Witt–vector structures on Nil–groups

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To my family

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Contents

Introduction

In geometric topology, the algebraic K-theory of the integral group ring, $\mathbb{Z}[\pi_1(X)]$, of the fundamental group of a CW-complex X has information about obstructions. For example, X is a finitely dominated CW-complex if there is a finite CW-complex Y and maps $f\colon X \to Y$ and $g\colon Y \to X$ such that there is a homotopy $g \circ f \simeq 1_X$. Recall that a finite CW-complex consists of finitely many cells. Wall's finiteness obstruction of a finitely dominated CW-complex X is an invariant $\tau(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1(X)])$ in the reduced K_0-group. The obstruction $\tau(X)$ is trivial if and only if X is homotopy equivalent to a finite CW-complex. A survey on applications of K-theory to geometric topology is [38].

The K-theory groups are hard to compute, but the Farrell–Jones conjecture is a powerful tool to get information about them. Let G be a torsion free group and R be a regular ring, i.e., a Noetherian ring such that any R-module has a finite-dimensional projective resolution. The Farrell–Jones conjecture predicts, an isomorphism

$$H_n(BG; \mathbb{K}^{alg}(R)) \longrightarrow K_n(R[G]), \qquad n \in \mathbb{Z}$$

where BG is the classifying space of G, $\mathbb{K}^{alg}(R)$ is the non–connective algebraic K-theory spectrum of R, as defined in [33], and $H_*(-; \mathbb{K}^{alg}(R))$ is the homology theory associated with the spectrum $\mathbb{K}^{alg}(R)$. For $G = C$, the infinite cyclic group, we get for all $n \in \mathbb{Z}$, $H_n(BC; \mathbb{K}^{alg}(R)) \cong K_n(R) \oplus K_{n-1}(R)$ and the conjecture above predicts an isomorphism

$$K_n(R) \oplus K_{n-1}(R) \cong K_n(R[C]).$$

However, the K-theory groups of $R[C]$ are given by the Fundamental Theorem of Algebraic K-theory in terms of the algebraic K-groups and Nil-groups of R

$$K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \longrightarrow K_n(R[C]).$$

For regular rings $NK_n(R) = 0$ for all $n \in \mathbb{Z}$ [37], but for arbitrary rings, due to Nil–phenomena, one expects that the homomorphism above fails to be an isomorphism. Before we describe the Nil–phenomena, the central problem of this thesis, we state the Farrell–Jones conjecture.

Let G be a group and $\mathcal{VCYC}$ be the family of virtually cyclic subgroups of G, the space $E_{\mathcal{VCYC}}(G)$ is a G-CW-complex, unique up to G-homotopy, such that $E_{\mathcal{VCYC}}(G)^H$ is contactible if $H \in \mathcal{VCYC}$ and empty if $H \notin \mathcal{VCYC}$ [29]. The Farrell–Jones conjecture predicts that the assembly map, $A_{\mathcal{VCYC}}$, induced by the projection $E_{\mathcal{VCYC}}(G) \to pt$

$$A_{\mathcal{VCYC}}\colon H_n^G(E_{\mathcal{VCYC}}(G); \mathbb{K}_R) \to H_n^G(pt; \mathbb{K}_R) = K_n(R[G])$$

is an isomorphism for all $n \in \mathbb{Z}$, all groups G and all rings R. Here $H_*^G(-; \mathbb{K}_R)$ is a G-homology theory associated with the non-connective K-theory spectrum of the ring R. We refer the reader to [30] for a survey on the Farrell–Jones conjecture.

In some sense, one transport the problem with cyclic (and finite) groups to the left hand side; it is a more complicated object, but accessible through spectral sequence type methods. The Farrell–Jones conjecture is known to hold for a large number of groups and at the moment of writing there is no counterexample to this conjecture.

Let us go back to Nil–phenomena. A well–known fact about the groups $NK_n(R)$, for all rings, is that they are either trivial or infinitely generated abelian groups. W. van der Kallen [42] proved this fact for all dimensions while T. Farrell [15] did it only for dimension 1. Nevertheless, the techniques of Farrell's work are interesting to us.

Let us fix $m \in \mathbb{N}$ and let $C^m \leq C$ be the subgroup of C of index m. The canonical ring inclusion $R[C^m] \hookrightarrow R[C]$ induces, in all dimensions, the restriction homomorphism $\mathrm{res}_{\sigma_m}\colon K_n(R[C]) \to K_n(R[C^m])$ and the induction homomorphism $\mathrm{ind}_{\sigma_m}\colon K_n(R[C^m]) \to K_n(R[C])$.

These homomorphisms restricts to Nil–groups, therefore it is possible to define an action, via restriction and induction, on $NK_n(R)$. This will extend to a $W(\mathbb{Z})$-module structure on $NK_n(R)$ (see [45]), where $W(\mathbb{Z})$ is the Witt vector ring of $\mathbb{Z}$. An interesting question is if the Bass Nil–groups are finitely generated as $W(\mathbb{Z})$-modules. There are only a few results about finite generation as $W(\mathbb{Z})$-modules, for example for $G = C_2$, the finite cyclic group of order 2, we have

- $NK_0(\mathbb{Z}[C_2 \times C_2])$ is non trivial and finitely generated as $W(\mathbb{Z})$-module [9];
- $NK_1(\mathbb{Z}[C_2 \times C_2])$ is not finitely generated as $W(\mathbb{Z})$-module [19];
- $NK_2(\mathbb{Z}[C_2])$ is a cyclic $W(\mathbb{Z})$-module [46].

Therefore, the $W(\mathbb{Z})$-module structure relevant in the study of Nil–groups.

Statement of results

We organized our results into 4 chapters. The classical approach to Witt–vector module structures on Nil–groups is in chapters 1 and 2 while chapters 3 and 4 offer a novel approach generalizing the classical methods. We also include two appendices for completeness, in particular, Appendix B offers a Nil–group oriented introduction to the ring of Witt vectors.

Chapter 1: Mackey functors, Burnside rings and K-theory

In this chapter, we introduce $\mathrm{B}(G)$, the Burnside ring of a group G. We use the finite G-set version of the Burnside ring [28], which is valid for all groups, instead of the classical definition [5], which is valid only for finite groups, because we have a particular interest in the infinite cyclic group $G = C$.

Of course, a vast amount of results for Burnside rings of finite groups does not hold for arbitrary groups. Hence, we state and prove facts about Burnside rings necessary for this thesis in Section 1.1.

We also define (cofinite) Mackey functors. From then on, Mackey functors become fundamental objects in our study. Whenever possible, we look for Mackey functor structures on the objects presented here. Among the Mackey functors, the Burnside ring is special because of its following universal property.

Proposition. 1.2.9 *If M is a Mackey functor with values in $\mathrm{MOD}(R)$, then M is in a canonical way a module over the Green functor given by the Burnside ring with respect to the canonical ring homomorphism $\phi\colon \mathbb{Z} \to R$.*

We end the chapter with Section 1.3. This section includes the proof that $K_*(R-)$ is a Mackey functor on the subgroups of G following [31]. For finite groups, the proper formulation and proof of this fact are in [12]. For arbitrary groups, we have to consider group homomorphisms $\alpha\colon H \hookrightarrow G$ that are injective, and whose image has a finite index. This is a necessary condition to have the contravariant (restriction) structure on K-theory of group rings. Notice that for the infinite cyclic group C and all $m > 0$, the canonical inclusion of $C^m \leq C$ satisfies the above condition.

Hence, $K_n(R[C])$ is, in a canonical way, a $B(C)$-module. This is a mere consequence of the universal property of the Burnside ring. In Chapter 2, we give a detailed description of this action.

Chapter 2: The Fundamental theorem and compatible actions

In the first section, we indicate, through the study of polynomial rings, the existence of NK-groups $NK_n(R)$. Then, we provide the formal definition of NK-groups and define the Nil–groups $\mathrm{Nil}_n(R)$. These groups are, up to dimension shifting, isomorphic. The Nil–group interpretation is better suited for computations than the NK–group interpretation and we take advantage of this fact to study the NK–groups.

In Section 2.2, we state the Fundamental Theorem of Algebraic K-theory 2.2.1 and summarize its proof, following [40]. We emphasize the localization sequences appearing there and provide an explicit description of the isomorphism

$$NK_n(R) \cong \mathrm{Nil}_{n-1}(R). \tag{0.0.1}$$

In Section 2.3, we describe two actions on Nil–groups. The Frobenius and Verschiebung endomorphisms define an action via by operations on nilpotent endomorphisms. We recover the following result from [45]

Proposition. 2.3.2 *The operations F_m and V_m, for $m \in \mathbb{N}$, defined in the category $\mathrm{NIL}(R)$, describe the $\mathrm{End}_0(\mathbb{Z})$-module structure on $\mathrm{Nil}_{n-1}(R)$ for all $n \in \mathbb{Z}$.*

We refer to the module structure given in the Proposition above as the $W(\mathbb{Z})$-module structure, since $\mathrm{End}_0(\mathbb{Z})$ is a dense subring of $W(\mathbb{Z})$. We extend the $\mathrm{End}_0(\mathbb{Z})$-module structure of Nil–groups to a $W(\mathbb{Z})$-module structure in Chapter 4.

The restriction and induction homomorphisms, arising from $R[C^m] \hookrightarrow R[C]$, give the second action, now described on NK–groups. This is the $\mathrm{B}(C)$-module structure mentioned at the end of Chapter 1, and we describe it in detail.

In order to have a well–defined action, we check that *localization commutes with restriction and induction.* This is a technical issue, since Farrell's original work [15] only considers the canonical inclusion $R[t^m] \hookrightarrow R[t]$ while we require the restriction and induction coming from $R[C^m] \hookrightarrow R[C]$. We prove

Theorem. 2.3.6 *For all $m \in \mathbb{N}$, the restriction and induction homomorphism in K-theory are compatible with the Bass–Heller–Swan decomposition.*

The last result of the chapter links the Witt ring module structure with the Burnside ring module structure.

Theorem. 2.3.7 *The $\mathrm{End}_0(\mathbb{Z})$-module structure on $\mathrm{Nil}_{n-1}(R)$ and the $\mathrm{B}(C)$-module structure of $NK_n(R)$ coincide, under the isomorphism 0.0.1.*

Chapter 3: Witt–Burnside ring

Chapter 3 generalizes Chapter 1. Dress and Siebeneicher introduced the concept of Witt–Burnside ring [13]. We follow the more recent work of J. Elliott [14] which is also more general. The Witt–Burnside ring is defined for profinite groups, in particular those presented in Appendix A.

In Section 3.1, we introduce, for a profinite group Γ and a commutative monoid M, the concept of Γ-strings over M. Then, we define $\mathbf{B}_M(\Gamma)$, the completed Burnside ring of Γ, as the Grothendieck ring of isomorphism classes of Γ-strings over M under disjoint union with product given by Cartesian product. We show that $\mathbf{B}_M(\_)$ is a Mackey functor on the open subgroups of Γ. The following theorem is important to establish our main result in the last chapter.

Theorem. 3.1.13 *For any profinite group Γ, there exists a ring isomorphism*

$$\begin{aligned} \mathbf{B}_M(\Gamma) &\longrightarrow \varprojlim_N \mathbf{B}_M(\Gamma/N) \\ \mathbf{X} &\mapsto \left(\mathbf{B}\mathrm{proj}_N^\Gamma(\mathbf{X})\right)_N \end{aligned}$$

where N runs over all open normal subgroups of Γ.

We use the existence of the Witt–Burnside ring over Γ of Theorem 3.2.1 and refer the reader to [13] and [14] for the proof of this fact. The following has important applications for K-theory.

Theorem. 3.2.3 *For any profinite group Γ and any commutative monoid M, there exists a ring isomorphism $\mathbf{W}_\Gamma(\mathbb{Z}[M]) \xrightarrow{\cong} \mathbf{B}_M(\Gamma)$.*

Finally, we show that $\mathbf{W}_\Gamma(\_)$ can be described as an inverse limit. This is based on a similar result for the completed Burnside rings.

Theorem. 3.2.9 *For all commutative rings R and all profinite groups Γ, there exists a ring isomorphism*

$$\mathbf{W}_\Gamma(R) \cong \varprojlim_N \mathbf{W}_{\Gamma/N}(R)$$

where N runs over all open normal subgroups.

Chapter 4: An equivariant homology theory with restriction

This chapter contains our main result, namely, we want to determine for, the trivial monoid $M = 1$ when a $\mathbf{B}_1(\_)$-module structure on an equivariant homology theory exists. Sections 4.1, 4.2 and 4.3 are a short digression about the assembly map in algebraic K-theory, the K-theory spectrum and G-homology theories.

We introduce the concept of an equivariant homology theory $\mathcal{H}_*^?$: it consists of a G-homology theory for each group G with an induction structure, linking the various G-homology theories. We require further that an equivariant homology theory has a restriction structure in the sense of [27].

The idea is that an equivariant homology theory with a restriction structure has a Mackey functor structure on it, thereby applying the universal property of Burnside ring we have a $\mathrm{B}(\_)$-module. We prove

Theorem. 4.5.2 *Let $\mathcal{H}_*^?(-)$ be an equivariant homology theory with restriction structure. Then for all group G, all G-CW-complexes X and all $n \in \mathbb{Z}$, $\mathcal{H}_n^?(res_G^? X)$ is a* $\mathrm{B}(G)$-module.

Finally, we look for conditions to extend the B(_)-module to a $\mathbf{B}_1(\_)$-module structure, where $M = 1$ denotes the trivial monoid. Our main result provides the criteria for such an extension to exist.

Theorem. 4.6.4 *Let $(I, \leq)$ be a direct set satisfying the condition that for every $j \in I$ the set $\{i \in I | i \leq j\}$ is finite, $\{G_i, \varphi_{ij}\}$ be a surjective inverse system of finite groups indexed by I and $\mathcal{H}$ be an abelian group. Assume that $\mathcal{H}$ is a $\mathbf{B}_M(G_i)$-module for all $i \in I$ and satisfies*

(a) for each $i \leq j$, the $\mathbf{B}_M(G_i)$-module structure of $\mathcal{H}$ is $\mathbf{B}\mathrm{res}_{\varphi_{ij}}$-compatible with the $\mathbf{B}_M(G_j)$-module structure of $\mathcal{H}$, i.e., the diagram

$$\begin{tikzcd}
\mathbf{B}_M(G_j) \times \mathcal{H} \arrow[r] & \mathcal{H} \\
\mathbf{B}_M(G_i) \times \mathcal{H} \arrow[r] \arrow[u, "\mathbf{B}\mathrm{res}_{\varphi ij} \times Id"] & \mathcal{H} \arrow[u, "Id"']
\end{tikzcd}$$

commutes;

(b) for every $x \in \mathcal{H}$ there exists $i = i(x) \in I$, such that for all j with $i \leq j$, then $\mathrm{Ker}(\mathbf{B}\mathrm{proj}_i^j) \subseteq \mathrm{Ann}_{\mathbf{B}_M(G_j)}(x) = \{T \in \mathbf{B}_M(G_j) \mid T.x = 0\}$.

Then $\mathcal{H}$ has a module structure over the ring $\varprojlim_i \mathbf{B}_M(G_i)$.

We conclude with an application of this result to algebraic K-theory using the formulation of the Farrell–Jones conjecture, to generalize the $W(R)$-module structure of $NK_*(R)$.

Theorem. 4.6.7 *Let R be a commutative ring with unit. For all $n \in \mathbb{Z}$ there exists a $\mathbf{B}_{R^\times}(C)$-module structure on $NK_n(R)$. Moreover, this structure is compatible with the $W(R)$-module structure of $NK_n(R)$.*

Conventions

We assume readers to be familiar with basic notions about K-theory, and a good introductory book to the topic is [37]. We also assume basic knowledge in Category Theory. Here the text [32] is sufficient. We also assume familiarity with concepts like homology, spectra arising in algebraic topology, see for example [22].

In this work, we consider associative rings with unit, usually denoted by R. In special cases, we require R to be commutative, and in such cases, we state it explicitly. The K-theory groups of a ring R are given by the homotopy groups of the non–connective algebraic K-theory spectrum of R, as defined in [33].

Since there is no consensus about 0 being a natural number, we denote by $\mathbb{N}$ the set of natural numbers $\{1, 2, 3, \ldots, \}$ and by $\mathbb{N}_0$ the set $\mathbb{N} \cup \{0\}$.

Chapter 1

Mackey functors, Burnside rings and K-theory

In Section 1.1, we define $\mathrm{B}(G)$, the Burnside ring of a (not necessarily finite) group G and define the restriction and the induction maps between Burnside rings. In section 1.2, we study cofinite Mackey functors and show that $\mathrm{B}(\_)$, the Burnside ring, is a cofinite Mackey functor with the universal property given by Proposition 1.2.9. In the last section, we sketch the proof that for any ring with unit R the algebraic K-theory groups $K_n(R[-])$ of group rings with coefficients in R is a Mackey functor following [31]. Last, we give an abstract description of the $\mathrm{B}(C)$-module structure of $K_n(R[C])$, with C the infinite cyclic group.

1.1 The Burnside ring

Let G be a group. If G is finite, the Burnside ring of G has been extensively studied, see for instance [5]. If G is a (discrete) infinite group there still exists the notion of Burnside ring of G. There are several generalizations of Burnside rings of groups and the choice of one version depends on the background problem. In our case, we chose the *finite G-set version* because it is related to Green functors and induction theory (see [28]).

We will consider (discrete) groups, not necessarily finite. As we will see, the induction map exists for all group homomorphism while the restriction map exists only for injective group homomorphisms whose image has finite index.

Definition 1.1.1 (G-sets)**.** A *G-set* T is a discrete set with a continuous left G-action. A *G-map*, $T \xrightarrow{f} S$, between two G-sets, is a G-equivariant map. Let G-SET be the category whose objects are G-sets and morphisms are G-maps. A *finite G-set* is a G-set whose underlying set is finite.

Let T and S be G-sets. The *disjoint union of T and S* is the set $T\coprod S$ endowed with the obvious G-action

$$\begin{aligned} G \times \left(T \coprod S\right) &\longrightarrow T \coprod S. \\ (g, z) &\mapsto gz \end{aligned}$$

The *product of T and S* is the set $T \times S$ endowed with diagonal G-action

$$\begin{aligned} G \times T \times S &\longrightarrow T \times S. \\ (g,(t,s)) &\mapsto (gt, gs) \end{aligned}$$

Remark 1.1.2. Let T, S and Z be G-sets, then $T \times S \cong S \times T$ and $T \times (S \coprod Z) \cong T \times S \coprod T \times Z$ are isomorphic as G-sets.

Definition 1.1.3 (Restriction of G-sets)**.** Let $H \xrightarrow{\alpha} G$ be a group homomorphism and T be G-set. The *restriction of T* is the H-set with underlying set $\mathrm{res}_{\alpha} T = T$ and H-action given by restriction of the G-action to H via α, i.e.,

$$\begin{aligned} H \times \mathrm{res}_{\alpha} T &\longrightarrow \mathrm{res}_{\alpha} T. \\ (h,t) &\mapsto \alpha(h)t \end{aligned}$$

Notice that for a G-map $T \xrightarrow{f} T'$, the restricted morphism $\mathrm{res}_{\alpha} T \to \mathrm{res}_{\alpha} T'$ is automatically an H-map. Hence, there is a functor

$$\mathrm{res}_{\alpha} \colon G\text{-SET} \longrightarrow H\text{-SET}. \tag{1.1.1}$$

Remark 1.1.4. Let $H \overset{\alpha}{\hookrightarrow} G$ be the inclusion of $H \leq G$ subgroup of G. Then we denote $\mathrm{res}_{\alpha} := \mathrm{res}^H_G$.

Definition 1.1.5 (Induction of an H-set)**.** Let $H \xrightarrow{\alpha} G$ be a group and S be an H-set. The *induction of S* is the G-set $\mathrm{ind}_{\alpha} S$ whose underlying set is the space of orbits $G \times_{\alpha} S$ of the right H-set $G \times S$ with H-action given by $(g,s).h := (g\alpha(h), h^{-1}s)$, for $g \in G$, $s \in S$ and $h \in H$. The G-action is given by

$$\begin{aligned} G \times (G \times_{\alpha} S) &\longrightarrow G \times_{\alpha} S. \\ (g',[g,s]) &\mapsto [g'g, s] \end{aligned}$$

For $S \xrightarrow{f} S'$ an H-map we define

$$\begin{aligned} \mathrm{ind}_{\alpha}(f) \colon \mathrm{ind}_{\alpha} S &\to \mathrm{ind}_{\alpha} S' \\ [g,s] &\mapsto [g, f(s)] \end{aligned}$$

which is a G-map. Hence, there is a functor

$$\mathrm{ind}_{\alpha} \colon H\text{-SET} \longrightarrow G\text{-SET}. \tag{1.1.2}$$

Remark 1.1.6. Let $H \overset{\alpha}{\hookrightarrow} G$ be the inclusion of $H \leq G$ subgroup of G. Then we denote $\mathrm{ind}_{\alpha} := \mathrm{ind}^G_H$

Example 1.1.7 (Transitive homogeneous G-sets)**.** Let $H \leq G$ be a subgroup and G/H be the set of (left) cosets of H in G. The group G acts on G/H by multiplication

$$\begin{aligned} G \times G/H &\longrightarrow G/H. \\ (g', gH) &\mapsto g'gH \end{aligned}$$

This action makes G/H into a *transitive homogeneous* G-set. A G-map between transitive homogeneous G-sets is easy to describe. Let $H, K \leq G$ and assume that

$$f \colon G/H \longrightarrow G/K$$

is a G-map. The image of the identity class eH determines f because G/H is transitive, thus assume that $f(eH) = gK$ for some $g \in G$. Then, for any $h \in H$, we have $gK = f(eH) = f(hH) = hf(eH) = hgK$. Hence, $g^{-1}hg \in K$ that is $g^{-1}Hg \subseteq K$ or equivalently $H \subseteq gKg^{-1}$.

We conclude that a G-map $G/H \xrightarrow{f} G/K$ exists if and only if there exists $g \in G$ such $H \subseteq {}^gK$, where ${}^gK = gKg^{-1}$. Moreover, f is an isomorphism, $f\colon G/H \xrightarrow{\cong} G/K$, if an only if H and K are conjugate.

Characterisation of G-sets

The homogeneous transitive G-sets are easy to visualize and they are the building blocks of G-sets because they generate any G-set in the following sense.

Lemma 1.1.8. *(a) Any G-set is a disjoint union of transitive homogeneous G-sets. If T is a transitive G-set and G_t denotes the stabilizer subgroup of $t \in T$, then there is an isomorphism of G-sets*

$$\begin{aligned} \mu\colon G/G_t &\longrightarrow T \\ gG_t &\mapsto gt. \end{aligned}$$

(b) Let $H, K \leq G$ be subgroups of G. Then there exists a one–to–one correspondence

$$\begin{aligned} \mathrm{mor}_G(G/H, G/K) &\longrightarrow \{gK \in G/K \mid H \subseteq {}^gK\} \\ f &\mapsto f(eH) \end{aligned}$$

where $\mathrm{mor}_G(G/H, G/K)$ is the set of G-maps from G/H to G/K.

Proof. (a) The proof that the map μ is an isomorphism follows from the fact that $T = \{gt \mid g \in G\}$ since T is transitive.

Let T be a G-set, the orbits of T form a partition $T = \coprod T_i$ where T_i is a transitive G-set. The result follows from the isomorphism μ.

(b) We discussed this in Example 1.1.7 above.

□

Remark 1.1.9. In particular, Lemma 1.1.8 states for any finite G-set T that there exists a G-isomorphism

$$T \cong \coprod_{(H)} \mu_H(T) \cdot G/H, \tag{1.1.3}$$

where (H) denotes the conjugacy class of $H \leq_f G$, subgroup of finite index in G, the disjoint union runs over all conjugacy classes; $\mu_H(T) \in \mathbb{N}_0$ is the number of G-orbits of T isomorphic to G/H and $\mu_H(T) \cdot G/H$ is short for the disjoint union of $\mu_H(T)$ copies of the finite transitive G-set G/H.

The ring structure

Definition 1.1.10 (Burnside ring of G)**.** Let G be a (discrete) group, not necessarily finite. Define $\mathrm{B}(G)$, the *finite G-set version of the Burnside ring of G*, as the Grothendieck ring of isomorphism classes of finite G-sets under disjoint union with product given by Cartesian product.

We think of $\mathrm{B}(G)$ as the free $\mathbb{Z}$-module with basis the set of isomorphism classes of G-sets. Further, we can consider $\mathrm{B}(G)$ as the free $\mathbb{Z}$-module with basis $\{G/H \mid H \leq_f G\}_{(H)}$ where (H) runs over conjugacy classes. Hence, the class of a G-set T in $\mathrm{B}(G)$ is

$$T = \sum_{(H)} \mu_H(T) \cdot G/H. \tag{1.1.4}$$

Remark 1.1.11. We write T for both, the G-set T and its isomorphism class T in $\mathrm{B}(G)$.

The multiplication in $\mathrm{B}(G)$ is determined by the multiplication of the basis elements.

Proposition 1.1.12. *Let $H, K \leq_f G$ be finite index subgroups of G.*

(a) ***Double coset formula****. If S is an H-set, then there is an isomorphism of K-sets*

$$\mathrm{res}^K_G \mathrm{ind}^G_H S \cong \coprod_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}^K_{H\cap K^\gamma} \mathrm{res}^{H\cap K^\gamma}_H S,$$

where $K^\gamma = \gamma^{-1}K\gamma$.

(b) ***Frobenius identity****. If T is a G-set and S is an H-set, then there is an isomorphism of G-sets*

$$T \times \mathrm{ind}^G_H S \cong \mathrm{ind}^G_H((\mathrm{res}^H_G T) \times S).$$

In particular, for any G-set T, there is an isomorphism of G sets

$$T \times G/H \cong \mathrm{ind}^G_H \mathrm{res}^H_G T.$$

Remark 1.1.13. We write $\mathrm{ind}^K_{H\cap K^\gamma}$ in the double coset formula above for the induction coming from the inner automorphism $c(\gamma)\colon H \cap K^\gamma \longrightarrow K$, sending $h \mapsto \gamma h \gamma^{-1}$. The notation $K\gamma H \in |K\backslash G/H|$ means a set of representatives of double cosets.

Proof. (a) The required K-isomorphism

$$\mathrm{res}^K_G \mathrm{ind}^G_H S \longrightarrow \coprod_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}^K_{H\cap K^\gamma} \mathrm{res}^{H\cap K^\gamma}_H S.$$

sends $[g, s] \mapsto [k, hs]$ if $g = k\gamma h$ for some $k \in K$ and $h \in H$, i.e., $[k, hs]$ lies in $\mathrm{ind}^K_{H\cap K^\gamma} \mathrm{res}^{H\cap K^\gamma}_H S$.

Surjectivity follows from the fact that $[k, s] \in \mathrm{ind}^K_{H\cap K^\gamma} \mathrm{res}^{H\cap K^\gamma}_H S$ can be written as $[k', hs]$, hence it comes from $[k'\gamma h, s] \in \mathrm{res}^K_G \mathrm{ind}^G_H S$.

To verify injectivity, let $[g', s']$ and $[g, s]$ such that $[k', h's'] = [k, hs]$. Then there exists $h'' = \gamma^{-1}k''\gamma = z \in H \cap K^\gamma$ such that $(k', h's') = (kc(\gamma)(z), z^{-1}hs)$. The element $\bar{h} = h^{-1}h''h'$ verifies $[g', s'] = [g, s]$.

(b) The required G-isomorphism is

$$\begin{aligned} T \times \mathrm{ind}^G_H S &\longrightarrow \mathrm{ind}^G_H((\mathrm{res}^H_G T) \times S) \\ (t, [g, s]) &\mapsto [g, (g^{-1}t, s)]. \end{aligned}$$

Surjectivity follows from the fact that $[g, (t, s)]$ comes from the element $(gt, [g, s])$.

To verify injectivity, let $(t', [g', s'])$ and $(t, [g, s])$ such that $[g', (g'^{-1}t', s')] = [g, (g^{-1}t, s)]$. Then there exists $h \in H$ such that $(g', (g'^{-1}t', s')) = (gh, (h^{-1}g^{-1}t, h^{-1}s))$ this is $g' = gh$, $s' = h^{-1}s$ and $t' = t$. Hence, $(t', [g', s']) = (t, [g, s])$.

The other statement is obtained with $S = H/H$. $\square$

Proposition 1.1.14. *Let $H, K \leq_f G$. The product of G/H and G/K in $B(G)$ is given by*

$$G/H \cdot G/K = \sum_{K\gamma H \in |K \backslash G/H|} G/(H \cap K^\gamma). \tag{1.1.5}$$

Proof. Frobenius identity and the double coset formula give the isomorphisms

$$\begin{aligned} G/H \times G/K &\cong \mathrm{ind}_K^G \mathrm{res}_G^K (\mathrm{ind}_H^G (H/H)) \\ &\cong \mathrm{ind}_K^G \left(\coprod_{K\gamma H \in |K \backslash G/H|} \mathrm{ind}_{H \cap K^\gamma}^K (H \cap K^\gamma / H \cap K^\gamma) \right) \\ &\cong \mathrm{ind}_K^G \left(\coprod_{K\gamma H \in |K \backslash G/H|} K/H \cap K^\gamma \right) \\ &\cong \coprod_{K\gamma H \in |K \backslash G/H|} G/H \cap K^\gamma \end{aligned}$$

□

Ring homomorphisms

The next argument holds for any injective group homomorphism $H \overset{\alpha}{\hookrightarrow} G$ whose image has finite index, but we only consider $H \leq_f G$. The functors

$$\mathrm{res}_G^H \colon G\text{-SET} \longrightarrow H\text{-SET}$$

and

$$\mathrm{ind}_H^G \colon H\text{-SET} \longrightarrow G\text{-SET}$$

preserve disjoint union of G-sets and H-sets respectively. Using this property, we define maps between Burnside rings

$$\mathrm{Bres}_G^H \colon \mathrm{B}(G) \longrightarrow \mathrm{B}(H)$$

and

$$\mathrm{Bind}_H^G \colon \mathrm{B}(H) \longrightarrow \mathrm{B}(G).$$

Indeed, Bres_G^H is a ring homomorphism while Bind_H^G is just an additive homomorphism. This is not the standard notation, but it keeps our exposition simpler. See Remark 1.1.16 below.

Proposition 1.1.15. *Let $H, K \leq_f G$.*

(a) ***Double coset formula****. If $S \in \mathrm{B}(H)$*, then in $\mathrm{B}(K)$

$$\mathrm{Bres}_G^K \circ \mathrm{Bind}_H^G(S) = \sum_{K\gamma H \in |K \backslash G/H|} \mathrm{Bind}_{H \cap K^\gamma}^K \circ \mathrm{Bres}_H^{H \cap K^\gamma}(S).$$

(b) **Frobenius identity**. If $T \in \mathrm{B}(G)$ and $S \in \mathrm{B}(H)$, then in $\mathrm{B}(G)$

$$T.\mathrm{Bind}_H^G(S) = \mathrm{Bind}_H^G(\mathrm{Bres}_G^H(T).S)$$

In particular, for $T \in \mathrm{B}(G)$

$$T.G/H = \mathrm{Bind}_H^G(\mathrm{Bres}_G^H(T)).$$

Proof. Both are direct consequence of Proposition 1.1.12. $\square$

Remark 1.1.16. We reserve the notation res_G^H and ind_H^G for functors over the corresponding categories and denote by Bres_G^H and Bind_H^G the corresponding Burnside ring homomorphisms. This will assist in the next chapters where we define $\mathbf{B}\mathrm{res}_G^H$ and $\mathbf{B}\mathrm{ind}_H^G$ and $\mathbf{W}\mathrm{res}_G^H$ and $\mathbf{W}\mathrm{ind}_H^G$ for different homomorphisms arising from res_G^H and ind_H^G.

1.2 Mackey functors

The principal examples of Mackey functors for us are the Burnside rings and the algebraic K-theory of group rings. Moreover, we prove that the Burnside ring functor is a Green functor acting over any Mackey functor. Let R be a ring with unit and $\mathrm{MOD}(R)$ the category of left R-modules with homomorphism of R-modules as morphisms.

Definition 1.2.1 (Cofinite Mackey functor)**.** Let GR be the category of groups with group homomorphisms. Let GRIFI be the subcategory with morphisms the injective group homomorphisms whose image has finite index in the target. A *cofinite Mackey functor with values in R-modules* is a pair of functors

$$\begin{aligned} M_*\colon \mathrm{GR} &\longrightarrow \mathrm{MOD}(R) \\ M^*\colon \mathrm{GRIFI} &\longrightarrow \mathrm{MOD}(R) \end{aligned}$$

such that:

(a) M_* is covariant, M^* is contravariant and agree on objects;

(b) for an inner automorphism $c(\gamma)\colon G \longrightarrow G$, with $\gamma \in G$, we have

$$M_*(c(\gamma)) = id\colon M(G) \longrightarrow M(G);$$

(c) let $\alpha\colon G \to H$ be a morphism in GRIFI and denote $M^*(\alpha) = \mathrm{res}_\alpha$ and $M_*(\alpha) = \mathrm{ind}_\alpha$. For an isomorphism of groups $\alpha\colon G \xrightarrow{\cong} H$ the composition $\mathrm{res}_\alpha \circ \mathrm{ind}_\alpha$ and $\mathrm{ind}_\alpha \circ \mathrm{res}_\alpha$ are the identity.

(d) **Double coset formula**. For $H, K \leq_f G$ subgroups

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G = \sum_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^K \circ \mathrm{res}_H^{H\cap K^\gamma}.$$

Remark 1.2.2. The reason to denote $M^*(\alpha) = \mathrm{res}_\alpha$ and $M_*(\alpha) = \mathrm{ind}_\alpha$ for a morphism $H \xrightarrow{\alpha} G$ in GRIFI is because our main example of Mackey functors are the Burnside rings and in the theory of Burnside rings it is common to use this notation.

Theorem 1.2.3. *The Burnside ring* $\mathrm{B}(\_)$ is a cofinite Mackey functor with values in $\mathrm{MOD}(\mathbb{Z})$.

Proof. (a) Let G be a group. Define $M_*(G) = M^*(G) = \mathrm{B}(G)$. If $H \xrightarrow{\alpha} G$ is a morphism in GR define $M_*(\alpha) = \mathrm{Bind}_\alpha$. If $H \overset{\alpha}{\hookrightarrow} G$ is a morphism in GRIFI define $M^*(\alpha) = \mathrm{Bres}_\alpha$.

(b) Let $\gamma \in G$ and $c(\gamma)\colon G \to G$ be conjugation by γ. For a G-set T the G-map

$$\begin{aligned} G \times_{c(\gamma)} T &\longrightarrow T \\ [g,t] &\mapsto g\gamma t \end{aligned}$$

is an isomorphism. Surjectivity follows immediately. For injectivity, notice that if $g'\gamma t' = g\gamma t$, then $t' = \gamma^{-1}g'^{-1}g\gamma t$ and $\bar{g} = \gamma^{-1}g'^{-1}g\gamma$ verifies $(g,t) = (g'c(\gamma)(\bar{g}), \bar{g}^{-1}t')$.

(c) Let $\alpha\colon H \xrightarrow{\cong} G$ be an isomorphism and T be a G-set. The G-map

$$\begin{aligned} G \times_\alpha \mathrm{res}_\alpha T &\longrightarrow T \\ [g,t] &\mapsto gt \end{aligned}$$

is an isomorphism. Surjectivity follows immediately. For injectivity, if $[g',t']$ and $[g,t]$ have the same image in T, then $t' = g'^{-1}gt$ and $h = \alpha^{-1}(g^{-1}g')$ verify $[g',t'] = [g,t]$. Hence, $\mathrm{Bind}_\alpha \circ \mathrm{Bres}_\alpha$ is the identity.

Let S be an H-set then, the H-map

$$\begin{aligned} \mathrm{res}_\alpha(G \times_\alpha S) &\longrightarrow S \\ [g,t] &\mapsto \alpha^{-1}(g)t \end{aligned}$$

is an isomorphism. The proof of this assertion is similar to the previous case. Hence, $\mathrm{Bres}_\alpha \circ \mathrm{Bind}_\alpha$ is the identity.

(d) We proved this in Proposition 1.1.15. □

Definition 1.2.4 (Pairing of Mackey functors)**.** Let M, N and P be Mackey functors with values in MOD(R). A *pairing of Mackey functors* is a family of bilinear maps

$$\begin{aligned} M(G) \times N(G) &\to P(G), \\ (x,y) &\mapsto x.y \end{aligned}$$

where G runs over the category GRIFI such that for any morphism $H \overset{\alpha}{\hookrightarrow} G$ in GRIFI the following relations hold

(i) $(x_1 + x_2).y = x_1.y + x_2.y$, for $x_1, x_2 \in M(G)$, and $y \in N(G)$;

(ii) $x.(y_1 + y_2) = x.y_1 + x.y_2$, for $x \in M(G)$, and $y_1, y_2 \in N(G)$;

(iii) $(rx).y = r(x.y) = x.(ry)$, for $r \in R$, $x \in M(G)$, and $y \in N(G)$;

(iv) $\mathrm{res}_\alpha(x.y) = \mathrm{res}_\alpha(x).\mathrm{res}_\alpha(y)$, for $x \in M(G)$ and $y \in N(G)$;

(v) $x.\mathrm{ind}_\alpha(y) = \mathrm{ind}_\alpha(\mathrm{res}_\alpha(x).y)$, for $x \in M(G)$ and $y \in N(H)$;

(vi) $\mathrm{ind}_\alpha(x).y = \mathrm{ind}_\alpha(x.\mathrm{res}_\alpha(y))$, for $x \in M(H)$ and $y \in N(G)$.

Remark 1.2.5. The notion of pairing is more general. If $\phi\colon R \to S$ is a homomorphism of associative (commutative) rings with unit, then we can define a pairing for M with values in MOD(R) and, N and P having values in MOD(S). One need to check that, in this case for (iii) above, $(rx).y = \phi(r)(x.y)$ and $x.sy = s(x.y)$, where $s \in S$.

Definition 1.2.6 (Green ring)**.** A *Green functor* with values in MOD(R) is a Mackey functor M with a pairing $M \times M \longrightarrow M$ and elements $1_G \in M(G)$ for each group G in GRIFI such that for each group G the pairing

$$M(G) \times M(G) \longrightarrow M(G)$$

induces the structure of an R-algebra on $M(G)$ with unit 1_G and for any morphism $H \overset{\alpha}{\hookrightarrow} G$ the map $M^*(\alpha) = \mathrm{res}_\alpha\colon M(G) \to M(H)$ is a homomorphism of R-algebras with unit.

Example 1.2.7. The natural example of a Green ring is the Burnside ring of a group viewed as a functor $\mathrm{B}\colon \mathrm{GRIFI} \to \mathrm{MOD}(\mathbb{Z})$. For any G in GRIFI the multiplicative structure of the ring $\mathrm{B}(G)$

$$m_G\colon \mathrm{B}(G) \times \mathrm{B}(G) \longrightarrow \mathrm{B}(G)$$

gives the Green ring structure.

Definition 1.2.8 (Green module)**.** Let ϕ be a ring homomorphism, M a Green functor with values in $\mathrm{MOD}(R)$ and P be a Mackey functor with values in $\mathrm{MOD}(S)$. A left M-module structure on P is a pairing with respect to ϕ such that any of the maps

$$M(G) \times P(G) \longrightarrow P(G)$$

induces the structure of a left module over the R-algebra $M(G)$ on the R-module $\phi^*(P(G))$ obtained from the S-module $P(G)$ by restriction along ϕ.

Proposition 1.2.9 (Universal Property of the Burnside ring)**.** *If M is a Mackey functor with values in $\mathrm{MOD}(R)$, then M is in a canonical way a module over the Green functor given by the Burnside ring with respect to the canonical ring homomorphism $\phi\colon \mathbb{Z} \to R$.*

Proof. Let G be a group and consider the pairing

$$\begin{aligned} \mathrm{B}(G) \times M(G) &\longrightarrow M(G) \\ (\sum_i n_i \cdot G/H_i, x) &\mapsto \sum_i n_i \mathrm{ind}_{H_i}^G \circ \mathrm{res}_G^{H_i}(x). \end{aligned}$$

The module structure follows since $\mathrm{res}_G^{H_i}$ and $\mathrm{ind}_{H_i}^G$ are, indeed, R-module homomorphisms.

□

1.3 Algebraic K-theory as Mackey functor

We include this section for completeness. In [12], Dress and Kuku showed that for finite groups G the functor $K_n(R[-])$ is a Mackey functor on the subgroups of G. Along similar lines, one can extend this result to infinite (discrete) groups (see [4]). We present here a proof of this fact following the more general approach of [31]. For the rest of the section, $\mathcal{C}$ will denote a small category and R will be an associative ring with unit.

Definition 1.3.1 ($R\mathcal{C}$-modules)**.** A *covariant $R\mathcal{C}$-module* is a covariant functor

$$M\colon \mathcal{C} \longrightarrow \mathrm{MOD}(R) \tag{1.3.1}$$

from the category $\mathcal{C}$ to the category $\mathrm{MOD}(R)$. A morphism, $M \longrightarrow N$, of $R\mathcal{C}$-modules is a natural transformation of functors. Let $\mathrm{MOD}(R\mathcal{C})$ be the category of covariant $R\mathcal{C}$-modules with natural transformation as morphisms. We define a *contravariant $R\mathcal{C}$-module* in a similar way.

Example 1.3.2. The *typical free $R\mathcal{C}$-module* is the following covariant $R\mathcal{C}$-module

$$\begin{aligned} R\mathcal{C}(c,?)\colon \mathcal{C} &\longrightarrow \mathrm{MOD}(R), \\ d &\mapsto R\mathrm{mor}_{\mathcal{C}}(c,d) \end{aligned}$$

where $R\mathrm{mor}_{\mathcal{C}}(c,d)$ denotes the free R-module with basis $\mathrm{mor}_{\mathcal{C}}(c,d)$ the set of morphisms from c to d in $\mathcal{C}$.

Let M be a contravariant $R\mathcal{C}$-module and N be a covariant $R\mathcal{C}$-module. The R-module $M \otimes_{R\mathcal{C}} N$ is the quotient

$$M \otimes_{R\mathcal{C}} N := \left(\bigoplus_{x \in \mathcal{C}} M(x) \otimes_R N(x) \right) /Q,$$

where Q is the R submodule generated by $\{M(f)(m) \otimes n - m \otimes N(f)(n) \mid m \in M(y),\ n \in N(x),\ f \in \mathrm{mor}_{\mathcal{C}}(x,y),\ x,y \in \mathcal{C}\}$.

An *$R\mathcal{C}$–$R\mathcal{D}$–bimodule* is a covariant functor

$$M \colon \mathcal{C} \times \mathcal{D}^{op} \longrightarrow \mathrm{MOD}(R).$$

Example 1.3.3. (a) Let $\mathcal{C} = \mathcal{D}$. There exists an $R\mathcal{C}$-$R\mathcal{C}$-bimodule

$$\begin{aligned} R\mathcal{C}(??,?) \colon \mathcal{C} \times \mathcal{C}^{op} &\longrightarrow \mathrm{MOD}(R) \\ (c,c') &\mapsto R\mathrm{mor}_{\mathcal{C}}(c',c). \end{aligned}$$

(b) Let $F \colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. There exists an $R\mathcal{D}$-$R\mathcal{C}$-bimodule

$$\begin{aligned} R\mathcal{D}(F(?),??) \colon \mathcal{D} \times \mathcal{C}^{op} &\longrightarrow \mathrm{MOD}(R) \\ (d,c) &\mapsto R\mathrm{mor}_{\mathcal{D}}(F(c),d) \end{aligned} \tag{1.3.2}$$

and there exists an $R\mathcal{C}$-$R\mathcal{D}$-bimodule

$$\begin{aligned} R\mathcal{D}(??,F(?)) \colon \mathcal{C} \times \mathcal{D}^{op} &\longrightarrow \mathrm{MOD}(R). \\ (c,d) &\mapsto R\mathrm{mor}_{\mathcal{D}}(d,F(c)) \end{aligned} \tag{1.3.3}$$

Definition 1.3.4 (Restriction and induction)**.** Let $F \colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. Define the covariant functors restriction and induction along F by

$$\begin{aligned} \mathrm{res}_F \colon \mathrm{MOD}(R\mathcal{D}) &\longrightarrow \mathrm{MOD}(R\mathcal{C}) \\ M &\mapsto M \otimes_{R\mathcal{D}} R\mathcal{D}(F(?),??) \end{aligned} \tag{1.3.4}$$

and

$$\begin{aligned} \mathrm{ind}_F \colon \mathrm{MOD}(R\mathcal{C}) &\longrightarrow \mathrm{MOD}(R\mathcal{D}) \\ N &\mapsto N \otimes_{R\mathcal{C}} R\mathcal{D}(??,F(?)). \end{aligned} \tag{1.3.5}$$

Definition 1.3.5. A *finitely generated free $R\mathcal{C}$-module* is a finite sum of typical free $R\mathcal{C}$-modules, i.e.,

$$\bigoplus_{c \in ob(\mathcal{C})} \bigoplus_{X(c)} R\mathrm{mor}_{\mathcal{C}}(c,?),$$

where $X(c)$ determines the multiplicity of $R\mathrm{mor}_{\mathcal{C}}(c,?)$ and $\sqcup_{c \in ob(\mathcal{C})} X(c)$ is finite. We denote by $\mathcal{F}_R(\mathcal{C})$ the full subcategory of $\mathrm{MOD}(R\mathcal{C})$ whose objects are finitely generated free $R\mathcal{C}$-modules.

We define

$$K_n(R\mathcal{C}) := K_n(\mathcal{F}_R(\mathcal{C})) \tag{1.3.6}$$

for all $n \in \mathbb{Z}$. We specialize the above definitions in order to show that for all $n \in \mathbb{Z}$, $K_n(R-)$ is cofinite Mackey functor.

Definition 1.3.6 (Groupoid)**.** A *groupoid* is a small category whose morphisms are isomorphisms. Let GROUPOID be the category of groupoids with functors of groupoids as morphisms.

Example 1.3.7. Let T be a left G-set. Define the groupoid $\mathscr{G}^G(T)$ with objects the elements of T and for $t, t' \in T$ the set of morphisms $\mathrm{mor}_{\mathscr{G}^G(T)}(t,t') = \{g \in G \mid gt = t'\}$. The group multiplication gives the composition of morphisms.

Every G-map $S \xrightarrow{f} T$ defines a covariant functor $\mathscr{G}^G(S) \xrightarrow{\mathscr{G}^G(f)} \mathscr{G}^G(T)$ given on objects by $\mathscr{G}^G(f)(s) = f(s)$ and on morphisms by $\mathscr{G}^G(f)(s \xrightarrow{g} s') = f(s) \xrightarrow{g} f(s')$. Hence, there is a covariant functor

$$\begin{aligned} \mathscr{G}^G\colon G\text{-SET} &\longrightarrow \text{GROUPOID} \\ S &\mapsto \mathscr{G}^G(S) \end{aligned} \tag{1.3.7}$$

called the *transport groupoid*.

Let us consider $\alpha\colon H \to G$ any group homomorphism. For any H-set S there is a functor

$$\overline{\alpha}\colon \mathscr{G}^H(S) \longrightarrow \mathscr{G}^G(\mathrm{ind}_\alpha S) \tag{1.3.8}$$

defined on objects by $s \mapsto [1, s]$ and on morphisms by $h \mapsto \alpha(h)$. Indeed, $\overline{\alpha}$ is a natural transformation

$$\begin{tikzcd} H\text{-SET} \arrow[r, "\mathscr{G}^H"] \arrow[d, "\mathrm{ind}_\alpha"'] & \text{GROUPOID} . \\ G\text{-SET} \arrow[ur, "\mathscr{G}^G"'] & \end{tikzcd}$$

Similarly, for any G-set T there is a functor

$$\underline{\alpha}\colon \mathscr{G}^H(\mathrm{res}_\alpha T) \longrightarrow \mathscr{G}^G(T) \tag{1.3.9}$$

which is the identity on objects and on morphisms $h \mapsto \alpha(h)$. Indeed, $\underline{\alpha}$ is a natural transformation

$$\begin{tikzcd} G\text{-SET} \arrow[r, "\mathscr{G}^G"] \arrow[d, "\mathrm{res}_\alpha"'] & \text{GROUPOID} . \\ H\text{-SET} \arrow[ur, "\mathscr{G}^H"'] & \end{tikzcd}$$

Induction

Let $\alpha\colon H \to G$ be any group homomorphism. For all H-set S the functor $\overline{\alpha}$ induces a functor

$$\mathrm{Ind}_\alpha\colon \mathcal{F}_R(\mathscr{G}^H(S)) \longrightarrow \mathcal{F}_R(\mathscr{G}^G(\mathrm{ind}_\alpha S)) \tag{1.3.10}$$

sending an object

$$\bigoplus_{s \in ob(\mathscr{G}^H(S))} \bigoplus_{X(s)} R\mathrm{mor}_{\mathscr{G}^H(S)}(s, ?)$$

to the object

$$\bigoplus_{[g,t] \in ob(\mathscr{G}^G(\mathrm{ind}_\alpha S))} \bigoplus_{s \in \overline{\alpha}^{-1}([g,t])} \bigoplus_{X(s)} R\mathrm{mor}_{\mathscr{G}^G(\mathrm{ind}_\alpha S)}([g,t], ?).$$

It is not hard to see that the target is finite if the sum in the source is finite.

Restriction

Let us consider $\alpha\colon H \to G$ injective such that $G/\alpha(H)$ is finite. For all G-set T the functor $\underline{\alpha}$ induces, by precomposition, a functor

$$\begin{aligned} \mathrm{Res}_\alpha\colon \mathcal{F}_R(\mathscr{G}^G(T)) &\longrightarrow \mathcal{F}_R(\mathscr{G}^H(\mathrm{res}_\alpha T)). \\ F &\mapsto F\circ\underline{\alpha} \end{aligned} \tag{1.3.11}$$

Let us briefly comment why the assumptions on α are necessary. It suffices to show that the image of a typical free $R\mathscr{G}^G(T)$-module under Res_α,

$$R\mathrm{mor}_{\mathscr{G}^G(T)}(t, \underline{\alpha}(?)),$$

is a typical free $R\mathscr{G}^H(\mathrm{res}_\alpha T)$-module. Let $s \in \mathrm{res}_\alpha T = T$, then

$$\mathrm{mor}_{\mathscr{G}^G(T)}(t, \underline{\alpha}(s)) = \{g \in G \mid gt = \underline{\alpha}(s) = s\}.$$

Since $G/\alpha(H)$, assume that $G = \alpha(H)g_1 \sqcup \cdots \sqcup \alpha(H)g_n$. Now we claim that there is a bijection between $\{g\in G \mid gt = s\} \cap \alpha(H)g_i$ and the set $\mathrm{mor}_{\mathscr{G}^H(\mathrm{res}^T_\alpha)}(g_i t, s)$ for $1 \le i \le n$. This is true because α is injective an therefore there is a bijection

$$\begin{aligned} \{g\in G \mid gt = s\} \cap \alpha(H)g_i &\xrightarrow{\simeq} \{h \in H \mid \alpha(h)g_i t = s\}. \\ g = \alpha(h)g_i &\mapsto h \end{aligned}$$

Hence

$$\mathrm{mor}_{\mathscr{G}^G(T)}(t, \underline{\alpha}(s)) \simeq \coprod_{i=1}^{n} \mathrm{mor}_{\mathscr{G}^H(\mathrm{res}_\alpha T)}(g_i t, s),$$

and since it is natural in $s \in ob(\mathscr{G}^H(\mathrm{res}_\alpha T))$ we obtain

$$R\mathrm{mor}_{\mathscr{G}^G(T)}(t, \underline{\alpha}(s)) \cong \bigoplus_{i=1}^{n} R\mathrm{mor}_{\mathscr{G}^H(\mathrm{res}_\alpha T)}(g_i t, s)$$

showing that Res_α is well defined.

Finally, let G be a group. Define the functor

$$M(G) := K_n(\mathcal{F}_R(\mathscr{G}^G(pt))),$$

where pt denotes the G-set consisting of one single point. Let $H \xrightarrow{\alpha} G$ be any group homomorphism, consider the composition of functors

$$\mathcal{F}_R(\mathscr{G}^H(pt)) \xrightarrow{\mathrm{Ind}_\alpha} \mathcal{F}_R(\mathscr{G}^G(\mathrm{ind}_\alpha(pt))) \xrightarrow{pr} \mathcal{F}_R(\mathscr{G}^G(pt)),$$

where pr is induced by the projection $G \times_\alpha (pt) \to pt$. Define $M_*(\alpha)$ as the induced map by $pr \circ \mathrm{Ind}_\alpha$ in K-theory. This yields

$$M_*\colon \mathrm{GR} \longrightarrow \mathrm{MOD}(\mathbb{Z}).$$

Let α be a morphism in GRIFI, then the functor Res_α induces a map $M^*(\alpha)$ in K-theory. This yields

$$M^*\colon \mathrm{GRIFI} \longrightarrow \mathrm{MOD}(\mathbb{Z}).$$

Theorem 1.3.8. *For all $n \in \mathbb{Z}$ the functor*

$$\begin{aligned} K_n(R-)\colon \mathrm{GR} &\longrightarrow \mathrm{MOD}(\mathbb{Z}) \\ G &\mapsto K_n(R\mathscr{G}^G(pt)) \end{aligned} \tag{1.3.12}$$

is a Mackey functor in the sense of Definition 1.2.1.

Proof. We have defined M_* and M^* above and coincide on objects. Parts (b) and (c) are easy to verify. The double coset formula requires a little work and we refer the reader to [31, Lemma 14.12] for details.

□

Remark 1.3.9. This section summarizes a particular case of the general setting considered in Chapters 13 and 14 on [31].

B(C)-module structure on K-theory

We are interested in C be the infinite cyclic group. The subgroups $C^m \leq C$ are indexed by $m \in \mathbb{N}$ with C^m the subgroup of index m. For all $n \in \mathbb{N}$, Theorem 1.3.8 gives a Mackey functor, this time on the subgroups of C

$$K_n(R-)\colon \mathrm{GR}_{\leq C} \longrightarrow \mathrm{MOD}(\mathbb{Z}),$$

where $\mathrm{GR}_{\leq C}$ is the subcategory of GR of subgroups of C. Identifying $K_n(R\mathscr{G}^{C^m}(pt)) = K_n(R[C^m])$ and $K_n(R\mathscr{G}^C(pt)) = K_n(R[C])$, for the inclusion map $\sigma_m\colon C^m \to C$ we have

$$\mathrm{res}_{\sigma_m}\colon K_n(R[C]) \longrightarrow K_n(R[C^m])$$

and

$$\mathrm{ind}_{\sigma_m}\colon K_n(R[C^m]) \longrightarrow K_n(R[C]).$$

Last, consider the Burnside ring $\mathrm{B}(C)$ with basis $\{C/C^m \mid m \in \mathbb{N}\}$. According to Proposition 1.2.9 we have

$$\begin{aligned} \mathrm{B}(C) \times K_n(R[C]) &\longrightarrow K_n(R[C]) \\ \left(\sum_m a_m \cdot C/C^m, x\right) &\mapsto \sum_m a_m \mathrm{ind}_{\sigma_m} \circ \mathrm{res}_{\sigma_m}(x). \end{aligned}$$

In Chapter 2, we will study this B(C)-module structure in detail. In particular, using the Bass–Heller–Swan decomposition theorem for $K_n(R[C])$ we will restrict this action to the copies of Bass Nil–groups.

Chapter 2

The Fundamental Theorem and Compatible actions

In the first section, we will define the NK–groups and the Nil–groups in K-theory and state an explicit isomorphism between them in lower dimensions. This isomorphism also illustrates exactly what occurs in higher dimensions. Section 2.2 contains the proof of the Fundamental Theorem of Algebraic K-theory following Swan's exposition [40] summarizing the results in [34] and [17]. We highlight the K-theory of the projective line and the localization sequences appearing in there.

The final section consists of the definition of the $\mathrm{End}_0(\mathbb{Z})$-module structure of $\mathrm{Nil}_n(R)$, using Frobenius and Verschiebung endomorphisms. It also comprises a review of the $\mathrm{B}(C)$-module structure of $K_n(R[C])$, briefly mentioned in Chapter 1. We show the compatibility of the $\mathrm{B}(C)$-module structure on $K_n(R[C])$ with the Bass–Heller–Swan decomposition to define a $\mathrm{B}(C)$-module structure of $NK_n(R)$. We complete the ideas in [39] asserting that both module structures are the same in Corollary 2.3.7

2.1 K-theory of rings and polynomial rings

Let us illustrate a desirable relation. A *Noetherian ring* R is a ring all whose ideals are finitely generated. The Hilbert Basis Theorem states that for a Noetherian ring R the polynomial ring, $R[t]$, and the Laurent polynomial ring, $R[t, t^{-1}]$, are Noetherian.

A *finite type projective resolution* of an R-module M is a finite length resolution by projective modules not necessarily finitely generated projective R-modules. A Noetherian ring R *is regular* if every R-module M having a *finite type projective resolution* is finitely generated R-module. The Hilbert Syzygies theorem states that for a regular ring R the polynomial ring, $R[t]$, and the Laurent polynomial ring, $R[t, t^{-1}]$, are regular.

A naive statement is that $K_n(R[t])$ and $K_n(R[t, t^{-1}])$ depend only on $K_n(R)$. This is false for general rings even for Noetherian rings, but it is partially true for regular rings. We make this statement clear. For the rest of the paragraph we only work with K-groups and G-groups in dimensions $n = 0, 1$.

Let R be a Noetherian ring and $\mathrm{MOD}(R)_{fg}$ denote the category of finitely generated R-modules. Define the groups G_n by

$$G_n(R) := K_n(\mathrm{MOD}(R)_{fg}).$$

In general, a ring homomorphism $R \xrightarrow{\phi} S$ does not induce a map $G_n(R) \to G_n(S)$, it depends on $- \otimes_R S$ being exact where S is considered as an R-module via ϕ. Nevertheless, the canonical ring injections $R \to R[t]$ and $R \to R[t, t^{-1}]$ induce homomorphisms $G_n(R) \to G_n(R[t])$ and $G_n(R) \to G_n(R[t, t^{-1}])$, because $R[t]$ and $R[t, t^{-1}]$ are projectives over R.

The splitting maps $R[t] \xrightarrow{t \mapsto 0} R$ and $R[t, t^{-1}] \xrightarrow{t \mapsto 1} R$, despite the fact that the corresponding tensors are not exact, also induce homomorphisms on corresponding G_n groups

$$\begin{aligned} G_n(R[t]) &\longrightarrow G_n(R), \\ [M] &\mapsto [R \otimes_{R[t]} M] - [\mathrm{Tor}_1^{R[t]}(R, M)] \end{aligned}$$

respectively for $R[t, t^{-1}]$ (see [37, p. 137]).

Proposition 2.1.1. *[37] Let R be a Noetherian ring. The natural homomorphisms*

$$\begin{aligned} G_n(R) &\longrightarrow G_n(R[t]), \quad n = 0, 1 \\ G_0(R) &\longrightarrow G_0(R[t, t^{-1}]). \end{aligned}$$

are isomorphisms with inverse the induce by corresponding splittings. There exists a well–defined homomorphism $G_1(R[t, t^{-1}]) \longrightarrow G_1(R)$ (see [37, p. 145]).

The augmentation map $R[t] \xrightarrow{t \mapsto 0} R$ induces homomorphisms in K-theory $K_n(R[t]) \to K_n(R)$. If R is regular, there are isomorphisms $K_n(R) \cong G_n(R)$ and $K_n(R[t]) \cong G_n(R[t])$ and the diagram

$$\begin{tikzcd} K_n(R[t]) \arrow[r, "\cong"] \arrow[d] & G_n(R[t]) \arrow[d] \\ K_n(R) \arrow[r, "\cong"] & G_n(R) \end{tikzcd}$$

commutes [37, Corollary 3.1.16].

The Laurent polynomial ring $R[t, t^{-1}]$ of a Noetherian ring R requires more effort, nonetheless, there exists a homomorphism [37, Proposition 3.2.18]

$$G_1(R[t, t^{-1}]) \longrightarrow G_0(R) \tag{2.1.1}$$

such that the following holds.

Theorem 2.1.2. *Let R be a Noetherian ring. There exists an isomorphism*

$$G_1(R) \oplus G_0(R) \longrightarrow G_1(R[t, t^{-1}])$$

with left inverse given by the homomorphism in Proposition 2.1.1 direct sum with the homomorphism above.

Corollary 2.1.3. *If R is regular the following are isomorphisms.*

(a) $K_0(R[t]) \cong K_0(R)$,

(b) $K_0(R[t, t^{-1}]) \cong K_0(R)$,

(c) $K_1(R[t]) \cong K_1(R)$,

(d) $K_1(R[t, t^{-1}]) \cong K_1(R) \oplus K_0(R)$.

2.1.1 NK–groups and Nil–groups

The augmentation map $R[t] \xrightarrow{t\mapsto 0} R$ induces homomorphisms in K-theory for all dimensions. The NK–groups are the kernel of the induced maps.

Definition 2.1.4 (NK–groups)**.** For all $n \in \mathbb{Z}$ the *NK–groups* of R are

$$NK_n(R) := \mathrm{Ker}\{K_n(R[t]) \xrightarrow{t\mapsto 0} K_n(R)\}. \tag{2.1.2}$$

The natural ring injection $i\colon R \hookrightarrow R[t]$ also induces homomorphisms in K-theory, hence for all $n \in \mathbb{Z}$ we can also define the NK–groups of R by

$$NK_n(R) := \mathrm{Coker}\{K_n(R) \xrightarrow{i} K_n(R[t])\}. \tag{2.1.3}$$

Example 2.1.5. The following example shows that the Bass Nil–groups are not trivial.

Let k be a field and t be an indeterminate over k. The *dual numbers over k* is the local ring $R = k[t]/(t^2)$, hence $K_0(R) = \mathbb{Z}$ and $K_1(R) = R^\times$, with $R^\times$ the units of R. Let s be an indeterminate over R and consider the split exact sequence

$$NK_1(R) \longrightarrow K_1(R[s]) \leftrightarrows K_1(R).$$

There is an inclusion of units $R^\times \hookrightarrow R[s]^\times$ and since $(1+ts)(1-ts) = 1 + t^2s^2 = 1$ we have that $R[s]^\times/R^\times \neq 0$. Furthermore, $R[s]^\times \hookrightarrow K_1(R[s])$ since $R[s]$ is commutative and the determinant splits this summand. Hence $K_1(R[s])/K_1(R)$ is not trivial and it is isomorphic to $NK_1(R)$.

Now, we introduce the concept of Nil–groups.

Definition 2.1.6 (Nilpotent category)**.** Let R be a ring. We define the category $\mathrm{NIL}(R)$ of pairs (Q,ν) where Q is in the category $\mathcal{P}(R)$ of finitely generated projective R-modules and $\nu\colon Q \to Q$ is a nilpotent endomorphism. A morphism $(Q,\nu) \xrightarrow{F} (Q',\nu')$ in $\mathrm{NIL}(R)$ is an R-module homomorphism $Q \xrightarrow{F} Q'$ such that $F \circ \nu = \nu' \circ F$.

A sequence $0 \longrightarrow (Q',\nu') \longrightarrow (Q,\nu) \longrightarrow (Q'',\nu'') \longrightarrow 0$ in $\mathrm{NIL}(R)$ is exact if its underlying sequence in $\mathcal{P}(R)$ is exact.

There exist exact functors

$$\begin{aligned} \mathrm{NIL}(R) &\longrightarrow \mathcal{P}(R) \\ (Q,\nu) &\mapsto Q \end{aligned} \tag{2.1.4}$$

and

$$\begin{aligned} \mathcal{P}(R) &\longrightarrow \mathrm{NIL}(R) \\ Q &\mapsto (Q,0), \end{aligned} \tag{2.1.5}$$

where 0 denotes the trivial endomorphism of Q.

Definition 2.1.7 (Nil–groups)**.** For all $n \in \mathbb{Z}$ define $\mathrm{Nil}_n(R)$, the *Nil–groups* of R, by the splitting induced by (2.1.5) in K-theory, i.e.,

$$K_n(\mathrm{NIL}(R)) = K_n(R) \oplus \mathrm{Nil}_n(R). \tag{2.1.6}$$

Remark 2.1.8. There is a subtle distinction in the literature between the groups $NK_n(R)$ and $\mathrm{Nil}_n(R)$. One should formally call $NK_n(R)$, *Bass Nil–groups* and call the groups $\mathrm{Nil}_n(R)$, *Nil–groups*. We avoid making an explicit differentation between them because they are isomorphic, up to a degree shifting, as Theorem 2.2.1 shows.

The isomorphism $NK_1 \cong \mathrm{Nil}_0$ below gives an insight for all degrees.

Lemma 2.1.9. *[37] Let R be a ring. Then any matrix $B \in GL(R[t])$ can be reduced, modulo $GL(R)$ and $E(R[t])$, to a matrix of the form $1+At$, where A is a nilpotent matrix with entries in R.*

Proof. Write B as $B = B_0 + B_1t + \cdots + B_dt^d$ with $B_j \in M(R)$. We reduce the degree d up to $d \leq 1$. Let us assume that $d \geq 1$, then the following reduction holds in $GL(R[t])$

$$B \sim \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} B & t^{d-1}B_d \\ 0 & 1 \end{pmatrix} \sim \begin{pmatrix} B - t^dB_d & t^{d-1}B_d \\ -t & 1 \end{pmatrix},$$

where $\sim$ stands for equal modulo multiplication by matrices in $E(R[t])$ and $GL(R)$. Then B is equivalent to a matrix of degree $d-1$. This reduction can be continued to get $B = 1 + At$ for $A \in M(R)$ since $B \in GL(R[t])$ and B_0 must be a unit. Let $B^{-1} = C_0 + tC_1 + \cdots + t^rC_r$ be the inverse of B then

$$1 = (1 + At)(C_0 + tC_1 + \cdots + t^rC_r) = (C_0 + tC_1 + \cdots + t^rC_r)(1 + At).$$

This yields the equations $C_0 = 1$ and $C_j = (-A)^j$. Since $A^{r+1} = 0$ we conclude that A is nilpotent. □

Proposition 2.1.10. *[37] Let R be a ring. Then $NK_1(R)$ is naturally isomorphic to* $\mathrm{Nil}_0(R)$.

Proof. Each $B \in GL(R[t])$ can be reduced to $1 + At$, with A nilpotent matrix by Lemma 2.1.9, then the image of $NK_1(R)$ in $K_1(R)$ consists of matrices of this form. Define

$$\begin{aligned} NK_1(R) &\longrightarrow \mathrm{Nil}_0(R) \\ [1 + At] &\mapsto [R^n, A]. \end{aligned}$$

This is a well–defined homomorphism since $1+At$ conjugated to $1+A't$ in $GL_n(R[t])$ implies, after sending $t \to 1$, that $1 + A$ is conjugated to $1 + A'$ in $GL_n(R)$, hence A is conjugated to A' in $GL_n(R)$.

A substitution of $1 + At$ by $(1 + At) \oplus (1_k)$, where 1_k is the identity in $M_k(R)$, corresponds replacing A by $A \oplus 0_k$ and $[R^n, A]$ by $[R^n, A] + [R^k, 0]$ that are the same in $\tilde{K}_0$ the reduced K_0 group.

Notice that $[1 + At] + [1 + A't] = [(1 + At) \oplus (1 + A't)] = [1 + (A \oplus A')t]$ and the last one is sent to $[R^n, A] + [R^m, A']$. The inverse is

$$\begin{aligned} \mathrm{Nil}_0(R) &\longrightarrow NK_1(R) \\ [R^n, A] &\mapsto [1 + At] \end{aligned}$$

and this concludes the proof. □

2.2 Fundamental Theorem of Algebraic K-theory

In this section, we outline the proof of the Fundamental Theorem of Algebraic K-theory. We focus on the K-theory of the projective line and the pair of localization sequences described in [40].

Theorem 2.2.1 (Fundamental Theorem of Algebraic K-theory)**.** *Let R be a ring. Then, for all $n \in \mathbb{Z}$ there exist natural isomorphisms*

(a) $K_n(R) \oplus NK_n(R) \cong K_n(R[t])$;

(b) $K_n(R) \oplus K_{n-1}(R) \oplus NK_n(R) \oplus NK_n(R) \cong K_n(R[t, t^{-1}])$;

(c) $NK_n(R) \cong \mathrm{Nil}_{n-1}(R)$.

We need the following results.

The K-theory of the projective line

Let $\mathrm{MOD}(\mathbb{P}^1_R)$ be the Abelian category of triples $\mathscr{M} = (M^+, M^-; \theta)$, where M^+ is an $R[t]$-module, M^- is an $R[t^{-1}]$-module, and $\theta\colon R[t, t^{-1}] \otimes_{R[t]} M^+ \xrightarrow{\cong} R[t, t^{-1}] \otimes_{R[t^{-1}]} M^-$ is an $R[t, t^{-1}]$-module isomorphism.

A morphism $\mathscr{M} \xrightarrow{F} \mathscr{N}$ in $\mathrm{MOD}(\mathbb{P}^1_R)$ is a pair (f^+, f^-) such that $f^+\colon M^+ \to N^+$ is an $R[t]$-module homomorphism, $f^-\colon M^- \to N^-$ is an $R[t^{-1}]$-module homomorphism, and the diagram of $R[t, t^{-1}]$-module homomorphisms

$$\begin{tikzcd}
R[t, t^{-1}] \otimes_{R[t]} M^+ \arrow[r, "\theta", "\cong"'] \arrow[d, "1 \otimes f^+"'] & R[t, t^{-1}] \otimes_{R[t]} M^- \arrow[d, "1 \otimes f^-"] \\
R[t, t^{-1}] \otimes_{R[t]} N^+ \arrow[r, "\varphi", "\cong"'] & R[t, t^{-1}] \otimes_{R[t]} N^-
\end{tikzcd}$$

commutes.

Let $\mathscr{M}$ be an object in $\mathrm{MOD}(\mathbb{P}^1_R)$, for $j \in \mathbb{Z}$ *the j-th Serre twist* of $\mathscr{M}$ is the object $\mathscr{M}(j) = (M^+, M^-; t^{-j}\theta)$. In particular, the j-th Serre twist of $\mathscr{O} = (R[t], R[t^{-1}]; 1)$ defines a functor

$$\begin{aligned} \mathrm{MOD}(R) &\longrightarrow \mathrm{MOD}(\mathbb{P}^1_R) \\ M &\mapsto \mathscr{O}(j) \otimes_R M := (M[t], M[t^{-1}]; t^{-j}), \end{aligned}$$

where $M[t]$ (resp. $M[t^{-1}]$) denotes $R[t] \otimes_R M$ (resp. $R[t^{-1}] \otimes_R M$).

Definition 2.2.2 (K-theory of projective line)**.** Let $\mathcal{P}(\mathbb{P}^1_R)$ be the full subcategory of $\mathrm{MOD}(\mathbb{P}^1_R)$ of objects $\mathscr{P} = (P^+, P^-; \theta)$ such that P^+ (resp. P^-) is finitely generated projective $R[t]$-module (resp. finitely generated $R[t^{-1}]$-module). The K-theory of the projective line $\mathbb{P}^1_R$ is

$$K_n(\mathbb{P}^1_R) = K_n(\mathcal{P}(\mathbb{P}^1_R)).$$

The j-th Serre twist defines a functor

$$\begin{aligned} u_j : \mathcal{P}(R) &\longrightarrow \mathcal{P}(\mathbb{P}^1_R) \\ P &\mapsto \mathscr{O}(-j) \otimes_R P \end{aligned} \tag{2.2.1}$$

for all $j \in \mathbb{Z}$ that induces the homomorphism u_{j_*} in K-theory. We only need u_{0*} and u_{1*} to calculate $K_n(\mathbb{P}^1_R)$.

Theorem 2.2.3. *[40, Theorem 9.11] Let R be a ring. Then*

$$(u_{0*}, u_{1*})\colon K_n(R) \oplus K_n(R) \to K_n(\mathbb{P}^1_R) \tag{2.2.2}$$

sending $(x, y) \mapsto u_{0*}(x) + u_{1*}(y)$ is an isomorphism.

For computational reasons, it is convenient to write the K-theory of the projective line in other basis, precomposing the map (u_{0*}, u_{1*}) of Theorem 2.2.3 with the isomorphism

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} : K_n(R) \oplus K_n(R) &\longrightarrow K_n(R) \oplus K_n(R) \\ (x, y) &\mapsto (x, x - y) \end{aligned}$$

The localization sequences

We will consider exact sequences of categories that give rise to long exact sequences in K-theory [40, Section 6].

Definition 2.2.4. Let R be a ring, $R[t]$ its polynomial ring in variable t, $T = \{t^i | i \in \mathbb{N}\}$ the closed multiplicative set of powers of t and consider the localization of $R[t]$ by T, $R[t]_T = R[t, t^{-1}]$.

Define $\mathcal{P}_T(R)$ to be the full subcategory of $\mathrm{MOD}(R[t])$ of objects M such that

(a) $M_T := R[t, t^{-1}] \otimes_{R[t]} M$ is in $\mathcal{P}(R[t, t^{-1}])$ and

(b) there exists a short exact sequence

$$0 \to Q \to P \to M \to 0$$

with Q, P in $\mathcal{P}(R[t])$.

Define $\mathcal{H}_T(R)$ to be the full subcategory of $\mathcal{P}_T(R)$ of those M such that $M_T = 0$.

The natural sequence of functors

$$\mathcal{H}_T(R) \to \mathcal{P}_T(R) \xrightarrow{L_T} \mathcal{P}(R[t, t^{-1}]), \tag{2.2.3}$$

with L_T localization of $R[t]$-modules by T, gives rise to a long exact sequence [40, Corollary 6.4] in K-theory

$$\cdots \xrightarrow{\partial} K_n(\mathcal{H}_T(R)) \to K_n(\mathcal{P}_T(R)) \to K_n(R[t, t^{-1}]) \xrightarrow{\partial_{n-1}} K_{n-1}(\mathcal{H}_T(R)) \to \cdots$$

The inclusion of categories $\mathcal{P}(R[t]) \hookrightarrow \mathcal{P}_T(R)$ satisfies the resolution theorem hypothesis [40, Lemma 3.7], which implies that $K_n(R[t]) \cong K_n(\mathcal{P}_T(R))$ and hence we have a long exact sequence

$$\cdots \xrightarrow{\partial} K_n(\mathcal{H}_T(R)) \to K_n(R[t]) \to K_n(R[t, t^{-1}]) \xrightarrow{\partial_{n-1}} K_{n-1}(\mathcal{H}_T(R)) \to \cdots \tag{2.2.4}$$

Definition 2.2.5. Define $\mathcal{P}^-(\mathbb{P}^1_R)$ the full subcategory of $\mathrm{MOD}(\mathbb{P}^1_R)$ of objects $\mathscr{M} = (M^+, M^-; \theta)$ such that

(a) M^- is in $\mathcal{P}(R[t^{-1}])$ and

(b) there exists a short exact sequence

$$0 \to \mathscr{Q} \to \mathscr{P} \to \mathscr{M} \to 0$$

with $\mathscr{Q}, \mathscr{P}$ in $\mathcal{P}(\mathbb{P}^1_R)$.

Define $\mathcal{H}^-(\mathbb{P}^1_R)$ the full subcategory of $\mathcal{P}^-(\mathbb{P}^1_R)$ of those $\mathscr{M}$ such that $M^- = 0$.

The natural sequence of functors

$$\mathcal{H}^-(\mathbb{P}^1_R) \to \mathcal{P}^-(\mathbb{P}^1_R) \xrightarrow{\alpha} \mathcal{P}(R[t^{-1}]), \tag{2.2.5}$$

with α the obvious projection functor, gives rise to a long exact sequence [40, Theorem 9.14] in K-theory

$$\cdots \xrightarrow{\partial} K_n(\mathcal{H}^-(\mathbb{P}^1_R)) \to K_n(\mathcal{P}^-(\mathbb{P}^1_R)) \to K_n(R[t^{-1}]) \xrightarrow{\partial} K_{n-1}(\mathcal{H}^-(\mathbb{P}^1_R)) \to \cdots$$

Since the inclusion of categories $\mathcal{P}(\mathbb{P}^1_R) \hookrightarrow \mathcal{P}^-(\mathbb{P}^1_R)$ satisfies the resolution theorem, there is an isomorphism $K_n(\mathbb{P}^1_R) \cong K_n(\mathcal{P}^-(\mathbb{P}^1_R))$ and hence we have a long exact sequence

$$\cdots \xrightarrow{\partial} K_n(\mathcal{H}^-(\mathbb{P}^1_R)) \to K_n(\mathbb{P}^1_R) \to K_n(R[t^{-1}]) \xrightarrow{\partial} K_{n-1}(\mathcal{H}^-(\mathbb{P}^1_R)) \to \cdots \tag{2.2.6}$$

2.2.1 Proof of the Fundamental Theorem

We sketch now the proof of the fundamental theorem.

Lemma 2.2.6. *The following categories are equivalent* $\mathcal{H}_T(R) \simeq \mathrm{NIL}(R)$ and $\mathcal{H}^-(\mathbb{P}^1_R) \simeq \mathrm{NIL}(R)$.

Proof of Lemma 2.2.6. The functor

$$\begin{aligned} \mathcal{H}_T(R) &\to \mathrm{NIL}(R) \\ M &\mapsto (M, t) \end{aligned} \tag{2.2.7}$$

is an equivalence with inverse $(Q, \nu) \to Q_\nu$ where Q_ν is Q considered as $R[t]$-module with ν acting via t. The functor 2.2.7 is well defined since for any $H \in \mathcal{H}_T(R)$ there exists $n \in \mathbb{N}$ such that $t^n M = 0$. Multiplying the terms of the resolution

$$0 \longrightarrow Q \longrightarrow P \longrightarrow M \longrightarrow 0$$

by t^n an application of the snake lemma yields

$$0 \longrightarrow M \longrightarrow Q/t^n Q \longrightarrow P/t^n P,$$

since $t^n M = 0$, then $t^n P \subseteq Q$. Therefore, there is an exact sequence

$$0 \longrightarrow M \longrightarrow Q/t^n Q \longrightarrow Q/t^n P \longrightarrow 0.$$

The module $Q/t^n P$ has projective dimension ≤ 1 and $Q/t^n Q$ is a projective R-module. This implies that M is a projective R-module.

For the second equivalence consider the functor

$$\begin{aligned} \mathcal{H}^-(\mathbb{P}^1_R) &\longrightarrow \mathrm{NIL}(R) \\ (P, 0; 0) &\mapsto (P, t) \end{aligned}$$

which is well–defined since P is an object in $\mathcal{P}(R[t])$ and $R[t,t^{-1}]\otimes_{R[t]}P\cong 0$ [40, Lemma 10.6]. The inverse of this functor is given by

$$\begin{aligned} \mathrm{H}\colon \mathrm{NIL}(R) &\longrightarrow \mathcal{H}^-(\mathbb{P}^1_R) \\ (Q,\nu) &\mapsto (Q_\nu,0;0). \end{aligned} \tag{2.2.8}$$

This is well defined since the required resolution is given by

$$0\longrightarrow (Q[t],Q[t^{-1}];t)\xrightarrow{v}(Q[t],Q[t^{-1}];1)\longrightarrow (Q_\nu,0;0)\longrightarrow 0, \tag{2.2.9}$$

where $v=(t-\nu,1-t^{-1}\nu)$. □

Outline of the proof of Theorem 2.2.1: The functor

$$\begin{aligned} \beta\colon \mathrm{MOD}(\mathbb{P}^1_R) &\longrightarrow \mathrm{MOD}(R[t]) \\ (M^+,M^-;\theta) &\mapsto M^+ \end{aligned} \tag{2.2.10}$$

links (2.2.5) and (2.2.3). It restricts to the subcategories $\mathcal{P}^-(\mathbb{P}^1_R)\to\mathcal{P}(R[t])$ and $\mathcal{H}^-(\mathbb{P}^1_R)\to\mathcal{H}_T(R[t])$, and induces $K_n(\mathbb{P}^1_R)\xrightarrow{\beta}K_n(R[t])$ linking (2.2.6) with (2.2.4) by

$$\begin{tikzcd} \cdots K_n(\mathrm{NIL}(R)) \arrow[r] \arrow[d, "\cong"] & K_n(\mathbb{P}^1_R) \arrow[r, "\alpha"] \arrow[d, "\beta"] & K_n(R[t^{-1}]) \arrow[r, "\partial_n"] \arrow[d] & K_{n-1}(\mathrm{NIL}(R))\cdots \arrow[d, "\cong"] \\ \cdots K_n(\mathrm{NIL}(R)) \arrow[r] & K_n(R[t]) \arrow[r] & K_n(R[t,t^{-1}]) \arrow[r, "\partial_n"] & K_{n-1}(\mathrm{NIL}(R))\cdots \end{tikzcd} \tag{2.2.11}$$

Part (a). The compositions $\beta u_j:\mathcal{P}(R)\to\mathcal{P}(R[t])$ are $\beta u_j(P)=P[t]$ $(i=0,1)$, hence induce, in K-theory, the same homomorphism as the canonical functor $\mathcal{P}(R)\longrightarrow\mathcal{P}(R[t])$ splitting after sending $t\mapsto 0$.

The compositions $\alpha u_j\colon\mathcal{P}(R)\to\mathcal{P}(R[t^{-1}])$ are $\alpha u_j(P)=P[t^{-1}]$ $(i=0,1)$, hence induce, in K-theory, the same homomorphism as the canonical functor $\mathcal{P}(R)\to\mathcal{P}(R[t^{-1}])$ splitting after sending $t\mapsto 1$.

Hence the top row of diagram (2.2.11) splits as:

$$0\to K_n(R)\overset{\leftarrow}{\longrightarrow}K_n(R[t^{-1}])\xrightarrow{\partial_n}K_{n-1}(\mathrm{NIL}(R))\to(u_{0*}-u_{1*})K_{n-1}(R)\to 0.$$

The last map coincides with the canonical split surjection and hence

$$0\to K_n(R)\leftrightarrows K_n(R[t^{-1}])\xrightarrow{\partial_n}\mathrm{Nil}_{n-1}(R)\to 0. \tag{2.2.12}$$

This last assertion is true because of the following. Extend the functors u_j of (2.2.1) to $\mathrm{NIL}(R)$ using the split surjection (2.1.4), this is

$$\begin{aligned} u_j\colon \mathrm{NIL}(R) &\longrightarrow \mathcal{P}(\mathbb{P}^1_R) \\ (Q,\nu) &\mapsto u_j(Q_\nu). \end{aligned}$$

In particular, for $j=1,0$ and by the resolution (2.2.9) there is an exact sequence of functors

$$0\longrightarrow u_1\longrightarrow u_0\longrightarrow \mathrm{H}\longrightarrow 0$$

from $\mathrm{NIL}(R)$ to $\mathcal{P}(\mathbb{P}^1_R)$. This shows that $\mathrm{H}_*=u_{o*}-u_{1*}$ coincides, in K-theory, with the canonical splitting (2.1.4) and hence it fits into the exact sequence (2.2.12).

This shows that $\mathrm{Nil}_{n-1}(R)$ coincides with the cokernel of the canonical split injection (2.1.3), then there is an isomorphism $NK_n(R) \cong \mathrm{Nil}_{n-1}(R)$. This proves **part (c)**.

Part (b). Follows from diagram chasing to get from 2.2.11 a Mayer–Vietoris sequence:

$$\cdots \to K_{n+1}(R[t,t^{-1}]) \xrightarrow{\partial_{n+1}} K_n(\mathbb{P}^1_R) \to K_n(R[t]) \oplus K_n(R[t^{-1}]) \to K_n(R[t,t^{-1}]) \xrightarrow{\partial_n} \cdots$$

By the observation made on αu_i and βu_i above, the map

$$K_n(\mathbb{P}^1_R) \to K_n(R[t]) \oplus K_n(R[t^{-1}])$$

factors by the same copy of $K_n(R)$ in the decompostion of $K_n(\mathbb{P}^1_R)$, thus the Mayer–Vietoris sequence splits as

$$0 \to K_n(R) \leftrightarrows K_n(R[t]) \oplus K_n(R[t^{-1}]) \xrightarrow{(i_+)+(i_-)} K_n(R[t,t^{-1}]) \xrightarrow{\partial_n} K_{n-1}(R) \to 0,$$

where $i_\pm$ come from the respective inclusions and ∂_n is a split surjection. $\square$

2.3 Two actions on Nil–groups

2.3.1 Action in terms of Frobenius and Verschiebung endomorphisms

We describe how $\mathrm{End}_0(\mathbb{Z})$ acts on $\mathrm{Nil}_{n-1}(R)$ using Frobenius and Verschiebung endomorphisms. Consider the categories $\mathrm{END}(\mathbb{Z})$ (see Definition B.4.1) and $\mathrm{NIL}(R)$. The tensor product of R-modules defines a pairing

$$\begin{aligned} \mathrm{END}(\mathbb{Z}) \times \mathrm{NIL}(R) &\longrightarrow \mathrm{NIL}(R) \\ ((P,f),(Q,\nu)) &\mapsto (P \otimes_{\mathbb{Z}} Q, f \otimes \nu). \end{aligned} \tag{2.3.1}$$

that induces a product in K-theory [43, Section 9]

$$K_i(\mathrm{END}(\mathbb{Z})) \times K_{n-1}(\mathrm{NIL}(R)) \longrightarrow K_{i+n-1}(\mathrm{NIL}(R))$$

valid for all $i, n \in \mathbb{Z}$. We consider only the case $i = 0$. The copy of $\mathcal{P}(R)$ inside $\mathrm{NIL}(R)$, reflects under pairing (2.3.1) to itself, then we have a well defined $\mathrm{End}_0(\mathbb{Z})$-module structure.

$$\mathrm{End}_0(\mathbb{Z}) \times \mathrm{Nil}_{n-1}(R) \longrightarrow \mathrm{Nil}_{n-1}(R). \tag{2.3.2}$$

The reverse characteristic polynomial χ_t (Section B.4) embeds the ring $\mathrm{End}_0(\mathbb{Z})$ as dense subring of the ring of Witt vectors $W(\mathbb{Z})$. We describe the $\mathrm{End}_0(\mathbb{Z})$-module structure using the Frobenius and Verschiebung endomorphism. This is the same module structure but described in an equivalent way.

Definition 2.3.1 (Frobenius and Verschiebung action)**.** Let $m \in \mathbb{N}$. The m-th Frobenius F_m and the m-th Verschiebung V_m act on $\mathrm{NIL}(R)$ by

$$F_m((Q,\nu)) := (Q, \nu^m)$$

$$V_m((Q,\nu)) := (Q^m, V_m(\nu))$$

where $V_m(\nu)$ is represented by the matrix

$$\begin{pmatrix} 0 & \dots & 0 & \nu \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

Proposition 2.3.2. *The operations F_m and V_m, for $m \in \mathbb{N}$, defined in the category* $\mathrm{NIL}(R)$ *describe the* $\mathrm{End}_0(\mathbb{Z})$*-module structure on* $\mathrm{Nil}_{n-1}(R)$ *given by (2.3.2) for all* $n \in \mathbb{Z}$*.*

Proof. Since the elements $1 - at^m$ suffices to describe $W(\mathbb{Z})$, it is enough to describe how $1-at^m$ acts in $\mathrm{Nil}_{n-1}(R)$. Let $A^m \colon \mathbb{Z}^m \to \mathbb{Z}^m$ be the endomorphism given by the matrix

$$\begin{pmatrix} 0 & \dots & 0 & a \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

whose reverse characteristic polynomial is $\chi_t(A^m) = 1 - at^m$. Let (Q, ν) be an element of $\mathrm{NIL}(R)$, under the pairing (2.3.1) we have

$$(\mathbb{Z}^m, A^m) \times (Q, \nu) \cong (Q^m, A^m \otimes \nu)$$

and $A^m \otimes \nu = V_m(a\nu)$. Hence, in $\mathrm{Nil}_{n-1}(R)$ we have

$$(1 - at^m).[Q, \nu] = [Q^m, V_m(a\nu)] = V_m([Q, a\nu]).$$

□

2.3.2 Action in terms of restriction and induction

Fix $m \in \mathbb{N}$, C the infinite cyclic group with generator t, and $C^m \leq C$ the only subgroup of index m. Identify $R[C] = R[t, t^{-1}]$. There exists ring homomorphisms

$$\begin{aligned} \sigma_m \colon R[C^m] &\longrightarrow R[C] \\ t^m &\mapsto t^m \end{aligned}$$

and

$$\begin{aligned} \sigma_m^+ \colon R[t^m] &\longrightarrow R[t] \\ t^m &\mapsto t^m. \end{aligned}$$

Remark 2.3.3. There exists also a ring homomorphism $\sigma_m^- \colon R[t^{-m}] \longrightarrow R[t^{-1}]$, but for simplicity we only consider σ_m^+. All the results we give for σ_m^+ hold for σ_m^-.

The ring homomorphisms σ_m and σ_m^+ induce restriction and induction homomorphisms in K-theory. These homomorphisms arise from the functors

$$\mathrm{res}_{\sigma_m} \colon \mathrm{MOD}(R[C]) \longrightarrow \mathrm{MOD}(R[C^m])$$

$$\mathrm{res}_{\sigma_m^+} \colon \mathrm{MOD}(R[t]) \longrightarrow \mathrm{MOD}(R[t^m])$$

coming from the *restriction of scalars along σ_m and σ_m^+* and, from the functors

$$\mathrm{ind}_{\sigma_m} \colon \mathrm{MOD}(R[C^m]) \longrightarrow \mathrm{MOD}(R[C])$$

$$\mathrm{ind}_{\sigma_m^+} \colon \mathrm{MOD}(R[t^m]) \longrightarrow \mathrm{MOD}(R[t])$$

coming from the *tensor product along* σ_m *and* σ_m^+.

Recall the localization functor

$$L_T \colon \mathrm{MOD}(R[t]) \longrightarrow \mathrm{MOD}(R[C])$$
$$M \mapsto M_T.$$

Theorem 2.3.4. *There exists a natural transformation of functors between* $L_T \circ \mathrm{ind}_{\sigma_m^+} \circ \mathrm{res}_{\sigma_m^+}$ *and* $\mathrm{ind}_{\sigma_m} \circ \mathrm{res}_{\sigma_m} \circ L_T$, from $\mathrm{MOD}(R[t])$ to $\mathrm{MOD}(R[C])$.

Proof. The localization of M comes equipped with a unique $R[t]$-module homomorphism $\rho_T \colon M \longrightarrow M_T$ such that for an $R[t]$-module homomorphism $M \xrightarrow{\varphi} N$, with N an $R[C]$-module, there exists a unique $R[C]$-module homomorphism $M_T \longrightarrow N$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "\rho_T"] \arrow[dr, "\varphi"'] & M_T \arrow[d, dashed, "\exists!"] \\ & N \end{tikzcd}$$

commutes.

The homomorphism ρ_T yields an $R[t^m]$-module homomorphism $r \colon \mathrm{res}_{\sigma_m^+}(M) \longrightarrow \mathrm{res}_{\sigma_m^+}(M_T)$.

Lemma 2.3.5. *There exists an* $R[t^m]$*-module homomorphism*

$$\mathrm{res}_{\sigma_m^+}(M_T) \xrightarrow{f} \mathrm{res}_{\sigma_m}(M_T)$$

Proof of lemma 2.3.5: As sets, $\mathrm{res}_{\sigma_m^+}(M_T) = M_T = \mathrm{res}_{\sigma_m}(M_T)$, define f as the identity and extend it by linearity to $R[t^m]$. $\square$

Now, the $R[t]$-module $\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}(M)$ comes with a unique homomorphism

$$\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}(M) \xrightarrow{\rho_T} (\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}(M))_T.$$

Since $\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}(M) := R[t] \otimes_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M)$, the unique homomorphism $R[t] \longrightarrow R[C]$ together with f from Lemma 2.3.5 yields

$$R[t] \otimes_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M) \xrightarrow{1 \otimes r} R[t] \otimes_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M_T) \xrightarrow{\rho_T \otimes f} R[C] \otimes_{\sigma_m} \mathrm{res}_{\sigma_m}(M_T).$$

It follows that any $R[t]$-module homomorphism $R[t] \otimes_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M) \longrightarrow N$, with N an $R[C]$-module, factors through $R[C] \otimes_{\sigma_m} \mathrm{res}_{\sigma_m}(M_T)$. In particular, it implies that

$$R[C] \otimes_{\sigma_m} \mathrm{res}_{\sigma_m}(M_T) \cong (R[t] \otimes_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M))_T$$

that is $\mathrm{ind}_{\sigma_m}\mathrm{res}_{\sigma_m}(M_T) \cong \mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}(M)_T$ as $R[C]$-modules. $\square$

The next theorem gives the $\mathrm{B}(C)$-module structure on $NK_n(R)$ by restriciting the action on the algebraic K-theory groups of $R[C]$ to the summands appearing in the Fundamental Theorem of Algebraic K-theory.

Theorem 2.3.6. *For all* $m \in \mathbb{N}$ *the restriction and induction homomorphism in* K*-theory are compatible with the Bass–Heller–Swan decomposition.*

Proof. Fix $m \in \mathbb{N}$ and consider $\mathrm{res}_{\sigma_m^+}$ (resp. res_{σ_m}) and $\mathrm{ind}_{\sigma_m^+}$ (resp. ind_{σ_m}). First, consider the localization sequence of categories (2.2.3)

$$\mathcal{H}_T(R) \to \mathcal{P}_T(R) \xrightarrow{L_T} \mathcal{P}(R[C]).$$

Let M be in $\mathcal{P}_T(R)$, then $L_T \mathrm{ind}_{\sigma_m^+} \mathrm{res}_{\sigma_m^+}(M) \cong \mathrm{ind}_{\sigma_m} \mathrm{res}_{\sigma_m}(L_T(M))$ is a projective $R[C]$-module; it is the trivial module if M was in $\mathcal{H}_T(R)$.

Theorem 2.3.4 yields the diagram

$$\begin{tikzcd}
\mathcal{H}_T(R) \arrow[r] \arrow[d, "\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}"] & \mathcal{P}_T(R) \arrow[r, "L_T"] \arrow[d, "\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+}"] & \mathcal{P}(R[C]) \arrow[d, "\mathrm{ind}_{\sigma_m}\mathrm{res}_{\sigma_m}"] \\
\mathcal{H}_T(R) \arrow[r] & \mathcal{P}_T(R) \arrow[r, "L_T"'] & \mathcal{P}(R[C])
\end{tikzcd}$$

that induces a map of exact sequences in K-theory

$$\begin{tikzcd}
\cdots \arrow[r] & K_{n+1}(R[C]) \arrow[r] \arrow[d] & K_n(\mathcal{H}_T(R)) \arrow[r] \arrow[d] & K_n(R[t]) \arrow[r] \arrow[d] & K_n(R[C]) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & K_{n+1}(R[C]) \arrow[r] & K_n(\mathcal{H}_T(R)) \arrow[r] & K_n(R[t]) \arrow[r] & K_n(R[C]) \arrow[r] & \cdots
\end{tikzcd}$$

where the vertical arrows correspond to the composition of restriction and induction in K-theory groups. Now consider the second localization sequence (2.2.5)

$$\mathcal{H}^-(\mathbb{P}^1_R) \to \mathcal{P}^-(\mathbb{P}^1_R) \xrightarrow{\alpha} \mathcal{P}(R[t^{-1}]),$$

where α is given on objects by $\alpha(\mathscr{M}) = M^-$ and is obvious on morphisms.

Now, consider the sequence 2.2.5. We have a functor $J\colon \mathcal{P}^-(\mathbb{P}^1_R) \longrightarrow \mathcal{P}^-(\mathbb{P}^1_R)$ given in objects by

$$(M^+, M^-; \theta) \mapsto (\mathrm{ind}_{\sigma_m^+}\mathrm{res}_{\sigma_m^+} M^-, \mathrm{ind}_{\sigma_m^-}\mathrm{res}_{\sigma_m^-} M^-; \mathrm{ind}_{\sigma_m}\mathrm{res}_{\sigma_m}\theta).$$

The functor J is well defined and restricts to the category $\mathcal{H}^-(\mathbb{P}^1_R)$. Therefore, Theorem 2.3.4 yields the commutative diagram

$$\begin{tikzcd}
\mathcal{H}^-(\mathbb{P}^1_R) \arrow[r] \arrow[d, "J"] & \mathcal{P}^-(\mathbb{P}^1_R) \arrow[r, "\alpha"] \arrow[d, "J"] & \mathcal{P}(R[t^{-1}]) \arrow[d, "\mathrm{ind}_{\sigma_m^-}\mathrm{res}_{\sigma_m^-}"] \\
\mathcal{H}^-(\mathbb{P}^1_R) \arrow[r] & \mathcal{P}^-(\mathbb{P}^1_R) \arrow[r, "\alpha"'] & \mathcal{P}(R[t^{-1}])
\end{tikzcd}$$

that induces a map of exact sequences in K-theory

$$\begin{tikzcd}
\cdots K_{n+1}(R[t^{-1}]) \arrow[r] \arrow[d] & K_n(\mathcal{H}^-(\mathbb{P}^1_R)) \arrow[r] \arrow[d] & K_n(\mathbb{P}^1_R) \arrow[r] \arrow[d] & K_n(R[t^{-1}]) \cdots \arrow[d] \\
\cdots K_{n+1}(R[t^{-1}]) \arrow[r] & K_n(\mathcal{H}^-(\mathbb{P}^1_R)) \arrow[r] & K_n(\mathbb{P}^1_R) \arrow[r] & K_n(R[t^{-1}]) \cdots
\end{tikzcd}$$

where the vertical arrows correspond to the composition of restriction and induction in K-theory groups. Last, the functors β and α respect $\mathrm{res}_{\sigma_m^+}$ and $\mathrm{ind}_{\sigma_m^+}$. This proves the theorem. $\square$

2.3.3 Compatibility of the actions

Let us define

$$\mathrm{res}_{\sigma_m} : NK_n(R) \longrightarrow NK_n(R)$$

and

$$\mathrm{ind}_{\sigma_m} : NK_n(R) \longrightarrow NK_n(R)$$

as the restriction of res_{σ_m} and ind_{σ_m}, defined on $K_n(R[C])$, to the summand $NK_n(R)$ coming from $K_n(R[t])$ giving by the Bass–Heller–Swan decomposition of $K_n(R[C])$. We could equally define the above maps on the other copy of $NK_n(R)$. The reason we use the one from $K_n(R[t])$ is for simplicity on our exposition.

Theorem 2.3.7. *The* $End_0(\mathbb{Z})$*-module structure on* $\mathrm{Nil}_{n-1}(R)$ *and the* $\mathrm{B}(C)$*-module structure on* $NK_n(R)$ *coincide.*

Proof. Fix $m \in \mathbb{N}$. Then the following diagram

$$\begin{tikzcd}
NK_n(R) \arrow[r, "\cong"] \arrow[d, "\mathrm{res}_{\sigma_m^+}"'] & \mathrm{Nil}_{n-1}(R) \arrow[d, "F_m"] \\
NK_n(R) \arrow[r, "\cong"] \arrow[d, "\mathrm{ind}_{\sigma_m^+}"'] & \mathrm{Nil}_{n-1}(R) \arrow[d, "V_m"] \\
NK_n(R) \arrow[r, "\cong"] & \mathrm{Nil}_{n-1}(R)
\end{tikzcd}$$

commutes. The horizontal arrows are the isomorphisms (c) of Theorem 2.2.1. We replaced res_{σ_m} and ind_{σ_m} with $\mathrm{res}_{\sigma_m^+}$ and $\mathrm{ind}_{\sigma_m^+}$ because of the compatibility with the Bass–Heller–Swan decomposition stated in Theorem 2.3.6 for $NK_n(R)$. $\square$

Chapter 3

Witt–Burnside ring

We start with the definition of $\mathbf{B}_M(\_)$, the completed Burnside ring of a profinite group Γ. The notation corresponds to Elliott's definition [14]. It includes, as part of the data, a commutative monoid M. If M is the trivial monoid, then we recover Dress–Siebeneicher's definition [13].

We show that $\mathbf{B}_M(\_)$ is a Mackey functor, indeed, for $H \leq_o \Gamma$, open subgroup of Γ, we define maps $\mathbf{B}\mathrm{res}_\Gamma^H$ and $\mathbf{B}\mathrm{ind}_H^\Gamma$ of the completed Burnside ring giving the Mackey functor structure. Moreover, for $N \trianglelefteq_o \Gamma$, open normal subgroup of Γ, we define a ring homomorphism $\mathbf{B}\mathrm{proj}_N^\Gamma$ and use it to prove the existence of a ring isomorphism

$$\mathbf{B}_M(\Gamma) \xrightarrow{\cong} \varprojlim_N \mathbf{B}_M(\Gamma/N). \tag{3.0.1}$$

In Section 3.2, we define $\mathbf{W}_\Gamma(\_)$, the Witt–Burnside ring over Γ. This ring is a generalization of the Witt vector ring (see Appendix B). The main result of the chapter is Theorem 3.2.3 giving a ring isomorphism $\mathbf{B}_M(\Gamma) \xrightarrow{\cong} \mathbf{W}_\Gamma(\mathbb{Z}[M])$ valid for all profinite groups.

In ordert to understand how $\mathbf{W}_\Gamma(\mathbb{Z})$ acts on $NK_n(R)$, we relate $\mathbf{W}_\Gamma(\mathbb{Z})$ to the ring $\mathrm{End}_0(\mathbb{Z})$ for Γ the profinite completion of the infinite cyclic group. Finally, Theorem 3.2.9 gives an interpretation of $\mathbf{W}_\Gamma(\_)$ in terms of inverse limits

$$\mathbf{W}_\Gamma(R) \xrightarrow{\cong} \varprojlim_N \mathbf{W}_{\Gamma/N}(R) \tag{3.0.2}$$

valid for all commutative rings.

3.1 The completed Burnside Ring

The theory of profinite groups is vast, we only need the results about profinite groups summarized in Appendix A. We follow the setting of Elliott [14] to define the completed Burnside ring of a profinite group Γ.

The basic objects to define the Burnside ring $\mathrm{B}(\Gamma)$ of a group Γ were finite Γ-sets. We replace finite Γ-sets by almost finite Γ-sets as basic objects. We use the notation of Appendix A for *open subgroups, closed subgroups* etc.; and we fix, once and for all, a commutative monoid M written multiplicatively and a profinite group Γ.

Definition 3.1.1 (Almost finite Γ-set). *A Γ-set X is a discrete topological space with a continuous Γ-action.* A Γ-set X is *almost finite* if each transitive Γ-set appears, up to isomorphism, finitely many times in the orbit decomposition of X.

Every element x of a Γ-set X lies in a finite orbit as the following proposition shows.

Proposition 3.1.2. *Any transitive Γ-set X is finite.*

Proof. Let $x \in X$ and Γ_x be its stabilizer. Then $\Gamma_x \leq_o \Gamma$ since X is discrete. Then $[\Gamma : \Gamma_x] < \infty$ because Γ is a profinite group. $\square$

The concept of Γ-string over M is the building block for Elliott's definition of the completed Burnside ring.

Definition 3.1.3 (Γ-string over M)**.** *A Γ-string over M* is a pair $(X, \|\_\|_X)$ where X is an almost finite Γ-set with a function

$$\|\_\|_X : \Gamma \setminus X \longrightarrow M$$

from the Γ-orbits of X to M. A morphism of Γ-strings $(X, \|\_\|_X) \xrightarrow{f} (Y, \|\_\|_Y)$ between two Γ-strings is a Γ-map $X \xrightarrow{f} Y$ satisfying

$$\|S\|_X = \|f(S)\|_Y^{\#S/\#f(S)}$$

for all orbits S in X where $\#$ denotes the cardinality of the orbit. Let $\Gamma\text{-STRING}_M$ be the category of Γ-strings over M with morphisms of Γ-strings.

Let $(X, \|\_\|_X)$ and $(Y, \|\_\|_Y)$ be a pair of Γ-strings over M. The *disjoint union of* $(X, \|\_\|_X)$ *and* $(Y, \|\_\|_Y)$ has $X \coprod Y$ as underlying almost finite Γ-set and the function given by

$$\|S\|_{X \coprod Y} = \begin{cases} \|S\|_X, & \text{if } S \in \Gamma \setminus X \\ \|S\|_Y, & \text{if } S \in \Gamma \setminus Y. \end{cases}$$

The *product of* $(X, \|\_\|_X)$ *and* $(Y, \|\_\|_Y)$ has $X \times Y$ as underlying almost finite Γ-set and the function given by

$$\|S\|_{X \times Y} = \|S_X\|_X^{\#S/\#S_X} \|S_Y\|_Y^{\#S/\#S_Y},$$

where S_X is the image of S in X under the projection to X and S_Y is the image of S in Y under the projection to Y.

Definition 3.1.4 (Restriction of Γ-string)**.** Let $H \leq_o \Gamma$. The restriction of a Γ-string $(X, \|\_\|_X)$ to an H-string is $(\mathrm{res}_\Gamma^H X, \|\_\|_{\mathrm{res}_\Gamma^H X})$, with underlying H-set $\mathrm{res}_\Gamma^H X$ and function

$$\|H \cdot x\|_{\mathrm{res}_\Gamma^H X} := \|\Gamma \cdot x\|_X^{[\Gamma_x : H_x]}.$$

Lemma 3.1.5. *If X is an almost finite Γ-set and $H \leq_o \Gamma$, then $\mathrm{res}_\Gamma^H X$ is an almost finite H-set.*

Proof. Notice first that each Γ-orbit S of X is a the disjoint union of finitely many H-orbits, say $S = \coprod S_i$. Let $S = \Gamma.x$ and $S' = \Gamma.x'$ be disjoint Γ-orbits of X, then the H-orbits S_i and S_i' are all disjoint. Thus the number of orbits after taking restriction remains finite. $\square$

Lemma 3.1.6. *Let $H \leq_o \Gamma$ and $(X, \|\_\|_X) \xrightarrow{f} (Y, \|\_\|_Y)$ be a map of Γ-strings, then*

$$(\mathrm{res}_\Gamma^H X, \|\_\|_{\mathrm{res}_\Gamma^H X}) \xrightarrow{\mathrm{res}(f)} (\mathrm{res}_\Gamma^H Y, \|\_\|_{\mathrm{res}_\Gamma^H Y})$$

is a map of H-strings.

Proof. The map $\operatorname{res}(f)$ is an H-map by definition. Fix $f(x) = y$, we want to proof that

$$\|H.x\|_{\operatorname{res}^H_\Gamma X} = \|H.y\|^{[H:H_x]/[H:H_y]}_{\operatorname{res}^H_\Gamma Y}. \tag{3.1.1}$$

By definition and because f is a map of Γ-strings we have for the left hand side

$$\|H.x\|_{\operatorname{res}^H_\Gamma X} = \|\Gamma.x\|^{[\Gamma_x:H_x]}_X = \left(\|\Gamma.y\|^{[\Gamma:\Gamma_x]/[\Gamma:\Gamma_y]}_Y\right)^{[\Gamma_x:H_x]}$$

and by definition

$$\|H.y\|_{\operatorname{res}^H_\Gamma Y} = \|\Gamma.y\|^{[\Gamma_y:H_y]}_Y.$$

Equality (3.1.1) follows from

$$\frac{[H:H_x][\Gamma_y:H_y]}{[H:H_y]} = \frac{[\Gamma:\Gamma_x][\Gamma_x:H_x]}{[\Gamma:\Gamma_y]}$$

since for indexes we have $[\Gamma : H_y] = [\Gamma : \Gamma_y][\Gamma_y : H_y]$ and $[\Gamma : H_x] = [\Gamma : H][H : H_x]$ and $[\Gamma : H_y] = [\Gamma : H][H : H_y]$. □

Thus, for $H \leq_o \Gamma$ we have a functor

$$\operatorname{res}^H_\Gamma : \Gamma\text{-STRING}_M \longrightarrow H\text{-STRING}_M \tag{3.1.2}$$

Definition 3.1.7 (Induction of Γ-string)**.** Let $H \leq_o \Gamma$. The induction of an H-string $(Y, \|\text{\_}\|_Y)$ to a Γ-string is $(\operatorname{ind}^\Gamma_H Y, \|\text{\_}\|_{\operatorname{ind}^\Gamma_H Y})$, with underlying Γ-set $\operatorname{ind}^\Gamma_H Y$ and function

$$\|\Gamma.[1,y]\|_{\operatorname{ind}^\Gamma_H Y} := \|H.y\|_Y.$$

Recall that $\operatorname{ind}^\Gamma_H Y = \Gamma \times_H Y$ is the quotient of $\Gamma \times Y$ modulo the relation $(\gamma, y) \sim (\gamma h, h^{-1}y)$ for all $\gamma \in \Gamma$, $y \in Y$ and $h \in H$ endowed with Γ-action

$$\begin{aligned} \Gamma \times \operatorname{ind}^\Gamma_H Y &\longrightarrow \operatorname{ind}^\Gamma_H Y \\ (\gamma', [\gamma, y]) &\mapsto [\gamma'\gamma, y]. \end{aligned}$$

Lemma 3.1.8. *Let $H \leq_o \Gamma$ and Y be an almost finite H-set, then* $\operatorname{ind}^\Gamma_H Y$ is an almost finite Γ-set.

Proof. Notice that each Γ-orbit of $\operatorname{ind}^\Gamma_H Y$ satisfies that $\Gamma.[1,y] \cong \Gamma/H_y \cong \Gamma \times_H H/H_y$. Assume that $Y = \coprod n_i \cdot H/H_i$ is the orbit decomposition of Y, where $n_i \cdot H/H_i$ short for disjoint union of n_i copies of H/H_i. Then we have

$$\begin{aligned} \Gamma \times_H Y &\cong \coprod_i n_i \cdot (\Gamma \times_H H/H_i) \\ &\cong \coprod_i n_i \cdot (\Gamma/H_i) \end{aligned}$$

showing that $\operatorname{ind}^\Gamma_H Y$ has only finitely many copies of each orbit type. □

Lemma 3.1.9. *Let $H \leq_o \Gamma$ and $(X, \|\text{\_}\|_X) \xrightarrow{f} (Y, \|\text{\_}\|_Y)$ be a map of H-strings, then*

$$(\operatorname{ind}^\Gamma_H X, \|\text{\_}\|_{\operatorname{ind}^\Gamma_H X}) \xrightarrow{\operatorname{ind}(f)} (\operatorname{ind}^\Gamma_H Y, \|\text{\_}\|_{\operatorname{ind}^\Gamma_H Y})$$

is a map of Γ*-strings.*

Proof. The map $\mathrm{ind}(f)$ is a Γ-map by definition. Fix $f(x) = y$, we want to proof that

$$\|\Gamma.[1,x]\|_{\mathrm{ind}_H^\Gamma X} = \|\Gamma.[1,y]\|_{\mathrm{ind}_H^\Gamma Y}^{[\Gamma:H_x]/[H:H_y]}. \tag{3.1.3}$$

By definition and because f is a map of H-strings we have for the left hand side

$$\|H.x\|_X = \|H.y\|_Y^{[H:H_x]/[H:H_y]}$$

and the right hand side of (3.1.3) is by definition

$$\|H.y\|_Y^{[\Gamma:H_x]/[\Gamma:H_y]}.$$

The equality follows from the index identities $[\Gamma : H_x] = [\Gamma : H][H : H_x]$ and $[\Gamma : H_y] = [\Gamma : H][H : H_y]$. $\square$

Thus, for $H \leq_o \Gamma$ we have a functor

$$\mathrm{ind}_H^\Gamma\colon H\text{-STRING}_M \longrightarrow \Gamma\text{-STRING}_M \tag{3.1.4}$$

Definition 3.1.10 (Projection of Γ-string)**.** Let $N \trianglelefteq_o \Gamma$. The projection of a Γ-string $(X, \|\text{\_}\|_X)$ to a Γ/N-string is $(\mathrm{proj}_N^\Gamma X, \|\text{\_}\|_{\mathrm{proj}_N^\Gamma X})$, with underlying Γ/N-set $\mathrm{proj}_N^\Gamma X = X^N$ *the N fixed points* of X and function

$$\|(\Gamma/N)\cdot x\|_{X^N} := \|\Gamma\cdot x\|_X.$$

We have by definition that proj_N^Γ gives a functor

$$\mathrm{proj}_N^\Gamma\colon \Gamma\text{-STRING}_M \longrightarrow \Gamma/N\text{-STRING}_M \tag{3.1.5}$$

for $N \trianglelefteq_o \Gamma$.

Definition 3.1.11 (Completed Burnside ring)**.** Let M be a commutative monoid and Γ a profinite group. The *completed Burnside ring* $\mathbf{B}_M(\Gamma)$ *of the profinite group* Γ is the Grothendieck ring of isomorphism classes of Γ-strings over M under disjoint union with product given by Cartesian product.

The functors (3.1.2), (3.1.4) and (3.1.5) define maps of completed Burnside rings

$\mathbf{B}\mathrm{res}_\Gamma^H\colon \mathbf{B}_M(\Gamma) \longrightarrow \mathbf{B}_M(H)$, for $H \leq_o \Gamma$;

$\mathbf{B}\mathrm{ind}_H^\Gamma\colon \mathbf{B}_M(H) \longrightarrow \mathbf{B}_M(\Gamma)$, for $H \leq_o \Gamma$;

$\mathbf{B}\mathrm{proj}_N^\Gamma\colon \mathbf{B}_M(\Gamma) \longrightarrow \mathbf{B}_M(\Gamma/N)$, for $N \trianglelefteq_o \Gamma$.

Indeed, $\mathbf{B}\mathrm{res}_\Gamma^H$ and $\mathbf{B}\mathrm{proj}_N^\Gamma$ are ring homomorphisms while $\mathbf{B}\mathrm{ind}_H^\Gamma$ is just an additive homomorphism.

Remark 3.1.12. Let $M = 1$ be the trivial monoid, then $\mathbf{B}_1(\Gamma) = \hat{\mathrm{B}}(\Gamma)$ is the completed Burnside ring of Γ given by Dress and Siebeneicher and it is the *completion of the Burnside ring* $\mathrm{B}(\Gamma)$. See [28] and [13] fur further details.

Notation. Let $(X, \|\text{\_}\|_X)$ be a Γ-string over M, we denote by $\mathbf{X} = [X, \|\text{\_}\|_X]$ the corresponding isomorphism class in $\mathbf{B}_M(\Gamma)$. The notations $\mathbf{B}\mathrm{res}_\Gamma^H(\mathbf{X})$ (respectively $\mathbf{B}\mathrm{ind}_H^\Gamma(\mathbf{X})$ and $\mathbf{B}\mathrm{proj}_N^\Gamma(\mathbf{X})$) denotes the class $[\mathrm{res}_\Gamma^H X, \|\text{\_}\|_{\mathrm{res}_\Gamma^H X}]$ (respectively $[\mathrm{ind}_H^\Gamma X, \|\text{\_}\|_{\mathrm{ind}_H^\Gamma X}]$ and $[\mathrm{proj}_N^\Gamma X, \|\text{\_}\|_{\mathrm{proj}_N^\Gamma X}]$)

The following result allows to study completed Burnside rings of profinite groups from Burnside rings of finite groups. It is also important because of the following, in Section 1.3, we mentioned that $\mathrm{B}(C)$ acts on $K_n(R[C])$ and studied this action in Chapter 2. It exists a $\mathbf{B}_1(C)$-module structure, for $M = 1$ the trivial monoid, on $NK_n(R)$. We will use Theorem 3.1.13 below to state conditions on $\mathrm{B}(C)$-modules such that they extend to $\mathbf{B}_1(C)$-modules.

Theorem 3.1.13. *For any profinite group Γ there exists a ring isomorphism*

$$\begin{aligned}\mathbf{B}_M(\Gamma) &\longrightarrow \varprojlim_N \mathbf{B}_M(\Gamma/N)\\ \mathbf{X} &\mapsto \left(\mathbf{B}\mathrm{proj}_N^\Gamma(\mathbf{X})\right)_N,\end{aligned}$$

where N runs over all open normal subgroups of Γ.

Proof. Recall that for $L, N \trianglelefteq_o \Gamma$ open normal subgroups of Γ with $L \subseteq N$ there exists a natural surjective homomorphism $\pi_N^L \colon \Gamma/L \to \Gamma/N$, by the functoriality of the Burnside ring, π_N^L yields a ring homomorphism

$$\begin{aligned}\mathbf{B}\mathrm{proj}_N^L \colon \mathbf{B}_M(\Gamma/L) &\longrightarrow \mathbf{B}_M(\Gamma/N)\\ \mathbf{X} &\mapsto \mathbf{B}\mathrm{proj}_N^L(\mathbf{X}).\end{aligned} \tag{3.1.6}$$

The homomorphism $\pi_N^\Gamma \colon \Gamma \to \Gamma/N$ gives the projection and yields a ring homomorphism

$$\begin{aligned}\mathbf{B}\mathrm{proj}_N^\Gamma \colon \mathbf{B}_M(\Gamma) &\longrightarrow \mathbf{B}_M(\Gamma/N)\\ \mathbf{X} &\mapsto \mathbf{B}\mathrm{proj}_N^\Gamma(\mathbf{X}).\end{aligned} \tag{3.1.7}$$

Homomorphisms (3.1.6) and (3.1.7) make the diagram

$$\begin{tikzcd}
\mathbf{B}_M(\Gamma) \arrow[r, "\mathbf{B}\mathrm{proj}_L^\Gamma"] \arrow[dr, "\mathbf{B}\mathrm{proj}_N^\Gamma"'] & \mathbf{B}_M(\Gamma/L) \arrow[d, "\mathbf{B}\mathrm{proj}_N^L"] \\
& \mathbf{B}_M(\Gamma/N)
\end{tikzcd}$$

commutative because for an almost finite Γ-set X and $L \subseteq N$ we have isomorphisms

$$\mathrm{proj}_N^\Gamma(X) \cong X^N \cong \mathrm{proj}_N^L(X^L) \cong \mathrm{proj}_N^L \mathrm{proj}_L^\Gamma(X)$$

of almost finite Γ/N-sets. Then we have a ring homomorphism

$$\begin{aligned}\mathbf{B}_M(\Gamma) &\longrightarrow \varprojlim_N \mathbf{B}_M(\Gamma/N)\\ \mathbf{X} &\mapsto \left(\mathbf{B}\mathrm{proj}_N^\Gamma(\mathbf{X})\right)_N.\end{aligned} \tag{3.1.8}$$

We construct the inverse. Let $\mathbf{X} = (X_N) \in \varprojlim_N \mathbf{B}_M(\Gamma/N)$. Since X_N is in $\mathbf{B}_M(\Gamma/N)$ there is a unique expression

$$X_N = \sum_{N \leq U \leq \Gamma} \mu_U(X_N) \cdot \Gamma/U.$$

By definition of inverse limit we have that $\mathbf{B}\mathrm{proj}_N^L(X_L) = X_N$, hence, for $N \subseteq U$, μ_U is independent of N and we simply write $\mu_U(\mathbf{X})$. Notice also that

$$\mathbf{B}\mathrm{proj}_N^L(\Gamma/U) = \begin{cases} \Gamma/U, & \text{if } N \subseteq U \\ 0, & \text{else.} \end{cases}$$

Combining these two observations, the inverse to (3.1.8) is

$$\begin{aligned} \varprojlim_N \mathbf{B}_M(\Gamma/N) &\longrightarrow \mathbf{B}_M(\Gamma) \\ \mathbf{X} = (X_N) &\mapsto \sum_{(U)} \mu_U(\mathbf{X}) \cdot \Gamma/U, \end{aligned} \tag{3.1.9}$$

where the sum runs over (conjugacy classes) open subgroups. □

Mackey functor structure of completed Burnside ring

Let PGR be the category of profinite groups with group homomorphisms as morphisms and PGRIFI be the subcategory of PGR whose morphisms are the injective group homomorphisms having finite index image in the target.

Theorem 3.1.14. *The functor* $\mathbf{B}_M(\_)$, defined for profinite groups, is a cofinite Mackey functor.

Proof. We check (a)-(d) from Definition 1.2.1. It suffices to give the isomorphisms at the underlying almost finite Γ-sets.

(a) Let $\Gamma \xrightarrow{\alpha} \Gamma'$ be a morphism in PGR, then $\mathbf{B}\mathrm{ind}_\alpha$ gives the covariant structure. Let $\Gamma \overset{\alpha}{\hookrightarrow} \Gamma'$ be a morphism in PGRIFI, then $\mathbf{B}\mathrm{res}_\alpha$ gives the contravariant structure.

(b) Let $\gamma \in \Gamma$ and $c(\gamma)\colon \Gamma \to \Gamma$ conjugation by γ. For $(X, \|\_\|_X)$, Γ-string over M, the Γ-map

$$\begin{aligned} \mathrm{ind}_{c(\gamma)} X &\longrightarrow X \\ [h, x] &\mapsto h\gamma x \end{aligned}$$

is an isomorphism of almost finite Γ-sets giving an isomorphism of Γ-strings over M.

(c) Let $\alpha\colon \Gamma \xrightarrow{\cong} \Gamma'$ be an isomorphism. For $(X, \|\_\|_X)$, Γ'-string over M, the Γ'-map

$$\begin{aligned} \Gamma' \times_\alpha \mathrm{res}_\alpha X &\longrightarrow X \\ [g', x] &\mapsto g'x \end{aligned}$$

is an isomorphism of almost finite Γ'-sets giving an isomorphsim of Γ'-strings over M. For $(Y, \|\_\|_Y)$, Γ-string over M, the Γ-map

$$\begin{aligned} \mathrm{res}_\alpha(\Gamma' \times_\alpha Y) &\longrightarrow Y \\ [g', y] &\mapsto \alpha^{-1}(g')y \end{aligned}$$

is an isomorphism of almost finite Γ-sets giving an isomorphsim of Γ-strings over M. These two isomorphisms verify (c).

(d) The Mackey formula holds because for $H, K \leq_o \Gamma$ the transitive Γ-sets Γ/H and Γ/K are finite and Proposition 1.1.12 is valid.

□

Remark 3.1.15. The multiplicative structure of $\mathbf{B}_M(\_)$ and the fact that it is a cofinite Mackey functor make $\mathbf{B}_M(\_)$ into a Green functor.

Assume that P is a Mackey functor on the open subgroups of Γ. The universal property of the Burnside ring 1.2.9 gives an action

$$\mathrm{B}(\Gamma) \times P(\Gamma) \longrightarrow P(\Gamma) \tag{3.1.10}$$

in terms of restriction and induction. We face the problem to determine if an action

$$\mathbf{B}_M(\Gamma) \times P(\Gamma) \longrightarrow P(\Gamma), \tag{3.1.11}$$

given in terms of restriction and induction and compatible with (3.1.10) exists or not. Theorem 3.1.13 suggests to look closer for actions

$$\mathbf{B}_M(\Gamma/N) \times P(\Gamma) \longrightarrow P(\Gamma)$$

for all $N \trianglelefteq_o \Gamma$. There are still some conditions that P must satisfy and we list them in the following chapter.

3.2 Witt–Burnside ring

In this section, we introduce the concept of Witt–Burnside ring over profinite group Γ as a commutative ring valued functor $\mathbf{W}_\Gamma(\_)$. The definition incorporates the features of the Witt ring with the combinatorial approach of the completed Burnside ring of a profinite group Γ. Indeed, for $\mathbb{Z}[M]$ the integral monoid ring of M there exists an isomorphism $\mathbf{W}_\Gamma(\mathbb{Z}[M]) \cong \mathbf{B}_M(\Gamma)$.

The Witt–Burnside ring over Γ of any commutative ring R is isomorphic to an inverse limit

$$\mathbf{W}_\Gamma(R) \longrightarrow \varprojlim_N \mathbf{W}_{\Gamma/N}(R), \tag{3.2.1}$$

where $N \trianglelefteq_o \Gamma$. We generalize the Frobenius and Verschiebung endomorphisms of Witt rings to endomorphisms of Witt–Burnside rings using the restriction and induction of Γ-sets.

We use the notation of Appendix A, in particular Section A.3 on orbit categories.

Theorem 3.2.1. *[13, 14] For any profinite group Γ, there is a unique functor $\boldsymbol{W}_\Gamma(\_)$ from the category of commutative rings to itself such that*

(a) for any ring R, the set $\mathbf{W}_\Gamma(R)$ coincides with the set $\prod_{\mathcal{O}r^f(\Gamma)} R$;

(b) for any ring R and $\Gamma/H \in \mathcal{O}r^f(\Gamma)$ the H-th Witt polynomial map

$$\begin{aligned} \omega_H \colon \mathbf{W}_\Gamma(R) &\longrightarrow R \\ \mathfrak{a} = (a_{\Gamma/K}) &\mapsto \sum_{(H) \leq_o (K)} \#\mathrm{mor}_\Gamma(\Gamma/K, \Gamma/H) a_{\Gamma/K}^{(K:H)} \end{aligned}$$

is a ring homomorphism. The sum runs over conjugacy classes of $K \leq_o \Gamma$ such that $H \leq \gamma K \gamma^{-1}$ for some $\gamma \in \Gamma$ and $(K : H)$ denotes the index of H as subgroup of $\gamma K \gamma^{-1}$.

Remark 3.2.2. The proof of Theorem 3.2.1 is beyond our purposes. The proof in [13] employs *universal polynomials*, while the proof in [14] employs the *extension problem of ring–valued functors.*

Theorem 3.2.3. *[14, Theorem 1.7] For any profinite group Γ and any commutative monoid M there exists a a ring isomorphism*

$$\mathbf{W}_\Gamma(\mathbb{Z}[M]) \xrightarrow{\cong} \mathbf{B}_M(\Gamma).$$

Outline of the proof: Let $\mathbb{Z}[M]^{(\Gamma)}$ denotes the product $\prod_{\mathcal{O}r^f(\Gamma)} \mathbb{Z}[M]$. We endow $\mathbb{Z}[M]^{(\Gamma)}$ with three different ring structures.

The first ring structure is $\mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}}$ with pointwise addition and multiplication. The second ring structure $\mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}}$ is given by transport structure with

$$\begin{aligned} b' \colon \mathbf{B}_M(\Gamma) &\longrightarrow \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}} \\ \mathbf{X} &\mapsto \left(\sum_{\Gamma.x \cong \Gamma/H} \|\Gamma.x\|_X \right)_{\Gamma/H} \end{aligned} \tag{3.2.2}$$

the counting orbits bijection.

For the third ring structure on $\mathbb{Z}[M]^{(\Gamma)}$ take $n \in \mathbb{N}$ and the monoid ring homomorphism $(\_)^{\{n\}} \colon \mathbb{Z}[M] \to [M]$ sending $m \mapsto m^n$. There exists an injective ring homomorphism

$$\begin{aligned} \omega' \colon \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}} &\longrightarrow \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}} \\ \mathfrak{a} &\mapsto \left(\sum_{H \leq_o K} \#\mathrm{mor}_\Gamma(\Gamma/K, \Gamma/H) a_{\Gamma/H}^{\{(K:H)\}} \right)_{\Gamma/H} \end{aligned} \tag{3.2.3}$$

and there exists a bijection [14, Proposition 1.9]

$$\begin{aligned} j \colon \mathbb{Z}[M]^{(\Gamma)} &\longrightarrow \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}} \\ \mathfrak{a} &\mapsto \left(\sum_{(L)} \sum_{\mathrm{ind}_L^\Gamma L/K \cong \Gamma/H} b'_K \mathbf{B}\mathrm{coind}_1^L (a_{\Gamma/L}) \right)_{\Gamma/H}, \end{aligned} \tag{3.2.4}$$

where (L) denotes the conjugacy class of $L \leq_o \Gamma$, b'_K is the Γ/K-th coordinate of b' and $\mathbf{B}\mathrm{coind}_1^L$ denotes the homomorphism $\mathbf{B}_1(1) \longrightarrow \mathbf{B}_M(L)$ induced by coinduction (see [14]). This bijection fits in a commutative diagram

$$\begin{tikzcd} \mathbb{Z}[M]^{(\Gamma)} \arrow[r, "j"] \arrow[dr, "\omega"'] & \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}} \arrow[d, "\omega'"] \\ & \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}}, \end{tikzcd}$$

where $\omega = \prod \omega_H$ is the product of the H-th Witt polynomials. Since ω' is a ring homomorphism the domain of ω is a ring by transport structure. This defines the third ring structure $\mathbb{Z}[M]^{(\Gamma)}_{\mathrm{W}}$.

Now, consider the *Burnside ring homomorphism* defined by

$$\begin{aligned} b \colon \mathbf{B}_M(\Gamma) &\longrightarrow \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}} \\ \mathbf{X} &\mapsto \left(\sum_{\varphi \in \mathrm{mor}_\Gamma(\Gamma/H, X)} \|\varphi(\Gamma/H)\|_X^{H/\varphi H} \right)_{\Gamma/H}, \end{aligned} \tag{3.2.5}$$

where $H/\varphi H$ is short for $\#(\Gamma/H)/\#\varphi(\Gamma/H)$. The Burnside ring homomorphism fits in the commutative diagram

$$\begin{tikzcd}
\mathbf{B}_M(\Gamma) \arrow[r, "b'"] \arrow[dr, "b"'] & \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{B}} \arrow[d, "\omega'"] \\
& \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}}.
\end{tikzcd}$$

Finally, the map

$$\begin{aligned} i\colon \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{W}} &\longrightarrow \mathbf{B}_M(\Gamma) \\ \mathfrak{a} &\mapsto \sum_{(H)} \mathbf{B}\mathrm{ind}^{\Gamma}_{H}(\mathbf{B}\mathrm{coind}^{H}_{1}(a_{\Gamma/H})) \end{aligned} \tag{3.2.6}$$

is the unique isomorphism for which the diagram

$$\begin{tikzcd}
\mathbb{Z}[M]^{(\Gamma)}_{\mathrm{W}} \arrow[r, "i"] \arrow[dr, "\omega"'] & \mathbf{B}_M(\Gamma) \arrow[d, "b"] \\
& \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{P}}
\end{tikzcd}$$

commutes [14, Proposition 1.10]. By the uniqueness of $\mathbf{W}_\Gamma(\_)$ we have $\mathbf{W}_\Gamma(\mathbb{Z}[M]) = \mathbb{Z}[M]^{(\Gamma)}_{\mathrm{W}}$. Since i is an isomorphism we conclude the proof. □

Corollary 3.2.4. *If M is the trivial monoid, then $\mathbf{B}_1(\Gamma) = \hat{\mathrm{B}}(\Gamma) \cong \mathbf{W}_\Gamma(\mathbb{Z})$ is the ring isomorphism given by Dress–Siebeneicher [13].*

Relation between Witt–Burnside ring and $\mathbf{End}_0(\mathbb{Z})$

Let C be the infinite cyclic group and $\hat{C}$ be its profinite completion, then $\mathcal{O}r^f(\hat{C}) = \{C/C^m \mid m \in \mathbb{N}\}$ and there exists a group homomorphism $C/C^n \longrightarrow C/C^m$ if and only if m divides n. The C^m-th polynomial of Theorem 3.2.1 coincides with the m-th polynomial of Theorem B.2.1, then it is easy to see that $W(\mathbb{Z}) \cong \mathbf{W}_{\hat{C}}(\mathbb{Z}) \cong \mathbf{B}_1(C)$.

The counting orbits bijection b' of (3.2.2), for the trivial monoid, has components b'_m such that

$$b'_m(C/C^n) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{else.} \end{cases}$$

Consider the set theoretical bijection

$$\begin{aligned} b'\colon \mathbf{B}_1(C) &\longrightarrow W(\mathbb{Z}) \\ \mathbf{X} &\mapsto (b'_m(\mathbf{X}))_m \end{aligned}$$

and define $\varpi_m(1) := b'(C/C^m)$ the vector having 1 in the C^m-th coordinate and zero elsewhere. The ring isomorphism of (B.2.2), $W(\mathbb{Z}) \xrightarrow{\Phi} \Lambda(\mathbb{Z})$, maps it to $\Phi(\varpi_m(1)) = (1 - t^m)^{-1}$.

Now, the endomorphism of free modules $T\colon \mathbb{Z}^m \longrightarrow \mathbb{Z}^m$, given by the matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1 \\ 1 & 0 & & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 1 & 0 \end{pmatrix}$$

satisfies that $[\mathbb{Z}^m, 0] - [\mathbb{Z}^m, T] \in \mathrm{End}_0(\mathbb{Z})$ and the reverse characteristic polynomial maps it to

$$\chi_t([\mathbb{Z}^m, 0] - [\mathbb{Z}^m, T]) = \chi_t(0) \cdot \chi_t(T)^{-1} = \frac{1}{1 - t^m}$$

i.e., the element $C/C^m \in \mathbf{B}_1(C)$ corresponds to the element $[\mathbb{Z}^m, 0] - [\mathbb{Z}^m, T]$ in $\mathrm{End}_0(\mathbb{Z})$.

3.2.1 Relation with Graham's definition

In [16], Graham defined the ring $\mathbf{F}_\Gamma(R)$ for a profinite group Γ and commutative ring R. This ring is, in our notation, $\mathbf{B}_{R_\times}(\Gamma)$ where $R_\times$ is the set R considered as a multiplicative monoid. The principal result in Graham's work is the existence of a ghost map [16, Proposition 1.1]

$$\begin{aligned} \psi \colon \mathbf{B}_{R_\times}(\Gamma) &\longrightarrow R_{\mathrm{P}}^{(\Gamma)}, \\ \mathbf{X} &\mapsto \left(\sum_{\varphi \in \mathrm{mor}_\Gamma(\Gamma/H, X)} \|\varphi(\Gamma/H)\|_X^{H/\varphi H} \right) \end{aligned} \tag{3.2.7}$$

with same notation as in (3.2.5). Consider $\mathbb{Z}[R_\times] \xrightarrow{\sigma} R$ the unique ring homomorphism that is the identity on $R_\times$, then there exists a commutative diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma) \arrow[r, "b"] \arrow[dr, "\psi"'] & \mathbb{Z}[R_\times]_{\mathrm{P}}^{(\Gamma)} \arrow[d, "\mathcal{O}r^f(\sigma)"] \\ & R_{\mathrm{P}}^{(\Gamma)} \end{tikzcd},$$

where $\mathcal{O}r^f(\sigma)$ is induced by σ. Elliott proved [14, Theorem 1.11] that there exists a surjective ring homomorphism $\pi \colon \mathbf{B}_{R_\times}(\Gamma) \longrightarrow \mathbf{W}_\Gamma(R)$ for which diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma) \arrow[d, "\pi"'] \arrow[dr, "\psi"] & \\ \mathbf{W}_\Gamma(R) \arrow[r, "\omega"'] & R_{\mathrm{P}}^{(\Gamma)} \end{tikzcd} \tag{3.2.8}$$

is commutative. The map π is, as set theoretical map, the *cleaning algorithm* given in [16, Section 6].

3.2.2 Witt–Burnside ring as inverse limit

Let Γ and Γ' be a pair of profinite groups and

$$\mathbf{B}F \colon \mathbf{B}_M(\Gamma) \longrightarrow \mathbf{B}_M(\Gamma') \tag{3.2.9}$$

be a set theoretical map that is functorial in M. We use the notation $\mathbf{B}F$ because our interest is in maps between completed Burnside rings induced by functors. In particular, those induced by the restriction (3.1.2), the induction (3.1.4) and the projection (3.1.5) functors.

Definition 3.2.5. A set theoretical map $\mathbf{B}F$ is *well–behaved with respect to Witt–Burnside ring* if there exists a set theoretical map $\mathbf{W}F$ for which diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma) \arrow[r, "\mathbf{B}F"] \arrow[d, "\pi"'] & \mathbf{B}_{R_\times}(\Gamma') \arrow[d, "\pi"] \\ \mathbf{W}_\Gamma(R) \arrow[r, dashed, "\mathbf{W}F"'] & \mathbf{W}_{\Gamma'}(R) \end{tikzcd}$$

commutes for all commutative ring R. The map π is that of diagram (3.2.8) above. The map $\mathbf{B}F$ is *well-behaved with respect to product rings* if there exists a set theoretical map $\mathrm{P}F$ for which diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma) \arrow[r, "\mathbf{B}F"] \arrow[d, "\psi"'] & \mathbf{B}_{R_\times}(\Gamma') \arrow[d, "\psi"] \\ R_{\mathrm{P}}^{(\Gamma)} \arrow[r, dashed, "\mathrm{P}F"'] & R_{\mathrm{P}}^{(\Gamma')} \end{tikzcd}$$

commutes for all commutative ring R. The map ψ is map (3.2.7) above.

Proposition 3.2.6. *[14, Proposition 4.1] If $\mathbf{B}F$ is well–behaved with respect to product rings, then $\mathbf{B}F$ is well–behaved with respect to Witt–Burnside rings and $\mathbf{W}F$ is the unique set theoretical map for which diagram*

$$\begin{tikzcd} \mathbf{W}_\Gamma(R) \arrow[r, "\mathbf{W}F"] \arrow[d, "\omega"'] & \mathbf{W}_{\Gamma'}(R) \arrow[d, "\omega"] \\ R_{\mathrm{P}}^{(\Gamma)} \arrow[r, "\mathrm{P}F"'] & R_{\mathrm{P}}^{(\Gamma')} \end{tikzcd}$$

is commutative for all commutative rings. Moreover, $\mathbf{B}F$, $\mathbf{W}F$ and $\mathrm{P}F$ are all additive (resp. multiplicative or ring homomorphism) or none of them. The map $\omega = \prod \omega_H$ is the product of the H-Witt polynomials.

We summarize propositions 4.7–4.10 of [14] into a single proposition.

Proposition 3.2.7. *The maps induced from the functors res_Γ^H, ind_H^Γ and proj_N^Γ are well–behaved with respect to product ring. In particular*

(a) $\mathbf{W}\mathrm{res}_\Gamma^H$ *is a ring homomorphism for $H \leq_o \Gamma$;*

(b) $\mathbf{W}\mathrm{ind}_H^\Gamma$ *is an additive map for $H \leq_o \Gamma$;*

(c) $\mathbf{W}\mathrm{proj}_\Gamma^H$ *is a ring homomorphism for $N \trianglelefteq_o \Gamma$.*

Remark 3.2.8. The homomorphisms $\mathbf{W}\mathrm{res}_C^{C^m}$ and $\mathbf{W}\mathrm{ind}_{C^m}^C$ correspond to the m-th Frobenius and m-th Verschiebung enodomorphisms defined in Section B.3. As before, C denotes the infinite cyclic group and C^m denotes the subgroup of C of index m.

Theorem 3.2.9. *For all commutative rings R and all profinite groups Γ there exists a ring isomorphism*

$$\mathbf{W}_\Gamma(R) \cong \varprojlim_N \mathbf{W}_{\Gamma/N}(R), \tag{3.2.10}$$

where N runs over all open normal subgroups of Γ.

Proof. Fix R, Γ and denote by $R_\times$ the underlying multiplicative monoid of R. For $L, N \trianglelefteq_o \Gamma$ with $L \subseteq N$ the ring homomorphism

$$\mathbf{B}\mathrm{proj}_N^L \colon \mathbf{B}_{R_\times}(\Gamma/L) \longrightarrow \mathbf{B}_{R_\times}(\Gamma/N)$$

gives a commutative diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma/L) \arrow[r, "\pi"] \arrow[d, "\mathbf{B}\mathrm{proj}_N^L"'] & \mathbf{W}_{\Gamma/L}(R) \arrow[d, "\mathbf{W}\mathrm{proj}_N^L"] \\ \mathbf{B}_{R_\times}(\Gamma/N) \arrow[r, "\pi"'] & \mathbf{W}_{\Gamma/N}(R). \end{tikzcd}$$

The ring homomorphism

$$\mathbf{B}\mathrm{proj}_N^\Gamma \colon \mathbf{B}_{R_\times}(\Gamma) \longrightarrow \mathbf{B}_{R_\times}(\Gamma/N)$$

also gives a commutative diagram

$$\begin{tikzcd} \mathbf{B}_{R_\times}(\Gamma) \arrow[r, "\pi"] \arrow[d, "\mathbf{B}\mathrm{proj}_N^\Gamma"'] & \mathbf{W}_{\Gamma}(R) \arrow[d, "\mathbf{W}\mathrm{proj}_N^\Gamma"] \\ \mathbf{B}_{R_\times}(\Gamma/N) \arrow[r, "\pi"'] & \mathbf{W}_{\Gamma/N}(R). \end{tikzcd}$$

for $N \trianglelefteq_o \Gamma$. Since $\mathbf{B}\mathrm{proj}_N^L \circ \mathbf{B}\mathrm{proj}_L^\Gamma = \mathbf{B}\mathrm{proj}_N^\Gamma$ for $L, N \trianglelefteq_o \Gamma$ with $L \subseteq N$ we have

$$\mathbf{W}\mathrm{proj}_N^L \circ \mathbf{W}\mathrm{proj}_L^\Gamma = \mathbf{W}\mathrm{proj}_N^\Gamma.$$

Then $\mathbf{W}_\Gamma(R)$ coincides with $\varprojlim_N \mathbf{W}_{\Gamma/N}(R)$. □

3.2.3 Nonexistence of inflation

For all profinite groups Γ and all commutative rings R we have ring homomorphisms

$$\mathbf{W}\mathrm{proj}_N^\Gamma \colon \mathbf{W}_\Gamma(R) \longrightarrow \mathbf{W}_{\Gamma/N}(R)$$

for $N \trianglelefteq_o \Gamma$. The question if there exists a splitting map

$$\mathbf{W}\mathrm{inf}_N^\Gamma \colon \mathbf{W}_{\Gamma/N}(R) \longrightarrow \mathbf{W}_\Gamma(R) \tag{3.2.11}$$

for any any profinite group Γ and any commutative ring R has a negative answer. We give a counterexample following [35].

Definition 3.2.10. Let p be a prime number. A ring R is called *p-strict ring* provided that R is complete and Hausdorff with respect to the p-adic topology, p is not zero divisor in R and the residue field $k = R/p$ is perfect, i.e., the map $x \mapsto x^p$ is bijective in k.

Example 3.2.11. Let p be a prime number. Then $R = \mathbb{Z}_p$, the p-adic numbers, is a p-strict ring whose residue field is $k = \mathbb{F}_p$ a field of characteristic p.

Theorem 3.2.12. *Let R be a p-strict ring with residue field k. Then*

(a) there exists a unique system of representatives $\tau \colon k \to R$ called Teichmüller representatives such that $\tau(xy) = \tau(x)\tau(y)$;

(b) *every element of* $x \in R$ *can be written uniquely in the form* $x = \sum_{n=0}^{\infty} \tau(x_n)p^n$ *for* $x_n \in k$.

Example 3.2.13. Let R and k be as in Example 3.2.11. The Teichmüller representatives are constructed as follows, $\mathbb{F}_p^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$, hence the non-zero elements of $\mathbb{F}_p^\times$ are the roots of the polynomial $t^{p-1}-1$. By the Hensel's lemma, each $x \in \mathbb{F}_p^\times$ can be uniquely lifted as $\tau(x) \in R$ satisfying $\tau(x)^{p-1}-1=0$. Making $\tau(0)=0$ we have the Teichmüller representatives.

Theorem 3.2.14. *[35, Theorem 2.13] Let* R *be a* p*-strict ring with residue field* k *and Teichmüller representatives* $\tau\colon k \to R$*. Then the map*

$$\begin{aligned}\Psi\colon \mathbf{W}_{\mathbb{Z}_p}(k) &\longrightarrow R\\ (x_{p^i})_i &\mapsto \sum_{n=0}^{\infty} \tau(x_{p^n}^{1/p^n})p^n\end{aligned}$$

is a ring isomorphism.

Corollary 3.2.15. *The rings* $\mathbf{W}_{\mathbb{Z}_p}(\mathbb{F}_p)$ *and* $\mathbb{Z}_p$ *are isomorphic.*

Consider $\Gamma = \mathbb{Z}_p = N$ in the formulation of homomorphism (3.2.11), this yields a ring homomorphism:

$$\mathbf{W}\mathrm{inf}_1^{\mathbb{Z}_p}\colon \mathbb{F}_p \longrightarrow \mathbb{Z}_p.$$

Hence the existence of an inflation homomorphism implies that $\mathbb{Z}_p$ is an $\mathbb{F}_p$-algebra which is false. Then, the ring homomorphism (3.2.11) does not always exist.

Chapter 4

Actions on equivariant homology theories

In this chapter, we introduce the concept of an equivariant homology theory with restriction structure following [27]. Our main example of such a homology theory comes from the Farrell-Jones conjecture as formulated in [10]. The idea behind such equivariant homology theories is that they have a Mackey functor structure. Then it is possible to study them using the module structure over the Burnside ring.

We start with a review of Loday's assembly map for K-theory and point out that this assembly map is compatible with the pairing giving by Pedersen and Weibel for the non-connective algebraic K-theory spectrum [33]. We define a G-homology theory and use the work in [10] to formulate the Farrell–Jones conjecture as the principal example of a G-homology theory.

In Section 4.4, we define $\mathcal{H}_*^?$ an equivariant homology theory. Again, the natural example of an equivariant homology theory is the one coming from the algebraic K-theory $\mathcal{O}r(G)$-spectra defined by Davis and Lück [10].

Lück in [27] used the restriction structure on equivariant homology theories to define an equivariant Chern character. Our example of an equivariant homology theory has also a restriction structure.

In Section 4.5, for $\mathcal{H}_*^?$ an equivariant homology theory with restriction structure, we show that $\mathcal{H}_*^G(X)$ is a $\mathrm{B}(G)$-module for all groups G and all G-CW-complex X.

In Section 4.6, we state and prove our main result, Theorem 4.6.4. It gives criteria to extend a $\mathrm{B}(\Gamma)$-module structure over an equivariant homology theory with restriction to a $\mathbf{B}_M(\Gamma)$-module structure. We close the chapter with applications to K-theory.

4.1 Assembly maps in K-theory

We consider SPACES the *category of compactly generated spaces* with continuous maps as morphisms, SPACES_+ the *category of pointed compactly generated spaces* with base point preserving continuous maps.

Definition 4.1.1 (Category of spectra)**.** A spectrum $\mathbb{E} = \{E_n, \sigma_n\}_{n\in\mathbb{N}_0}$ is a sequence of spaces E_n in SPACES_+ with base-point-preserving maps $\sigma_n\colon \Sigma E_n \to E_{n+1}$, called structure maps.

The space ΣE_n denotes the suspension space of E_n. A map of spectra $\mathbb{E} \longrightarrow \mathbb{E}'$ consists of a sequence of maps $E_n \xrightarrow{f_n} E_n'$ such that $f_{n+1} \circ \sigma_n = \sigma_{n+1} \circ \Sigma f_n$, where Σf_n denotes the induced map in suspension spaces. We denote by SPECTRA the category of spectra with maps of spectra as morphisms.

Definition 4.1.2 (Ω-spectrum)**.** A spectrum $\mathbb{E} = \{E_n, \sigma_n\}$ is called an Ω-spectrum if for each structure map $\sigma_n \colon \Sigma E_n \to E_{n+1}$ its adjoint map $E_n \to \Omega E_{n+1}$ is a weak homotopy equivalence of spaces. We denote by Ω-SPECTRA the subcategory of SPECTRA consisting of Ω-spectra.

For a spectrum $\mathbb{E}$ there is a sequence of homomorphisms

$$\pi_{i+n}(E_n) \xrightarrow{S} \pi_{i+n+1}(E_n \wedge S^1) \xrightarrow{\sigma_{n*}} \pi_{i+n+1}(E_{n+1}), \tag{4.1.1}$$

where S is induced by the suspension map and σ_{n*} is induced from the structure maps. The compositions of the homomorphisms in (4.1.1) form a directed system $\{\pi_{i+n}(E_n)\}$ and the homotopy groups of $\mathbb{E}$ are defined by

$$\pi_i(\mathbb{E}) := \operatorname{colim}_{n\to\infty} \pi_{i+n}(E_n). \tag{4.1.2}$$

Example 4.1.3. Let $\mathbb{E}$ be spectrum and X be a pointed CW-complex. The spectrum $X \wedge \mathbb{E}$ with underlying spaces $(X \wedge \mathbb{E})_n = X \wedge E_n$ and structure maps $1 \wedge \sigma_n$ defines $H_i(X) := \pi_i(X \wedge \mathbb{E})$, a reduced homology theory [22, Proposition 4F.2].

Definition 4.1.4. A *homotopy of maps between spectra $f_k \colon \mathbb{E} \to \mathbb{F}$* is a map of spectra $h \colon [0,1]_+ \wedge \mathbb{E} \to \mathbb{F}$ whose composition with the inclusion $i_k \colon \mathbb{E} \to [0,1]_+ \wedge \mathbb{E}$, given by $e \mapsto k \wedge e$ is f_k for $k = 0, 1$.

Let us illustrate the idea of an assembly map in a naive way. Consider a functor $\mathbb{F}\colon$ SPACES $\to$ SPECTRA. We want to compute the homotopy groups $\pi_i(\mathbb{F}(X))$, but this might be hard. The idea of an assembly map is to replace $\mathbb{F}$ by a nicer functor $\mathbb{E}$ such that $\pi_i(\mathbb{E}(X))$ is a generalized homology theory. The advantage is that, at least, we can use spectral sequences to compute $\pi_i(\mathbb{E}(X))$ and get information about $\pi_i(\mathbb{F}(X))$. We refer the reader to [21] for further details about the definition below.

Definition 4.1.5 (Assembly map)**.** Let $\mathbb{F}\colon$ SPACES $\to$ SPECTRA be a covariant functor. An *assembly map* is a natural transformation $\alpha \colon \mathbb{E} \to \mathbb{F}$ where $\mathbb{E}$ is a homotopy invariant and strongly excisive functor with an homotopy equivalence $\mathbb{E}(*) \simeq \mathbb{F}(*)$.

Loday's product

We consider associative rings with unit and summarize here the results in [25, Chapter 2]. Let R and S be rings. The $R \otimes_{\mathbb{Z}} S$-module isomorphism $R^p \otimes S^q \cong (R \otimes S)^{pq}$ defines, through the tensor product of matrices, a group homomorphism $GL_p(R) \times GL_q(S) \longrightarrow GL_{pq}(R \otimes S)$, hence a continuous map

$$f_{p,q} \colon \mathrm{B}GL_p(R)^+ \times \mathrm{B}GL_q(S)^+ \longrightarrow \mathrm{B}GL_{pq}(R \otimes S)^+. \tag{4.1.3}$$

Let us consider the inclusion map $i_k \colon \mathrm{B}GL_k(R \otimes S)^+ \hookrightarrow \mathrm{B}GL(R \otimes S)^+$ and the stabilizing maps $i_{pq} \circ f_{p,q} \colon \mathrm{B}GL_p(R)^+ \times \mathrm{B}GL_q(S)^+ \longrightarrow \mathrm{B}GL(R \otimes S)^+$ to define

$$\gamma_{p,q} \colon \mathrm{B}GL_p(R)^+ \times \mathrm{B}GL_q(S)^+ \longrightarrow \mathrm{B}GL(R \otimes S)^+$$

by $\gamma_{p,q}(x,y) = i_{pq} \circ f_{p,q}(x,y) - i_{pq} \circ f_{p,q}(x_0,y) - i_{pq} \circ f_{p,q}(x,y_0)$. The maps $\gamma_{p,q}$ are compatible with the stabilizing maps [25, Lemma 2.1.3], i.e., there exists an, up to homotopy, commutative diagram

$$\begin{tikzcd} \mathrm{B}GL_p(R)^+ \times \mathrm{B}GL_q(S)^+ \arrow[rr, hook] \arrow[dr, "\gamma_{p,q}"'] & & \mathrm{B}GL_{p+1}(R)^+ \times \mathrm{B}GL_{q+1}(S)^+ \arrow[dl, "\gamma_{p+1,q+1}"] \\ & \mathrm{B}GL(R\otimes S)^+. & \end{tikzcd}$$

The maps $\gamma_{p,q}$ are trivial on $\mathrm{B}GL_p(R)^+ \vee \mathrm{B}GL_q(S)^+$. The space $\mathrm{B}GL(R\otimes S)^+$ is a connected *H-space* [25, Theorem 1.2.6], hence there is a continuous map

$$\hat{\gamma}_{p,q}\colon \mathrm{B}GL_p(R)^+ \wedge \mathrm{B}GL_q(S)^+ \longrightarrow \mathrm{B}GL(R\otimes S)^+$$

unique up to homotopy such that the diagram

$$\begin{tikzcd} \mathrm{B}GL_p(R)^+ \times \mathrm{B}GL_q(S)^+ \arrow[rr, two heads] \arrow[dr, "\gamma_{p,q}"'] & & \mathrm{B}GL_p(R)^+ \wedge \mathrm{B}GL_q(S)^+ \arrow[dl, "\hat{\gamma}_{p,q}"] \\ & \mathrm{B}GL(R\otimes S)^+ & \end{tikzcd}$$

commutes up to homotopy. Loday defined $\gamma\colon \mathrm{B}GL(R)^+ \times \mathrm{B}GL(S)^+ \longrightarrow \mathrm{B}GL(R\otimes S)^+$ ([25, p. 333]), unique up to weak homotopy equivalence. Similar arguments yields a map

$$\hat{\gamma}\colon \mathrm{B}GL(R)^+ \wedge \mathrm{B}GL(S)^+ \longrightarrow \mathrm{B}GL(R\otimes S)^+$$

such that the following diagram

$$\begin{tikzcd} \mathrm{B}GL(R)^+ \times \mathrm{B}GL(S)^+ \arrow[rr, two heads] \arrow[dr, "\gamma"'] & & \mathrm{B}GL(R)^+ \wedge \mathrm{B}GL(S)^+ \arrow[dl, "\hat{\gamma}"] \\ & \mathrm{B}GL(R\otimes S)^+ & \end{tikzcd}$$

commutes up to weak homotopy equivalence [25, Lemma 2.1.8]. We use the map $\hat{\gamma}$ to induce a a pairing

$$\begin{aligned} \pi_n(\mathrm{B}GL(R)^+) \times \pi_m(\mathrm{B}GL(S)^+) &\xrightarrow{\star} \pi_{n+m}(\mathrm{B}GL(R\otimes S)^+) \\ ([f],[f']) &\mapsto [f]\star[f'] := [\hat{\gamma}(f\wedge f')] \end{aligned}$$

Theorem 4.1.6. *[25, Theorem 2.1.11] There exists a bilinear and associative product, natural in R and S*

$$K_n(R) \times K_m(S) \xrightarrow{\star} K_{n+m}(R\otimes S)$$

for $n, m \geq 1$.

Loday's spectra

Definition 4.1.7 (Suspension ring)**.** The *cone ring* of $\mathbb{Z}$, $\Lambda\mathbb{Z}$, is the ring of column and row finite $\mathbb{N}\times\mathbb{N}$-matrices over $\mathbb{Z}$. The *suspension ring* $\Sigma\mathbb{Z}$ is the quotient of $\Lambda\mathbb{Z}$ by the ideal of finite matrices. For any associative ring with unit R we define $\Lambda R := \Lambda\mathbb{Z}\otimes_{\mathbb{Z}} R$ and $\Sigma R := \Sigma\mathbb{Z}\otimes_{\mathbb{Z}} R$.

Definition 4.1.8 (Loday's K-theory spectra)**.** Let R be a ring with unit. The algebraic K-theory spectrum of R, $\mathcal{K}(R)$, is the sequence of spaces

$$\mathcal{K}(R)_n := \{K_0(\Sigma^n R)\times \mathrm{B}GL(\Sigma^n R)^+\}_{n\in\mathbb{N}_0}$$

with structure maps given in [25, Definition 2.3.4].

Theorem 4.1.9. *[25, Theorem 2.3.5] Let R be a ring. Then*

(a) $\mathcal{K}(R)$ is an Ω-spectrum;

(b) $K_n(R)=\pi_n(\mathcal{K}(R))$ for $n\in\mathbb{Z}$.

This spectrum has multiplicative properties

Proposition 4.1.10. *[25, Proposition 2.4.2] Let R and S be rings. The continuous map $BGL(R)^+\wedge BGL(S)^+ \xrightarrow{\hat{\gamma}} BGL(R\otimes S)^+$ extends to a pairing of spectra*

$$\mathcal{K}(R)\wedge\mathcal{K}(S)\longrightarrow\mathcal{K}(R\otimes S).$$

Hence, for $n,m\in\mathbb{Z}$ there exists a well defined associative product

$$K_n(R)\times K_m(S)\longrightarrow K_{n+m}(R\otimes S)$$

For any ring R the spectra $\mathcal{K}(R)$ defines a generalized homology theory, namely, for a pointed CW-space X consider

$$\tilde{H}_n(X;\mathcal{K}(R)) := \mathrm{colim}_k[S^{n+k}, X\wedge\mathcal{K}(R)_k] = \pi_n(X\wedge\mathcal{K}(R)).$$

The assembly map offers a way to compare $\tilde{H}_n(\mathrm{B}G_+;\mathcal{K}(\mathbb{Z}))$ with $K_n(\mathbb{Z}[G])$, that is, let G be a group and consider the injective group homomorphism

$$\begin{aligned} G &\longrightarrow GL(R[G]) \\ g &\mapsto \begin{pmatrix} g & 0 & \cdots \\ 0 & 1 & 0 \\ \vdots & 0 & \ddots \end{pmatrix}. \end{aligned}$$

It induces $j^+\colon \mathrm{B}G\longrightarrow \mathrm{B}GL(\mathbb{Z}[G])^+$. Further, we compose it with the injective map $\mathrm{B}GL(\mathbb{Z}[G])^+\hookrightarrow(\mathcal{K}(\mathbb{Z}[G]))_0$, sending $x\mapsto(0,x)$, to obtain a map

$$j^+\colon \mathrm{B}G\longrightarrow\mathcal{K}(\mathbb{Z}[G])_0.$$

Proposition 4.1.11. *[25, Proposition 4.1.1] Let R be a ring with unit and G be a group. The composition*

$$\mathrm{B}G_+\wedge\mathcal{K}(R)_n \xrightarrow{j^+\wedge 1} \mathcal{K}(\mathbb{Z}[G])_0\wedge\mathcal{K}(R)_n \xrightarrow{\hat{\gamma}} \mathcal{K}(R[G])_n$$

defines a map of spectra

$$\mathrm{B}G_+\wedge\mathcal{K}(R)\longrightarrow\mathcal{K}(R[G]). \tag{4.1.4}$$

Definition 4.1.12 (Loday's assembly map)**.** Let G be a group. Loday's assembly map, A_G, is the homomorphism induced on homotopy groups by (4.1.4), i.e.,

$$A_G\colon \pi_n(BG_+\wedge\mathcal{K}(R))\longrightarrow\pi_n(\mathcal{K}(R[G]))=K_n(R[G]). \tag{4.1.5}$$

4.2 Algebraic K-theory spectrum

The Algebraic K-theory of additive categories is easier to manage when it arises from a spectra valued functor rather than from a space valued functor. Quillen's original work [34] to define higher algebraic K-theory is through the homotopy groups of a space which happened to be an infinite loop space, therefore it defines a spectrum. Unfortunately, delooping is not functorial. This can be fixed with an iterated application of Waldhausen's $S_\bullet$-construction [44] assigning to an additive category a spectrum.

We follow and refer the reader for further details to [6]. Waldhausen considered categories $\mathcal{C}$ with a distinguished zero object $0_\mathcal{C}$, and *cofibrations* $co\mathcal{C}$ and *weak equivalences* $w\mathcal{C}$, both subcategories of $\mathcal{C}$. A functor $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ between categories with cofibrations and weak equivalences is an *exact functor* if it preserves $co\mathcal{C}$ and $w\mathcal{C}$, and $F(0_\mathcal{C}) = 0_\mathcal{D}$.

Example 4.2.1. The following are standard examples of categories with cofibrations and weak equivalences.

(a) Any exact category $\mathcal{C}$ in the sense of Quillen [34] is a category with cofibrations and weak equivalences. The zero object $0_\mathcal{C}$ is any zero object, the cofibrations are the *admissible monomorphisms*, and the weak equivalences are the *isomorphisms.*

(b) An additive category $\mathcal{C}$ is a category with cofibrations and weak equivalences. The zero object $0_\mathcal{C}$ is any zero object, the cofibrations are inclusions of direct summands, up to isomorphism, and the weak equivalences are the isomorphisms.

Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. For all $n \geq 0$ there is an associated category $S_n\mathcal{C}$ of sequences of cofibrations of length $n+1$, starting at $0_\mathcal{C}$, with a choice of quotients for each cofibration [6, Remark 18]. The category $S_n\mathcal{C}$ is also a category with cofibrations and weak equivalences.

The collection $S_\bullet\mathcal{C} = \{S_n\mathcal{C}\}$ is a *simplicial category*, i.e., it is a functor

$$\begin{aligned} S_\bullet\mathcal{C}\colon \Delta^{op} &\longrightarrow \mathrm{CAT} \\ [n] &\mapsto S_n\mathcal{C} \end{aligned}$$

where Δ is the *simplex category* and CAT is the category of small categories.

Let CAT^c_w be the category of *categories with cofibrations, weak equivalences and exact functors* and SCAT^c_w the category of *simplicial categories with cofibrations, weak equivalences and levelwise exact functors.*

Definition 4.2.2 ($S_\bullet$-category)**.** The $S_\bullet$ *construction* is a functor

$$\begin{aligned} S_\bullet\colon \mathrm{CAT}^c_w &\longrightarrow \mathrm{SCAT}^c_w \\ \mathcal{C} &\mapsto S_\bullet\mathcal{C} \end{aligned} \tag{4.2.1}$$

where $S_\bullet\mathcal{C}[n] := S_n\mathcal{C}$.

We can iterate the $S_\bullet$ construction. In particular, $S^{(2)}_\bullet\mathcal{C} = S_\bullet S_\bullet\mathcal{C}$ is a bisimplicial category with cofibrations and weak equivalences; in general, we have a sequence of k-simplicial categories with cofibrations and weak equivalences $\{S^{(k)}_\bullet\mathcal{C}\}$ and a sequence of topological spaces $\{|wS^{(k)}_\bullet\mathcal{C}|\}$ obtained by *geometric realization.* These are the spaces we need to define the K-theory spectrum of $\mathcal{C}$. The structure maps comes from Waldhausen's observation that for any category $\mathcal{C}$ with cofibrations and weak equivalences there exists an inclusion from the reduced suspension

$$\Sigma|w\mathcal{C}| \longrightarrow |wS_\bullet\mathcal{C}|.$$

and by adjointness

$$|w\mathcal{C}| \longrightarrow \Omega|wS_\bullet\mathcal{C}|.$$

We replace $\mathcal{C}$ with the category $S_\bullet^{(n)}\mathcal{C}$ to get the structure maps

$$|wS_\bullet^{(n)}\mathcal{C}| \xrightarrow{w_n} \Omega|wS_\bullet^{(n+1)}\mathcal{C}|. \tag{4.2.2}$$

The structure maps are homotopy equivalences for $n \geq 1$ [6, Theorem 19].

Definition 4.2.3 (Algebraic K-theory spectrum)**.** The (connective) *algebraic K-theory spectrum* of a category $\mathcal{C}$ with cofibrations and weak equivalences, $\mathbb{K}^{alg}(\mathcal{C})$, is given by $\{|wS_\bullet^{(n)}\mathcal{C}|\ , w_n\}_{n\in\mathbb{N}_0}$.

There are two observations. We make the convention $|wS_\bullet^{(0)}\mathcal{C}| := |w\mathcal{C}|$. The second observation is that, indeed, it is an Ω-spectrum except in dimension $n = 0$, where it can be described as a homotopy theoretic group completion (see [6]).

Remark 4.2.4. Example 4.2.1 shows that every exact category has cofibrations and exact equivalences. Applying the algebraic K-theory spectrum to this category we recover Quillen's Q-construction [6, Theorem 26].

We have that the an iterated application of $S_\bullet$-construction yields a connective spectrum

$$\mathbb{K}^{alg}\colon \text{AdCAT} \longrightarrow \text{SPECTRA} \tag{4.2.3}$$

from the additive categories AdCAT to spectra whose homotopy groups are the K-theory groups of the additive category.

Pedersen and Weibel in [33] use the spectrum $\mathbb{K}^{alg}$ to construct a non–connective spectrum. First, they consider categories of possibly infinitely generated free modules over a ring R equipped with a basis, further they assumed that the elements of this basis are in a metric space X. Finally, they put restrictions on both, the objects an the morphism. These are known as *control conditions.*

In our case, for an additive category $\mathcal{A}$ we consider the category $\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})$ of objects parametrized by the Euclidean space $\mathbb{R}^i$ and bounded morphisms. The natural inclusion $\mathbb{R}^i \times \{0\} \hookrightarrow \mathbb{R}^{i+1}$ yields an inclusion of categories

$$\mathcal{C}_{\mathbb{R}^i}(\mathcal{A}) \longrightarrow \mathcal{C}_{\mathbb{R}^{i+1}}(\mathcal{A})$$

and a map of spectra

$$\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})) \longrightarrow \mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathcal{A}))$$

which is naturally null–homotopic in two ways [7, p. 737]. Recall that for any spectrum $\mathbb{E}$ the suspension spectrum, $\Sigma\mathbb{E}$, is the spectrum with n-th space $(\Sigma\mathbb{E})_n := (\mathbb{E})_{n+1}$. Pedersen and Weibel produced a functorial map of spectra

$$\Sigma\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})) \longrightarrow \mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathcal{A}))$$

or dually

$$\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})) \longrightarrow \Omega\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^{i+1}}(\mathcal{A})).$$

which induces an isomorphism on π_n except possibly on π_0.

Using the directed system

$$\mathbb{K}^{alg}(\mathcal{A}) \to \Omega\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}}(\mathcal{A})) \to \cdots \to \Omega^i\mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})) \to \cdots \tag{4.2.4}$$

we can define the non-connective algebraic K-theory spectrum.

Definition 4.2.5. Let $\mathcal{A}$ be an additive category. The non-connective algebraic K-theory spectrum of $\mathcal{A}$ is given by

$$\mathbb{K}^{-\infty}(\mathcal{A}) := \operatorname{hocolim}_{i\to\infty} \Omega^i \mathbb{K}^{alg}(\mathcal{C}_{\mathbb{R}^i}(\mathcal{A})).$$

Remark 4.2.6. The associated spectrum $\mathbb{K}^{-\infty}(\mathcal{A})$ of an additive category $\mathcal{A}$, with homotopy groups the K-theory groups of $\mathcal{A}$ is, indeed, an Ω-spectrum [33].

The Pedersen–Weibel construction, described above, has a *tensor product pairing* in the sense of [21, Lemma 4.1] i.e., for any *tensor pairing of categories* $\mathcal{A} \times \mathcal{B} \to \mathcal{D}$ the natural tensor pairing

$$\mathcal{C}_{\mathbb{R}^n}(\mathcal{A}) \times \mathcal{C}_{\mathbb{R}^m}(\mathcal{B}) \longrightarrow \mathcal{C}_{\mathbb{R}^{n+m}}(\mathcal{D})$$

induces a pairing of spectra

$$\mathbb{K}^{-\infty}(\mathcal{A}) \wedge \mathbb{K}^{-\infty}(\mathcal{B}) \longrightarrow \mathbb{K}^{-\infty}(\mathcal{D}).$$

Theorem 4.2.7. *[21, Theorem 4.2] For any rings R and S, there is a homotopy equivalence between the Loday pairing $\mathcal{K}(R) \wedge \mathcal{K}(S) \to \mathcal{K}(R \otimes S)$ and the Pedersen-Weibel pairing $\mathbb{K}^{-\infty}(R) \wedge \mathbb{K}^{-\infty}(S) \to \mathbb{K}^{-\infty}(R \otimes S)$.*

Notice that we write $\mathbb{K}^{-\infty}(R)$ for the K-theory of the category of finitely generated free R-modules.

Now we state the assembly map in a more general setting. Let $\mathscr{G}$ be a groupoid and $R\mathscr{G}_{\oplus}$ the symmetric monoidal category associated to $R\mathscr{G}$ (defined in the next section) then there is a natural transformation

$$A_{\mathscr{G}} \colon \mathrm{B}\mathscr{G}_{+} \wedge \mathbb{K}^{-\infty}(R) \longrightarrow \mathbb{K}^{-\infty}(R\mathscr{G}_{\oplus}) \tag{4.2.5}$$

such that A_* is a homotopy equivalence [21, Theorem 4.3].

Theorem 4.2.8. *[21, Theorem 4.3] If $\mathscr{G}$ is a group G considered as a groupoid, then the natural transformation $A_{\mathscr{G}}$ induces in homotopy groups a map that is isomorphic with Loday's assembly map A_G.*

4.3 G-homology theories

We follow [10] to extend the functor $\mathbb{K}^{-\infty}$ from additive categories to the category $\mathcal{O}r(G)$, the orbit category of G. We use it as example of a G-homology theory.

Let us fix a small category $\mathcal{C}$. The next step is to generalize the concept of a space and spectrum to *pointed $\mathcal{C}$-space* and *$\mathcal{C}$-spectrum* as functors.

Definition 4.3.1 ($\mathcal{C}$-space)**.** A *covariant pointed $\mathcal{C}$-space* X is a covariant functor

$$X \colon \mathcal{C} \longrightarrow \mathrm{SPACES}_{+}.$$

A map between $\mathcal{C}$-spaces is a natural transformation of functors. A *contravariant pointed $\mathcal{C}$-space* is defined in a similar way.

Example 4.3.2. Let G be a (discrete) group and $\mathcal{O}r(G)$ the orbit category. Let X be a pointed G-space, i.e., a pointed topological Hausdorff space with a continuous G-action. Then, there is a contravariant pointed $\mathcal{O}r(G)$-space

$$\operatorname{map}_G(\_, X) \colon \mathcal{O}r(G) \longrightarrow \mathrm{SPACES}_{+} \tag{4.3.1}$$
$$G/H \mapsto \operatorname{map}_G(G/H, X) = X^H$$

assigning to G/H the H-fixed points of X. Recall that $\operatorname{map}_G(Y, X)$ denotes the G-equivariant maps from Y to X.

Let X be a contravariant pointed $\mathcal{C}$-space and Y be a covariant pointed $\mathcal{C}$-space. Their *balanced smash product* is the space

$$X \wedge_{\mathcal{C}} Y = \left(\bigvee_{c \in \mathrm{Ob}(\mathcal{C})} X(c) \wedge Y(c) \right) / \sim,$$

where $(X(\phi)(x), y) \sim (x, Y(\phi)(y))$ for all morphism $\phi\colon c \to d$ in $\mathcal{C}$ and points $x \in X(d)$, $y \in Y(c)$.

Definition 4.3.3 ($\mathcal{C}$-spectrum)**.** A covariant $\mathcal{C}$-spectrum $\mathbb{E}$ is a covariant functor

$$\mathbb{E}\colon \mathcal{C} \longrightarrow \mathrm{SPECTRA}.$$

A map of $\mathcal{C}$-spectra is a natural transformation of functors. The notion of $\mathcal{C}$-Ω-spectrum is obtained by replacing the target category with Ω-SPECTRA.

Let X be a contravariant pointed $\mathcal{C}$-space and $\mathbb{E}$ be a covariant $\mathcal{C}$-spectrum. We define their balanced *smash product spectrum* $X \wedge_{\mathcal{C}} \mathbb{E}$ by considering their level-wise smash product $X \wedge_{\mathcal{C}} E_n$ with obvious structure maps.

Example 4.3.4. Let X be a left G-space and $\mathbb{E}$ be a covariant $\mathcal{O}r(G)$-spectrum. We have the balanced product of $\mathrm{map}_G(\_, X_+)$ with $\mathbb{E}$, $\mathrm{map}_G(\_, X_+) \wedge_{\mathcal{O}r(G)} \mathbb{E}$.

Definition 4.3.5 (G-homology theory)**.** Let G be a (discrete) group and R be a ring. Denote by G-CW-PAIR the category of pairs of G-CW-complexes in the sense of [30, p. 738]. A G-homology theory $\mathcal{H}_*^G$ with values in the category of R-modules is a collection of covariant functors

$$\mathcal{H}_n^G : G\text{-}CW\text{-PAIR} \longrightarrow \mathrm{MOD}(R)$$

indexed by $n \in \mathbb{Z}$ together with natural transformations

$$\partial_n^G : \mathcal{H}_n^G(X, A) \longrightarrow \mathcal{H}_{n-1}^G(A) := \mathcal{H}_{n-1}^G(A, \emptyset)$$

for $n \in \mathbb{Z}$ such that the following axioms are satisfied:

(a) **G-homotopy invariance**. If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW-pairs, then $\mathcal{H}_n^G(f_0) = \mathcal{H}_n^G(f_1)$ for all $n \in \mathbb{Z}$.

(b) **Long exact sequence of a pair**. Given a G-CW-pair (X, A) there is a long exact sequence

$$\cdots \to \mathcal{H}_{n+1}^G(X, A) \xrightarrow{\partial} \mathcal{H}_n^G(A) \xrightarrow{\mathcal{H}_n^G(i)} \mathcal{H}_n^G(X) \xrightarrow{\mathcal{H}_n^G(j)} \mathcal{H}_n^G(X, A) \to \cdots$$

where $i\colon A \to X$ and $j\colon X \to (X, A)$ are the inclusions.

(c) **Excision**. Let (X, A) be a G-CW-pair and $f\colon A \to B$ a cellular map of G-CW-complexes. Equip $(X \cup_f B, B)$ with the induced structure of a G-CW-pair. Then the canonical map $(F, f) : (X, A) \to (X \cup_f B, B)$ induces for each $n \in \mathbb{Z}$ an isomorphism:

$$\mathcal{H}_n^G(F, f)\colon \mathcal{H}_n^G(X, A) \xrightarrow{\cong} \mathcal{H}_n^G(X \cup_f B, B)$$

(d) **Disjoint union axiom**. Let $\{X_i \mid i \in I\}$ be a family of G-CW-complexes. Denote by $j_i : X_i \to \coprod_i X_i$ the canonical inclusion. Then the map

$$\bigoplus_i \mathcal{H}_n^G(j_i) : \bigoplus_i \mathcal{H}_n^G(X_i) \longrightarrow \mathcal{H}_n^G(\coprod_i X_i)$$

is bijective for each $n \in \mathbb{Z}$.

Remark 4.3.6. A G-homology theory for $G = \{1\}$ is a homology theory satisfying the disjoint union axiom in the non-equivariant case.

Example 4.3.7. [10, Lemma 4.4] For any G-CW-pair (X, A) consider the space $X_+ \cup_{A_+} \mathrm{cone}(A_+)$, where $\mathrm{cone}(A_+)$ denotes the reduced cone of the pointed space A_+. If $\mathbb{E}$ is a covariant $\mathcal{O}r(G)$-spectrum, then

$$H_n^G(X, A; \mathbb{E}) := \pi_n(\mathrm{map}_G(\,\_\,, X_+ \cup_{A_+} \mathrm{cone}(A_+)) \wedge_{\mathcal{O}r(G)} \mathbb{E})$$

is a G-homology theory.

4.3.1 Isomorphism conjecture for algebraic K-theory

Recall that a *weak homotopy equivalence* of spectra is a map of spectra $\mathbb{E} \to \mathbb{F}$ inducing an isomorphism on all homotopy groups. We want to extend $\mathbb{K}^{-\infty}$: AdCAT $\to$ SPECTRA to an $\mathcal{O}r(G)$-spectra, $\mathbb{K}_R$, in such a way that $\mathbb{K}_R(G/H)$ has the weak homotopy type of $\mathbb{K}^{-\infty}(R[H])$.

First, we extend $\mathbb{K}^{-\infty}$ to the category GROUPOID and then we precompose it with the transport groupoid (see 1.3.7). The following definitions are from Section 2 of [10].

Definition 4.3.8 (R-category)**.** A small category $\mathcal{C}$ is an *R-category* if for any two objects x and y the set $\mathrm{mor}_{\mathcal{C}}(x, y)$ of morphisms from x to y carries the structure of an R-module such that composition induces an R-bilinear map $\mathrm{mor}_{\mathcal{C}}(x, y) \times \mathrm{mor}_{\mathcal{C}}(y, z) \longrightarrow \mathrm{mor}_{\mathcal{C}}(x, z)$ for all objects x, y and z in $\mathcal{C}$.

Example 4.3.9 (Associated $R\mathcal{C}$ category)**.** Given a category $\mathcal{C}$ there is an *associated R-category $R\mathcal{C}$* with the same objects as $\mathcal{C}$ and morphism set $\mathrm{mor}_{R\mathcal{C}}(x, y)$ from x to y is given by the free R-module $R\mathrm{mor}_{\mathcal{C}}(x, y)$ generated by the set $\mathrm{mor}_{\mathcal{C}}(x, y)$. The composition is induced by the composition in $\mathcal{C}$ in the obvious way. The functor $\mathcal{C} \mapsto R\mathcal{C}$ is the left adjoint of the forgetful functor from the category of R-categories to the category of small categories.

Definition 4.3.10 (Symmetric monoidal category)**.** Let $\mathcal{C}$ be a R-category. We define a new R-category $\mathcal{C}_\oplus$, called the *symmetric monoidal R-category associated to $\mathcal{C}$* with an associative and commutative sum $\oplus$ as follows.

The objects in $\mathcal{C}_\oplus$ are n-tuples $x = (x_1, x_2, \ldots, x_n)$ consisting of objects x_i in $\mathcal{C}$ for $n = 0, 1, 2, \ldots$. We will think of the empty set as 0-tuple which we denote by 0. The R-module of morphisms from $x = (x_1, \ldots, x_m)$ to $y = (y_1, \ldots, y_n)$ is given by

$$\mathrm{mor}_{\mathcal{C}_\oplus}(x, y) := \bigoplus_{1 \leq i \leq m, 1 \leq j \leq n} \mathrm{mor}_{\mathcal{C}}(x_i, y_j).$$

Given a morphism $x \xrightarrow{f} y$, we denote by $x_i \xrightarrow{f_{ij}} y_j$ the component which belongs to $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$. If x or y is the empty tuple, then $\mathrm{mor}_{\mathcal{C}_\oplus}(x, y)$ is defined to be the trivial R-module. The composition of $x \xrightarrow{f} y$ and $y \xrightarrow{g} z$ for objects $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_n)$ and $z = (z_1, \dots, z_p)$ is defined by

$$(g \circ f)_{ik} = \sum_{j=1}^{n} g_{jk} \circ f_{ij}$$

The sum on $\mathcal{C}_\oplus$ is defined on objects by sticking the tuples together, i.e. for $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_n)$ define

$$x \oplus y := (x_1, \dots, x_m, y_1, \dots, y_n).$$

The sum of morphism is the obvious one.

Define a GROUPOID-spectrum by

$$\begin{aligned} \mathbb{K}^{-\infty} \colon \text{GROUPOID} &\longrightarrow \Omega\text{-SPECTRA} \\ \mathscr{G} &\mapsto \mathbb{K}^{-\infty}((R\mathscr{G})_\oplus) \end{aligned} \tag{4.3.2}$$

Definition 4.3.11 ($\mathcal{O}r(G)$-Ω-spectrum for K-theory)**.** We define $\mathbb{K}_R$, the *$\mathcal{O}r(G)$-Ω-spectrum over the ring R*, for algebraic K-theory as the composition

$$\mathcal{O}r(G) \xrightarrow{\mathscr{G}^G} \text{GROUPOID} \xrightarrow{\mathbb{K}^{-\infty}} \Omega\text{-SPECTRA}.$$

There is a G-homology theory for K-theory. For any G-CW-pair (X, A) we have

$$H_n^G(X, A; \mathbb{K}_R) := \pi_n(\operatorname{map}_G(\,\_\,, X_+ \cup_{A_+} \operatorname{cone}(A_+)) \wedge_{\mathcal{O}r(G)} \mathbb{K}_R). \tag{4.3.3}$$

The generalized assembly map is the map

$$A_X \colon H_n^G(X; \mathbb{K}_R) \longrightarrow H_n^G(pt; \mathbb{K}_R) \tag{4.3.4}$$

induced by the projection map $X \to pt$ from X to the single point space pt.

Example 4.3.12 ($(\mathbb{K}_R, \mathfrak{F})$-assembly map)**.** Let $X = E_{\mathfrak{F}}(G)$ the classifying space of the family $\mathfrak{F}$, a G-CW-complex, unique up to G-homotopy, such that $E_{\mathfrak{F}}(G)^H$ is contractible if $H \in \mathfrak{F}$ and empty otherwise. The map

$$A_{\mathfrak{F}} \colon H_n^G(E_{\mathfrak{F}}(G); \mathbb{K}_R) \longrightarrow H_n^G(pt; \mathbb{K}_R) = K_n(R[G]) \tag{4.3.5}$$

is called the $(\mathbb{K}_R, \mathfrak{F})$-assembly map.

Conjecture 4.3.13 (Farrell-Jones conjecture)**.** *The Farrell-Jones conjecture for K-theory states that the $(\mathbb{K}_R, \mathcal{VCYC})$-assembly map is an isomorphism, where $\mathcal{VCYC}$ is the family of virtually cyclic subgroups of G.*

At the moment of writing there is no counterexample to to the Farrell-Jones conjecture.

4.4 Equivariant homology theories

An equivariant homology theory consists of a collection of G-homology theories, one for each group G, with the additional property that homomorphisms of groups give morphisms among G-homology theories. We follow the ideas in [27].

Definition 4.4.1 (Equivariant homology theory)**.** An *equivariant homology theory* $\mathcal{H}_*^?$ *with values in* $\mathrm{MOD}(R)$ consists of a G-homology theory $\mathcal{H}_*^G$ for each group G together with an *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and an H-CW-pair (X, A) where $\mathrm{Ker}(\alpha)$ acts freely on X, then for all $n \in \mathbb{Z}$ there exists natural isomorphisms

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \xrightarrow{\cong} \mathcal{H}_n^G(\mathrm{ind}_\alpha(X,A))$$

satisfying:

(a) **Compatibility with the boundary homomorphism**.

$$\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^H.$$

(b) **Functoriality**. Let $\beta\colon G \to K$ be another group homomorphism such that $\mathrm{Ker}(\beta\circ\alpha)$ acts freely on X. Then we have for all $n \in \mathbb{Z}$

$$\mathrm{ind}_{\beta\circ\alpha} = \mathcal{H}_n^K(f) \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X,A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X,A))$$

where $f\colon \mathrm{ind}_\beta\mathrm{ind}_\alpha(X,A) \xrightarrow{\cong} \mathrm{ind}_{\beta\circ\alpha}(X,A)$, $(k,g,x) \mapsto (k\beta(g),x)$ is the natural K-homeomorphism.

(c) **Compatibility with conjugation**. For $n \in \mathbb{Z}$, any $g \in G$, conjugation $c(g)\colon G \to G$ and G-CW-pair (X,A) the homomorphism

$$\mathrm{ind}_{c(g)}\colon \mathcal{H}_n^G(X,A) \longrightarrow \mathcal{H}_n^G(\mathrm{ind}_{c(g)}(X,A))$$

agrees with the homomorphism induced by the G-homeomorphism $f'\colon (X,A) \to \mathrm{ind}_{c(g)}(X,A)$ which sends $x \mapsto [1, g^{-1}x]$ in $G \times_{c(g)} (X,A)$.

Example 4.4.2. [30, Proposition 157] Let $\mathbb{K}_R : \mathcal{O}r(G) \longrightarrow \mathrm{SPECTRA}$ be the algebraic K-theory spectra of Definition 4.3.11. Then, $H_*^?(-;\mathbb{K}_R)$ is an equivariant homology theory.

4.4.1 Restriction structure on equivariant homology theories

Definition 4.4.3 (Restriction structure)**.** A *restriction structure* on an equivariant homology theory $\mathcal{H}_*^?$ consists of the following data. For any injective group homomorphism $\alpha\colon H \hookrightarrow G$, whose image has finite index in G, we require in G-CW-pairs (X,A) natural homomorphisms

$$\mathrm{res}_\alpha\colon \mathcal{H}_n^G(X,A) \longrightarrow \mathcal{H}_n^H(\mathrm{res}_\alpha(X,A)).$$

We require:

(a) **Compatibility with boundary homomorphism**.

$$\mathrm{res}_\alpha \circ \partial_n^G = \partial_n^H \circ \mathrm{res}_\alpha.$$

(b) **Functoriality**. If $\beta\colon G \to K$ is another injective group homomorphism whose image has finite index in K, then $\mathrm{res}_{\beta\circ\alpha} = \mathrm{res}_\alpha \circ \mathrm{res}_\beta$.

(c) **Compatibility of induction and restriction for isomorphisms**. If $\alpha\colon H \xrightarrow{\cong} G$ is an isomorphism of groups, then the composition

$$\mathcal{H}_n^G(X) \xrightarrow{\mathrm{res}_\alpha} \mathcal{H}_n^H(\mathrm{res}_\alpha X) \xrightarrow{\mathrm{ind}_\alpha} \mathcal{H}_n^G(\mathrm{ind}_\alpha \mathrm{res}_\alpha X) \xrightarrow{T(X)} \mathcal{H}_n^G(X) \tag{4.4.1}$$

is the identity, where $T(X)\colon \mathrm{ind}_\alpha \mathrm{res}_\alpha X \to X$ is the canonical G-homeomorphism.

(d) **Double coset formula**. Let $H, K \leq G$ be subgroups such that K has finite index in G. Notice that $|K\backslash G/H|$ is finite in this case. For an H-CW-pair (X, A) let

$$f\colon \coprod_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma}(X, A) \xrightarrow{\cong} \mathrm{res}_G^K \mathrm{ind}_H^G (X, A)$$

be the canonical K-homeomorphism. Then, the following homomorphism agrees for all $n \in \mathbb{Z}$,

$$\mathcal{H}_n^H(X, A) \xrightarrow{\prod_{K\gamma H\in|K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma}} \prod_{K\gamma H\in|K\backslash G/H|} \mathcal{H}_n^K(\mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma}(X, A)) \xrightarrow{\cong}$$

$$\xrightarrow{\cong} \mathcal{H}_n^K\Big(\coprod_{K\gamma H\in|K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma}(X, A)\Big) \xrightarrow{\mathcal{H}_n^K(f)} \mathcal{H}_n^K(\mathrm{res}_G^K \mathrm{ind}_H^G(X, A))$$

with the homomorphism

$$\mathrm{res}_G^K \circ \mathrm{ind}_H^G\colon \mathcal{H}_n^H(X, A) \longrightarrow \mathcal{H}_n^K(\mathrm{res}_G^K \mathrm{ind}_H^G(X, A)).$$

Remark 4.4.4. The notation $\mathrm{ind}_{H\cap K^\gamma}^K$, as in Remark 1.1.13, indicates the induction coming from the inner automorphism $c(\gamma)\colon H \cap K^\gamma \longrightarrow K$, conjugation by γ.

Example 4.4.5. Let $\mathbb{K}_R : \mathcal{O}r(G) \longrightarrow$ SPECTRA be the K-theory spectra. Then $H_*^?(-; \mathbb{K}_R)$ is an equivariant homology theory with a restriction structure. This can be done adapting the proof in [31, Section 14] to our case.

4.5 The $\mathrm{B}(G)$-module structure of $\mathcal{H}_*^G(X)$

In the previous sections, we introduced the concept of equivariant homology theory with restriction structure. We now see that these two structures are sufficient to have a Mackey functor structure.

Let G be any group. Denote by $\mathrm{GR}_{\leq G}$ be the subcategory of GR with objects the subgroups of G and group homomorphisms of subgroups of G as morphisms. Let $\mathrm{GRIFI}_{\leq G}$ the subcategory of $\mathrm{GR}_{\leq G}$ whose morphisms are injective group homomorphisms having finite image in the target.

Proposition 4.5.1. *Let $\mathcal{H}_*^?$ be an equivariant homology theory with a restriction structure. Then for all groups G, all G-CW-complexes X and all $n \in \mathbb{Z}$ there is a cofinite Mackey functor $\mathcal{H}_n^?(\mathrm{res}_G^? X)$ with values in* $\mathrm{MOD}(\mathbb{Z})$.

Proof. Let us fix a group G, a G-CW-complexes X and $n \in \mathbb{Z}$. Define the functor on objects, $H \leq G$, by $\mathcal{H}_n^H(\mathrm{res}_G^H X)$.

(a) Let $H \to G$ be a morphism in $\mathrm{GR}_{\leq G}$. Then we have

$$\mathcal{H}_n^H(\mathrm{res}_G^H X) \xrightarrow{\mathrm{ind}_H^G} \mathcal{H}_n^K(\mathrm{ind}_H^K \mathrm{res}_G^H X) \to \mathcal{H}_n^K(\mathrm{res}_G^K X),$$

where the last map is induced by adjunction. This gives the covariant structure. If $H \to K$ is a morphism in $\mathrm{GRIFI}_{\leq G}$, then

$$\mathcal{H}_n^K(\mathrm{res}_G^K X) \xrightarrow{\mathrm{res}_K^H} \mathcal{H}_n^H(\mathrm{res}_K^H \mathrm{res}_G^K X) = \mathcal{H}_n^H(\mathrm{res}_G^H X)$$

gives the contravariant structure.

(b) Let $H \leq G$ and $\gamma \in G$ such that $\gamma H \gamma^{-1} = H$. Then, conjugation $c(\gamma)\colon H \to H$ is a map in $\mathrm{GR}_{\leq G}$ and the isomorphism

$$\begin{aligned} \mathrm{ind}_{c(\gamma)} \mathrm{res}_G^H X &\longrightarrow \mathrm{res}_G^H X \\ [h, x] &\mapsto h\gamma x \end{aligned}$$

verifies the second axiom of a cofinite Mackey functor.

(c) Follows since the restriction structure on $\mathcal{H}_*^?$ implies compatibility of restriction and induction for isomorphism.

(d) Follows since $\mathcal{H}_*^?$ satisfies the double coset formula by definition.

$\square$

Theorem 4.5.2. *Let $\mathcal{H}_*^?$ be as in Proposition 4.5.1. Then for all group G, all G-CW-complexes X and all $n \in \mathbb{Z}$, $\mathcal{H}_n^?(res_G^? X)$ is a* B(G)-module.

Proof. Let $H \leq_f G$ and consider $L_{G/H}$ the composition

$$\mathcal{H}_n^G(X) \xrightarrow{\mathrm{res}_G^H} \mathcal{H}_n^H(\mathrm{res}_G^H X) \xrightarrow{\mathrm{ind}_H^G} \mathcal{H}_n^G(\mathrm{ind}_H^G \mathrm{res}_G^H X) \xrightarrow{T(X)} \mathcal{H}_n^G(X)$$

where $T(X)\colon \mathrm{ind}_H^G \mathrm{res}_G^H X \to X$ is the canonical G-homeomorphism i.e., the one corresponding to the identity by adjunction. Define an action

$$\begin{aligned} \mathrm{B}(G) \times \mathcal{H}_n^G(X) &\longrightarrow \mathcal{H}_n^G(X) \\ (G/H, x) &\mapsto L_{G/H}(x) \end{aligned}$$

and extend it by linearity. As usual, the difficult part is to prove that $L_{G/K} \circ L_{G/H}$ coincides with $L_{G/K \cdot G/H}$. Recall Proposition 1.1.14 for the product of basic elements in B(G)

$$G/H \cdot G/K = \sum_{K\gamma H \in |K \backslash G/H|} G/(H \cap K^\gamma).$$

We need the following result.

Lemma 4.5.3. *Let $H, K \leq_f G$ and X be a G-CW-complex. Then*

(a) for all $\gamma \in G$, $\mathrm{res}_H^{H\cap K^\gamma} \mathrm{res}_G^H X$ and $\mathrm{res}_G^{H\cap K^\gamma} X$ are $H \cap K^\gamma$-homeomorphic,

(b) $\mathrm{ind}_K^G(\coprod_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_G^{H\cap K^\gamma} X)$ and $\coprod_{K\gamma H \in |K\backslash G/H|} \mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X$ are G-homeomorphic.

Proof of Lemma 4.5.3: (a) This observation is obvious.

(b) Holds because induction distributes over disjoint union and for all $H \cap K^\gamma$-space Y there exists G-homeomorphism

$$\mathrm{ind}_K^G \mathrm{ind}_{H\cap K^\gamma}^K Y = G \times_K K \times_{H\cap K^\gamma} Y \cong G \times_{H\cap K^\gamma} Y = \mathrm{ind}_{H\cap K^\gamma}^G Y.$$

□

The proof will follow after we show the commutativity of the following diagram

$$\begin{tikzcd}
\mathcal{H}_n^G(X) \arrow[r, "\mathrm{res}_G^H"] \arrow[dd, "(4)"'] & \mathcal{H}_n^H(\mathrm{res}_G^H X) \arrow[r, "\mathrm{ind}_H^G"] & \mathcal{H}_n^G(\mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[d, "\mathrm{res}_G^K"'] \\
 & \mathcal{H}_n^K(\coprod \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_G^{H\cap K^\gamma} X) \arrow[d, "\mathrm{ind}_K^G"'] & \mathcal{H}_n^K(\mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[l, "\cong", "(2)"'] \arrow[d, "\mathrm{ind}_K^G"'] \\
\prod \mathcal{H}_n^G(\mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X) \arrow[r, "\cong", "(1)"'] & \mathcal{H}_n^G(\coprod \mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X) & \mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[l, "\cong", "(3)"']
\end{tikzcd}$$

where the vertical homomorphism (4) is

$$\prod \mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} : \mathcal{H}_n^G(X) \to \prod \mathcal{H}_n^G(\mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X)$$

and all indexes run over double cosets $K\gamma H \in |K\backslash G/H|$.

The isomorphism (1) is the natural isomorphism; the isomorphism (2) follows from part (a) of Lemma 4.5.3 to have the K-homeomorphism

$$\begin{aligned}
\mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X &\cong \coprod \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma} (\mathrm{res}_G^H X) \\
&\cong \coprod \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_G^{H\cap K^\gamma} X.
\end{aligned}$$

The isomorphism (3) is a consequence of double coset formula. The map

$$\mathcal{H}_n^K(\coprod \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_G^{H\cap K^\gamma} X) \xrightarrow{\mathrm{ind}_K^G} \mathcal{H}_n^G(\coprod \mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X)$$

in the middle part follows from (b) of Lemma 4.5.3. The small inner square is commutative by direct inspection. To see that the bigger square commutes, recall the isomorphism

$$\begin{aligned}
\mathrm{ind}_K^G \mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X &\cong \mathrm{ind}_K^G \left(\sum_{K\gamma H \in |K\backslash \Gamma/H|} \mathrm{ind}_{H\cap K^\gamma}^K \mathrm{res}_H^{H\cap K^\gamma} \right) \mathrm{res}_G^H X \\
&\cong \sum_{K\gamma H \in |K\backslash \Gamma/H|} \mathrm{ind}_{H\cap K^\gamma}^G \mathrm{res}_G^{H\cap K^\gamma} X
\end{aligned}$$

of G-CW-complexes. The diagram extends to the right as follows

$$\begin{tikzcd}
\mathcal{H}_n^G(\mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[r, "T(X)"] \arrow[d, "\mathrm{res}_G^K"'] & \mathcal{H}_n^G(X) \arrow[d, "\mathrm{res}_G^K"] & \\
\mathcal{H}_n^G(\mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[r, "\mathrm{res}_G^K \circ T(X)"] \arrow[d, "\mathrm{ind}_K^G"'] & \mathcal{H}_n^G(\mathrm{res}_G^K X) \arrow[d, "\mathrm{ind}_K^G"] & \\
\mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{res}_G^K \mathrm{ind}_H^G \mathrm{res}_G^H X) \arrow[r, "F"] & \mathcal{H}_n^G(\mathrm{ind}_K^G \mathrm{res}_G^K X) \arrow[r, "T(X)"] & \mathcal{H}_n^G(X)
\end{tikzcd}$$

where $F = \mathrm{ind}_G^K \circ \mathrm{res}_G^K \circ T(X)$. The comoposition of the homomorphism (4), (1), (3), F and $T(X)$ of the lower part of the diagram corresponds to $L_{G/K\cdot G/H}$, by commutativity is the same as $L_{G/K} \circ L_{G/H}$ given by the upper part of the diagram. □

4.6 Extension to $\mathbf{B}_M(\_)$

In this section, we give, for profinte groups Γ, a criterion to extend modules over the Burnside ring $\mathrm{B}(\Gamma)$ to modules over the completed Burnside ring $\mathbf{B}_1(\Gamma)$. The main result, Theorem 4.6.4, applies to $NK_n(R)$.

We also show that for commutative rings R if we take $M = R^\times$, the units of R, and $\Gamma = \hat{C}$, the profinite completion of C, there exists a $\mathbf{B}_{R^\times}(C)$-module structure on $NK_n(R)$ that is compatible with the $W(R)$-module structure on Nil–groups defined in [45].

Consider $(I, \leq)$ a *directed set*, i.e., a partially ordered set such that for every $i, j \in I$ exists $k \in I$ with $i \leq k$ and $j \leq k$. Assume further that for every $j \in I$ the set $\{i \in I | i \leq j\}$ is finite.

Let $\{G_i\}_{i\in I}$ be a surjective inverse system of finite groups indexed by I and for $i \leq j$ let $\varphi_{ij} \colon G_j \to G_i$ denotes the corresponding surjective group homomorphism.

Fix $i, j \in I$ with $i \leq j$ and denote by $K_{ij} := \mathrm{Ker}(\varphi_{ij})$ the kernel of the homomorphism $\varphi_{ij} \colon G_j \to G_i$. We have an isomorphism $G_i \cong G_j/K_{ij}$, a commutative diagram

$$\begin{tikzcd} G_j \arrow[r, "\varphi_{ij}"] \arrow[dr, "\tilde{\varphi}_{ij}"'] & G_i \\ & G_j/K_{ij} \arrow[u, "\cong"'] \end{tikzcd}$$

where $\tilde{\varphi}_{ij}$ is the natural homomorphism and a bijection on subgroups

$$\begin{aligned} \{H \leq G_j \mid K_{ij} \subseteq H\} &\longleftrightarrow \{L \mid L \leq G_i\}. \\ H &\mapsto \varphi_{ij}(H) \end{aligned}$$

For $i \leq j$ we have the projection homomorphism

$$\begin{aligned} \mathbf{B}\mathrm{proj}_i^j \colon \mathbf{B}_M(G_j) &\longrightarrow \mathbf{B}_M(G_i) \\ S &\mapsto S^{K_{ij}} \end{aligned} \tag{4.6.1}$$

of completed Burnside rings.

Lemma 4.6.1. *For all $i, j \in I$ such that $i \leq j$, the projection homomorphism $\mathbf{B}\mathrm{proj}_i^j$ is surjective. Moreover, it is given on generators by*

$$\mathbf{B}\mathrm{proj}_i^j(G_j/H) = \begin{cases} G_i/\varphi_{ij}(H), & K_{ij} \subseteq H \\ 0, & \text{else.} \end{cases}$$

Proof. It suffices to proof the second statement. Since $K_{ij} \trianglelefteq G_j$ we have

$$\mathrm{proj}_i^j(G_j/H) = \begin{cases} G_j/H, & K_{ij} \subseteq H \\ \emptyset, & \text{else.} \end{cases}$$

If $K_{ij} \subseteq H$, then the G_j-action on G_j/H descends to a well defined G_i-action and φ_{ij} induces an isomorphism of G_i-sets $G_j/H \xrightarrow{\cong} G_i/\varphi_{ij}(H)$. □

The inverse system $\{G_i\}_{i\in I}$ defines the system $\{\mathbf{B}_M(G_i), \mathbf{B}\mathrm{proj}_i^j\}$ and we denote its limit by $\varprojlim_i \mathbf{B}_M(G_i)$.

Recall that there is also the restriction homomorphism of Burnside rings

$$\begin{aligned} \mathbf{B}\mathrm{res}_{\varphi_{ij}} : \mathbf{B}_M(G_i) &\to \mathbf{B}_M(G_j) \\ S &\mapsto \mathbf{B}\mathrm{res}_{\varphi_{ij}}(S). \end{aligned} \tag{4.6.2}$$

We describe $\mathbf{B}\mathrm{res}_{\varphi_{ij}}$ on generators using the following Lemma.

Lemma 4.6.2. *For all $i, j \in I$ such that $i \leq j$, there exists a G_j-isomorphism $\mathrm{res}_{\varphi_{ij}}(G_i/L) \cong G_j/\varphi_{ij}^{-1}(L)$.*

Proof. The G_j-set $\mathrm{res}_{\varphi_{ij}}(G_i/L)$ is transitive because φ_{ij} is surjective. The stabilizer of $e.L$ is $\{g \in G_j \mid \varphi_{ij}(g)L = L\} = \varphi_{ij}^{-1}(L)$. □

Example 4.6.3. Consider Γ a profinite group. The collection $\{\Gamma/N \mid N \trianglelefteq_o \Gamma\}$ of finite groups is partially ordered by inclusion of the subgroups N. For Γ/N the set $\{\Gamma/L \subset \Gamma/N\}$ is finite, since the index $[\Gamma : N]$ is finite.

For $L \leq N$ there exists a canonical surjective homomorphism $\Gamma/L \to \Gamma/N$ defining an inverse system of groups. The corresponding system of Burnside rings $\{\mathbf{B}_M(\Gamma/N), \mathbf{B}\mathrm{proj}_N^L\}$ is exactly that of Theorem 3.1.13, hence its inverse limit is $\mathbf{B}_M(\Gamma)$ the completed Burnside ring of Γ.

Theorem 4.6.4. *Let $(I, \leq)$ be a direct set satisfying that for every $j \in I$ the set $\{i \in I | i \leq j\}$ is finite, $\{G_i, \varphi_{ij}\}$ be a surjective inverse system of finite groups indexed by I and $\mathcal{H}$ be an abelian group. Assume that $\mathcal{H}$ is a $\mathbf{B}_M(G_i)$-module for all $i \in I$ and satisfies*

(a) for each $i \leq j$, the $\mathbf{B}_M(G_i)$-module structure of $\mathcal{H}$ is $\mathbf{B}\mathrm{res}_{\varphi_{ij}}$-compatible with the $\mathbf{B}_M(G_j)$-module structure of $\mathcal{H}$, i.e., the diagram

$$\begin{tikzcd} \mathbf{B}_M(G_j) \times \mathcal{H} \arrow[r] & \mathcal{H} \\ \mathbf{B}_M(G_i) \times \mathcal{H} \arrow[r] \arrow[u, "\mathbf{B}\mathrm{res}_{\varphi_{ij}} \times Id"] & \mathcal{H} \arrow[u, "Id"'] \end{tikzcd}$$

commutes;

(b) for every $x \in \mathcal{H}$ there exists $i = i(x) \in I$, such that for all j with $i \leq j$, then $\mathrm{Ker}(\mathbf{B}\mathrm{proj}_i^j) \subseteq \mathrm{Ann}_{\mathbf{B}_M(G_j)}(x) = \{T \in \mathbf{B}_M(G_j) \mid T.x = 0\}$.

Then $\mathcal{H}$ has a module structure over the ring $\varprojlim_i \mathbf{B}_M(G_i)$.

Proof. Fix $\mathbf{S} = (S_i) \in \varprojlim_i \mathbf{B}_M(G_i)$ and $x \in \mathcal{H}$ and pick $i(x)$ as in (b).

Define the multiplication of $\mathbf{S}$ on x by

$$\mathbf{S}.x = S_{i(x)}.x \tag{4.6.3}$$

Notice that $S_{i(x)} \in \mathbf{B}_M(G_{i(x)})$ and the multiplication uses the $\mathbf{B}_M(G_{i(x)})$-module structure of $\mathcal{H}$.

For $i(x) \leq k$, we have $\mathbf{B}\mathrm{proj}_{i(x)}^k(\mathbf{B}\mathrm{res}_{\varphi_{i(x)k}}(S_{i(x)}) = S_{i(x)}$, by Lemmas 4.6.1 and 4.6.2, thus $S_k = \mathbf{B}\mathrm{res}_{\varphi_{i(x)k}}(S_{i(x)}) + S'$ by the structure maps and $S' \in \mathrm{Ker}(\mathbf{B}\mathrm{proj}_{i(x)}^k)$. Thus, $S'.x = 0$ by condition (b). This shows that $S_{i(x)}.x = S_k.x$ for $i(x) \leq k$, hence the action is well defined since for a different choice $i'(x)$ there exists a k such that $i(x) \leq k$ and $i'(x) \leq k$.

The identity element $\mathbf{1} = (G_i/G_i)_i \in \varprojlim_i \mathbf{B}_M(G_i)$ acts as the identity element of $\mathbf{B}_M(G_{i(x)})$ by definition.

Let $x, y \in \mathcal{H}$. Then there exists $k \in I$ such that $i(x) \leq k$ and $i(y) \leq k$, moreover, $S_{i(x)}.x = S_k.x$ and $S_{i(y)}.y = S_k.y$. Then we have $\mathbf{S}.(x+y) = S_k.(x+y) = S_k.x + S_k.y = \mathbf{S}.x + \mathbf{S}.y$.

Finally, the existence, for $x \in \mathcal{H}$, of the ring homomorphism $\mathbf{B}\mathrm{proj}_{i(x)}$ from the inverse limit to $\mathbf{B}_M(G_{i(x)})$ implies that $(\mathbf{S}+\mathbf{S}').x = \mathbf{S}.x + \mathbf{S}'.x$ and $\mathbf{S}.(\mathbf{S}'.x) = (\mathbf{S}\mathbf{S}').x$. This concludes the proof.

□

4.6.1 Applications to K-theory

Let $\Gamma = \hat{C}$ be the profinite completion of the infinite cyclic group C and consider $M = 1$ the trivial monoid. Example 4.6.3 above gives a surjective inverse system of finite quotient groups of $\hat{C}$. We use Theorem A.3.3 to describe this system as $\{C/C^n \mid n \in \mathbb{N}\}$ where C^n is the subgroup of C of index $n \in \mathbb{N}$.

We describe now the projection and restriction homomorphisms for $m, n \in \mathbb{N}$ with m dividing n. Recall that $\mathbf{B}_1(C_n)$ is the free $\mathbb{Z}$-module with basis $\{C_n/C_q \mid q \text{ divides } n\}$.

The projection homomorphism is given by

$$\begin{aligned} \mathbf{B}\mathrm{proj}^n_m \colon \mathbf{B}_1(C_n) &\longrightarrow \mathbf{B}_1(C_m) \\ C_n/C_q &\mapsto \begin{cases} C_m/C_{mq/n}, & n/q \text{ divides } m \\ 0, & \text{else.} \end{cases} \end{aligned} \tag{4.6.4}$$

The restriction homomorphism is given by

$$\begin{aligned} \mathbf{B}\mathrm{res}_{\varphi_{mn}} \colon \mathbf{B}_1(C_m) &\longrightarrow \mathbf{B}_1(C_n) \\ C_m/C_q &\mapsto C_m/C_q = C_n/C_{qn/m}. \end{aligned} \tag{4.6.5}$$

Notice that for all $n \in \mathbb{N}$ the rings $\mathbf{B}_1(C_n)$ act on the K-theory groups of $R[C]$, by restiction and induction, i.e.,

$$\begin{aligned} \mathbf{B}_1(C_n) \times K_m(R[C]) &\longrightarrow K_m(R[C]) \\ (\sum a_q \cdot C_n/C_q, x) &\mapsto \sum a_q \mathrm{ind}_{\sigma_{n/q}} \circ \mathrm{res}_{\sigma_{n/q}}(x) \end{aligned}$$

where $\sigma_{n/q} \colon C^{n/q} \to C$ is the canonical inclusion. The homomorphism 4.6.5 shows that, for $n \in \mathbb{N}$, the actions of $\mathbf{B}_1(C_n)$ are $\mathbf{B}\mathrm{res}_{\varphi_{mn}}$ compatible in the sense of Theorem 4.6.4.

Finally, recall that in Theorem 2.3.7 we used the Bass–Heller–Swan decomposition Theorem to prove the compatibility of the actions defined in terms of $\mathrm{res}_{\varphi_{mn}}$ and $\mathrm{ind}_{\varphi_{mn}}$ with the action defined in terms of F_{mn} and V_{mn} on Nil–groups. The next result is well known in the literature.

Lemma 4.6.5. *[20] For every $n \in \mathbb{Z}$ and each $x \in NK_n(R)$, there exists a positive integer $i(x)$ such that $F_m(x) = 0$ for $m \geq i(x)$*

Proof. The Frobenius homomorphism $F_m \colon NK_n(R) \longrightarrow NK_n(R)$ is induced by the functor sending $(Q, \nu) \in \mathrm{NIL}(R)$ to (Q, ν^m), for (Q, ν) there exists a positive integer $i(\nu)$ with $(Q, \nu^m) = (Q, 0)$ for $m \geq i(\nu)$. The claim follows from compatibility of the actions. □

Lemma 4.6.5 gives condition (b) of Theorem 4.6.4. Hence we conclude

Theorem 4.6.6. *There exists a $\mathbf{B}_1(C)$-module structure on $NK_n(R)$ for all rings R. Moreover, this structure is compatible with the usual $W(\mathbb{Z})$-module structure of $NK_n(R)$.*

4.6.2 The $\mathbf{B}_{R^\times}(C)$-module structure of Nil–groups

Let R be a commutative ring with unit and $M = R^\times$ the units of R. Using Theorem 4.6.4 for $M = R^\times$ we offer another description of the $W(R)$-module structure of Nil–groups described in [45].

First, recall that for $R^\times \hookrightarrow R_\times$, monoid map from the units of R to the underlying multiplicative monoid of R, there exists a commutative diagram (3.2.8)

$$\begin{tikzcd}
\mathbf{W}_\Gamma(\mathbb{Z}[R^\times]) \arrow[r] \arrow[d] & \mathbf{B}_{R^\times}(\Gamma) \arrow[d] \\
\mathbf{W}_\Gamma(\mathbb{Z}[R_\times]) \arrow[r, "\cong"] \arrow[dr, "\mathbf{W}_\Gamma(\sigma)"'] & \mathbf{B}_{R_\times}(\Gamma) \arrow[d, "\pi"] \\
 & \mathbf{W}_\Gamma(R)
\end{tikzcd}$$

where the isomorphism is given by Theorem 3.2.3, the ring homomorphism $\mathbf{W}_\Gamma(\sigma)$ is induced by $\sigma\colon \mathbb{Z}[R_\times] \to R$ the ring homomorphism that extends the identity $R_\times \to R$, and π is a surjective ring homomorphism with set–theoretical section (see [14]):

$$\begin{aligned} \mathbf{W}_\Gamma(R) &\longrightarrow \mathbf{B}_{R_\times}(\Gamma) \\ (a_H) &\mapsto \sum_{(H)} [\Gamma/H, a_H] \end{aligned} \tag{4.6.6}$$

where (H) is the conjugacy class of $H \leq_o \Gamma$ and $[\Gamma/H, a_H]$ is the isomorphism class of the Γ-string Γ/H with constant function $\|\_\| = a_H$. We restrict to $\Gamma = \hat{C}$ the profinite completion of C to state our result.

Fix $r \in R_\times$. The ring homomorphism

$$\begin{aligned} [r]\colon R[t] &\to R[t] \\ t &\mapsto rt \end{aligned}$$

induces a map in K-theory groups $K_n(R[t]) \to K_n(R[t])$. The effect of $[r]$ is that the $R[t]$-module structure is now multiplication by rt.

We have a functor

$$\begin{aligned} [r]\colon \mathrm{END}(R) &\to \mathrm{END}(R) \\ (P, f) &\mapsto (P, rf) \end{aligned}$$

where rf denotes the endomorphism f followed by multiplication by r. The functor

$$\begin{aligned} \mathrm{END}(R) &\to \mathcal{P}(R[t]) \\ (P, f) &\to P_f, \end{aligned}$$

where P_f denotes the $R[t]$-module P with t acting via f, gives, up to isomorphism, a commutative diagram of categories

$$\begin{tikzcd} \mathrm{END}(R) \arrow[r] \arrow[d, "{[r]}"'] & \mathcal{P}(R[t]) \arrow[d, "{[r]}"] \\ \mathrm{END}(R) \arrow[r] & \mathcal{P}(R[t]). \end{tikzcd}$$

The functor $[r]$ restricts to the category $\mathrm{NIL}(R)$, moreover, it is compatible with the splitting of $\mathcal{P}(R)$ from $\mathrm{NIL}(R)$. This defines the homomorphism

$$[r]\colon \mathrm{Nil}_n(R) \longrightarrow \mathrm{Nil}_n(R)$$

that corresponds, via $NK_n(R) \cong \mathrm{Nil}_{n-1}(R)$, with the homothety operator.

Now, if $r \in R^\times$ the same arguments holds for the ring homomorphism

$$\begin{aligned} [r]\colon R[t,t^{-1}] &\to R[t,t^{-1}]. \\ t^{\pm 1} &\mapsto r^{\pm 1}t^{\pm 1} \end{aligned}$$

.

Theorem 4.6.7. *Let R be a commutative ring with unit. For all $n \in \mathbb{Z}$ there exists a $\mathbf{B}_{R^\times}(C)$-module structure on $NK_n(R)$. Moreover, this structure is compatible with the $W(R)$-module structure of $NK_n(R)$.*

Proof. Let $[C/C^m, r]$ be the class of C/C^m, i.e., the C-string over $R^\times$ with constant function $r \in R^\times$ and underlying almost finite C-set C/C^m. Recall the m-th Frobenius, F_m, and the m-th Verschiebung, V_m, operators. The homothety operator satisfies $[r][s] = [rs]$, $[r]V_m = V_m[r^m]$ and $F_m[r] = [r^m]F_m$ for all $r, s \in R^\times$.

Define

$$\begin{aligned} \mathbf{B}_{R^\times}(C) \times \mathrm{Nil}_{n-1}(R) &\longrightarrow \mathrm{Nil}_{n-1}(R), \\ ([C/C^m, r], x) &\mapsto L^r_{C/C^m}(x) := V_m[r]F_m(x) \end{aligned}$$

and extend by linearity. We only need to show they are compatible with the product in $\mathbf{B}_{R^\times}(C)$. For $m, n \in \mathbb{N}$, $r, s \in R$ a direct computation on C-strings yields

$$[C/C^m, r][C/C^n, s] = \coprod_d [C/C^{mn/d}, r^{n/d}s^{m/d}],$$

where $d = \mathrm{g.c.d}(m,n)$ is the greatest common divisor of m and n. For $x \in \mathrm{Nil}_{n-1}(R)$ we have

$$\begin{aligned} L^{r^{n/d}s^{m/d}}_{C/C^m.C/C^n}(x) &= dL^{r^{n/d}s^{m/d}}_{C/C^{mn/d}}(x) \\ &= dV_{mn/d}[r^{n/d}s^{m/d}]F_{mn/d}(x). \end{aligned}$$

On the other hand we have

$$\begin{aligned} L^r_{C/C^m}L^s_{C/C^n}(x) &= V_m[r]F_mV_n[s]F_n(x) \\ &= V_m[r]dV_{n/d}F_{m/d}[s]F_n(x) \\ &= dV_{mn/d}[r^{n/d}s^{m/d}]F_{mn/d}(x), \end{aligned}$$

since Proposition B.3.4 states that $F_mV_n = dV_{n/d}F_{m/d}$, $V_mV_n = V_{mn}$ and $F_mV_n = F_{mn}$. This shows that

$$L^{r^{n/d}s^{m/d}}_{C/C^m.C/C^n} = L^r_{C/C^m}L^s_{C/C^n}$$

and concludes the proof. □

Appendix A

Profinite Groups

A.1 Basic definitions

A profinite group is an inverse limit of finite groups. Various collections of finite groups give various profinite groups. We make this more precise.

Definition A.1.1 (Class of finite groups)**.** A class of finite groups $\mathfrak{C}$ is a non-empty collection of finite groups that contains all isomorphic images of groups in $\mathfrak{C}$, i.e., if $G \in \mathfrak{C}$ and $H \cong G$, then $H \in \mathfrak{C}$.

Example A.1.2. The following classes of groups are important to our research. They are the easiest classes to handle.

(a) The class $\mathfrak{C}$ of *all finite groups.*

(b) Let p be a prime number. The class $\mathfrak{C}_p$ of *all finite p-groups.*

Definition A.1.3 (Pro-$\mathfrak{C}$ group)**.** Let $\mathfrak{C}$ be a class of finite groups. *A pro-$\mathfrak{C}$ group G* is an inverse limit

$$G = \varprojlim_i G_i \tag{A.1.1}$$

of an inverse system $\{G_i, \varphi_{ij} \colon G_j \to G_i\}$ with surjective structure maps of groups $G_i \in \mathfrak{C}$. Here each G_i has the discrete topology.

The surjectivity hypothesis in Definition A.1.3 is necessary to work in vast generality, nevertheless, we can weaken this hypothesis if we restrict to subgroup closed classes of finite groups.

Definition A.1.4 (Subgroup closed class)**.** A class of finite groups $\mathfrak{C}$ is *subgroup closed* if for any $G \in \mathfrak{C}$ and $H \subseteq G$ then $H \in \mathfrak{C}$.

Any inverse limit taken over a subgroup closed class $\mathfrak{C}$ is a pro-$\mathfrak{C}$ group [36, p. 19]. We consider only closed subgroups classes without explicit mention.

Example A.1.5. The class of all finite groups $\mathfrak{C}$ and the class of all p-groups $\mathfrak{C}_p$ are subgroup closed.

Notation. If $\mathfrak{C}$ is the class of all finite groups we call pro-$\mathfrak{C}$ groups *profinte groups* and we call a pro-$\mathfrak{C}_p$ groups *pro-p groups.*

Topology of profinite groups

The properties of a class $\mathfrak{C}$ determine the topology of of a pro-$\mathfrak{C}$ group G. We formulate properties of a class that ensure that the topology is rich enough for our purposes.

Definition A.1.6. Let $\mathfrak{C}$ be a class of finite groups.

(a) $\mathfrak{C}$ is *closed under quotients* if for any $G \in \mathfrak{C}$ and $H \trianglelefteq G$ then $G/H \in \mathfrak{C}$.

(b) $\mathfrak{C}$ is *closed under subdirect product* if for any finite group G and normal subgroups $N_1, N_2 \trianglelefteq G$ such that $G/N_1, G/N_2 \in \mathfrak{C}$ then $G/N_1 \cap N_2 \in \mathfrak{C}$.

A class $\mathfrak{C}$ that is closed under quotients and subdirect products is a *formation*.

Example A.1.7. The classes $\mathfrak{C}$ and $\mathfrak{C}_p$ are formations.

A pro-$\mathfrak{C}$ group is an inverse limit of compact Hausdorff totally disconnected spaces G_i in some formation $\mathfrak{C}$, this makes G itself into a compact Hausdorff totally disconnected space [47, Proposition 1.1.5]. The following results characterize the open subsets of G.

Lemma A.1.8. *[36, Lemma 2.1.2] In a compact topological group G, a subgroup U is open if and only if U is closed of finite index.*

Theorem A.1.9. *[36, Theorem 2.1.3] Let $\mathfrak{C}$ be a formation of finite groups. The following conditions on a topological group G are equivalent.*

(a) G is a pro-$\mathfrak{C}$ group,

(b) G is compact Hausdorff totally disconnected and for each open normal subgroup $U \trianglelefteq G$ the quotient G/U is in $\mathfrak{C}$.

Notation. Let G be a topological group and $H \leq G$ a subgroup.

(a) We write $H \leq_o G$ if H is an open subgroup of G, respectively, $H \leq_c G$ if H is a closed subgroup of G;

(b) we write $H \trianglelefteq_o G$ if H is an open normal subgroup of G, respectively, $H \trianglelefteq_c G$ if H is a closed normal subgroup of G;

(c) we write $H \leq_f G$ if H has finite index in G, respectively, and $H \trianglelefteq_f G$ if H is a normal subgroup of finite index in G.

We introduce the concept of pro-$\mathfrak{C}$ completion of a group. Let $\mathfrak{C}$ be a formation, G be a group and consider the collection

$$\mathfrak{N} = \{N \trianglelefteq_f G \mid G/N \in \mathfrak{C}\}. \tag{A.1.2}$$

This collection is not empty since $G \in \mathfrak{N}$ and it defines an inverse system $\{G/N\}$ where the structure maps $\varphi_{MN} : G/N \to G/M$ are given by the canonical projection map whenever $N \leq M$. The *pro-$\mathfrak{C}$ completion of G* is the inverse limit

$$G_{\hat{\mathfrak{C}}} = \varprojlim_{N \in \mathfrak{N}} G/N.$$

Example A.1.10. (a) Let $\mathfrak{C}$ the formation of all finite groups and $G = \mathbb{Z}$ be the infinite cyclic group. The collection $\mathfrak{N} = \{n\mathbb{Z} \mid n \in \mathbb{N}\}$ defines the profinite completion

$$\hat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$$

i.e., the profinite completion of the infinite cyclic group is the inverse limit of all finite cyclic groups.

(b) Let $\mathfrak{C}_p$ the formation of all finite p-groups and $G = \mathbb{Z}$ be the infinite cyclic group. The collection $\mathfrak{N}_p = \{p^n\mathbb{Z} \mid n \in \mathbb{N}_0\}$ defines the pro-p completion

$$\mathbb{Z}_p = \varprojlim_{n \in \mathbb{N}_0} \mathbb{Z}/p^n\mathbb{Z}.$$

The notation we use here is the standard notation of Number Theory to denote the p-adic integers.

A.2 The pro-$\mathfrak{C}$ completion

Definition A.2.1 (Bounded from below)**.** A non-empty collection $\mathfrak{N}$ of normal subgroups of finite index of a group G is *bounded from below* if for any $N_1, N_2 \in \mathfrak{N}$ there exists $N \in \mathfrak{N}$ such that $N \leq N_1 \cap N_2$.

Example A.2.2. Let G be a group and $\mathfrak{C}$ be a formation. The family

$$\mathfrak{N}_{\mathfrak{C}}(G) = \{N \trianglelefteq_f G \mid G/N \in \mathfrak{C}\}$$

that we used to define the profinite completion of G is bounded from below, because formations are closed under subdirect products, compare with Definition A.1.3, (b).

A collection $\mathfrak{N}$ in G bounded from below defines a topology on G, namely, by making $\mathfrak{N}$ into a fundamental system of neighborhoods of the identity element $1 \in G$.

Remark A.2.3. We call the topology of G associated to $\mathfrak{N}_{\mathfrak{C}}(G)$, as in Example A.2.2, the *pro-$\mathfrak{C}$ topology of G* or the *full pro-$\mathfrak{C}$ topology of G*.

Example A.2.4. The collections $\mathfrak{N}_{\mathfrak{C}}(\mathbb{Z})$ and $\mathfrak{N}_{\mathfrak{C}_p}(\mathbb{Z})$ of Example A.1.10 define the *full profinite topology of* $\mathbb{Z}$ and the *full pro-p topology of* $\mathbb{Z}$ respectively. Both topologies are Hausdorff as the following result states.

Proposition A.2.5. *The pro-$\mathfrak{C}$ completion of G is Hausdorff if and only if*

$$\bigcap_{N \in \mathfrak{N}_{\mathfrak{C}}(G)} N = 1$$

Remark A.2.6. The Example A.1.10 describes $\hat{\mathbb{Z}}$ and $\mathbb{Z}_p$ as an inverse limit of finite groups, by definition, both are profinite groups. The Example A.2.4 endows $\mathbb{Z}$ with a profinite topology and a pro-p topology.

Definition A.2.7 (Completion)**.** Let G be a group and $\mathfrak{C}$ be a formation of finite groups. The *completion of G with respect to its pro-$\mathfrak{C}$ topology* is

$$G_{\hat{\mathfrak{C}}} = \varprojlim_{N \in \mathfrak{N}_{\mathfrak{C}}(G)} G/N.$$

Notice immediately that the completion $G_{\hat{\mathfrak{C}}}$ is a pro-$\mathfrak{C}$ group. It is almost immediate that there exists a continuous homomorphism

$$\begin{aligned} i\colon G &\longrightarrow G_{\hat{\mathfrak{C}}} \\ g &\mapsto (gN)_{N\in\mathfrak{N}_{\mathfrak{C}}(G)} \end{aligned} \tag{A.2.1}$$

that is injective if and only if G is Hausdorff, with respect to the $\mathfrak{C}$-topology defined in Remark A.2.3.

The group G is dense in $G_{\hat{\mathfrak{C}}}$, this allows us to establish a relation between subgroups. The following lemma characterizes such relation.

Lemma A.2.8. *[36, Proposition 3.2.2] Let G be Hausdorff in its pro-$\mathfrak{C}$ topology. Identify G with its dense image $i(G)$ in $G_{\hat{\mathfrak{C}}}$. For any $X \subseteq G$ let $\overline{X}$ be its closure in $G_{\hat{\mathfrak{C}}}$.*

(a) There exists a bijection

$$\begin{aligned} \Phi : \{N \mid N \leq_o G\} &\longrightarrow \{U \mid U \leq_o G_{\hat{\mathfrak{C}}}\} \\ H &\mapsto \overline{H} \end{aligned}$$

with inverse

$$\begin{aligned} \Phi^{-1} : \{U \mid U \leq_o G_{\hat{\mathfrak{C}}}\} &\longrightarrow \{N \mid N \leq_o G\} \\ V &\mapsto V \cap G. \end{aligned}$$

(b) Φ sends normal subgroups to normal subgroups.

(c) If $H, K \leq_o G$ with $H \leq K$ then $[K : H] = [\overline{K} : \overline{H}]$. Moreover, if $H \trianglelefteq K$ then $K/H \cong \overline{K}/\overline{H}$.

A.3 The orbit category of a profinite group

The main result of this appendix is Theorem A.3.3. It simplifies the exposition in Chapter 3 of the Witt-Burnside ring.

Definition A.3.1 (Orbit category)**.** The *orbit category* $\mathcal{O}r(G)$ of G consists of homogeneous transitive G-spaces G/H with G-maps as morphisms. The *finite orbit subcategory* $\mathcal{O}r^f(G)$ is the subcategory of finite homogeneous transitive G-spaces G/H of $\mathcal{O}r(G)$, i.e., $[G : H] < \infty$.

Remark A.3.2. Recall that if T is a transitive G-set, then only after choosing a point $t \in T$ we can identify T with G/H, with $H = G_t$ the stabilizer of t, so $T \cong G/H$ is not canonical. See Lemma 1.1.8.

Let $\mathfrak{C}$ be a formation and assume that G is Hausdorff in its full pro-$\mathfrak{C}$ topology. Our interest focus in the subcategory $\mathcal{O}r^f(G_{\hat{\mathfrak{C}}})$ and we describe it in terms of $\mathcal{O}r^f(G)$.

Theorem A.3.3. *Let G be Hausdorff in its pro-$\mathfrak{C}$ topology and $G_{\hat{\mathfrak{C}}}$ be its pro-$\mathfrak{C}$ completion. Then there exists an equivalence of categories*

$$\mathcal{O}r^f(G) \longrightarrow \mathcal{O}r^f(G_{\hat{\mathfrak{C}}})$$

Proof. First, we give a the functors on objects.

Let G/H be in $\mathcal{O}r^f(G)$, then $H \leq_c G$ with finite index, hence, $H \leq_o G$ [36, Proposition 2.3.2]. Thus, by Lemma A.2.8, there exists a unique open subgroup $\overline{H} \leq_o G_{\hat{\mathfrak{C}}}$ associated with H. Since $G_{\hat{\mathfrak{C}}}$ is compact, $\overline{H}$ is closed of finite index by Lemma A.1.8. This shows that $G_{\hat{\mathfrak{C}}}/\overline{H}$ is an object in $\mathcal{O}r^f(G_{\hat{\mathfrak{C}}})$.

The inverse is given as follows. Let $G_{\hat{\mathfrak{C}}}/V$ be an object in $\mathcal{O}r^f(G_{\hat{\mathfrak{C}}})$, since $V \leq_o G_{\hat{\mathfrak{C}}}$, then $G \cap V$ is open in G, thus $G \cap V$ is closed in G. Since $[G_{\hat{\mathfrak{C}}} : V] = [G : V \cap G]$ is finite, then $G/G \cap V$ is an object in $\mathcal{O}r^f(G)$.

We use Lemma 1.1.8 to define these functors on homomorphisms. That is, $\{gK \in G/K \mid H \subseteq {}^gK\}$ the morphisms in G from G/H to G/K, correspond to the set $\{g\overline{K} \in G_{\hat{\mathfrak{C}}}/\overline{K} \mid \overline{H} \subseteq {}^g\overline{K}\}$ of morphisms in $G_{\hat{\mathfrak{C}}}$ from $G_{\hat{\mathfrak{C}}}/\overline{H}$ to $G_{\hat{\mathfrak{C}}}/\overline{K}$. $\square$

Example A.3.4. The finite orbit category of the pro-$\mathfrak{C}$ completion of $\mathbb{Z}$ is uniquely determined by $\mathcal{O}r^f(\mathbb{Z})$. Hence, it is characterized by $\mathcal{O}r^f(\hat{\mathbb{Z}}) = \{\mathbb{Z}/n\mathbb{Z} \mid n \in \mathbb{N}\}$. A morphism $\mathbb{Z}/n\mathbb{Z} \longrightarrow \mathbb{Z}/m\mathbb{Z}$ exists if and only in m divides n.

Appendix B

Witt vector ring

The concept of Witt vector ring was introduced by Ernst Witt [48] and later generalized by Jean Pierre Cartier [8]. An extensive work about the Witt vector rings is that of M. Hazewinkel [23].

For those who have been in touch with the theory of Witt vectors it is not surprising the complexity of performing explicit computations with formulas, they become involved even for an explicit description of the product. We privilege applications over strictly formal definitions.

We motivate Witt vectors from linear algebra and generalized this to projective modules over a ring R. Proceeding in this way we link $W(R)$, the Witt vectors of R, with the algebraic K-theory of the category of endomorphisms over the ring R. This is a convenient point of view to describe the action of the Witt ring on Bass Nil–groups.

We conclude this appendix with a review of the Frobenius and Verschiebung endomorphisms of $W(R)$ using the ideas of Almkvist [1–3] and Grayson [18] to give matrix expressions for Frobenius and Verschiebung endomorphisms. Recall that Frobenius is a ring endomorphisms while Verschiebung is only an additive endomorphism.

B.1 Motivation

Let k be a field of characteristic zero, V a finite dimensional vector space over k and $V \xrightarrow{L} V$ a linear transformation. The vector space V is characterized, up to isomorphism, by its dimension over k, $\dim_k V$.

The linear transformation L has trace $\mathrm{Tr}(L)$ and *reverse characteristic polynomial* $\chi_t(L) = \det(1 - tL)$ in variable t. Of course, $\chi_t(L)$ as well as the trace are not complete invariants. There is no deeper reasons to chose one or the other than merely convenience as they both are related by the exponential trace formula.

Proposition B.1.1 (Exponential–Trace formula)**.** *Let V be a finite dimensional vector space over k and $L\colon V \to V$ be a linear transformation. Then we have*

$$-t\frac{d}{dt}\log(\chi_t(L)) = \sum_{i=1}^{\infty} Tr(L^k)t^k.$$

in the ring of formal power series $k[\![t]\!]$.

Proof. Take $V = k^n$ and assume that k is an algebraically closed field. Then $\chi_t(L) = \prod_{i=1}^n (1 - v_i t)$ where v_i are the eigenvalues of L counted with multiplicity. Then we have

$$\begin{aligned}
-t\frac{d}{dt}\log(\chi_t(L)) &= \sum_{i=1}^{n} \frac{v_i t}{1 - v_i t} \\
&= \sum_{i=1}^{n} (v_i t)(1 + v_1 t + v_2^2 t^2 + \cdots) \\
&= \sum_{i=1}^{n} \sum_{j=1}^{\infty} v_i^j t^j \\
&= \sum_{j=1}^{\infty} (\sum_{i=1}^{n} v_i^j) t^j \\
&= \sum_{j=1}^{\infty} \mathrm{Tr}(L^j) t^j.
\end{aligned}$$

For a general field k there exists a surjection $p\colon A \longrightarrow k$ where A is a subring of a closed field, lift L to $L'\colon A^n \longrightarrow A^n$ and since the formula holds for L' by applying p it holds for L. $\square$

Let $V \xrightarrow{L} V$ and $V' \xrightarrow{L'} V'$ as before. There exists direct sum $V \oplus V' \xrightarrow{L \oplus L'} V \oplus V'$ and tensor product $V \otimes_k V' \xrightarrow{L \otimes L'} V \otimes_k V'$ operations with vector spaces and $\dim_k$, Tr and χ_t satisfies:

(a) $\dim_K(V \oplus V') = \dim_K(V) + \dim_K(V')$ and $\dim_K(V \otimes V') = \dim_K(V) \cdot \dim_K(V')$;

(b) $\mathrm{Tr}(L \oplus L') = \mathrm{Tr}(L) + \mathrm{Tr}(L')$ and $\mathrm{Tr}(L \otimes L') = \mathrm{Tr}(L) \cdot \mathrm{Tr}(L')$;

(c) $\chi_t(L \oplus L) = \chi_t(L) \cdot \chi_t(L')$.

We postpone the description of $\chi_t(L \otimes L')$ until we have show the existence of Witt vectors. The previous list suggests an additive invariant behavior of trace and characteristic polynomial. Indeed, this was the departing point of Almkvist to define characteristic polynomial for the category of endomorphisms of a small subcategory of an abelian category [3].

A more general setting is the following. We change the field k by a commutative ring with unit R, the vector space V by a finitely generated projective R-module P and the linear transformation L by an R-module endomorphism $P \xrightarrow{f} P$. We try to classify, up to isomorphism, the pairs (P, f). This is our departing point to study $W(R)$.

B.2 Construction of the Witt vector ring

Recall that E. Witt introduced the concept of *p-Witt vectors* or *p-typical Witt vectors* for a prime number p, Cartier generalized this and introduced the concept of *(big) Witt vectors* or *generalized Witt vectors.* We work only with big Witt vectors. Let us state Cartier's theorem.

Theorem B.2.1. *[8] There exists a functor W from the category of commutative rings with unit to itself such that:*

(a) *for any commutative ring R, the set $W(R)$ consists of sequences $\mathfrak{a} = (a_n)_{n\geq 1}$ of elements in R and for all ring homomorphism $f\colon R \to S$, the homomorphism $W(f)$ sends $\mathfrak{a}$ to the sequence $(f(a_n))_{n\geq 1}$,*

(b) *for every commutative ring R and every integer $n \geq 1$ the map*

$$\begin{aligned}\omega_n\colon W(R) &\longrightarrow R\\ \mathfrak{a} &\mapsto \omega_n(\mathfrak{a}) := \sum_{d|n} d a_d^{n/d}\end{aligned}$$

is a ring homomorphism.

Outline of the proof: We reproduce parts of the proof in [24].

Part (a). Fix R and let $R[\![t]\!]$ be its ring of power series over t. The augmentation homomorphism $R[\![t]\!] \xrightarrow{t\mapsto 0} R$, restricts to a unit group homomorphism $R[\![t]\!]^\times \to R^\times$ and this defines the abelian group (under series multiplication)

$$\Lambda(R) = \mathrm{Ker}(R[\![t]\!]^* \to R^*) = 1 + tR[\![t]\!]. \tag{B.2.1}$$

We endow the abelian group $\Lambda(R)$ with a multiplication $*_R\colon \Lambda(R) \times \Lambda(R) \to \Lambda(R)$ such that:

- $*_R$ is right and left distributive with respect to multiplication of series,
- for all $a, b \in R$, $(1-at)^{-1} *_R (1-bt)^{-1} = (1-abt)^{-1}$, where $(1-at)^{-1}$ denotes the geometric progression $\sum_{i\geq 0} a^i t^i$ in the formal sense.
- $*$ is functorial in R, i.e., if $R \xrightarrow{\varphi} S$ is a ring homomorphism, then the diagram

$$\begin{tikzcd}\Lambda(R)\times\Lambda(R) \arrow[r, "*_R"] \arrow[d] & \Lambda(R) \arrow[d]\\ \Lambda(S)\times\Lambda(S) \arrow[r, "*_S"] & \Lambda(S)\end{tikzcd}$$

  commutes.

Once this multiplication $*_R$ is defined in $\Lambda(R)$ we use the bijection

$$\begin{aligned}\Phi\colon W(R) &\longrightarrow \Lambda(R)\\ \mathfrak{a} &\mapsto \prod_{d\geq 1}(1-a_d t^d)^{-1}\end{aligned} \tag{B.2.2}$$

to define $\mathfrak{a}+\mathfrak{b} = \Phi^{-1}(\Phi(\mathfrak{a})\cdot\Phi(\mathfrak{b}))$ and $\mathfrak{a}*\mathfrak{b} = \Phi^{-1}(\Phi(\mathfrak{a}) *_R \Phi(\mathfrak{b}))$ in $W(R)$.

We show existence of $*_R$.

Let $n \geq 0$ an integer and consider the augmentation map on the truncated polynomial ring over R, $R[t]/(t^{n+1}) \to R$. As before, this induces a multiplicative group homomorphism in the units and we define

$$\Lambda_n(R) = \ker(R[t]/(t^{n+1})^* \to R^*).$$

Notice that $\Lambda(R) = \varprojlim_n \Lambda_n(R)$. Consider $M_n(R) = \langle\{1-at \mid a \in R\}\rangle$ the subgroup of $\Lambda_n(R)$ generated by the linear polynomials $1-at$. Each $a \in R$ defines an R-algebra homomorphism:

$$\begin{aligned}R[t]/(t^{n+1}) &\longrightarrow R[t]/(t^{n+1})\\ t &\mapsto at\end{aligned}$$

and induces $\varphi_a\colon \Lambda_n(R) \to \Lambda_n(R)$ satisfying $\varphi_a\varphi_b = \varphi_{ab}$. Consider the subring $E_n = \langle\{\varphi_a \mid a \in R\}\rangle$ of the $\mathbb{Z}$-linear endomorphisms ring of $\Lambda_n(R)$, $\mathrm{End}(\Lambda_n(R))$ and the E_n-module homomorphism

$$\begin{aligned} E_n &\longrightarrow \Lambda_n(R) \\ \varphi_a &\mapsto (1-at) \end{aligned} \tag{B.2.3}$$

whose image is precisely $M_n(R)$ hence isomorphic to the ring E_n/I, where I is the kernel of the homomorphism (B.2.3) so $M_n(R)$ has a ring structure.

H. Lenstra ([24], Lemma 2) constructed for any ring R a R-algebra $\overline{R}$ such that for all $n \geq 0$, $\Lambda_n(\overline{R}) = M_n(\overline{R})$ and such that as R-module, $\overline{R}$, it has a basis that contains the unit element to show that $\Lambda_n(R) \subseteq \Lambda_n(\overline{R}) = M_n(\overline{R})$ is a subring and hence itself a ring.

Finally, Λ_n is a functor from the category of commutative rings to itself and all the natural maps $\Lambda_{n+1} \to \Lambda_n$ are morphisms of funtors showing that $\Lambda(R) = \varprojlim_n \Lambda_n(R)$ has a ring structure.

Part (b). Let $\mathbf{a} = \prod_d (1-a_d t^d)^{-1}$ and consider the logaritmic derivative

$$\begin{aligned} t\frac{d}{dt}\log\colon \Lambda(R) &\longrightarrow tR[\![t]\!] \\ \mathbf{a} &\mapsto t\frac{d}{dt}\log(\prod_d (1-a_d t^d)^{-1}). \end{aligned} \tag{B.2.4}$$

The image of $\mathbf{a}$ is:

$$\begin{aligned} t\frac{d}{dt}\log(a) &= t\sum_{d\geq 1}\frac{d}{dt}\log(1-a_d t^d)^{-1} \\ &= \sum_{d\geq 1} d a_d t^d \left(\sum_{n\geq 0}(a_d t^d)^n\right) \\ &= \sum_{d\geq 1}\sum_{n\geq 1} d(a_d^n t^{nd}) \\ &= \sum_{n\geq 1}\left(\sum_{d\mid n} d a_d^{n/d}\right) t^n \\ &= \sum_{n\geq 1}\omega_n(\mathfrak{a})t^n. \end{aligned}$$

Since $\omega_n((1-at)^{-1}) = a^n$ is multiplicative and logarithmic derivative takes multiplication into addition we have that each ω_n is a ring homomorphism.

$\square$

We close this section with a commutative diagram of rings that summarizes the maps used in the proof

$$\begin{tikzcd} W(R) \arrow[r, "\Phi"] \arrow[d, "\omega"'] & \Lambda(R) \arrow[d, "t\frac{d}{dt}\log"] \\ \prod_{n\geq 1} R \arrow[r, "\cong"] & tR[\![t]\!] \end{tikzcd} \tag{B.2.5}$$

where $\prod_{n\geq 1} R$ has pointwise sum and product. The ring isomorphism Φ is given by (B.2.2), the ghost map $\omega = \prod \omega_n$ where ω_n are the homomorphisms given by (b) Theorem B.2.1. The bottom line isomorphism sends $(a_n)_{n\geq 1}$ to $\sum_{n\geq 1} a_n t^n$. In elements, diagram (B.2.5) looks like:

$$\begin{tikzcd} \mathfrak{a} \arrow[r, mapsto] \arrow[d, mapsto] & \prod_d (1-a_d t^d)^{-1} \arrow[d, mapsto] \\ (\sum_{d|n} d a_d^{n/d})_n \arrow[r, mapsto] & \sum_{n=1}^{\infty} (\sum_{d|n} d a_d^{n/d}) t^n. \end{tikzcd}$$

B.3 Frobenius and Verschiebung

We review the Frobenius and Verschiebung endomorphisms defined on Witt vector rings. Our approach is intended to clarify the correspondence of Frobenius and Verschiebung endomorphism with matrix expressions.

Fix $n \in \mathbb{N}$ and a commutative ring R we first define n-th Frobenius and n-the Verschiebung

$$F_n \colon \Lambda(R) \longrightarrow \Lambda(R)$$

and

$$V_n \colon \Lambda(R) \longrightarrow \Lambda(R).$$

and then transport this back to $W(R)$ via the isomorphism (B.2.2).

Frobenius

In order to keep simple formulas we introduce a *change of coordinates* for $\Lambda(R)$. For further details see paragraphs 9.10 and 9.61 in [23]. The functor $\Lambda(-)$, as set–theoretical functor, is representable by the free polynomial ring $\mathbb{Z}[\mathbf{h}] := \mathbb{Z}[h_1, \ldots, h_d, \ldots]$. We can think of $\mathbb{Z}[\mathbf{h}]$ as the usual ring of symmetric functions on infinitely many variables $X_1, \ldots, X_d, \ldots$ by writting

$$h_d := h_d(\mathbf{X}) = \sum_{j_1 \leq \cdots \leq j_d} X_{j_1} \cdots X_{j_d}.$$

The $h_d(\mathbf{X})$ are known as complete symmetric functions. Using a geometric series expansion and perfoming the product it is easy to see that

$$\sum_{d\geq 0} h_d t^d = \prod_{d\geq 1} (1 - X_d t)^{-1}.$$

Now, assume that $\Phi(\mathfrak{a}) = \prod_{d\geq 1}(1-a_d t^d)^{-1} = 1 + a_1' t + a_2' t^2 + \cdots$ in $\Lambda(R)$, and that for all $d \in \mathbb{N}$

$$a_d' = h_d(\varepsilon_1, \ldots, \varepsilon_d, \ldots),$$

where ε_i are elements in a larger ring containing R. This condition is necessary because one may need to invert elements to write a_i' in terms of ε_i explicitely. In this way we could write

$$\prod_{d\geq 1} (1 - a_d t^d)^{-1} = \prod_{d\geq 1} (1 - \varepsilon_d t)^{-1}.$$

In this new coordinate the n-th Frobenius endomorphism F_n is given by

$$\begin{aligned} F_n\colon \Lambda(R) &\longrightarrow \Lambda(R) \\ \prod_{d\geq 1}(1-\varepsilon_d t)^{-1} &\longrightarrow \prod_{d\geq 1}(1-\varepsilon_d^n t)^{-1}. \end{aligned} \tag{B.3.1}$$

We can obtain the expression of F_n in ghost coordinates if we take logarithmic derivative in both sides of (B.3.1) i.e.,

$$\begin{aligned} t\frac{d}{dt}\log(\prod_{d\geq 1}(1-\varepsilon_d t)^{-1}) &= -t\frac{d}{dt}\sum_{d\geq 1}\log(1-\varepsilon_d t) \\ &= -t\frac{d}{dt}\sum_{d\geq 1}\left(-\sum_{j\geq 1}(\frac{\varepsilon_d^j t^j}{j})\right) \\ &= \sum_{j\geq 1}(\sum_{d\geq 1}\varepsilon_d^j)t^j \end{aligned}$$

and similarly

$$t\frac{d}{dt}\log(\prod_{d\geq 1}(1-\varepsilon_d^n t)^{-1}) = \sum_{j\geq 1}(\sum_{d\geq 1}\varepsilon_d^{nj})t^j.$$

Define $p_j = \sum_{d\geq 1}\varepsilon_d^j$, then $p_{nj} = \sum_{d\geq 1}\varepsilon_d^{nj}$ and hence, $F_n(p_j) = p_{nj}$.

Remark B.3.1. We can think of $F_n\colon W(R)\to W(R)$ as the unique endomorphism such that $\omega_m(F_n(\mathfrak{a})) = \omega_{mn}(\mathfrak{a})$.

Verschiebung

The n-th Verschiebung endomorphism V_n is defined as

$$\begin{aligned} V_n\colon \Lambda(R) &\longrightarrow \Lambda(R) \\ f(t) &\longrightarrow f(t^n) \end{aligned} \tag{B.3.2}$$

i.e., taking the n-th power of t. Assume that $f(t) = \prod_d(1-a_d t^d)^{-1}$, then $V_n(f(t)) = \prod_d(1-a_d t^{nd})^{-1}$. The expression for V_n in ghost coordinates is obtainead after taking logarithmic derivative on both sides of (B.3.2) this yields

$$t\frac{d}{dt}\log(f(t)) = \sum_{m\geq 1}\omega_m(\mathfrak{a})t^m$$

and

$$t\frac{d}{dt}\log(f(t^n)) = \sum_{m\geq 1}n\omega_m(\mathfrak{a})t^{nm}$$

where, as bijection (B.2.2) states, $\mathfrak{a}$ is determined by the coefficients of $f(t)$. In ghost coordinates we have

$$V_n(\omega_m) = \begin{cases} n\omega_{m/n}, & \text{if } n \text{ divides } m \\ 0, & \text{else.} \end{cases} \tag{B.3.3}$$

Remark B.3.2. We can think of $V_n\colon W(R)\to W(R)$ as the unique homomorphism such that $V_n((a_m)) = (b_m)$ where

$$b_m = \begin{cases} a_{m/n}, & \text{if } n \text{ divides } m \\ 0, & \text{else.} \end{cases}$$

The following lemma characterizes F_n and V_n and establishes a general principle in the theory of Witt vectors, *to demonstrate certain equations it suffices to check them on vectors of the form* $1 - at$.

Lemma B.3.3. *[8] Let* R *be a ring. The endomorphisms* V_n *and* F_n *of* $\Lambda(R)$ *are characterized by the formulas:*

(a) $V_n(1 - at) = 1 - at^n$,

(b) $F_n(1 - at) = 1 - a^n t$.

The following list summarizes standard relations between Frobenius and Verschiebung endomorphisms.

Proposition B.3.4. *The following are identities between the endomorphisms* F_n *and* V_n *of* $W(R)$ *and* $\Lambda(R)$.

(a) $V_m \circ V_n = V_{mn}$ *for all* $m, n \in \mathbb{N}$,

(b) $F_m \circ F_n = F_{mn}$ *for all* $m, n \in \mathbb{N}$,

(c) if $d = g.c.d(m, n)$ *then* $F_m \circ V_n = dV_{n/d} \circ F_{m/d}$. *In particular, if* $d = 1$ *they commute,*

(d) (Frobenius reciprocity) for all $f(t), h(t) \in \Lambda(R)$ *we have*

$$V_n(f(t) * F_n(h(t))) = (V_n(f(t))) * h(t).$$

B.4 Relations with the K-theory of endomorphisms

We mentioned that characteristic polynomial and trace familiar from linear algebra have analogues for projective modules over a ring. We present here the work of G. Almkvist [1–3] and D. Grayson [18]. From now on we consider only commutative rings with unit.

Definition B.4.1. We define the category $\mathrm{END}(R)$ of pairs (P, f) where P is an object in $\mathcal{P}(R)$ and $P \xrightarrow{f} P$ is an endomorphism of P. A morphism between two objects $(P, f) \xrightarrow{F} (P', f')$ in $\mathrm{END}(R)$ is an R-module homomorphism $F\colon P \to P'$ such that $F \circ f = f' \circ F$. A sequence in $\mathrm{END}(R)$ is exact if its underlying sequence in $\mathcal{P}(R)$ is exact.

Reverse characteristic polynomial

Let (P, f) be an object in $\mathrm{END}(R)$. Since P is finitely generated projective R-module there exists Q such that $P \oplus Q \cong R^n$ is a free module of rank n, thus f can be extended to all R^n by the zero endomorphism in Q. The *reverse characteristic polynomial* of f is

$$\chi_t(f) = \det(1_n - t(f \oplus 0_Q)) \in R[t],$$

where 1_n is the identity endomorphism of R^n. This definition is independent of the choice of Q and for any exact sequence in $\mathrm{END}(R)$

$$\begin{tikzcd} 0 \arrow[r] & P' \arrow[r] \arrow[d, "f'"'] & P \arrow[r] \arrow[d, "f"'] & P'' \arrow[r] \arrow[d, "f''"'] & 0 \\ 0 \arrow[r] & P' \arrow[r] & P \arrow[r] & P'' \arrow[r] & 0 \end{tikzcd} \tag{B.4.1}$$

satisfies

$$\chi_t(f) = \chi_t(f')\chi_t(f''). \tag{B.4.2}$$

Recall that there exists an exact pairing of categories

$$\begin{aligned} \mathrm{END}(R) \times \mathrm{END}(R) &\longrightarrow \mathrm{END}(R) \\ ((P, f), (P', f')) &\mapsto (P \otimes_R P', f \otimes f') \end{aligned}$$

inducing a product in K-theory [43, Section 9]. In particular, $K_0(\mathrm{END}(R))$ is a ring. Moreover, there are split functors

$$\begin{aligned} \mathrm{END}(R) &\longrightarrow \mathcal{P}(R) \\ (P, f) &\mapsto P \end{aligned} \tag{B.4.3}$$

and

$$\begin{aligned} \mathcal{P}(R) &\longrightarrow \mathrm{END}(R) \\ P &\mapsto (P, 0) \end{aligned} \tag{B.4.4}$$

where 0 denotes the trivial endomorphism of P. Showing that $K_0(R)$ is a direct summand of $K_0(\mathrm{END}(R))$. Indeed, since tensorizing with the zero endomorphism gives the zero endomorphism, $K_0(R)$ is an ideal in $K_0(\mathrm{END}(R))$. We define $\mathrm{End}_0(R)$ to be the quotient ring

$$K_0(\mathrm{END}(R))/K_0(R).$$

We use the reverse characteristic polynomial χ_t in order to describe a map out of $\mathrm{End}_0(R)$. We want the target of this map to be a group such that, according to (B.4.2), for exact sequences in $\mathrm{END}(R)$ we also have $\chi_t(f') = \chi_t(f)\chi_t(f'')^{-1}$. We need to extend the target of χ_t in order that it constitutes an invariant for endomorphisms. To solve this consider the multiplicative group of rational functions with constant term 1

$$\tilde{R} = \left\{ \frac{1 + a_1 t + \cdots + a_n t^n}{1 + b_1 t + \cdots + b_m t^m} | a_i, b_j \in R \right\}.$$

G. Almkvist endowed $\tilde{R}$ with a product $*_R$ defined in Section B.2 and considered the ring homomorphism

$$\begin{aligned} \mathrm{End}_0(R) &\longrightarrow \tilde{R} \\ [P, f] &\mapsto \chi_t(f). \end{aligned}$$

For a diagram (B.4.1) it satisfies, $[P, f] - [P', f'] \mapsto \chi_t(f)/\chi_t(f')$. Moreover, it is an isomorphism [3, Theorem 3.3] and $\chi_t(f \otimes f') = \chi_t(f) * \chi_t(f')$ and $\chi_t(f \oplus f') = \chi_t(f)\chi_t(f')$. This completes the description of $\chi_t(f \otimes f') = \chi_t(f) * \chi_t(f')$ started in Section B.1.

Matrix interpretation of F_n and V_n

The ring $\tilde{R} \hookrightarrow \Lambda(R)$ embeds as subring and we have

$$\mathrm{End}_0(R) \xrightarrow[\chi_t]{\cong} \tilde{R} \hookrightarrow \Lambda(R) \xleftarrow[\varphi]{\cong} W(R).$$

This interpretation allows us to consider $\mathrm{End}_0(R)$ as a subring of $W(R)$, as such, we define F_n and V_n on the ring $\mathrm{End}_0(R)$.

Lemma B.3.3 characterizes F_n and V_n on elements $1 - at$. Using χ_t we define

- The n-th Frobenius map F_n is given by:

$$\begin{aligned} F_n\colon \mathrm{End}_0(R) &\longrightarrow \mathrm{End}_0(R)\\ [P,f] &\mapsto [P,f^n]. \end{aligned}$$

- The n-th Verschiebung map V_n is given by:

$$\begin{aligned} V_n\colon \mathrm{End}_0(R) &\longrightarrow \mathrm{End}_0(R)\\ [P,f] &\mapsto [P^n,V_n f] \end{aligned}$$

  where P^n is the direct sum of n copies of P and the endomorphism $V_n f$ is represented by matrix:

$$\begin{pmatrix} 0 & \dots & 0 & f\\ 1 & 0 & & \vdots\\ \vdots & \ddots & \ddots & 0\\ 0 & \dots & 1 & 0 \end{pmatrix}.$$

The following proposition illustrates how we proceed to prove formulas for $W(R)$ in terms of endomorphisms.

Proposition B.4.2. *[18]*

(a) $F_n(\mathfrak{a} * \mathfrak{b}) = F_n(\mathfrak{a}) * F_n(\mathfrak{b})$ *corresponds to the equality* $(f \otimes g)^n = f^n \otimes g^n$ *for endomorphisms* f *and* g*.*

(b) $F_n V_n = n$ *corresponds to the equality*

$$(V_n(f))^n = \begin{pmatrix} f & \dots & 0\\ \vdots & \ddots & \vdots\\ 0 & \dots & f \end{pmatrix}.$$

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Abstract

In this thesis, we study the Nil–groups appearing in the Bass–Heller–Swan decomposition on K-theory. These groups are hard to compute, but accessible through their module structure over $W(\mathbb{Z})$, the Witt vector ring of the integers $\mathbb{Z}$. We generalize this structure.

For a profinite group Γ, we endow $\mathcal{H}_*^?$, an equivariant homology theory with restriction, with a module structure over $\mathrm{B}(\Gamma)$, the Burnside ring of Γ. Then, we give conditions on $\mathcal{H}_*^?$ to extend its $\mathrm{B}(\Gamma)$-module structure to a module structure over $\mathbf{B}_M(\Gamma)$, the completed Burnside ring of Γ. We show that when $M = 1$ is the trivial monoid, Γ is the profinite completion of the infinite cyclic group and $\mathcal{H}_*^?$ is the equivariant homology theory in the formulation of the Farrell-Jones conjecture, then the $\mathbf{B}_M(\Gamma)$-module structure coincides with the $W(\mathbb{Z})$-module structure.

Zusammenfassung

In der vorliegenden Dissertation wurden die Nilgruppen untersucht, die in der Bass–Heller–Swan Zerlegung der K-Theorie auftauchen. Diese Gruppen sind zwar schwer zu berechnen, jedoch zugänglich durch ihre Modulstruktur über $W(\mathbb{Z})$, dem Wittvektor Ring der ganzen Zahlen $\mathbb{Z}$. Wir verallgemeinern diese Struktur.

Für eine profinit Gruppe Γ statten wir $\mathcal{H}_*^?$, eine äquivariante Homologietheorie mit Einschränkungen, mit einer Modulstruktur über $\mathrm{B}(\Gamma)$, dem Burnside Ring von Γ, aus. Dann geben wir Bedingungen für $\mathcal{H}_*^?$ an, um ihre $\mathrm{B}(\Gamma)$-Modulstruktur zu einer Modulstruktur über $\mathbf{B}_M(\Gamma)$, dem vervollständigte Burnside Ring von Γ, zu erweitern. Wir zeigen, dass, wenn $M = 1$ das triviale Monoid ist, Γ die profinit Vervollständigung der unendlichen zyklischen Gruppen ist und $\mathcal{H}_*^?$ die äquivariante Homologytheorie aus der Formulierung der Farrell-Jones Vermutung, dann die $\mathbf{B}_M(\Gamma)$-Modulstruktur mit der $W(\mathbb{Z})$-Modulstruktur übereinstimmt.

Curriculum Vitae

For privacy reasons, the curriculum vitae is not included in the electronic version of this thesis.

Erklärung

Hiermit erkläre ich, dass ich die vorliegende Arbeit selbständig und ohne unerlaubte Hilfe angefertigt habe. Alle verwendeten Hilfsmittel und Quellen sind im Literaturverzeichnis vollständig aufgefuhrt und die aus den benutzten Quellen wörtlich oder inhaltlich entommenen Stellen als solche kenntlich gemacht.

Berlin, 4. November 2015

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