{HMA}(T_i^+) = \text{HMA}(T_i + x)$ then $A - x$ is an $(i-1)$-antichain in T_i. The assumption $A - x \neq A_{i-1}$ is contradictory by Lemma 3. $\triangle$

Fact D. If S_i^+ is determined by Case D then $A_i = \text{HMA}(T_i)$ and $T_{i-1}^+ = T_{i-1}$.
Proof. Obvious. $\triangle$

Lemma 4 *Let $\mathcal{S}$ satisfy the invariance properties for P and let $P^+ = P + x$ with x maximal in P^+. The algorithm defines a structure $\mathcal{S}^+$ satisfying the invariance properties.*

Proof. Note that for $i = 1,..,l$ the set T_i^+ matches the condition of at most one of cases A–D It remains to show that there is always a matching condition.

If $width(P^+) = l + 1$ then $A_l \subseteq T_l^+ \subseteq P = T_l$, hence, after Step 1 we continue with Case D. If $width(P^+) = l$ then $T_l^+ = T_l + x$ and after Step 1 we continue with one of the Cases A,B or C. Suppose that S_j^+ for $j \geq i$ have been defined. If S_i^+ was defined in Case A then $T_{i-1}^+ = T_{i-1} + x$ by Fact A and we continue with one of the Cases A,B or C. If S_i^+ was defined in Case B then by Fact B $x \in \text{HMA}(T_{i-1}^+) = \text{HMA}(T_{i-1} + x)$ and we continue with Case B or C. If S_i^+ was defined in Case C then by Fact C $A_{i-1} \subseteq T_{i-1}^+ \subseteq T_{i-1}$ this is the condition for continuation with Case D. Finally, if S_i^+ was defined in Case D then as consequence of Fact D all S_j^+ with $j < i$ will be defined in Case D.

With Fact A and Fact D we have shown that in all cases $A_i^+ = \text{HMA}(T_i^+)$ and $a_i^+ \in A_i^+$, i.e., invariance properties (1) and (2) hold. □

We summarize the result.

Theorem 4 *An up-growing on-line order of width at most k can be partitioned on-line into $\binom{k+1}{2}$ chains.*

3 The on-line dimension of up-growing orders

Kierstead, McNulty and Trotter [3] investigate the on-line dimension of orders. In the corresponding game Bob builds an on-line order of given width k while Alice maintains a realizer of the order. The minimum number of linear extensions in an on-line realizer of an order P is called the on-line dimension of P. Since the dimension of an order never exceeds its width it is natural to compare the on-line dimension of an order to the width of the order. The main negative result of [3] is.

Theorem 5 *For every positive integer n there are on-line orders of width three whose on-line dimension exeeds n.*

If the on-line order is up-growing we obtain a different result as an easy corollary of Theorem 4.

Theorem 6 *The on-line dimension of an up-growing on-line order of width k is at most $\binom{k+1}{2}$.*

Proof. Construct an on-line chain partition with at most $\binom{k+1}{2}$ chains. With each chain c associate a linear extension L_c according to two rules. (1) If the new element belongs to c then put it on top of L_c. (2) If the new element x is not in c then let x go as deep in L_c as possible, i.e., x is positioned immediately above the highest y in L_c with $y < x$. For every pair x, y of incomparable elements with $x \in c$ and $y \notin c$ we have x above y in L_c. Therefore, the family of linear extensions L_c forms an on-line realizer. □

For the above proof to work it is not really necessary that c is a chain. Consider the following chain covering game in which the rules governing the moves of Alice are relaxed compared to the chain partitioning game of the previous section: It is allowed to assign a set $C(x)$ of colors to the new element x, i.e., assign x to several chains. Moreover, colors may be removed from $C(x)$ in subsequent moves subject to two conditions. $C(x) \neq \emptyset$ for all elements x and for every color γ the set $\{x : \gamma \in C(x)\}$ is a chain. Call a game where Alice obeys these rules an adaptive chain covering game.

Theorem 7 *For up-growing on-line orders the value of the adaptive chain covering game and the on-line dimension equal each other.*

Proof. The idea for converting an on-line chain covering into an on-line realizer is exactly as in Theorem 6.

For the converse let $\{L_1, .., L_t\}$ be an on-line realizer. We use the numbers $1, .., t$ as colors and assign to an element x the set $C(x) = \{i :$ all elements above x in L_i are greater then x in the on-line order $\}$. It is clear that the set $C(x)$ can only shrink during the game. It remains to show $C(x) \neq \emptyset$. This follows from the fact that the algorithm constructing the on-line realizer has to be able to handle an element z with $y < z$ exactly if $x \not\leq y$. Such an element y can go below x in L_i only if $i \in C(x)$. $\square$

It would be very interesting to have good bounds for this on-line chain covering game. The author has not been able to make progress towards this goal. However, there are some indications that the on-line dimension for up-growing orders is substantially smaller than their on-line width, i.e., the number of chains in an on-line chain partition. This would complement the situation for general on-line orders and somehow tell us that having a linear extension helps more for dimension then for chain partitioning.

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