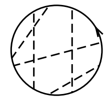
3 On the number of chord diagrams

In this section we treat some enumeration problems of certain kinds of chord diagrams. Recall, that a chord diagram (a CD) is an object like this,



i. e. an oriented circle with finitely many dashed chords in it and considered up to isotopy.

The essential difficulty of this enumeration is determining their linearized relatives, called LCD's, fixed by a certain cyclic permutation of the basepoints. This we achieve by introducing some new objects called generalized linearized chord diagrams or short GLCD's.

It should be mentioned, that similar enumeration problems have been treated in another way in several other papers, e. g. [Be, S, NW, Bo, DP, HS].

3.1 Notations

For two numbers $m, n \in \mathbf{N}$ their g.c.d. is denoted (m, n) and $m\%n$ is $m \bmod n$.

If P is a finite set, by the symbol $\#P$ we will denote its cardinality and by $\mathcal{P}(P)$ its power set (set of all subsets).

In the following we will need some number-theoretic functions. $\varphi(n)$ will denote the EULER function, which can be defined by

$$\varphi(n) := \#\{ 0 < n' \leq n ;\ (n, n') = 1 \} = n \cdot \prod_{\substack{p \text{ prime} \\ p|n}} \left(1 - \frac{1}{p}\right) .$$

A well-known property of these values is that for all $n \in \mathbf{N}_+$

$$\sum_{d|n} \varphi(d) = n . \tag{3.1}$$

Let

$$(n)_d := \frac{n!}{(n-d)!}$$

denote the number of d-fold ordered choices out of n elements.

The bifactorial $n!!$ of an integral number n is defined by

$$n!! := \prod_{i=0}^{\lfloor \frac{n-1}{2} \rfloor} n - 2i$$

for $n > 0$ and by convention we set $0!! := 1$, $(-1)!! := 1$ and $n!! = 0$ for $n \leq -2$.

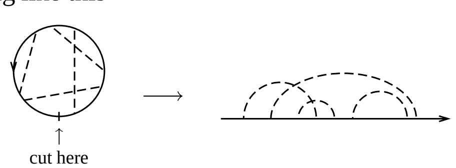
$[P(x)]_d$ will denote the coefficient of x^d in the polynomial (or power series) P in the formal variable x.

By $\lfloor n \rfloor$ we will mean the greatest integer not greater than n.

For two sequences of numbers $(a_i)_{i=1}^{\infty}$ and $(b_i)_{i=1}^{\infty}$, the expression $a_i \asymp b_i$ denotes $\lim_{n\to\infty} a_i/b_i = 1$, or, in words, that (a_i) and (b_i) are asymptotically equivalent.

3.2 Linearized chord diagrams

One can obtain a linearized chord diagram (an LCD) from a usual chord diagram by "cutting" the solid line somewhere. Then one has something like this



Both chord diagrams and LCD's are graded by the number of their chords[3], so the picture above is of degree 4.

Let us use the following notations.

$$\begin{array}{rcl} C_D & := & \{ \text{CD's of deg } D \} \\ L_D & := & \{ \text{LCD's of deg } D \} \end{array} \qquad \begin{array}{rcl} \sigma_D & := & \#C_D \\ \lambda_D & := & \#L_D \end{array}$$

A generalization of the LCD's with more than one solid line are the so called *string link* diagrams (for pictures look e. g. in [BN4]). Let

$$L_{D,k} \quad := \quad \{ \text{ string link diagrams with } k \text{ strands of deg } D \} \qquad \lambda_{D,k} \quad := \quad \#L_{D,k}$$

The motivation to start these considerations was for me the fact, that the number λ_D of LCD's of deg D can be computed very easily. In fact, it is a simple exercise to show the following

Lemma 3.1

$$\lambda_D \;=\; (2D-1)!!$$

As a generalization of this fact, one can prove the following statement about $\lambda_{D,k}$.

Lemma 3.2

$$\lambda_{D,k} \;=\; \binom{2D+k-1}{2D}(2D-1)!!$$

Hint: Glue all strands into one and place a mark on the point of each gluing.

The symmertric group S_{2D} acts on L_D by permuting the order of the base points of the D chords, and in this sense C_D is isomorphic to the orbit space of the cyclic subgroup $\mathbf{Z}_{2D} \subset S_{2D}$ generated by the cycle $z_D := (1\,2\,3\,\ldots 2D)$ on L_D. So, we shall consider the behaviour of LCD's under this action.

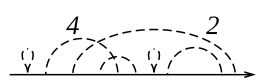
Let for $\sigma \in S_{2D}$

$$R_\sigma \quad := \quad \{ \text{ LCD's } Y \text{ of deg } D \text{ with } \sigma(Y) = Y \} \quad r_\sigma \;:=\; \#R_\sigma$$

3.3 Cyclic CD's and GLCD's

Definition 3.3 *A generalized linearized chord diagram (GLCD) is a pair of the following form*

$$\Bigg(\qquad\qquad\qquad\qquad , \; n \Bigg)$$



where $n \in \mathbf{N}_+$ *and the first component is something like an LCD, but has the following 2 additional features*

- *If* n *is even, it may contain self-loops* *, i. e. chords starting and ending onto the same basepoint*
- *Each real chord (a chord which is not a self-loop) is equipped with a number between 0 (in this case we drop the number for convenience) and* $n-1$*. We will say that it's* coloured *or* labeled *by this number.*

Let the GLCD's be graded by the number of the basepoints (not chords !) and the cyclicity of a GLCD be its second component. So the LCD's are exactly GLCD's with cyclicity 1. Then the above picture has degree 10 and cyclicity n.

It will be sometimes convenient to drop the cyclicity and take only the first part (what is meant will be clear from the context).

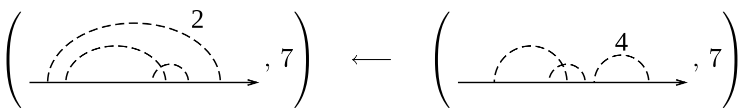
Let

$$\Gamma_{d,c} \quad := \quad \{ \text{ GLCD's of deg } d \text{ and cyclicity } c \} \quad \text{and} \quad \gamma_{d,c} \quad := \quad \#\Gamma_{d,c}\,.$$

Furthermore, we introduce an action of $\mathbf{Z}_d$ on $\Gamma_{d,c}$ by letting $1 \in \mathbf{Z}_d$ act on a GLCD in the following manner.

[3] From the point of view of the following considerations it might appear more natural to take the number of basepoints (i. e. the double number of chords) as graduation but I preferred to keep the original definition of this notion from knot theory.

- It flips self-loops and real chord ends from the right-most position to the left-most
- Each time it flips one of the ends of a real chord, its number changes from k to $n-1-k$, e. g.



It will turn out as useful to know the cardinality $\gamma_{d,c}$ of $\Gamma_{d,c}$. This is an easy combinatorial task.

If c is odd, then self-loops don't exist and the only non-vanishing case is if d is even. Then we are left with counting LCD's with numbered chords.
If c is even then sum over all possible numbers of  and over the choices to put them between the real chords. Then

$$\gamma_{d,c} \quad = \quad \begin{cases} 0 & 2 \not| c, d \\ c^{\frac{d}{2}}(d-1)!! & 2 \not| c, 2|d \\ \sum_{i=0}^{\lfloor \frac{d}{2} \rfloor} \binom{d}{2i} \cdot c^i \cdot (2i-1)!! & 2|c \end{cases} \quad . \tag{3.2}$$

Let

$$\begin{aligned} L_{D,c} &:= \left\{ \text{LCD's } Y \text{ of deg } D \text{ with } \mathbf{Z}_c \subset \text{stab}_{\mathbf{Z}_{2D}}(Y) \subset \mathbf{Z}_{2D} \right\} \\ C_{D,c} &:= \left\{ \text{CD's } Y \text{ of deg } D \text{ with } \mathbf{Z}_c \subset \text{stab}_{\mathbf{Z}_{2D}}(Y) \subset \mathbf{Z}_{2D} \right\}, \end{aligned}$$

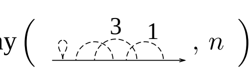
where $C_{D,c}$ counts the chord diagrams obtained by closing up the LCD's counted by $L_{D,c}$. In other words

$$L_{D,c} = R_{z_D^{2D/c}} \quad \text{and} \quad C_{D,c} = R_{z_D^{2D/c}} \Big/ \mathbf{Z}_c \, .$$

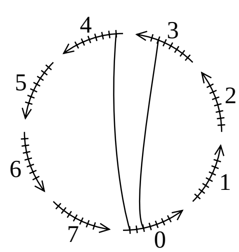
A relation between GLCD's and CD's is given by the following

Theorem 3.4 *There is a bijection*

$$C_{D,c} \quad \overset{\Phi_{D,c}}{\longleftrightarrow} \quad \Gamma_{\frac{2D}{c},c} \Big/ \mathbf{Z}_{\frac{2D}{c}} \quad .$$

Proof. We should best use an example to demonstrate what we are going to do. Given a GLCD, say  construct the corresponding chord diagram in $C_{\frac{7}{2}n,n}$ followingly :

1. Separate an oriented circle into n arcs and mark on each arc 7 basepoints. Number the arcs from 0 to $n-1$ and the basepoints on each arc from 1 to 7.
2. For each real chord in your GLCD and $0 \le m \le n-1$ connect the left end basepoint on the arc m with the right end basepoint on the arc $(m+\mu) \bmod n$, where μ is the number of the chord, e. g. for $n=8$ in the case of arc 0 and the chord numbered by 3 in the example we use we get the right chord in the following picture



3. For a self-loop, connect the basepoint in arc m with the same basepoint in arc $(m+\frac{n}{2}) \bmod n$, e. g. for arc 0 we get the left chord in the picture above.
4. Glue all the arcs together and remove all markings on them.

Now it is easy to see how to construct the inverse of Φ – separate for a cyclic chord diagram in $C_{D,c}$ the circle into c pieces with $\frac{2D}{c}$ basepoints and assign the unique numbers to the chords in your GLCD, counting the difference between the arc numbers. If chords in your chord diagram start and end on the same position in different arcs (i. e. you obtain a self-loop), then c must be even and the arcs opposite in order the chord diagram to be cyclic. Now check that the action of $\mathbf{Z}_{\frac{2D}{c}}$ factors out exactly the arbitrarity how to choose the splitting of the baseline into arcs. □

3.4 Counting all chord diagrams

Using theorem 3.4 we see that LCD's invariant under $d \in \mathbf{Z}_{2D}$ bijectively correspond to GLCD's with cyclicity $c = 2D/d$. Noticing that an LCD is invariant under $d \in \mathbf{Z}_{2D}$ exactly if it is under $d \cdot l$ where $(l, 2D/d) = 1$, we have

$$r_{z_D^c} \;=\; \gamma_{(2D,c),\frac{2D}{(2D,c)}}$$

and by BURNSIDE's lemma on counting orbits of a finite group action we get the following combinatorial expression for σ_D.

Theorem 3.5 *With (3.2) one has*

$$\sigma_D \;=\; \frac{1}{2D} \sum_{d\cdot c = 2D} \varphi(c)\, \gamma_{d,c}\,. \tag{3.3}$$

This formula is probably originally due to JEAN BÉTRÉMA [Sl].

3.5 Symmetric chord diagrams

A variation of the enumeration problem is to count chord diagrams up to mirror images (or equivalently, up to change of orientation of the solid line). Let

$$\hat\sigma_D \;:=\; \#\,\{\text{ CD's of degree } D\ \}\Big/\text{symmetry}$$

and

$$\sigma_D^{\text{sym}} \;:=\; \#\,\{\text{ symm. CD's s of degree } D\ \}\,.$$

Then clearly,

$$\hat\sigma_D \;=\; \frac{\sigma_D + \sigma_D^{\text{sym}}}{2}\,. \tag{3.4}$$

$\hat\sigma_D$ can also be computed using Burnside's lemma. In view of (3.4) it is more convenient to give $\hat\sigma_D$ in terms of σ_D^{sym}, since the resulting formula for σ_D^{sym} turns out to be surprisingly simple.

Theorem 3.6 *For $D > 0$ we have*

$$\sigma_D^{\text{sym}} \;=\; \sum_{i=0}^{\lfloor\frac{D}{2}\rfloor} \frac{(D-1)_{2i}}{i!}\,(D-2i)\,.$$

The resulting formula for $\hat\sigma_D$ is originally due to V. Liskovets [Li]. See [S, §4] for discussion of symmetric LCD's.

Proof. We are looking for the orbits of the dihedral group[4]

$$D_{2D} := \langle \omega_D, z_D\rangle \;\subset\; S_{2D}\,,$$

where $\omega_D(i) := 2D + 1 - i$, $i \le i \le 2D$. We have

$$\#D_{2D} \;=\; \begin{cases} 4D & D > 1 \\ 2 & D = 1 \end{cases}$$

For $D > 1$ we have by Burnside's lemma

$$\hat\sigma_D = \frac{1}{4D} \sum_{i=0}^{2D-1} r_{z_D^i} + r_{\omega_D\cdot z_D^i}\,.$$

This is however also true for $D = 1$ (since we count both elements twice and divide by twice the group order).

Then

$$\sigma_D^{\text{sym}} = 2\hat\sigma_D - \sigma_D = \frac{1}{2D} \sum_{i=0}^{2D-1} r_{\omega_D\cdot z_D^i}$$

[4] This definition of the dihedral group is in fact orrect only for $D > 1$.

Lemma 3.7 *There is a bijection from $R_{\omega_D \cdot z_D^i}$ to $\Gamma_{D-i\%2,2}$.*

Using this lemma we get

$$\sigma_D^{\text{sym}} = \frac{1}{2}\{\, \gamma_{D,2} + \gamma_{D-1,2} \,\}$$

from which the formula follows by an easy transformation. □

Proof of lemma. The bijection from $\Gamma_{D-i\%2,2}$ to $R_{\omega_D \cdot z_D^i}$ can be described as follows. Note that $\omega_D \cdot z_D^i$ acts as a transposition on $\{1,\ldots,i\}$ and $\{i+1,\ldots,2D\}$.

Define a map $m_i : \{1,\ldots,D-i\%2\} \to \mathcal{P}(\{1,\ldots,2D\})$ by

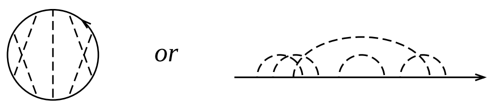
$$m_i(j) := \left\{ \begin{array}{ll} \{j, 1+i-j\} & j \le \lfloor \frac{i}{2} \rfloor \\ \{\lfloor \frac{i+1}{2} \rfloor + j, 1+2D+\lfloor \frac{i}{2} \rfloor - j\} & D-i\%2 \ge j > \lfloor \frac{i}{2} \rfloor \end{array} \right\}.$$

For a self-loop at basepoint j make a chord between the two points in $m_i(j)$. For a chord between basepoints j_1 and j_2 connect the four basepoints of $m_i(\{j_1, j_2\})$ in two pairs by connecting $\min(m_i(j_1))$ with $\min(m_i(j_2))$ for a chord labeled by 1 and to $\max(m_i(j_2))$ if the label is 2. Finally, if i is odd, connect basepoints $\lfloor \frac{i+1}{2} \rfloor$ and $\lfloor \frac{i+1}{2} \rfloor + D$.

One can check that this procedure indeed describes a bijection. □

3.6 Degenerate CD's and LCD's

Definition 3.8 *Let a chord diagram (or linearized chord diagram) be* degenerate*, if it has an* isolated *chord, i. e. one not crossed by any other, e. g. like in the following diagrams*



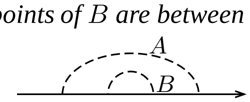
The count of FI relations is the same as the count of degenerate CD's or LCD's. Let

$$\omega_D := \#\{\text{ degenerate CD's of deg } D \}\ , \qquad \psi_D := \#\{\text{ degenerate LCD's of deg } D \}$$

and $\bar{\omega}_D$ and $\bar{\psi}_D$ the corresponding counts of non-degenerate CD's and LCD's.

For the explanation of our approach it will be helpful to introduce the following appealing notions.

Definition 3.9 *We will say that a chord A of an LCD (or GLCD)* encloses *another chord (or a self-loop) B (or B is* enclosed *by A, or B is* inside *of A), if the endpoints of B are between the endpoints of A.*



Conversely, we will say that B is appearing outside *of A if A does not enclose B (which does* not *mean that B encloses A !). A chord (with a certain property) will be called* minimal *(with this property) if it does not enclose another chord (with the same property). In the same way it will be called* maximal *if there is no other chord enclosing it.*

Another definition we will need later is the following one.

Definition 3.10 *The* length *of a chord A in an LCD is the by 1 augumented number of basepoints of other chords between the two basepoints of A. The length of a chord in a chord diagram will be the minimum of its lengths counted on both circle segments between its endpoints. E.g., the chord diagram in definition 3.8 has 4 chords of length 2 and one of length 5.*

We will start by counting degenerate LCD's of degree D. Applying the inclusion-exclusion principle [Ai, §2.4] and grouping by the number of minimal isolated chords on LCD's we get a recursive formula for ψ_D.

$$\psi_D = \sum_{i=1}^{D} (-1)^{i-1} \sum_{\substack{(j_1,\ldots,j_i,k) \\ j_l, k \ge 0 \\ \sum j_l + k = D-i}} \lambda_{k,i+1} \prod_{l=1}^{i} (\lambda_{j_l} - \psi_{j_l}) \tag{3.5}$$

and $\psi_0 = 0$. Here $\lambda_{k,i}$'s are the numbers of string link diagrams, introduced in lemma 3.2, i is the number of choices of minimal isolated chords, $j_1, \ldots, j_i$ are the degrees of the LCD's enclosed by the i chords, and k is the degree of the remaining string link diagram.

Using the characteristic series $P_{\bar\psi}$ and P_λ defined by

$$P_{\bar\psi}(x) := \sum_{i=0}^{\infty} \bar\psi_i \, x^i$$

and

$$P_\lambda(x,y) := \sum_{i,l=0}^{\infty} x^i y^l \, \lambda_{i,l+1} \,,$$

(3.5) can be rewritten more nicely as

$$P_{\bar\psi}(x) = P_\lambda(x, -x\, P_{\bar\psi}(x)) \,.$$

For determining ω_D we have to work a little harder. We will calculate the number $\tilde\gamma_{d,c}$ of GLCD's of degree d and cyclicity c, which produce[5] degenerate chord diagrams.

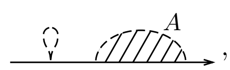
In the case $c = 1$ we have

$$\tilde\gamma_{d,1} := \left\{ \begin{array}{cc} \psi_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\} .$$

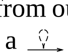
So, from now on let $c \geq 2$. We will distinguish 2 cases.

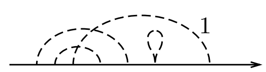
Case 1. All isolated chords in the CD come from a self-loop in the GLCD.

In this case we must have $c = 2$ and exactly one . If we cut the chord diagram just before the chord coming from the self-loop, we get (applying $\Phi^{-1}_{D,2}$ of theorem 3.4) a GLCD which looks like



,

where A is a non-degenerate LCD[6]. We see that such a GLCD is of odd degree. All other GLCD's producing the same GLCD are generated by the action of $\mathbf{Z}_d$ (described in §3.3) from our special one above. They can be described as follows: Put between the basepoints of a non-degenerate LCD a  and colour the chords by 1 if they enclose the self-loop and by 0 otherwise, e. g.

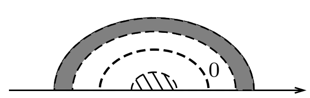


.

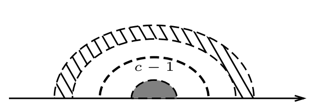
So for odd d there are $d \cdot \bar\psi_{(d-1)/2}$ such GLCD's.

Case 2. There is an isolated chord in the chord diagram coming from a chord in the GLCD.

Let $\eta_{c,d}$ be the number of such GLCD's. Take a minimal chord C in the GLCD producing an isolated chord. Then it must be coloured by 0 or $c-1$ (else its c copies would mutually intersect in the chord diagram). If it has label 0 then it only encloses 0-labeled chords, which have to build up a non-degenerate LCD.



If it is labeled $c-1$, we can use the $\mathbf{Z}_d$ action to transform it into a 0-labeled chord. This way we see that outside of a $c-1$-labeled chord there are no self-loops and the chords have a unique labelling: $c-1$ if they enclose C, and 0 otherwise. Furthermore, by forgetting the labels they build up a non-degenerate LCD.



[5] From now on we will always mean this in the sense described in the proof of theorem 3.4.
[6] From now on a gray filled part in a diagram stands for an *arbitrary* LCD and a shaded part for a *non-degenerate* LCD.

We will count GLCD's of both types by the inclusion-exclusion principle over minimal chords producing isolated chords, that is, we have to count a GLCD with $\geq k$ such chords, so that each GLCD with exactly n chords is counted $\binom{n}{k}$ times.

There are two cases.

Case 2.1. All k chosen minimal chords are labeled by 0. Let $\xi^k_{c,d}$ be the resulting number. We can calculate it by contracting the chords and taking into account the LCD's they enclose. So we have

$$\begin{aligned}\xi^k_{c,d} &= \sum_{(e_1,\dots,e_k)\geq 0} \prod_{j=1}^{k} \bar{\psi}_{e_j} \cdot \lambda^c_{d-2k-2\sum e_j,k+1} \\ &= \Big[\big(P_{\bar{\psi}}(x^2)\big)^k P_{\lambda^c_{*,k+1}}(x)\Big]_{d-2k} \text{ for } 0\leq 2k\leq d\,,\end{aligned}$$

where

$$\lambda^c_{e,d} := \binom{e+d-1}{e}\cdot\gamma_{e,c}$$

is the number of generalized string link diagrams of d strands, cyclicity c and degree e (with the obvious definition and counted by the same idea as in lemma 3.2) and $P_{\lambda^c_{*,k}}(x)$ is the characteristic series in x of $\lambda^c_{d,k}$ over the degree[7] d

$$P_{\lambda^c_{*,k}}(x) := \sum_{d=0}^{\infty} x^d \lambda^c_{d,k}\,.$$

Case 2.2. There are $k-1$ chords with colour 0 and one chord with colour $c-1$. Let $\bar{\xi}^k_{c,d}$ be this number. Such a GLCD we can describe by the GLCD outside of the $c-1$-coloured chord (whose degree we will call e) with a position marked between its basepoints (where the $c-1$-coloured chord and what it encloses is attached) and by the GLCD enclosed by the chord, where $k-1$ chords of colour 0 remain, and which has to be counted as in case 2.1. So

$$\begin{aligned}\bar{\xi}^k_{c,d} &= \sum_{e\geq 0}(2e+1)\,\bar{\psi}_e\cdot\xi^{k-1}_{c,d-(2e+2)} \\ &= \left[\frac{\partial}{\partial x}\big(x\cdot P_{\bar{\psi}}(x^2)\big)\cdot P_{\xi^{k-1}_{c,*}}(x)\right]_{d-2}.\end{aligned}$$

Let

$$\zeta^k_{c,d} := \xi^k_{c,d} + \bar{\xi}^k_{c,d}\,. \tag{3.6}$$

Now by the inclusion-exclusion principle we get

$$\eta_{c,d} = \sum_{k=1}^{\lfloor d/2\rfloor}(-1)^{k-1}\zeta^k_{c,d} \tag{3.7}$$

Putting it all together, we find that

$$\begin{aligned}\tilde{\gamma}_{d,2} &= \eta_{2,d} + \left\{\begin{array}{cc} d\,\bar{\psi}_{(d-1)/2} & \text{if } 2\not|d \\ 0 & \text{else}\end{array}\right\} \\ \tilde{\gamma}_{d,c} &= \eta_{c,d} \text{ for } c>2\,.\end{aligned}$$

Having obtained $\tilde{\gamma}_{d,c}$, we can now apply Burnside's lemma [HC, lemma 14.3 on p. 1058] and get

$$\omega_D = \frac{1}{2D}\sum_{d\cdot c=2D}\varphi(c)\,\tilde{\gamma}_{d,c}\,. \tag{3.8}$$

[7]By P with a subscript containing a '$*$' we will always denote the characteristic series of the expression in the subscript over the variable replacing the '$*$'.

3.7 Chord diagrams with chords of length 1

Let ω_D^1 be this number[8]. Determining it is nothing but a slight modification of what we did above.

Following the same strategy, first we compute $\bar{\psi}_D^1$.

We look at an LCD whose closure produces a chord diagram with an isolated minimal chord of length 1. Such an LCD either has such a chord or it has a maximal chord, which is isolated[9]. Once again we apply the inclusion-exclusion principle. For a fixed number k of chords we have again as in subsection 3.6 two cases.

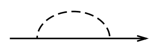
Case 1. All k chosen chords are minimal. By removing them we are left with a string link count.

Case 2. There are $k-1$ minimal chords and one maximal chord. By removing the maximal chord we get back to case 1.

So we get

$$\bar{\psi}_D^1 = \sum_{k=0}^{D} (-1)^k (\lambda_{D-k,k+1} + \lambda_{D-k,k}) . \tag{3.9}$$

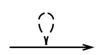
For $D = 1$ the formula gives $\psi_0^1 = -1$, since in



the chord is both of length 1 and maximal and is counted twice. So set $\psi_1^1 := 0$.

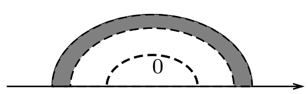
For the computation of $\tilde{\gamma}_{d,c}^1$ for $c \geq 2$ we make the same case distinction as above.

Case 1. There is an isolated chord on the chord diagram coming from a self-loop in the GLCD. This is possible in only one case – $c = 2$, $d = 1$ and the GLCD is

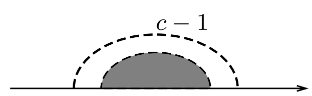
$\Big($  $, \ 2 \Big) .$

Case 2. All isolated chords in the chord diagram come from chords in the GLCD.

This enumeration process is as in subsection 3.6, but here we have no enclosed or enclosing LCD's of minimal or maximal chords. The GLCD's look like



and



.

We obtain

$$\begin{aligned} \xi_{c,d}^{k,1} &= \lambda_{d-2k,k+1}^c \text{ for } 0 \leq 2k \leq d \\ \bar{\xi}_{c,d}^{k,1} &= \xi_{c,d-2}^{k-1,1} = \lambda_{d-2k,k}^c \end{aligned}$$

and

$$\begin{aligned} \tilde{\gamma}_{d,1}^1 &= \left\{ \begin{array}{cc} \bar{\psi}_{d/2}^1 & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}_{d,2}^1 &= \eta_{2,d}^1 + \left\{ \begin{array}{cc} 1 & \text{if } d = 1 \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}_{d,c}^1 &= \eta_{c,d}^1 \text{ for } c > 2 \end{aligned}$$

with the analogous formulas as (3.6) and (3.7) for $\zeta_{c,d}^{k,1}$ and $\eta_{c,d}^1$. The formula for ω_D^1 is then the same as (3.8).

Remark 3.1 The sequence ω_D^1 (and probably also σ_D) was first calculated for $D \leq 9$ without a formula by direct enumeration by D. BAR-NATAN [BN2]. It appeared in the algorithm he uses to compute the dimension of the space of weight systems (see therein the table in section 6.1, 2nd last row).

[8] We will henceforth denote the equivalents of the symbols in subsection 3.6 by an additional superscript.

[9] i.e., not enclosed by *any* chord, not only by non-isolated chords!

3.8 Chord diagrams with isolated chords only

We will call such chord diagrams also *fully-degenerate* and will denote their number by ω_D^2.

This enumeration problem and the formula for it are classical. However, I include it here, because we will just see how easily it can be reproduced using our approach.

Let's start once again with the linear case.

It is a classical combinatorial fact, that the number of LCD's of degree D with isolated chords only is the Catalan number

$$\psi_D^2 \;=\; C_D \;:=\; \frac{(2D)!}{D!\,(D+1)!}\,.$$

This number is the number of ways to parenthesize $D+1$ factors in a non-associative algebra or the number of binary planar trees with a basepoint and $D+1$ leaves, as considered in [Lo2, §2.1][10].

To see this, group such LCD's by maximal chords and prove for the generating (or characteristic) series P_C

$$P_C(x) \;=\; \frac{1}{1-xP_C(x)}\,.$$

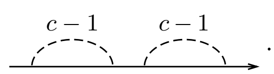
For calculating $\tilde{\gamma}_{d,c}^2$ for $c \geq 2$ make the following case distinction.

Case 1. There is a self-loop in the GLCD. The argument is analogous to the one in subsection 3.6. We have $c = 2$ and d odd, and the count is $d \cdot \psi^2_{(d-1)/2}$.

Case 2. There is no self-loop in the GLCD.

This means that d is even. By the same argument as in subsection 3.6 all chords must be labeled either by 0 or by $c-1$. Furthermore, the GLCD has the following two properties

- a chord of colour 0 encloses only chords of colour 0.
- for each 2 chords of colour $c-1$ one encloses the other, i. e. we never have something like this



Then distinguish once again 2 cases.

Case 2.1. All chords are coloured by 0. There are $\psi^2_{d/2}$ such GLCD's.

Case 2.2. There is a chord coloured by $c-1$.

Then by the above properties there exists a *unique* minimal chord coloured by $c-1$ and once given this the colouring of the others is uniquely determined. So such GLCD's correspond to LCD's with a distinguished chord, and their number is $\frac{d}{2} \cdot \psi^2_{d/2}$.

So

$$\eta^2_{c,d} \;=\; \left\{ \begin{array}{cc} \left(\frac{d}{2}+1\right)\psi^2_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\}.$$

Hence we get

$$\begin{aligned} \tilde{\gamma}^2_{d,1} &= \left\{ \begin{array}{cc} \psi^2_{d/2} & \text{if } 2|d \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}^2_{d,2} &= \eta^2_{2,d} + \left\{ \begin{array}{cc} d\,\psi^2_{(d-1)/2} & \text{if } 2 \not| d \\ 0 & \text{else} \end{array} \right\} \\ \tilde{\gamma}^2_{d,c} &= \eta^2_{c,d} \text{ for } c > 2 \end{aligned}$$

[10] These are planar trees with a root vertex of valence 2, internal vertices of valence 3 and leaves of valence 1 modulo isotopies in the plane, which preserve a distinguished order of the 2 edges adjacent to the root vertex.

and

$$\omega_D^2 = \frac{1}{2D} \sum_{d\cdot c=2D} \varphi(c)\,\tilde\gamma_{d,c}^2 \tag{3.10}$$

$$= \frac{1}{2D}\Big\{ \sum_{\substack{c\cdot d=D\\ c\geq 2}} \varphi(c)\,(d+1)\,C_d \,+\, C_D \Big\} + \left\{ \begin{array}{cc} \frac{1}{2}\,C_{(D-1)/2} & \text{if } 2\not| D \\ 0 & \text{else} \end{array} \right\},$$

which is the classical formula for the number of planar trees with $D+1$ nodes [Sl]. (If you like, find a direct bijection between the latter set and the set of fully degenerate chord diagrams.)

Remark 3.2 Using similar arguments it should also be possible to count the various kinds of Gauß diagrams (chord diagrams with oriented chords) [Po]. There we have to orient each chord in the GLCD and we have no self-loops.

3.9 Some computations

With the previous formulas it is not hard to compute the beginning of the various integer sequences above[11]. The first 10 values are given in the following table.

D	1	2	3	4	5	6	7	8	9	10
σ_D	1	2	5	18	105	902	9,749	127,072	1,915,951	32,743,182
σ_D^{sym}	1	2	5	16	53	206	817	3,620	16,361	80,218
$\hat\sigma_D$	1	2	5	17	79	554	5,283	65,346	966,156	16,411,700
ω_D	1	1	3	11	70	607	6,577	85,198	1,276,563	21,695,178
ω_D^1	1	1	3	11	69	602	6,531	84,737	1,271,143	21,623,667
ω_D^2	1	1	2	3	6	14	34	95	280	854

(If you like, find the two mutually (but not self-) symmetric chord diagrams in degree 4. Which is the only degenerate chord diagram of degree 5 with no chord of length 1?)

3.10 Asymptotics

A first fact to mention is the (not very surprising) observation

Lemma 3.11

$$\sigma_D \asymp \frac{(2D-1)!!}{2D}\,.$$

This is, the contribution to σ_D in (3.3) coming from $\gamma_{2D,1}=\lambda_D$ is the dominating one.

Proof. Using the bound

$$\gamma_{d,c} \leq \left(1+\sqrt{c}\right)^d (d-1)!!\,,$$

following directly from (3.2), and that the function $\sqrt[n]{1+\sqrt{n}}$ is monotonously decseasing for $n>0$ we obtain

$$\sum_{\substack{c|2D\\ c\geq 2}} \varphi(c)\,\gamma_{2D/c,c} \leq \left(1+\sqrt{2}\right)^D (2D-1)\left(2\left\lfloor\frac{D}{2}\right\rfloor-1\right)!!$$

$$= \left(\frac{1+\sqrt{2}}{\sqrt{2}}\right)^D \left(\frac{\sqrt{2}}{D}\right)^{D\%2} (2D-1)\,\frac{D!}{\lfloor\frac{D}{2}\rfloor!}\,.$$

[11] A MATHEMATICA™ package doing this is available on my WWW page.

Therefore

$$\frac{\sum_{c|2D,\, c\geq 2} \varphi(c)\, \gamma_{2D/c,c}}{\gamma_{2D,1}} \;\leq\; \frac{\left(2+\sqrt{2}\right)^{D}(2D-1)}{\binom{2D}{D}\left\lfloor \frac{D}{2}\right\rfloor !} \;\xrightarrow[D\to\infty]{}\; 0 \qquad \square$$

Something more interesting happens in the case ω_D^1, the number of chord diagrams with chords of length 1. Looking at (3.9) we see that we can write the ratio between λ_D and the k^{th} term in the sum on the r.h.s.

$$\frac{\lambda_{D-k,k+1} + \lambda_{D-k,k}}{\lambda_D} \;=\; \frac{1}{k!}\, P(D)\,,$$

where $P(D)$ is a polynomial fraction of degree 0 in D bounded above by 1 and converging to 1 for $D \to \infty$. This means that

$$\frac{\bar{\psi}_D^1}{\lambda_D} \;\xrightarrow[D\to\infty]{}\; \frac{1}{e}\,,$$

where e is the Euler number $2.71828\ldots$, and together with lemma 3.11 we get the same result for chord diagrams:

$$\frac{\bar{\omega}_D^1}{\lambda_D} \;\xrightarrow[D\to\infty]{}\; \frac{1}{e}\,.$$

Lemma 3.12 *Asymptotically* $\frac{1}{e}$ *of all chord diagrams and LCD's have no isolated chord (or isolated chord of length 1).*

In fact, it is an easy exercise to convince oneself that there are "very few" degenerate chord diagrams with no chord of length 1, that is

$$\frac{\omega_D - \omega_D^1}{\sigma_D} \;\xrightarrow[D\to\infty]{}\; 0\,.$$

Unfortunately, computing more carefully the difference

$$\frac{1}{e} - \frac{\bar{\psi}_D^1}{\lambda_D}$$

we see that the dominating term is

$$\frac{1}{2(2D-1)}\,,$$

so we cannot hope for a fast convergence.

Problem. At present I don't know the asymptotics of σ_D^{sym}.

However, although unimportant for our context, one can obtain the following alternative expression for it.

$$\sigma_D^{\text{sym}} \;=\; \frac{d^{D-1}}{dx^{D-1}}\Big((1+x)\, e^{x+x^2}\Big)(0)\,.$$

To see this, look at the nomalized generating series of $\gamma_{*,2}$

$$\tilde{P}_{\gamma_{*,2}}(x) \;:=\; \sum_{k=0}^{\infty} \frac{\gamma_{k,2}}{k!}\, x^k$$

and prove that $\tilde{P}_{\gamma_{*,2}}$ is a solution of the differential equation

$$x f'(x) \;=\; x f(x) + 2x^2 f(x), \qquad f(0) = 1\,.$$

4 Connected and tree-connected chord diagrams

Using the methods of section 3, in this section we discuss the enumeration of two more classes of chord diagrams.