Finally, as always by BURNSIDE's lemma,

$$\omega_D^{\mathrm{tc}} = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\mathrm{tc}}_{2D/c,c} \,.$$

4.3 Some computations

With the previous formulas one can compute the beginning of the various integer sequences above[15]. Here are the first 10 values of each one.

D	1	2	3	4	5	6	7	8	9	10
ω_D^{tc}	1	1	1	2	7	25	108	492	2,431	12,371
ω_D^{nc}	0	1	3	12	74	647	6,961	89,739	1,337,152	22,609,111
$\bar{\omega}_D^{\mathrm{nc}}$	1	1	2	6	31	255	2,788	37,333	578,799	10,134,071

Remark 4.3 By similar arguments as in §3 one can show that

$$\frac{\bar{\omega}_D^{\mathrm{nc}}}{\sigma_D} \xrightarrow[D\to\infty]{} \frac{1}{e} \,.$$

This was first proven for the linear case by STEIN and EVERETT [SE], see also [S, p. 362].

Remark 4.4 The sequence of tree–connected chord diagrams was previously known. According to SLOANE [Sl] it appeared in an enumeration of planar alcohol molecules [LMi] (which correspond to tree–connected chord diagrams by putting on each chord crossing a 'C' and on each chord basepoint an 'H' atom and forgetting the solid line). Leroux and Miloudi used some more general and more elegant PÓLYA theory arguments. Our approach is however geometrically more understandable.

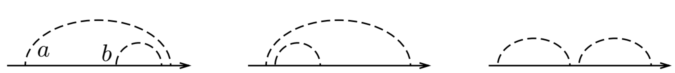
5 Enumeration of chord diagrams and an upper bound for Vassiliev invariants

In this section, we treat an enumeration problem of chord diagrams, which is shown to yield an upper bound for the dimension of the space of Vassiliev invariants for knots. We give an asymptotic estimate for this bound. More precisely, we prove the upper bound $D!$/(any given polynomial in D) for both $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ (where $\mathcal{A}$ and $\mathcal{A}^r$ are the graded algebras of chord diagrams introduced in §1) and later improve it to $D!/1.1^D$. As an aside, we present a trivial proof for the bound $D!$.

In the next section we will discuss further results on the dimension of $\mathcal{A}$.

5.1 Factoring out 4T relations

There are three types of mutual positions of non-intersecting chords in the LCD's with neighboring basepoints.



type I type II type III

It is clear that one can resolve one *fixed* type I and II intervals using the 4T-relation. But this also works with *all* such intervals. This means, we have

[15] A MATHEMATICA™ package with all formulas is available on my WWW page.

Theorem 5.1 *$\mathcal{A}$ is generated by LCD's without type I and II intervals.*

We will call such LCD's *regular*.

Proof. We have to show that we can prform the removing of type I and II intervaals using the 4T relation so that this does not lead to an infinite loop. For this it suffices to define a partial ordering relation on LCD's such that it is always possible to transform an LCD modulo 4T into a linear combination of smaller LCD's.

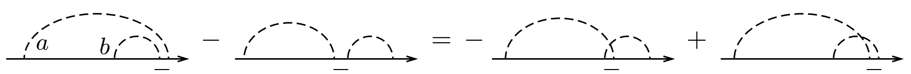
We will use the following relation.

Definition 5.2 *The* length *of a chord in an LCD is the number of enclosed intervals.*

Definition 5.3 *Define a half-ordering relation on LCD's by $a \prec b$ exactly if $(\chi_1(a), -\chi_2(a))$ is lexicographically less than $(\chi_1(b), -\chi_2(b))$, where*

$$
\begin{aligned}
\chi_1(L) &:= \sum_{i=1}^{D} \textit{len(chord } i \textit{)} \\
\chi_2(L) &:= \#\ \{ \textit{ chord intersections in } L \ \} \ .
\end{aligned}
$$

Look at the 4 LCD's of the 4T relation ($4T(a, b)$).



$(4T(a, b))$

where term 1 has a type I interval. (For the type II interval apply an analogous argument). We have

$$
\begin{aligned}
&\chi_1(\text{term } 4) \ < \ \chi_1(\text{term } 1)\,, \\
&\chi_1(\text{term } 2) \ = \ \chi_1(\text{term } 1) \quad \text{ and } \quad \chi_2(\text{term } 2) = \chi_2(\text{term } 1) + 1
\end{aligned}
$$

and

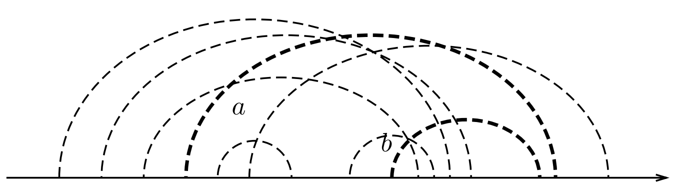
$$\chi_1(\text{term } 3) \ \leq \ \chi_1(\text{term } 1)\,,$$

with inequality unless all basepoints inside chord b are basepoints of left-outgoing arcs with respect to b (i. e., their left basepoint is left from b and their right basepoint is within b).

I. e., each time we can find a and b of type I so that not all basepoints of b belong to left-outgoing arcs (with respect to b) we have a *simplifying* 4T relation. We can also exclude the case len$(b) = 1$, as then term 2 & 3 cancel.

Assume we have a 4T relation to apply but no simplifying one (i.e., each type I looks as in figure 2).

Figure 2. A non-simplifying 4T relation I



Now look at the left-most type I interval. All arcs having an endpoint within b are left-outgoing with respect to b and all mutually intersect. I. e., we have something like figure 3, which we will call A.

Apply in A a 4T relation at a_1, a_2. Term 2 and 4 are simpler as above. Look at term 3, which we will call A_1.

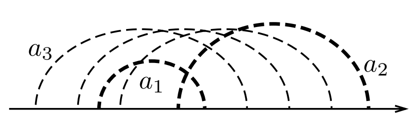
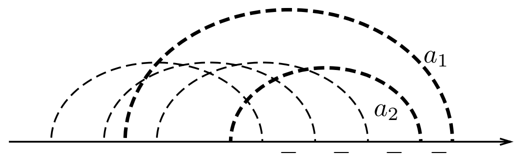
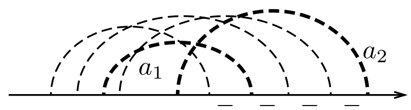


Figure 3. A non-simplifying 4T relation II



We have $\chi_1(A_1) = \chi_1(A)$. Now the chord a_1 in A_1 has a right-outgoing arc, i. e. 4T(a_1, a_3) is simplifying, and term 3 and 4 have a χ_1 less than $\chi_1(A_1) = \chi_1(A)$ (we will just see that in case there is no a_3 the step we perform is unnecessary). There remains term 2, which we will call A_2.



Using the same argument, repeat sliding the right basepoint of a_1 again to the right (a_2 remains always right-outgoing for a_1) until a_1 intersects all chords intersecting a_2. The resulting LCD has χ_1 equal and χ_2 higher than A, since now a_1 intersects all chords it was intersecting in A, and additionally at least a_2. (Here we would have immediately arrived if we had no a_3). Now we are done. □

From the proof one can see, that one can use only $-\chi_1$ to prove that $\mathcal{A}$ is generated modulo 4T by LCD's without type III.

Now comes the hard combinatorial part – the enumeration of regular linearized chord diagrams.

5.2 Regular linearized chord diagrams

We will number in an LCD the chords by the order (from left to right) of their left basepoints.

Let's start with a simple observation: The number of LCD's of degree D without one of the types I, II or III is $D!$.

To see this, for type I and III place the chords in their reverse order and show that there are exactly $D + 1 - k$ ways to place chord k. For type II apply mirroring of the LCD.

This way we obtain with the observation, that LCD's without type III are generating, a simple proof of the bound $D!$.

Let

$$\Xi_D \quad := \quad \{ \text{ regular LCD's of deg } D \, \} \qquad \xi_D \quad := \quad \#\Xi_D$$

We would like to calculate the numbers ξ_D of regular LCD's of degree D and compare them with Ng's bound $(D-2)!/2$ [Ng].

Unfortunately the enumeration of regular LCD's is much harder. Let us start.

We will consider a certain map

$$m_D \, : \, \Xi_D \, \longrightarrow \, \Xi_{D-1} \, .$$

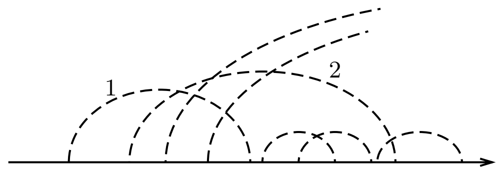
The image of a regular LCD L with D chords under m_D is obtained as follows:

1. Remove the first chord (i. e., the one, whose left basepoint is the leftmost one).

2. The resulting LCD might not be regular. But if not it has exactly one type II interval. Note, that arcs with left basepoint between the left basepoint of chord 2 and the right basepoint of chord 1 are right-outgoing with respect to chord 2, so removing chord 1 we do not get a new type I interval.

3. Now mark all the chords whose left basepoints are between the left basepoint of chord 1 and the first right

basepoint of a chord on the right of the right basepoint of chord 1. E. g., in an LCD beginning like this



the marked chords will be 2 – 6.

4. Separate the marked chords into a left and a right part depending on whether their left basepoint was left or right from the right basepoint of chord 1, e. g. in the above picture the left part would consist of chords 2 – 4, and the right one – of chords 5 and 6. Within both parts we have the property, that each two of the chords intersect, i. e. their right basepoints lie in the same order as the left ones.

5. Now apply a shuffle permutation[16] on the left basepoints of both parts, so that *all* marked chords mutually intersect. In our example this will be the $(3,2)$-shuffle 34512. The resulting LCD of degree $D-1$ is regular. This is by definition $m_D(L)$.

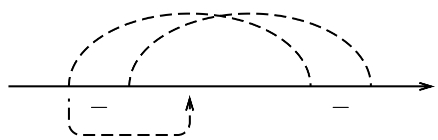
Consider

$$\Xi_{D,k} \quad := \quad \big\{ \text{ regular LCD's of deg } D \text{ with len(chord 1)} = k \,\big\} \qquad \xi_{D,k} \quad := \quad \#\Xi_{D,k}$$

We can construct all preimages under m_{D+1} of $L \in \Xi_{D,k}$ by *re*shuffling the left basepoints of the leftmost k chords with an (α,β)-shuffle (for some $\alpha+\beta=k$) and putting a new $(D+1)^{\text{st}}$ chord, which becomes in the resulting LCD chord number 1, of length $\alpha+1$.

Now comes the unpleasant detail. Not for all LCD's in $\Xi_{D,k}$ and all (α,β)-shuffles this procedure gives a regular LCD. The problem is that the reshuffling can produce type I intervals, if both the left and right basepoints of two marked arcs were neighbored in L. In this case it will be forbidden to permute the left basepoint of the lefter arc to the right of the left basepoint of its right neighbor (see figure 4).

Figure 4. A forbidden move.



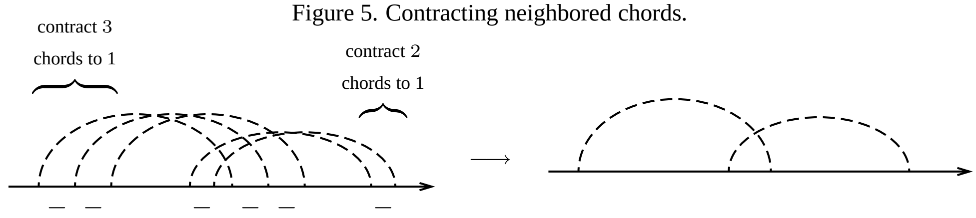
Define between the first k arcs in L a (de)composition (a representation as sum of ordered numbers) by setting chords l and $l+1$ to belong to the same part iff their right basepoints are also neighbored, i. e. if no right basepoints are neighbored we get the decomposition of k into k parts of length 1, whereas in the case that all right basepoints are neighbored we get the decomposition of k into 1 single part of length k. We will call this decomposition the *induced (de)composition* of L.

Definition 5.4 *A composition P' is* rougher *than P if it can be obtained from P by a sequence of additions of neighbored parts of P. Conversely, P' is* finer *than P. E. g., the composition $(1,2,1,1,1,3,2,1)$ of 12 is finer than $(1,2+1,1,1+3+2,1)=(1,3,1,6,1)$. The length len$(P)$ of a composition P is the number of numbers appeing in it.*

For a fixed decomposition P of k the number $\hat{\xi}_{D,k,l}$ of diagrams in $\Xi_{D,k}$ whose induced decomposition is exactly P, depends only on $l=\text{len}(P)$ and can be computed using the combinatorial inclusion-exclusion principle and the simple idea, that each diagram T in $\Xi_{D,k}$ with induced decomposition rougher than one fixed decomposition P' of k of length l bijectively corresponds to a diagram in $\Xi_{D+l-k,l}$ by contracting chords in T belonging to the same component of the decomposition to one (see figure 5).

[16] For the definition and algebraic properties of shuffles see e. g. [St3, Lo, Re, BN].

Figure 5. Contracting neighbored chords.



The result is

$$\hat{\xi}_{D,k,l} = 0 \text{ for } l \le 0 \text{ or } l > k$$
$$\hat{\xi}_{D,k,l} = \sum_{i=0}^{l-1} \binom{l-1}{i} (-1)^i \xi_{D+l-k-i,l-i}\,,$$

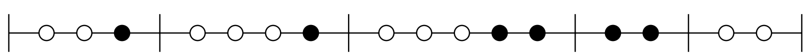
where D is the degree, k is the length of chord 1 and l is the length of the induced decomposition.

Now we determine the number of applicable shuffles for a fixed $L \in \Xi_{D,k}$. Note that to have a preimage of L in $\Xi_{D+1,k'}$ we must have $k \ge k'-1$ (since the new to be installed chord 1 must intersect the previous chord 1). In each such case we apply the inverse of a $(k'-1, k-k'+1)$-shuffle on the first k basepoints of L. A $(k'-1, k-k'+1)$-shuffle corresponds to a coloring of k chords by $k-k'+1$ black (to be reshuffled to the right) and $k'-1$ white (to be reshuffled to the left) chords. The restriction is that within each component of the induced decomposition there must be no black chord followed to the right by a white one. Since the number of LCD's with induced composition P depends only on $p = \text{len}(P)$, let us sum over all compositions of length p and compute the number

$$\eta_{l,k,p} := \# \left\{ \begin{array}{l} \text{choices of } k \text{ out of a decomposition } P \text{ of } l \\ \text{elements of length } p\text{, such that in each com-} \\ \text{ponent of } P \text{ all chosen elements are to the} \\ \text{left} \end{array} \right\}.$$

Note that in $\eta_{l,k,p}$ a fixed choice is counted sometimes multiple times for different decompositions!

Here is an example (where chords are replaced for simplicity by beads):



$$(l = 16,\ P = 3+4+5+2+2,\ p = 5,\ k = 10)$$

We see that such a picture can be described uniquely by the following procedure:

1. Choose l' components out of p, where we will have at least one white mark.
2. Build a composition of k of length l' with the white marks and a composition with zeros of the same length (of some number d) with the black marks appearing in the l' chosen components.
3. Join the compositions of the white and black marks, i. e. add parts at the same position.
4. Complete the composition of l by a composition of $l-k-d$ of $l-l'$ parts.

In the above example

- The composition of $k = 10$ is $2+3+3+2$,
- the choice of $l' = 4$ components out of $p = 5$ is $\{1, 2, 3, 5\}$,
- the composition with zeros of $d = 4$ is $1+1+2+0$,
- the composition of $l-k-d = 16-10-4 = 2$ of length $l-l' = 5-4 = 1$ parts is 2.

The number $P_{k,l}$ of compositions of k elements into l parts is

$$P_{k,l} = \binom{k-1}{l-1}.$$

It makes sense to set

$$P_{0,0} := 1 \quad \text{and} \quad P_{0,\cdot} := 0 \ \text{else}\,.$$

The number of compositions with zeros of k elements into l parts is $P_{k+l,l}$.

So we get

$$\eta_{l,k,p} = \sum_{l'=0}^{k} \sum_{d=0}^{l-k} P_{k,l'} \cdot P_{d+l',l'} \cdot P_{l-k-d,p-l'} \binom{p}{l'}\,.$$

Remark 5.1 Using some combinatorial formulas the expression for the $\eta_{l,k,p}$'s can be a little bit simplified:

$$\eta_{l,k,p} \; = \; \begin{cases} P_{l,p} & k=0 \\ \sum_{l'=0}^{p} \binom{k-1}{l'-1}\binom{l-k+l'-1}{p-1}\binom{p}{l'} & k>0 \end{cases}\,.$$

With some effort one can find an alternative characterization of these numbers (which is unimprtant in our context).

$$\eta_{l,k,p} = \left[\left(\sum_{n=1}^{l} \frac{1-x^{n+1}}{1-x}\, y^n\right)^p\right]_{x^k\cdot y^l}\,,$$

where $[\text{polynomial}]_{\text{monomial}}$ denotes the coefficient of 'monomial' in 'polynomial', and we obtain

$$\sum_{l,k,p=0}^{\infty} \eta_{l,k,p}\, x^k\, y^l\, z^p \; = \; f(x,y,z) \; := \; \frac{1-x}{1-x-z\left(\dfrac{y}{1-y} \; - \; \dfrac{x^2 y}{1-xy}\right)}\,,$$

i. e., the numbers $\eta_{l,k,p}$ appear as the TAYLOR coefficients of $f(x,y,z)$ at $(0,0,0)$.

With the previous preparations we are ready to write down the recursive formula for $\xi_{D,k}$

$$\begin{aligned} \xi_{1,1} &= 1\,, \\ \xi_{D,k} &= \sum_{l=k-1}^{D-1} \sum_{p=1}^{l} \hat{\xi}_{D-1,l,p}\eta_{l,k-1,p}\,. \end{aligned} \tag{5.1}$$

Finally, what we seek is

$$\xi_D = \sum_{k=1}^{D} \xi_{D,k}\,.$$

5.3 Connected regular LCD's

With some more effort one can attack a further combinatorial task to give a way to compute the number of *connected* regular LCD's. By the facts recalled in section 1 this gives a bound for the primitive (additive) VI's.

The first observation we make is the following: if in a regular LCD there is one connected component enclosing another one, then the latter is attached at a type III interval to the first one.

So first we shall count regular LCD's by their number of type III intervals. For the later calculation it will turn out more useful to use the *characteristics*

$$\chi(L) \; := \; \#\,\{\,\text{type III intervals of } L\,\} \, + \, 1$$

instead. So let

$$\begin{aligned} \Xi^2_{D,n} \quad &:= \quad \left\{\,\text{regular LCD's } L \text{ of deg } D \text{ with } \chi(L)=n\,\right\} & \xi^2_{D,n} \quad &:= \quad \#\Xi^2_{D,n} \\ \Xi^2_{D,k,n} \quad &:= \quad \left\{ \begin{array}{c} \text{regular LCD's } L \text{ of deg } D \text{ with} \\ \text{len(chord 1)} = k \text{ and } \chi(L)=n \end{array} \right\} & \xi^2_{D,k,n} \quad &:= \quad \#\Xi^2_{D,k,n} \end{aligned}$$

The counting ia analogous to that of $\xi_D, \xi_{D,k}$ once one has noticed that the reattachment of chord 1 done in the construction of the various preimages in $\Xi_{D+1,k'}$ of a regular LCD $L \in \Xi_{D,k}$ under m_{D+1} increases the number of type III intervals of the LCD by 1 unless in the case $k'-1=k$.

Then the formulas for $\xi^2_{D,k,n}$ follow directly from those of $\xi_{D,k}$.

We obtain

$$\xi^2_{D,k,n} = 0 \text{ for } k \le 0 \text{ or } k > D \text{ or } n < 0$$

and else

$$\xi^2_{D,k,n} \;=\; \sum_{l=k}^{D-1} \sum_{p=1}^{l} \hat{\xi}^2_{D-1,l,p,n-1} \eta_{l,k-1,p} \;+ \xi^2_{D-1,k-1,n}\,,$$

where

$$\hat{\xi}^2_{D,k,l,n} = 0 \text{ for } l \le 0 \text{ or } l > k \text{ or } n < 0, \text{ and else}$$
$$\hat{\xi}^2_{D,k,l,n} \;=\; \sum_{i=0}^{l-1} \binom{l-1}{i} (-1)^i \xi^2_{D+l-k-i,l-i,n}$$

with $\hat{\xi}_{D,k,l,n}$ being explained similarly to $\hat{\xi}_{D,k,l}$.

In order to make the recursive computation start correctly, it makes sense to set for $D=0$

$$\xi^2_{0,0,1} := 1 \quad \text{and}$$
$$\xi^2_{0,k,n} := 0 \text{ for } (k,n) \neq (0,1)\,.$$

Finally,

$$\xi^2_{D,n} \;=\; \sum_{k=0}^{D} \xi^2_{D,k,n}\;.$$

Now consider

$$\Lambda_{D,n,l} \quad := \quad \left\{ \begin{array}{c} \text{regular LCD's } L \text{ of deg } D \text{ with } l \text{ conn. comp.} \\ \text{and } \chi(L) = n \end{array} \right\} \qquad \lambda_{D,n,l} \quad := \quad \#\Lambda_{D,n,l}$$

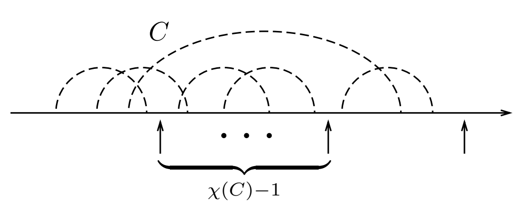
Let

$$\lambda^{\mathrm{c}}_{D,n} \;:=\; \lambda_{D,n,1}\,.$$

For the calculation of these numbers we will proceed followingly. First we have

$$\lambda_{D,n,1} \;=\; \xi^2_{D,n} - \sum_{l=2}^{D} \lambda_{D,n,l}\;.$$

For the non-connected case look at the following picture



Choose the connected component C of L containing chord 1. Then all remaining components are attached either to the right of C or at type III intervals within C, i. e. we have exactly $\chi(C)$ positions of attachment. From this we find the following recursive formula

$$\lambda_{D,n,l} \;=\; \sum_{d'=1}^{D-1} \sum_{n'=1}^{\min(d',n)} \lambda^{\mathrm{c}}_{d',n'} \cdot \Big[P_\lambda(x,y,z)^{n'} \Big]_{x^{D-d'}y^{l-1}z^{n-n'}} \quad \text{for } l > 1 \tag{5.2}$$

with

$$P_\lambda(x,y,z) := \sum_{D=0}^{\infty}\sum_{l=0}^{\infty}\sum_{n=0}^{\infty} \lambda_{D,n,l}\cdot x^D\cdot y^l\cdot z^n\ .$$

Here d' and n' are the degree and characteristics of C. We use for this formula the observation that attaching (whereever!) an LCD with $\chi = k$ the total number of type III intervals is augumented exactly by k. There is however one important exception – the attachment of the empty LCD does not augument this number. So it makes sense to set for $D = 0$

$$\lambda_{0,0,0} := 1 \qquad \text{and} \qquad \lambda_{0,n,l} := 0 \text{ for } (n,l) \neq (0,0)$$

in order for the recursive formula (5.2) to work correctly.

The number of connected regular LCD's of degree D is then

$$\lambda^{\mathrm{c}}_D = \sum_{n=0}^{D} \lambda^{\mathrm{c}}_{D,n}\ .$$

5.4 Numerical and asymptotical results

After all the considerations it is now possible to calculate ξ_D. Let us compare it with the bound $(D-2)!/2$ obtained by Ng [Ng] for $D \leq 20$ (see table 1)[17].

We see that ξ_D does not grow as fast as $(D-2)!/2$ and our bound is better for $D \geq 18$. It appears obvious that this is true in general. At the end of this section we will give a (non-straight forward) proof for this fact.

Using the bound λ^{c}_D for the primitive VI's, we get a sharper bound β_D for $\mathcal{A}$ and β^r_D for $\mathcal{A}^r$ than ξ_D by summing over partitions of the degree of products of various λ^{c}_D's and omitting symmetry in factors of equal degree[18]. But the situation with asymptotical behaviour is even more complicated. However,

Remark 5.2 It is very likely that

$$\frac{\lambda^{\mathrm{c}}_D}{\xi_D} \xrightarrow[D\to\infty]{} \frac{1}{e}\,,$$

where e is the EULER number $2.71828\ldots$ (although a proof appears very messy). This limit appears in some other enumeration problems of chord diagrams, see [St2, St9]. (So, I don't expect a big qualitative improvement of the bound in this way.)

Remark 5.3 The computation of λ^{c}_D is the hardest one ($\lambda^{\mathrm{c}}_{30}$ took ≈ 30 h total time with C++ on a Sparc10). For $D \leq 14$ the bound is still worse than Ng's, but for $D = 15$ we already have an improvement in both the framed and non-framed case.

The following table 1 compares our numbers with Ng's bound χ_D for primitive VI in [Ng, theorem 4.2] and the resulting bounds α_D and α^r_D for $\mathcal{A}$ and $\mathcal{A}^r$.

Here is a further challenge:

Problem. Is $\mathcal{A}$ generated by diagrams which do not posess any chord exclosing another one?

Note that for such diagrams the recursion formula (5.1) would change to

$$\bar\xi_{D,k} = \sum_{l\geq k-1} \bar\xi_{D-1,l}$$

(since there is only one way to (re)shuffle, i. e. the identity), and one easily shows by induction

$$\bar\xi_{D,k} \leq \binom{2D-k}{D}$$

[17] A MATHEMATICA™ package with all formulas will soon be available on my web page.

[18] This relation is extensively treated in [St10].

Table 1. Numerical values for the various bounds.

D	$(D-2)!/2$	χ_D	α^r_D	α_D	ξ_D	λ^c_D	β^r_D	β_D
1	0	1	0	1	1	1	0	1
2	1/2	1	1	2	2	1	1	2
3	1/2	1	1	3	5	2	2	4
4	1	2	3	6	15	5	6	10
5	3	4	5	11	53	16	18	28
6	12	14	18	29	217	63	72	100
7	60	54	61	90	1, 014	293	321	421
8	360	332	357	447	5, 335	1, 561	1, 680	2, 101
9	2, 520	2, 246	2, 330	2, 777	31, 240	9, 321	9, 852	11, 953
10	20, 160	18, 264	18, 715	21, 492	201, 608	61, 436	64, 168	76, 121
11	181, 440	164, 950	167, 783	189, 275	1, 422, 074	442, 134	457, 659	533, 780
12	1, 814, 400	1, 664, 354	1, 686, 327	1, 875, 602	10, 886, 503	3, 446, 077	3, 543, 785	4, 077, 565
13	19, 958, 400	18, 423, 144	18, 615, 873	20, 491, 475	89, 903, 100	28, 905, 485	29, 578, 097	33, 655, 662
14	239, 500, 800	222, 406, 776	224, 310, 263	244, 801, 738	796, 713, 190	259, 585, 900	264, 623, 046	298, 278, 708
15	3, 113, 510, 400	2, 905, 943, 328	2, 926, 679, 435	3, 171, 481, 173	7, 541, 889, 195	2, 485, 120, 780	2, 525, 897, 073	2, 824, 175, 781
16	43, 589, 145, 600	40, 865, 005, 494	41, 112, 186, 396	44, 283, 667, 569	75, 955, 177, 642	25, 267, 283, 367	25, 622, 275, 565	28, 446, 451, 346
17	653, 837, 184, 000	615, 376, 173, 184	618, 573, 033, 369	662, 856, 700, 938	810, 925, 547, 354	271, 949, 606, 805	275, 257, 714, 175	303, 704, 165, 521
18	$1.046, 139, 4 \cdot 10^{13}$	$9.880, 209, 2 \cdot 10^{12}$	$9.924, 788, 8 \cdot 10^{12}$	$1.058, 764, 5 \cdot 10^{13}$	$9.148, 832, 1 \cdot 10^{12}$	$3.089, 330, 1 \cdot 10^{12}$	$3.122, 190, 8 \cdot 10^{12}$	$3.425, 895, 0 \cdot 10^{12}$
19	$1.778, 437, 1 \cdot 10^{14}$	$1.684, 835, 1 \cdot 10^{14}$	$1.691, 501, 2 \cdot 10^{14}$	$1.797, 377, 7 \cdot 10^{14}$	$1.087, 597, 5 \cdot 10^{14}$	$3.694, 347, 7 \cdot 10^{13}$	$3.729, 011, 4 \cdot 10^{13}$	$4.071, 600, 9 \cdot 10^{13}$
20	$3.201, 186, 8 \cdot 10^{15}$	$3.041, 127, 5 \cdot 10^{15}$	$3.051, 766, 5 \cdot 10^{15}$	$3.231, 504, 3 \cdot 10^{15}$	$1.358, 836, 1 \cdot 10^{15}$	$4.639, 430, 0 \cdot 10^{14}$	$4.678, 128, 6 \cdot 10^{14}$	$5.085, 288, 7 \cdot 10^{14}$
25	$1.292, 600, 8 \cdot 10^{22}$	$1.240, 896, 8 \cdot 10^{22}$	$1.243, 475, 7 \cdot 10^{22}$	$1.300, 029, 2 \cdot 10^{22}$	$8.090, 291, 0 \cdot 10^{20}$	$2.812, 171, 4 \cdot 10^{20}$	$2.826, 415, 8 \cdot 10^{20}$	$3.021, 075, 6 \cdot 10^{20}$
30	$1.524, 441, 7 \cdot 10^{29}$	$1.473, 626, 9 \cdot 10^{29}$	$1.475, 655, 8 \cdot 10^{29}$	$1.530, 326, 3 \cdot 10^{29}$	$1.263, 341, 6 \cdot 10^{27}$	$4.439, 023, 6 \cdot 10^{26}$	$4.454, 162, 5 \cdot 10^{26}$	$4.707, 888, 4 \cdot 10^{26}$

and hence

$$\bar{\xi}_D \le \binom{2D+1}{D+1} < 2^{2D+1},$$

i. e., we would have an *exponential* bound.

Conversely, it is possible to show that ξ_D grows stronger than any exponential.

Theorem 5.5 *We have*

$$\xi_D \ge B(D),$$

where the $B(D)$'s are the BELL *numbers [Ri], satisfying the recursive formula*

$$B(0) = 1 \quad \text{and} \quad B(d) := \sum_{k=1}^{d} \binom{d-1}{k-1} B(d-k).$$

It is possible to show using the recursive formula of the Bell numbers that they grow stronger than any exponential. E. g., since

$$B(d) \ge (d-1)\,B(d-2)$$

we have

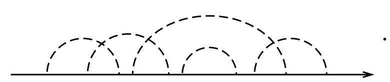
$$B(d) \ge (d-1)!!\,.$$

More generally one has asymptotically $B(d) > (d!)^{1-\epsilon}$ for each $\epsilon > 0$.

Theorem 5.5 follows from the following

Theorem 5.6 *The Bell number $B(D)$ is the number of regular LCD's L of degree D with the following property: Consider the sequence $L_0 = L, L_1, L_2, \dots$ of LCD's, where L_{i+1} is obtained from L_i by removing its first chord and all chords intersecting or enclosed by it. Then all L_i are regular.*

Not all regular LCD's have this property. The most simple example of an LCD without this property is



Proof. Denote by d the degree of an LCD with the above property. If k is the number of marked chords then the right endpoints of the last $k-1$ marked chords can be attached at arbitrary positions in the remaining regular LCD of degree $d-k$ except at intervals whose left end is a right basepoint of one of the $d-k$ non-marked chords. So there are $d-k+1$ positions of attachment (the position of the right end of chord 1 is determined – left from all non-marked chords). Since the order of the right basepoints of the marked chords is equal to the order of their left basepoints and at an interval you can put more than one new right basepoint, we have to count ordered choices with repetitions of $k-1$ out of $d-k+1$ elements. Since this number is

$$\binom{(d-k+1)+(k-1)-1}{k-1} = \binom{d-1}{k-1},$$

in this way we obtain exactly the recursion formula for the Bell numbers. □

Here is an upper bound for ξ_D which shows the improvement we have achieved.

Theorem 5.7 *ξ_D grows more slowly than $D!$/any polynomial in D.*

Proof. Fix a k with $1 \le k \le D$ and consider a regular LCD L of degree D. This means especially that L has no type I interval at all and no type II intervals among the first k intervals (i. e., between the $k+1$ leftmost basepoints). Make the following case distinction for L.

Case 1. Among the first $k+1$ chords there exist two consecutive chords with neighbored right basepoints. The number of such LCD's L is bounded by the contraction described in section 5.2 (and explained in figure 5) by[19] $k\xi_{D-1}$.

[19] Here is the only time we use the non-existence of *any* type II intervals in L; else we will use their non-existence only among the first k intervals.

Case 2. There exist two chords which have both their left and right basepoints among the first $k+1$. Then chord 1 is one of these chords. Since, applying m_D, we shuffle maximally k basepoints and the number of shuffles with k elements is 2^k, we obtain each LCD in Ξ_{D-1} maximally 2^k times, i. e., the number of such LCD's L is bounded by $2^k\xi_{D-1}$.

Case 3. There exists no chord with both left and right basepoint among the first $k+1$ (i. e., all are left and all chords mutually intersect). In this case the symmetric group S_{k+1} acts freely on the first $k+1$ basepoints and all LCD's obtained have (by exclusion of case 1) no type I intervals. Since the number of LCD's with no type I is $D!$, the number of these LCD's L is bounded by $D!/(k+1)!$.

Case 4. There exists exactly one chord with right basepoint among the first $k+1$. Then this chord has to be chord 1. The right basepoint of chord 1 separates the left basepoint of the $k-1$ remaining chords starting at the first $k+1$ basepoints into two parts, where within each part all chords mutually intersect. Now apply the argument in case 3 for $D-1$ instead of D and each of the two parts. For len(chord 1) $= l$ you get the bound

$$\frac{(D-1)!}{(l-1)!\,(k-l)!}\,.$$

Summing over l you get

$$\frac{(D-1)!}{(k-1)!}\,2^{k-1}$$

as a bound for this case.

Putting it all together, we find that

$$\xi_D \,\le\, (k+2^k)\,\xi_{D-1} \,+\, \frac{D!}{k!}\,\left(2^{k-1}+1\right) \tag{5.3}$$

for each $1\le k\le D$.

Fix some polynomial $P(D)$ and assume that we find arbitrary larde D with

$$\xi_D \,\ge\, \frac{D!}{P(D)} \tag{5.4}$$

Choose k_D minimal with $k_D! > 2(2^{k_D-1}+1)P(D)$. This k_D grows very slowly (more slowly than $\log_C D$ for any $C>1$). Then

$$\xi_D \,\le\, (k_D+2^{k_D})\,\xi_{D-1} \,+\, \frac{1}{2}\xi_D$$

and

$$\xi_D \,\le\, 2(k_D+2^{k_D})\,\xi_{D-1}\,. \tag{5.5}$$

Note, that the expression $k_D+2^{k_D}$ grows less fast than any positive power of D. Furthermore, we do *not* use (5.4) for $D-1$ instead of D. This will be crucial later.

To finish the proof we consider two cases.

Case 1. For almost all D we have (5.4). By (5.5) you get for sufficiently large D, say,

$$\frac{\xi_D}{\xi_{D-1}} \,<\, \frac{D}{2}\,,$$

which gives an asymptotical contradiction.

Case 2. There exists a sequence $\{D_i\}$ such that

$$\xi_{D_i} \,\le\, \frac{D_i!}{P(D_i)}\,.$$

W.l.o.g. you can choose $\{D_i\}$ so that additionally

$$\xi_{D_i+1} \,\ge\, \frac{(D_i+1)!}{P(D_i+1)}$$

(if this is not possible, you are done). Then by (5.5) you get once again for sufficiently large i

$$\xi_{D_i+1} \,\le\, 2\left(k_{D_i+1}+2^{k_{D_i+1}}\right)\,\xi_{D_i}\,.$$

But by the choice of $\{D_i\}$ you get on the other hand for each $\epsilon > 0$

$$\frac{\xi_{D_i+1}}{\xi_{D_i}} \;>\; (D_i+1)(1-\epsilon)$$

for sufficiently large i and in this way a contradiction. □

Remark 5.4 The maximal result we can achieve with our proof is a little better. Fix some $\epsilon' > 0$. Then

$$\xi_D \;\le\; \delta_D \;:=\; 2\,\frac{D!}{k_D!}\,\left(2^{k_D-1}+1\right)$$

with

$$k_D = \lfloor \log_2 D - \epsilon - 1 \rfloor$$

The proof works as above using

$$\frac{\delta_{D+1}}{(D+1)\,\delta_D} \;\xrightarrow[D\to\infty]{}\; 1$$

and

$$2\left(k_D + 2^{k_D}\right) \;<\; D(1-\epsilon')$$

for some $\epsilon' > 0$ and sufficiently high D.

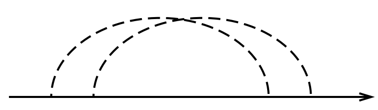
5.5 A further improvement

It appears that our asymptotical result is very weak. The numerical computation suggests as an estimate something like $D!/1.5^D$. I did not manage to obtain this, but achieved a qualitatively similar result.

Theorem 5.8 *ξ_D is bounded by $D!$ over an exponential in D.*

The basic idea for our further improvement is to consider *reduced* regular LCD's.

Definition 5.9 *An LCD is called* reduced *if there is no pair of contractable chords (that is, intersecting chords with both left and right basepoints neighbored).*



Recall the STIRLING formula

$$n! \;\asymp\; \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \qquad n\to\infty\,,$$

basically saying that asymptotically $n! \asymp \left(\frac{n}{e}\right)^n$ modulo a polynomial in n.

Definition 5.10 *Define the* path *of an LCD L of degree D as the sequence of pairs $\{(i,l_i)\}_{i=1}^{D}$, where l_i is the length of chord 1 in L_i, and L_i is obtained from L by removing chords $1,\dots,D-i$.*

If L is regular we have by the previous discussion $l_{i+1} \le l_i + 1$. Call

$$\alpha(L) := \#\{\, i \,:\, l_{i+1} = l_i + 1 \,\}$$

the *ascend* of L.

Recall the map m_D, which mapped regular LCD's of degree D to those of degree $D-1$ by removing chord 1 and reshuffling the leftmost left basepoints. We carefully studied the number of applicable shuffles for each image under m_D. If we simply take *all* possible shuffles, we obtain the inequality

$$\xi_{D,k} \;\le\; \sum_{l\ge k-1}\binom{l}{k-1}\,\xi_{D-1,l}\,. \tag{5.6}$$

Applied this way, this does not give much because it is an expression of forgetting the type II condition, and therefore we get $D!$ as bound. In fact, the numbers satisfying the equality in (5.6) are the STIRLING numbers (see [Ri]).

So we will use this formula with a certain constraint.

Proof of theorem 5.8. Fix some $0 \le \delta \le 1$ and distinguish two cases.

Case 1. Consider first LCD's L with $\alpha(L) < \delta D$. Let $k = \text{len}(\text{ chord } 1\ )$ in L. Call $\xi^{\delta}_{D,k}$ the number of such LCD's. Expanding (5.6) recursively we obtain

$$\xi^{\delta}_{D,k} \le \sum_{\substack{\text{paths } \{(i,k_i)\}_{i=1}^{D} \text{ of} \\ L \text{ with } \alpha(L) < \delta D}} \prod_{j=2}^{D} \binom{k_{j-1}}{k_j - 1}.$$

We always have $k_1 = 1$ and $k_D = k$.

Expanding the binomial coefficients we obtain for the above product

$$\prod_{j=2}^{D} \frac{(k_{j-1})!\, k_j}{k_j!\,(k_{j-1} - k_j + 1)!} = \frac{1}{k!} \prod_{j=2}^{D} \frac{k_j}{(k_{j-1} - k_j + 1)!}. \tag{5.7}$$

Set $k'_j := k_{j-1} - k_j + 1$ for $j = 2, \dots, n$. We have $k'_j \ge 0$ with $\#\left\{\, j \,:\, k'_j = 0 \,\right\} = \alpha(L) < \delta D$ and

$$k_j = j - \sum_{l=2}^{j} k'_l \ge 0 \qquad \forall j = 2, \dots, D \tag{5.8}$$

So the term in (5.7) is

$$\prod_{j=2}^{D} \frac{j - \sum_{l=2}^{j} k'_l}{k'_j!}. \tag{5.9}$$

Since we have that more than $(1-\delta)D$ of the k'_j are at least equal to 1, you see that no numerators in (5.9) can be greater than δD, while the first δD ones are less than or equal to their indexing number. So as an estimate of the product of numerators in (5.9) we obtain[20]

$$(\delta D)^{(1-\delta)D}\,(\delta D)!$$

which is by Stirling formula less than

$$(\delta D)^{D}\, e^{-\delta D}\, P(D)$$

for a certain polynomial P in D.

As an estimate for the r.h.s. of (5.9) we obtain

$$\frac{D!}{k!} \sum_{(k'_i)} \prod_{l=2}^{D} \frac{1}{k'_l!} \cdot \frac{(\delta D)^D}{D!\, e^{\delta D}}\, P(D),$$

where the summation is over all (k'_i) with $k'_i \ge 0$, $\sum k'_i = D - k$ and $\#\left\{\, j \,:\, k'_j = 0 \,\right\} = \alpha(L)$. Noticing that

$$\sum_{l=2}^{D} k'_l = D - k$$

(which is (5.8) for $j = D$) and $\frac{1}{k'_l!} = \left[e^x\right]_{k'_l}$ we have

$$\sum_{(k'_i)} \prod_{l=2}^{D} \frac{1}{k'_l!} \le \sum_{(l'_i)} \prod_{l=2}^{D} \frac{1}{l'_l!} = \left[(e^x)^D\right]_{D-k} = \frac{D^{D-k}}{(D-k)!}.$$

[20] The reader might forgive us that we write down factorials of non-(necessarily) integral numbers, but as this can be corrected by a polynomial(ly bounded) term in D and polynomials will not matter to us, this does not spoil anything.

where the summation in the second term is over all (l'_i) with $l'_i \geq 0$ and $\sum l'_i = D - k$. So (5.9) is less than

$$\binom{D}{k} D^{D-k} \frac{(\delta D)^D}{D!\, e^{\delta D}} P(D)\,.$$

We obtain a bound for ξ_D^δ by summing over $k = 1, \ldots, D$.

$$\xi_D^\delta \leq (D+1)^D \frac{(\delta D)^D}{D!\, e^{\delta D}} P(D)\,. \tag{5.10}$$

The r.h.s. is by Stirling formula less than or equal to

$$\delta^D e^{(2-\delta)D} D!\, P'(D)$$

for a certain polynomial P' in D.

Case 2. For $\alpha(L) > \delta D$ consider first only reduced regular LCD's with path l. Let $\mathring{\xi}_D^{\bar{\delta}}$ be this number.

By artificially putting to the left of such a LCD a chord of length 1 we obtain a reduced regular LCD of degree $D+1$ with len(chord 1) $= 1$ and with a path with ascend $\geq \delta D$. The ascend of this LCD is equal to its *total descend*

$$\sum_{i\,:\,k_{i+1} \leq k_i} (k_i - k_{i+1}) \geq \delta D\,.$$

Each descend of $k'_i - 1$ means letting a segment of k'_i left neighbored basepoints to the right of the right basepoint of the newly placed chord (which is preserved by the addition of all remaining chords) by applying m_i^{-1}. Let the symmetric groups of corresponding order act on such segments simultaneously. The orbits are free and, since we consider reduced LCD's, all LCD's in the orbits have no type I. As we noted, the number of LCD's with no type I is $D!$. The lowest cardinality of the direct product of symmetric groups is in the case where we have 1-descends only, which is $\geq 2^{\delta D}$. So the total number of regular reduced LCD's L with $\alpha(L) \geq \delta D$ is

$$\mathring{\xi}_D^{\bar{\delta}} \leq \frac{D!}{2^{\delta D}}\,.$$

Combining both cases we find that for reduced regular LCD's their number $\mathring{\xi}_D$ is

$$\begin{aligned} \mathring{\xi}_D &\leq P''(D) \left(\delta^D e^{(2-\delta)D} D! + \frac{D!}{2^{\delta D}} \right) \\ &\leq P''(D) \frac{D!}{\lambda(\delta)^D}, \qquad \lambda(\delta) := \frac{1}{\max(\delta e^{2-\delta}, \frac{1}{2^\delta})}\,. \end{aligned}$$

Now choose δ so that $\delta < \frac{1}{e^2}$ and you are done for reduced LCD's.

The best choice of δ would be one for which

$$\delta e^{2-\delta} = \frac{1}{2^\delta}\,.$$

Call δ_0 this number and $\lambda_0 := \lambda(\delta_0)$.

However, our choice of δ will be somewhat worse in order to remove all polynomial mess in D by the exponential correction.

The result for all regular LCD's is now easy. We have

$$\begin{aligned} \xi_D &= \sum_{i=0}^{D-1} \binom{D-1}{i} \mathring{\xi}_{D-i} \\ &\leq P(D) \cdot \sum_i \frac{\lambda_0^i}{i\,!} \frac{D!}{\lambda_0^D} \\ &\leq P(D) \cdot e^{\lambda_0} \frac{D!}{\lambda_0^D}\,. \end{aligned}$$

Again, kill the polynomial by perturbing λ_0. □

Numerically we have $\delta_0 \approx 0.141334$, where $\lambda_0 = 2^{\delta_0} \approx 1.10292$. So we have

Theorem 5.11 *The number ξ_D (and therefore the number of linearly independent Vassiliev invariants of order D) is asymptotically bounded by*

$$\xi_D < \frac{D!}{1.1^D}.$$

5.6 The segment length inequality

After having estabilished our result it is perhaps worth saying a word about some possibilities left open in the proof of our bound.

The observation made in case 2 of the proof can be generalized somewhat.

Definition 5.12 *Call a* segment *of an LCD a maximal piece of the solid line containing left basepoints only and its length the number of such basepoints.*

Then by the argument above we have the

Theorem 5.13 (Segment length inequality)

$$\sum_{\substack{\text{reduced reg.} \\ \text{LCD's } L \text{ of deg } D}} \prod_{\text{segments of } L} \big(\textit{ length of segment }\big)! \quad \leq D!$$

Basically our proof was that we bounded the L's with $\leq \delta D$ factors equal to one in the product and used that the rest appears with multiplicity at least $2^{\delta D}$ in the sum. However, many reduced regular LCD's appear with much higher factors and if one were able to control their number (which probably requires much labour) this would improve the base in the denominator we obtain in case 2 (and the total one for all regular LCD's). One might even hope that one can achieve *each* base in case 2 (and therefore as well for the total bound). But, to put an end to our dreams, recall that we will never be able to prove $(D!)^{1-\epsilon}$ (for some $\epsilon > 0$) this way!

6 The dimension of a graded commutative algebra and asymptotics of Vassiliev invariants

Here we discuss the relation between the dimension of a symmetric algebra (with the induced grading) over a graded vector space (latter called henceforth the primitive part of the algebra), and apply it to deduce a lower bound for the number of all Vassiliev invariants.

One of the combinatorial aspects of a commutative graded algebra (CGA) A is the relation between the asymptotical behaviour of its gradeded pieces A_D depending on their primitive parts P_D (d and D will denote the degree). We will make two assumptions on such an algebra:

1) $\deg(a \cdot b) = \deg(a) + \deg(b) \quad \forall a, b \in A$,
2) prime factorization is unique in A.

Consider the commutative graded Hopf algebra $A = \mathcal{A}$ of chord diagrams.

Recently, Chmutov and Duzhin [CD2] obtained the following result for $\mathcal{A}$.

Theorem 6.1 *The dimension of primitive elements in A_D has the asymptotical lower bound $D^{\log_{4+\varepsilon} D}$ for each $\varepsilon > 0$.*

As it was not explained by the authors which base of the logarithm we can choose, we should do this here.