Therefore

$$\frac{\sum_{c|2D,\,c\geq 2}\varphi(c)\,\gamma_{2D/c,c}}{\gamma_{2D,1}} \;\leq\; \frac{\left(2+\sqrt{2}\right)^{D}(2D-1)}{\binom{2D}{D}\left\lfloor\frac{D}{2}\right\rfloor!} \quad \xrightarrow[D\to\infty]{} \; 0 \qquad \square$$

Something more interesting happens in the case ω_D^1, the number of chord diagrams with chords of length 1. Looking at (3.9) we see that we can write the ratio between λ_D and the k^{th} term in the sum on the r.h.s.

$$\frac{\lambda_{D-k,k+1}+\lambda_{D-k,k}}{\lambda_D} \;=\; \frac{1}{k!}\,P(D)\,,$$

where $P(D)$ is a polynomial fraction of degree 0 in D bounded above by 1 and converging to 1 for $D\to\infty$. This means that

$$\frac{\bar\psi_D^1}{\lambda_D} \;\xrightarrow[D\to\infty]{}\; \frac{1}{e}\,,$$

where e is the Euler number $2.71828\ldots$, and together with lemma 3.11 we get the same result for chord diagrams:

$$\frac{\bar\omega_D^1}{\lambda_D} \;\xrightarrow[D\to\infty]{}\; \frac{1}{e}\,.$$

Lemma 3.12 *Asymptotically* $\frac{1}{e}$ *of all chord diagrams and LCD's have no isolated chord (or isolated chord of length 1).*

In fact, it is an easy exercise to convince oneself that there are "very few" degenerate chord diagrams with no chord of length 1, that is

$$\frac{\omega_D-\omega_D^1}{\sigma_D} \;\xrightarrow[D\to\infty]{}\; 0\,.$$

Unfortunately, computing more carefully the difference

$$\frac{1}{e}-\frac{\bar\psi_D^1}{\lambda_D}$$

we see that the dominating term is

$$\frac{1}{2(2D-1)}\,,$$

so we cannot hope for a fast convergence.

Problem. At present I don't know the asymptotics of σ_D^{sym}.

However, although unimportant for our context, one can obtain the following alternative expression for it.

$$\sigma_D^{\text{sym}} \;=\; \frac{d^{D-1}}{dx^{D-1}}\Big((1+x)\,e^{x+x^2}\Big)(0)\,.$$

To see this, look at the nomalized generating series of $\gamma_{*,2}$

$$\tilde P_{\gamma_{*,2}}(x) \;:=\; \sum_{k=0}^{\infty}\frac{\gamma_{k,2}}{k!}\,x^k$$

and prove that $\tilde P_{\gamma_{*,2}}$ is a solution of the differential equation

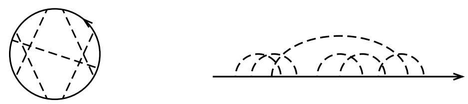
$$xf'(x) \;=\; xf(x)+2x^2f(x), \qquad f(0)=1\,.$$

4 Connected and tree-connected chord diagrams

Using the methods of section 3, in this section we discuss the enumeration of two more classes of chord diagrams.

4.1 Connected CD's and LCD's

Definition 4.1 *A CD (or LCD) is called* connected, *if the graph remaining after removing the solid circle line is connected, or equivalently, the intersection graph of the dashed chords is connected.*



Recall the enumeration of degenerate chord diagrams in §3. The present enumeration problem of

$$\omega_D^{\text{nc}} := \#\{\text{ non-connected CD's of deg } D\ \}\ , \quad \bar{\omega}_D^{\text{nc}} := \#\{\text{ connected CD's of deg } D\ \}$$

is very similar to this[12].

Start with the linear case and look for a formula for

$$\psi_D^{\text{nc}} := \#\{\text{ non-connected LCD's of deg } D\ \}\ , \quad \bar{\psi}_D^{\text{nc}} := \#\{\text{ connected LCD's of deg } D\ \}\ .$$

Non-connected LCD's have a *virtual* separating arc (i. e., they can be extended by such an arc), which means that the arc is isolated and the enclosed and outside pieces of the LCD (with respect to this arc) are both non-empty.

Apply the inclusion-exclusion principle over the *minimal* virtual separating arcs (i. e., such ones which enclose *connected* LCD's). Note, that such arcs lie beside each other.

The recursive formula you obtain for $\bar{\psi}_D^{\text{nc}}$ is

$$\bar{\psi}_0^{\text{nc}} = 1$$

and

$$\begin{aligned}\bar{\psi}_D^{\text{nc}} &= \sum_{i=0}^{D} (-1)^i \sum_{\substack{j_1,\,...,\,j_i,\,k \\ j_i > 0 \\ k \geq 0,\, k>0 \text{ for } i=1 \\ \sum j_i + k = D}} \lambda_{k,i+1} \prod_{l=1}^{i} \bar{\psi}_{j_l}^{\text{nc}} \\ &= \sum_{i=0}^{D} (-1)^i \left[(P_{\bar{\psi}^{\text{nc}}}(x) - 1)^i \cdot \left(P_{\lambda_{*,i+1}}(x) - \left\{ \begin{array}{ll} 1 & \text{if } i=1 \\ 0 & \text{else} \end{array} \right\} \right) \right]_{x^D} \quad \text{for } D > 0\,,\end{aligned}$$

using the characteristic series

$$P_{\bar{\psi}^{\text{nc}}}(x) := \sum_{i=0}^{\infty} \bar{\psi}_i^{\text{nc}} x^i\ .$$

Here i is the number of choice of minimal virtual separating chords, $j_1,\ \ldots, j_i$ are the degrees of the LCD's enclosed by the i chords, and k is the degree of the remaining string link diagram. (Recall §3, that string link diagrams arise from LCD's by cutting the solid line.)

Note, that the condition of the non-emptyness of the enclosing string link diagram is necessary only in the case of 1 virtual separating chord.

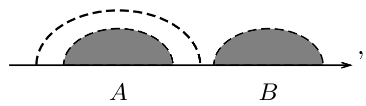
The number of non-connected LCD's is then

$$\psi_D^{\text{nc}} = \lambda_D - \bar{\psi}_D^{\text{nc}}\ .$$

Now look at the number $\tilde{\gamma}_{d,c}^{\text{nc}}$ of GLCD's of degree d and cycl. $c \geq 2$, producing (in the sense described in §3) non-connected chord diagrams.

[12] All further references to §3 in this section should be understood as references to this enumeration.

Look at the GLCD producing such a chord diagram obtained by cutting the solid line just before one of the endpoints of such an arc[13]. Such a GLCD can be incorporated into the following pattern (where the arc drawn is the virtual one):



where B is arbitrary and A has only 0-colored chords[14].

So we need to enumerate such GLCD's as above and the ones resulting from them by the cyclic group action described in §3.

Let us apply as in the case of degenerate chord diagrams the inclusion-exclusion principle on minimal virtual separating chords in the GLCD. As in the case in §3, minimal virtual separating chords can have two *virtual* colorings – 0 (as in the above picture) and $c-1$ (if the action of the cyclic group has reversed the endpoint order; in this case such a chord contains outside of itself a connected GLCD with chords colored by 0 if both endpoints are on the same side of the minimal virtual separating chord and by $c-1$ otherwise).

Let us recompute the numbers $\xi^k_{c,d}$ and $\bar{\xi}^k_{c,d}$ of §3 for our new case. The formulas now look this way:

$$\begin{aligned} \xi^k_{c,d} &= \sum_{(e_1,\ldots,e_k)>0} \prod_{j=1}^{k} \bar{\psi}^{\text{nc}}_{e_j} \cdot \lambda^c_{d-2\sum e_j,k+1} \\ &= \left[\left(P_{\bar{\psi}^{\text{nc}}}(x^2)-1\right)^k P_{\lambda^c_{*,k+1}}(x) \right]_{x^d} \quad \text{for } 0 \le k \le d \\ \xi^k_{c,d} &= 0 \text{ for } k>d \text{ or } k<0 \end{aligned} \tag{4.1}$$

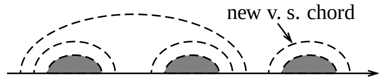
and with

$$P_{\xi^k_{c,*}}(x) := \sum_{d=0}^{\infty} \xi^k_{c,d} x^d$$

$$\begin{aligned} \bar{\xi}^k_{c,d} &= \sum_{e>0} (2e-1)\,\bar{\psi}^{\text{nc}}_e \cdot \xi^{k-1}_{c,d-2e} \\ &= \left[\left(\frac{\partial}{\partial x} \left(x \cdot P_{\bar{\psi}^{\text{nc}}}(x^2)\right) - 2P_{\bar{\psi}^{\text{nc}}}(x^2) + 1 \right) \cdot P_{\xi^{k-1}_{c,*}}(x) \right]_{x^d} . \end{aligned} \tag{4.2}$$

The main differences to the previous case are

1. LCD's enclosed by virtual separating chords are *non-empty*, so $e_j > 0$ in (4.1) and analogously $e > 0$ in (4.2).
2. You get an index translation for the string link diagram in (4.1), since the chords are *virtual*, i. e. they do not really belong to the GLCD.
3. In the case of $\bar{\xi}^k_{c,d}$ you must replace the factor "$(2e+1)$" in §3 by "$(2e-1)$". The reason is the following: If you allow a minimal virtual separating chord of color $c-1$ to contain all basepoints of the LCD outside of it on *one* side only, then you can also draw a minimal virtual separating chord of color 0 around this component.



This way you obtain diagrams with positions of minimal virtual separating chords which are not incorporated in either counts $\xi^k_{c,d}$ (which counts pictures like) and $\bar{\xi}^k_{c,d}$ (which counts pictures like ).

[13] "before" we mean with respect to the orientation of the solid line.

[14] This is clear in the case where the length of the separating arc is shorter than the degree of the GLCD. In the other case note that the separating arc intersects the one from the next cycle, and smoothing out this intersection we obtain a new separating arc whose length is exactly the degree of the GLCD, so we get back to the picture above with B equal to the empty GLCD.

You can avoid this by declaring that you draw a virtual separating chord of color $c-1$ *only* if the component outside of this chord would *enclose* it, i. e., has basepoints on *both* sides of the virtual separating chord (else you draw a virtual separating chord of color 0 around the component).

4. Note, that the non-emptyness of the enclosing string link diagram is not required here.

The rest of the formulas then remain the same as for degenerate chord diagrams.

$$\zeta^k_{c,d} = \xi^k_{c,d} + \bar{\xi}^k_{c,d}$$
$$\eta_{c,d} = \sum_{k=1}^{d} (-1)^{k-1} \zeta^k_{c,d}$$

Then we have

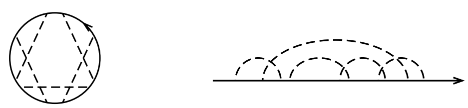
$$\tilde{\gamma}^{\text{nc}}_{d,1} = \psi^{\text{nc}}_{d/2} \text{ for } 2|d$$
$$\tilde{\gamma}^{\text{nc}}_{d,1} = 0 \text{ for } 2 \not| d$$
$$\tilde{\gamma}^{\text{nc}}_{d,c} = \eta_{c,d} \text{ for } c \geq 2\,,$$

and finally

$$\omega^{\text{nc}}_D = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\text{nc}}_{2D/c,c}\,.$$

4.2 Tree–connected CD's and LCD's

Definition 4.2 *A chord diagram (or LCD) of degree D is called* tree–connected, *if it is connected and there are exactly $D-1$ chord intersections, i. e., the intersection graph of the dashed chords is a tree.*



This definition appears in a somewhat different form in [NW, §3] and [Be, §2]. See [BG] for a different application of the labelled intersection graph of a LCD.

Remark 4.1 Don't confuse tree–connected chord diagrams with tree–*like* chord diagrams (which are the ones with no chord intersections, i. e. only with isolated chords, and are called in §3 "fully degenerate").

Let

$$\omega^{\text{tc}}_D \quad := \quad \#\{ \text{ tree–connected CD's of deg } D \ \} \ .$$

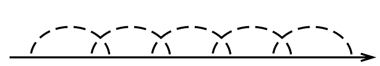
Here the difference to the counting in section 3 is more substantial than in the previous subsection.

Start with the linear case. To enumerate tree–connected LCD's

$$\bar{\psi}^{\text{tc}}_D \quad := \quad \#\{ \text{ tree–connected LCD's of deg } D \ \} \ ,$$

use the following idea.

Start with the left-most chord of a tree–connected LCD and walk along the chord (in the direction given by the orientation of the solid line) until you intersect a second chord, then walk (in the same direction) along this chord until you get to a third chord, and repeat this way until you walk to the end of the last chord and land back on the solid line. The chords (segments of which) you have passed, form a sub–LCD C which looks like this



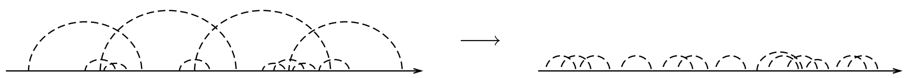
(4.3)

Let us denote for a moment its degree by d. All the remaining chords are associateded at the $2d-2$ inner basepoints of this sub–LCD in such a way that you obtain a (non-empty) tree-connected LCD of lower degree for the fixed basepoint P by applying the following procedure:

1. Call a chord associated to P if it is intersecting the chord ending on P and is enclosed by the chord in (4.3), enclosing P, or if it is intersecting another chord associated to P.

2. For each of the $2d-2$ inner basepoints P build the diagram made up of all chords associated to P and the chord ending on P, putting latter's rightmost basepoint to the left, if the basepoint P was a left basepoint of a chord in (4.3).

Here is an example:



Conversely, from any collection of $2d-2$ non-empty tree-connected LCD's you can reconstruct a tree-conn. LCD by inverting this procedure. The recursive formula for $\bar{\psi}^{\mathrm{tc}}_D$ is then immediate:

$$\begin{aligned} \bar{\psi}^{\mathrm{tc}}_0 &= 1 \\ \bar{\psi}^{\mathrm{tc}}_D &= \sum_{i=1}^{D} \left[\left(P_{\bar{\psi}^{\mathrm{tc}}}(x) - 1 \right)^{2i-2} \right]_{x^{D+i-2}} \text{ for } D > 0\,, \end{aligned} \tag{4.4}$$

with

$$P_{\bar{\psi}^{\mathrm{tc}}}(x) := \sum_{i=0}^{\infty} \bar{\psi}^{\mathrm{tc}}_i x^i\,.$$

Remark 4.2 LEROUX and MILOUDI [LMi, (5.16)] and also DULUCQ and PENAUD [DP, theorem 2.2] found the explicite formula

$$\bar{\psi}^{\mathrm{tc}}_D = \frac{1}{D-1} \binom{3D-3}{D-2}, \qquad D \geq 2\,.$$

In fact, one can derive from (4.4) that

$$g(x) := \frac{P_{\bar{\psi}^{\mathrm{tc}}}(x) - 1}{x}$$

is the solution of the equation (5.15) of [LMi]

$$g(x) = 1 + x g^3(x)\,.$$

Let us now consider cyclicity $c \geq 2$ and count GLCD's producing tree–connected CD's. Let us adopt the convention that we separate the solid line of a chord diagram into $2D$ *equally* long arcs and draw chords always as *straight* lines.

First look at the tree–connected chord diagram (drawn this way) coming from the GLCD. Temporarily remove all chords connecting opposite basepoints (i. e., *self-loop chords*). Then the center of the circle belongs to a component of the complement of the chord diagram.

If this component is not bounded by any piece of the solid line then we have an intersection loop of chords and our chord diagram is not tree-connected.

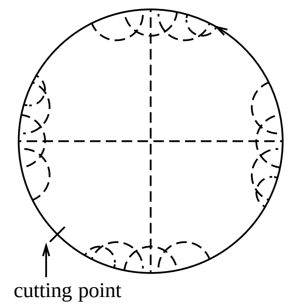
If there is such a piece, then there are at least c of them and the chord diagram has $\geq c$ components. In order for the chord diagram to be connected we must connect all components by reinstalling the self-loop chords. So we need at least 1 self-loop chord. On the other hand there are at most two self-loop chords (since three self-loop chords produce an intersection cycle among themselves).

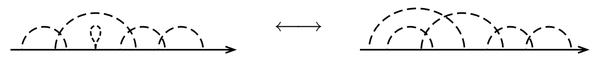
We have two cases.

Case 1. One self-loop . Cyclicity must be 2 (1 self-loop chord) or 4 (2 self-loop chords). Degree must be *odd*.

In this case look at the GLCD produced by cutting the chord diagram at a position which you choose to lie on a piece of the solid line bounding the component of the circle center created after removing the self-loop chords (see example).

There is a bijection between such GLCD's and tree–connected LCD's by replacing the self-loop by a chord whose left end becomes the leftmost basepoint and whose right end replaces the previous position of the .

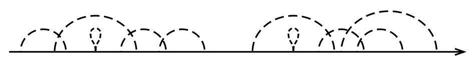




The orbits of the cyclic group action on these GLCD's are free (e. g., since the appears always at different positions if you enumerate all basepoints from left to right), i. e., we get $d\bar{\psi}^{\mathrm{tc}}_{(d+1)/2}$ GLCD's.

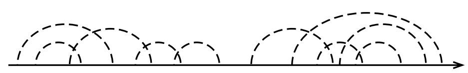
Case 2. Two . Cyclicity must be 2. Degree is *even*.

By the procedure of case 1 you obtain a GLCD looking like this:



(4.5)

If you transform the left self-loop into a chord ending on the left–most position and the right self-loop into a chord ending on the right–most position, you get a bijection to diagrams obtained by multiplication of two tree–connected LCD's. E. g., the diagram corresponding to (4.5) is



.

The number of such diagrams is

$$\sum_{\substack{e_1+e_2=d,\\ 2\,\not|\,e_1,e_2}} \bar{\psi}^{\mathrm{tc}}_{(e_1+1)/2}\bar{\psi}^{\mathrm{tc}}_{(e_2+1)/2}\,.$$

Now let the cyclic group $\mathbf{Z}_d$ act on such diagrams and note that each GLCD obtained in this way is obtained always in exactly two ways.

- If first and second component in (4.5) are equal, the action of $\mathbf{Z}_d$ has period $d/2$.
- If first and second component in (4.5) are distinct, $\mathbf{Z}_d$ acts freely, but the orbit is equal to the one of the diagram obtained from (4.5) by swapping the components.

In full completeness, we obtain the following values for $\tilde{\gamma}^{\mathrm{tc}}_{d,c}$ of in our case (compare with §3.7).

$$\begin{aligned}
\tilde{\gamma}^{\mathrm{tc}}_{d,1} &= \bar{\psi}^{\mathrm{tc}}_{d/2} && \text{for } 2|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,1} &= 0 && \text{for } 2\not|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,2} &= \frac{d}{2}\cdot\sum_{e=1}^{d/2}\bar{\psi}^{\mathrm{tc}}_{e}\bar{\psi}^{\mathrm{tc}}_{d/2+1-e}\\
&= \frac{1}{2}\cdot\left[x\cdot\frac{\partial}{\partial x}\left(\frac{1}{x^2}\cdot\left(P_{\bar{\psi}^{\mathrm{tc}}}(x^2)-1\right)^2\right)\right]_{x^d} && \text{for } 2|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,2} &= d\cdot\bar{\psi}^{\mathrm{tc}}_{(d+1)/2} && \text{for } 2\not|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,4} &= d\cdot\bar{\psi}^{\mathrm{tc}}_{(d+1)/2} && \text{for } 2\not|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,4} &= 0 && \text{for } 2|d\\
\tilde{\gamma}^{\mathrm{tc}}_{d,c} &= 0 && \text{else}
\end{aligned}$$

Finally, as always by BURNSIDE's lemma,

$$\omega_D^{\text{tc}} = \frac{1}{2D} \cdot \sum_{c|2D} \varphi(c) \cdot \tilde{\gamma}^{\text{tc}}_{2D/c,c}\,.$$

4.3 Some computations

With the previous formulas one can compute the beginning of the various integer sequences above[15]. Here are the first 10 values of each one.

D	1	2	3	4	5	6	7	8	9	10
ω_D^{tc}	1	1	1	2	7	25	108	492	2,431	12,371
ω_D^{nc}	0	1	3	12	74	647	6,961	89,739	1,337,152	22,609,111
$\bar{\omega}_D^{\text{nc}}$	1	1	2	6	31	255	2,788	37,333	578,799	10,134,071

Remark 4.3 By similar arguments as in §3 one can show that

$$\frac{\bar{\omega}_D^{\text{nc}}}{\sigma_D} \xrightarrow[D\to\infty]{} \frac{1}{e}\,.$$

This was first proven for the linear case by STEIN and EVERETT [SE], see also [S, p. 362].

Remark 4.4 The sequence of tree–connected chord diagrams was previously known. According to SLOANE [Sl] it appeared in an enumeration of planar alcohol molecules [LMi] (which correspond to tree–connected chord diagrams by putting on each chord crossing a 'C' and on each chord basepoint an 'H' atom and forgetting the solid line). Leroux and Miloudi used some more general and more elegant PÓLYA theory arguments. Our approach is however geometrically more understandable.

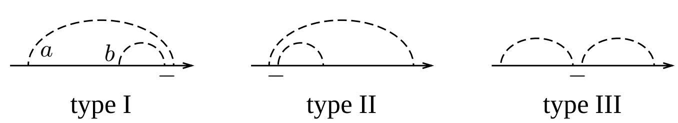
5 Enumeration of chord diagrams and an upper bound for Vassiliev invariants

In this section, we treat an enumeration problem of chord diagrams, which is shown to yield an upper bound for the dimension of the space of Vassiliev invariants for knots. We give an asymptotic estimate for this bound. More precisely, we prove the upper bound $D!$/(any given polynomial in D) for both $G_D\mathcal{A}$ and $G_D\mathcal{A}^r$ (where $\mathcal{A}$ and $\mathcal{A}^r$ are the graded algebras of chord diagrams introduced in §1) and later improve it to $D!/1.1^D$. As an aside, we present a trivial proof for the bound $D!$.

In the next section we will discuss further results on the dimension of $\mathcal{A}$.

5.1 Factoring out 4T relations

There are three types of mutual positions of non-intersecting chords in the LCD's with neighboring basepoints.



It is clear that one can resolve one *fixed* type I and II intervals using the 4T-relation. But this also works with *all* such intervals. This means, we have

[15] A MATHEMATICA™ package with all formulas is available on my WWW page.