Lemma 6.7 *If p_d primitive elements produce a_d total elements, then p_d^2 primitive elements produce $\geq \frac{a_d^2}{2}$ total elements. As a consequence, $p_d^{2^k}$ primitive elements produce $\geq a_d^{2^k-\varepsilon''}$ total elements for all $\varepsilon'' > 0$.*

Proof of lemma. Expand the binomial coefficients on the r.h.s. of (6.2) as a product of k_j factors as in (6.5). Now you see that replacing $p_d \mapsto p_d^2$ each factor at least squares itself. It only remains to apply the standard inequality

$$2\sum_{i\vdash d} a_i^2 \geq \left(\sum_{i\vdash d} a_i\right)^2,$$

with a_i being the contributions from the partitions of the degree. □

Of course, for the k we have to choose in the proof of theorem 6.6, when applying lemma 6.7 we have

$$\hat{C}^{d/\sqrt{f(d)}} \geq \left(\hat{C}^{d/f(d)}\right)^{2^k}$$

only for *almost* all d, say for $d \geq d_0$, but omitting factors of the first d_0 degrees divides the contribution by a term bounded above by a polynomial in d, e.g. you can take

$$d^{\sum_{i=1}^{d_0} \hat{C}^{i/f(i)}},$$

which can be compensated by choosing ε'' in the lemma a little bigger (since $d/f(d)$ grows faster than d^ε for some $\varepsilon > 0$), so the argument still works. □

We see that the exponential growth is a very strong "barrier". If the growth of p_d is less than exponential, then there is a qualitative difference between the growth of the total dimension and the one of the primitive part.

And, once it is broken, the primitive elements become dominating in each degree, so the asymptotics of a_D and p_D is (up to a negligable factor) equal.

In view of all this the decisive question is

Question. Is the exponential asymptotics a lower or an upper bound for Vassiliev invariants?

Answering this question will surely be hard. We saw why for the lower bound it will be so – we are much further away from the exponential bound than theorem 6.5 suggests. On the other hand, for an upper bound the best we can offer at present is something like $D!/1.1^D$ [St6]. Thus also in this case hard work is in store for us …

7 The braid index and the growth of Vassiliev invariants

In this section, we use the new approach of braiding sequences to prove exponential upper bounds for the number of Vassiliev invariants on knots with bounded braid index and arborescent knots.

Diagrams refer henceforth to knot diagrams (and not to chord diagrams).

7.1 Braiding sequences

Recall the basic definitions in the context of braiding sequences from §1.7.

Definition 7.1 *For some odd $k \in \mathbf{Z}$, a k-braiding of a crossing p in a diagram D is a replacement of (a neighborhood of) p by the braid σ_1^k (see figure 6). A braiding sequence (associated to a numbered set P of crossings in a diagram D; all crossings by default) is a family of diagrams, parametrized by $|P|$ odd numbers $x_1, \dots, x_{|P|}$, each one indicating that at crossing number i an x_i-braiding is done.*

Any Vassiliev invariant v of degree at most k behaves on a braiding sequence as a polynomial of degree at most k in $x_1, \dots, x_{|P|}$ (see [St4] and [Tr]), and this polynomial is called braiding polynomial of v on this braiding sequence.

Let C be a class of knots and $v : C \longrightarrow \mathbf{Q}$ a map. Extend v to singular knots as described in section 1, equation (1.1). This extension is well-defined on those singular knots, all of whose resolutions result in knots from C.

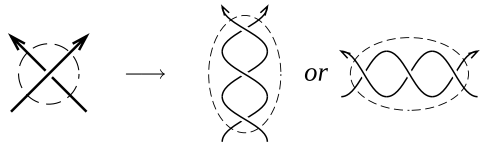


Figure 6. Two ways to do a -3-braiding at a crossing.

Definition 7.2 *Let C be a knot class. Then a Vassiliev invariant of deg $\le m$ on C is a map $v : C \longrightarrow \mathbf{Q}$ with v vanishing on all $(m+1)$-singular knots, whose all 2^{m+1} possible resolutions give knots in C.*

Question. Is any Vassiliev invariant on some knot class C the restriction of a Vassiliev invariant (on all knots) to C? Or, in other words, does any Vassiliev invariant on C admit an extension to a Vassiliev invariant (on all knots)?

Remark 7.1 It is maybe a question of personal taste to which degree this question is interesting. However, while the classical approach fails to give any statement about counterexample invariants to question 7.1 (if such exist), such counterexample invariants would be naturally incorporated into the braiding sequence approach. Therefore, until the non-existence of such counterexamples is not proved, it is not clear, that the finite-dimensionality results of [St4] are consequences of [Ko, Dr, BN2]. However, the braiding sequence approach fails to give any indication if (or for which classes) such examples exist.

Definition 7.3 *Let C be a class of knots. The set of Vassiliev invariants on C is called finitely-determined, if for all $n \in \mathbf{N}$ there exists a constructible finite set $C_n \subset C$, such that any Vassiliev invariant on C of degree $\le n$ is uniquely determined by its values on C_n.*

In [St4] I proved that Vassiliev invariants are finitely-determined on arborescent knots (and connected sums thereof) and closed 3 braids.

Let us briefly recall the idea in [St4, §10], how braiding sequences can be used to prove finite-determination and upper bounds for the number of Vassiliev invariants on some knot classes. We covered such a class C by braiding sequences and worked inductively over the length/weight (in some specified sense, which depends on the context, see lemma 7.7 and definition 7.13) of these braiding sequences. Assume that for a certain set $C' \subset C$ a Vassiliev invariant v of degree p is 0 and use induction. Special cases of a braiding sequence B give shorter braiding sequences, on which v is 0 by induction, and each such special case on the level of braiding polynomials gives a *simplifying* or recursive relation (an equality between special values of a braiding polynomial P and values of a braiding polynomial of a shorter braiding sequence, which are 0 by induction). That is, a simplifying or recursive relation is an equality of the kind

$$P_B\big|_{x_{i_1}:=a_{i_1},\dots,x_{i_n}:=a_{i_n}} = P_{B'}\big|_{x_{i'_1}:=b_{i'_1},\dots,x_{i'_m}:=b_{i'_m}}$$

for some $m, n, i_j, i'_j \in \mathbf{N}$ and $a_{i_j}, b_{i'_j} \in \mathbf{Z}$, and $P_B, P_{B'}$ being the braiding polynomials of some fixed Vassiliev invariant v on braiding sequences B and B' with $\mathrm{len}(B') < \mathrm{len}(B)$.

A simplifying or recursive relation augments the degree of a non-trivial solution for the braiding polynomial P of v on B. This happens, because if P turns to zero setting $x_{i_j} := a_{i_j}$ for some $i_j, j = 1, \dots, k, a_{i_j} \in \mathbf{Z}$, then any top degree monomial of P contains at least one of the x_{i_j}'s. If the degree of P is pushed higher than the degree of v by finding sufficiently many such (disjoint) sets $\{x_{i_j}\}$, only the trivial solution for the braiding polynomial P of v on B is possible, and so the induction step shows that v is 0 on the whole class C. Then Vassiliev invariants are finite-determined on C by setting $C_p := C'$, and in particular, the number of linearly independent Vassiliev invariants of degree $\le p$ on C is maximally $|C'|$.

7.2 Arborescent knots

Rational (2-bridge) knots are a nice example of such a class which can be dealt with by the above idea – the braiding sequences are given by (the parities of) the coefficients in the Conway notation.

The Conway notation [Ad, Co] can be formally considered as a map of formal expressions built up of the letter set $\mathbf{Z} \cup \{\infty\}$ and binary operations '$\cdot$' (called product and often omitted and replaced by concatenating the factors) and ',' (called sum) to tangle diagrams, as shown on figure 7.

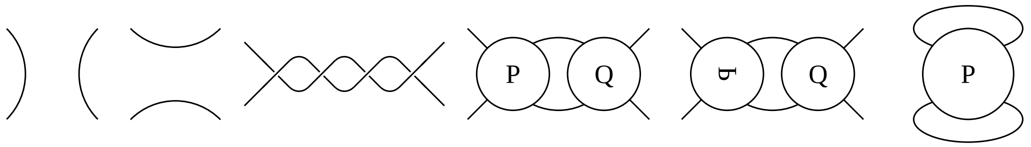


Figure 7. Operations with arborescent tangles

Such tangles are called arborescent. Note, that the product operation is not associative. The omission of parentheses in the notation means left parenthesation, that is $abcd$ is treated as $((ab)c)d$. Rational tangles are arborescent tangles, whose notation does not contain parentheses and sum operations. Rational/arborescent knots are closed rational/arborescent tangles, as shown to the right of figure 7.

Now any summation can be replaced by a product, where the first factor (tangle) has been mirrored with respect to its NW-SE axis. More generally, it is possible to rotate a tangle through an angle of $k \cdot \pi/2, k \in \mathbf{Z}$ or mirror it in the plane or in projection direction producing another tangle. Such an operation we will call $flip$. Replacing in the Conway notation sums by products and flips, we obtain the *notation with flip*.

Now, in all the Conway notations or notations with flip varying some coefficient by ± 2 corresponds to a braiding and so any notation is a member of a braiding sequence given by the parities ('$+$' for even parity and '$-$' for odd parity) of its coefficients, e. g.,

$$(2,3)\cdot(5,6) \,\in\, (+,-)\cdot(-,+)\,.$$

Using this method, in [St4] we proved:

Theorem 7.4 *([St4]) Vassiliev invariants on rational knots are exponentially bounded in the degree, that is, there exists some $C > 1$ with*

$$\dim\{\text{ Vassiliev invariants of degree} \le k \text{ on rational knots }\} \,\le\, C^k\,.$$

We also proved in the same way:

Theorem 7.5 *([St4]) Vassiliev invariants on arborescent knots with all Conway coefficients even are exponentially bounded in the degree.*

Our first aim is the obvious extension both of these theorems.

Theorem 7.6 *Vassiliev invariants on arborescent knots are exponentially bounded in the degree.*

Proof. We split it into two lemmas. The first one is [St4, lemma 8.1], but we include the proof here.

Lemma 7.7

$$\#\{\text{ simplifying relations }\} \ge \frac{\sqrt{n}}{2},\;\; n = \text{len}(\text{ notation })\,,$$

where the length of a braiding sequence is the number of integers (or rather their parities) in the notation with flip, and such that each variable appears maximally in one such relation.

Proof of lemma 7.7. Start induction with $n = 2$ and $n = 3$.
For $n = 2$ there are 4 choices: $--, -+, +-, ++$, with possible 'flip's applied to some signs. In all cases one of the following relations applies (note, that $\infty = $ flip (0))

$$\begin{gathered} 1\cdot 1 = 2 \\ 1\cdot 0 = 1 \\ 0\cdot 1 = \infty \\ 0\cdot 0 = \infty \end{gathered}$$

and analogously for ∞. So there is always 1 simplifying relation. In the same way argue for $n = 3$.

Now do the induction step.

Assume the inequality of the lemma is true for all notations with flip of len $< n$ and consider a notation A with len $= n$. There are 4 choices for A

$$(ab)c \quad \text{flip}\,(ab)c \quad a(bc) \quad a\text{flip}\,(bc) \tag{7.1}$$

or 'flip's of these 4 expressions, but it is unnecessary to consider $A = \text{flip}\,(\text{ something })$ as all the simplifications we can achieve in 'something' carry over after flips.

Denote by $l_i, i \in \{a, b, c\}$ the lengths of the subexpressions a, b, c in (7.1).

Case 1. Assume, that maximally one of a, b, c has length 1. Then

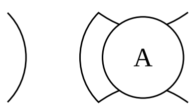
$$\#\{\text{ simplifying relations }\} \geq \frac{\sqrt{l_a}}{2} + \frac{\sqrt{l_b}}{2} + \frac{\sqrt{l_c}}{2} - \frac{1}{2},$$

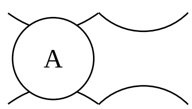
latter term standing to equilibrate a possible uncorrect contribution from one of l_a, l_b, l_c being 1. So, as $l_a + l_b + l_c = n$,

$$\frac{\sqrt{l_a} + \sqrt{l_b} + \sqrt{l_c}}{2} - \frac{1}{2} \geq \frac{\sqrt{n}}{2} \quad \text{for } n \geq 3\,.$$

Case 2. So two of a, b, c must be of length 1.

Assume, one of these two would be a '$+$' (or a 'flip' thereof). Then set 0 into the Conway notation. There are basically 2 possibilities (any 'flip's of the factors do not change qualitatively the picture):

2.1) $0 \cdot A$ 

2.2) $A \cdot 0$ 

In case 1 your diagram decomposes. You obtain

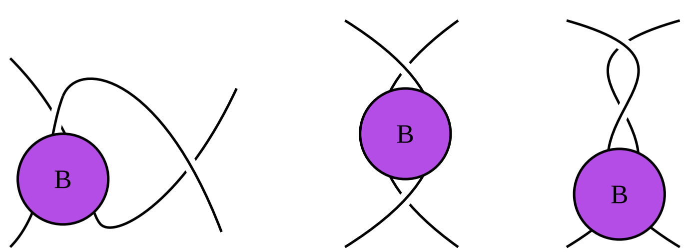
$$\ldots (0 \cdot A) \ldots \;=\; \ldots 0 \ldots \#\,\text{flip}\,(A)\,.$$

By the additivity of v under '$\#$', it suffices to consider the (possibly trivial) factors separately. But both factors have a shorter notation and are hence dealt with by induction premise.

In case 2 we just have $A \cdot 0 = \text{flip}\,(A)$, which is also simplifying.

So in both cases there are $\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$ $(n \geq 3)$ recursive relations.

Case 3. Two of a, b, c are of length 1 and both are '$-$' (or flip $(-)$). Then by inserting appropriately 1 and -1 into these '$-$'es, you obtain modulo flips one of the following tangles



where B is some flip of the remaining tangle of len > 1 in the notation (7.1). But, after performing a flype, in all 3 cases you can simplify the tangles to ones, having as a notation with flip this of B and only *one* additional number (with some 'flip's performed on subexpressions). But this notation is again simpler, so it produces a relation, and you have

$$\geq \frac{\sqrt{n-2}}{2} + 1 \geq \frac{\sqrt{n}}{2}$$

simplifying relations.

Now the case distinction and the proof of the lemma are complete (note, that in our inductive procedure we never involved an entry into two recursive relations). □

The main point now is to prove the following improved

Lemma 7.8

$$\#\{\ \textit{simplifying relations}\ \} \geq C \cdot \textit{len}(\ \textit{notation}\ ), \quad C > 0\,.$$

Then by the argument at the end of subsection 7.1, any Vassiliev invariant v of degree n on arborescent knots is uniquely determined by its values on braiding sequences of length $\leq {}^1\!/_C \cdot n$, and the number of such braiding sequences is exponentially bounded in the degree. Finally, the number of knots in each braiding sequence which suffice to determine v on it is exponentially bounded in n as well, and theorem 7.6 is proved. □

Proof of lemma. Let us start with a definition.

Definition 7.9 *A critical entry in a notation with flip C is a '$-$', which appears as one of a, b, c in one subexpression of C of the form*

$$(ab)c \quad \text{flip}\,(ab)c \quad a(bc) \quad a\text{flip}\,(bc) \tag{7.2}$$

such that the other 2 expressions have length > 1 (where length is the number of signs '$+$' and '$-$').

We proved in lemma 8.1 of [St4], that every pair of signs of non-critical entries gives rise to a relation, so that each variable appears maximally in one such relation.

Therefore, to prove the lemma 7.8, it suffices to see that

$$\#\{\ \text{critical entries}\ \} \leq C \cdot \text{len}(\ \text{notation}\ ), \quad C < 1\,.$$

This, however, follows from the observation, that in any notation with flip, we have

$$\#\{\ \text{critical entries}\ \} \leq \#\{\ \text{non-critical entries}\ \} - 1\,,$$

which can be proved straightforwardly by induction over the construction of the notation, and this completes the proof of lemma 7.8. □

7.3 Bounds for braid representations

In the following we will use closed braid representations of knots combined with the braiding sequences approach to Vassiliev invariants to give upper bounds for the number of Vassiliev invariants on classes of knots with special braid representations. Although it is my feeling (because of the special shape of diagrams it restricts us to) that this is not the best attack for solving the question of finite-determination of Vassiliev invariants on all knots, they give rise to a considerable generalization of the results on closed 3 braids in [St4].

Theorem 7.10 *For all $k \in \mathbf{N}$ the space of Vassiliev invariants of degree $\leq n$ on knots with braid index $\leq k$ is finite-dimensional and exponentially bounded in n.*

Although a little restrictive, I hope this statement provides a flavour of the capabilities of the new approach and we hope it proves to be useful in a later attempt to settle the question of finite-determination of Vassiliev invariants on all knots using the closed braid approach.

For its proof we need to introduce a specific, but very appealing notation for braid words.

Definition 7.11 *A braid scheme is a checkerboard diagram with integers put on the black fields, e. g.*

$$\begin{array}{ccccccc}
 & & -5 & & & & 6 \\
 & -1 & & -2 & & 3 & \\
1 & & 3 & & 2 & & 4 \\
\\
\sigma_1 & & \ldots\ldots & & \ldots & & \sigma_7
\end{array}\ .$$

If integers are omitted, they are assumed to be 0*. The braid word corresponding to the diagram is the concatenation of powers of interchangingly with the row of the scheme (rows are numbered from bottom to top) odd and even index (Artin) generators, the powers given by the entries in the scheme. E. g., the braid word corresponding to the above scheme is*

$$(\sigma_1\sigma_3^3\sigma_5^2\sigma_7^4)\,(\sigma_2^{-1}\sigma_4^{-2}\sigma_6^3)\,(\sigma_3^{-5}\sigma_7^6)\,.$$

Definition 7.12 *A reducing move in a braid scheme is the "pulling down" of an entry (in row* ≥ 3*) two rows below, if one row below its 2 neighbors (or its 1 neighbor, if it is the first or last generator), are equal to* 0*,*

$$\begin{array}{ccc} & x & \\ 0 & & 0 \\ & y & \end{array} \longrightarrow \begin{array}{ccc} & 0 & \\ 0 & & 0 \\ & x+y & \end{array},$$

adding it to the below entry (if it is not 0*).*

Example 7.1

$$\begin{array}{ccccccc} & 3 & & -2 & & & \\ 0 & & 0 & & -4 & & \\ & 1 & & 0 & & & \\ 1 & & 2 & & 3 & & \end{array} \longrightarrow \begin{array}{ccccc} & 4 & & -2 & \\ 1 & & 2 & & -1 \end{array}.$$

Definition 7.13 *The weight of a scheme is the sum of all generator indices in the corresponding braid word (i. e.,* $(1+3+5+7)+(2+4+6)+(3+7)=38$ *in the example of definition 7.11).*

Definition 7.14 *A scheme is called reduced if it does not admit a reducing move. Any scheme can be reduced by finitely many reducing moves, not augmenting its weight.*

Proof of theorem 7.10. The idea is always the same as used for all the previous theorems: to any braid scheme there corresponds a braiding sequence according to the parity of its entries. Then prove, that inserting special values into this scheme at sufficiently many different positions gives braids, admitting a scheme representation of smaller weight.

So the key point is as always before the following

Lemma 7.15

$$\#\{\textit{ recursive relations }\} \geq C\cdot\frac{1}{k^3},\quad \textit{weight}(\textit{ scheme }),\quad C>0\,.$$

Using this lemma, we know, that to determine a Vassiliev invariant of deg $\leq n$ on knots with braid index $\leq k$, we only need to consider braiding sequences of weight $\leq C'\cdot k^3\cdot n$ (for some constant $C'>0$). As such schemes have $\leq C'\cdot k^3\cdot n$ entries and each entry of the braiding sequence (or rather the corresponding sequence of braid words) is given by the index of the generator and the parity of the exponent, we have $\leq \big(2(k-1)\big)^{C'\cdot k^3\cdot n}$ braiding sequences to consider. The number of monomials of the braiding polynomials of all such braiding sequences is at most $\binom{C'\cdot k^3\cdot n+n}{n}$, which is exponentially bounded in n, and so the theorem follows. $\square$

Proof of lemma. We will show, that we have at least in every $2k$-th row of each reduced scheme of a braiding sequence (that is, a braid scheme with $\pm$ signs instead of numbers) a situation like this

$$\begin{array}{ccc} & x & \\ x & & 0 \\ & x & \end{array},\qquad \text{where all } x\in\{+,-\},$$

(and the x's are not necessarily equal).

Such setting always gives rise to a recursive relation: if one of the x's is '$+$', then set 0 and the entry disappears. If all x's are '$-$', set into the upper two entries 1 and apply a Reidemeister III move (or Yang-Baxter relation) to slide the lower generator to the top, reducing its index by 1 (and so also the weight of the scheme). Then eventually bring the resulting scheme into its reduced form, which does not augment its weight.

Consider the function, assigning to each row in the reduced scheme the maximal index of its entries (i. e., the position of the right-most entry in this row of the scheme). This function has odd values on odd rows and even values on even rows. Each k rows it has an ascent. As the scheme is reduced, this ascent is by 1 (in row ≥ 3).

Then you have a picture like this

$$\begin{array}{ccccc} & & x & & \\ & x & & 0 & \\ ① & & ② & & \end{array}$$

If ② is not empty, then you are done. If it is, then ① cannot be empty as the scheme is reduced. Therefore, you obtain the same picture one row below and with the index of the generators decreased by 1. If you repeat this argument, at least after k steps, when your index of the generator has become 1 and ① does not exist, you must be able to find the desired picture.

As the contribution (index sum) to the weight of the scheme, coming from $2k$ rows, is cubically bounded in k, the lemma follows. □

Evidently, the problem to extend the theorem to arbitrary braid index is the possibility to have large parts in the scheme, consisting of '$-$'es only, so that there are no empty entries (see [St11, remark 3.4] for an example).

Whenever we can avoid this, we are almost immediately done. Here is an example of a theorem which does not involve a bound on the braid index. It can easily be proven in the same way. The details are always the same and therefore I leave them to the reader.

Theorem 7.16 *Vassiliev invariants are finitely-determined and exponentially bounded on all knots, which are closed braids β with*

$$\beta = \prod_{i=1}^{r} \sigma_{p_i}^{r_i}\,,$$

$p_i > 0, r_i \in \mathbf{Z}$*, such that even (index) generators appear with exactly one odd power (that is, for each even $p \leq \max_{j=1}^{r} p_i$ $\exists!i$ with $p_i = p$ and r_i odd).*

Remark 7.2 Any generator must appear with at least one odd power, $\hat{\beta}$ to be a knot.

Proof sketch. Consider formal connected sums of reduced schemes (i. e., expressions like

$$<\text{ scheme 1 }>\#<\text{ scheme 2 }>\#<\text{ scheme 3 }>,$$

where "scheme i" stands for some scheme) with the weight defined by the number of all entries in all schemes.

It easily follows from the premise and the reducedness of the scheme that

$$\frac{\#\{\text{ odd entries }\} - \#\{\text{ generators }\}}{2} \leq \#\{\text{ even entries }\}\,,$$

and isolated (odd) occurences of a generator in a scheme can be made into isolated crossings in the braid diagram by setting ± 1, and then turning the entry into a '#', which is also simplifying.

This shows, that

$$\#\{\text{ recursive relations }\} \geq {}^1\!/\!_9 \cdot \text{ weight(scheme sum)}\,.$$

So for degree $\leq n$ Vassiliev invariants you only need to consider schemes with $k \leq 9n$ entries. Now for each such scheme/braiding sequence, the braiding polynomial of deg $\leq n$ is uniquely determined by its values on tuples $(x_1, \dots, x_k)$ with $0 \leq x_i$, $\sum_{i=1}^{k} x_i \leq n$ (see remark 7.3 below), and all braids corresponding to such $(x_i)_{i=1}^{k}$ have $\leq 11n$ crossings. To finish with, use the result of D. Welsh [We], as quoted in [Ad, p. 49], that the number of knots with n crossings is exponentially bounded (above) in n. □

Remark 7.3 The fact, that a polynomial P of deg $\leq n$ in k variables $x_1, \dots, x_k$ is uniquely determined by its values on $(x_1, \dots, x_k)$ with $0 \leq x_i$, $\sum_{i=1}^{k} x_i \leq n$, can be observed from the multivariable Newton formula (see [Tr] and *loc. cit.*) by remarking, that $\Delta_{x_1}^{a_1} \dots \Delta_{x_k}^{a_k} P(0)$ can be computed out of $P(x_1, \dots, x_k)$ with $0 \leq x_i \leq a_i$, and only such difference sequences appear in the formula, where $\sum_{i=1}^{k} a_i \leq n$. (Here $\Delta_{x_i}^{a_i}$ means taking the a_i-th difference sequence of P with respect to x_i.)

Remark 7.4 As quoted in [Ad, p. 49], D. Welsh proved the exponential upper bound only for prime knots. From this, however, it follows for composite knots as well. The key point in this argument is to notice, that it suffices to see it for alternating composite knots. This is, because the number of crossing changes is exponentially bounded, and it follows from the Tait flyping conjecture [MT] and the fact, that crossing changes commute with flypes, that you can achieve a minimal diagram of any knot by crossing changes in alternating diagrams, such that for each alternating knot you just take *one* alternating diagram. The exponential upper bound for alternating knots follows from the fact for alternating prime knots, which is proved by Welsh, using the additivity of the crossing number for alternating knots under connected sum (which follows from results of Menasco [Me] on the one hand and Kauffman [Ka3], Murasugi [Mu] and Thistlethwaite [Th] on the other hand, see [Ad, §6.2] for details), and some standard combinatorial arguments (e. g., that the partition function [An, p. 70] is subexponentially growing; see e. g. [St10] for details).

Remark 7.5 Theorem 7.10 can also be proved via the Fundamental theorem for braids [Hu]. However, the way via (proving first) the Fundamental theorem is of course more tedious, and theorem 7.16 shows that our arguments appear more universally applicable even for braids.

Question. Which knots have braid representations as in theorem 7.16? Maybe all knots?

7.4 The growth of the number of knots and Vassiliev invariants

Finally, we will try to outline a possible strategy to prove the finite-determination of Vassiliev invariants on all knots, relating their growth to the growth of the number of knots.

Here is a project, which will lead to a very simple proof of the finite-determination of Vassiliev invariants on all knots using braiding sequences.

Conjecture 7.17 *There exists a function $f:\mathbf{N}\to\mathbf{R}$ with*

1) $\lim_{n\to\infty} f(n)=\infty$, $f(n+1)>f(n)$.
2) *In any minimal diagram of a knot of n crossings, there exists a Gousarov scheme [G, Ng, NS] (a family of disjoint sets of crossings in the diagram) N with $\#N\geq f(n)$, such that $\forall S\in N$, after changing of a set of crossings $S'\subset S$ in the diagram, the resulting diagram can be (crossing) reduced by Reidemeister moves, involving crossings of S only.*

Theorem 7.18 *Assume conjecture 7.17 is true. Then Vassiliev invariants are finite-determined on all knots. More exactly, any Vassiliev invariant of degree $\leq n$ is uniquely determined by its values on alternating knots with $\leq f^{-1}(n)+2n$ crossings. In particular, if in conjecture 7.17 we can choose $f(n)\geq C\cdot n$ for some constant $C>0$, then Vassiliev invariants are exponentially bounded in the degree.*

Proof. Consider the braiding sequences, associated to the minimal knot diagrams ordered by the number of crossings. Then choose for a fixed diagram (and its associated braiding sequence) the scheme N, and inserting special values into the variables corresponding to crossings in any set in $S\in N$, you obtain a recursive relation (we need the additional condition to the simplification by Reidemeister moves to ensure that the recursive relation holds for *all* values of the remaining variables!). The rest of the argument is as usual.

Now note, that we only need to consider braiding sequences of diagrams with $\leq f^{-1}(n)$ crossings. Combining the observations in the proof of theorem 7.16 with the one that appropriate sign choice of the arguments of the braiding polynomial corresponds to alternating diagrams only, we deduce the second assertion. For the third assertion, use again the above cited result of D. Welsh. □

One motivatation for this conjecture is to note, that all our previous proofs were basically proofs of special cases of this conjecture, with the difference that we distinguished for any considered knot a set of diagrams and proved that simplification of a distinguished diagram gives again a distinguished diagram, such that the simplification commutes with the other braidings (i. e., the insertions of the braids σ_1^{2k+1} into the distinguished rooms of the braiding sequence, see figure 1). This way we can relax our conjecture.