A NOTE ON THE TRANSITIVE HURWITZ ACTION ON DECOMPOSITIONS OF PARABOLIC COXETER ELEMENTS

BARBARA BAUMEISTER, MATTHEW DYER, CHRISTIAN STUMP,
AND PATRICK WEGENER

(Communicated by Patricia L. Hersh)

ABSTRACT. In this note, we provide a short and self-contained proof that the braid group on n strands acts transitively on the set of reduced factorizations of a Coxeter element in a Coxeter group of finite rank n into products of reflections. We moreover use the same argument to also show that all factorizations of an element in a parabolic subgroup of W also lie in this parabolic subgroup.

1. INTRODUCTION

Let (W, T) be a dual Coxeter system of finite rank m in the sense of [Bes03]. This is to say that there is a subset $S \subseteq T$ with $|S| = m$ such that (W, S) is a Coxeter system, and $T = \{wsw^{-1} : w \in W, s \in S\}$ is the set of reflections for the Coxeter system (W, S). We then call (W, S) a simple system for (W, T) and S a set of simple reflections. Such simple systems for (W, T) were studied by several authors, see e.g. [FHM06] and the references therein. In particular, if S is a simple system for (W, T) then so is wSw^{-1} for any $w \in W$. It is moreover shown in [FHM06] that for important classes, all simple systems for (W, T) are conjugate to one another in this sense.

A reflection subgroup W' is a subgroup of W generated by reflections. It is well known that $(W', W' \cap T)$ is again a dual Coxeter system, see e.g. [Dye90]. For $w \in W$, a reduced T-factorization of w is a shortest length factorization of w into reflections, and we denote by $\mathrm{Red}_T(w)$ the set of all such reduced T-factorizations. Similarly for a given simple system (W, S), a reduced S-factorization of w is a shortest length factorization of w into simple reflections. An element $c \in W$ is called a parabolic Coxeter element for (W, T) if there is a simple system $S = \{s_1, \ldots, s_m\}$ such that $c = s_1 \cdots s_n$ for some $n \leq m$. We call the reflection subgroup generated by $\{s_1, \ldots, s_n\}$ a parabolic subgroup. The element c is moreover called a standard parabolic Coxeter element for the Coxeter system (W, S).

Remark 1.1. Observe that this definition of parabolic Coxeter elements and parabolic subgroups is more general than usual. The simplest Coxeter group for which this definition is indeed more general than considering conjugates of a fixed simple

system is the finite Coxeter group of type $H_2 = I_2(5)$ given by all linear transformations of the plane that leave a regular pentagon invariant. One choice for a simple system is given by two reflections through two consecutive vertices of the pentagon, another choice is the product of two reflections through two vertices with distance two. Both choices of simple systems generate the same set of reflections, even though they are not conjugate.

The braid group on n strands is the group B_n with generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the relations

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for } |i - j| > 1,$$
$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

It acts on the set T^n of n-tuples of reflections as

$$\sigma_i(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \ t_i t_{i+1} t_i, \quad t_i \quad, \ t_{i+2}, \ldots, t_n),$$
$$\sigma_i^{-1}(t_1, \ldots, t_n) = (t_1, \ldots, t_{i-1}, \quad t_{i+1}, \ t_{i+1} t_i t_{i+1}, \ t_{i+2}, \ldots, t_n).$$

For example, if $n = 2$, then the action of σ_1 is described by

$$\ldots \mapsto (srs, srsrs) \mapsto (s, srs) \mapsto (r, s) \mapsto (rsr, r) \mapsto (rsrsr, rsr) \mapsto \ldots$$

for any $r, s \in T$. Note that in this case, the B_2-orbit of (r, s) is the set of all pairs (t_1, t_2) of reflections of the subgroup $\langle r, s \rangle$, such that $t_1 t_2 = rs$.

The following lemma is a direct consequence of the definition.

Lemma 1.2. *Let W' be a reflection subgroup of W and let $T' = T \cap W'$ be the set of reflections in W'. For an element $w \in W'$ with reduced T-factorization $w = t_1 \cdots t_n$, the braid group on n strands acts on $\mathrm{Red}_{T'}(w)$.*

Proof. Let $\sigma_i(t_1, \ldots, t_n) = (t_1', \ldots, t_n')$. The lemma follows from the two observations that $t_1 \cdots t_n = t_1' \cdots t_n'$ and $\{t_1, \ldots, t_n\} \subseteq T'$ if and only if $\{t_1', \ldots, t_n'\} \subseteq T'$. □

This action on $\mathrm{Red}_{T'}(w)$ is also known as the Hurwitz action. For finite Coxeter systems, the Hurwitz action was first shown to act transitively on $\mathrm{Red}_T(c)$ for a Coxeter element c in a letter from P. Deligne to E. Looijenga [Del74]. The first published proof is due to D. Bessis and can be found in [Bes03]. K. Igusa and R. Schiffler generalized this result to arbitrary Coxeter groups of finite rank; see [IS10, Theorem 1.4]. This transitivity has important applications in the theory of Artin groups, see [Bes03, Dig06], and as well as in the representation theory of algebras; see [IS10, Igu11, HK13].

The aim of this note is to provide a simple proof of K. Igusa and R. Schiffler's theorem, based on arguments similar to those in [Dye01]. We moreover emphasize that the condition on the Coxeter element $c \in W$ in this note is slightly relaxed from the condition in the original theorem; compare [IS10, Theorem 1.4].

Theorem 1.3. *Let (W, T) be a dual Coxeter system of finite rank m and let $c = s_1 \cdots s_n$ be a parabolic Coxeter element in W. The Hurwitz action on $\mathrm{Red}_T(c)$ is transitive. In symbols, for each $(t_1, \ldots, t_n) \in T^n$ such that $c = t_1 \cdots t_n$, there is a braid group element $b \in B_n$ such that*

$$b(t_1, \ldots, t_n) = (s_1, \ldots, s_n).$$

By the observation in Lemma 1.2, this theorem has the direct consequence that the parabolic subgroup $\langle s_1, \ldots, s_n \rangle$ of W does indeed not depend on the particular S-factorization $c = s_1 \cdots s_n$ but only on the parabolic Coxeter element c itself. We thus denote this parabolic by $W_c := \langle t_1, \ldots, t_n \rangle$ for any T-factorization $c = t_1 \cdots t_n$. We moreover obtain that $\mathrm{Red}_T(c) = \mathrm{Red}_{T'}(c)$ with $T' = W_c \cap T$ being the set of reflections in the parabolic subgroup W_c. The main argument in the proof of this theorem (see Proposition 2.2 below) will also imply the following theorem that extends this direct consequence to all elements in a parabolic subgroup.

Theorem 1.4. *Let W' be a parabolic subgroup of W. Then for any $w \in W'$,*

$$\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w),$$

where $T' = W' \cap T$ is the set of reflections in W'.

2. The proof

For the proof of the two theorems, we fix a Coxeter system (W, S). Denote by $\ell = \ell_S$ and by ℓ_T the length function on W with respect to the simple generators S and with respect to the generating set T, respectively. Since $S \subseteq T$, we have that $\ell_T(w) \leq \ell(w)$ for all $w \in W$.

The following lemma provides an alternative description of standard parabolic Coxeter elements.

Lemma 2.1. *An element $w \in W$ is a standard parabolic Coxeter element for (W, S) if and only if $\ell_T(w) = \ell(w)$.*

Proof. Given a reduced S-factorization $w = s_{i_1} \ldots s_{i_k}$, it was shown in [Dye01, Theorem 1.1] that $\ell_T(w)$ is given by the minimal number of simple generators that can be removed from $s_{i_1} \ldots s_{i_k}$ to obtain the identity. This yields that $\ell_T(w) = k = \ell(w)$ if and only if $s_{i_1} \ldots s_{i_k}$ does not contain any generator twice. $\square$

Define the Bruhat graph Ω for the dual Coxeter system (W, T) as the undirected graph on vertex set W with edges given by $w - wt$ for $t \in T$. For any factorization $w = t_1 \cdots t_n \in W$ with $t_i \in T$ and any $x \in W$, there is a corresponding path

$$x - xt_1 - xt_1t_2 - \ldots - xt_1 \cdots t_n = xw$$

from x to xw in Ω. It is clear that this T-factorization of w is reduced if and only if the corresponding path from x to xw has minimal length among paths from x to xw for some (equivalently, every) $x \in W$. The simple system (W, S) induces an orientation on Ω given by $w \to wt$ if $\ell(w) < \ell(wt)$. We denote the resulting directed Bruhat graph by Ω_{dir}.

The proof of the two main results is based on the case $x = e$ of the following proposition.

Proposition 2.2. *Let (W, S) be a Coxeter system. Moreover, let $w = t_1 \cdots t_n \in W$ be a reduced T-factorization of an element in W, and let*

$$x - xt_1 - xt_1t_2 - \ldots - xt_1 \cdots t_n = xw$$

be the corresponding path in Ω starting at an element $x \in W$. Then there is a T-factorization $w = t'_1 \cdots t'_n$ in the Hurwitz orbit of the T-factorization $t_1 \cdots t_n$ such that the corresponding path in Ω_{dir} starting at x is first decreasing in length, then increasing; more precisely, it is of the form

$$x \leftarrow xt'_1 \leftarrow xt'_1t'_2 \leftarrow \ldots \leftarrow xt'_1 \cdots t'_i \to \ldots \to xt'_1 \cdots t'_n = xw$$

for some (unique) integer i with $0 \leq i \leq n$. In the special case $x = e$, this gives a directed path

$$e \to t_1' \to t_1' t_2' \to \ldots \to t_1' \cdots t_n' = w$$

in Ω_{dir}.

Proof. First consider two distinct reflections t_1 and t_2 and an element $z \in W$ such that $z \to zt_1 \leftarrow zt_1t_2$ in Ω_{dir}. We claim that there exist reflections $t_1', t_2' \in \langle t_1, t_2 \rangle$ with $t_1 t_2 = t_1' t_2'$ such that $z \to zt_1' \to zt_1't_2'$ or $z \leftarrow zt_1' \leftarrow zt_1't_2'$ or $z \leftarrow zt_1' \to zt_1't_2'$. This implies, by the comment before Lemma 1.2, that one can get from the factorization $t_1 t_2$ to the factorization $t_1' t_2'$ inside $W' = \langle t_1, t_2 \rangle$ by braid moves, and hence in particular that $W' = \langle t_1', t_2' \rangle$. Moreover, one has $\ell(zt_1') < \max(\ell(z), \ell(zt_1t_2)) < \ell(zt_1)$.

To prove the claim, consider the coset $z\langle t_1, t_2 \rangle$ in W. By [Dye01, Theorem 2.1], the proof immediately reduces to the case $z = e$ and $W' = \langle t_1, t_2 \rangle$ dihedral. We check this case directly. To this end, let s_1', s_2' be the Coxeter generators of W', and observe that any reflection (element of odd length) and any rotation (element of even length) in W' are joined by an edge in Ω, which in Ω_{dir} is oriented towards the element of greater length with respect to the generating set s_1', s_2'. Given $z \in W'$, there are three situations: either $\ell(z) < \ell(zt_1t_2)$, or $\ell(z) > \ell(zt_1t_2)$, or $\ell(z) = \ell(zt_1t_2)$. This implies that one can choose t_1' and t_2' with $t_1' t_2' = t_1 t_2$ in the three situations such that $z \to zt_1' \to zt_1't_2'$, $z \leftarrow zt_1' \leftarrow zt_1't_2'$ or $z \leftarrow zt_1' \to zt_1't_2'$ respectively (note that in the third case, one has $z \neq e$ or else $l(zt_1t_2) = l(z)$ implies $t_1 = t_2$, contrary to assumption).

Consider the path in Ω attached to $w = t_1 \cdots t_n$ and beginning at x. Any subpath $z - zt_1 - zt_1t_2$ as in the claim may be replaced by a path $z - zt_1' - zt_1't_2'$ as there, to give a new path from x to xw of the same length n; we call this a "replacement." Apply to the original path a sequence of successive replacements. Any path so obtained corresponds to the path beginning at x attached to some reduced T-factorization of w in the same Hurwitz orbit as $t_1 \cdots t_n$, and is a shortest path in Ω from x to xw. Note that a replacement of any subpath $z \to zt_1' \leftarrow zt_1't_2'$ of such a path is possible since the path's minimal length implies that $t_1' \neq t_2'$. Each replacement decreases the total sum of the ℓ-lengths of the vertices of the path, so eventually one obtains a path in which no further replacements are possible i.e. of the desired decreasing-then-increasing form. Finally, if $x = e$, then $i = 0$ since there are no paths $e \leftarrow t$. $\qquad \square$

Given this proposition, we are finally in the position to prove the two main results of this note.

Proof of Theorem 1.4. Consider a Coxeter system (W, S), a reflection subgroup W' of W. It is known that the directed Bruhat graph for W' corresponding to the simple system of W' induced by S is the full subgraph $\Omega_{\mathrm{dir}}\big|_{W'}$ of Ω_{dir} on vertex set W'; see [Dye01, Theorem 2.1].

Let $w \in W'$. Then Lemma 1.2, Proposition 2.2 (with $x = e$), and the discussion before Proposition 2.2 imply that $\mathrm{Red}_T(w) = \mathrm{Red}_{T'}(w)$ if and only if every shortest directed path from e to w in Ω_{dir} lies inside $\Omega_{\mathrm{dir}}\big|_{W'}$.

Now assume that W' is a standard parabolic subgroup generated by some subset of S. Then it is well known that every reduced S-factorization for $w \in W'$ is actually inside W'. (To see this, recall that Coxeter systems have the property that all reduced S-factorizations for w are related by braid relations.) It therefore

follows that in this situation any shortest directed path from e to w in Ω_{dir} indeed lies inside $\Omega_{\mathrm{dir}}\big|_{W'}$. The theorem follows by the above equivalence. $\qquad\square$

Proof of Theorem 1.3. Again, fix a parabolic Coxeter element $c = s_1 \cdots s_n \in W$ and a corresponding simple system (W, S), and denote by Ω and Ω_{dir} the undirected and directed version of the Bruhat graph for (W, S). By Proposition 2.2, it is left to show that any two directed paths from e to c in Ω_{dir} are in the same Hurwitz orbit. Let therefore

$$e \rightarrow t_1 \rightarrow t_1 t_2 \rightarrow \ldots \rightarrow t_1 \cdots t_n = c$$

be such a path. We have seen in Lemma 2.1 that $\ell(c) = \ell_T(c) = n$. It thus follows that $\ell(t_1 \cdots t_i) = \ell_T(t_1 \cdots t_i) = i$ for any i. The strong exchange condition, see e.g. [Hum90, Theorem 5.8], then yields that $t_1 \cdots t_{i+1}$ is obtained from $t_1 \cdots t_i$ by adding a single simple generator into its position within $s_1 \cdots s_n$. Therefore, such a path is (bijectively) encoded by a permutation $\pi = [\pi_1, \ldots, \pi_n]$ where π_i is the index of the simple generator added at the i-th step. Given the factorization corresponding to such a path, it is straightforward to see that the embedding of the permutation into the braid group (by sending a simple transposition $(i, i+1)$ to the generator σ_i of B_n) yields a braid that turns the given factorization into the factorization $s_1 \cdots s_n$. To this end, observe that given two factorizations encoded by two permutations π_1 and π_2 with $\ell(\pi_1) = \ell(\pi_2) + 1$ and such that these differ only by a single simple transposition $\pi_2^{-1}\pi_1 = (i, i+1)$ for some index i. Then the given factorizations are obtained from each other by applying the braid group generator σ_i to the factorization corresponding to π_1 to obtain the factorization corresponding to π_2. As the factorization corresponding to the identity permutation $[1, \ldots, n]$ is $(s_1, \ldots, s_n)$, the claim follows. $\qquad\square$

As an example of the construction in the previous proof, consider the path

$$e \rightarrow s_2 \rightarrow s_2 s_5 \rightarrow s_2 s_3 s_5 \rightarrow s_1 s_2 s_3 s_5 \rightarrow s_1 s_2 s_3 s_4 s_5 = c.$$

The corresponding factorization of c is given by

$$c = s_2 \cdot s_5 \cdot s_5 s_3 s_5 \cdot s_5 s_3 s_2 s_1 s_2 s_3 s_5 \cdot s_5 s_4 s_5,$$

and the permutation is $\pi = [2, 5, 3, 1, 4] = (1, 2)(2, 3)(4, 5)(3, 4)(2, 3)$. On the other hand,

$$\sigma_1 \sigma_2 \sigma_4 \sigma_3 \sigma_2 (s_2, s_5, s_5 s_3 s_5, s_5 s_3 s_2 s_1 s_2 s_3 s_5, s_5 s_4 s_5) = (s_1, s_2, s_3, s_4, s_5),$$

as desired.

References

[Bes03]   David Bessis, *The dual braid monoid* (English, with English and French summaries), Ann. Sci. École Norm. Sup. (4) **36** (2003), no. 5, 647–683, DOI 10.1016/j.ansens.2003.01.001. MR2032983 (2004m:20071)

[Del74]   P. Deligne, letter to E. Looijenga, 9/3/1974. available at `http://homepage.univie.ac.at/christian.stump/Deligne_Looijenga_Letter_09-03-1974.pdf`.

[Dig06]   F. Digne, *Présentations duales des groupes de tresses de type affine $\widetilde{A}$* (French, with English summary), Comment. Math. Helv. **81** (2006), no. 1, 23–47, DOI 10.4171/CMH/41. MR2208796 (2006k:20075)

[Dye01]   Matthew J. Dyer, *On minimal lengths of expressions of Coxeter group elements as products of reflections*, Proc. Amer. Math. Soc. **129** (2001), no. 9, 2591–2595 (electronic), DOI 10.1090/S0002-9939-01-05876-2. MR1838781 (2002e:20075)

[Dye90] Matthew Dyer, *Reflection subgroups of Coxeter systems*, J. Algebra **135** (1990), no. 1, 57–73, DOI 10.1016/0021-8693(90)90149-I. MR1076077 (91j:20100)

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990. MR1066460 (92h:20002)

[Igu11] Kiyoshi Igusa, *Exceptional sequences, braid groups and clusters*, Groups, algebras and applications, Contemp. Math., vol. 537, Amer. Math. Soc., Providence, RI, 2011, pp. 227–240, DOI 10.1090/conm/537/10578. MR2799103 (2012f:16001)

[IS10]   Kiyoshi Igusa and Ralf Schiffler, *Exceptional sequences and clusters*, J. Algebra **323** (2010), no. 8, 2183–2202, DOI 10.1016/j.jalgebra.2010.02.003. MR2596373 (2011b:20118)

[HK13]   A. Hubery, H. Krause, A categorification of non-crossing partitions, arXiv:1310.1907 (2013), 30 pages

[FHM06] W. N. Franzsen, R. B. Howlett, and B. Mühlherr, *Reflections in abstract Coxeter groups*, Comment. Math. Helv. **81** (2006), no. 3, 665–697, DOI 10.4171/CMH/69. MR2250859 (2007e:20080)

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 33501 BIELEFELD, GERMANY
E-mail address: b.baumeister@math.uni-bielefeld.de

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME, 255 HURLEY, NOTRE DAME, INDIANA 46556
E-mail address: matthew.j.dyer.1@nd.edu

INSTITUT FÜR MATHEMATIK, FREIE UNIVERSITÄT BERLIN, 14195 BERLIN, GERMANY
E-mail address: christian.stump@fu-berlin.de

FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT BIELEFELD, 33501 BIELEFELD, GERMANY
E-mail address: patrick.wegener@math.uni-bielefeld.de