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Brownian motions on metric graphs

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Brownian motions on a metric graph are defined. Their generators are characterized as Laplace operators subject to Wentzell boundary at every vertex. Conversely, given a set of Wentzell boundary conditions at the vertices of a metric graph, a Brownian motion is constructed pathwise on this graph so that its generator satisfies the given boundary conditions. © *2012 American Institute of Physics*. [http://dx.doi.org/10.1063/1.4714661]

Dedicated to Elliott Lieb on the occasion of his 80th birthday

I. INTRODUCTION AND MAIN RESULTS

Since the groundbreaking works of Bachelier,[2] Einstein,[21,22] and Smoluchowski,[86] the theory of the Brownian movement had been established as a central, recurrent theme in mathematics and physics.[87] In the sequel, the Brownian phenomenon stimulated the development of many important ideas and theories. A complete description of the history is beyond the scope of this introduction, but in keywords, we want to mention the following: The construction of Wiener space[90] and Wiener's approach of statistical mechanics and chaos,[91] Itô's theory of stochastic integration[39] and stochastic differential equations,[40] Lévy's analysis of the fine structure of Brownian motion and his theory of the Brownian local time,[61,62] Feynman's path integral[27] with its new view towards quantum mechanics, and Kac' work[44,45] on path integrals. Towards the middle of the last century, there were the works by Feller[24–26] and Itô–McKean[41,42] on Brownian motions on intervals (see also below), Gross' abstract Wiener spaces,[34] Nelson's work[65–68] on functional integration and on the relation between quantum and stochastic dynamics, giving new momentum to Euclidean and to constructive quantum field theory, e.g., Refs. 33, 78, 79, 81, and 83 and nonrelativistic quantum physics, e.g., Ref. 82. Further we want to mention the asymptotics of Wiener integrals and large deviation theory,[16,75] the theory of Dirichlet forms,[1,32,80] the development of the Malliavin[63,64] and Hida calculi,[36,37] and Bismuth's approach[8,9] to the Atiyah–Singer index theorem. In addition, there were important developments in other fields, such as engineering, biology or mathematical finance, which were triggered by the theory of Brownian motion.

The present article is directly linked to the above quoted works by Feller and Itô–McKean. So, we want to sketch these in little more detail. In his pioneering articles,[24–26] Feller raised the problem of characterizing and constructing all Brownian motions on a finite or on a semi-infinite interval. In the sequel, this problem stimulated very important research in the field of stochastic processes, and the problem of constructing all such Brownian motions found a complete solution in the work of Itô and McKean[41,42] via the combination of the theory of the local time of Brownian motion,[62] and the theory of (strong) Markov processes.[10,17–19,38] The central result of these investigations is

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that the most general Brownian motion on the half line $\mathbb{R}_+$ is determined by a generator, which is (one half times) the Laplace operator on $\mathbb{R}_+$ with *Wentzell boundary conditions* at the origin, i.e., linear combinations of the function value with the values of the first and second derivative at the origin (with coefficients satisfying certain restrictions, see below). Itô and McKean showed in Ref. 41—partly based on the ideas of Feller[24–26]—how to construct the paths of such motions: The boundary conditions are implemented by a combination of reflection at the origin with a slow down and killing, both on the scale of the local time at zero. The ideas contained this article became one of the roots of their highly influential book.[42]

In recent years, there has been a growing interest in *metric graphs*, that is, piecewise linear spaces with singularities formed by the vertices of the graph. Metric graphs arise naturally as models in many domains, such as physics, chemistry, computer science, and engineering to mention just a few—we refer the interested reader to Ref. 59 for a review of such models and for further references. Therefore, it is natural to extend Feller's problem to metric graphs. Stochastic processes, in particular Brownian motions and diffusions, on locally one-dimensional structures, notably on graphs and networks, have already been studied in a number of articles of which we want to mention[6,13,23,29–31,35,58] in this context.

In previous articles,[54–57] two of the current authors studied the self-adjointness of Laplace operators on metric graphs and discussed their spectra. This allowed a discussion of the associated quantum scattering matrices. Further properties of the semigroups generated by Laplace operators on metric graphs, including a Selberg–type trace formula and the problem whether these semigroups are positivity preserving or contractive, have been studied in Refs. 49 and 51. This is one of the motivations of our present study since semigroups with these properties typically show up in Markov processes. Below we will return to this point, see Remark 3.5. The wave equation on metric graphs and its finite propagation speed have been discussed in Ref. 53. For suitable Laplacians, free quantum fields on metric graphs satisfying the Klein-Gordon equation and Einstein causality were constructed in Ref. 76.

In Ref. 52, the authors have constructed the paths of all possible Brownian motions (in the sense defined below) on single vertex graphs using the well-known Walsh process[89,3] (see also Refs. 6, 72–74, and 85) as the starting point. Furthermore, the relation to the quantum mechanical scattering is discussed in detail there. The latter article provides an essential input for the construction of all possible Brownian motions on a general metric graph in the sense of Definition 2.1 (see below), which we carry out here.

The article is organized in the following way. In Sec. II we set up our framework and prove our main results: Theorem 2.5 characterizes all possible Brownian motions (in the sense of Definition 2.1) on a metric graph $\mathcal{G}$ in terms of Wentzell boundary conditions at the vertices. Conversely, Theorem 2.8 states that for every choice of a set of Wentzell boundary conditions at the vertices as described in Theorem 2.5, one can construct a Brownian motion on $\mathcal{G}$ implementing these conditions. Theorem 2.5 is proved in Sec. III. As a preparation of the proof of Theorem 2.8, we consider in Sec. IV the situation where one is given two metric graphs $\mathcal{G}_1$, $\mathcal{G}_2$ with Brownian motions X_2, X_2 in the sense of Definition 2.1 thereon. If one joins some of the external edges of $\mathcal{G}_1$ and $\mathcal{G}_2$ to form a new metric graph $\mathcal{G}$, it is shown how to construct the paths of a Brownian motion X on $\mathcal{G}$ by appropriately gluing the paths of X_1 and X_2 together. Theorem 2.8 is proved in Sec. V via the procedure of Sec. IV and the results in Ref. 52, where the paths of Brownian motions on star graphs are constructed with methods similar to those of Feller[24–26] and Itô–McKean.[41,42] The article is concluded in Sec. VI by a discussion of the inclusion of tadpoles. Furthermore, there are two appendices: one with a technical result on the crossover times, which is used in Sec. IV, the other about Feller semigroups and resolvents.

Given these results, it would be interesting to see whether known results for special cases of Brownian motion or diffusions on metric graphs can be extended to all Feller processes. For example, an arcsine law has been proved in Ref. 4 for the case of a Walsh process on a single vertex graph, for the case of a general metric graph, we refer to Refs. 7 and 15 (for a discussion of local time distributions see also Ref. 12). In a similar vein: What about occupation times on edges for the case of general (local) boundary conditions of the type (2.5) at the vertices? Can one say something about large deviations as done, for example, for Brownian motions without killing and more generally for conservative diffusion processes in?[30] What form does the Itô formula take in the case of a diffusion process on a metric graph with a generator subject to the boundary conditions (2.5)?

II. MAIN RESULTS

In the present article, we shall only treat *finite* metric graphs, and consider a metric graph $(\mathcal{G}, d)$ as being defined by a finite collection of finite or semi-infinite closed intervals, some of their endpoints—the *vertices* of the graph—being identified. See Figure 1 for an example of a simple, typical metric graph. The metric d is then defined in the canonical way as the length of a shortest path between two points along the *edges* (formed by the intervals), and the length along each edge is measured with the usual metric on the real line. For a formal definition of metric graphs within the context of graph theory, we refer the interested reader, e.g., to Refs. 54 and 55. Within that context, our definition above means that we identify—as we may without any loss of generality—an abstract metric graph with its *geometric graph* (see, e.g., Ref. 43). Moreover, in the sequel, it will often be convenient and without any danger of confusion to identify an edge of a metric graph with the corresponding interval of the real line. Edges isomorphic to $\mathbb{R}_+$ are called *external*, while those isomorphic to a finite interval—that is, those edges connecting two vertices—are called *internal*. The set of vertices of $\mathcal{G}$ is denoted by V, the set of internal edges by $\mathcal{I}$ and the set of external edges by $\mathcal{E}$. Moreover, we set $\mathcal{L} = \mathcal{I} \cup \mathcal{E}$. The combinatorial structure of the graph $\mathcal{G}$ is described by a map δ from $\mathcal{L}$ into $V \cup (V \times V)$, which associates with every internal edge i an ordered pair $\bigl(\partial^-(i), \partial^+(i)\bigr) \in V \times V$, $\partial^-(i)$ is called the *initial vertex* of i while $\partial^+(i)$ is its *terminal vertex*. If $i \in \mathcal{I}$ is isomorphic to the interval $[a, b]$ then $\partial^-(i)$ corresponds to a, while $\partial^+(i)$ corresponds to b. An external edge e is mapped by ∂ to $\partial(e) \in V$, which is the vertex to which e is incident, and also in this case, we call the vertex the *initial vertex* of e.

For the definition of a Brownian motion on the metric graph $(\mathcal{G}, d)$, we take a standpoint similar to the one that of Knight's[48] for the semi-line or a finite interval:

Definition 2.1: A Brownian motion on a metric graph $(\mathcal{G}, d)$ *is a diffusion process* $(X_t,\ t \in \mathbb{R}_+)$ *such that when X starts on an edge e of* $\mathcal{G}$ *then the process X with absorption in the vertex, vertices respectively, to which e is incident is equivalent to a standard one-dimensional Brownian motion on the interval e with absorption in the endpoint(s) of e.*

Remarks 2.2: By saying that X is a *diffusion process,* we mean that X is a normal, strong Markov process (in the sense of Ref. 11), almost surely with paths which are càdlàg and continuous on $[0, \zeta)$, where ζ is the lifetime of X. We shall always assume that the filtration for X satisfies the "usual conditions." With the help of the well-known first passage time formula (e.g., Ref. 70 or 42) for the resolvent of X, it is not hard to show as in Ref. 48 that every Brownian motion on a metric graph $\mathcal{G}$ is a Feller process.

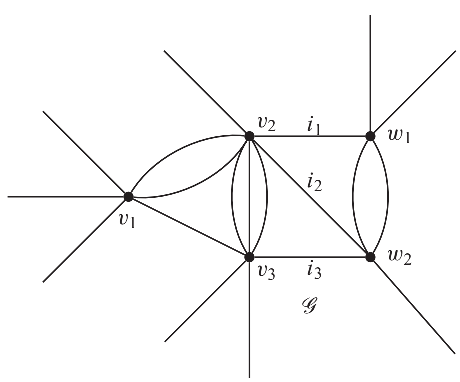

FIG. 1. A metric graph $\mathcal{G}$ with 5 vertices, 9 external, and 11 internal edges.

The first crucial problem is then to characterize the behavior of the stochastic process when it reaches one of the vertices of the graph $\mathcal{G}$, or in other words, the characterization of the boundary conditions at the vertices of the Laplace operator, which generates the stochastic process. We want to mention in passing that in an L^2-setting all boundary conditions for Laplace operators on $\mathcal{G}$, which make them self-adjoint have been characterized in Refs. 54 and 55. The first main result of the present article is Feller's theorem for metric graphs. In order to state this theorem, we have to introduce some notation.

The Banach space of real valued, continuous functions on $\mathcal{G}$ vanishing at infinity, equipped with the sup-norm, is denoted by $C_0(\mathcal{G})$. We let Δ denote a universal cemetery point for all stochastic processes considered, and make the usual convention that every $f \in C_0(\mathcal{G})$ is extended to $\mathcal{G} \cup \{\Delta\}$ by setting $f(\Delta) = 0$.

Consider the generator A of X on $C_0(\mathcal{G})$ with domain $\mathcal{D}(A)$. Define the space $C_0^2(\mathcal{G})$ to consist of those functions f in $C_0(\mathcal{G})$, which are twice continuously differentiable in the *open interior* $\mathcal{G}^\circ = \mathcal{G} \setminus V$ of $\mathcal{G}$, and which are such that their second derivative f'' extends from $\mathcal{G}^\circ$ to a function in $C_0(\mathcal{G})$.

The next lemma, which can be proved with the fundamental theorem of calculus and the mean value theorem, states some of the properties of functions in $C_0^2(\mathcal{G})$. $\mathcal{L}(v)$ denotes the set of edges incident with $v \in V$.

Lemma 2.3: Assume that f belongs to $C_0^2(\mathcal{G})$, and consider $v \in V$, $l \in \mathcal{L}(v)$. Then the inward directional derivatives $f^{(i)}(v_l)$, $i = 1, 2$, of f of first and second order at v in direction of the edge l exist, and

$$f'(v_l) = \begin{cases} \lim\limits_{\xi \to v,\, \xi \in l^\circ} f'(\xi), & \text{if } v \text{ is an initial vertex of } l, \\ -\lim\limits_{\xi \to v,\, \xi \in l^\circ} f'(\xi), & \text{if } v \text{ is a terminal vertex of } l, \end{cases} \tag{2.1}$$

$$f''(v_l) = \lim_{\xi \to v,\, \xi \in l^\circ} f''(\xi) \tag{2.2}$$

hold true. Moreover, f' (defined on $\mathcal{G}^\circ$) vanishes at infinity.

Remark 2.4: If $f \in C_0^2(\mathcal{G})$ then by definition of $C_0^2(\mathcal{G})$, $f''(v_k) = f''(v_l)$ for every $v \in V$, and all $k, l \in \mathcal{L}(v)$, and we shall simply write $f''(v)$. On the other hand, in general $f'(v_k) \neq f'(v_l)$ for $k \neq l$.

Let $V_{\mathcal{L}}$ denote the subset of $V \times \mathcal{L}$ given by

$$V_{\mathcal{L}} = \big\{(v, l),\ v \in V \text{ and } l \in \mathcal{L}(v)\big\}.$$

We shall also write v_l for $(v, l) \in V_{\mathcal{L}}$. Consider data of the following form:

$$\begin{aligned} a &= (a_v,\ v \in V) \in [0, 1)^V, \\ b &= (b_{v_l},\ v_l \in V_{\mathcal{L}}) \in [0, 1]^{V_{\mathcal{L}}}, \\ c &= (c_v,\ v \in V) \in [0, 1]^V, \end{aligned} \tag{2.3}$$

subject to the condition

$$a_v + \sum_{l \in \mathcal{L}(v)} b_{v_l} + c_v = 1, \qquad \text{for every } v \in V. \tag{2.4}$$

Define a subspace $\mathcal{H}_{a,b,c}$ of $C_0^2(\mathcal{G})$ as the space of those functions f in $C_0^2(\mathcal{G})$, which at every vertex $v \in V$ satisfy the *Wentzell boundary condition*

$$a_v f(v) - \sum_{l \in \mathcal{L}(v)} b_{v_l} f'(v_l) + \frac{1}{2} c_v f''(v) = 0. \tag{2.5}$$

Now we can state our first main result:

Theorem 2.5: (Feller's theorem for metric graphs): *Let X be a Brownian motion on $\mathcal{G}$, and let A be its generator on $C_0(\mathcal{G})$ with domain $\mathcal{D}(A)$. Then there are a, b, c as in* (2.3), (2.4), *so that* $\mathcal{D}(A) = \mathcal{H}_{a,b,c}$. *For* $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.

Remark 2.6: The boundary conditions in $\mathcal{H}_{a,b,c}$ are *local* in the sense that only *one vertex* enters each of the conditions (2.5). This is a direct consequence of the path properties of X, namely, of the condition that the only jumps X may have are those from $\mathcal{G}$ (actually from a vertex) to the cemetery point.

Remark 2.7: Boundary conditions with $a_v = 0 = c_v$ are often called *standard boundary conditions* (see, e.g., Refs. 56 and 59) giving rise to what is called a *skew Brownian motion*,[42] for a recent survey see, e.g., Ref. 60 Killing occurs when $a_v \neq 0$, and Ref. 52 provides a detailed discussion of the process for single vertex graphs. When $a_v = 0$, the process is conservative and has been studied extensively in Refs. 29 and 30.

Our second main result is converse of Theorem 2.5, namely,

Theorem 2.8: *For any choice of the data as in* (2.3), (2.4), *there is a Brownian motion X on the metric graph $\mathcal{G}$ so that its generator A has $\mathcal{H}_{a,b,c}$ as its domain.*

In order to rephrase the statements of Theorems 2.5 and 2.8 in a concise way, we bring in some additional notation. With a slight abuse of language, we shall also call any quadruple $\mathbb{X} = (\Omega, \mathcal{A}, P, X)$ a *Brownian motion on* $\mathcal{G}$ whenever $(\Omega, \mathcal{A}, P)$ is a complete probability space, and $X = (X_t,\ t \in \mathbb{R})$ defined thereon is a Brownian motion on $\mathcal{G}$ as in Definition 2.1. $\mathcal{X}(\mathcal{G})$ denotes the set of all Brownian motions in this sense, subject to the equivalence relation, which is defined by equality of all finite dimensional distributions. In $\mathbb{R}^{n+1}$, consider the (compact, convex) n-simplex

$$\sigma^n = \left\{ x \in \mathbb{R}^{n+1},\ x_i \geq 0, \sum_{i=1}^{n+1} x_i = 1 \right\}.$$

Let σ_0^n be the simplex σ^n, with the point $(1, 0, \ldots, 0)$ removed

$$\sigma_0^n = \sigma^n \setminus \{(1, 0, \ldots, 0)\}.$$

It is still convex but not closed. Given a fixed but arbitrary ordering of $\mathcal{L}(v)$ any triple $\big(a_v, (b_{v_l}, l \in \mathcal{L}(v)), c_v\big)$ satisfying (2.4) can be viewed as an element in $\sigma_0^{n(v)+1}$ with $n(v) = |\mathcal{L}(v)|$. With $N(\mathcal{G}) = \sum_v (n(v) + 2)$ set

$$\Sigma(\mathcal{G}) = \bigtimes_{v \in V} \sigma_0^{n(v)+1} \subset \mathbb{R}^{N(\mathcal{G})}.$$

Let ι be the mapping defined via Theorem 2.5 by associating to every Brownian motion $\mathbb{X}$ the data $(a, b, c) \in \Sigma(\mathcal{G})$. Since any two Brownian motions on $\mathcal{G}$, which have the same finite dimensional distributions, define the same semigroup, and therefore have the same generator, it follows that ι maps these to the same data, that is, ι can be viewed as mapping from $\mathcal{X}(\mathcal{G})$ to $\Sigma(\mathcal{G})$. Theorem 2.8 states that ι is surjective. To see its injectivity, suppose that $[\mathbb{X}_1]$, $[\mathbb{X}_2]$ are different elements in $\mathcal{X}(\mathcal{G})$, where $[\mathbb{X}_i]$, $i = 1, 2$, denotes the equivalence class of a representative $\mathbb{X}_i$. Assume that $\iota([\mathbb{X}_1]) = \iota([\mathbb{X}_2])$. By Theorem 2.5, the generator A_i, $i = 1, 2$, of $\mathbb{X}_i$ is uniquely determined by the data $\iota([\mathbb{X}_i])$, and therefore we get $A_1 = A_2$. It follows, that $\mathbb{X}_1$ and $\mathbb{X}_2$ define the same semigroup, and therefore all their finite dimensional distributions coincide, which is a contradiction. Thus, we have proved that ι is a bijection from $\mathcal{X}(\mathcal{G})$ onto $\Sigma(\mathcal{G})$:

Corollary 2.9: The set $\mathcal{X}(\mathcal{G})$ of all Brownian motions on $\mathcal{G}$ is in one-to-one correspondence with the set $\Sigma(\mathcal{G})$.

III. PROOF OF THEOREM 2.5

The following notation will be useful throughout this article: If ξ is a point in $\mathcal{G}^\circ = \mathcal{G} \setminus V$, then it is in one-to-one correspondence with its *local coordinates* (l, x), where $l \in \mathcal{L}$ is the edge to which ξ belongs, while x is the point corresponding to ξ in the interval to which l is isomorphic. Then we simply write $\xi = (l, x)$. If f is a function on the graph $\mathcal{G}$, we shall also denote $f(\xi)$ by $f(l, x)$ or $f_l(x)$.

We denote by $U = (U_t,\ t \in \mathbb{R}_+)$ the semigroup generated by a Brownian motion X on $\mathcal{G}$ acting on the Banach space $B(\mathcal{G})$ of bounded measurable functions on $\mathcal{G}$, equipped with the sup-norm, that is, for $f \in \mathbb{B}(\mathcal{G})$,

$$U_t f(\xi) = E_\xi\big(f(X_t)\big), \qquad t \in \mathbb{R}_+,\ \xi \in \mathcal{G}.$$

Clearly, U is a positivity preserving contraction semigroup. In the sequel, we shall notationally not distinguish between the semigroup U acting on $B(\mathcal{G})$ and its restriction to the subspace $C_0(\mathcal{G})$ of $B(\mathcal{G})$.

The proof of the following lemma can be taken over with minor modifications from the standard literature, e.g., from Ref. [48, Chap. 6.1]. Therefore, it is omitted here.

Lemma 3.1: For every Brownian motion X on the metric graph $\mathcal{G}$, the generator A of its semigroup U acting on $C_0(\mathcal{G})$ has a domain $\mathcal{D}(A)$ contained in $C_0^2(\mathcal{G})$. Moreover, for every $f \in \mathcal{D}(A)$, $Af = 1/2 f''$.

The preceding lemma implies the second statement of Theorem 2.5. The proof of the first statement of Theorem 2.5 has two rather distinct parts, and therefore we split it by proving the following two lemmas:

Lemma 3.2: Suppose that X is a Brownian motion on a metric graph $\mathcal{G}$, and that $\mathcal{D}(A)$ is the domain of the generator A of its semigroup. Then there are a, b, c as in (2.3), (2.4), *so that $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$.*

Lemma 3.3: Suppose that A is the generator of a Brownian motion X on $\mathcal{G}$ with domain $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ for some a, b, c as in (2.3), (2.4). *Then $\mathcal{D}(A) = \mathcal{H}_{a,b,c}$.*

Proof of Lemma 3.2: Our proof follows the one in Ref. [48, Chap. 6.1] quite closely—actually, it is sufficient to consider a special case of the proof given there.

We show that for every vertex $v \in V$ there are constants $a_v \in [0, 1)$, $b_{v_l} \in [0, 1]$, $l \in \mathcal{L}(v)$, $c_v \in [0, 1]$ satisfying (2.4), and such that all f in the domain $\mathcal{D}(A)$ of the generator satisfy the boundary condition (2.5). To this end, we let $f \in \mathcal{D}(A)$, fix a vertex $v \in V$, and compute $Af(v)$.

Consider the exit time from v, i.e., the stopping time $S_v = H(\mathcal{G}^\circ)$, where for any subset $M \subset \mathcal{G}$, $H(M) \equiv H_M$ denotes the hitting time of M. It is well known (e.g., Refs. 20, 48, and 71) that because of the strong Markov property of X, S_v is under P_v exponentially distributed with a rate $\beta_v \in [0, +\infty]$. Consequently, we discuss three cases:

Case $\beta_v = 0$: X is absorbed at v, i.e., v is a *trap*. Thus, $U_t f(v) = f(v)$ for all $t \geq 0$. Consequently, $Af(v) = 0$, and therefore $1/2\, f''(v) = 0$. Thus, f satisfies the boundary condition (2.5) at v with $a_v = 0$, $c_v = 1$, and $b_{v_l} = 0$ for all $l \in \mathcal{L}(v)$.

Case $0 < \beta_v < +\infty$: In this case, the process stays at v P_v–almost surely for a strictly positive, finite moment of time, i.e., v is *exponentially holding*. It is well known (cf., e.g., Refs. [48, p. 154] and [71, p. 104, Proposition 3.13]) that then the process has to leave v by a jump, and by our assumption of path continuity on $[0, \eta)$, the process has to jump to the cemetery Δ. Therefore, we get for $t > 0$, $U_t f(v) = \exp(-\beta t) f(v)$, and thus $Af(v) + \beta f(v) = 0$, and the boundary condition (2.5) holds for the choice

$$a_v = \frac{\beta}{1+\beta}, \quad c_v = \frac{1}{1+\beta}, \quad b_{v_l} = 0,\ l \in \mathcal{L}(v). \tag{3.1}$$

Case $\beta_v = +\infty$: In this case, the X leaves the vertex v immediately, and it begins a Brownian excursion into one of the edges incident with the vertex v. In particular, v is not a trap. Therefore, we may compute $Af(v)$ in Dynkin's form, e.g., Refs. [18, p. 140, ff] and [42, p. 99]. For $\epsilon > 0$ let $H_{v,\epsilon}$ denote the hitting time of the complement of the ball $B_\epsilon(v)$ of radius ϵ around v. Then

$$Af(v) = \lim_{\epsilon\downarrow 0} \frac{E_v\Big(f\big(X(H_{v,\epsilon})\big)\Big) - f(v)}{E_v(H_{v,\epsilon})}. \tag{3.2}$$

Now

$$\begin{aligned} E_v\Big(f\big(X(H_{v,\epsilon})\big)\Big) &= \sum_{l\in\mathcal{L}(v)} f_l(\epsilon)\, P_v\big(X(H_{v,\epsilon})\in l\big) + f(\Delta)\, P_v\big(X(H_{v,\epsilon}) = \Delta\big) \\ &= \sum_{l\in\mathcal{L}(v)} f_l(\epsilon)\, P_v\big(X(H_{v,\epsilon})\in l\big), \end{aligned}$$

where the last equality follows from $f(\Delta) = 0$. Let us denote

$$r_l(\epsilon) = \frac{P_v\big(X(H_{v,\epsilon})\in l\big)}{E_v(H_{v,\epsilon})},\ l\in\mathcal{L}(v), \qquad r_\Delta(\epsilon) = \frac{P_v\big(X(H_{v,\epsilon}) = \Delta\big)}{E_v(H_{v,\epsilon})},$$

$$K(\epsilon) = 1 + r_\Delta(\epsilon) + \epsilon \sum_{l\in\mathcal{L}(v)} r_l(\epsilon).$$

The continuity of the paths of X up to the lifetime ζ yields

$$\sum_{l\in\mathcal{L}(v)} P_v\big(X(H_{v,\epsilon})\in l\big) + P_v\big(X(H_{v,\epsilon}) = \Delta\big) = 1,$$

and therefore Eq. (3.2) can be rewritten as

$$\lim_{\epsilon\downarrow 0}\Big(Af(v) + r_\Delta(\epsilon) f(v) - \sum_{l\in\mathcal{L}(v)} r_l(\epsilon)\big(f_l(\epsilon) - f(v)\big)\Big) = 0.$$

Since for all $\epsilon > 0$, $K(\epsilon)^{-1} \le 1$, it follows that

$$\lim_{\epsilon\downarrow 0}\Big(\frac{1}{K(\epsilon)}\, Af(v) + \frac{r_\Delta(\epsilon)}{K(\epsilon)}\, f(v) - \sum_{l\in\mathcal{L}(v)} \frac{\epsilon\, r_l(\epsilon)}{K(\epsilon)}\, \frac{f_l(\epsilon) - f(v)}{\epsilon}\Big) = 0,$$

which by Lemma 3.1, we may rewrite as

$$\lim_{\epsilon\downarrow 0}\Big(a_v(\epsilon) f(v) + \frac{1}{2}\, c_v(\epsilon) f''(v) - \sum_{l\in\mathcal{L}(v)} b_{v_l}(\epsilon)\, \frac{f_l(\epsilon) - f(v)}{\epsilon}\Big) = 0,$$

where we have introduced the non-negative quantities

$$a_v(\epsilon) = \frac{r_\Delta(\epsilon)}{K(\epsilon)}, \quad c_v(\epsilon) = \frac{1}{K(\epsilon)}, \quad b_{v_l}(\epsilon) = \frac{\epsilon\, r_l(\epsilon)}{K(\epsilon)},\ l\in\mathcal{L}(v).$$

Observe that for every $\epsilon > 0$,

$$a_v(\epsilon) + c_v(\epsilon) + \sum_{l\in\mathcal{L}(v)} b_{v_l}(\epsilon) = 1.$$

Therefore, every sequence $(\epsilon_n,\ n\in\mathbb{N})$ with $\epsilon_n > 0$ and $\epsilon_n\downarrow 0$ has a subsequence so that $a_v(\epsilon)$, $c_v(\epsilon)$, and $b_{v_l}(\epsilon)$, $l\in\mathcal{L}(v)$, converge along this subsequence to numbers a_v, c_v, and b_{v_l}, respectively in [0, 1], and the relation (2.4) holds true. It is not hard to check that for every $f\in C_0^2(\mathcal{G})$

$$\frac{f_l(\epsilon) - f(v)}{\epsilon}$$

converges with $\epsilon\downarrow 0$ to $f'(v_l)$, and therefore we obtain that for every vertex $v\in V$, $f\in\mathcal{D}(A)$ satisfies the boundary condition (2.5) with data a, b, c as in (2.3), (2.4). □

Before we can prove Lemma 3.3, we have to introduce some additional formalism.

We define the subspace $C_0^{0,2}(\mathcal{G})$ of functions f in $C_0(\mathcal{G})$, which are twice continuously differentiable on $\mathcal{G}^\circ$, such that f'' (as defined on $\mathcal{G}^\circ$) vanishes at infinity, and furthermore for every $v \in V$ and all $l \in \mathcal{L}$ the limit

$$\lim_{\xi \to v,\, \xi \in l^\circ} f''(\xi)$$

exists. Similarly, as in the statement of Lemma 2.3, the last limit is equal to the second order derivative $f''(v_l)$ of f at v in direction of l.

For given data a, b, c as in (2.3), (2.4), it will be convenient to consider $\mathcal{H}_{a,b,c}$ equivalently as being the subspace of $C_0^{0,2}(\mathcal{G})$ so that for its elements f at every $v \in V$ the boundary conditions (2.5) as well as the boundary condition

$$f''(v_l) = f''(v_k), \qquad \text{for all } l,\ k \in \mathcal{L}(v) \tag{3.3}$$

hold true. Relation (3.3) is just another way to express that f'' extends continuously from $\mathcal{G}^\circ$ to $\mathcal{G}$.

We consider the sets V, $\mathcal{E}$, and $\mathcal{I}$ as being ordered in some arbitrary way. With the convention that in $\mathcal{L}$ the elements of $\mathcal{E}$ come first, this induces also an order relation on $\mathcal{L}$.

Suppose that $f \in C_0^{0,2}(\mathcal{G})$. With the given ordering of $\mathcal{E}$ and $\mathcal{I}$, we define the following column vectors of length $|\mathcal{E}| + 2|\mathcal{I}|$:

$$f(V) = \Big(\big(f_e(0),\ e \in \mathcal{E}\big), \big(f_i(0),\ i \in \mathcal{I}\big), \big(f_i(\rho_i)\, i \in \mathcal{I}\big)\Big)^t,$$
$$f'(V) = \Big(\big(f_e'(0),\ e \in \mathcal{E}\big), \big(f_i'(0),\ i \in \mathcal{I}\big), \big(-f_i'(\rho_i)\, i \in \mathcal{I}\big)\Big)^t,$$
$$f''(V) = \Big(\big(f_e''(0),\ e \in \mathcal{E}\big), \big(f_i''(0),\ i \in \mathcal{I}\big), \big(f_i''(\rho_i)\, i \in \mathcal{I}\big)\Big)^t,$$

where the superscript "t" indicates transposition. Moreover, here and below we assume without loss of generality that the internal edge $i \in \mathcal{I}$ is isomorphic to the interval $[0, \rho_i]$, $\rho_i > 0$.

We want to write the boundary conditions (2.5), (3.3) in a compact way. To this end, we introduce the following order relation on $V_\mathcal{L}$: For v_l, $v'_{l'} \in V_\mathcal{L}$, we set $v_l \preceq v'_{l'}$ if and only if $v \prec v'$ or $v = v'$ and $l \preceq l'$ (where for V and $\mathcal{L}$, we use the order relations introduced above). For f, as above set

$$\tilde{f}(V) = \big(f(v_l),\ v_l \in V_\mathcal{L}\big)^t,$$
$$\tilde{f}'(V) = \big(f'(v_l),\ v_l \in V_\mathcal{L}\big)^t,$$
$$\tilde{f}''(V) = \big(f''(v_l),\ v_l \in V_\mathcal{L}\big)^t.$$

Then there exists a permutation matrix P so that

$$\tilde{f}(V) = Pf(V), \qquad \tilde{f}'(V) = Pf'V), \qquad \tilde{f}''(V) = Pf''(V).$$

In particular, P is an orthogonal matrix, which has in every row and in every column exactly one entry equal to one while all other entries are zero.

For every $v \in V$, we define the following $|\mathcal{L}(v)| \times |\mathcal{L}(v)|$ matrices:

$$\tilde{A}(v) = \begin{pmatrix} a_v & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\tilde{B}(v) = \begin{pmatrix} -b_{v_{l_1}} & -b_{v_{l_2}} & -b_{v_{l_3}} & \cdots & -b_{v_{l_{|\mathcal{L}(v)|}}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$\tilde{C}(v) = \begin{pmatrix} 1/2\, c_v & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & -1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 \end{pmatrix},$$

where we have labeled the elements in $\mathcal{L}(v)$ in such a way that in the above defined ordering, we have $l_1 \prec l_2 \prec \cdots \prec l_{|\mathcal{L}(v)|}$. Observe that $\tilde{C}(v)$ is invertible if and only if $c_v \neq 0$. Define block matrices $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ by

$$\tilde{A} = \bigoplus_{v\in V} A(v), \quad \tilde{B} = \bigoplus_{v\in V} B(v), \quad \tilde{C} = \bigoplus_{v\in V} C(v).$$

Then we can write the boundary conditions (2.5), (3.3) simultaneously for all vertices as

$$\tilde{A}\tilde{f}(V) + \tilde{B}\tilde{f}'(V) + \tilde{C}\tilde{f}''(V) = 0. \tag{3.4}$$

Consequently, the boundary conditions can equivalently be written in the form

$$Af(V) + Bf'(V) + Cf''(V) = 0, \tag{3.5}$$

with

$$A = P^{-1}\tilde{A}P, \quad B = P^{-1}\tilde{B}P, \quad C = P^{-1}\tilde{C}P. \tag{3.6}$$

We bring in the following two matrix-valued functions on the complex plane:

$$\hat{Z}_\pm(\kappa) = A \pm \kappa B + \kappa^2 C, \qquad \kappa \in \mathbb{C}. \tag{3.7}$$

Lemma 3.4: There exists $R > 0$ so that for all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$, the matrices $\hat{Z}_\pm(\kappa)$ are invertible, and there are constants $C, p > 0$ so that

$$\|\hat{Z}_\pm(\kappa)^{-1}\| \leq C\,|\kappa|^p, \qquad |\kappa| \geq R. \tag{3.8}$$

Proof of Lemma 3.4: Since we have

$$\hat{Z}_\pm(\kappa) = P^{-1}\big(\tilde{A} \pm \kappa\tilde{B} + \kappa^2\tilde{C}\big)P \tag{3.9}$$

for an orthogonal matrix P, for the proof of the first statement, it suffices to show that there exists $R > 0$ such that

$$\tilde{A} \pm \kappa \tilde{B} + \kappa^2 \tilde{C}$$

are invertible for complex κ outside of the open ball of radius R. For this, in turn, it suffices to show that for every vertex $v \in V$ the matrices

$$\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)$$

$$= \begin{pmatrix} a_v \pm \kappa b_{v_{l_1}} + \kappa^2/2\, c_v & \pm\kappa b_{v_{l_2}} & \pm\kappa b_{v_{l_3}} & \pm\kappa b_{v_{l_4}} & \cdots & \pm\kappa b_{v_{l_{|\mathcal{L}(v)|}}} \\ \kappa^2 & -\kappa^2 & 0 & 0 & \cdots & 0 \\ 0 & \kappa^2 & -\kappa^2 & 0 & \cdots & 0 \\ 0 & 0 & \kappa^2 & -\kappa^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\kappa^2 \end{pmatrix}$$

are invertible for all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$. An elementary calculation gives

$$\det\bigl(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)\bigr) = \Bigl(a_v \pm \kappa \sum_{l \in \mathcal{L}(v)} b_{v_l} + \frac{\kappa^2}{2} c_v\Bigr)\bigl(-\kappa^2\bigr)^{|\mathcal{L}(v)|-1}.$$

The choices $\kappa = \pm 1$ together with condition (2.4) show that the polynomial of second order in κ in the first factor on the right-hand side does not vanish identically. Therefore, it is non-zero in the exterior of an open ball with some radius $R_v > 0$. Hence, we obtain the first statement for the choice $R = \max_{v \in V} R_v$. Moreover, from the calculation of the determinants above, we also get for every $v \in V$ and all $\kappa \in \mathbb{C}$ with $|\kappa| \geq R$ an estimate of the form

$$\Bigl|\det\bigl(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)\bigr)\Bigr|^{-1} \leq \text{const.} \tag{3.10}$$

Thus, using the co-factor formula for

$$\bigl(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)\bigr)^{-1}$$

we find with (3.10) the estimate

$$\bigl\|\bigl(\tilde{A}(v) \pm \kappa \tilde{B}(v) + \kappa^2 \tilde{C}(v)\bigr)^{-1}\bigr\| \leq C_v |\kappa|^{p_v}, \qquad |\kappa| \geq R,$$

for some constants C_v, $p_v > 0$. Consequently, we get

$$\bigl\|\bigl(\tilde{A} \pm \kappa \tilde{B} + \kappa^2 \tilde{C}\bigr)^{-1}\bigr\| \leq C |\kappa|^{p}, \qquad |\kappa| \geq R,$$

for some constants C, $p > 0$, and by (3.9), we have proved inequality (3.8). □

With these preparations, we can enter the

Proof of Lemma 3.3: Let the data a, b, c be given as in (2.3), (2.4). We have to show that the inclusion $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ is not strict. Assume to the contrary that the inclusion $\mathcal{D}(A) \subset \mathcal{H}_{a,b,c}$ is strict. We will derive a contradiction.

Let $R = (R_\lambda, \lambda > 0)$ be the resolvent of A. Then for every $\lambda > 0$, R_λ is a bijection from $C_0(\mathcal{G})$ onto $\mathcal{D}(A)$, that is, R_λ^{-1} is a bijection from $\mathcal{D}(A)$ onto $C_0(\mathcal{G})$. For $\lambda > 0$, consider the linear mapping $H_\lambda\colon f \mapsto \lambda f - 1/2 f''$ from $\mathcal{H}_{a,b,c}$ to $C_0(\mathcal{G})$. On $\mathcal{D}(A)$, this mapping coincides with the bijection R_λ^{-1} from $\mathcal{D}(A)$ onto $C_0(\mathcal{G})$. Therefore, our assumption entails that H_λ cannot be injective. Hence, for any $\lambda > 0$, there exists $f(\lambda) \in \mathcal{H}_{a,b,c}, f(\lambda) \neq 0$, with

$$H_\lambda f(\lambda) = \lambda f(\lambda) - \frac{1}{2} f''(\lambda) = 0. \tag{3.11}$$

We will show that $f(\lambda) \in \mathcal{H}_{a,b,c}$ satisfying (3.11) can only hold when $f(\lambda) = 0$ on $\mathcal{G}$. It will be convenient to change the variable λ to $\kappa = \sqrt{2\lambda}$, and there will be no danger of confusion that we shall simply write $f(\kappa)$ for $f(\lambda)$ from now on. Then the solution of (3.11) is necessarily of the form given by

$$f_e(\kappa, x) = r_e(\kappa)\, e^{-\kappa x} \qquad e \in \mathcal{E},\ x \in \mathbb{R}_+, \tag{3.12}$$

$$f_i(\kappa, x) = r_i^+(\kappa)\, e^{\kappa x} + r_i^-(\kappa)\, e^{\kappa(\rho_i - x)} \qquad i \in \mathcal{I},\ x \in [0, \rho_i], \tag{3.13}$$

and we want to show that for some $\kappa > 0$, the boundary conditions (2.5) and (3.3) entail that $r_e(\kappa) = r_i^+(\kappa) = r_i^-(\kappa) = 0$ for all $e \in \mathcal{E}$, $i \in \mathcal{I}$. For $\kappa > 0$, define a column vector $r(\kappa)$ of length $|\mathcal{E}| + 2|\mathcal{I}|$ by

$$r(\kappa) = \big((r_e(\kappa),\ e \in \mathcal{E}), (r_i^+(\kappa),\ i \in \mathcal{I}), (r_i^-(\kappa),\ i \in \mathcal{I})\big)^t,$$

and introduce the $(|\mathcal{E}| + 2|\mathcal{I}|) \times (|\mathcal{E}| + 2|\mathcal{I}|)$ matrices

$$X_\pm(\kappa) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm e^{\kappa\rho} \\ 0 & \pm e^{\kappa\rho} & 1 \end{pmatrix}$$

—appropriately modified in case that $\mathcal{E}$ or $\mathcal{I}$ is the empty set—with the $|\mathcal{I}| \times |\mathcal{I}|$ diagonal matrices

$$e^{\kappa\rho} = \operatorname{diag}\big(e^{\kappa\rho_i},\ i \in \mathcal{I}\big).$$

Then the boundary conditions (2.5), (3.3) for $f(\kappa)$ read

$$Z(\kappa) r(\kappa) = 0, \tag{3.14}$$

with

$$Z(\kappa) = (A + \kappa^2 C) X_+(\kappa) + \kappa B X_-(\kappa). \tag{3.15}$$

Thus, if we can show that for some $\kappa > 0$ the matrix $Z(\kappa)$ is invertible, the proof of the lemma is finished. Note that the matrix-valued function Z is entire in κ, and therefore, so is its determinant. Thus, if it can show that $\kappa \mapsto \det Z(\kappa)$ does not vanish identically then it can only vanish on a discrete subset of the complex plane, and for κ in the complement of this set $Z(\kappa)$ is invertible. Write

$$X_\pm(\kappa) = 1 \pm \delta X(\kappa),$$

with

$$\delta X(\kappa) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e^{\kappa\rho} \\ 0 & e^{\kappa\rho} & 0 \end{pmatrix},$$

so that we can write

$$Z(\kappa) = \hat{Z}_+(\kappa)\big(1 + \delta Z(\kappa)\big),$$

with

$$\delta Z(\kappa) = \hat{Z}_+(\kappa)^{-1}\, \hat{Z}_-(\kappa)\, \delta X(\kappa).$$

Observe that in case that $\mathcal{I} = \emptyset$, we obtain $\delta Z(\kappa) = 0$, and in this case the invertibility of $Z(\kappa)$ for all κ with $\kappa \geq R$ follows from Lemma 3.4. Hence, we assume from now on that $\mathcal{I} \neq \emptyset$. Lemma 3.4 provides us with the bound

$$\big\| \hat{Z}_+(\kappa)^{-1} \hat{Z}_-(\kappa) \big\| \leq \text{const.}\, |\kappa|^q,$$

for all $\kappa \in \mathbb{C}$, $|\kappa| \geq R$, and for some $q > 0$. On the other hand, we get

$$\|\delta X(\kappa)\| \leq e^{\kappa\rho_0},$$

for all $\kappa \leq 0$ where $\rho_0 = \min_{i\in\mathcal{I}} \rho_i$. Therefore, there exists a constant $R' > 0$ so that for all $\kappa \leq -R'$ we have $\|\delta Z(\kappa)\| < 1$, and therefore for such κ, $Z(\kappa)$ is invertible, i.e., $\det Z(\kappa) \neq 0$. Hence, there also exists $\kappa > 0$ so that $Z(\kappa)$ is invertible, and the proof is finished. □

Remark 3.5: The special (and only) choice of boundary conditions in the form $c_v = 0$ and $b_{v_l} = (1 - a_v)/n(v) > 0$ for all v also gives rise to a self-adjoint nonpositive Laplace operator on $L^2(\mathcal{G})$, see Theorem 5.1 in Ref. 56. The associated semigroup is positivity preserving and agrees on $C_0(\mathcal{G}) \cap L^2(\mathcal{G})$ with the semigroup associated to a corresponding Brownian motion on $\mathcal{G}$. In turn, the theorem just quoted also provides examples of self-adjoint Laplace operators, whose semigroups are positivity preserving but which are not linked to a Brownian motion process in the above way.

IV. JOINING TWO METRIC GRAPHS

For what follows, it will be convenient to write $\mathcal{G} = (V, \mathcal{I}, \mathcal{E}, \partial)$ for a metric graph $\mathcal{G}$ in order to make explicit that V is its set of vertices, $\mathcal{I}$ its set of internal and $\mathcal{E}$ its set of external edges. δ is the map from $\mathcal{L}$ into $V \cup (V \times V)$ as defined in Sec. I. For simplicity, we assume from now on—with the exception of the discussion in Sec. VI— that the metric graphs under consideration do not have any *tadpoles*, that is, internal edges i so that $\partial^-(i) = \partial^+(i)$.

Throughout this section, we suppose that $\mathcal{G}_k = (V_k, \mathcal{I}_k, \mathcal{E}_k, \partial_k)$, $k = 1, 2$, are two finite metric graphs. In Subsection IV A, we shall construct a new metric graph $\mathcal{G} = (V, \mathcal{I}, \mathcal{E}, \partial)$ from $\mathcal{G}_1$ and $\mathcal{G}_2$ by connecting some of their external edges.

It will be convenient to consider the metric graphs $\mathcal{G}_1, \mathcal{G}_2$ as subgraphs of the metric graph $\mathcal{G}_0 = \mathcal{G}_1 \uplus \mathcal{G}_2$, which is their (disjoint) union: $\mathcal{G}_0 = (V_0, \mathcal{I}_0, \mathcal{E}_0, \partial_0)$, with $V_0 = V_1 \cup V_2$, $\mathcal{I}_0 = \mathcal{I}_1 \cup \mathcal{I}_2$, $\mathcal{E}_0 = \mathcal{E}_1 \cup \mathcal{E}_2$, and where the map ∂_0 comprises the maps ∂_1, ∂_2 in the obvious way.

A. Construction of the graph $\mathcal{G}$

Suppose that N is a natural number such that $N \leq \min(|\mathcal{E}_1|, |\mathcal{E}_2|)$. For $k = 1, 2$, select subsets $\mathcal{E}'_k \subset \mathcal{E}_k$ of edges with $|\mathcal{E}'_1| = |\mathcal{E}'_2| = N$ to be joined. Let these sets be labeled as follows:

$$\mathcal{E}'_1 = \{e_1, \ldots, e_N\}, \quad \mathcal{E}'_2 = \{l_1, \ldots, l_N\}.$$

In addition, we assume that we are given strictly positive numbers $b_1, \ldots, b_N$, which will serve as the lengths of the new internal edges, as well as $\sigma_k \in \{-1, 1\}$, $k = 1, \ldots, N$, which will determine their orientations. For every $k \in \{1, \ldots, N\}$, we associate with the interval $[0, b_k]$ an abstract edge i_k (not in $\mathcal{I}_0$), which is isomorphic to $[0, b_k]$. Set $\mathcal{I}_c = \{i_1, \ldots, i_N\}$, and

$$V = V_0,$$

$$\mathcal{I} = \mathcal{I}_0 \cup \mathcal{I}_c,$$

$$\mathcal{E} = \mathcal{E}_0 \setminus (\mathcal{E}'_1 \cup \mathcal{E}'_2).$$

The map ∂ is constructed in two steps: Let ∂' be the restriction of ∂_0 to $\mathcal{I}_0 \cup \mathcal{E}_0 \setminus (\mathcal{E}'_1 \cup \mathcal{E}'_2)$. Then ∂ is the extension of ∂' to $\mathcal{I} \cup \mathcal{E}$, which is defined by

$$\partial(i_k) = \begin{cases} \big(\partial_1(e_k), \partial_2(l_k)\big), & \text{if } \sigma_k = 1, \\ \big(\partial_2(l_k), \partial_1(e_k)\big), & \text{if } \sigma_k = -1, \end{cases} \qquad k = 1, \ldots, N.$$

Figure 1 shows an example of a metric graph, which is constructed from the two metric graphs in Figure 2 by joining the $N = 3$ pairs of external edges (e_1, l_1), (e_2, l_2), and (e_3, l_3). The new internal edges i_1, i_2, and i_3 have the lengths 1, $\sqrt{2}$, and 1, respectively (in some scale).

Conversely, let a metric graph $\mathcal{G}$ be given. Associate with every vertex $v \in V$ of $\mathcal{G}$ a single vertex graph $\mathcal{G}(v)$ with vertex v and $n(v)$ external edges, where $n(v)$ is the number of edges incident with v in $\mathcal{G}$. Then it is clear that we can reconstruct $\mathcal{G}$ from the single vertex graphs $\mathcal{G}(v)$, $v \in V$, by finitely many applications of the joining procedure described above.

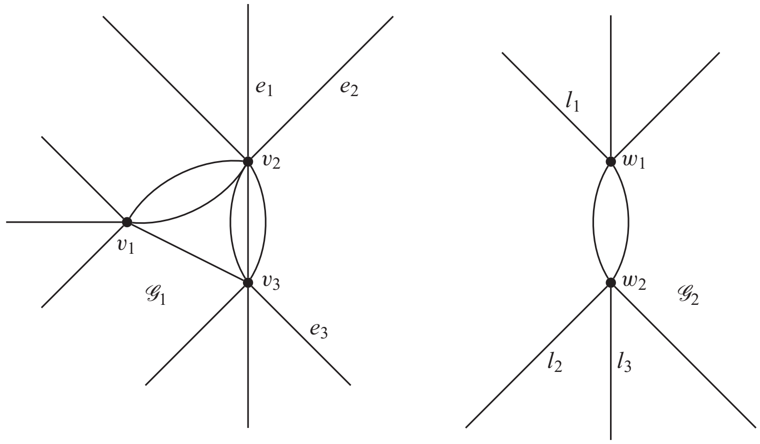

FIG. 2. Two metric graphs $\mathcal{G}_1$, $\mathcal{G}_2$, to be joined by connecting the pairs of external lines (e_1, l_1), (e_2, l_2), and (e_3, l_3).

For the purposes below, it will be convenient to introduce some additional notation. We let $V_c \subset V$ denote the subset of vertices of $\mathcal{G}$, which are connected to each other by the new internal edges in $\mathcal{I}_c$. That is, $v \in V_c$ is such that there exists at least one $i \in \mathcal{I}_c$ with $v \in \partial(i)$. For notational simplicity, here and below, we also use $\partial(l)$ to denote the set consisting of $\partial^-(l)$ and $\partial^+(l)$ if $l \in \mathcal{I}$, and of $\partial(l)$ if $l \in \mathcal{E}$. In the example of Figures 1 and 2, $V_c = \{v_2, v_3, w_1, w_2\}$.

Consider a vertex $v \in V_c$, which belongs to $\mathcal{G}_1$, and let $i_k \in \mathcal{I}_c$, $k \in \{1, \ldots, N\}$, be an internal edge connecting v to $\mathcal{G}_2$, i.e., $v \in \partial(i_k)$. Then the point $\eta \in \mathcal{G}_2^\circ$ with local coordinates (l_k, b_k) is called a *shadow vertex* of the vertex v. $\text{shad}(v) \subset \mathcal{G}_2^0$ is the set of all shadow vertices of v. If $v \in V_c \cap \mathcal{G}_2$, its set of shadow vertices (which are points in $\mathcal{G}_1^\circ$) is defined analogously. $V_s = \text{shad}(V_c) = \cup_{v \in V_c} \text{shad}(v)$ is the set of all shadow vertices. If $\xi \in V_s$ then there exists a unique $v \in V_c$ so that $\xi \in \text{shad}(v)$. We put $\kappa(\xi) = v$ and thereby define a mapping from V_s onto V_c. Of course, in general, κ is not injective. In Figure 3, the shadow vertices of the example above are depicted as small circles on the external edges, i.e., $V_s = \{\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3\}$. For example, $\text{shad}(v_2) = \{\eta_1, \eta_2\}$, $\text{shad}(w_1) = \{\xi_1\}$, and $\kappa(\xi_2) = w_2$, $\kappa(\eta_2) = v_2$.

B. Construction of a preliminary version of the Brownian motion

From now, we suppose that we are given a family of probability spaces

$$(\Xi^0, \mathcal{C}^0, Q^0_\xi), \qquad \xi \in \mathcal{G}_0,$$

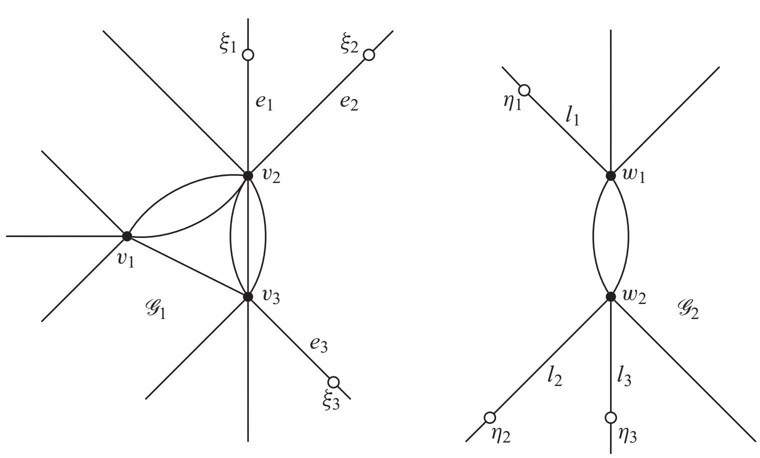

FIG. 3. The graphs $\mathcal{G}_1$, $\mathcal{G}_2$ with the shadow vertices (small circles).

and that thereon, a Brownian motion with state space $\mathcal{G}_0$ in the sense of Definition 2.1 is defined. This Brownian motion is denoted by $Z^0 = (Z^0(t),\ t \in \mathbb{R}_+)$. Actually, since $\mathcal{G}_0 = \mathcal{G}_1 \cup \mathcal{G}_2$ and $\mathcal{G}_1$, $\mathcal{G}_2$ are disconnected, this is the same as saying that we are given a Brownian motion on $\mathcal{G}_1$ and one on $\mathcal{G}_2$. However, notationally it will be more convenient to view this as one stochastic process. We assume, as we may, that Z^0 has exclusively càdlàg paths, which are continuous up to the lifetime ζ^0 of Z^0. $\mathcal{F}^0 = (\mathcal{F}^0_t,\ t \in \mathbb{R}_+)$ denotes the natural filtration of Z^0. The hitting time of V_s by Z^0 is denoted by τ^0, i.e.,

$$\tau^0 = \inf\{t > 0,\ Z^0(t) \in V_s\}.$$

Furthermore, we assume that $\vartheta = (\vartheta_t,\ t \in \mathbb{R}_+)$ is a family of shift operators for Z^0 acting on Ξ^0.

For any topological space $(T, \mathcal{T})$ denote by $C_\Delta(\mathbb{R}_+, T)$ the space of mappings ω from $\mathbb{R}_+$ into $T \cup \{\Delta\}$, which are right continuous, have left limits in T, are continuous up to the lifetime

$$\zeta_\omega = \inf\{t > 0,\ \omega(t) = \Delta\},$$

and which are such that $\omega(t) = \Delta$ implies $\omega(s) = \Delta$ for all $s \geq t$. In particular and in the present context, $\omega \in C_\Delta(\mathbb{R}_+, \mathcal{G}_0)$ is either continuous from $\mathbb{R}_+$ into $\mathcal{G}_0$ or it has a jump from $\mathcal{G}_1$ or $\mathcal{G}_2$ to Δ, but there can be no jump from $\mathcal{G}_1$ to $\mathcal{G}_2$ or vice versa.

We shall make use of some special versions of the process Z^0, which we introduce now. For every $v \in V_c$, $Z^1_v = (Z^1_v(t),\ t \in \mathbb{R}_+)$ denotes a Brownian motion on $\mathcal{G}_0$ defined on another probability space $(\Xi^1_v, \mathcal{C}^1_v, \mu^1_v)$ such that under μ^1_v, Z^1_v is equivalent to Z^0 under Q^0_v. We suppose that Z^1_v exclusively has paths, which start in v and which belong to $C_\Delta(\mathbb{R}_+, \mathcal{G}_0)$. (For example, one can use a standard path space construction to obtain such a version from $(\Xi^0, \mathcal{C}^0, Q^0_v, Z^0)$.) The hitting time of V_s by Z^1_v is denoted by τ^1_v, its lifetime by ζ^1_v.

The idea to define the preliminary version $Y = (Y(t),\ t \in \mathbb{R}_+)$ of the Brownian motion on $\mathcal{G}$ is to construct its paths as follows. Let $\xi \in \mathcal{G}$ be a given starting point. $\mathcal{G}$ (viewed as a set) has the following decomposition (cf. Figure 4):

$$\mathcal{G} = \hat{\mathcal{G}}_1 \uplus \hat{\mathcal{G}}_2,$$

with

$$\hat{\mathcal{G}}_1 = \mathcal{G}_1 \setminus \big(e_1^\circ \cup \cdots \cup e_N^\circ\big),$$

$$\hat{\mathcal{G}}_2 = \big(\mathcal{G}_2 \setminus \big(l_1^\circ \cup \cdots \cup l_N^\circ\big)\big) \cup \big(i_1^\circ \cup \ldots \cup i_N^\circ\big).$$

Thus, we may consider ξ instead as a point in $\hat{\mathcal{G}}_1 \uplus \hat{\mathcal{G}}_2 \subset \mathcal{G}_0$.

We pause here for the following remark: Of course, the convention we make that all new open inner edges $i_1^\circ, \ldots, i_N^\circ$ are attached to $\hat{\mathcal{G}}_2$ is somewhat arbitrary. Just as well any subset of them could have been attached to $\hat{\mathcal{G}}_1$ instead. Even though different conventions lead to processes with

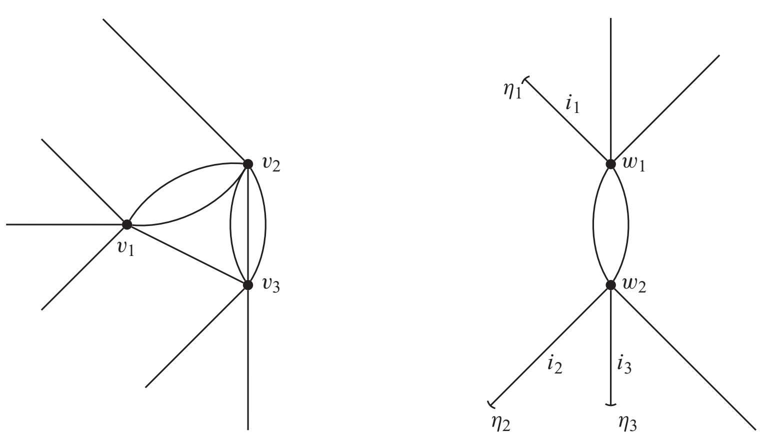

FIG. 4. The starting points of Y.

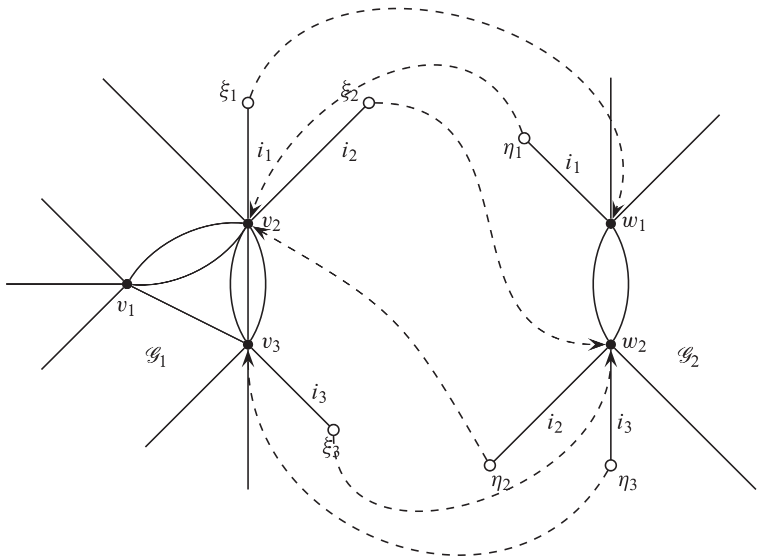

FIG. 5. The construction of the process Y.

different paths, the main result of this section, Theorem 4.13, remains unchanged. It follows that all resulting processes are equivalent to each other.

Let Y start as Z^0 in $\xi \in \hat{\mathcal{G}}_1 \uplus \hat{\mathcal{G}}_2$, and consider one trajectory. (In order to avoid any confusion, let us point out that even though $\xi \in \hat{\mathcal{G}}_k$, $k = 1, 2$, the process Z^0 moves in $\mathcal{G}_k$.) If this trajectory reaches the cemetery point Δ before hitting the set V_s of shadow vertices, it is the complete trajectory of Y and it stays forever at the cemetery. If the trajectory hits a shadow vertex $\eta \in V_s$ before its lifetime expires, this piece of the trajectory of Y ends at the hitting time τ^0. Set $v = \kappa(\eta)$, and let the trajectory of Y continue with an (independent) trajectory of Z_v^1 until its lifetime expires or it hits a shadow vertex, and so on. Figure 5 explains the idea.

The construction described above is formalized in the following way. Define

$$\Xi^1 = \bigtimes_{v \in V_c} \Xi_v^1, \quad \mathcal{C}^1 = \bigotimes_{v \in V_c} \mathcal{C}_v^1, \quad Q^1 = \bigotimes_{v \in V_c} \mu_v^1,$$

and view each of the stochastic processes Z_v^1, $v \in V_c$, as well as the random variables τ_v^1, ζ_v^1, as defined on this product space. Let

$$\left(\Xi^n, \mathcal{C}^n, Q^n, Z^n, \tau^n, \zeta^n\right), \qquad n \in \mathbb{N},\ n \geq 2,$$

be an independent sequence of copies of

$$\left(\Xi^1, \mathcal{C}^1, Q^1, Z^1, \tau^1, \zeta^1\right),$$

where $Z^1 = (Z_v^1,\ v \in V_c)$ and similarly for τ^1, ζ^1. Next set

$$\Xi = \bigtimes_{n=0}^{\infty} \Xi^n, \quad \mathcal{C} = \bigotimes_{n=0}^{\infty} \mathcal{C}^n, \quad Q_\xi = Q_\xi^0 \otimes \Big(\bigotimes_{n=1}^{\infty} Q^n\Big),\ \xi \in \mathcal{G}.$$

The procedure sketched above of pasting together pieces of the trajectories of the various processes Z_v^n is controlled by a Markov chain $(K_n,\ n \in \mathbb{N})$, which moves at random times $(S_n,\ n \in \mathbb{N})$ in the state space $V_c \cup \{\Delta\}$. We set out to construct this chain $\big((S_n, K_n),\ n \in \mathbb{N}\big)$. Define $S_1 = \tau^0$. On $\{S_1 = +\infty\}$, i.e., in the case when $\zeta^0 < \tau^0$, set $K_1 = \Delta$. Otherwise, define

$$K_1 = \kappa\big(Z^0(\tau^0)\big).$$

Observe that since all processes considered have right continuous paths, they are all measurable stochastic processes, and therefore the evaluation of their time argument at a random time yields a well-defined random variable. Set $S_2 = +\infty$ on $\{S_1 = +\infty\}$, while

$$S_2 = S_1 + \tau^1_{K_1}$$

on $\{S_1 < +\infty\}$. On $\{S_2 = +\infty\}$, put $K_2 = \Delta$, and on its complement

$$K_2 = \kappa\big(Z^1_{K_1}(\tau^1_{K_1})\big).$$

These construction steps are iterated in the obvious way: The sequence

$$\big((S_n, K_n),\ n \in \mathbb{N}\big)$$

is inductively defined by $S_n = +\infty$ and $K_n = \Delta$ on $\{S_{n-1} = +\infty\}$, while

$$S_n = S_{n-1} + \tau^{n-1}_{K_{n-1}},$$

$$K_n = \kappa\big(Z^{n-1}_{K_{n-1}}(\tau^{n-1}_{K_{n-1}})\big)$$

on $\{S_{n-1} < +\infty\}$.

Note that by construction $K_n = \Delta$, $n \in \mathbb{N}$, if and only if $S_n = +\infty$, and in that case $K_{n'} = \Delta$, $S_{n'} = +\infty$ for all $n' \geq n$. Thus, $(+\infty, \Delta)$ is a cemetery state for the chain $((S_n, K_n),\ n \in \mathbb{N})$.

For example, with a Borel–Cantelli argument, it is not hard to see (cf. also Ref. 50) that there exists a set $\Xi' \in \mathcal{C}$ so that for all $\xi \in \mathcal{G}$, $Q_\xi(\Xi') = 0$, and for all $\omega \in \Xi\backslash\Xi'$ the sequence $(S_n(\omega),\ n \in \mathbb{N})$ increases to $+\infty$ in such a way that for all $n \in \mathbb{N}$, $S_n(\omega) < S_{n+1}(\omega)$ holds when $S_n(\omega) < +\infty$.

Now we are ready to construct $Y = (Y(t), \in \mathbb{R}_+)$. Let $\xi \in \mathcal{G} = \hat{\mathcal{G}}_1 \uplus \hat{\mathcal{G}}_2$ be a given starting point, and suppose that $t \in \mathbb{R}_+$ is given. On Ξ' set $Y(t) = \Delta$. On $\Xi\backslash\Xi'$ there is a unique $n \in \mathbb{N}_0$ so that $t \in [S_n, S_{n+1})$, with the convention $S_0 = 0$. If $t \in [0, S_1)$, define $Y(t) = Z^0(t)$. If $t \in [S_n, S_{n+1})$ for $n \in \mathbb{N}$, then necessarily S_n is finite, so that $K_n \in V_c$, and we define

$$Y(t) = Z^n_{K_n}(t - S_n).$$

In addition, we make the convention $Y(+\infty) = \Delta$. The natural filtration generated by Y will be denoted by $\mathcal{F}^Y = (\mathcal{F}^Y_t,\ t \in \mathbb{R}_+)$.

It follows from the construction of Y that Δ is a cemetery state for Y. Indeed, suppose that $\omega \in \Xi\backslash\Xi'$, and that the trajectory $Y(\,\cdot\,, \omega)$ reaches the point Δ at a finite time $\zeta^Y(\omega)$. This implies that there is an $n \in \mathbb{N}_0$ such that $\zeta^Y(\omega) \in [S_n(\omega), S_{n+1}(\omega))$. Then $S_n(\omega)$ is finite, and therefore $K_n(\omega) \in V_c$ so that $Y(\,\cdot\,, \omega)$ is equal to $Z^n_{K_n}(\,\cdot\, - S_n(\omega), \omega)$ on the interval $[S_n(\omega), S_{n+1}(\omega))$, and this trajectory reaches Δ before hitting a shadow vertex. Hence, $\tau^n_{K_n} = +\infty$, which entails that $S_{n+1}(\omega) = +\infty$. Consequently, after $S_n(\omega)$, there are no finite crossover times for this trajectory, and therefore $Y(\,\cdot\,, \omega)$ stays at Δ forever. Furthermore, note that the left limit $Y(\zeta^Y(\omega)\ -, \omega)$ at $\zeta^Y(\omega)$ belongs to V_0.

In terms of the stochastic process Y, the random times S_n, $n \in \mathbb{N}$, have the following description. Suppose that Y starts in $\xi \in \mathcal{G}$. Then S_1 is the hitting time of V_c. But if $\xi \in V_c$, then actually it is the hitting time of $V_c \setminus \{\xi\}$, because it hits a vertex in V_c, which corresponds to the first hitting of a shadow vertex, i.e., a point in $\mathcal{G}_0$ different from ξ, by Z^0. In particular, $S_1 > 0$. Similarly, S_n is the hitting time of $V_c \setminus \{K_{n-1}\}$ by Y after time S_{n-1}. In Appendix A, it is shown that for every $n \in \mathbb{N}$, S_n is a stopping time with respect to $\mathcal{F}^Y$.

It follows from its construction that Y is a normal process, that is, for every $\xi \in \mathcal{G}$, $Q_\xi(Y(t = 0) = \xi) = 1$. Furthermore, all paths of Y belong to $C_\Delta(\mathbb{R}_+, \mathcal{G})$. Let S_V be the hitting time of the set of vertices V of $\mathcal{G}$ by Y. Then $S_V \leq S_1$, because $V_c \subset V$ and therefore we find that $Y(\,\cdot\, \wedge S_V)$ is pathwise equal to $Z^0(\,\cdot\, \wedge S^0_V)$, where S^0_V denotes the hitting time of V by Z^0. Suppose that the starting point ξ belongs to l°, $l \in \mathcal{I} \cup \mathcal{E}$, and l is isomorphic to the interval I. Then by definition of Z^0 (cf. Definition 2.1), the stopped process $Z^0(\,\cdot\, \wedge S^0_V)$ is equivalent to a standard Brownian motion on the interval I with absorption at the endpoint(s) of I. Hence, the same is true for Y: $Y(\,\cdot\, \wedge S_V)$ is equivalent to a standard Brownian motion on I with absorption at the endpoint(s) of I.

C. Markov property of *Y*

For any measurable space $(M, \mathcal{M})$, $B(M)$ denotes the space of bounded, measurable functions on M. Every $f \in B(\mathcal{G}^n)$, $n \in \mathbb{N}$, is extended to $(\mathcal{G} \cup \{\Delta\})^n$ by $f(\xi_1, \ldots, \xi_n) = 0$, $(\xi_1, \ldots, \xi_n) \in (\mathcal{G} \cup \{\Delta\})^n$, whenever there is an index $k \in \{1, \ldots, n\}$ so that $\xi_k = \Delta$. In this subsection, we shall prove the following:

Proposition 4.1: Y has the simple Markov property: For all $f \in B(\mathcal{G})$, $s, t \in \mathbb{R}_+$, $\xi \in \mathcal{G}$,

$$E_\xi\big(f\big(Y(s+t)\big)\,\big|\,\mathcal{F}_s^Y\big) = E_{Y(s)}\big(f\big(Y(t)\big)\big) \tag{4.1}$$

holds true Q_ξ*–almost surely on* $\{Y(s) \neq \Delta\}$.

The proof of Proposition 4.1 is somewhat technical and lengthy. Therefore, it will be broken up into a sequence of lemmas.

For every $n \in \mathbb{N}$, the probability space $(\Xi, \mathcal{C}, Q_\xi)$, $\xi \in \mathcal{G}$, underlying the construction of the process Y may be written as the product of the probability spaces $(\Xi^{\le n-1}, \mathcal{C}^{\le n-1}, Q_\xi^{\le n-1})$ and $(\Xi^{\ge n}, \mathcal{C}^{\ge n}, Q^{\ge n})$ with

$$\Xi^{\le n-1} = \mathop{\times}_{j=0}^{n-1} \Xi^j, \qquad \Xi^{\ge n} = \mathop{\times}_{j=n}^{\infty} \Xi^j,$$

$$\mathcal{C}^{\le n-1} = \bigotimes_{j=0}^{n-1} \mathcal{C}^j, \qquad \mathcal{C}^{\ge n} = \bigotimes_{j=n}^{\infty} \mathcal{C}^j,$$

$$Q_\xi^{\le n-1} = Q_\xi^0 \otimes \Big(\bigotimes_{j=1}^{n-1} Q^j\Big), \qquad Q^{\ge n} = \bigotimes_{j=n}^{\infty} Q^j.$$

Introduce a family $\mathcal{B} = (\mathcal{B}_n,\ n \in \mathbb{N}_0)$ of sub–σ–algebras of $\mathcal{C}$ by setting

$$\mathcal{B}_n = \mathcal{C}^{\le n-1} \times \Xi^{\ge n}.$$

Obviously, the family $\mathcal{B}$ forms a filtration. Furthermore, from the construction of K_n and S_n it is easy to see that the chain $((S_n, K_n),\ n \in \mathbb{N})$ is adapted to $\mathcal{B}$.

First, we study the chain $((S_n, K_n),\ n \in \mathbb{N})$ in more detail. Recall our convention that $S_0 = 0$. We set $\mathcal{B}_0 = \{\emptyset, \Xi\}$, and under the law Q_v, $v \in V_c$, we put $K_0 = v$. $g \in B(\mathbb{R}_+ \times V_c)$ is extended to $\overline{\mathbb{R}}_+ \times \big(V_c \cup \{\Delta\}\big)$ by $g(+\infty, \cdot\,) = g(\,\cdot\,, \Delta) = g(+\infty, \Delta) = 0$. For $g \in B(\mathbb{R}_+ \times V_c)$, $n \in \mathbb{N}_0$, define

$$(U_n g)(s, v) = E_v\big(g(s + S_n, K_n)\big), \qquad s \in \mathbb{R}_+,\ v \in V_c. \tag{4.2}$$

Note that $U_0 = \mathrm{id}$, and that for every $g \in B(\mathbb{R}_+ \times V_c)$ and all $n \in \mathbb{N}$, $U_n g \in B(\mathbb{R}_+ \times V_c)$. In particular, the convention mentioned above applies to $U_n g$, too.

Lemma 4.2:

(a) *For all* $m, n \in \mathbb{N}$, $m \le n$, $\xi \in \mathcal{G}$, $s \ge 0$, *and every* $g \in B(\mathbb{R}_+ \times V_c)$ *the following formula holds true* Q_ξ*–almost surely:*

$$E_\xi\big(g(s + S_n, K_n)\,\big|\,\mathcal{B}_m\big) = (U_{n-m} g)(s + S_m, K_m). \tag{4.3}$$

(b) $(U_n,\ n \in \mathbb{N}_0)$ *forms a semigroup of linear maps on* $B(\mathbb{R}_+ \times V_c)$. *In particular, for all* $(s, v) \in \mathbb{R}_+ \times V_c$ *under* Q_v *the chain* $\big((s + S_n, K_n),\ n \in \mathbb{N}_0\big)$ *is a homogeneous Markov chain with transition kernel*

$$P\big((s, v), A\big) = Q_v\big((s + S_1, K_1) \in A\big), \qquad A \in \mathcal{B}(\mathbb{R}_+ \times V_c).$$

Proof: For $m = n$ formula (4.3) is trivial. Consider the case when $n \geq 2$, $m = n - 1$. Let $\Lambda \in \mathcal{B}_{n-1}$, $v \in V_c$, and put $\Lambda_v = \Lambda \cap \{K_{n-1} = v\} \in \mathcal{B}_{n-1}$. From the construction of S_n and K_n

$$\begin{aligned}
&E_\xi\big(g(s + S_n, K_n); \Lambda_v\big)\\
&= \int_{\Xi^{\leq n-2}} 1_{\Lambda_v}\Big(\int_{\Xi^{\geq n-1}} g\big(s + S_{n-1} + \tau_v^{n-1}, \kappa\big(Z_v^{n-1}(\tau_v^{n-1})\big)\big)\, dQ^{\geq n-1}\Big)\, dQ_\xi^{\leq n-2}\\
&= \int_{\Xi^{\leq n-2}} 1_{\Lambda_v}\Big(\int_{\Xi^0} g\big(s + u + \tau^0, \kappa\big(Z^0(\tau^0)\big)\big)\, dQ_v^0\Big)\Big|_{u=S_{n-1}}\, dQ_\xi^{\leq n-2}\\
&= E_\xi\Big(E_v\big(g(s + u + S_1, K_1)\big)\big|_{u=S_{n-1}}; \Lambda_v\Big)\\
&= E_\xi\Big(E_{K_{n-1}}\big(g(s + u + S_1, K_1)\big)\big|_{u=S_{n-1}}; \Lambda_v\Big)\\
&= E_\xi\Big((U_1 g)(s + S_{n-1}, K_{n-1}); \Lambda_v\Big),
\end{aligned}$$

where in the last step, we used Definition (4.2). If in the preceding calculation we replace the event $\{K_{n-1} = v\}$ by $\{K_{n-1} = \Delta\}$, we get zero on both sides because $K_{n-1} = \Delta$ implies $K_n = \Delta$ (see Subsection IV B). Thus, summation over $v \in V_c$ gives

$$E_\xi\big(g(s + S_n, K_n); \Lambda\big) = E_\xi\big((U_1 g)(s + S_{n-1}, K_{n-1}); \Lambda\big),$$

and Eq. (4.3) is proved for the case where $n \geq 2$ and $m = n - 1$. As a consequence, we get

$$\begin{aligned}
(U_n g)(s, v) &= E_v\big(E_v\big(g(s + S_n, K_n)\,\big|\,\mathcal{B}_{n-1}\big)\big)\\
&= E_v\big((U_1 g)(s + S_{n-1}, K_{n-1})\big)\\
&= \big(U_{n-1} \circ U_1 g\big)(s, v).
\end{aligned}$$

Now the general semigroup relation $U_{n+m} = U_n \circ U_m$, $m, n \in \mathbb{N}_0$, follows by an application of Fubini's theorem.

Finally, we show formula (4.3) in the general case:

$$\begin{aligned}
&E_\xi\big(g(s + S_n, K_n)\,\big|\,\mathcal{B}_m\big)\\
&\qquad = E_\xi\big(E_\xi\big(g(s + S_n, K_n)\,\big|\,\mathcal{B}_{n-1}\big)\,\big|\,\mathcal{B}_m\big)\\
&\qquad = E_\xi\big((U_1 g)(s + S_{n-1}, K_{n-1})\,\big|\,\mathcal{B}_m\big)\\
&\qquad = \quad \ldots\\
&\qquad = E_\xi\big((U_1 \circ \cdots \circ U_1 g)(s + S_m, K_m)\,\big|\,\mathcal{B}_m\big),
\end{aligned}$$

where the multiple composition in the last expression involves $n - m$ operators U_1. The semigroup property of $(U_n,\ n \in \mathbb{N}_0)$ implies formula (4.3). □

It will be useful to introduce some additional notation. For $r \in \mathbb{N}$, let $\hat{\mathbb{R}}_+^r$ denote the set of all increasingly ordered r–tuples with entries in $\mathbb{R}_+$. If $u \in \hat{\mathbb{R}}_+^r$ and $s \in \mathbb{R}$, we set $u + s = (u_1 + s, \ldots, u_r + s) \in \mathbb{R}^r$. $u < s$ means that $u_i < s$ for all $i = 1, \ldots, r$ or equivalently $u_r < s$. The relations $u > s$, $u \leq s$, and $u \geq s$ are defined analogously. In particular, when $s \leq u$, then

$u - s \in \hat{\mathbb{R}}^r_+$. For $r, q \in \mathbb{N}$, and $u \in \hat{\mathbb{R}}^r_+$, $w \in \hat{\mathbb{R}}^q_+$ with $u \le w_1$, define

$$(u, w) = (u_1, \dots, u_r, w_1, \dots, w_q) \in \hat{\mathbb{R}}^{r+q}_+.$$

Furthermore, $Y(u)$ stands for $(Y(u_1), \dots, Y(u_r))$, and similarly for $Z^n_v(u)$, $n \in \mathbb{N}_0$, $v \in V_c$.

In the sequel, we shall consider random variables $W_m(h, g, u)$ of the following form:

$$W_m(h, g, u) = h\big(Y(u)\big)\, \chi_m(u)\, g(S_{m+1}, K_{m+1}), \tag{4.4}$$

where $m \in \mathbb{N}$, h belongs to $B(\mathcal{G}^r)$, $r \in \mathbb{N}$, g to $B(\mathbb{R}_+ \times V_c)$, and $u \in \hat{\mathbb{R}}^r_+$. Here, we have set

$$\chi_m(u) = 1_{\{S_m \le u < S_{m+1}\}}.$$

For $s \ge 0$ with $s \le u$ define

$$W_{m,s}(h, g, u) = h\big(Y(u - s)\big)\, \chi_m(u - s)\, g(s + S_{m+1}, K_{m+1}), \tag{4.5}$$

so that $W_{m,s=0}(h, g, u) = W_m(h, g, u)$. Moreover, set

$$R_m(h, g, u)(s, v) = E_v\big(W_{m,s}(h, g, u)\big), \qquad s \in \mathbb{R}_+,\ s \le u,\ v \in V_c. \tag{4.6}$$

For the following, it will be convenient to let $W_{m,s}(h, g, u)$ and $R_m(h, g, u)(s, v)$, $v \in V_c$, be defined for all $s \in \mathbb{R}_+$. To this end, we make the convention that $Y(t) = \Delta$ for all $t < 0$. Then by $W_{m,s}(h, 1, u) = W_m(h, 1, u - s)$, the following formula:

$$R_m(h, 1, u)(s + t, v) = R_m(h, 1, u - s)(t, v) \tag{4.7}$$

holds for all $m \in \mathbb{N}$, $h \in B(\mathcal{G}^r)$, $r \in \mathbb{N}$, $u \in \hat{\mathbb{R}}^r_+$, $s, t \in \mathbb{R}_+$, $v \in V_c$.

Suppose that $r, q \in \mathbb{N}$, and that $h \in B(\mathcal{G}^r)$, $f \in B(\mathcal{G}^q)$. Then $h \otimes f$ denotes the function in $B(\mathcal{G}^{r+q})$ given by

$$\begin{aligned} &h \otimes f(\eta_1, \dots, \eta_{r+q}) \\ &\quad = h(\eta_1, \dots, \eta_r)\, f(\eta_{r+1}, \dots, \eta_{r+q}), \qquad (\eta_1, \dots, \eta_{r+q}) \in \mathcal{G}^{r+q}. \end{aligned} \tag{4.8}$$

Lemma 4.3: Suppose that $r \in \mathbb{N}$, $u \in \hat{\mathbb{R}}^r_+$, and that $h \in B(\mathcal{G}^r)$.

(a) *If $q \in \mathbb{N}$, $w \in \hat{\mathbb{R}}^q_+$ with $u_r \le w$, and $f \in B(\mathcal{G}^q)$, then*

$$R_0\big(h \otimes f, 1, (u, w)\big) = R_0\big(M(f, w, u_r)h, 1, u\big) \tag{4.9a}$$

holds true, where $M(f, w, s)h \in B(\mathcal{G}^r)$ is given by

$$\big(M(f, w, s)h\big)(\eta) = h(\eta)\, E_{\eta_r}\big(W_{0,s}(f, 1, w)\big), \quad \eta \in \mathcal{G}^r,\ 0 \le s \le w. \tag{4.9b}$$

(b) *If $g \in B(\mathbb{R}_+ \times V_c)$, then*

$$R_0\big(h, g, u\big) = R_0\big(N(g, u_r)h, 1, u\big) \tag{4.10a}$$

holds, where $N(g, s)h \in B(\mathcal{G}^r)$ is given by

$$\big(N(g, s)h\big)(\eta) = h(\eta)\, E_{\eta_r}\big(g(s + S_1, K_1)\big), \qquad \eta \in \mathcal{G}^r,\ s \ge 0. \tag{4.10b}$$

Proof: Both statements follow from the Markov property of the Brownian motion Z^0 on $\mathcal{G}_0$ underlying the construction of Y. We only prove statement (b), the proof of (a) is similar and therefore omitted. Using the definition of R_0 and the construction of Y, we compute for $s \in \mathbb{R}_+$, $v \in V_c$, as follows:

$$\begin{aligned} R_0\big(h, g, u\big)(s, v) &= E_v\Big(h\big(Y(u - s)\big)\, \chi_0(u - s)\, g(s + S_1, K_1)\Big) \\ &= E_v\Big(h\big(Z^0(u - s)\big)\, 1_{\{0 \le u - s < \tau^0\}}\, g\big(s + \tau^0, \kappa(Z^0(\tau^0))\big)\Big). \end{aligned}$$

Recall that $\mathcal{F}^0$ denotes the natural filtration of Z^0, and ϑ is a family of shift operators for Z^0. It follows from the definition of the stopping time τ^0 and the path properties of Z^0, that on $\{\tau^0 \ge u_r$

$- \; s\}$ the relation $\tau^0 = u_r - s + \tau^0 \circ \vartheta_{u_r - s}$ holds true. Moreover, it is easy to check that on this event we have $Z^0(\tau^0) = Z^0(\tau^0)\circ\vartheta_{u_r-s}$. Therefore,

$$\begin{aligned}
R_0\big(h, g, u\big)(s, v) &= E_v\Big(h\big(Z^0(u-s)\big)\, 1_{\{0\le u-s<\tau^0\}} \\
&\qquad \times E_v\big(g(u_r+\tau^0, \kappa(Z^0(\tau^0)))\circ\vartheta_{u_r-s}\,\big|\, \mathcal{F}^0_{u_r-s}\big)\Big) \\
&= E_v\Big(h\big(Z^0(u-s)\big)\, 1_{\{0\le u-s<\tau^0\}} \\
&\qquad \times E_{Z^0(u_r-s)}\big(g(u_r+\tau^0, \kappa(Z^0(\tau^0)))\big)\Big) \\
&= E_v\Big(h\big(Y(u-s)\big)\, \chi_0(u-s)\, E_{Y(u_r-s)}\big(g(u_r+S_1, K_1)\big)\Big) \\
&= R_0\big(N(g, u_r)h, 1, u\big)(s, v),
\end{aligned}$$

and the proof is concluded. □

Lemma 4.4: For all $m, r \in \mathbb{N}$, $h \in B(\mathcal{G}^r)$, $g \in B(\mathbb{R}_+ \times V_c)$, $u \in \hat{\mathbb{R}}^r_+$, $\xi \in \mathcal{G}$, *the formula*

$$E_\xi\big(W_m(h, g, u)\,\big|\,\mathcal{B}_m\big) = R_0(h, g, u)(S_m, K_m) \tag{4.11}$$

holds Q_ξ*–almost surely.*

Proof: Observe that both side of Eq. (4.11) vanish on the set $\{K_m = \Delta\}$. Let $\Lambda \in \mathcal{B}_m$, $v \in V_c$, and set $\Lambda_v = \Lambda \cap \{K_m = v\} \in \mathcal{B}_m$. Then

$$\begin{aligned}
&E_\xi\big(W_m(h, g, u); \Lambda_v\big) \\
&= \int_{\Xi^{\le m-1}} 1_{\Lambda_v}\Big(\int_{\Xi^{\ge m}} h\big(Y(u)\big)\, 1_{\{S_m\le u<S_{m+1}\}} \\
&\qquad \times g(S_{m+1}, K_{m+1})\, dQ^{\ge m}\Big)\, dQ^{\le m-1}_\xi \\
&= \int_{\Xi^{\le m-1}} 1_{\Lambda_v}\Big(\int_{\Xi^{m}} h\big(Z^m_v(u-s)\big)\, 1_{\{0\le u-s<\tau^m_v\}} \\
&\qquad \times g\big(s+\tau^m_v, \kappa(Z^m_v(\tau^m_v))\big)\, dQ^{m}\Big)\Big|_{s=S_m}\, dQ^{\le m-1}_\xi \\
&= \int_{\Xi^{\le m-1}} 1_{\Lambda_v}\Big(\int_{\Xi^{0}} h\big(Z^0(u-s)\big)\, 1_{\{0\le u-s<\tau^0\}} \\
&\qquad \times g\big(s+\tau^0, \kappa(Z^0(\tau^0))\big)\, dQ^{0}_v\Big)\Big|_{s=S_m}\, dQ^{\le m-1}_\xi \\
&= E_\xi\Big(E_v\big(h\big(Y(u-s)\big)\, \chi_0(u-s)\, g(s+S_1, K_1)\big)\big|_{s=S_m};\, \Lambda_v\Big) \\
&= E_\xi\Big(E_{K_m}\big(h\big(Y(u-s)\big)\, \chi_0(u-s)\, g(s+S_1, K_1)\big)\big|_{s=S_m};\, \Lambda_v\Big) \\
&= E_\xi\big(R_0(h, g, u)(S_m, K_m);\, \Lambda_v\big).
\end{aligned}$$

Summation over $v \in V_c$ finishes the proof. □

Lemma 4.5: For all $m, r \in \mathbb{N}$, $u \in \hat{\mathbb{R}}_+^r$, $h \in B(\mathcal{G}^r)$, $g \in B(\mathbb{R}_+ \times V_c)$,

$$R_m(h, g, u)(s, v) = \big(U_m R_0(h, g, u)\big)(s, v), \qquad s \in \mathbb{R}_+,\ v \in V_c, \tag{4.12}$$

holds.

Proof: By definition of U_m

$$\big(U_m R_0(h, g, u)\big)(s, v) = E_v\big(R_0(h, g, u)(s + S_m, K_m)\big).$$

With formula (4.7) and Lemma 4.4, we find

$$\begin{aligned}\big(U_m R_0(h, g, u)\big)(s, v) &= E_v\big(R_0(h, g(s + \cdot\,), u - s)(S_m, K_m)\big)\\ &= E_v\big(E_v\big(W_m(h, g(s + \cdot\,), u - s)\,\big|\,\mathcal{B}_m\big)\big)\\ &= E_v\big(W_m(h, g(s + \cdot\,), u - s)\big)\\ &= E_v\big(W_{m,s}(h, g, u)\big)\\ &= R_m(h, g, u)(s, v). \qquad \square\end{aligned}$$

Corollary 4.6: For all $m, n, r \in \mathbb{N}$, $u \in \hat{\mathbb{R}}_+^r$, $h \in B(\mathcal{G}^r)$, $g \in B(\mathbb{R}_+ \times V_c)$,

$$U_n R_m(h, g, u) = R_{n+m}(h, g, u). \tag{4.13}$$

is valid.

Proof: By Lemmas 4.5 and 4.2, statement (b), we obtain

$$U_n R_m(h, g, u) = U_n \circ U_m R_0(h, g, u) = U_{n+m} R_0(h, g, u) = R_{n+m}(h, g, u). \qquad \square$$

Lemma 4.7: For all $m, n \in \mathbb{N}$, $m \le n$, $r \in \mathbb{N}$, $u \in \hat{\mathbb{R}}_+^r$, $h \in B(\mathcal{G}^r)$, $g \in B(\mathbb{R}_+ \times V_c)$, $\xi \in \mathcal{G}$, *the following formula holds true:*

$$E_\xi\big(W_n(h, g, u)\,\big|\,\mathcal{B}_m\big) = R_{n-m}(h, g, u)(S_m, K_m). \tag{4.14}$$

Proof: Apply Lemma 4.4 to compute as follows:

$$\begin{aligned}E_\xi\big(W_n(h, g, u)\,\big|\,\mathcal{B}_m\big) &= E_\xi\big(E_\xi\big(W_n(h, g, u)\,\big|\,\mathcal{B}_n\big)\,\big|\,\mathcal{B}_m\big)\\ &= E_\xi\big(R_0(h, g, u)(S_n, K_n)\,\big|\,\mathcal{B}_m\big)\\ &= \big(U_{n-m} R_0(h, g, u)\big)(S_m, K_m),\end{aligned}$$

where we used Lemma 4.2, formula (4.3), in the last step. An application of Lemma 4.5 concludes the proof. $\square$

With these preparations, we are ready for the

Proof of Proposition 4.1: Assume that $f \in B(\mathcal{G})$, $s, t \in \mathbb{R}_+$, and that $\xi \in \mathcal{G}$. Since $(S_m,\ m \in \mathbb{N})$ Q_ξ–almost surely, strictly increases to $+\infty$, and since S_m, $m \in \mathbb{N}$, is an $\mathcal{F}^Y$–stopping time (cf., lemma A.1 in Appendix A), it suffices to prove that equation (4.1) holds Q_ξ–almost surely for every $m \in \mathbb{N}_0$ on $\{S_m \le s < S_{m+1}, Y(s) \ne \Delta\} \in \mathcal{F}_s^Y$. We fix an arbitrary $m \in \mathbb{N}_0$. Clearly, the family of random variables of the form

$$g\big(Y(w)\big)\, 1_{\{0 \le w < S_m\}}\, W_m(h, 1, u)$$

with $r, q \in \mathbb{N}$, $u \in \hat{\mathbb{R}}_+^r$, $u_r = s$, $w \in \hat{\mathbb{R}}_+^q$, $h \in B(\mathcal{G}^r)$, and $g \in B(\mathcal{G}^q)$, generates the σ–algebra $\mathcal{F}_s^Y \cap \{S_m \le s < S_{m+1}\}$. Therefore, it is sufficient to show that

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,f\big(Y(s+t)\big)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,E_{Y(s)}\big(f(Y(t))\big)\big), \end{aligned} \tag{4.15}$$

holds for all $r, q \in \mathbb{N}$, $u \in \hat{\mathbb{R}}_+^r$ with $u_r = s$, $w \in \hat{\mathbb{R}}_+^q$, $h \in B(\mathcal{G}^r)$, $g \in B(\mathcal{G}^q)$ and $f \in B(\mathcal{G})$. (Since the random variables under the expectation signs of both sides of equation (4.15) vanish on the set $\{Y(s) = \Delta\}$, we can henceforth safely ignore the condition $Y(s) \neq \Delta$.) Expand the left-hand side of Eq. (4.15) as follows:

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,f\big(Y(s+t)\big)\big) \\ &\qquad = \sum_{n=m}^{\infty} E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_n(f,1,s+t)\big). \end{aligned} \tag{4.16}$$

Consider the summand with $n = m$, which is of the form

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h\otimes f,1,(u,s+t))\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,E_\xi\big(W_m(h\otimes f,1,(u,s+t))\,\big|\,\mathcal{B}_m\big)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,R_0(h\otimes f,1,(u,s+t))(S_m,K_m)\big), \end{aligned}$$

where we made use of formula (4.11). Now we apply statement (a) of Lemma 4.3 with the choice $q = 1$, which yields (recall that $u_r = s$)

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h\otimes f,1,(u,s+t))\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,R_0(M(f,s+t,s)h,1,u)(S_m,K_m)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,E_\xi\big(W_m(M(f,s+t,s)h,1,u)\,\big|\,\mathcal{B}_m\big)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(M(f,s+t,s)h,1,u)\big), \end{aligned}$$

where we used formula (4.11) again. Combining (4.4) with (4.9b) in $W_m(M(f, s+t, s)h, 1, u)$, we thus have shown

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_m(f,1,s+t)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,E_{Y(s)}\big(f(Y(t))\,1_{\{0<t<S_1\}}\big)\big). \end{aligned} \tag{4.17}$$

Next consider a generic summand with $n > m$ on the right-hand side of (4.16). Then

$$\begin{aligned} &E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_n(f,1,s+t)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,E_\xi\big(W_n(f,1,s+t)\,\big|\,\mathcal{B}_{m+1}\big)\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,R_{n-m-1}(f,1,s+t)(S_{m+1},K_{m+1})\big) \\ &\qquad = E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,R_{n-m-1}(f,1,s+t),u)\big), \end{aligned}$$

where we used Lemma 4.7 in the second step. Conditioning on $\mathcal{B}_m$ gives

$$\begin{aligned}
&E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_n(f,1,s+t)\big)\\
&\quad= E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,E_\xi\big(W_m(h,\,R_{n-m-1}(f,1,s+t),u)\,\big|\,\mathcal{B}_m\big)\big)\\
&\quad= E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,R_0(h,\,R_{n-m-1}(f,1,s+t),u)(S_m,K_m)\big)\\
&\quad= E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,R_0\big(N(R_{n-m-1}(f,1,s+t),s)h,1,u\big)(S_m,K_m)\big).
\end{aligned}$$

Here, we used Lemmas 4.4, 4.3.b, and in the last step also $u_r = s$. Applying Lemma 4.4, we get

$$\begin{aligned}
&E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_n(f,1,s+t)\big)\\
&\quad= E_\xi\Big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\\
&\qquad\quad\times E_\xi\big(W_m(N(R_{n-m-1}(f,1,s+t),s)h,1,u)\,\big|\,\mathcal{B}_m\big)\Big)\\
&\quad= E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m\big(N(R_{n-m-1}(f,1,s+t),s)h,1,u\big)\big)\\
&\quad= E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,h\big(Y(u)\big)\,\chi_m(u)\\
&\qquad\quad\times E_{Y(s)}\big(R_{n-m-1}(f,1,s+t)(s+S_1,K_1)\big)\big).
\end{aligned}$$

For $\eta\in\mathcal{G}$, relation (4.7) yields

$$\begin{aligned}
E_\eta\big(R_{n-m-1}(f,1,s+t)(s+S_1,K_1)\big) &= E_\eta\big(R_{n-m-1}(f,1,t)(S_1,K_1)\big)\\
&= E_\eta\big(E_\eta\big(W_{n-m}(f,1,t)\,\big|\,\mathcal{B}_1\big)\big)\\
&= E_\eta\big(W_{n-m}(f,1,t)\big),
\end{aligned}$$

with another application of formula (4.14). With the choice $\eta = Y(s)$ this relation therefore gives

$$\begin{aligned}
&E_\xi\big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,W_m(h,1,u)\,W_n(f,1,s+t)\big)\\
&\quad= E_\xi\Big(g\big(Y(w)\big)\,1_{\{0\le w<S_m\}}\,h\big(Y(u)\big)\,\chi_m(u) \qquad (4.18)\\
&\qquad\quad\times E_{Y(s)}\big(f(Y(t))\,1_{\{S_{n-m}\le t<S_{n-m+1}\}}\big)\Big).
\end{aligned}$$

Formulae (4.17) and (4.18) entail

$$\sum_{n=m}^{\infty} E_\xi\Big(g\big(Y(w)\big)\, 1_{\{0\le w<S_m\}}\, W_m(h,1,u)\, W_n(f,1,s+t)\Big)$$

$$= E_\xi\Big(g\big(Y(w)\big)\, 1_{\{0\le w<S_m\}}\, W_m(h,1,u)\, E_{Y(s)}\big(f(Y(t))\, 1_{\{0\le t<S_1\}}\big)\Big)$$

$$+ \sum_{n=m+1}^{\infty} E_\xi\Big(g\big(Y(w)\big)\, 1_{\{0\le w<S_m\}}\, h\big(Y(u)\big)\, \chi_m(u)$$

$$\times E_{Y(s)}\big(f(Y(t))\, 1_{\{S_{n-m}\le t<S_{n-m+1}\}}\big)\Big)$$

$$= E_\xi\Big(g\big(Y(w)\big)\, 1_{\{0\le w<S_m\}}\, W_m(h,1,u)\, E_{Y(s)}\big(f(Y(t))\big)\Big),$$

which proves equation (4.15). □

D. A Brownian motion on $\mathcal{G}$ and its generator

The stochastic process Y and its underlying probability family $(\Xi, \mathcal{C}, Q)$, $Q = (Q_\xi,\ \xi \in \mathcal{G})$, are not very convenient to work with. Therefore, we introduce another version in this subsection. As the underlying sample space Ω, we choose the path space $C_\Delta(\mathbb{R}_+, \mathcal{G})$ of Y endowed with the σ–algebra $\mathcal{A}$ generated by the cylinder sets of $C_\Delta(\mathbb{R}_+, \mathcal{G})$. Obviously, Y is a measurable mapping from $(\Xi, \mathcal{C})$ into $(\Omega, \mathcal{A})$. For $\xi \in \mathcal{G}$, let P_ξ denote the image measure of Q_ξ under Y. Set $P = (P_\xi,\ \xi \in \mathcal{G})$. Moreover, let the canonical coordinate process on $(\Omega, \mathcal{A})$ be denoted by $X = (X_t,\ t \in \mathbb{R}_+)$. Clearly, X is a version of Y. We set $X_{+\infty} = \Delta$, and denote the natural filtration of X by $\mathcal{F} = (\mathcal{F}_t,\ t \in \mathbb{R}_+)$. As usual $\mathcal{F}_\infty$ stands for $\sigma(\mathcal{F}_t,\ t \in \mathbb{R}_+)$. Whenever it is notationally more convenient, we shall also write $X(t)$ for X_t, $t \in \mathbb{R}_+$.

Let H_V denote the hitting time of the set V of vertices of $\mathcal{G}$ by X:

$$H_V = \inf\{t > 0,\ X_t \in V\}.$$

Suppose that X starts in $\xi \in l^\circ$, $l \in \mathcal{I} \cup \mathcal{E}$, and that l is isomorphic to the interval I. Then it follows directly from the discussion at the end of Subsection IV B that the stopped process $X(\,\cdot\, \wedge H_V)$ is equivalent to a standard Brownian motion on I with absorption in the endpoint(s) of I. The necessary path properties of X being obvious, we therefore find that X satisfies all defining properties of a Brownian motion on $\mathcal{G}$ (cf. Definition 2.1), except that we still have to prove its strong Markov property. This will be done next.

Let $\theta = (\theta_t,\ t \in \mathbb{R}_+)$ denote the natural family of shift operators on Ω: $\theta_t(\omega) = \omega(t + \,\cdot\,)$ for $\omega \in \Omega$. Thus in particular, θ is a family of shift operators for X. Since the simple Markov property is a property of the finite dimensional distributions of a stochastic process, and the finite dimensional distributions of X and Y coincide, it immediately follows from Proposition 4.1 that X is a Markov process. Then standard monotone class arguments (e.g., Refs. 47 and 71) give the Markov property in the familiar general form:

Proposition 4.8: Assume that $\xi \in \mathcal{G}$, $t \in \mathbb{R}_+$, and that W is an $\mathcal{F}_\infty$–measurable, positive or integrable random variable on $(\Omega, \mathcal{A}, P)$. Then

$$E_\xi\big(W\circ\theta_t \,\big|\, \mathcal{F}_t\big) = E_{X_t}\big(W\big), \tag{4.19}$$

holds true P_ξ–almost surely on $\{X_t \neq \Delta\}$.

A routine argument based on the path properties of X (similar to, but much easier than the one used in the proof of Lemma A.1 in Appendix A) shows that H_V is an $\mathcal{F}$–stopping time. We have the following:

Lemma 4.9: X has the strong Markov property with respect to the hitting time H_V. That is, for all $\xi \in \mathcal{G}$, $t \in \mathbb{R}_+$, $f \in B(\mathcal{G})$,

$$E_\xi\big(f(X_{t+H_V})\,\big|\,\mathcal{F}_{H_V}\big) = E_{X_{H_V}}\big(f(X_t)\big) \tag{4.20}$$

holds true P_ξ–almost surely.

Proof: To begin with, observe that since $\Omega = C_\Delta(\mathcal{G})$ and X is the canonical coordinate process, there is a natural family $\alpha = (\alpha_t,\ t \in \mathbb{R}_+)$ of stopping operators for X, namely $\alpha_t(\omega) = \omega(\,\cdot \wedge t)$, $t \in \mathbb{R}_+$. Therefore, we get that $\mathcal{F}_T = \sigma(X_{s\wedge T},\ s \in \mathbb{R}_+)$ for any stopping time T relative to $\mathcal{F}$. Indeed, one can show this along the same lines used to prove Galmarino's theorem (e.g., Refs. [5, p. 458], [42, p. 86], [48, p. 43, ff], and [71, p. 45]). Therefore, it is sufficient to prove that for all $n \in \mathbb{N}$, $s_1, \dots, s_n \in \mathbb{R}_+$, $t \in \mathbb{R}_+$, $\xi \in \mathcal{G}$, and all $g \in B(\mathcal{G}^n)$, $f \in B(\mathcal{G})$, the following formula:

$$\begin{aligned} &E_\xi\big(g(X(s_1\wedge H_V),\dots,X(s_1\wedge H_V))\,f(X(t+H_V))\big)\\ &\qquad = E_\xi\big(g(X(s_1\wedge H_V),\dots,X(s_1\wedge H_V))\,E_{X(H_V)}\big(f(X(t))\big)\big)\end{aligned} \tag{4.21}$$

holds. Recall that S_V denotes the hitting time of V by Y. Since P_ξ is the image of Q_ξ under Y, and since $S_V = H_V \circ Y$, Eq. (4.21) is equivalent to

$$E_\xi\big(G\,f(Y(t+S_V))\big) = E_\xi\big(G\,E_{Y(S_V)}\big(f(Y(t))\big)\big), \tag{4.22}$$

where we have set

$$G = g\big(Y(s_1\wedge S_V),\dots,Y(s_n\wedge S_V)\big).$$

Recall that S_1 denotes the hitting time of $V_c \subset V$ by Y, so that $S_V \le S_1$. Y is progressively measurable relative to $\mathcal{F}^Y$, which entails that G is measurable with respect to $\mathcal{F}^Y_{S_V} \subset \mathcal{F}^Y_{S_1} \subset \mathcal{B}_1$ (see also the corresponding argument in the proof of Lemma A.1). Using the notation of Subsection IV C, we write

$$\begin{aligned} &E_\xi\big(G\,f(Y(t+S_V))\big)\\ &\quad = E_\xi\big(G\,f(Y(t+S_V);\ S_V \le t+S_V < S_1)\big)\\ &\qquad + \sum_{n=1}^{\infty} E_\xi\big(G\,E_\xi\big(f(Y(t+u))\,\chi_n(t+u)\,\big|\,\mathcal{B}_1\big)\big|_{u=S_V}\big).\end{aligned} \tag{4.23}$$

For the last equality — similarly as in the proof of Lemma 4.2—we made use of the product structure of the probability space $(\Xi, \mathcal{C}, Q_\xi)$, and the fact that $S_V \le S_1$, which entails that S_V only depends on the variable $\omega^0 \in \Xi^0$. By Lemma 4.7 and formula (4.7), we get for $u \le S_1$, $n \in \mathbb{N}$,

$$\begin{aligned} E_\xi\big(f(Y(t+u))\,\chi_n(t+u)\,\big|\,\mathcal{B}_1\big) &= R_{n-1}(f,1,t+u)(S_1,K_1)\\ &= R_{n-1}(f,1,t)(S_1-u,K_1).\end{aligned}$$

Then

$$
\begin{aligned}
E_\xi\big(G\, R_{n-1}(f,1,t)(S_1 - S_V, K_1)\big)
&= E_\xi\big(G\, R_{n-1}(f,1,t)(0, K_1);\ S_V = S_1\big)\\
&\qquad + E_\xi\big(G\, R_{n-1}(f,1,t)(S_1 - S_V, K_1);\ S_V < S_1\big)\\
&= E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t))\,\chi_{n-1}(t)\big);\ S_V = S_1\big)\\
&\qquad + E_\xi\big(G\, R_{n-1}(f,1,t)(S_1 - S_V, K_1);\ S_V < S_1\big),
\end{aligned}
$$

because on $S_V = S_1$, $Y(S_V) = Y(S_1) = K_1$. The second term on the right-hand side of the last equality only involves the random variables $Y(s_i \wedge S_V)$, S_1, S_V, and K_1. They are all defined in terms of the strong Markov process Z^0 underlying the construction of Y (cf. Sec. IV B). Moreover, on the event $\{S_V < S_1\}$, we get from the definition of S_1 as the hitting time of V_c that $S_1 = S_V + S_1 \circ \vartheta_{S_V}$. Also, on $\{S_V < S_1\}$, $K_1 = K_1 \circ \vartheta_{S_V}$ holds true. On the other hand, G is measurable with respect to $\mathcal{F}^0_{S_V}$, where $\mathcal{F}^0$ is the natural filtration of Z^0. Thus, the strong Markov property of Z^0 gives

$$
\begin{aligned}
E_\xi\big(G\, R_{n-1}(f,1,t)(S_1 - S_V, K_1);\ S_V < S_1\big)
&= E_\xi\big(G\, E_{Y(S_V)}\big(R_{n-1}(f,1,t)(S_1, K_1)\big);\ S_V < S_1\big)\\
&= E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t))\,\chi_n(t)\big);\ S_V < S_1\big).
\end{aligned}
$$

In the last step, we used for $\eta \in \mathcal{G}$,

$$
\begin{aligned}
E_\eta\big(R_{n-1}(f,1,t)(S_1, K_1)\big) &= E_\eta\big(E_\eta\big(W_n(f,1,t)\,\big|\,\mathcal{B}_1\big)\big)\\
&= E_\eta\big(f(Y(t))\,\chi_n(t)\big),
\end{aligned}
$$

with another application of Lemma 4.7, and then we made the choice $\eta = Y(S_V)$. Similarly, for the first term on the right-hand side of Eq. (4.23), we can use the strong Markov property of Z^0 (together with $\{t + S_V < S_1\} = \{t < S_1 \circ \vartheta_{S_V}\} \cap \{S_V < S_1\}$) to show that

$$
\begin{aligned}
E_\xi\big(G\, f(Y(t + S_V));\, S_V \le t + S_V < S_1\big)
&= E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t);\ t < S_1)\big);\ S_V < S_1\big).
\end{aligned}
$$

Inserting these results into the right-hand side of formula (4.23), we find

$$
\begin{aligned}
E_\xi\big(G\, f(Y(t + S_V))\big)
&= E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t));\ t < S_1\big);\ S_V < S_1\big)\\
&\qquad + \sum_{n=1}^{\infty} E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t))\,\chi_n(t)\big);\ S_V < S_1\big)\\
&\qquad + \sum_{n=0}^{\infty} E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t))\,\chi_n(t)\big);\ S_V = S_1\big)\\
&= E_\xi\big(G\, E_{Y(S_V)}\big(f(Y(t))\big)\big).
\end{aligned}
$$

Thus, Eq. (4.22) holds and Lemma 4.9 is proved. □

Proposition 4.10: X is a Feller process.

Proof: It is well known that it is sufficient to prove (i) that the resolvent of X preserves $C_0(\mathcal{G})$, and (ii) that for all $f \in C_0(\mathcal{G})$, $\xi \in \mathcal{G}$, $E_\xi\big(f(X_t)\big)$ converges to $f(\xi)$ as t decreases to 0. (A complete proof can be found in Appendix B.) Statement (ii) immediately follows by an application of the dominated convergence theorem and the fact that X is a normal process with right continuous paths.

To prove statement (i), consider the resolvent $R = (R_\lambda, \lambda > 0)$ of X, and let $\lambda > 0$. Since X is strongly Markovian with respect to the hitting time H_V of the set of vertices V (Lemma 4.9), we get for $\xi \in \mathcal{G}$, $f \in B(\mathcal{G})$ the first passage time formula (e.g., Ref. 70 or 42)

$$(R_\lambda f)(\xi) = (R_\lambda^D f)(\xi) + E_\xi\big(e^{-\lambda H_V}\,(R_\lambda f)(X_{H_V})\big), \tag{4.24}$$

where R^D is the Dirichlet resolvent. That is, R^D is the resolvent of the process X with killing at the moment of reaching a vertex of $\mathcal{G}$. Recall the equivalence of the stopped process $X(\,\cdot\, \wedge H_V)$ with the Brownian motion with absorption on the corresponding interval I stated at the beginning of this subsection. Then we can give explicit expressions for all entities appearing in the first passage time formula (4.24). Using the well-known formulae for Brownian motions on the real line (see, e.g., Refs. [20, p. 73, ff] and [42, p. 29, ff]), we find for $\xi, \eta \in \mathcal{G}^\circ$,

$$r_\lambda^D(\xi,\eta) = \sum_{i\in\mathcal{I}} r_{\lambda,i}^D(\xi,\eta)\,1_{\{\xi,\eta\in i\}} + \sum_{e\in\mathcal{E}} r_{\lambda,e}^D(\xi,\eta)\,1_{\{\xi,\eta\in e\}}, \tag{4.25a}$$

with

$$r_{\lambda,i}^D(\xi,\eta) = \frac{1}{\sqrt{2\lambda}} \sum_{k\in\mathbb{Z}} \Big(e^{-\sqrt{2\lambda}|x-y+2ka_i|} - e^{-\sqrt{2\lambda}|x+y+2ka_i|}\Big), \tag{4.25b}$$

where in local coordinates $\xi = (i, x)$, $\eta = (i, y)$, $x, y \in (0, a_i)$. In the case of an external edge e, we get

$$r_{\lambda,e}^D(\xi,\eta) = \frac{1}{\sqrt{2\lambda}} \Big(e^{-\sqrt{2\lambda}|x-y|} - e^{-\sqrt{2\lambda}(x+y)}\Big), \tag{4.25c}$$

with $\xi = (e, x)$, $\eta = (e, y)$, $x, y \in (0 + \infty)$. Remark that both kernels vanish whenever ξ or η converge from the interior of any edge to a vertex to which the edge is incident.

Consider the second term on the right-hand side of Eq. (4.24). Suppose that $\xi \in i^\circ$, $i \in \mathcal{I}$, and that i is isomorphic to $[0, a_i]$. Assume furthermore that v_1, v_2 are the vertices in V to which i is incident, and that under this isomorphism v_1 corresponds to 0, while v_2 corresponds to a_i. Then we get

$$\begin{aligned} E_\xi\big(e^{-\lambda H_V}\,(R_\lambda f)(X_{H_V})\big) &= E_\xi\big(e^{-\lambda H_{v_1}};\, H_{v_1} < H_{v_2}\big)\,(R_\lambda f)(v_1) \\ &\quad + E_\xi\big(e^{-\lambda H_{v_2}};\, H_{v_2} < H_{v_1}\big)\,(R_\lambda f)(v_2), \end{aligned}$$

because X has paths, which are continuous up to the lifetime of X, and X cannot be killed before reaching a vertex. Here, H_{v_k}, $k = 1, 2$, denotes the hitting time of the vertex v_k. The expectation values in the last line are those of a standard Brownian motion and they are well-known, too (see, e.g., Refs. [20, p. 73, ff] and [42, p. 29, ff]). Thus, for $\xi = (i, x)$, $x \in [0, a_i]$,

$$\begin{aligned} &E_\xi\big(e^{-\lambda H_V}\,(R_\lambda f)(X_{H_V})\big) \\ &\quad = \frac{\sinh\big(\sqrt{2\lambda}(a_i - x)\big)}{\sinh\big(\sqrt{2\lambda}a_i\big)}\,(R_\lambda f)(v_1) + \frac{\sinh\big(\sqrt{2\lambda}x\big)}{\sinh\big(\sqrt{2\lambda}a_i\big)}\,(R_\lambda f)(v_2). \end{aligned} \tag{4.26}$$

Similarly, for $\xi \in e^\circ$ with local coordinates $\xi = (e, x)$, $x \in (0, +\infty)$ we find

$$E_\xi\big(e^{-\lambda H_V}\,(R_\lambda f)(X_{H_V})\big) = e^{-\sqrt{2\lambda}x}\,(R_\lambda f)(v), \tag{4.27}$$

where v is the vertex to which e is incident.

With the formulae (4.24)–(4.27), it is straightforward to check that R_λ maps $C_0(\mathcal{G})$ into itself, and the proof is complete. □

Since X has right continuous paths, standard results (see, e.g, Ref. [71, Theorem 3.3.1], or [92, Theorem 3.15.3]) provide the

Corollary 4.11: X is strongly Markovian.

Thus, we have also proved the

Corollary 4.12: X is a Brownian motion on $\mathcal{G}$ in the sense of Definition 2.1.

It remains to calculate the domain of the generator of X, i.e., the boundary conditions at the vertices. Let $v \in V$, and assume that X starts in v. Then by construction of X and Y, X is equivalent to Z^0 with start in v up to its first hitting of a shadow vertex. That is, X is equivalent to Z^0 up to the first time X hits a vertex different from v. It follows that if v is absorbing for Z^0, then it is so for X, and if v is an exponential holding point with jump to Δ, then it is also so for X with the same exponential rate. In particular, v is a trap for X if and only if it is a trap for Z^0. If v is not a trap, then we can use Dynkin's formula (Ref. [18, p. 140, ff]) to calculate the boundary condition implemented by X. Clearly, this gives the same boundary conditions as for Z^0, because Dynkin's formula only involves an arbitrary small neighborhood of the vertex. (See also the corresponding arguments in Sec. III.) Thus, we have proved the following:

Theorem 4.13: *X is a Brownian motion on $\mathcal{G}$ whose generator has a domain characterized by the same boundary conditions as the generator of Z^0.*

V. PROOF OF THEOREM 2.8

Suppose that $\mathcal{G} = (V, \mathcal{I}, \mathcal{E}, \partial)$ is a metric graph without tadpoles. Let data a, b, c as in (2.3) be given, which satisfy Eq. (2.4).

With every $v \in V$, we associate a single vertex graph $\mathcal{G}(v)$ consisting of the vertex v and $|\mathcal{L}(v)|$ external edges. In Ref. 52, the authors have shown how the construction of Brownian motions on a finite or semi-infinite interval by Feller[24–26] and Itô–McKean[41,42] (cf. also Ref. 48) can be extended to the case of single vertex graphs. For the convenience of the reader, we quickly sketch the method. If $b_v = 0$ and $c_v = 1$ then this trivially is a collection of $|\mathcal{L}(v)|$ many standard Brownian motions on the real line with absorption at the origin (corresponding to the vertex v), mapped onto the external edges of $\mathcal{G}(v)$. If $b_v = 0$ and $c_v < 1$, these Brownian motions are killed by a jump to Δ after holding the processes at the origin for an independent exponentially distributed time of rate a/c. For $b_v \neq 0$, one uses a Walsh process on $\mathcal{G}(v)$ (see Refs. 89 and 3), and builds in a time delay as well as killing, both on the scale of the local time at the vertex. With appropriately chosen parameters for these two mechanisms, Theorem 5.7 in Ref. 52 states that the so constructed process X^v is a Brownian motion on $\mathcal{G}(v)$ such that its generator is the 1/2 times the Laplace operator acting on $f \in C_0^2(\mathcal{G}(v))$ with boundary conditions at the vertex v given by (2.5).

Next, we build the graph $\mathcal{G}$ by successively connecting appropriately chosen external edges of the single vertex graphs $\mathcal{G}(v)$, $v \in V$, as in Sec. IV A. Consider the stochastic process X, which is successively constructed from the Brownian motions X^v as in Subsections IV B–IV D. Theorem 4.13 states that X is a Brownian motion on $\mathcal{G}$, which is such that its generator has a domain, which is characterized by the same boundary conditions at each vertex $v \in V$ as for the single vertex graphs $\mathcal{G}(v)$. Therefore, X is a Brownian motion on $\mathcal{G}$ as in the statement of Theorem 2.8, whose proof is therefore complete.

VI. DISCUSSION OF TADPOLES

Suppose that $\mathcal{G}$ is a metric graph, which has one tadpole i_t connected to a vertex $v \in V$. That is, v is simultaneously the initial and final vertex of i_t: $\partial(i_t) = (v, v)$. Figure 6 shows a metric graph with a tadpole attached to the vertex v. Let b_t be the length of i_t. Assume furthermore that we are

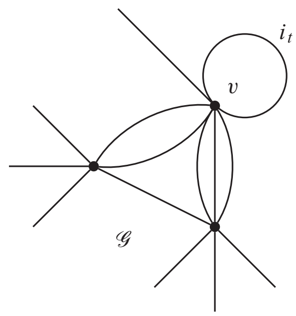

FIG. 6. A metric graph with a tadpole i_t attached to v.

given data a, b, c as in Eqs. (2.3) and (2.4). We want to construct a Brownian motion X on $\mathcal{G}$ the implementing boundary conditions corresponding to these data.

Let $\mathcal{G}_1$ be the metric graph obtained from $\mathcal{G}$ by replacing the tadpole by two external edges e_1, e_2, incident with v. Construct a Brownian motion X_1 on $\mathcal{G}_1$ corresponding to the data a, b, c as above.

Consider the real line $\mathbb{R}$ as a single vertex graph $\mathcal{G}_2$ with the origin as the vertex v_0, and edges l_1, l_2, which are isomorphic to $[0, +\infty)$, $(-\infty, 0]$. Take a Walsh process X_2 on this graph, which with probability 1/2 chooses either edge for the next excursion when at the origin. Then X_2 is just a "skew" Brownian motion as in Ref. [42, p. 115], which actually is not skew. That is, it is equivalent to a standard Brownian motion on the real line.

Now join $\mathcal{G}_1$ and $\mathcal{G}_2$ by connecting the pairs (e_1, l_1), (e_2, l_2) via two new internal edges of length $b_t/2$. Denote the resulting metric graph by $\hat{\mathcal{G}}$. Figure 7 shows this construction. Construct a Brownian motion $\hat{X}$ on $\hat{\mathcal{G}}$ from X_1 and X_2 as in Sec. III. By construction, $\hat{X}$ is equivalent to a standard Brownian motion in every neighborhood of v_0, which is small enough such that it does not include the vertex v. Therefore, the additional vertex v_0 of $\mathcal{G}$ does not yield any non-trivial boundary condition. Thus, if we identify the open tadpole edge i_t^0 with the subset of $\mathcal{G}_2$ isomorphic to $(-b_t/1, b_t/2)$, then we obtain a Brownian motion X on $\mathcal{G}$ implementing the desired boundary conditions.

Obviously, any (finite) number of tadpoles can be handled in the same way.

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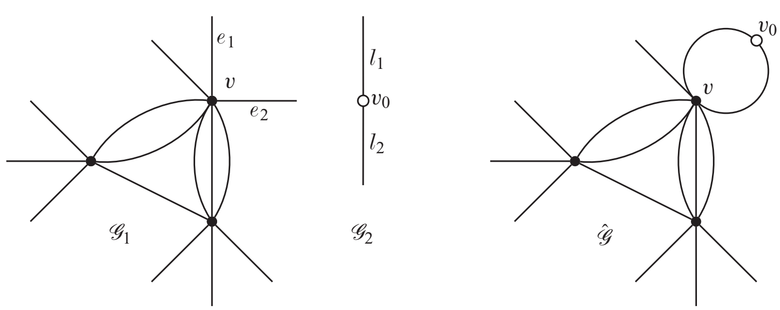

FIG. 7. $\hat{\mathcal{G}}$ constructed from $\mathcal{G}_1$ and $\mathcal{G}_2$.

APPENDIX A: ON THE CROSSOVER TIMES S_n

We recall from Sec. III that in terms of the process Y, the crossover times S_n, $n \in \mathbb{N}$, can be described as follows. Let Y start in $\xi \in \mathcal{G}$. Then S_1 is the hitting time of $V_c \setminus \{\xi\}$ by Y. In particular, $S_1 > 0$ for all paths of Y. For $n \geq 2$, S_n is the hitting time after S_{n-1} of $V_c \setminus \{K_{n-1}\}$ by Y. Since by construction $Y(S_{n-1}) = K_{n-1}$ and the paths of Y are continuous on $[0, \zeta)$, we get $S_n > S_{n-1}$ for all paths of Y. Therefore,

$$S_n = \inf\bigl\{t > S_{n-1},\ Y(t) \in V_c \setminus \{K_{n-1}\}\bigr\}$$

$$= \inf\bigl\{t \geq S_{n-1},\ Y(t) \in V_c \setminus \{K_{n-1}\}\bigr\}$$

holds.

In this appendix, we prove the following:

Lemma A.1: For every $n \in \mathbb{N}$, S_n is a stopping time relative to $\mathcal{F}^Y$.

Proof: Set $S_0 = 0$, and for $n \in \mathbb{N}$,

$$V_n = \begin{cases} V_c \setminus \{\xi\}, & \text{if } n = 1, \\ V_c \setminus \{K_{n-1}\}, & \text{otherwise.} \end{cases}$$

For $n \in \mathbb{N}$, $r \in \mathbb{R}_+$ define

$$A_{n,r} = \{S_n \leq r < \zeta\}$$

$$B_{n,r} = \bigcap_{m \in \mathbb{N}} \bigcup_{u \in \mathbb{Q},\, 0 \leq u \leq r} \bigl\{S_{n-1} \leq u,\ d\bigl(Y(u), V_n\bigr) \leq 1/m,\ r < \zeta\bigr\}.$$

We claim that for all $n \in \mathbb{N}$, $r \in \mathbb{R}_+$,

$$A_{n,r} = B_{n,r}. \tag{A.1}$$

To prove this claim, suppose first that $\omega \in A_{n,r}$. Then $S_n(\omega)$ is finite, and therefore the set

$$\bigl\{t \geq S_{n-1}(\omega),\ Y(t, \omega) \in V_n(\omega)\bigr\} \subset [0, \zeta(\omega))$$

is non-empty. Thus, there exists a sequence $(u_i,\ i \in \mathbb{N})$ in this set, which decreases to $S_n(\omega)$. Since $Y(\,\cdot\,, \omega)$ is continuous on $[0, \zeta(\omega))$, it follows that $Y(S_n(\omega), \omega) \in V_n(\omega)$. By assumption, $S_n(\omega) \leq r < \zeta(\omega)$, and therefore the continuity of $Y(\,\cdot\,, \omega)$ on $[0, \zeta(\omega))$ and $S_{n-1}(\omega) < S_n(\omega)$ imply that for every $m \in \mathbb{N}$ there exists $u \in \mathbb{Q} \cap [S_{n-1}(\omega), r]$ so that $d(Y(u, \omega), V_n(\omega)) \leq 1/m$. Hence, $\omega \in B_{n,r}$.

As for the converse, suppose now that $\omega \in B_{n,r}$. Then there exists a sequence $(u_m,\ m \in \mathbb{N})$ in $\mathbb{Q} \cap [S_{n-1}(\omega), r]$ so that $d(Y(u_m, \omega), V_n(\omega))$ converges to zero as m tends to infinity. Since $(u_m,\ m \in \mathbb{N})$ is bounded, we may assume, by selecting a subsequence if necessary, that $(u_m,\ m \in \mathbb{N})$ converges to some $u \in [S_{n-1}(\omega), r]$ as $m \to +\infty$. Thus, we find that $Y(u, \omega) \in V_n(\omega)$, and therefore $Y(\,\cdot\,, \omega)$ hits $V_n(\omega)$ in the interval $[S_{n-1}(\omega), r]$. Consequently, $S_n(\omega) \leq r$, and hence $\omega \in A_{n,r}$, concluding the proof of the claim.

Next, we prove by induction that for every $n \in \mathbb{N}$, S_n is an $\mathcal{F}^Y$–stopping time. Let $n = 1$. By (A.1) for every $r \geq 0$,

$$A_{1,r} = \bigcap_{m \in \mathbb{N}} \bigcup_{u \in \mathbb{Q},\, 0 \leq u \leq r} \bigl\{d\bigl(Y(u), V_1\bigr) \leq 1/m\bigr\} \cap \bigl\{r < \zeta\bigr\}.$$

Clearly, $\{r < \zeta\} = \{Y(r) \in \mathcal{G}\} \in \mathcal{F}^Y_r$. Moreover, since V_1 is a deterministic set, $d(\,\cdot\,, V_1)$ is measurable from $\mathcal{G}$ to $\mathbb{R}_+$, and therefore $\{d(Y(u), V_1) \leq 1/m\} \in \mathcal{F}^Y_u \subset \mathcal{F}^Y_r$. Hence, $A_{1,r} \in \mathcal{F}^Y_r$. Let $t \geq 0$, and write

$$\begin{aligned}\{S_1 \le t\} &= \{S_1 < \zeta \le t\} \cup A_{1,t} \\ &= \Big(\bigcup_{r\in\mathbb{Q},\, 0\le r\le t} \{S_1 \le r < \zeta\} \cap \{\zeta \le t\}\Big) \cup A_{1,t} \\ &= \Big(\bigcup_{r\in\mathbb{Q},\, 0\le r\le t} A_{1,r} \cap \{\zeta \le t\}\Big) \cup A_{1,t}.\end{aligned}$$

Therefore, $\{S_1 \le t\} \in \mathcal{F}^Y_t$, and hence S_1 is a stopping time relative to $\mathcal{F}^Y$.

Now suppose that $n \in \mathbb{N}$, $n \ge 2$, and that S_{n-1} is an $\mathcal{F}^Y$–stopping time. We show that for all $r \in \mathbb{R}_+$, $A_{n,r} \in \mathcal{F}^Y_r$. First, we remark that since $\mathcal{G}$ is a separable metric space, the metric d on $\mathcal{G}$ is a measurable mapping from $(\mathcal{G}\times\mathcal{G}, \mathcal{B}_d\otimes\mathcal{B}_d)$ to $(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+))$. For example, this follows from Theorem 1.1.10 in Ref. 69 and the continuity of $d : \mathcal{G}\times\mathcal{G} \to \mathbb{R}_+$ when $\mathcal{G}\times\mathcal{G}$ is equipped with the product topology. Consider $K_{n-1} = Y(S_{n-1})$. Since Y has right continuous paths, it is progressively measurable relative to $\mathcal{F}^Y$ (e.g., Ref. 71, Proposition 1.4.8]). Thus, by Proposition 1.4.9 in Ref. 71 it follows that K_{n-1} is $\mathcal{F}^Y_{S_{n-1}}$–measurable. Consequently, on $\{S_{n-1} \le u\}$, K_{n-1} is $\mathcal{F}^Y_u$–measurable. Equation (A.1) reads

$$A_{n,r} = \bigcap_{m\in\mathbb{N}} \bigcup_{u\in\mathbb{Q},\, 0\le u\le r} \{S_{n-1} \le u\} \cap \Big\{d\big(Y(u), V_c \setminus K_{n-1}\big) \le 1/m\Big\} \cap \{r < \zeta\}.$$

It follows that $A_{n,r} \in \mathcal{F}^Y_r$, as claimed. But then $\{S_n \le t\} \in \mathcal{F}^Y_t$ for all $t \in \mathbb{R}_+$ is proved with the same argument as at the end of the discussion of the case $n = 1$. □

APPENDIX B: FELLER SEMIGROUPS AND RESOLVENTS

In this appendix, we give an account of the Feller property of semigroups and resolvents. The material here seems to be quite well known, and our presentation of it owes very much to Ref. 70, most notably the inversion formula for the Laplace transform, Eq. (B.3) in connection with Lemma B.7. On the other hand, we were not able to locate a reference where the results are collected and stated in the form in which we employ them in the present article. This applies in particular to the "mixed" forms of the statements (iii)–(vi) in Theorem B.3, which we find especially convenient to use in this article. Therefore, we also provide proofs for some of the statements.

Assume that (E, d) is a locally compact separable metric space with Borel σ–algebra denoted by $\mathcal{B}(E)$. $B(E)$ denotes the space of bounded measurable real valued functions on E, $C_0(E)$ the subspace of continuous functions vanishing at infinity. $B(E)$ and $C_0(E)$ are equipped with the sup-norm $\|\cdot\|$.

The following definition is as in:[71]

Definition B.1: A Feller semigroup is a family $U = (U_t, t \ge 0)$ of positive linear operators on $C_0(E)$ such that

(i) $U_0 = id$ *and* $\|U_t\| \le 1$ *for every* $t \ge 0$;
(ii) $U_{t+s} = U_t \circ U_s$ *for every pair* $s, t \ge 0$;
(iii) $\lim_{t\downarrow 0}\|U_t f - f\| = 0$ *for every* $f \in C_0(E)$.

Analogously, we define

Definition B.2: A Feller resolvent is a family $R = (R_\lambda, \lambda > 0)$ *of positive linear operators on* $C_0(E)$ *such that*

(i) $\|R_\lambda\| \leq \lambda^{-1}$ *for every* $\lambda > 0$;
(ii) $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda \circ R_\mu$ *for every pair* $\lambda, \mu > 0$;
(iii) $\lim_{\lambda \to \infty} \|\lambda R_\lambda f - f\| = 0$ *for every* $f \in C_0(E)$.

In the sequel, we shall focus our attention on semigroups U and resolvents R associated with an E–valued Markov process, and which are *a priori* defined on $B(E)$. (In our notation, we shall not distinguish between U and R as defined on $B(E)$ and their restrictions to $C_0(E)$.)

Let $X = (X_t, t \geq 0)$ be a Markov process with state space E, and let $(P_x, x \in E)$ denote the associated family of probability measures on some measurable space $(\Omega, \mathcal{A})$ so that $P_x(X_0 = x) = 1$. $E_x(\,\cdot\,)$ denotes the expectation with respect to P_x. We assume throughout that for every $f \in B(E)$ the mapping

$$(t, x) \mapsto E_x\big(f(X_t)\big)$$

is measurable from $\mathbb{R}_+ \times E$ into $\mathbb{R}$. The semigroup U and resolvent R associated with X act on $B(E)$ as follows. For $f \in B(E)$, $x \in E$, $t \geq 0$, and $\lambda > 0$ set

$$U_t f(x) = E_x\big(f(X_t)\big), \tag{B.1}$$

$$R_\lambda f(x) = \int_0^\infty e^{-\lambda t} U_t f(x)\, dt. \tag{B.2}$$

Property (i) of Definitions B.1 and B.2 is obviously satisfied. The semigroup property, (ii) in Definition B.1, follows from the Markov property of X, and this, in turn, implies the resolvent equation, (ii) of Definition B.2. Moreover, it follows also from the Markov property of X that the semigroup and the resolvent commute. On the other hand, in general, neither the property that U or R map $C_0(E)$ into itself, nor the strong continuity property (iii) in Definitions B.1, B.2 hold true on $B(E)$ or on $C_0(E)$.

If W is a subspace of $B(E)$, the resolvent equation shows that the image of W under R_λ is independent of the choice of $\lambda > 0$, and in the sequel, we shall denote the image by RW. Furthermore, for simplicity, we shall write $UC_0(E) \subset C_0(E)$, if $U_t f \in C_0(E)$ for all $t \geq 0, f \in C_0(E)$.

Theorem B.3. *The following statements are equivalent:*

(i) *U is Feller.*
(ii) *R is Feller.*
(iii) $UC_0(E) \subset C_0(E)$, *and for all* $f \in C_0(E)$, $x \in E$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$.
(iv) $UC_0(E) \subset C_0(E)$, *and for all* $f \in C_0(E)$, $x \in E$, $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$.
(v) $RC_0(E) \subset C_0(E)$, *and for all* $f \in C_0(E)$, $x \in E$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$.
(vi) $RC_0(E) \subset C_0(E)$, *and for all* $f \in C_0(E)$, $x \in E$, $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$.

Remark B.4: The equivalence of statements (i) and (ii) has been shown in Ref. [14, Vol. 81, p. 291] based on an application of the Hille–Yosida–theorem.

We prepare a sequence of lemmas. The first one follows directly from the dominated convergence theorem:

Lemma B.5: Assume that for $f \in B(E)$, $U_t f \to f$ as $t \downarrow 0$. *Then* $\lambda R_\lambda f \to f$ *as* $\lambda \to +\infty$.

Lemma B.6: The semigroup U *is strongly continuous on* $RB(E)$.

Proof: If strong continuity at $t = 0$ has been shown, strong continuity at $t > 0$ follows from the semigroup property of U, and the fact that U and R commute. Therefore, it is enough to show strong continuity at $t = 0$.

Let $f \in B(E)$, $\lambda > 0$, $t > 0$, and consider for $x \in E$, the following computation:

$$
\begin{aligned}
&U_t R_\lambda f(x) - R_\lambda f(x)\\
&\quad = \int_0^\infty e^{-\lambda s} E_x\big(f(X_{t+s})\big)\, ds - \int_0^\infty e^{-\lambda s} E_x\big(f(X_s)\big)\, ds\\
&\quad = e^{\lambda t} \int_t^\infty e^{-\lambda s} E_x\big(f(X_s)\big)\, ds - \int_0^\infty e^{-\lambda s} E_x\big(f(X_s)\big)\, ds\\
&\quad = \big(e^{\lambda t} - 1\big) \int_t^\infty e^{-\lambda s} E_x\big(f(X_s)\big)\, ds - \int_0^t e^{-\lambda s} E_x\big(f(X_s)\big)\, ds,
\end{aligned}
$$

where we used Fubini's theorem and the Markov property of X. Thus, we get the following estimation:

$$
\begin{aligned}
\big\| U_t R_\lambda f - R_\lambda f \big\| &\le \left(\big(e^{\lambda t} - 1\big) \int_t^\infty e^{-\lambda s}\, ds + \int_0^t e^{-\lambda s}\, ds \right) \| f \|\\
&= \frac{2}{\lambda} \big(1 - e^{-\lambda t}\big) \| f \|,
\end{aligned}
$$

which converges to zero as t decreases to zero. □

For $\lambda > 0$, $t \ge 0$, $f \in B(E)$, $x \in E$ set

$$
U_t^\lambda f(x) = \sum_{n=1}^\infty \frac{(-1)^{n+1}}{n!}\, n\lambda\, e^{n\lambda t}\, R_{n\lambda} f(x). \tag{B.3}
$$

Observe that, because of $n\lambda \|R_{n\lambda} f\| \le \|f\|$, the last sum converges in $B(E)$.

For the proof of the next lemma, we refer the reader to Ref. [70, p. 477, ff]:

Lemma B.7: For all $t \ge 0$, $f \in RB(E)$, $U_t^\lambda f$ converges in $B(E)$ to $U_t f$ as λ tends to infinity.

Lemma B.8: If $U_t C_0(E) \subset C_0(E)$ for all $t \ge 0$, then $R_\lambda C_0(E) \subset C_0(E)$, for all $\lambda > 0$. If $R_\lambda C_0(E) \subset C_0(E)$, for some $\lambda > 0$, and $R_\lambda C_0(E)$ is dense in $C_0(E)$, then $U_t C_0(E) \subset C_0(E)$ for all $t \ge 0$.

Proof: Assume that $U_t C_0(E) \subset C_0(E)$ for all $t \ge 0$, let $f \in C_0(E)$, $x \in E$, and suppose that $(x_n,\, n \in \mathbb{N})$ is a sequence converging in (E, d) to x. Then a straightforward application of the dominated convergence theorem shows that for every $\lambda > 0$, $R_\lambda f(x_n)$ converges to $R_\lambda f(x)$. Hence, $R_\lambda f \in C_0(E)$.

Now assume that $R_\lambda C_0(E) \subset C_0(E)$, for some and therefore for all $\lambda > 0$, and that $R_\lambda C_0(E)$ is dense in $C_0(E)$. Consider $f \in RC_0(E)$, $t > 0$, and for $\lambda > 0$ define $U_t^\lambda f$ as in Eq. (B.3). Because $R_{n\lambda} f \in C_0(E)$ and the series in formula (B3) converges uniformly in $x \in E$, we get $U_t^\lambda f \in C_0(E)$. By Lemma B.7, we find that $U_t^\lambda f$ converges uniformly to $U_t f$ as $\lambda \to +\infty$. Hence, $U_t f \in C_0(E)$. Since $RC_0(E)$ is dense in $C_0(E)$, U_t is a contraction and $C_0(E)$ is closed, we get that $U_t C_0(E) \subset C_0(E)$ for every $t \ge 0$. □

The following lemma is proved as a part of Theorem 17.4 in Ref. 46 (cf. also the proof of Proposition 2.4 in Ref. 71).

Lemma B.9: Assume that $RC_0(E) \subset C_0(E)$, and that for all $x \in E$, $f \in C_0(E)$, $\lim_{\lambda \to \infty} \lambda R_\lambda f(x) = f(x)$. Then $RC_0(E)$ is dense in $C_0(E)$.

If for all $f \in C_0(E)$, $x \in E$, $U_t f(x)$ converges to $f(x)$ as t decreases to zero, then similarly as in the proof of Lemma B.5, we get that $\lambda R_\lambda f(x)$ converges to $f(x)$ as $\lambda \to +\infty$. Thus, we obtain the following:

Corollary B.10: Assume that $RC_0(E) \subset C_0(E)$, *and that for all* $x \in E$, $f \in C_0(E)$, $\lim_{t \downarrow 0} U_t f(x) = f(x)$. *Then* $RC_0(E)$ *is dense in* $C_0(E)$.

Now we can come to the

Proof of Theorem B.3: We show first the equivalence of statements (i), (ii), (iv), and (vi):

"(i)⇒ (ii)" Assume that U is Feller. From Lemma B.8, it follows that $R_\lambda C_0(E) \subset C_0(E)$, $\lambda > 0$. Let $f \in C_0(E)$. Since U is strongly continuous on $C_0(E)$, Lemma B.5 implies that $\lambda R_\lambda f$ converges to f as λ tends to $+\infty$. Hence, R is Feller.

"(ii)⇒ (vi)" This is trivial.

"(vi)⇒ (iv)" By Lemma B.9, $RC_0(E)$ is dense in $C_0(E)$, and therefore Lemma B.8 entails that $UC_0(E) \subset C_0(E)$.

"(iv)⇒ (i)" By Lemmas B.8 and B.9, $RC_0(E)$ is dense in $C_0(E)$, and therefore by Lemma B.6 U is strongly continuous on $C_0(E)$. Thus, U is Feller.

Now we prove the equivalence of (i), (iii), and (v):

"(i)⇒ (iii)" This is trivial.

"(iii)⇒ (v)" This follows directly from Lemma B.8.

"(v)⇒ (i)" By Corollary B.10, $RC_0(E)$ is dense in $C_0(E)$, hence it follows from Lemma B.8 that $UC_0(E) \subset C_0(E)$. Furthermore, Lemma B.6 implies the strong continuity of U on $RC_0(E)$, and by density therefore on $C_0(E)$. (i) follows. □

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