# Bose-Einstein Condensate in Weak $3d$ Isotropic Speckle Disorder

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(Dated: Received January 17, 2014)



The effect of a weak three-dimensional ($3d$) isotropic laser speckle disorder on various thermodynamic properties of a dilute Bose gas is considered at zero temperature. First, we summarize the derivation of the autocorrelation function of laser speckles in $1d$ and $2d$ following the seminal work of Goodman. The goal of this discussion is to show that a Gaussian approximation of this function, proposed in some recent papers, is inconsistent with the general background of laser speckle theory. In this context we also point out that the concept of a quasi-three dimensional speckle, which appears due to an extension of the autocorrelation function in the longitudinal direction of a transverse $2d$ speckle, is not applicable for the true $3d$ speckle, since it requires an additional space dimension. Then we propose a possible experimental realization for an isotropic $3d$ laser speckle potential and derive its corresponding autocorrelation function. Using a Fourier transform of that function, we calculate both condensate depletion and sound velocity of a Bose-Einstein condensate as disorder ensemble averages of such a weak laser speckle potential within a perturbative solution of the Gross-Pitaevskii equation. By doing so, we reproduce the expression of the normalfluid density obtained earlier within the treatment of Landau. This physically transparent derivation shows that condensate particles, which are scattered by disorder, form a gas of quasiparticles which is responsible for the normalfluid component.

PACS numbers: 67.85.Hj, 46.65.+g

## I. INTRODUCTION

The study of interacting bosonic atoms in a disordered potential landscape, called in the literature as "dirty boson problem" [1], has originally been introduced in the context of the motion of superfluid helium in porous Vycor glass [2]. Due to the frozen environment, disorder ensembles averages of physical observables have to be determined, which depend on many system parameters as, for instance, the strength of a repulsive interaction between two particles of the Bose gas as well as the strength and the correlation length which characterize the disorder potential. The main and intriguing part of the problem is the competition between the repulsive two-particle interaction and the localization property of disorder. From a theoretical point of view, the disorder potential was introduced by investigating the Anderson localization phenomenon for fermions [3]. Much attention has recently been paid for the Anderson localization and the propagation of bosonic matter waves in random external potentials [4]. Experimentally, the bosonic matter waves have been studied in the random potential produced either by laser speckles [5] or by an incommensurable optical lattice [6]. Whereas the laser speckle disorder potential is created by a laser beam scattered from a diffusive glass plate [7], the incommensurable optical lattice is produced through two interfering laser beams with incommensurable wavelengths. However, one needs to remark that such lattices exhibit certain pathological features, which do not occur in genuinely random lattices, such as a transition between localized and delocalized states, even in one spatial dimension [8]. In that sense the quasi-periodic lattices should be considered as to be quasi-random ones. Recent progress in different experimental realizations of laser speckle disorder is reported in Refs. [9, 10].

According to the laser speckle theory described in the seminal work of Goodman [7, 11], the monochromatic light reflected from a rough surface on the scale of an optical wavelength yields many independent dephased but coherent wavelets which interfere at a distance, which is essentially larger than the wavelength. This results in a granular pattern of intensity that is called Gaussian speckle as the real and imaginary parts of the field amplitude form a circular complex Gaussian distribution at any fixed spatial point. Details of the speckle formation will be considered in the next section of the paper. Here, we note that this distribution consists of the first-order statistics of the speckle disorder, while the second-order statistics of disorder is represented by its autocorrelation function.

In order to understand the underlying physics of laser speckles, let us briefly describe their formation in $2d$. Object waves are fields, which are a result of the incident polarized monochromatic field reflection from a rough surface, and they are described in a plane $\alpha, \beta$ immediately adjacent to the surface in terms of a complex function $a(\alpha, \beta)$ [12]. The Huygens-Fresnel principle establishes in the Fresnel approximation a relation between these object waves $a(\alpha, \beta)$ and the complex waves $A(x, y)$ in the observation plane $x, y$ through an integral which resembles a Fourier transformation. Hence, the wave $A(x, y)$ is a result of the interference of all object waves in the $x, y$ plane. As in the Fresnel approximation one assumes the condition $z \gg (\alpha^2 + \beta^2)_{\max}/\lambda$, where $z$ denotes the distance between the object wave $\alpha, \beta$ plane as well as the observation wave $x, y$ plane and $\lambda$ denotes the light wavelength, the waves $A(x, y)$ are called to be in far field [12].

In the Fourier mapping of object waves for the formation of far fields both the form and the finite size of the diffraction aperture $\mathcal{A}$ in the $\alpha, \beta$ plane plays a central role. It determines the form of the autocorrelation function as well as its correlation length, which characterizes the average size of the speckle, i.e. a grain of the above mentioned intensity pattern. Typically, the expression for the autocorrelation function consists of a constant and a spatially varying part. The latter, which is of interest for various speckle applications, has one central maximum and a set of side maxima of decaying height, which are separated from each other by zeros. This analytical structure is principal in the theory of laser speckles, since it is the result of the Fourier transformation of the finite-size diffraction aperture $\mathcal{A}$. Due to the existence of zeros, it can qualitatively not be approximated by a function of a Gaussian form as was assumed and even numerically derived in Refs. [13–15]. It is interesting that the experiment demonstrates an ambiguity in the following respect: whereas the function with zeros is exploited in the papers [5, 16, 17], the spatial autocorrelation function is fitted by a Gaussian in Refs. [9, 10, 18–21]. Calculating a standard deviation of the second-order moment of the random intensity, it was shown in Ref. [22] that for $1d$ the autocorrelation function derived in Ref. [7, 11] can be well approximated by a Gaussian form. However, a Fourier transform of this autocorrelation function, the power spectral density, which is essential for the theory of a Bose-Einstein condensate (BEC) in an external disorder potential, behaves, unlike the Gaussian function, as the triangle function $\text{tri}(x) = 1 - |x|$ for $|x| \leq 1$ and otherwise zero for any dimensionality. For $1d$ and $2d$ this was shown by Goodman in Refs. [7, 11], the corresponding $3d$ case is dealt with below in the text. This triangle function makes the upper limit of the integration in momentum space finite. For those reasons the recently proposed Gaussian autocorrelation function for the laser speckle is not suitable for a comprehensive description of a BEC in laser speckle disorder.

The present paper is organized as follows. We start with describing the basic principles of the laser speckle theory in Sec. II. Following a scheme described in Refs. [7, 11], we will then derive in Sec. III the expressions for the autocorrelation function of laser speckles and their Fourier transforms ranging from $1d$ to $3d$ with special emphasize on discussing both isotropic and anisotropic cases. The scheme of the possible experimental realization of the $3d$ isotropic speckle will be outlined in Sec. IV. Note, however, that we consider in our paper a true $3d$ speckle pattern, not a quasi-three dimensional one of a transverse $2d$ speckle with a longitudinal depth in the autocorrelation function as described in Ref. [23] and section 4.4.3 of the Goodman book [11], which has been applied in many experiments (see, for instance, Ref. [22]). This depth autocorrelation function concept assumes the existence of an additional spatial direction for the relevant speckle and can only be valid for $1d$ or $2d$ speckles. As is further discussed in Refs. [11, 23], the depth size is essentially larger than ones in other dimensions. Here we consider a $3d$ volume speckle with compatible speckle grain sizes in all spatial directions, which was already simulated in Refs. [13, 14]. Since the existing speckle patterns are experimentally produced mainly in a $2d$ geometry, we will propose a special scheme for its possible realization in a $3d$ volume. In the subsequent Sec. V the effect of a weak $3d$ isotropic speckle on various thermodynamic properties of a dilute Bose gas will be considered at zero temperature. To this end we calculate both condensate depletion and sound velocity of a BEC within a perturbative solution of the Gross-Pitaevskii equation. Afterwards, in Sec. VI, we reproduce the expression of the normalfluid density of a BEC in an external disorder potential obtained earlier within the treatment of Landau. From this rederivation we realize that condensate particles, which are scattered by a disorder potential, form a gas of quasiparticles, which is responsible for the normalfluid component. Finally, we summarize and analyze the results obtained in the paper in Sec. VII.

## II. FUNDAMENTALS OF LASER SPECKLE THEORY

According to Refs. [7, 11, 12] the circular Gaussian probability density function

$$p(A_R, A_I) = \frac{1}{2\pi\eta^2} \exp\left(-\frac{A_R^2 + A_I^2}{2\eta^2}\right), \tag{1}$$

for the real $A_R$ and imaginary $A_I$ parts of a far-field $A(x, y)$ at each point $x, y$ with the variance $\eta = \sqrt{\langle |A|^2 \rangle}$ represents the background of the theory of laser speckles. Another basis of the theory is the $M$-fold joint Gaussian probability density function

$$p([A]) = \frac{1}{(2\pi)^M |C_A|} \exp\left(-\frac{[A^*][A]}{[C_A]}\right) \tag{2}$$

for far-fields $A(x, y)$ at different points $x, y$. Here $[C_A]$ is a Hermitian symmetric matrix with determinant $|C_A|$, whose elements are given by $(C_A)_{i,j} =$

$\langle A^*(x_i, y_i)A(x_j, y_j)\rangle$ for a set of far-fields $[A] \equiv \{A(x_1, y_1), A(x_2, y_2), \ldots A(x_M, y_M)\}$ at $M$ points of the $x, y$ plane. Note that the notation $\langle\cdots\rangle$ in the expressions for $\eta^2$ and $(C_A)_{i,j}$ and throughout below in the text means the disorder ensemble average. Furthermore, one assumes that the indices $i, j$ at $(C_A)_{i,j}$ are taken for adjacent spatial positions.

Expressions (1) and (2) are the result of the central limit theorem of probability theory [24], which claims the following: if complex random variables are the sum of other independent complex random variables then, at the increase of the number of second ones, the first ones are distributed according to the Gaussian law. Applying the theorem for our case we have far fields $A(x, y)$ as result of the interference of independent object waves at all positions of $x, y$ plane. As we will see below, there are mainly two physical conditions for the validity of the central limit theorem. They are related to the physics of providing independence of the object waves and to the method of their summation within the interference process. We will now describe both of them in more detail.

A requirement for the object wave $a(\alpha, \beta)$ to be independent leads to some limitations for its statistical properties [12]. First of all, formed after the reflection of monochromatic light from the rough surface, the individual wavelet $a(\alpha, \beta)$ should be completely polarized. Second, the first-order probability density of its phase should be uniform in the interval $-\pi$ to $\pi$. And at last, the object wave $a(\alpha, \beta)$ should be quasi-homogeneous, which means that its autocorrelation function $C_a$ consists of a slowly-varying intensity $I_a$ envelope and a short-range normalized correlation function $C'_a$:

$$C_a(\alpha_1, \beta_1; \alpha_2, \beta_2) \equiv \langle a^*(\alpha_1, \beta_1)a(\alpha_2, \beta_2)\rangle$$
$$= I_a\left(\frac{\alpha_1+\alpha_2}{2}, \frac{\beta_1+\beta_2}{2}\right) C'_a(\alpha_2-\alpha_1, \beta_2-\beta_1)\,. \quad (3)$$

If we increase in this expression the range of variation of the correlation part, i.e. the correlation length of the object wave, the changing range of the intensity becomes smaller. However, in order to entirely satisfy the independence condition of the object waves, their correlation length in Eq. (3) should be as short as possible, which means that $C'_a(\alpha,\beta)$ has to be delta correlated. The latter introduces some demands upon the properties of the random light scatterer, which is called in the literature as the diffusor. Typically, a diffusor is an optically homogeneous transparent glass plate with no reflection centers for light in the volume and a geometrically inhomogeneous distribution of reflection centers with random heights on its surface. As mentioned in the introductory section of the paper, the scattering rough surface generates object waves in a plane $\alpha, \beta$, which is closely situated at the surface, when the monochromatic polarized incident light transmits through the plate. Another realization of object waves is considered in Refs. [7, 11], where the lateral monochromatic light was directly incident on the rough surface. For our purpose to calculate the normalized speckle autocorrelation function, the optical property of the medium, from which light falls on the rough surface, is merely dropped from the consideration. Assuming a Gaussian probability density of the surface height $h(\alpha, \beta)$ with the autocorrelation function $C_h(\alpha, \beta)$ and a variance $\eta_h^2$ and assuming also a Gaussian probability density of object wave phases with a variance $\eta_\phi^2$, Goodman derived the relation [7, 11]

$$C'_a(\alpha,\beta) = \exp\left(-\eta_\phi^2[1-C_h(\alpha,\beta)]\right)\,, \quad (4)$$

where $\eta_\phi = 2\pi\eta_h/\lambda$. This function can be approximated by a delta function $\delta(\alpha,\beta)$ when $\eta_\phi > 1$ and thus $\eta_h > \lambda/2$ and the mean distance between two inhomogeneous $h(\alpha, \beta)$ is larger than $\lambda$. The delta functional autocorrelation of object waves provides their independence from each other. On the other hand, the Gaussian probability density of phases reduces to a uniform one for $\eta_\phi > 1$ supporting the second requirement for the object waves outlined above. Therefore, the requirements for object waves described in the previous paragraph can be experimentally realized if the size of the surface inhomogeneities and the distance between them are larger than the wave length of the light. As a next implication of the presented analysis we can suppose that for the outlined system parameters the object wave probability density itself may have a circular Gaussian form for the real and imaginary components of the wavelet $a(\alpha, \beta)$.

The next important step of the theory of Gaussian laser speckles is the formation of far fields for a given set of object waves. It is based on the Huygens-Fresnel principle of optics, which preserves the individual wavelet picture, i.e., works in the limit of optics, where deviations from geometrical optics are small. The interference of the object waves yields an amplitude $A(x, y)$, which reads in the above mentioned far field Fresnel approximation as follows

$$A(x,y) = \int_{\mathcal{A}} a(\alpha,\beta)\exp\left[-\frac{2\pi i}{\lambda z}(x\alpha+y\beta)\right]d\alpha d\beta\,, \quad (5)$$

where we have omitted unimportant multipliers in front and inside of the integral. This expression resembles, indeed, a Fourier transformation and, thus, conserves the principle that each object wave contributes individually to the interference.

Using Eqs. (3) and (5) it is straight-forward to derive the expression for the autocorrelation function

$$C_A(x_1, y_1; x_2, y_2) = \langle A^*(x_1, y_1)A(x_2, y_2)\rangle\,. \quad (6)$$

It turns out to be given by

$$C_A(x_1, y_1; x_2, y_2) = I_A\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) C'_A(\Delta x, \Delta y) \quad (7)$$

for $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$ with

$$I_A(x,y) = \int_{\mathcal{A}} C'_a(\alpha'',\beta'')\exp\left[-\frac{2\pi i}{\lambda z}(x\alpha''+y\beta'')\right]d\alpha'' d\beta'' \quad (8)$$

and

$$C'_A(x,y) = \int_{\mathcal{A}} I_a(\alpha',\beta') \exp\left[-\frac{2\pi i}{\lambda z}(x\alpha' + y\beta')\right] d\alpha' d\beta' \tag{9}$$

for the novel variables $\alpha' = (\alpha_1+\alpha_2)/2$, $\beta' = (\beta_1+\beta_2)/2$ and $\alpha'' = \alpha_2 - \alpha_1$, $\beta'' = \beta_2 - \beta_1$. Eq. (7) shows that the far fields $A(x,y)$ are quasi-homogeneous like the object waves $a(\alpha,\beta)$. This is a direct result of the Fourier transformation (5). Another important result of this linear transformation is the implicit proof of the above made supposition that object waves are circularly Gaussian distributed. Indeed, only Gaussian distributed object waves can contribute through the linear mapping to Gaussian far fields. As we will see below the role of the so far uninvestigated parameter, the aperture $\mathcal{A}$, will lead to the formation of a correlation length of the correlation function $C'_A(x,y)$.

The autocorrelation function $C_I(x,y)$ of the far-field intensity $I(x,y) = |A(x,y)|^2$ can be calculated using the Wick theorem for variables distributed according to the Gaussian law. A simple calculation gives the expression

$$C_I(x,y) = \langle I\rangle^2 \left[1 + |C'_A(x,y)|^2\right] , \tag{10}$$

where the normalized autocorrelation function for the far-field $C'_A(x,y)$ is defined as $C'_A(x,y)/C'_A(0,0)$, where $C'_A(0,0) = \langle I\rangle$.

As already mentioned above, if in Eq. (3) the autocorrelation function of object waves $C'_a(\alpha,\beta)$ is delta correlated, then the intensity function of these waves $I_a(\alpha,\beta)$ can be approximated as a constant. Assuming that the $\alpha,\beta$ plane is close to the rough surface, one can write $|a(\alpha,\beta)| = \kappa|P(\alpha,\beta)|$, where $P(\alpha,\beta)$ is the incident to the glass plate light wave and $\kappa$ is the average reflectivity of surface, for each position $\alpha,\beta$. Then the intensity of the object waves at $\alpha,\beta$ is determined by the relation $I_a(\alpha,\beta) = \kappa^2|P(\alpha,\beta)|^2$. Therefore, the expression for the normalized far-field autocorrelation function reads

$$C'_A(x,y) = \frac{\int_{\mathcal{A}} |P(\alpha,\beta)|^2 \exp\left[-\frac{2\pi i}{\lambda z}(x\alpha + y\beta)\right] d\alpha d\beta}{\int_{\mathcal{A}} |P(\alpha,\beta)|^2 d\alpha d\beta} . \tag{11}$$

## III. SPECKLE AUTOCORRELATION FUNCTION FOR APERTURES IN $1d$ TO $3d$ DIMENSIONS

As already mentioned in the introductory section, the investigation of a BEC in the laser speckle disorder has found much attention from both a theoretical and an experimental point of view. In particular, a variety of isotropic and anisotropic speckles have been the subject of these works. Motivated by this interest, we will describe in the present section the derivation of the speckle autocorrelation function for different apertures ranging from one to three dimensions by generalizing the appropriate expressions from the previous section to these dimensions.

### A. Real space

Due to the analytic form of Eq. (11), we can take the intensity of the incident wave $|P(\alpha,\beta)|^2$ to be unity over the whole aperture region of the $\alpha,\beta$ plane. Writing the function $|P|^2$ in the form $|P|^2_{d,\mathcal{A}}$, where $d$ is the space of dimensionality and $\mathcal{A}$ is the form of the aperture, we have the following expressions:

$$|P(\alpha,\beta)|^2_{2d,\mathrm{rct}} = \mathrm{rect}\left(\frac{\alpha}{L_\alpha}\right) \mathrm{rect}\left(\frac{\beta}{L_\beta}\right) \tag{12}$$

for the $2d$ anisotropic rectangular aperture with sizes $L_\alpha$ and $L_\beta$, where the function $\mathrm{rect}(x) = 1$ for $|x| \leq 1/2$ and zero otherwise; retaining in Eqs. (12) only the first $\mathrm{rect}(x)$ function and equating $L_\alpha = L$ one obtains the expression of $|P(\alpha)|^2_{1d,\mathrm{inv}}$ for the $1d$ interval aperture of the size $L$; the analytic form of $|P(\alpha,\beta)|^2_{2d,\mathrm{qdt}}$ for the $2d$ quadratic aperture of the size $L$ is obtained from Eqs. (12) if we specialize this equation according to $L_\alpha = L_\beta = L$;

$$|P(\alpha,\beta)|^2_{2d,\mathrm{crc}} = \mathrm{circ}\left(\frac{2r}{D}\right) \tag{13}$$

for the $2d$ isotropic circular aperture with the diameter $D$ and $r = \sqrt{\alpha^2+\beta^2}$, where the function $\mathrm{circ}(x) = 1$ for $|x| \leq 1$ and zero otherwise;

$$|P(\alpha,\beta,\gamma)|^2_{3d,\mathrm{rcpl}} = \mathrm{rect}\left(\frac{\alpha}{L_\alpha}\right) \mathrm{rect}\left(\frac{\beta}{L_\beta}\right) \mathrm{rect}\left(\frac{\gamma}{L_\gamma}\right) \tag{14}$$

for the $3d$ anisotropic rectangular parallelepiped aperture of sizes $L_\alpha$, $L_\beta$ and $L_\gamma$; the expression $|P(\alpha,\beta,\gamma)|^2_{3d,\mathrm{cub}}$ of the $3d$ cubic aperture of the size $L$ is obtained from Eq. (14) by setting $L_\alpha = L_\beta = L_\gamma = L$;

$$|P(\alpha,\beta,\gamma)|^2_{3d,\mathrm{sph}} = \mathrm{circ}\left(\frac{2r}{D}\right) \tag{15}$$

for the $3d$ isotropic sphere aperture with the diameter $D$ and $r = \sqrt{\alpha^2+\beta^2+\gamma^2}$;

$$|P(r,\gamma)|^2_{3d,\mathrm{cyl}} = \mathrm{circ}\left(\frac{2r}{D}\right) \mathrm{rect}\left(\frac{\gamma}{L_\gamma}\right) \tag{16}$$

for the $3d$ anisotropic cylinder aperture with the diameter of circle $D$, $r = \sqrt{\alpha^2+\beta^2}$ and size $L_\gamma$ along the $\gamma$ axis. Substituting the expressions (12)–(16) of the $|P|^2_{d,\mathcal{A}}$ function in Eq. (11) and calculating the respective integrals, we obtain the corresponding expressions for the correlation function $|C'_A|^2$:

$$|C'_A(\Delta x,\Delta y)|^2_{2d,\mathrm{rct}} = \mathrm{sinc}^2\left(\frac{L_\alpha \Delta x}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{L_\beta \Delta y}{\lambda z}\right) \tag{17}$$

where $\mathrm{sinc}(y) = \sin(\pi y)/(\pi y)$, for the $2d$ anisotropic rectangular aperture with $z$ being the distance between object wave and far field planes; retaining in this equation only first $\mathrm{sinc}^2(y)$ function, the dependence on $\Delta x$ and assuming $L_\alpha = L$ one obtains the expression $|C'_A(\Delta x)|^2_{1d,\mathrm{inv}}$ for the $1d$ interval aperture with $z$ being the distance between object wave and far field intervals; the expression $|C'_A(\Delta x, \Delta y)|^2_{2d,\mathrm{qdt}}$ for the $2d$ quadratic aperture one can derive from Eq. (17) for $L_\alpha = L_\beta = L$;

$$|C'_A(r)|^2_{2d,\mathrm{crc}} = \left| 2\frac{J_1\left(\dfrac{\pi D r}{\lambda z}\right)}{\dfrac{\pi D r}{\lambda z}} \right|^2 , \qquad (18)$$

where $J_1(x)$ is a Bessel function of the first kind and of the first order, for the $2d$ isotropic circular aperture with $r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$;

$$|C'_A(\Delta x, \Delta y, \Delta z)|^2_{3d,\mathrm{rcpl}} = \mathrm{sinc}^2\left(\frac{L_\alpha \Delta x}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{L_\beta \Delta y}{\lambda z}\right) \mathrm{sinc}^2\left(\frac{L_\gamma \Delta z}{\lambda z}\right) \qquad (19)$$

for the $3d$ anisotropic rectangular parallelepiped aperture with $z$ as the distance between the object wave and the far field volumes; the expression $|C'_A(\Delta x, \Delta y, \Delta z)|^2_{3d,\mathrm{cub}}$ for the $3d$ cubic aperture is obtained from Eq. (19) by assuming $L_\alpha = L_\beta = L_\gamma = L$;

$$|C'_A(r)|^2_{3d,\mathrm{sph}} = \left| 3\left(\frac{\lambda z}{\pi D r}\right)^3 \left[ \sin\left(\frac{\pi D r}{\lambda z}\right) - \left(\frac{\pi D r}{\lambda z}\right) \cos\left(\frac{\pi D r}{\lambda z}\right) \right] \right|^2 \qquad (20)$$

for the $3d$ isotropic sphere aperture with $r = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2}$;

$$|C'_A(r, \Delta z)|^2_{3d,\mathrm{cyl}} = \left| 2\frac{J_1\left(\dfrac{\pi D r}{\lambda z}\right)}{\dfrac{\pi D r}{\lambda z}} \right|^2 \mathrm{sinc}^2\left(\frac{L_\gamma \Delta z}{\lambda z}\right) \qquad (21)$$

for the $3d$ anisotropic cylinder aperture with $r = \sqrt{(\Delta x)^2 + (\Delta y)^2}$.

Expressions of the autocorrelation function for a $2d$ quadratic aperture $|C'_A(\Delta x, \Delta y)|^2_{2d,\mathrm{qdt}}$ and for a $2d$ circular aperture in Eq. (18) are derived by Goodman in Refs. [7, 11]. As can be seen from the formulas for other cases of the aperture, they are closely related to both of these Goodman cases of the aperture. However, the derivation of the $3d$ isotropic sphere autocorrelation function (20), which is a result of the present paper, required some additional effort.

The analytical forms of the autocorrelation functions $|C'_A|^2$ are similar in every spatial direction. They have one central maximum and a set of side maxima of decaying height, which are separated from each other by zeros. As was pointed out in the introductory section, it is obvious that these forms can not be fitted by a Gaussian. The argument of the autocorrelation function, which corresponds to its first zero, provides the correlation length of the disorder, i.e. the average size of the speckle grain, for the appropriate spatial direction. Denoting it by $\delta x$ we have, for instance, for $1d$ speckle

$$\delta x = \frac{\lambda z}{L} . \qquad (22)$$

The main interest of the present paper is the $3d$ spherical aperture of Eq. (20) since we will carry out the calculation of BEC properties for this particular case of laser speckles. Numerically solving the equation $\sin(x) - x\cos(x) = 0$ we find first its solution to be at $x_c = 4.493$, thus the disorder correlation length is given by $r_c = 1.4302\ \lambda z/D$.

In order to establish a physical meaning of $\delta x$ we introduce the "wave number" $k_{\mathrm{eff}}$, which is related to the vector $\alpha, \beta$ in the above Fourier transform formulas, by the relation $k_{\mathrm{eff}} = 2\pi x/(\lambda z)$. If we substitute in it $\delta x$ from Eq. (22) then we obtain $k_{\mathrm{eff}} = 2\pi/L$. For a circular and a spherical aperture the "wave number" is $k_{\mathrm{eff}} = 2\pi/D$. However, the sense of $k_{\mathrm{eff}}$ is in an uncertainty of the wave vector when the problem of wave propagation is solved in the restricted area. It is well known that in this area the wave vector is determined within the resolution $k_{\mathrm{eff}}$. Therefore, we can say that the origin of a speckle grain with a correlation length $\delta x$ as its size represents the spatial uncertainty in the determination of far fields, which is introduced by the finite size of the aperture.

### B. Fourier space

For many applications the Fourier transform of the far-field intensity autocorrelation function, or the power spectral density, of the speckle is of considerable interest. In the literature on laser speckle theory [7, 11] it is defined according to

$$C_I(\mathbf{k}) = \int C_I(\mathbf{x}) e^{-i2\pi \mathbf{k}\mathbf{x}} d^d x . \qquad (23)$$

Substituting in it Eq. (10) for $C_I(\mathbf{x})$ one obtains

$$C_I(\mathbf{k}) = \langle I \rangle^2 \left[ \delta(\mathbf{k}) + |C'_A(\mathbf{k})|^2 \right] . \qquad (24)$$

In the perturbative considerations of BEC in the speckle potential the Fourier transform $|C'_A(\mathbf{k})|^2$ plays the central role. It has the following expressions for the real space autocorrelation functions taken from Eqs. (17)–(21):

$$|C'_A(\mathbf{k})|^2_{2d,\mathrm{rct}} = \frac{(\lambda z)^2}{L_\alpha L_\beta} \mathrm{tri}\left(\frac{k_x \lambda z}{L_\alpha}\right) \mathrm{tri}\left(\frac{k_y \lambda z}{L_\beta}\right) \qquad (25)$$

where the triangle function is defined as $\mathrm{tri}(x) = 1 - |x|$ for $|x| \le 1$ and zero otherwise, for the $2d$ anisotropic rectangular aperture; the expression $|C'_A(\mathbf{k})|^2_{1d,\mathrm{inv}}$ for the $1d$

interval aperture one can get from Eq. (25) by assuming $L_\alpha = L_\beta = L$, $k_x = k_y = k$ and taking the square root of its right-hand side; the expression $|C'_A(\mathbf{k})|^2_{2d,\mathrm{qdt}}$ for the $2d$ quadratic aperture is obtained from Eq. (25) with the assumption $L_\alpha = L_\beta = L$;

$$|C'_A(\mathbf{k})|^2_{2d,\mathrm{crc}} = 2\left(\frac{2\lambda z}{\pi D}\right)^2 \times\left[\cos^{-1}\left(\frac{k\lambda z}{D}\right) - \frac{k\lambda z}{D}\sqrt{1-\left(\frac{k\lambda z}{D}\right)^2}\,\right] \tag{26}$$

for the $2d$ isotropic circular aperture with $k = \sqrt{k_x^2 + k_y^2}$;

$$|C'_A(\mathbf{k})|^2_{3d,\mathrm{rcpl}} = \frac{(\lambda z)^3}{L_\alpha L_\beta L_\gamma} \times \mathrm{tri}\left(\frac{k_x\lambda z}{L_\alpha}\right)\mathrm{tri}\left(\frac{k_y\lambda z}{L_\beta}\right)\mathrm{tri}\left(\frac{k_z\lambda z}{L_\gamma}\right) \tag{27}$$

for the $3d$ anisotropic rectangular parallelepiped aperture; the expression $|C'_A(\mathbf{k})|^2_{3d,\mathrm{cub}}$ for the $3d$ cubic aperture is obtained from Eq. (27) by specializing $L_\alpha = L_\beta = L_\gamma = L$;

$$|C'_A(\mathbf{k})|^2_{3d,\mathrm{sph}} = \frac{3}{\pi}\left(\frac{2\lambda z}{4D}\right)^3 (b^3 - 12b + 16) \tag{28}$$

for the $3d$ isotropic sphere aperture with $b = 2k\lambda z/D$ and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$;

$$|C'_A(\mathbf{k})|^2_{3d,\mathrm{cyl}} = \left(\frac{\lambda z}{L_\gamma}\right)\mathrm{tri}\left(\frac{k_z\lambda z}{L_\gamma}\right) \times 2\left(\frac{2\lambda z}{\pi D}\right)^2\left[\cos^{-1}\left(\frac{k\lambda z}{D}\right) - \frac{k\lambda z}{D}\sqrt{1-\left(\frac{k\lambda z}{D}\right)^2}\,\right] \tag{29}$$

for the $3d$ anisotropic cylinder aperture with $k = \sqrt{k_x^2 + k_y^2}$.

The equation for the quadratic aperture $|C'_A(\mathbf{k})|^2_{2d,\mathrm{qdt}}$ and Eq. (26) have been derived by Goodman in Refs. [7, 11]. Other expressions of $|C'_A(\mathbf{k})|^2$, except for the $3d$ isotropic sphere aperture case, can be obtained by using these formulas. Eq. (28) is a result of this paper. In all our formulas for the anisotropic aperture we have assumed that the size deviation of the aperture with respect to its average isotropic size is essentially less than the distance $z$.

As is seen from the formulas of $|C'_A(\mathbf{k})|^2$ expressed through the triangle function their value becomes zero when their argument is unity. For the $2d$ circle and the $3d$ sphere apertures $|C'_A(\mathbf{k})|^2$ is zero for $k\lambda z/D = 1$. Hence, the wave vector of the Fourier transform autocorrelation function only varies in a finite interval from zero, in contrast to the case for a Gaussian function. This fact is another reason why the speckle autocorrelation function can not be approximated by a Gaussian form.

It is worth to discuss the expression $|C'_A(r)|^2 = \mathrm{sinc}^2(k_L r)$ with $k_L = D/\lambda z$ for the autocorrelation function used in Ref. [25] for the $3d$ isotropic aperture. It is similar to our correlation function $|C'_A(\Delta x)|^2_{1d,\mathrm{inv}}$ for the $1d$ interval aperture. The authors of Ref. [25] claim that this expression is valid for $z \sim (\alpha^2 + \beta^2)_{\max}/\lambda$, which is outside of the far-field limit. However, that limit destroys the fundamentals of the Gaussian speckle theory as they are described in Sec. II. Therefore, it is unclear whether $|C'_A(r)|^2$ of Ref. [25] is related to laser speckles or not.

Furthermore, we discuss the definition of the speckle correlation length to be the width at the half value of the maximum of $|C'_A(r)|^2$ for $r = 0$, when the last one is approximated by a Gaussian function. Probably, this definition was introduced first by Modugno in Ref. [26], when he considered $|C'_A(\Delta x)|^2_{1d,\mathrm{inv}}$. It was found in Ref. [26] that the correlation length is given by $\delta x = 0.88\lambda z/L$, while from Eq. (22) the exact value turns out to be $\delta x = \lambda z/L$. It is interesting that the Gaussian $|C'_A(r)|^2$ has been obtained in the numerical simulation of the $3d$ isotropic laser speckle in Refs. [13, 14] which should be compared with the exact $|C'_A(r)|^2_{3d,\mathrm{sph}}$ in Eq. (20), with the correlation length $r_c = 1.1\lambda z/D$, however, the exact one is $r_c = 1.4302\lambda z/D$, see the discussion after Eq. (22). It seems that we can explain the reason why the authors of Refs. [13, 14] obtained the Gaussian form of $|C'_A(r)|^2$. They used the speckle simulation method proposed by Huntley in Ref. [27] which we briefly review for the $2d$ case. Let us consider to this end two square planes $\alpha, \beta$ and $x, y$ with the same size $L$. According to the Huntley method one uses Eq. (5) in order to perform a double Fourier transformation. In the first inverse Fourier transformation the complex object waves $a(\alpha, \beta)$ on the mesh points in the $\alpha, \beta$ plane are simulated through the given Gaussian distributed complex random waves $A(x, y)$ on the mesh points in the $x, y$ plane. Afterwards, one cuts by a circle with radius $D/2$ the $\alpha, \beta$ region of the obtained $a(\alpha, \beta)$ such that it vanishes outside of this region. In the second direct Fourier transformation the derived complex waves $a(\alpha, \beta)$ form the final complex far-fields $A(x, y)$. Huntley has investigated in Ref. [27] only the first-order statistical property of the simulated pattern, i.e. the probability density of the intensity, and showed that it corresponds to the theoretical laser speckles of Ref. [7]. However, the proposed simulation method can drastically deviate in the second-order statistical property of a speckle, i.e. its autocorrelation function, from the theoretical one.

Indeed, in accordance with the theory of a speckle autocorrelation function as presented in this section, after the first Fourier mapping the object waves $a(\alpha, \beta)$ acquire a correlation with the correlation length $\delta\alpha = \delta\beta = \lambda z/L$, where $L$ is size of the square $x, y$ plane. More precisely, now the function $C'_a(\alpha, \beta)$ is not delta correlated. However, according to Eq. (3), the broadening of the $C'_a(\alpha, \beta)$ function reduces to a changing of a constant character of the $I_a(\alpha, \beta)$ function to one of varying in space in the $\alpha, \beta$ plane. Substituting this function of

$I_a(\alpha, \beta)$ in Eq. (9) and integrating over $\alpha$ and $\beta$ gives the function $C'_A(x, y)$ which may qualitatively be different from the one discussed in this section.

A simulation method, which is consistent with the above laser speckle theory, is described in the book of Goodman [11]. There are other numerical methods in Refs. [28, 29], in which the exact form of the real space autocorrelation function is used to generate the speckle pattern. In particular, one of such methods was exploited for the simulation of $1d$ speckle in Ref. [30].

## IV. EXPERIMENTAL REALIZATION OF $3d$ ISOTROPIC SPECKLE

As was already mentioned in the introductory section, we consider here a true $3d$ speckle, not the quasi-three dimensional one consisting of a transverse $2d$ speckle with a longitudinal depth in the autocorrelation function as described in details in Ref. [23] and section 4.4.3 of the Goodman book [11] and applied in many experiments. At a first glance, it seems exotic and unrealistic to experimentally realize such a $3d$ volume speckle pattern. However, in the present section we will describe the physical principle how it can be generated.

In the typical $2d$ geometry of the experimental realization of a speckle a lens, which collects the incident light, is installed close to the glass plate such that its focal plane coincides with the far-field plane [22]. This idea of a speckle formation in the focal plane can be generalized to a full $3d$ geometry, when the speckle is formed in the focal point, i.e. the focus, of an empty ellipsoidal optic cavity according to the scheme displayed in Fig. 1. Let us consider that cavity, whose inside surface reflects absolutely the light emitted from one of its focus (point $A$) and collects it at the second focus (point $B$). Two laser beams (thick yellow arrows) are incident through holes in the cavity surface into the small metallic sphere, i.e. the reflector, with absolute light reflection, located in the center of the glass sphere $A$. It is assumed that laser beams cover the entire surface of this reflector and a BEC is deposited in the sphere $B$, which is located at the second focus of the ellipsoid.

The glass sphere $A$ additionally contains the located randomly light scattering centers, for instance, absolutely light reflective metallic polyhedrons with a random average size of each facet. The theory how to derive the $2d$ object wave autocorrelation function $C'_a(\alpha, \beta)$, described in Refs. [7, 11], can be easily generalized to the derivation of $C'_a(\alpha, \beta, \gamma)$ for such a $3d$ case with the same expression (4). However, now this expression is a function of $3d$ other quantities. The condition, at which $C'_a(\alpha, \beta, \gamma)$ becomes delta correlated and the object waves are independent, is the same as for $C'_a(\alpha, \beta)$. Therefore, if the mean distance between these light scattering centers and the average size of polyhedrons are larger than the light wavelength, then $C'_a(\alpha, \beta, \gamma)$ will be delta correlated. On the other hand, each sphere with radius $(\alpha^2 + \beta^2 + \gamma^2)^{1/2}$ and with the same center as the sphere $A$ will be the object wave volume, whereas for comparison for $2d$ we had a $\alpha, \beta$ object wave plane.

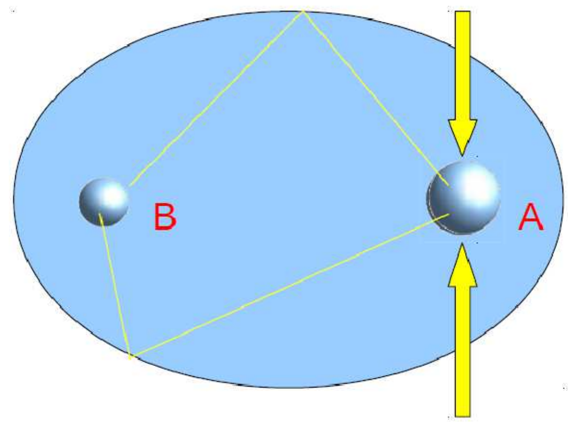


FIG. 1: Cross section of the ellipsoidal reflective cavity with spheres $A$ and $B$ in its focuses. Focus $A$ contains a small size absolute spherical light reflector in the center and volume optic inhomogeneities. Incident laser beams (thick yellow arrows), after reflection from the reflector, scatter additionally from inhomogeneities producing individual wavelets (thin yellow lines), which are collected in focus $B$, where a BEC is deposited.

Incident laser beams, after reflection from the reflector, scatter additionally from scattering centers and produce individual and independent wavelets, the object waves $a(\alpha, \beta, \gamma)$, indicated via thin yellow lines in Fig. 1, which are collected in the sphere $B$, where a BEC is deposited. For the presented geometry the far-field condition is satisfied, since a distance $z$ between the object wave and the far-field volumes, i.e. the length of each wavelet trajectory between two focuses of the ellipsoid, is larger than the size of the object wave sphere $A$.

The described scheme can be generalized for the experimental realization of any $3d$ anisotropic speckle. To this end one only needs to change the form of the spherical aperture $A$, which contains the glass and the light scattering centers, into a suitable one listed in the previous section. The spherical form of the metallic reflector retains unchanged.

At the end of this section, it is worthwhile to discuss the possible realization of a $3d$ volume speckle pattern using $2d$ plane speckles. Such a scenario presumes a $3d$ speckle as a result of the sum or, more clearly, as a linear interference of two and more $2d$ speckles. While theoretically this scenario is discussed by Pilati *at al.* in Ref. [14], the experiment, in which two perpendicular $2d$ speckle planes form a $3d$ speckle pattern, was realized in Ref. [31] by Jendrzejewski *et al.* Instantly the question arises whether the random pattern realized in such a way belongs to the class of speckles or not. In spite of an additional theoretical analysis, which is required to answer that question in detail, the following argument shows that the possible conclusion is negative. Indeed, according to the fundamentals of the laser speckle theory of Goodman, Refs. [7, 11], and Dainty, Ref. [12], see also Secs. II and III of this paper, the correlated speckle pattern in any dimension, except the one described in Ref. [23] and its analogue for $1d$ (see next paragraph), is a result of the Fourier transform over the restricted aperture object wave region in the same dimension. This means that a $3d$ volume speckle can be obtained only by a $3d$ object wave volume. Physically it means that the single connected spatial domain of each $3d$ speckle grain, which is a result of $3d$ correlations, can not be obtained by a linear combination of randomly sized and independent $2d$ speckle grains. By that reason, the true $3d$ speckle cannot be obtained even by a combination of quasi-three dimensional speckles, which we discussed at the beginning of this section.

## V. BEC DEPLETION AND SOUND VELOCITY IN WEAK $3d$ ISOTROPIC SPECKLE

The interaction potential of light with an atom at position $\mathbf{r}$ is determined by the far-field intensity $I(\mathbf{r}) = |A(\mathbf{r})^2|$ and has the form $V(\mathbf{r}) = tI(\mathbf{r})$, see for instance Refs. [22, 25], where the constant $t$ is a function of the atomic and light characteristics. At the derivation of $V(\mathbf{r})$ it was assumed that the incident laser wave does not induce an atomic electron interlevel transition, but merely deforms the atomic ground state.

It is convenient to define the interaction potential as $V(\mathbf{r}) = V_0 + \Delta V(\mathbf{r})$, where $\Delta V(\mathbf{r}) = V(\mathbf{r}) - V_0$ and $V_0 = \langle I \rangle$. Using the obvious property $\langle \Delta V(\mathbf{r})\rangle = 0$, a simple calculation shows that

$$\langle V(\mathbf{r}')V(\mathbf{r}'+\mathbf{r})\rangle = V_0^2\left[1+\frac{\langle \Delta V(\mathbf{r}')\Delta V(\mathbf{r}'+\mathbf{r})\rangle}{V_0^2}\right] \quad (30)$$

and, therefore, we have the following relationships between the laser speckle autocorrelation and the disorder potential correlation functions:

$$\begin{aligned} |C_I(\mathbf{r})|^2 &= \langle V(\mathbf{r}')V(\mathbf{r}'+\mathbf{r})\rangle , \\ |C_A'(\mathbf{r})|^2 &= \frac{\langle \Delta V(\mathbf{r}')\Delta V(\mathbf{r}'+\mathbf{r})\rangle}{V_0^2} . \end{aligned} \quad (31)$$

Our interest is a Bose gas with a contact interaction. Taking into account that, according to the novel definition of $V(\mathbf{r})$, the chemical potential for the ground state of BEC will be renormalized according to $\mu \to \mu - V_0$, the Gross-Pitaevskii equation (GPE) reads

$$\left[-\frac{\hbar^2}{2m}\boldsymbol{\nabla}^2 + \Delta V(\mathbf{r}) + g|\Psi(\mathbf{r})|^2 - \mu\right]\Psi(\mathbf{r}) = 0 . \quad (32)$$

Here $g = 4\pi\hbar^2 a/m$ denotes the strength of the contact interaction with the scattering length $a$.

Under the assumption that the disorder potential is weak, one can expand the solution

$$\Psi(\mathbf{r}) = \psi_0 + \psi_1(\mathbf{r}) + \psi_2(\mathbf{r}) + \cdots \quad (33)$$

and solve the GPE (32) perturbatively in the respective order of $\Delta V(\mathbf{r})$ [32]. For the ground state all functions of the expansion as well as $\Psi(\mathbf{r})$ are real. In this way the problem is reduced to find the total particle density $n = \langle \Psi(\mathbf{r})^2\rangle$ and the condensate density $n_0 = \langle \Psi(\mathbf{r})\rangle^2$. In particular, the lowest order expression for the condensate depletion reads

$$n - n_0 = n_0 \int \frac{d^3k}{(2\pi)^3}\frac{R(\mathbf{k})}{[\hbar^2\mathbf{k}^2/2m + 2ng]^2} + \cdots , \quad (34)$$

where we introduced the literature notation $R(\mathbf{k}) = R|C_A'(\mathbf{k})|^2$ with $R = V_0^2$.

In order to further apply our formula Eq. (28) for the $3d$ isotropic autocorrelation function $|C_A'(\mathbf{k})|^2_{3d,\mathrm{sph}}$, one needs to make a remark. According to the definition in Eq. (23), the Fourier transforms of autocorrelation functions carry a physical dimension. In particular, the correlation function $|C_A'(\mathbf{k})|^2_{3d,\mathrm{sph}}$, calculated with Eq. (23), is proportional to the inverse volume of the $3d$ isotropic aperture $3/(4\pi)(2/D)^3$ times $(\lambda z)^3$. If we introduce the correlation length as $\sigma = \lambda z/D$, then the proportionality factor is $3(2\sigma)^3/(4\pi)$. In the following, we assume that $|C_A'(\mathbf{k})|^2_{3d,\mathrm{sph}}$ is already normalized by that factor.

It is convenient to introduce also the BEC coherence length according to $\xi = [\hbar^2/(2mng)]^{1/2} = 1/\sqrt{8\pi n a}$. Substituting the normalized correlation function $R|C_A'(\mathbf{k})|^2_{3d,\mathrm{sph}}$ from Eq. (28) in Eq. (34) and performing the integration, we get the expression $n - n_0 = n_{\mathrm{HM}} f(\sigma/\xi)$, where the depletion $n_{\mathrm{HM}} = [m^2R/(8\pi^{3/2}\hbar^4)]\sqrt{n/a}$ was obtained by Huang and Meng in Ref. [33] (see also Ref. [34]) for delta correlated disorder $R(\mathbf{r})$ and the condensate depletion function is defined via

$$\begin{aligned} f\left(\frac{\sigma}{\xi}\right) &= \frac{1}{\sqrt{2}\pi}\frac{\sigma}{\xi}\left[4 - \left(\frac{8\sigma^2}{\xi^2}+6\right)\ln\left(1+\frac{\xi^2}{2\sigma^2}\right)\right. \\ &\left. + \frac{4}{\sqrt{2}}\frac{\xi}{\sigma}\arctan\left(\frac{\xi}{\sqrt{2}\sigma}\right)\right] . \end{aligned} \quad (35)$$

The function $f(\sigma/\xi)$, which is depicted in Fig. 2, has the following asymptotics for small $\sigma/\xi$

$$f\left(\frac{\sigma}{\xi}\right) \approx 1 - \frac{14\sqrt{2}}{3\pi}\left(\frac{\sigma}{\xi}\right)^3 - \frac{18\sqrt{2}}{5\pi}\left(\frac{\sigma}{\xi}\right)^5 + \cdots \quad (36)$$

and, correspondingly, for large $\sigma/\xi$

$$f\left(\frac{\sigma}{\xi}\right) \approx \frac{1}{2^{5/2}\pi}\left[\frac{1}{3}\left(\frac{\xi}{\sigma}\right)^3 - \frac{1}{10}\left(\frac{\xi}{\sigma}\right)^5\right] + \cdots \quad (37)$$

Introducing the appropriate correlation length for each aperture, as described in Sec. III, one can show that,

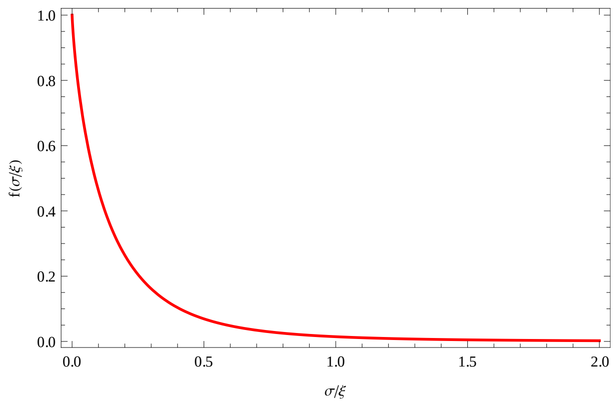


FIG. 2: Condensate depletion function $f(\sigma/\xi)$ from Eq. (35).

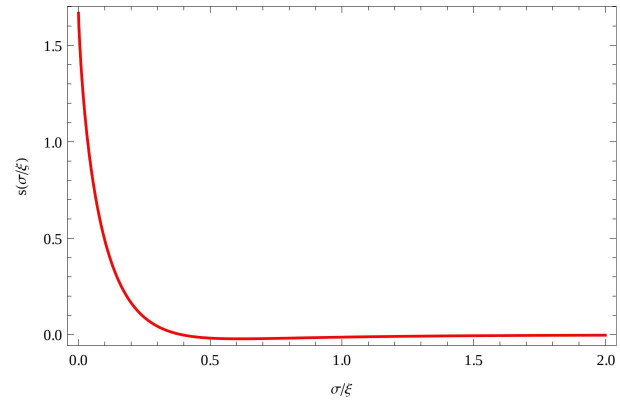


FIG. 3: Sound velocity function $s(\sigma/\xi)$ from Eq. (39).

when this correlation length tends to zero, then the corresponding correlation function $|C'_A(\mathbf{r})|^2$ tends to the delta function. The same behavior has our function $|C'_A(\mathbf{r})|^2_{3d,\mathrm{sph}}$ in the limit $\sigma \to 0$. Therefore, we should reproduce the Huang and Meng result $n_{\mathrm{HM}}$ for the condensate depletion in this limit. Indeed, when $\sigma/\xi \to 0$ we read off from Eq. (36) that one obtains $f(\sigma/\xi) \to 1$.

For the $3d$ isotropic Bose gas with contact interaction the normalfluid density $n_N$ is determined by the equation $n_N = 4(n - n_0)/3$ (see Sec. VI below and Ref. [32] as well as the references therein), from which $n_N$ is proportional to the function of $f(\sigma/\xi)$.

In Ref. [32] the sound velocity of a dipolar BEC in a weak external disorder potential is calculated within a hydrodynamic approach. To this end a general derivation was performed which is applicable for an arbitrary interaction potential. For an isotropic $3d$ system with contact interaction it has the form:

$$\frac{c}{c_0} = 1 + \int \frac{d^3k}{(2\pi)^3} \frac{R(\mathbf{k})}{(\hbar^2\mathbf{k}^2/2m + 2ng)^2} \times \left\{ \frac{\hbar^2\mathbf{k}^2/2m}{(\hbar^2\mathbf{k}^2/2m + 2ng)} - (\hat{\mathbf{q}}\hat{\mathbf{k}})^2 \right\} + \cdots , \quad (38)$$

where $c_0 = (ng/m)^{1/2}$ is the sound velocity in a system without disorder and the scalar product between the sound direction $\hat{\mathbf{q}}$ and the direction of wave propagation $\hat{\mathbf{k}}$ has the form $\hat{\mathbf{q}}\hat{\mathbf{k}} = \cos\vartheta$ for an isotropic system.

Calculating the integral in Eq. (38), we obtain $c/c_0 = 1 + n_{\mathrm{HM}} s(\sigma/\xi)/(2n)$, where the sound velocity function reads

$$s\left(\frac{\sigma}{\xi}\right) = \frac{2^{3/2}}{\pi}\frac{\sigma}{\xi}\left[\frac{14}{3} - \left(\frac{28\sigma^2}{3\xi^2} + 4\right)\ln\left(1 + \frac{\xi^2}{2\sigma^2}\right) + \frac{5}{3\sqrt{2}}\frac{\xi}{\sigma}\arctan\left(\frac{\xi}{\sqrt{2}\sigma}\right)\right] \quad (39)$$

It is depicted in Fig. 3 and has the following asymptotics for small $\sigma/\xi$

$$s\left(\frac{\sigma}{\xi}\right) \approx \frac{5}{3} + \frac{2^{3/2}3}{\pi}\left(\frac{\sigma}{\xi}\right) - \frac{2^{3/2}62}{9\pi}\left(\frac{\sigma}{\xi}\right)^3 + \cdots \quad (40)$$

and for large $\sigma/\xi$

$$s\left(\frac{\sigma}{\xi}\right) \approx \frac{2^{1/2}}{\pi}\left[-\frac{7}{3}\left(\frac{\xi}{\sigma}\right) + \frac{13}{18}\left(\frac{\xi}{\sigma}\right)^3\right] + \cdots , \quad (41)$$

respectively.

Again, when the correlation length $\sigma \to 0$ and thus the correlation function $|C'_A(\mathbf{r})|^2_{3d,\mathrm{sph}}$ is delta correlated, we reproduce the result $s(\sigma/\xi) \approx 5/3$ of Ref. [35], obtained for delta correlated $R(\mathbf{r})$.

As shown in this section, the finite range of integration for the vector $\mathbf{k}$, when a Fourier transform of a speckle correlation function is being applied, essentially simplifies the analytic calculation of the BEC properties. This is an essential advantage of applying the laser speckle theory to the BEC investigation. Conversely, due to the infinite limit of integration on $\mathbf{k}$, the Gaussian disorder correlation function, which is often used in the literature, introduces some difficulties in its application to the BEC theory.

## VI. LANDAU DERIVATION OF NORMALFLUID DENSITY

Let $K_0$ be a reference frame, and $K$ a second frame with relative velocity $-\mathbf{v}$ with respect to $K_0$. According to the Galilean transformation in classical mechanics, the energy $E_0$ of a system in the frame $K_0$ and its energy $E$ in the frame $K$ are related to each other by:

$$E = E_0 - \mathbf{P}_0\mathbf{v} + \frac{M}{2}\mathbf{v}^2 , \quad (42)$$

where $\mathbf{P}_0$ and $M$ are the total momentum and the mass of the system, respectively.

Following to Refs. [36, 37] let us assume that, at temperature $T = 0$, the condensate is in rest, i.e., in the frame $K_0$, and its energy is $E_0 = 0$ with momentum $\mathbf{P}_0 = 0$. If one quasiparticle with mass $m$ appears in the condensate with energy $\varepsilon(\mathbf{p})$, where $\mathbf{p}$ is a momentum of the quasiparticle, then in the frame $K_0$ energy and momentum now become $E_0 = \varepsilon(\mathbf{p})$ and $\mathbf{P}_0 = \mathbf{p}$. Hence, from Eq. (42) the energy $E$ in frame $K$ will be $E = \varepsilon(\mathbf{p}) - \mathbf{pv} + M\mathbf{v}^2/2$ and the energy of the quasiparticle in frame $K$ after a Galilean transformation has a form $\varepsilon(\mathbf{p}) - \mathbf{pv}$.

According to the Landau two-fluid theory [36, 37] of liquid helium II, a gas of quasiparticles, for instance phonons, constitutes the normalfluid density at low temperatures. For $T = 0$ no quasiparticles exist, thus the helium is entirely superfluid. If a gas of quasiparticles appears in the system for finite but low temperatures, which has zero center mass velocity in the frame $K_0$ and moves with constant velocity $-\mathbf{v}$ with respect to the frame $K$, in which the helium liquid is in the rest, then the total momentum of the gas per volume in the frame $K$ is given by

$$\frac{\mathbf{P}}{V} = \int \mathbf{p}\, N(\varepsilon(\mathbf{p}) - \mathbf{pv}) \frac{d^3p}{(2\pi\hbar)^3}\,, \tag{43}$$

where $N(\varepsilon(\mathbf{p}))$ is the average occupation number of states by phonons with energy $\varepsilon(\mathbf{p})$. Eq. (43) describes the thermodynamic property of a gas of phonons. However, it can be generalized to our BEC system in the external disorder potential at $T = 0$, if we assume that, after scattering with the disorder, particles of the condensate become the quasiparticles of the normalfluid density. It is clear that it occurs when the disorder is attached to the frame $K_0$. To this end we replace in Eq. (43) the thermodynamic quantity $N(\varepsilon(\mathbf{p})-\mathbf{pv})$ by the quantum one $|\Psi(\mathbf{p}-m\mathbf{v})|^2$, where the wave function is a solution of the GPE with disorder and written in momentum representation. After that we average both sides of Eq. (43) over the disorder ensemble. The obtained mean square of the modulo of the wave function is now homogeneous in space, so it can be expressed in terms of the energy of a quasiparticle, as the Hamiltonian is commutative with the momentum operator and thus the eigenfunction of the latter can be taken as the eigenfunction of the former [38]. Recalling that the expression for the total density is $n = \langle \Psi^2 \rangle$, we obtain

$$\frac{\langle \mathbf{P} \rangle}{V} = \int \mathbf{p}\, n(\varepsilon(\mathbf{p}) - \mathbf{pv}) \frac{d^3p}{(2\pi\hbar)^3}\,. \tag{44}$$

In order to derive the expression for the normalfluid density we expand the integrand of Eq. (44) in power of $\mathbf{pv}$ and, in the limit $\mathbf{v} \to \mathbf{0}$, retain only its first two terms. After integrating over the directions of the vector $\mathbf{p}$ the zeroth order term of this expansion disappears. Thus one obtains

$$\frac{\langle \mathbf{P} \rangle}{V} = -\int \mathbf{p}\,(\mathbf{pv}) \frac{dn(\varepsilon(\mathbf{p}))}{d\varepsilon(\mathbf{p})} \frac{d^3p}{(2\pi\hbar)^3}\,. \tag{45}$$

This expression is the main result of the normalfluid density Landau theory, when the two replacements $\langle \mathbf{P} \rangle$ by $\mathbf{P}$ and $n(\varepsilon(\mathbf{p}))$ by $N(\varepsilon(\mathbf{p}))$ are performed.

Taking into account that $\mathbf{p}(\mathbf{pv}) = p_z^2 \mathbf{v}$, the expression for the normalfluid density reduces to

$$\rho_n = -\int p_z^2 \frac{dn(\varepsilon(\mathbf{p}))}{d\varepsilon(\mathbf{p})} \frac{d^3p}{(2\pi\hbar)^3}\,. \tag{46}$$

From Eq. (34) we have the expression of the total density Fourier transform

$$n(\varepsilon(\mathbf{p})) = (2\pi)^3 n_0 \delta(\mathbf{k}) + \frac{n_0 R(\mathbf{k})}{(\hbar^2\mathbf{k}^2/2m + 2ng)^2}\,, \tag{47}$$

in first order of $R(\mathbf{k})$, from which the energy of the quasiparticles follows to be $\varepsilon(\mathbf{p}) = \mathbf{p}^2/2m + 2ng$, where $\mathbf{p} = \hbar\mathbf{k}$. Substituting $\varepsilon(\mathbf{p})$ in Eq. (46) and performing its integral by parts and using in the obtained expression $n(\varepsilon(\mathbf{p}))$ from Eq. (47), one gets

$$\rho_n = \rho_0 \int \frac{d^3k}{(2\pi)^3} \frac{p_z^2 R(\mathbf{k})}{\mathbf{p}^2(\hbar^2\mathbf{k}^2/2m + 2ng)^2}\,, \tag{48}$$

where $\rho_0 = mn_0$.

It is interesting that there is the relationship $\varepsilon_{\mathrm{B}}(\mathbf{p}) = \varepsilon^{1/2}(\mathbf{p})\mathbf{p}/(2m)^{1/2}$ between our $\varepsilon(\mathbf{p})$ and the Bogoliubov quasiparticle energy $\varepsilon_B(\mathbf{p})$. If we use this relation, then we obtain

$$\rho_n = \frac{\rho_0}{4} \int \frac{d^3k}{(2\pi)^3} \frac{\mathbf{p}^2\, p_z^2 R(\mathbf{k})}{m^2 \varepsilon_B^4(\mathbf{p})}\,. \tag{49}$$

This expression without the prefactor $1/4$ coincides with Eq. (19) of Ref. [35] for the normalfluid density $\rho_{n,LR}$, obtained within the linear response approach, if we replace $V \int d^3k/(2\pi)^3$ by $\sum_{\mathbf{k}}$. The prefactor $1/4$ appears from the relation between $\varepsilon(\mathbf{p})$ and $\varepsilon_{\mathrm{B}}(\mathbf{p})$. For a $3d$ isotropic BEC system we have $p_x^2 = p_y^2 = p_z^2$ and $\mathbf{p}^2 = 3p_z^2$. Multiplying the right-hand side of Eq. (48) with 3 and canceling $3p_z^2$ and $\mathbf{p}^2$ in the numerator and the denominator, we obtain Eq. (34), therefore, $n - n_0 = 3\rho_{n,LR}/(4m)$ [35].

It is worth to discuss the validity to use the Landau approach for BEC with the disorder. According to a remark in the text book [39] the Landau approach should not be applicable for such a system. Indeed, the applied Landau derivation of the normalfluid density presumes the validity of the quasiparticle concept (see, for instance, Refs. [36, 37]), in which there are no collisions not only between quasiparticles but also of last ones with the external disorder potential. More exactly, according to this concept quasiparticles should be well defined and their gas should be ideal.

In our case, effective quasiparticles with the mean-field energy $\varepsilon(\mathbf{p})$ and the quantum state distribution at temperature $T = 0$, represented by the total density $n(\varepsilon(\mathbf{p}))$, appear in the system after the disorder ensemble average. However, after this averaging the real space is homogeneous and there is no reason for the gas of effective quasiparticles to be not ideal. Hence, if for the conventional quasiparticles the source of their appearance is the low temperature, here it is the scattering of the condensate particles with the disorder and then their excitation and departure from the condensate. This physical conclusion naturally arises from the Landau derivation of the normalfluid density.

## VII. SUMMARY AND CONCLUSION

At first, we have summarized the derivation of the autocorrelation function of the laser speckle in $1d$ and $2d$ following the seminal work of Goodman. We showed that a Gaussian approximation of this function, proposed in some recent papers, is inconsistent with the background of laser speckle theory. Then we have proposed a possible experimental realization for an isotropic $3d$ laser speckle potential and derived its corresponding autocorrelation function. Using a Fourier transform of that function, we calculated both condensate depletion and sound velocity of a BEC in a weak speckle disorder within a perturbative solution of the Gross-Pitaevskii equation. At the end, we reproduced the expression of the normalfluid density obtained earlier within the treatment of Landau. This physically transparent derivation showed that condensate particles, which are scattered by disorder, form a gas of quasiparticles which is responsible for the normalfluid component. We have justified the validity of the Landau approach to our BEC system with disorder.

## VIII. ACKNOWLEDGEMENTS

One of the authors, B. A., thanks the Volkswagen Foundation for partial support of the work. B. A. is also grateful to Center for International Cooperation at the Freie Universität Berlin for its hospitality. Both authors appreciate Hagen Kleinert and the members of his group for many discussions.

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